Total Labellings of Some Connected Graphs

By

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A Thesis Presented to

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MS in Mathematics

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Declaration

I **<u>Mujahid Shehzad</u>** registration number **CIIT/SP17-RMT-014/LHR** hereby declare that I have produced the work presented in this thesis, during the scheduled period of study. I also declare that I have not taken any material from any source except referred to wherever due, that amount of plagiarism is within acceptable range. If a violation of HEC rules on research has occurred in this thesis, I shall be liable to punishable action under the plagiarism rules of the HEC.

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Certificate

It is certified that **<u>Mujahid Shehzad(CIIT/SP17-RMT-014/LHR)</u>** has carried out all the work related to this report under my supervision at the Department of Mathematics, COMSATS University Islamabad, Lahore Campus and the work fulfills the requirement for award of MS Mathematics degree.

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DEDICATED

To

My Family

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First of all, **I** am gratefull to **Almighty Allah**, Most Gracious and Merciful, for enabling me to complete research thesis.

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ABSTRACT

Ashraf et al. in [3] introduced the concept of *vertex H irregular labelling* and *edge H irregular labelling.* Let H be sub graph of given graph G. If each edge of graph G is contained in any subgraph H_i where $i = \{ 1,2,3,\ldots, k \}$ of G and if each H_i is isomorphic to given subgraph H. then the collection H_i of subgraphs of G is said to be *H-covering* of G.

They also introduced the concept of edge H irregular strength *ehs(G)* that is the minimum k for which the graph has total H irregular k labelling and vertex H irregular strength *vhs(G)* that is the minimum k for which the graph has total H irregular k labelling

The thesis is devoted to study the *total vertex* C_3 *irregular strength wheel graph* and *triangular ladder graph* taking subgraph C_3. We also found the *total vertex* C_4 *irregular strength ladder graph* and g*rid graph*. Furthermore, we find the *total vertex* C_8 *irregular strength* of a *connected graph* Q_n.

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Chapter 1

Introduction

In this chapter we will also focus on some notations and basic definitions to understand this study.

1.1 Graph

A *Graph G* is an ordered pair of set of vertices *V(G)* and a set of edges *E(G)* between vertices. Symbolically a graph written as $G = (V(G), E(G))$. Both sets are finite if not mentioned. If v_1 and v_2 are two vertices and the pair $\{v_1 , v_2\}$ *is an edge*. It can also be denoted as $e_1=v_1v_2$, it indicates that v_1 and v_2 have an edge e_1 between them.If two vertices are associated by an edge, they are said to be adjacent vertices. Edge connected by vertices are said to be *incident* on each of the vertex and *adjacent edges* are those edges which are incident on same end points and a vertex with zero incident edges is said to be *isolated vertex*.

A graph is shown in Figure 1.1.

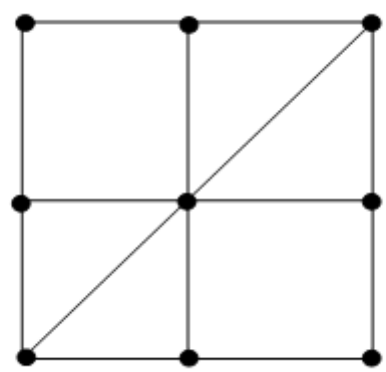


Figure 1.1: Graph

Two or more than two edges which have same end points are called *parallel edges*. When the edge is joined with the same vertex a *loop* is formed. A graph in which no loop exists is called multi graph and if neither loops nor parallel edges exist in a graph then it is said to be *simple* graph. Moreover the *order* of a graph in a graph is the number of vertices. Similarly *size* of a graph in the same graph numbers of all edges and number of edges incident on a vertex is called *degree* of the vertex and we denote it by *d(v)*. All the vertices adjacent to *v* is said to be neighborhood of *v* and is denoted by *N(v)*. An isolated vertex has degree zero. The smallest degree of a vertex among all vertices is said to be minimum degree of a graph a denoted by $\delta(G) = min\ d(v)$ where $v \in V(G)$ and similarlay maximum degree is denoted by $\Delta(G)$ i.e.

Let G be a simple Graph having n vertices and v be the vertex of G then the following inequality shows that graph can have minimum degree 0 and maximum degree *n-1*

$$0 \le \delta(G) \le d(v) \le \Delta(G) \le n-1$$

In a graph if we check the degrees of its components, each edge gives degree *2* because every edge is made by joining of two vertices. With the help of this information *Handshaking lemma.*

Let G be a graph then

$$\sum_{u \in V} d(u) = 2|E(G)|$$

1.2 Isomorphic Graphs

Let G and H be two graphs and If there exist a mapping s.t.

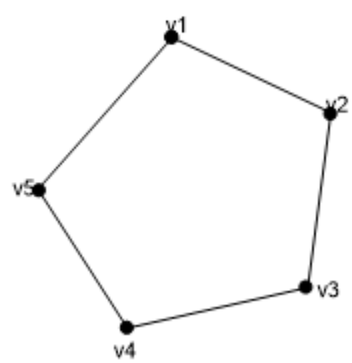
$$\beta : V(G) \to V(H) \;\; \text{i.e.}$$

$$uv \in E(H) \textit{ if and only if } \beta(u)\beta(v) \in E(G)$$

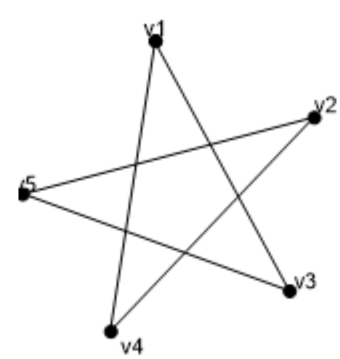
Then G and H are said to be isomorphic

We use notion $G \cong H$ to show that G is isomorphic to H.If there does not exist any bijective function between two graphs then graphs are said to be *non isomorphic* graphs. More generally for isomorphism if one graph has some property then the other graph must also have this property. For example if G contain a cycle having length k then H should also must contain the cycle having length k. Degrees of vertices of G should also be the degrees of vertices of H. Vertices of G should be the equal to the number of vertices of H. Edges of G should be equal to the number of edges of H . If a single property does not hold in two graphs they are not *isomorphic* [9].

Isomorphic graphs are shown in Figure 1.2.



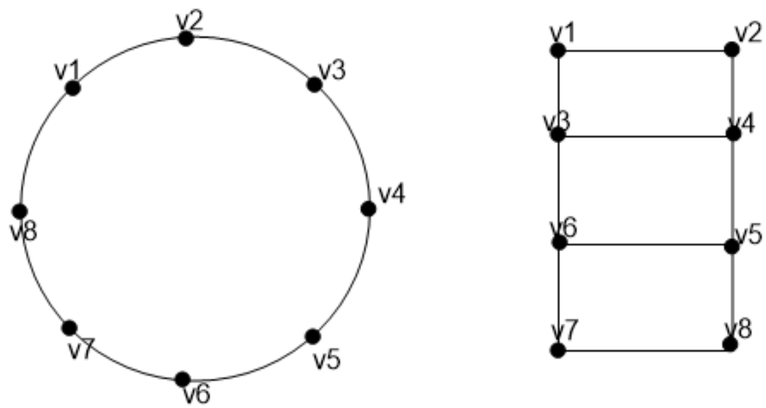
G_1



G_2

Figure 1.2 : isomorphic graphs

As the degrees of vertices , number of edges , number of vertices and number of cycle are same in G_1 and G_2. So both graphs are isomorphic. Now Figure 1.3.



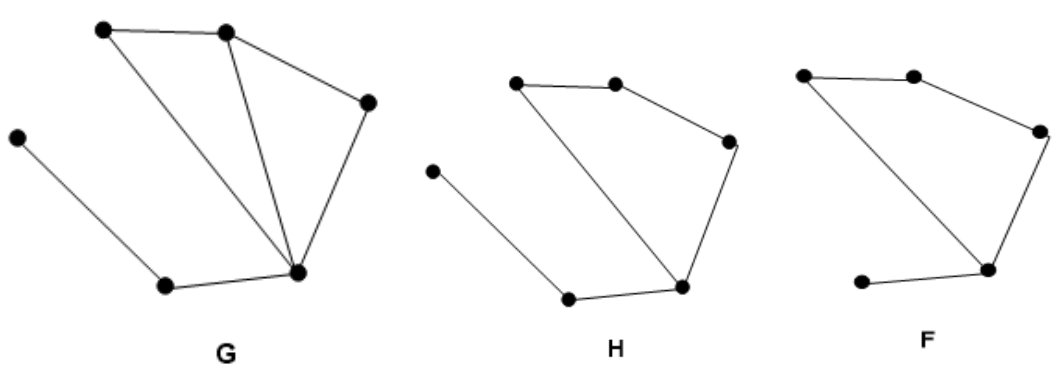
G *H*

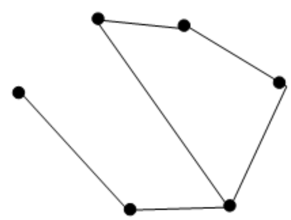
Figure 1.3 : non isomorphic graphs (*G* and *H*)

The above graphs are non isomorphic because their sizes are not equal. Where *H* contains three cycle of C_4 and there is no cycle of C_4 in *G*. So *G* and *H* are not isomorphic.

1.3 Subgraphs

Suppose *G* and *H* be two graphs where *V(G)* is the vertex set of *G* and *E(G)* is the edge set of *G*, *V(H)* is the vertex set of *H* and *E(H)* is the edge set of two graphs respectively. If the edges of *H* is contained in edges of *G* and vertices of *H* are contained in vertices of *G*. Then we will say that *H* is the *subgraph* of *G*. Moreover consider another subgraph *L* of *G* and let vertices of *L* is the proper subset of vertices of *G* and edges of *L* is also proper subset of edges of *G* then *L* is called *proper subgraph* of *G*. if *V(L) = V(G)* then L is called *spanning* subgraph of *G*. As shown in Figure 1.4.





K

Figure 1.4 : Subgraphs of Graph G

1.4 Operations in Graphs

Consider two graphs G_1 and G_2, we can find sum and product of two graphs by following definitions. Sum of G_1 and G_2 is a graph in which vertex set consist of $V(G_1) \cup V(G_2)$ and edge set consists of edges of G_1, edges of G_2 and edges which are made by joining all vertices of G_1 and G_2 [23] . Similarly cartesian product of these two graphs

$$G_1 = (V_1 , E_1) \text{ and } G_2 = (V_2 , E_2)$$

is a graph G such that $G = G_1 \times G_2$ has a vertex set

$$V_1 \times V_2 = \{(v_1 , v_2) \mid v_1 \in V_1, v_2 \in V_2\}$$

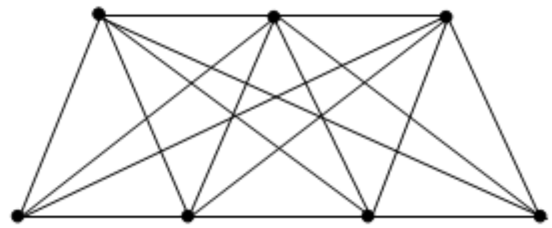
And has a edge set E consisting of elements uv such that

$$uv = \{(u_1 , v_1) (u_2 , v_2) \text{ where either } u_1 = v_1 \text{ and } u_2v_2 \in E_2 \text{ or } u_2 = v_2 \text{ and } u_1v_1 \in E_1\}$$

lets take two graphs P_3 and P_4, we calculate sum of these graphs. As shown in Figure 1.5.

P_3

P_4



P_3+P_4

Figure 1.5: Sum of Path graph

1.5 Walks in Graphs

A $(m-n)$ *walk* is a graph G which is made by a finite sequence of vertices in G which starts from m and ends at n such that consective vertices are adjacent. *i.e* $W : u_0, u_1, u_2, u_3, \ldots, u_r = n$ where $r \geq 0$ and u_i and u_{i+1} are adjacent in G. Total numbers of edgesin a walk is said to be *length* of a walk. Vertices and edges can be repeated in a walk and the $m-n$ walk is open if $m \neq n$ or walk is closed if $m = n$.

Walk : i, r, k, t, n, L, s, m, k, r, i, o, q, j, s

A walk is said to be *trail* if no edge is repeated but vertex can be repeated.

Trail : i, o, q, j, p, r, k, t, n, L, s, m, k

Moreover a walk is said to be *path* if no vertex is repeated. An isolated vertex is itself a triavial $(m-m)$ path.

Path : i, o, q, j, p , r, k, t, n, L, s, m

A path in which first and last vertex is same is called *Cycle* .See Figure 1.6.

Cycle : i, o, t, k, m, s, j, q, n, i

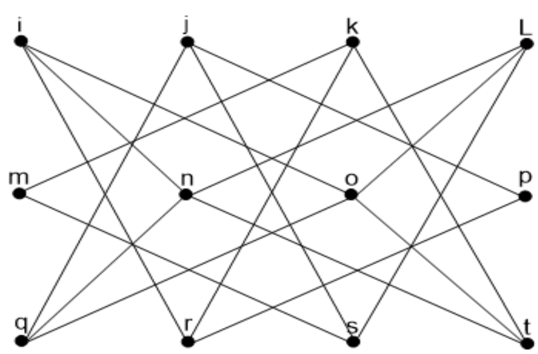


Figure 1.6 : Walk , Trail , Path and Cycle

A cycle is called *Even cycle* if its length is even and a cycle having odd length is said to be *Odd cycle*. A graph which consists of no cycles is known as *Acyclic graph*.

Tree

A connected graph having no cycles is said to be *tree*. *Caterpiller graph* is the example of tree graph. In Caterpiller Graph if we remove its endpoints it gives us a path.

There are some results about tree graphs.

- Tree consisting of a single edge has only two vertices.
- A tree containing k vertices has exactly $k-1$ edges.
- A connected graph G must have a spanning tree.

Connected Graphs

Consider a Graph $G = (V, E)$ and $o, p \in V(G)$ and o is *Connected* to p if ∃ a $(o - p)$ path in *G*. *Connected* graph is a graph in which a path exist between every adjacent vertices of *G*. Similarly in a graph *G* if there does not exist a path between any two vertices of *G* then it is said to be *Disconnected graph*. Some graphs are constructed in such a way that remowal of single edge or vertex can produce disconnected graph. A vertex *m* is said to be *Cut Vertex* if by removing it graph *G* is changed to disconnected graph. Similarly an edge is said to be *Bridge* if by removing it given graph becomes disconnected. See Figure 1.7 if we remove the edge *bc* then the graph is disconnected so *bc* is *bridge* and see Figure 1.8 the vertex *d* is *cut vertex*.

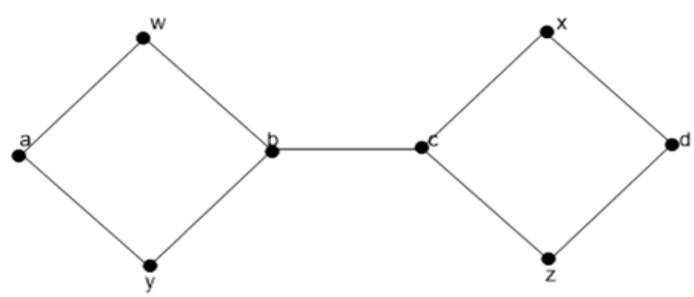


Figure: 1.7: Edge Connected Graph

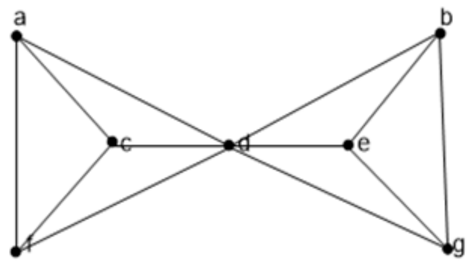


Figure 1.8: Vertex Connected Graph

1.6 Standard Families of Graphs

There exist many families of graphs and structures of graphs. Here some of them has been discussed. P_n is the family of *Paths.* A path has k vertices and one less than k edges. as shown in Figure 1.9

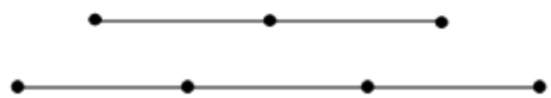
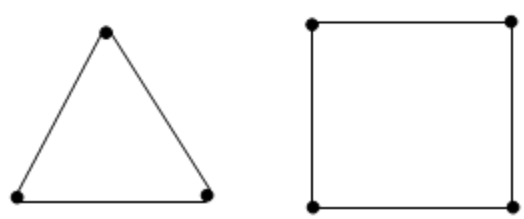


Figure 1.9: Family of Path Graph

Cycles

C_n denotes the family of *Cycles* which contains n vertices and n edges, $n \geq 3$. As shown in the figure 1. cycles C_n , $3 \leq n \leq 6$**.** Figure 1.10 shows the Family of Cycles



C_3 C_4

Figure 1.10 : Family of Cycles

k-Regular Graphs

Consider we have a graph G and if degree of each vertex in G is same then we can say that G is *Regular*. In a graph if every vertex has degree k then the graph is asaid to be k-regular graph. It is mathematically written as $d(v) = k$**.** There are some examples of *k-Regular Graph*. See Figure 1.11

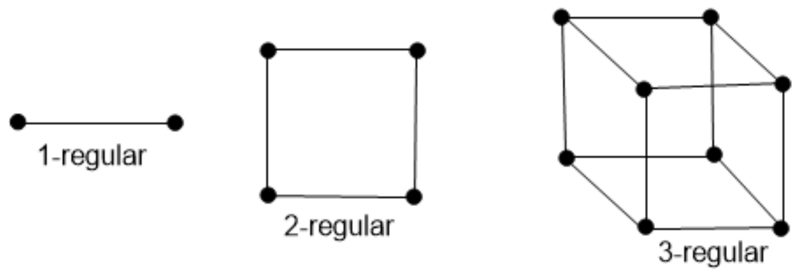


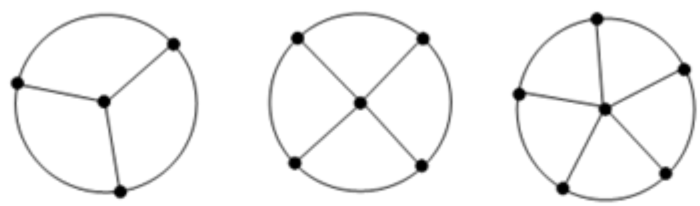
Figure 1.11: *k*-Regular graphs

Complete Graph

K_n denotes the family of complete graphs which is consisted of *m* vertices have one edge between all distinct vertices of *G*.

Wheel Graph

When we an additional vertex is added to C_n we obtain *Wheel graph*, add join this with all other vertices with new edges. It is denoted by W_n . the Figure 1.12 shows W_n , $3 \leq n \leq 5$.

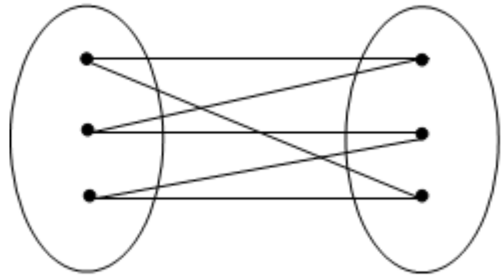


W_3 W_4 W_5

Figure 1.12: Family of Wheel graph

Bipartite graph

A *Bipartite graph* is a graph if the set of vertices is devided into two sets *A* and *B* such that there does not exist edge between the vertices of *A* and no edge between the vertices of *B*. We will show in the next example that C_6 is bipartite graph. We can devide the vertex set of C_6 into two sets $V_1 = \{v_1, v_3, v_5\}$ and $V_2 = \{v_2, v_4, v_6\}$ and each edge of C_6 is connected with a vertex of V_1 and a vertex of V_2.



V_1 V_2

Figure 1.13 : Showing that C_6 is Bipartite

Ladder Graph

A *ladder* graph is obtained by taking cartesian product of two path graphs P_2 and P_n . It is denoted by L_n . It is mathematically written as $L_n = P_2 \times P_n$. It has *2n* vertices and *3n-2* edges as shown in Figure 1.14,

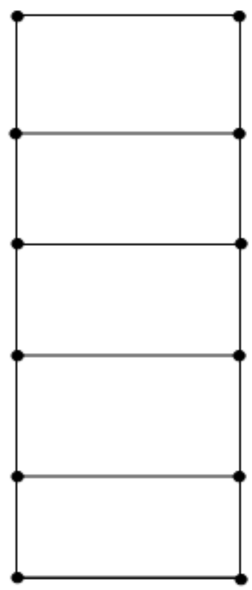


Figure 1.14: Ladder Graph

Grid Graph

A *grid* graph is obtained by taking cartesian product of two path graphs P_n and P_m . It is denoted by G_{mn} . it is mathematically written as $G_{mn} = P_m \times P_n$. It has *mn* vertices and *2mn-m-n* edges as shown in Figure 1.15.

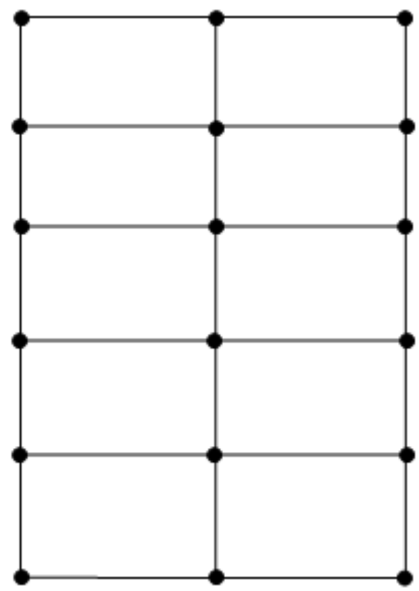


Figure 1.15: Grid Graph

Triangular Ladder Graph

A *triangular ladder* graph is obtained by adding one edge as a diagonal in each C_4 of ladder graph. It is denoted by TL_n .It has $2n$ vertices and $4n-3$ edges as shown in Figure 1.16.

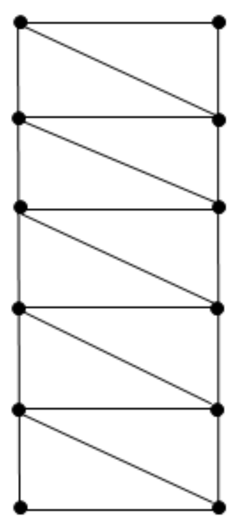


Figure 1.16: Triangular Ladder Graph

Connected Graph

Q_n be a connected which is obrained by an edge amalgamation of $n \geq 2$ C_8-cycles as shown in Figure 1.17.

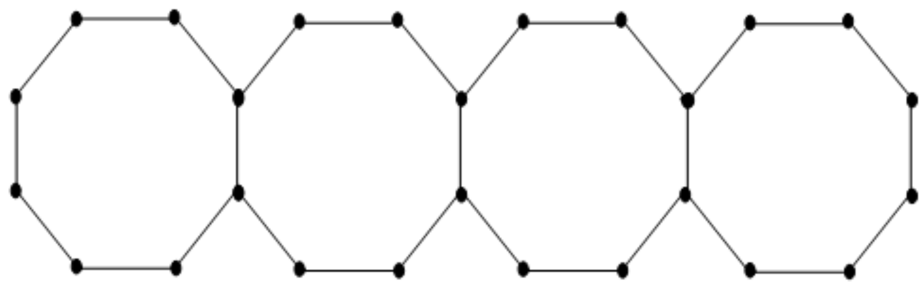


Figure 1.17: Q_n

Chapter 2

Graph Labelling

2.1 Introduction

The thought of graph labelling is presented over forty years back. Many papers on graph labelling has been flashed and many results and publications are blared. There were many mathematitians who took many problems related to graph theory. However , in high innovations the significance of graph labelling is additionally developing and these structures are studied seriously at present.

In this second unit graph labelling of different kinds are talked about. We will clarify their definitions and verifiable history. The majority of the work on graph labelling and techniques for graph labelling started in mid 1960s [21]. Dynamic survey of graph labelling written by [13] is a very good source of information.

2.2 Labelling

A function in which usually positive integers are assigned to the graph elements is said to be labelling. If the domain of the function are vertices then it is said to be *Vertex Labelling* and if the domain is set of edges then it is said to be *Edge Labelling* and if both edges and vertices are used in domain of the function then it is said to be *Total Labelling*.Generally mapping creates the partial sum of labeled elements. Set of vertex weight or a set of edge weight are called *Partial sum*. Where we can calculate *Vertex Weight* generally by partial sum of labels of vertices and its adjacent labeled edges and *Edge Weight* can be calculated generally by partial sum of labels of edges and its adjacent labeled vertices

2.3 Magic Type Labelling

Almost forty five years ago, Sedlacek in [24] gave the idea of *Magic Labelling*. In the defined labelling if the weights of all edges are not different then graph is said to be *Edge Magic Labeled* and similarly a graph is said to be *Vertex Magic* if the weight of all vertices is same. We can study many new results which give us a comprehensive and satisfactory approach to magic labelling in ([5-7]). A characterization of regular magic graph was given by Doob in [11].Now we will discuss the total magic labelling which is consisted of the edge magic total labelling and the vertex magic total labelling.

2.3.1 Edge-Magic Total Labelling

Most importantly edge magic total labelling was clarified by Ringle and L Llado [23]. It was presented in *1970*. Initially it was described as magic labelling but later mathematitians have same opinion with the stewart to consider magic labelling as They described this as magic labelling as total edge magic labelling. A bijectitive map α is called *edge magic total labelling* of graph G is a map from $V(G) \cup E(G)$ onto the set of integers $\{1, 2, 3, \ldots, |V| + |E|\}$, such that $\alpha(m) + \alpha(mn) + \alpha(n) = p$ for some magic constatnt p , for every edge $mn \in$ E(G). Rosa and Kotzig described described that all cycles C_n and all path P_n are edge magic total. As shown in Figure 2.1 and 2.2.

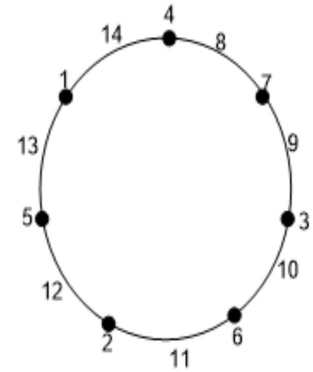


Figure 2.1 : Edge Magic Total Labelling of C_7

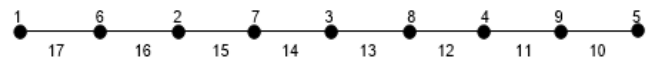


Figure 2.2 : Edge Magic Total Labelling of P_9

2.3.2 Vertex-Magic Total Labelling

In 1999 Macdougall et al. [18] discussed of vertex-magic labelling. A bijective mapping ϕ is called *Vertex-Magic Total Labelling* of a graph G from $V(G) \cup E(G) \to \{1, 2, 3, \ldots, k\}$, for every $v \in V(G)$ if $\phi(v) + \phi(vr) = q$ for some magic constant q, where sum is all of incident edges to vertex *v*. A *Vertex Magic Total Graph* is a graph which posses the vertex-magic labelling [14]. Grey has calculated new construction Methods for vertex-magic total labellings of graphs [15]. Magic labellings are shown in Figures 2.3 and 2.4.

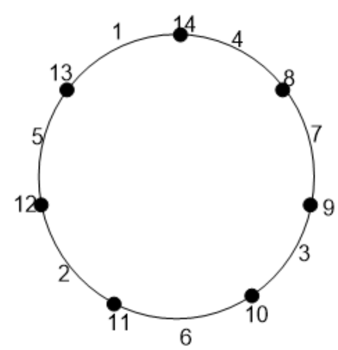


Figure 2.3 : Vertex Magic Total Labelling Of C_7

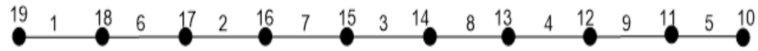


Figure 2.4 : Vertex Magic Total Labelling Of P_9

2.3.3 Super Magic Total Labelling

Super edge magic total labelling is defined by Stewart. A *super edge magic total labelling* is that edge magic total labelling in which the smallest positive labels appears on the edges and A *super vertex magic total labelling* is that vertex magic total labelling in which the smallest positive labels appears on the vertices. Jeyanthi has determined Supermagic coverings of some simple graphs [16].

2.4 Irregular Labelling

A labelling of the elements of a graph in which weights of the elements of the graph are different is called irregular labelling. It has two types. First is vertex irregular labelling and second is edge irregular labelling and third is total irregular labelling. Aigner et al. calculated irregular labelling of trees and forests [2]. Anholcer and Palmer has found irregular labelling of circulant graphs [4].

2.4.1 Edge-Irregular Labelling

Propelled by the papers on the irregularity strength Jendrol , Baca ,Ryan and Miller began to examine about *Total Edge Irregularity Strength* of the graphs and characterized as pursues :

A total labelling φ of a graph *G* is called a *Total Edge Irregular K Labelling* if for any two edges *e and f wt(e)* and *wt(f)* are distinct. Where the weight of edge *e* is labels of vertices *m* and *n* adjacent to *e*.

$$wt(e) = \varphi(e) + \varphi(m) + \varphi(n)$$

The smallest integer *k* for which a graph has an edge irregular total k labelling is called *Total Edge Irregularity Strength of G*. The *total edge irregularity strength* is denoted by *tes(G)*. Bohman et al. has found irregular strength of trees [8].

2.4.2 Vertex-Irregular Labelling

Chartland et al in [19] proposed the problem, assigns positive integers labels to the edges of a connected graph of order at least *3* in such a way that graphs become irregular , i.e the weight at each vertex is different. Motivated by the notion of the irregularity strength of Graph and various kinds of other total labelling, Jendrol , Baca ,Ryan and Miller [10] introduced the *Total Vertex Irregular Labelling* as follows: We define labelling such that a labelling $\emptyset: V \cup E \rightarrow \{1, 2, 3, \ldots, k\}$ is said to be *total vertex k labelling* if

$wt(x) \neq wt(y)$ for all x,y ∈ V. Where the weight of any vertex x in the labelling $\emptyset$ is given as

$$wt(x) = \emptyset(x) + \textstyle\sum_{y \in N(x)} \emptyset(xy),$$

Here $N(x)$ is the set of neibour of *x.* the smallest integer k which arrives intotal vertex k labelling is called *total vertex irregularity strength* which is denoted by *tvs (G).* Irregular labelling of different graphs can be seen in Figure 2.5.

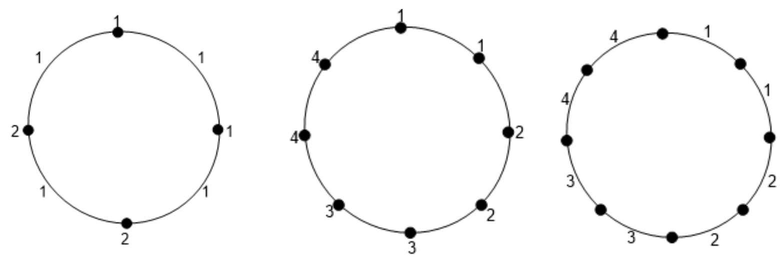


Figure 2.5: Irregular labelling

Baca et al. has calculated many results on different families of graphs in [10]. Where *tvs(G)* is the total vertex irregularity strength of graph *G*. The real motivation for the definition of total vertex irregularity strength of graph *G* was defined in [19] by Chartland. Baca et al. calculated lower bound which is given below

$$s(G) \geq \max_{1 \leq i \leq \Delta} \frac{i - 1 + n_i}{i}$$

Where n_i indicates quantity of vertices having degree *n* where Δ is maximum degree of the graph .

Lehel and Faudree [12] proved that if G is *m-Regular*, $m \geq 2$,

$$\lceil n + m - 1)/m \rceil \leq s(G) \leq \left\lceil \frac{n}{2} + 9 \right\rceil$$

and they showed that $s(G) \leq \left\lceil \frac{n}{m} + c \right\rceil$ for some constant *c*.

Przybylo [20] demonstrate that $s(G) \leq \left\lceil \frac{16n}{m} + 6 \right\rceil$

Kalkowski, Karonski and Pfender [17] show to be true that $s(G) \leq \frac{16n}{\delta} + 6$. where δ is the minimum degree of graph G.

Ahmad, Al-Mushayt and Bača has found many results on *tes* and *tvs* in [1]. They calculated the lower bound as follows.

$$s(G) \leq \left\{ \left\lceil \frac{E(G)+1}{2} \right\rceil, \Delta(G) \right\}$$

In [1] edge strength of path *(P_n)* , star *($K_{1,n}$)* , double star *($K_{m,n}$)* and cross product of two paths *($P_n x P_m$)* is determined as

$$es(P_n) = \left\lceil \frac{n}{2} \right\rceil$$

$$es(K_{1,n}) = n$$

$$es(K_{n,m}) = n$$

$$es(P_n \; x \; P_m) = \left\lceil \frac{(2mn - m - n + 1)}{2} \right\rceil$$

2.5 H-Covering

Let *H* be sub graph of given graph G. If each edge of graph *G* is contained in any subgraph H_i where $i = \{ 1,2,3,\ldots, k \}$ of *G* and if each H_i is isomorphic to given subgraph *H*. then the collection H_i of subgraphs of *G* is said to be *H-covering* of *G*.

2.6 Vertex H-Irregularity Strength

A new term *vertex H- irregularity strength* of a graph was defined by Ashraf et al in [3]. They defined in such a way that let G be a graph having properties of H covering and $H \subseteq G$. A vertex k labelling ω is defined in such a way that

$$\omega : V \to \{ 1, 2, 3, \ldots k \}$$

And associated weight of subgraph H is defined as

$$wt(H) = \sum_{v \in v(H)} \omega(v)$$

Then a vertex *k* labelling is said to be *H-Irregular Vertex k Labelling* if the weights of two distinct subgraphs H_i and H_j are different where $i \neq j$. The smallest integer *k* label which arrives in H-irregular vertex *k* labelling is said to be *Vertex H-Irregular Strength* and it is denoted by *vhs(G).*

2.7 Edge H-Irregularity Strength

Similarly A edge k labelling σ is defined in such a way that

$$\sigma: E \to \{ 1, 2, 3, \dots k \}$$

And associated weight of subgraph *H* is defined as

$$wt(H) = \sum_{e \in E(H)} \sigma(e)$$

Then a edge k labelling is said to *H-irregular Edge k Labelling* if the weights of two distinct subgraphs H_i and H_j are different where $i \neq j$. The smallest integer *k* label which arrives in H-irregular edge k labelling is said to be *Edge H-Irregular Strength* and it is denoted by *ehs(G).*

2.8 Lower Bounds to Calculate ehs and vhs:

Ashraf et al calculated *ehs(G)* and *vhs(G)* and gave the following results.

$$ehs\ (G, H) \geq \left\lceil 1 + \frac{t-1}{E(H)} \right\rceil$$

$$vhs\ (G, H) \geq \left\lceil 1 + \frac{t-1}{V(H)} \right\rceil$$

Here *t* is the number of isomorphic subgraphs of *G* and $E(H)$ and $V(H)$ are number of edges and vertices respectively.

Theorem: Let *m* and *n* be two integers, $2 \leq m \leq n$, then

$$ehs(P_n, P_m) = \left\lceil \frac{n-1}{m-1} \right\rceil$$

Theorem: Let m and n be two integers, $2 \leq m \leq n$, then

$$vhs(P_n, P_m) = \left\lceil \frac{n}{m} \right\rceil$$

Theorem: Let Ladder graph $L_n = P_n x\ P_2\ \forall\ n \geq 2$,, then

$$ehs(L_n, C_m) = \left\lceil \frac{m+2n}{2m} \right\rceil$$

Chapter 3

Total Labellings of Some Connected Graphs

Theorem: Let $L_n = P_n x\ P_2 be\ a\ Ladder\ graph, \forall\ n \geq 2$, Then

$$vhs(\ L_n, C_4) = \left\lceil \frac{n+2}{4} \right\rceil$$

Proof: Let $L_n = P_n x\ P_2$ be a Ladder graph. This graph has $2n$ vertices. We denotes the vertices by u_β^α , $\alpha = 1{,}2$ and $\beta = \{\ 1\ ,2\ ,3\ , \dots, n\ \}$ and $|V(L_n)| = 2n$

We denotes the edges by $u_\beta^\alpha u_{\beta+1}^\alpha$.

$$u_\beta^\alpha u_{\beta+1}^\alpha : \beta = \{\alpha = 1{,}2\}\ and\ \{\ 1\ ,2\ ,3\ , \dots , n-1\ \} \cup$$

$$\{u_\beta^\alpha u_{\beta+1}^\alpha : \{\alpha = 1{,}2\} and\ \beta = \{\ 1\ ,2\ ,3\ , \dots , n\ \}$$

$$|E(L_n)| = 2n$$

We define a labelling such that $\sigma :\ u_\beta^\alpha\ \rightarrow \{\ 1\ ,2\ ,3\ , \dots , k\ \}$ i.e

$$\sigma(u_\beta^\alpha) = \left\lfloor \frac{\beta+(-1)^\alpha}{4} + 1 \right\rfloor$$

After giving labels of the vertices of the ladder graph. Weight of every C_4 is,

$$w(C_4^\gamma) = \sigma(u_\gamma^1) + \sigma(u_{\gamma+1}^1) + \sigma(u_\gamma^2) + \sigma(u_{\gamma+1}^2)$$

$$w(C_4^\gamma) = \left\lfloor \frac{\gamma-1}{4} \right\rfloor + \left\lfloor \frac{\gamma}{4} \right\rfloor + \left\lfloor \frac{\gamma+1}{4} \right\rfloor + \left\lfloor \frac{\gamma+2}{4} \right\rfloor$$

This shows that

$$w(C_4^\gamma) = \gamma - 1 \text{, where } \gamma = \{\ 1\ ,2\ ,3\ , \dots , n-1\ \}$$

Thus for the ladder graph $\mathrm{L_n = P_n x\ P_2}$

$$vhs(\ L_n, C_4) \leq \left\lceil \frac{n+2}{4} \right\rceil \qquad (1)$$

Ashraf et al. proved the lower bound of ladder graph, which is given below

$$vhs(\ L_n, C_4) \geq \left\lceil \frac{n+2}{4} \right\rceil \qquad (2)$$

Thus from (1) and (2)

$$vhs(\ L_n, C_4) = \left\lceil \frac{n+2}{4} \right\rceil$$

Theorem: Let TL_n *be a triangular ladder graph* , $\forall\, n \geq 2$, Then

$$vhs(\,\mathrm{T}L_n, C_3) \;=\; \left\lceil \frac{2n}{3} \right\rceil$$

Proof: let TL_n be a triangular Ladder graph. This graph has *2n* vertices. We denote the vertices by u_β^α , $\alpha = 1{,}2$ and $\beta = \{\,1\,,2\,,3\,,\ldots,n\,\}$ and $|V(L_n)| = 2n$

We denote the edges by $u_\beta^\alpha u_{\beta+1}^\alpha$.

$$u_\beta^\alpha u_{\beta+1}^\alpha : \{\alpha = 1{,}2\}\ and\ \beta = \{\,1\,,2\,,3\,,\ldots,n-1\,\} \cup$$

$$u_\beta^\alpha u_\beta^{\alpha+1} : \{\alpha = 1\}\ and\ \beta = \{\,1\,,2\,,3\,,\ldots,n\,\} \;\cup$$

$$\{u_{\beta+1}^\alpha u_{\beta+1}^{\alpha+1} : \{\alpha = 1\} and\ \beta = \{\,1\,,2\,,3\,,\ldots,n-1\,\}$$

$$|E(TL_n)| = 4n - 3$$

We define a labelling such that $\sigma :\ u_\beta^\alpha\ \rightarrow \{1\,,2\,,3\,,\ldots,k\,\}$ i.e

$$\sigma(u_\beta^\alpha) = \left\lfloor 1 + \frac{\alpha+2\beta-3}{3} \right\rfloor$$

After giving labels of the vertices of the triangular ladder graph . Weight of every C_3 is,

$$w(C_3^\gamma) = \sigma\left(u^1_{\left\lfloor\frac{\gamma+2}{2}\right\rfloor}\right) + \sigma\left(u^1_{\left\lfloor\frac{\gamma+3}{2}\right\rfloor}\right) + \sigma\left(u^2_{\left\lfloor\frac{\gamma+2}{2}\right\rfloor}\right)$$

This shows that

$$w(C_3^\gamma) =\ \gamma + 2, \text{ where } \gamma = \{\,1\,,3\,,\ 5\,,\ldots,2n-3\,\}$$

$$w(C_3^\gamma) = \sigma\left(u^1_{\left\lfloor\frac{\gamma+2}{2}\right\rfloor}\right) + \sigma\left(u^2_{\left\lfloor\frac{\gamma}{2}\right\rfloor}\right) + \sigma\left(u^2_{\left\lfloor\frac{\gamma+2}{2}\right\rfloor}\right)$$

This shows that

$$w(C_3^\gamma) =\ \gamma + 2\ , \text{ where } \gamma = \{\,2\,,4\,,\ 6\,,\ldots,2n-2\,\}$$

Thus for the triangular ladder graph

$$vhs(\,\mathrm{T}L_n, C_3) \leq\ \left\lceil \frac{2n}{3} \right\rceil \ldots\ldots\ldots\ldots..(1)$$

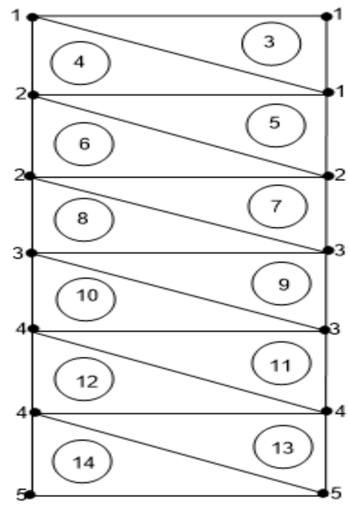
Ashraf et al. proved the lower bound of triangular ladder graph, which is given below

$$vhs(\mathrm{T}L_n, C_3) \geq \left\lceil \frac{2n}{3} \right\rceil \ldots\ldots\ldots\ldots\ldots(2)$$

Thus from (1) and (2)

$$vhs(\mathrm{T}L_n, C_3) = \left\lceil \frac{2n}{3} \right\rceil$$

Example:



Theorem: The *wheel graph* W_n, admits C_3-irregular labelling for $n \geq 6$.

Proof: Let W_n be a Wheel graph. This graph has *n* vertices. We denotes the vertices by v_a, $a = \{1, 2, 3, \ldots, n\}$ and $|V(W_n)| = n$

We denote the edges by $v_a v_c$.

$$v_a v_c : a = \{1, 2, 3, \ldots, n-1\} \cup$$

$$v_a v_c : a = \{1, 2, 3, \ldots, n-2\} \cup v_{n-1} v_n$$

$$|E(W_n)| = 2n - 2$$

Case 1: W_n with $n = 4m + 2$ admits C_3-irregular labelling with *c*. which is given below

$$c = \frac{n}{2}$$

We define a labelling such that $\alpha : v_a \rightarrow \{1,2,3, \ldots, k\}$ i.e

For the vertices of wheel graph with $n = 4m + 2$

$$\alpha(v_a) = \begin{cases} a, & \text{if } 1 \leq a \leq \frac{n}{2} \quad ; a - odd \\ a-1, & \text{if } 2 \leq a \leq \frac{n+2}{2} \quad ; a - even \\ n-a+1, & \text{if } \frac{n+4}{2} \leq a \leq n-1 \\ 1, & \text{if } n = a, \end{cases}$$

After giving labels of the vertices of the wheel graph. Weight of every C_3 is,

$$w(C_3^{\gamma}) = \begin{cases} \alpha(v_n) + \alpha(v_\gamma) + \alpha(v_{\gamma+1}), & \text{if } 1 \leq \gamma \leq n-2 \\ \alpha(v_n) + \alpha(v_{n-1}) + \alpha(v_1), & \text{if } \gamma = \text{n} - 1 \end{cases}$$

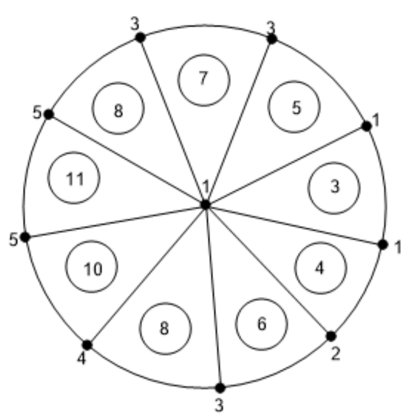
This shows that

$$w(C_3^{\gamma}) = \begin{cases} 2\gamma + 1, & \text{if } 1 \leq \gamma \leq \frac{n}{2} \\ 2(n-\gamma) + 2, & \text{if } \frac{\text{n}+2}{2} \leq \gamma \leq n-1 \end{cases}$$

Thus for the wheel graph

$$c = \frac{n}{2}$$

Example:



Case 2: W_n with $n = 4m + 3$ admits C_3-irregular labelling with c . which is given below

$$c = \frac{n+1}{2}$$

We define a labelling such that $\alpha\colon v_a \rightarrow \{1,2,3,\ldots,k\}$i.e

For the vertices of wheel graph with $n = 4m + 3$

$$\alpha(v_a) = \begin{cases} a, & \text{if } 1 \le a \le \frac{n-1}{2} \quad ; a - odd \\ a-1, & \text{if } 2 \le a \le \frac{n-3}{2} \quad ; a - even \\ \frac{\mathrm{n}+1}{2}, & \text{if } \frac{\mathrm{n}+1}{2} \le a \le \frac{\mathrm{n}+3}{2} \\ n-a+1 & \text{if } \frac{\mathrm{n}+5}{2} \le a \le \mathrm{n}-1 \\ 1, & \text{if } n = a \end{cases}$$

After giving labels of the vertices of the wheel graph . Weight of every C_3 is,

$$w(C_3^{\gamma}) = \begin{cases} \alpha(v_n) + \alpha(v_{\gamma}) + \alpha(v_{\gamma+1}), & \text{if } 1 \le \gamma \le n-2 \\ \alpha(v_n) + \alpha(v_{n-1}) + \alpha(v_1)\,, & \text{if } \gamma = \mathrm{n}-1 \end{cases}$$

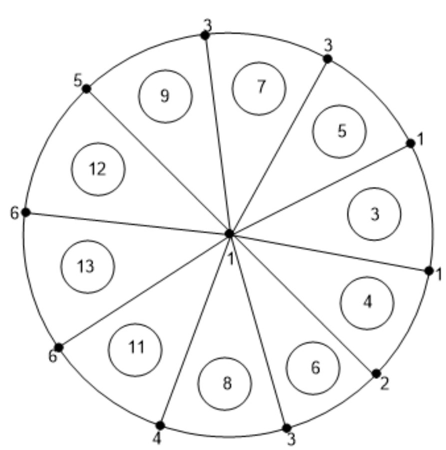
This shows that

$$w(C_3^{\gamma}) = \begin{cases} 2\gamma + 1, & \text{if } \quad 1 \le \gamma \le \frac{\mathrm{n}-3}{2} \\ n+1, & \text{if } \quad \gamma = \frac{n-1}{2} \\ 2(n-\gamma) + 3, & \text{if } \quad \frac{\mathrm{n}+1}{2} \le \gamma \le \frac{n+3}{2} \\ 2(n-\gamma) + 2, & \text{if } \quad \frac{\mathrm{n}+5}{2} \le \gamma \le n-1 \end{cases}$$

Thus for the wheel graph

$$c = \frac{n+1}{2}$$

Example:



Case 3: W_n with $n = 4m + 4$ admits C_3-irregular labelling with c. which is given below

$$c = \frac{n+2}{2}$$

We define a labelling such that α: $v_a \rightarrow \{1,2,3, \dots, k\}$i.e

For the vertices of wheel graph with $n = 4m + 4$

$$\alpha(v_a) = \begin{cases} a, & \text{if } 1 \leq a \leq \frac{n+2}{2} \quad ; a - odd \\ a - 1 & \text{if } 2 \leq a \leq \frac{n+4}{2} \quad ; a - even \\ n - a + 1 & \text{if } \frac{n+6}{2} \leq a \leq n - 1 \\ 1 & \text{if } a = n \end{cases}$$

After giving labels of the vertices of the wheel graph . Weight of every C_3 is,

$$w(C_3^\gamma) = \begin{cases} \alpha(v_n) + \alpha(v_\gamma) + \alpha(v_{\gamma+1}), & \text{if } 1 \leq \gamma \leq n - 2 \\ \alpha(v_n) + \alpha(v_{n-1}) + \alpha(v_1), & \text{if } \gamma = \mathrm{n} - 1 \end{cases}$$

This shows that

$$w(C_3^\gamma) = \begin{cases} 2\gamma + 1, & \text{if} \quad 1 \leq \gamma \leq \frac{n+2}{2} \\ 2\gamma - 4, & \text{if} \quad \gamma = 2s + 4;\ \forall\, s \in N \\ 2(n - \gamma) + 2, & \text{if} \quad \frac{n+6}{2} \leq \gamma \leq n - 1 \end{cases}$$

Thus for the wheel graph

$$vhs(W_n, C_3) = \frac{n+1}{2}$$

Hence wheel graph is C_3 irregular $\forall\, n \geq 6$

Case 4: W_n with $n = 4m + 5$ admits C_3-irregular labelling with c. which is given below

$$c = \frac{n+1}{2}$$

We define a labelling such that α: $v_a \rightarrow \{1,2,3, \dots, k\}$i.e

For the vertices of wheel graph with $n = 4m + 5$

$$\alpha(v_a) = \begin{cases} a & \text{if } 1 \le a \le \frac{n-3}{2} \quad ; a-odd \\ a-1 & \text{if } \ 2 \le a \le \frac{n-1}{2} \quad ; a-even \\ n-a+1 & \text{if } \ \frac{n+1}{2} \le a \le n-1 \\ 1 & \text{if } \quad a = n \end{cases}$$

After giving labels of the vertices of the wheel graph . Weight of every C_3 is,

$$w(C_3^\gamma) = \begin{cases} \alpha(v_n) + \alpha(v_\gamma) + \alpha(v_{\gamma+1}), & if\ 1 \le \gamma \le n-2 \\ \alpha(v_n) + \alpha(v_{n-1}) + \alpha(v_1)\,, & if\ \gamma = n-1 \end{cases}$$

This shows that

$$w(C_3^\gamma) = \begin{cases} 2\gamma + 1, & \text{if} \quad 1 \le \gamma \le \frac{n-1}{2} \\ 2(n-\gamma) + 2, & \text{if} \ \frac{\mathrm{n}+1}{2} \le \gamma \le n-1 \end{cases}$$

Thus for the wheel graph

$$c = \frac{n+1}{2}$$

Theorem: The *grid graph* $G_{qr} = P_q x\, P_r$ admits *C4* irregular labelling.

Proof: Let $G_{qr} = P_q x\, P_r$ be a grid graph. This graph has qr vertices. We denotes the vertices by u_β^α , $\alpha = \{1,2,3, \dots, q\}$ and $\beta = \{\,1\,,2\,,3\,, \dots, r\}; \forall\, m \ge 1, n \ge 1$

$$|V(G_{qr})| = qr$$

$$|E(G_{qr})| = 2qr - q - r$$

Now a mapping is defined as

$$\varphi : V \to \{1,2,3, \dots, c\}$$

Case 1: G_{qr} with $q = 2m\,, r = 4n+1$ admits C_3-irregular labelling with *c*. which is given below

$$c = 2mn - n + 1$$

We define the labelling of vertices as

$$\varphi(u_\beta^\alpha) = \left\lfloor \frac{\beta + (-1)^\alpha}{4} + 1 \right\rfloor + \left\lfloor \frac{\alpha - 1}{2} \right\rfloor (2n-1)$$

After labelling the vertices we can find the weight of every *C4* denoted by *w(C4)* is,

$$w\left(C_{4,\delta}^{\gamma}\right) = \varphi\left(u_{\delta}^{\gamma}\right) + \varphi\left(u_{\delta+1}^{\gamma}\right) + \varphi\left(u_{\delta}^{\gamma+1}\right) + \varphi\left(u_{\delta+1}^{\gamma+1}\right)$$

$$\gamma = 1,2,3, \dots, 2m; \; \delta = 1,2,3, \dots, 4n+1$$

This shows that

$$w\left(C_{4,\delta}^{\gamma}\right) = \sum_{a,b=0}^{1} \left\lfloor \frac{\delta + a + (-1)^{\gamma+b}}{4} \right\rfloor + \left(\sum_{a=0}^{1} \left\lfloor \frac{\gamma - a}{2} \right\rfloor\right)(4n-2) + 3$$

$$\gamma = 1,2,3, \dots, 2m; \; \delta = 1,2,3, \dots, 4n+1$$

Thus for the grid graph $G_{qr} = P_q x\, P_r$ having $q = 2m\,, r = 4n - 1 \;\; \forall\, m \geq 1, n \geq 1$ admits C_4 irregular labelling with

$$c = 2mn - m + 1$$

Case 2 : If $q = 2m\,, r = 4n - 1 \;\; \forall\, m \geq 1, n \geq 1$ admits C_4 irregular labelling with *c*. Which is given below

$$c = 2mn - m - n + 2$$

Now a mapping is defined as

$$\varphi : V \rightarrow \{1,2,3, \dots, c\}$$

And we define the labelling of vertices as

$$\varphi(u_{\beta}^{\alpha}) = \left\lfloor \frac{\beta+(-1)^{\alpha}}{4} + 1 \right\rfloor + \left\lfloor \frac{\alpha-1}{2} \right\rfloor (2n-1)$$

After labelling the vertices we can find the weight of every C_4 denoted by $w(C_4)$ is,

$$w\left(C_{4,\delta}^{\gamma}\right) = \varphi\left(u_{\delta}^{\gamma}\right) + \varphi\left(u_{\delta+1}^{\gamma}\right) + \varphi\left(u_{\delta}^{\gamma+1}\right) + \varphi\left(u_{\delta+1}^{\gamma+1}\right)$$

$$\gamma = 1,2,3, \dots, 2m; \; \delta = 1,2,3, \dots, 4n-1$$

This shows that

$$w\left(C_{4,\delta}^{\gamma}\right) = \sum_{a,b=0}^{1} \left\lfloor \frac{\delta + a + (-1)^{\gamma+b}}{4} \right\rfloor + \left(\sum_{a=0}^{1} \left\lfloor \frac{\gamma - a}{2} \right\rfloor\right)(4n) + 4$$

$$\gamma = 1,2,3, \dots, 2m; \; \delta = 1,2,3, \dots, 4n-1$$

Thus for the grid graph $G_{qr} = P_q x\, P_r$ having $q = 2m\,, r = 4n+1 \;\; \forall\, m \geq 1, n \geq 1$ admits C_4 irregular labelling with

$$c = 2mn - m - n + 2.$$

Case 3: If $q = 2m + 1\,, r = 2n + 1\ \ \forall\, m \geq 1, n \geq 1$ admits *C_4* irregular labelling with *c*. Which is given below

$$c = mn + 1$$

Now a mapping is defined as

$$\varphi : V \rightarrow \{1,2,3, \dots, c\}$$

And we define the labelling of vertices as

$$\varphi(u_\beta^\alpha) = \begin{cases} \frac{n}{2}(\alpha - 1) + 1 & \text{if } \alpha \in O \\ \frac{n}{2}(\alpha - 1) + \left\lfloor \frac{\beta - 1}{2} \right\rfloor & \text{if} \alpha \in E \end{cases}$$

After labelling the vertices we can find the weight of every C_4 denoted by $w(C_4)$ is,

$$w\left(C_{4,\delta}^{\gamma}\right) = \varphi\left(u_\delta^\gamma\right) + \varphi\left(u_{\delta+1}^\gamma\right) + \varphi\left(u_\delta^{\gamma+1}\right) + \varphi\left(u_{\delta+1}^{\gamma+1}\right)$$

$$\gamma = 1,2,3, \dots ,2m;\ \delta = 1,2,3, \dots ,2n$$

This shows that

$$w\left(C_{4,\delta}^{\gamma}\right) = 2n(\gamma - 1) + \left\lfloor \frac{\delta}{2} \right\rfloor + \left\lfloor \frac{\delta - 1}{2} \right\rfloor \delta + (4n - 1)(\gamma - 1) + 3$$

$$\gamma = 1,2,3, \dots ,2m + 1;\ \delta = 1,2,3, \dots ,2n + 1$$

Thus for the grid graph $G_{qr} = P_q x\, P_r$ having $q = 2m\,, r = 4n - 1\ \ \forall\, m \geq 1, n \geq 1$ admits C_4 irregular labelling with

$$c = mn + 1$$

Case 4 : If $q = 4m - 3\,, r = 4n - 1\ \ \forall\, m \geq 1, n \geq 1$ admits *C_4* irregular labelling with *c*. Which is given below

$$c = 4mn - m - 3n + 1$$

Now a mapping is defined as

$$\varphi : V \rightarrow \{1,2,3, \dots, c\}$$

And we define the labelling of vertices as

$$\varphi(u_\beta^\alpha) = \begin{cases} \left\lfloor\frac{\beta-1}{4}\right\rfloor + \frac{(4n-1)(\alpha-1)}{4} + 1 & \text{if } \alpha = 4s-3 \\ \left\lfloor\frac{\beta+1}{4}\right\rfloor + \frac{(4n-1)(\alpha-2)}{4} + 1 & \text{if } \alpha = 4s-2 \\ \left\lfloor\frac{\beta+8n-4}{4}\right\rfloor + \frac{(4n-1)(\alpha-3)}{4} + 1 & \text{if } \alpha = 4s-1\,, \beta \neq 4t-2 \\ \frac{\beta+8n-2}{4} + \frac{(4n-1)(\alpha-1)}{4} + 1 & \text{if } \alpha = 4s-1\,, \beta = 4t-2 \\ \left\lfloor\frac{\beta+8n-2}{4}\right\rfloor + \frac{(4n-1)(\alpha-4)}{4} + 1 & \text{if } \alpha = 4s, \beta \neq 4t \\ \frac{\beta+8n-2}{4} + \frac{(4n-1)(\alpha-4)}{4} + 1 & \text{if } \alpha = 4s, \beta = 4t \end{cases}$$

After labelling the vertices we can find the weight of every C_4 denoted by $w(C_4)$ is,

$$w\left(C_{4,\delta}^{\gamma}\right) = \varphi\left(u_\delta^\gamma\right) + \varphi\left(u_{\delta+1}^\gamma\right) + \varphi\left(u_\delta^{\gamma+1}\right) + \varphi\left(u_{\delta+1}^{\gamma+1}\right)$$

$$\gamma = 1,2,3, \dots ,4m-3;\ \delta = 1,2,3, \dots ,4n\ -1$$

This shows that

$$w\left(C_{4,\delta}^{\gamma}\right) = \delta + (4n-1)(\gamma-1) + 3$$

$$\gamma = 1,2,3, \dots ,4m-2;\ \delta = 1,2,3, \dots ,4n$$

Thus for the grid graph $G_{qr} = P_q x\ P_r$ having $q = 2m\ , r = 4n-1\ \ \forall\, m \geq 1, n \geq 1$ admits C_4 irregular labelling with $c = 4mn - m - 3n + 1$.

Case 5 : If $q = 4m-1\ , r = 4n-1\ \ \forall\, m \geq 1, n \geq 1$ admits C_4 irregular labelling with c. Which is given below

$$c = 4mn - m - n + 1$$

Now a mapping is defined as

$$\varphi : V \to \{1,2,3, \dots , c\}$$

And we define the labelling of vertices as

$$\varphi(u_\beta^\alpha)=\begin{cases}\left\lfloor\frac{\beta}{4}\right\rfloor+\frac{(4n-1)(\alpha-1)}{4}+1 & \text{if } \alpha=4s-3\,,\beta\neq 4t+1\\ \left\lfloor\frac{\beta-5}{4}\right\rfloor+\frac{(4n-1)(\alpha-1)}{4}+1 & \text{if } \alpha=4s-3,\beta=4t+1\\ \frac{(4n-1)(\alpha-2)}{4}+1 & \text{if } \alpha=4s-2\,,\beta=1\\ \left\lfloor\frac{\beta+3}{4}\right\rfloor+\frac{(4n-1)(\alpha-2)}{4}+1 & \text{if } \alpha=4s-2\,,\beta\neq 1,\beta\in O\\ \left\lfloor\frac{\beta-2}{4}\right\rfloor+\frac{(4n-1)(\alpha-2)}{4}+1 & \text{if } \alpha=4s-2,\beta\in E\\ 2n+\frac{(4n-1)(\alpha-3)}{4}+1 & \text{if } \alpha=4s-1\,,\beta=1\\ \left\lfloor\frac{\beta+8n-3}{4}\right\rfloor+\frac{(4n-1)(\alpha-3)}{4}+1 & \text{if } \alpha=4s-1\,,\beta\neq 1\\ \left\lfloor\frac{\beta+8n-1}{4}\right\rfloor+\frac{(4n-1)(\alpha-4)}{4}+1 & \text{if } \alpha=4s,\beta\neq 1\\ 2n+\frac{(4n-1)(\alpha-1)}{4} & \text{if } \alpha=4s,\beta=1\end{cases}$$

After labelling the vertices we can find the weight of every C_4 denoted by $wt(C_4)$ is,

$$w\left(C_{4,\delta}^{\gamma}\right)=\varphi\left(u_\delta^\gamma\right)+\varphi\left(u_{\delta+1}^\gamma\right)+\varphi\left(u_\delta^{\gamma+1}\right)+\varphi\left(u_{\delta+1}^{\gamma+1}\right)$$

$$\gamma=1,2,3,\dots,4m;\ \delta=1,2,3,\dots,4n$$

This shows that

$$w\left(C_{4,\delta}^{\gamma}\right)=\delta+3+(4n-1)(\gamma-1)$$

$$\gamma=1,2,3,\dots,4m-1;\ \delta=1,2,3,\dots,4n-1$$

Thus for the grid graph $G_{qr}=P_q x\, P_r$ having $q=4m-1\,,r=4n-1\ \forall\, m\geq 1,n\geq 1$ admits C_4 irregular labelling with $c=4mn-m-n+1$.

Case 6: If $q=4m-3,r=4n-3\ \forall\, m\geq 1,n\geq 1$ admits C_4 irregular labelling with c. Which is given below

$$c=4mn-3m+n$$

Now a mapping is defined as

$$\varphi:V\rightarrow\{1,2,3,\dots,c\}$$

And we define the labelling of vertices as

$$\varphi(u_\beta^\alpha) = \begin{cases} \left\lfloor \frac{\beta-1}{4} \right\rfloor + \frac{(4n-3)(\alpha-1)}{4} + 1 & \text{if } \alpha = 4s-3, \\ \left\lfloor \frac{\beta+1}{4} \right\rfloor + \frac{(4n-3)(\alpha-2)}{4} + 1 & \text{if } \alpha = 4s-2, \\ \left\lfloor \frac{\beta+8n-8}{4} \right\rfloor + \frac{(4n-3)(\alpha-3)}{4} + 1 & \text{if } \alpha = 4s-1\,, \beta \neq 4t-2 \\ \frac{\beta+8n-6}{4} + \frac{(4n-3)(\alpha-3)}{4} + 1 & \text{if } \alpha = 4s-1\,, \beta = 4t-2 \\ \left\lfloor \frac{\beta+8n-6}{4} \right\rfloor + \frac{(4n-3)(\alpha-4)}{4} + 1 & \text{if } \alpha = 4s, \beta \neq 4t \\ \frac{\beta+8n-4}{4} + \frac{(4n-3)(\alpha-4)}{4} + 1 & \text{if } \alpha = 4s-3, \beta = 4t \end{cases}$$

After labelling the vertices we can find the weight of every C_4 denoted by $wt(C_4)$ is,

$$w(C_{4,\delta}^{\gamma}) = \varphi(u_\delta^\gamma) + \varphi(u_{\delta+1}^\gamma) + \varphi(u_\delta^{\gamma+1}) + \varphi(u_{\delta+1}^{\gamma+1})$$

$$\gamma = 1,2,3, \ldots ,4m-3;\ \delta = 1,2,3, \ldots ,4n-1$$

This shows that

$$w(C_{4,\delta}^{\gamma}) = \delta + 3 + (4n-3)(\gamma-1)$$

$$\gamma = 1,2,3, \ldots ,4m-3;\ \delta = 1,2,3, \ldots ,4n-1$$

Thus for the grid graph $G_{qr} = P_q x\, P_r$ having $q = 4m-3\,, r = 4n-1\ \ \forall\, m \geq 1, n \geq 1$ admits C_4 irregular labelling with $c = 4mn - 3m + n$

Case 7 : If $q = 4m-2\,, r = 4n\ \ \forall\, m \geq 1, n \geq 1$ admits C_4 irregular labelling with *c*. Which is given below

$$c = 2mn - m - n + 1$$

Now a mapping is defined as

$$\varphi : V \to \{1,2,3, \ldots , c\}$$

And we define the labelling of vertices as

$$\varphi(u_\beta^\alpha) = \begin{cases} \frac{(2n-1)(\alpha-1)}{4} & \text{if } \alpha = 4s-3 \\ \left\lfloor \frac{\beta-1}{4} \right\rfloor + \frac{(2n-1)(\alpha-2)}{4} & \text{if } \alpha = 4s-2 \\ n-1+\frac{(2n-1)(\alpha-3)}{4} & \text{if } \alpha = 4s-1\,, \beta \in O \\ n+\frac{(2n-1)(\alpha-3)}{4} & \text{if } \alpha = 4s-1\,, \beta \in E \\ \left\lfloor \frac{\beta+2n-2}{4} \right\rfloor + \frac{(2n-1)(\alpha-4)}{4} & \text{if } \alpha = 4s-3 \end{cases}$$

After labelling the vertices we can find the weight of every C_4 denoted by *wt(*C_4*)* is,

$$w\left(C_{4,\delta}^{\gamma}\right) = \varphi\left(u_\delta^\gamma\right) + \varphi\left(u_{\delta+1}^\gamma\right) + \varphi\left(u_\delta^{\gamma+1}\right) + \varphi\left(u_{\delta+1}^{\gamma+1}\right)$$

$$\gamma = 1,2,3, \dots ,4m-2;\ \delta = 1,2,3, \dots ,2n+1$$

This shows that

$$w\left(C_{4,\delta}^{\gamma}\right) = \delta + 3 + (2n-1)(\gamma-1)$$

$$\gamma = 1,2,3, \dots ,4m-2;\ \delta = 1,2,3, \dots ,2n+1$$

Thus for the grid graph $G_{qr} = P_q x\ P_r$ having $q = 2m\,, r = 4n-1\ \forall\, m \geq 1, n \geq 1$ admits C_4 irregular labelling with $c = 2mn - m - n + 1$.

Theorem: Let Q_n be a connected which is obrained by an edge amalgamtion of $n \geq 2$ C_8-cycles. Then

$$vhs(\,G, C_8) = \left\lceil \frac{n+7}{8} \right\rceil$$

Proof: let G a connected C_8 graph. This graph has *6n+2* vertices. We denotes the first two vertices by u_0 and u_1 which is equal to *1* and all other vertices is denoted by v_a and $a = \{1,2,3, \dots ,6n\}$.

Lets us define a new labelling for this such that

$$\beta :\ v_a\ \rightarrow \{\,1\,,2\,,3\,, \dots , 6n\,\}$$

$$\beta(v_a) = \begin{cases} i & \forall\, v_{6j+1} \quad 8(i-2)+6 \ \le j \le 8(i-1)+5 \\ i & \forall\, v_{6j+2} \quad 8(i-2)+5 \ \le j \le 8(i-1)+4 \\ i & \forall\, v_{6j+3} \quad 8(i-2)+1 \ \le j \le 8(i-1) \\ i & \forall\, v_{6j+4} \quad 8(i-2)+3 \ \le j \le 8i+2 \\ i & \forall\, v_{6j+5} \quad 8(i-2)+7 \ \le j \le 8(i-1)+6 \\ i & \forall\, v_{6j+6} \quad 8(i-1) \ \le j \le 8i-1 \end{cases}$$

And we know that

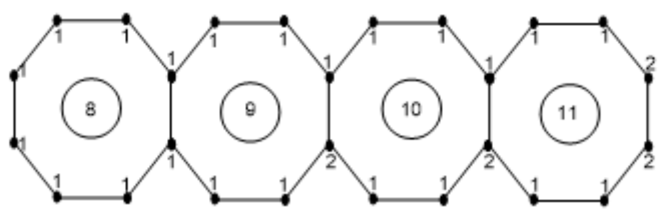
After labelling the vertices we can find the weight of every C_8 denoted by $w(C_8)$ is,

$$w(C_i) = \beta(v_i) + \beta(v_{i+1}) + \beta(v_{i+2}) + \beta(v_{i+3} + \beta(v_{i+4}) + \beta(v_{i+5}) + \beta(v_{i+6}) + \beta(v_{i+7})$$

This shows that

$$w(C_i) = i + 7 \quad \forall\, i \ge 1$$

$$vhs\ (K_n\,, C_8) = \left\lceil \frac{n-7}{8} \right\rceil$$



Theorem: The $ladder\ graph\ \ L_n$ admits total C_4 irregular labelling with c for $n\ \ge\ 2$.

$$c = \left\lceil \frac{n+2}{4} \right\rceil$$

Proof: Let $L_n = P_n x\, P_2$ be a Ladder graph. This graph has $2n$ vertices. We denotes the vertices by u_β^α , $\alpha = 1{,}2$ and $\beta = \{\,1\,,2\,,3\,,\dots,n\,\}$ and $|V(L_n)| = 2n$

We denotes the edges by $u_\beta^\alpha u_{\beta+1}^\alpha$.

$$u_\beta^\alpha u_{\beta+1}^\alpha\!: \beta = \{\alpha = 1{,}2\}\ and\ \{\,1\,,2\,,3\,,\dots,n-1\,\} \cup$$

$$\{u_\beta^\alpha u_{\beta+1}^\alpha\!: \{\alpha = 1{,}2\} and\ \beta = \{\,1\,,2\,,3\,,\dots,n\,\}$$

$$|E(L_n)| = 2n$$

We define a labelling such that $\sigma : u_\beta^\alpha \rightarrow \{ 1 , 2 , 3 , \dots , k \}$ and $\gamma: E(G) \rightarrow \{1\}$

$$\sigma(u_\beta^\alpha) = \left\lfloor \frac{\beta+(-1)^\alpha}{4} + 1 \right\rfloor$$

After giving labels of the vertices of the ladder graph. Weight of every C_4 is,

$$w(C_4^\gamma) = \sigma(u_\gamma^1) + \sigma(u_{\gamma+1}^1) + \sigma(u_\gamma^2) + \sigma(u_{\gamma+1}^2)$$

This shows that

$$w(C_4^\gamma) = \gamma + 7 \text{ , where } \gamma = \{ 1 , 2 , 3 , \dots , n-1 \}$$

Thus for the ladder graph $\mathrm{L_n} = \mathrm{P_n x\ P_2}$

$$vhs(L_n, C_4) \leq \left\lceil \frac{n+2}{4} \right\rceil \qquad (1)$$

Ashraf et al. proved the lower bound of ladder graph, which is given below

$$vhs(L_n, C_4) \geq \left\lceil \frac{n+2}{4} \right\rceil \qquad (2)$$

Thus from (1) and (2)

$$vhs(L_n, C_4) = \left\lceil \frac{n+2}{4} \right\rceil$$

Theorem: The *triangular ladder graph* TL_n admits total C_3 irregular labelling with c for $n \geq 2$.

$$vhs(\mathrm{T}L_n, C_3) = \left\lceil \frac{2n}{3} \right\rceil$$

Proof: let TL_n be a triangular Ladder graph. This graph has *2n* vertices. We denote the vertices by u_β^α , $\alpha = 1{,}2$ and $\beta = \{ 1 , 2 , 3 , \dots , n \}$ and $|V(L_n)| = 2n$

We denote the edges by $u_\beta^\alpha u_{\beta+1}^\alpha$.

$$u_\beta^\alpha u_{\beta+1}^\alpha : \{\alpha = 1{,}2\} \ and \ \beta = \{ 1 , 2 , 3 , \dots , n-1 \} \cup$$

$$u_\beta^\alpha u_\beta^{\alpha+1} : \{\alpha = 1\} \ and \ \beta = \{ 1 , 2 , 3 , \dots , n \} \ \cup$$

$$\{u_{\beta+1}^\alpha u_{\beta+1}^{\alpha+1} : \{\alpha = 1\} and \ \beta = \{ 1 , 2 , 3 , \dots , n-1 \}$$

$$|E(TL_n)| = 4n - 3$$

We define a labelling such that $\sigma: u_{\beta}^{\alpha} \rightarrow \{1, 2, 3, \dots, k\}$ and $\gamma: E(G) \rightarrow \{1\}$

$$\sigma(u_{\beta}^{\alpha}) = \left\lfloor 1 + \frac{\alpha+2\beta-3}{3} \right\rfloor$$

After giving labels of the vertices of the triangular ladder graph . Weight of every C_3 is,

$$w(C_3^{\gamma}) = \sigma\left(u^1_{\left\lfloor\frac{\gamma+2}{2}\right\rfloor}\right) + \sigma\left(u^1_{\left\lfloor\frac{\gamma+3}{2}\right\rfloor}\right) + \sigma\left(u^2_{\left\lfloor\frac{\gamma+2}{2}\right\rfloor}\right)$$

This shows that

$$w(C_3^{\gamma}) = \gamma + 5, \text{ where } \gamma = \{1, 3, 5, \dots, 2n-3\}$$

$$w(C_3^{\gamma}) = \sigma\left(u^1_{\left\lfloor\frac{\gamma+2}{2}\right\rfloor}\right) + \sigma\left(u^2_{\left\lfloor\frac{\gamma}{2}\right\rfloor}\right) + \sigma\left(u^2_{\left\lfloor\frac{\gamma+2}{2}\right\rfloor}\right)$$

This shows that

$$w(C_3^{\gamma}) = \gamma + 5, \text{ where } \gamma = \{2, 4, 6, \dots, 2n-2\}$$

Thus for the triangular ladder graph

$$vhs(\mathrm{T}L_n, C_3) \leq \left\lceil\frac{2n}{3}\right\rceil \dots\dots\dots\dots\dots (1)$$

Ashraf et al. proved the lower bound of triangular ladder graph, which is given below

$$vhs(\mathrm{T}L_n, C_3) \geq \left\lceil\frac{2n}{3}\right\rceil \dots\dots\dots\dots\dots (2)$$

Thus from (1) and (2)

$$vhs(\mathrm{T}L_n, C_3) = \left\lceil\frac{2n}{3}\right\rceil$$

Theorem: The *wheel graph* W_n admits total C_3 irregular labelling.

Proof: Let W_n be a Wheel graph. This graph has *n* vertices. We denotes the vertices by v_a , $a = \{1, 2, 3, \dots, n\}$ and $|V(W_n)| = n$

We denote the edges by $v_a v_c$.

$$v_a v_c: a = \{1, 2, 3, \dots, n-1\} \cup$$

$$v_a v_c: a = \{1, 2, 3, \dots, n-2\} \cup v_{n-1} v_n$$

$$|E(W_n)| = 2n - 2$$

Case 1: W_n with $n = 4m + 2$ admits total C_3-irregular labelling with c. which is given below

$$c = \frac{n}{2}$$

We define a labelling such that $\alpha: v_a \rightarrow \{1,2,3, \dots, k\}$ and $\gamma: E(G) \rightarrow \{1\}$

For the vertices of wheel graph with $n = 4m + 2$

$$\alpha(v_a) = \begin{cases} a, & \text{if } 1 \leq a \leq \frac{n}{2} \quad ; a - odd \\ a - 1, & \text{if } 2 \leq a \leq \frac{n+2}{2} \quad ; a - even \\ n - a + 1, & \text{if } \frac{n+4}{2} \leq a \leq n - 1 \\ 1, & \text{if } n = a, \end{cases}$$

After giving labels of the vertices of the wheel graph. Weight of every C_3 is,

$$w(C_3^\gamma) = \begin{cases} \alpha(v_n) + \alpha(v_\gamma) + \alpha(v_{\gamma+1}), & \text{if } 1 \leq \gamma \leq n - 2 \\ \alpha(v_n) + \alpha(v_{n-1}) + \alpha(v_1), & \text{if } \gamma = \text{n} - 1 \end{cases}$$

This shows that

$$w(C_3^\gamma) = \begin{cases} 2\gamma + 4, & \text{if } 1 \leq \gamma \leq \frac{n}{2} \\ 2(n - \gamma) + 5, & \text{if } \frac{\text{n}+2}{2} \leq \gamma \leq n - 1 \end{cases}$$

Thus for the wheel graph

$$c = \frac{n}{2}$$

Case 2: W_n with $n = 4m + 3$ admits total C_3-irregular labelling with c . which is given below

$$c = \frac{n + 1}{2}$$

We define a labelling such that $\alpha: v_a \rightarrow \{1,2,3, \dots, k\}$and $\gamma: E(G) \rightarrow \{1\}$

For the vertices of wheel graph with $n = 4m + 3$

$$\alpha(v_a) = \begin{cases} a, & \text{if } 1 \le a \le \frac{n-1}{2} \quad ; a - odd \\ a-1, & \text{if } 2 \le a \le \frac{n-3}{2} \quad ; a - even \\ \frac{n+1}{2}, & \text{if } \frac{n+1}{2} \le a \le \frac{n+3}{2} \\ n-a+1 & \text{if } \frac{n+5}{2} \le a \le n-1 \\ 1, & \text{if } n = a \end{cases}$$

After giving labels of the vertices of the wheel graph . Weight of every C_3 is,

$$w(C_3^\gamma) = \begin{cases} \alpha(v_n) + \alpha(v_\gamma) + \alpha(v_{\gamma+1}), & \text{if } 1 \le \gamma \le n-2 \\ \alpha(v_n) + \alpha(v_{n-1}) + \alpha(v_1)\,, & \text{if } \gamma = n-1 \end{cases}$$

This shows that

$$w(C_3^\gamma) = \begin{cases} 2\gamma + 4, & \text{if } \quad 1 \le \gamma \le \frac{n-3}{2} \\ n+4, & \text{if } \quad \gamma = \frac{n-1}{2} \\ 2(n-\gamma)+6, & \text{if } \quad \frac{n+1}{2} \le \gamma \le \frac{n+3}{2} \\ 2(n-\gamma)+5, & \text{if } \quad \frac{n+5}{2} \le \gamma \le n-1 \end{cases}$$

Thus for the wheel graph

$$c = \frac{n+1}{2}$$

Case 3: W_n with $n = 4m+4$ admits total C_3-irregular labelling with *c*. which is given below

$$c = \frac{n+2}{2}$$

We define a labelling such that $\alpha: v_a \to \{1,2,3,\dots,k\}$i.e $\gamma: E(G) \to \{1\}$

For the vertices of wheel graph with $n = 4m+4$

$$\alpha(v_a) = \begin{cases} a, & \text{if } 1 \le a \le \frac{n+2}{2} \quad ; a - odd \\ a-1 & \text{if } 2 \le a \le \frac{n+4}{2} \quad ; a - even \\ n-a+1 & \text{if } \frac{n+6}{2} \le a \le n-1 \\ 1 & \text{if } a = n \end{cases}$$

After giving labels of the vertices of the wheel graph . Weight of every C_3 is,

$$w(C_3^\gamma) = \begin{cases} \alpha(v_n) + \alpha(v_\gamma) + \alpha(v_{\gamma+1}), & \text{if } 1 \le \gamma \le n-2 \\ \alpha(v_n) + \alpha(v_{n-1}) + \alpha(v_1), & \text{if } \gamma = \mathrm{n} - 1 \end{cases}$$

This shows that

$$w(C_3^\gamma) = \begin{cases} 2\gamma + 4, & \text{if } \quad 1 \le \gamma \le \frac{n+2}{2} \\ 2\gamma - 1, & \text{if } \quad \gamma = 2s + 4;\ \forall\, s \in N \\ 2(n-\gamma) + 5, & \text{if } \quad \frac{n+6}{2} \le \gamma \le n-1 \end{cases}$$

Thus for the wheel graph

$$c = \frac{n+1}{2}$$

Hence wheel graph is C_3 irregular $\forall\, n \ge 6$

Case 4: W_n with $n = 4m + 5$ admits C_3-irregular labelling with *c*. which is given below

$$c = \frac{n+1}{2}$$

We define a labelling such that $\alpha: v_a \rightarrow \{1,2,3, \dots, k\}$and $\gamma: E(G) \rightarrow \{1\}$

For the vertices of wheel graph with $n = 4m + 5$

$$\alpha(v_a) = \begin{cases} a, & \text{if } 1 \le a \le \frac{n-3}{2} \quad ; a - odd \\ a-1, & \text{if } 2 \le a \le \frac{n-1}{2} \quad ; a - even \\ n-a+1, & \text{if } \frac{n+1}{2} \le a \le n-1 \\ 1 & \text{if } a = n \end{cases}$$

After giving labels of the vertices of the wheel graph . Weight of every C_3 is,

$$w(C_3^\gamma) = \begin{cases} \alpha(v_n) + \alpha(v_\gamma) + \alpha(v_{\gamma+1}), & if\ 1 \le \gamma \le n-2 \\ \alpha(v_n) + \alpha(v_{n-1}) + \alpha(v_1), & if\ \gamma = n-1 \end{cases}$$

This shows that

$$w\left(C_3^{\gamma}\right)=\begin{cases}2\gamma+4, & \text{if} \quad 1\le\gamma\le\frac{n-1}{2}\\ 2(n-\gamma)+5, & \text{if} \quad \frac{n+1}{2}\le\gamma\le n-1\end{cases}$$

Thus for the wheel graph

$$c=\frac{n+1}{2}$$

Theorem: The *grid graph* $G_{qr}=P_q x\, P_r$ admits total C_4 irregular labelling.

Proof: Let $G_{qr}=P_q x\, P_r$ be a grid graph. This graph has qr vertices. We denotes the vertices by u_β^α , $\alpha=\{1,2,3,\dots,q\}$ and $\beta=\{1,2,3,\dots,r\}$; $\forall\, m\ge 1, n\ge 1$

$$\left|V\left(G_{qr}\right)\right|=qr$$

$$\left|E\left(G_{qr}\right)\right|=2qr-q-r$$

Now a mapping is defined as

$$\varphi: V\to\{1,2,3,\dots,c\} \text{ and } \gamma: E(G)\to\{1\}$$

Case 1: G_{qr} with $q=2m, r=4n+1$ admits C_3-irregular labelling with c. which is given below

$$c=2mn-n+1$$

We define the labelling of vertices as

$$\varphi(u_\beta^\alpha)=\left\lfloor\frac{\beta+(-1)^\alpha}{4}+1\right\rfloor+\left\lfloor\frac{\alpha-1}{2}\right\rfloor(2n-1)$$

After labelling the vertices we can find the weight of every C_4 denoted by $w(C_4)$ is,

$$w\left(C_{4,\delta}^{\gamma}\right)=\varphi\left(u_\delta^{\gamma}\right)+\varphi\left(u_{\delta+1}^{\gamma}\right)+\varphi\left(u_\delta^{\gamma+1}\right)+\varphi\left(u_{\delta+1}^{\gamma+1}\right)$$

$$\gamma=1,2,3,\dots,2m;\ \delta=1,2,3,\dots,4n+1$$

This shows that

$$w\left(C_{4,\delta}^{\gamma}\right)=\sum_{a,b=0}^{1}\left\lfloor\frac{\delta+a+(-1)^{\gamma+b}}{4}\right\rfloor+\left(\sum_{a=0}^{1}\left\lfloor\frac{\gamma-a}{2}\right\rfloor\right)(4n-2)+7$$

$$\gamma=1,2,3,\dots,2m;\ \delta=1,2,3,\dots,4n+1$$

Thus for the grid graph $G_{qr}=P_q x\, P_r$ having $q=2m, r=4n-1\ \forall\, m\ge 1, n\ge 1$ admits C_4 irregular labelling with

$$c = 2mn - m + 1$$

Case 2 : If $q = 2m$, $r = 4n - 1$ $\forall\, m \geq 1, n \geq 1$ admits total C_4 irregular labelling with *c.* Which is given below

$$c = 2mn - m - n + 2$$

Now a mapping is defined as

$$\varphi : V \to \{1,2,3, \dots , c\} \text{ and } \gamma : E(G) \to \{1\}$$

And we define the labelling of vertices as

$$\varphi(u_\beta^\alpha) = \left\lfloor \frac{\beta + (-1)^\alpha}{4} + 1 \right\rfloor + \left\lfloor \frac{\alpha - 1}{2} \right\rfloor (2n - 1)$$

After labelling the vertices we can find the weight of every C_4 denoted by $w(C_4)$ is,

$$w\left(C_{4,\delta}^\gamma\right) = \varphi\left(u_\delta^\gamma\right) + \varphi\left(u_{\delta+1}^\gamma\right) + \varphi\left(u_\delta^{\gamma+1}\right) + \varphi\left(u_{\delta+1}^{\gamma+1}\right)$$

$$\gamma = 1,2,3, \dots ,2m;\ \delta = 1,2,3, \dots ,4n - 1$$

This shows that

$$w\left(C_{4,\delta}^\gamma\right) = \sum_{a,b=0}^{1} \left\lfloor \frac{\delta + a + (-1)^{\gamma+b}}{4} \right\rfloor + \left(\sum_{a=0}^{1} \left\lfloor \frac{\gamma - a}{2} \right\rfloor\right)(4n) + 8$$

$$\gamma = 1,2,3, \dots ,2m;\ \delta = 1,2,3, \dots ,4n - 1$$

Thus for the grid graph $G_{qr} = P_q x\, P_r$ having $q = 2m$, $r = 4n+1$ $\forall\, m \geq 1, n \geq 1$ admits C_4 irregular labelling with

$$c = 2mn - m - n + 2.$$

Case 3 : If $q = 2m + 1$, $r = 2n + 1$ $\forall\, m \geq 1, n \geq 1$ admits total C_4 irregular labelling with *c.* Which is given below

$$c = mn + 1$$

Now a mapping is defined as

$$\varphi : V \to \{1,2,3, \dots , c\} \text{ and } \gamma : E(G) \to \{1\}$$

And we define the labelling of vertices as

$$\varphi(u_\beta^\alpha) = \begin{cases} \frac{n}{2}(\alpha - 1) + 1 & \text{if } \alpha \in O \\ \frac{n}{2}(\alpha - 1) + \left\lfloor \frac{\beta-1}{2} \right\rfloor & \text{if} \alpha \in E \end{cases}$$

After labelling the vertices we can find the weight of every C_4 denoted by $w(C_4)$ is,

$$w(C_{4,\delta}^{\gamma}) = \varphi(u_\delta^\gamma) + \varphi(u_{\delta+1}^\gamma) + \varphi(u_\delta^{\gamma+1}) + \varphi(u_{\delta+1}^{\gamma+1})$$

$$\gamma = 1,2,3, \dots ,2m;\ \delta = 1,2,3, \dots ,2n$$

This shows that

$$w(C_{4,\delta}^{\gamma}) = 2n(\gamma - 1) + \left\lfloor \frac{\delta}{2} \right\rfloor + \left\lfloor \frac{\delta - 1}{2} \right\rfloor \delta + (4n - 1)(\gamma - 1) + 7$$

$$\gamma = 1,2,3, \dots ,2m + 1;\ \delta = 1,2,3, \dots ,2n + 1$$

Thus for the grid graph $G_{qr} = P_q x\, P_r$ having $q = 2m\ , r = 4n - 1\ \ \forall\, m \geq 1, n \geq 1$ admits C_4 irregular labelling with

$$c = mn + 1$$

Case 4 : If $q = 4m - 3\ , r = 4n - 1\ \ \forall\, m \geq 1, n \geq 1$ admits total *C_4* irregular labelling with *c*. Which is given below

$$c = 4mn - m - 3n + 1$$

Now a mapping is defined as

$$\varphi : V \to \{1,2,3, \dots, c\} \text{ and } \gamma : E(G) \to \{1\}$$

And we define the labelling of vertices as

$$\varphi(u_\beta^\alpha) = \begin{cases} \left\lfloor \frac{\beta-1}{4} \right\rfloor + \frac{(4n-1)(\alpha-1)}{4} + 1 & \text{if } \alpha = 4s - 3 \\ \left\lfloor \frac{\beta+1}{4} \right\rfloor + \frac{(4n-1)(\alpha-2)}{4} + 1 & \text{if } \alpha = 4s - 2 \\ \left\lfloor \frac{\beta+8n-4}{4} \right\rfloor + \frac{(4n-1)(\alpha-3)}{4} + 1 & \text{if } \alpha = 4s - 1\ , \beta \neq 4t - 2 \\ \frac{\beta+8n-2}{4} + \frac{(4n-1)(\alpha-1)}{4} + 1 & \text{if } \alpha = 4s - 1\ , \beta = 4t - 2 \\ \left\lfloor \frac{\beta+8n-2}{4} \right\rfloor + \frac{(4n-1)(\alpha-4)}{4} + 1 & \text{if } \alpha = 4s, \beta \neq 4t \\ \frac{\beta+8n-2}{4} + \frac{(4n-1)(\alpha-4)}{4} + 1 & \text{if } \alpha = 4s, \beta = 4t \end{cases}$$

After labelling the vertices we can find the weight of every *C_4* denoted by *$w(C_4)$* is,

$$w\left(C_{4,\delta}^{\gamma}\right) = \varphi\left(u_{\delta}^{\gamma}\right) + \varphi\left(u_{\delta+1}^{\gamma}\right) + \varphi\left(u_{\delta}^{\gamma+1}\right) + \varphi\left(u_{\delta+1}^{\gamma+1}\right)$$

$$\gamma = 1,2,3, \dots ,4m - 3;\ \delta = 1,2,3, \dots ,4n\ - 1$$

This shows that

$$w\left(C_{4,\delta}^{\gamma}\right) = \delta + (4n - 1)(\gamma - 1) + 7$$

$$\gamma = 1,2,3, \dots ,4m - 2;\ \delta = 1,2,3, \dots ,4n$$

Thus for the grid graph $G_{qr} = P_q x\, P_r$ having $q = 2m\ , r = 4n - 1\ \ \forall\, m \geq 1, n \geq 1$ admits C_4 irregular labelling with $c = 4mn - m - 3n + 1$.

Case 5 : If $q = 4m - 1\ , r = 4n - 1\ \ \forall\, m \geq 1, n \geq 1$ admits total C_4 irregular labelling with *c*. Which is given below

$$c = 4mn - m - n + 1$$

Now a mapping is defined as

$$\varphi : V \to \{1,2,3, \dots, c\} \text{ and } \gamma : E(G) \to \{1\}$$

And we define the labelling of vertices as

$$\varphi(u_{\beta}^{\alpha}) = \begin{cases} \left\lfloor \frac{\beta}{4} \right\rfloor + \frac{(4n-1)(\alpha-1)}{4} + 1 & \text{if } \alpha = 4s - 3\ , \beta \neq 4t + 1 \\ \left\lfloor \frac{\beta-5}{4} \right\rfloor + \frac{(4n-1)(\alpha-1)}{4} + 1 & \text{if } \alpha = 4s - 3, \beta = 4t + 1 \\ \frac{(4n-1)(\alpha-2)}{4} + 1 & \text{if } \alpha = 4s - 2\ , \beta = 1 \\ \left\lfloor \frac{\beta+3}{4} \right\rfloor + \frac{(4n-1)(\alpha-2)}{4} + 1 & \text{if } \alpha = 4s - 2\ , \beta \neq 1, \beta \in O \\ \left\lfloor \frac{\beta-2}{4} \right\rfloor + \frac{(4n-1)(\alpha-2)}{4} + 1 & \text{if } \alpha = 4s - 2, \beta \in E \\ 2n + \frac{(4n-1)(\alpha-3)}{4} + 1 & \text{if } \alpha = 4s - 1\ , \beta = 1 \\ \left\lfloor \frac{\beta+8n-3}{4} \right\rfloor + \frac{(4n-1)(\alpha-3)}{4} + 1 & \text{if } \alpha = 4s - 1\ , \beta \neq 1 \\ \left\lfloor \frac{\beta+8n-1}{4} \right\rfloor + \frac{(4n-1)(\alpha-4)}{4} + 1 & \text{if } \alpha = 4s, \beta \neq 1 \\ 2n + \frac{(4n-1)(\alpha-1)}{4} & \text{if } \alpha = 4s, \beta = 1 \end{cases}$$

After labelling the vertices we can find the weight of every C_4 denoted by *wt(*C_4*)* is,

$$w\left(C_{4,\delta}^{\gamma}\right) = \varphi\left(u_{\delta}^{\gamma}\right) + \varphi\left(u_{\delta+1}^{\gamma}\right) + \varphi\left(u_{\delta}^{\gamma+1}\right) + \varphi\left(u_{\delta+1}^{\gamma+1}\right)$$

$$\gamma = 1,2,3, \dots ,4m;\ \delta = 1,2,3, \dots ,4n$$

This shows that

$$w\left(C_{4,\delta}^{\gamma}\right) = \delta + (4n-1)(\gamma-1) + 7$$

$$\gamma = 1,2,3, \dots ,4m-1;\ \delta = 1,2,3, \dots ,4n-1$$

Thus for the grid graph $G_{qr} = P_q x\ P_r$ having $q = 4m-1\ , r = 4n-1\ \ \forall\, m \geq 1, n \geq 1$ admits C_4 irregular labelling with $c = 4mn - m - n + 1$.

Case 6: If $q = 4m-3, r = 4n-3\ \ \forall\, m \geq 1, n \geq 1$ admits total C_4 irregular labelling with c. Which is given below

$$c = 4mn - 3m + n$$

Now a mapping is defined as

$$\varphi : V \to \{1,2,3, \dots, c\} \text{ and } \gamma : E(G) \to \{1\}$$

And we define the labelling of vertices as

$$\varphi(u_\beta^\alpha) = \begin{cases} \left\lfloor \frac{\beta-1}{4} \right\rfloor + \frac{(4n-3)(\alpha-1)}{4} + 1 & \text{if } \alpha = 4s-3, \\ \left\lfloor \frac{\beta+1}{4} \right\rfloor + \frac{(4n-3)(\alpha-2)}{4} + 1 & \text{if } \alpha = 4s-2, \\ \left\lfloor \frac{\beta+8n-8}{4} \right\rfloor + \frac{(4n-3)(\alpha-3)}{4} + 1 & \text{if } \alpha = 4s-1\ , \beta \neq 4t-2 \\ \frac{\beta+8n-6}{4} + \frac{(4n-3)(\alpha-3)}{4} + 1 & \text{if } \alpha = 4s-1\ , \beta = 4t-2 \\ \left\lfloor \frac{\beta+8n-6}{4} \right\rfloor + \frac{(4n-3)(\alpha-4)}{4} + 1 & \text{if } \alpha = 4s, \beta \neq 4t \\ \frac{\beta+8n-4}{4} + \frac{(4n-3)(\alpha-4)}{4} + 1 & \text{if } \alpha = 4s-3, \beta = 4t \end{cases}$$

After labelling the vertices we can find the weight of every C_4 denoted by $wt(C_4)$ is,

$$w\left(C_{4,\delta}^{\gamma}\right) = \varphi\left(u_\delta^\gamma\right) + \varphi\left(u_{\delta+1}^\gamma\right) + \varphi\left(u_\delta^{\gamma+1}\right) + \varphi\left(u_{\delta+1}^{\gamma+1}\right)$$

$$\gamma = 1,2,3, \dots ,4m-3;\ \delta = 1,2,3, \dots ,4n-1$$

This shows that

$$w\left(C_{4,\delta}^{\gamma}\right) = \delta + 7 + (4n-3)(\gamma-1)$$

$$\gamma = 1,2,3, \dots ,4m-3;\ \delta = 1,2,3, \dots ,4n-1$$

Thus for the grid graph $G_{qr} = P_q x\ P_r$ having $q = 4m-3\ , r = 4n-1\ \ \forall\, m \geq 1, n \geq 1$ admits C_4 irregular labelling with $c = 4mn - 3m + n$

Case 7 : If $q = 4m - 2$, $r = 4n$ $\forall\, m \geq 1, n \geq 1$ admits C_4 irregular labelling with c. Which is given below

$$c = 2mn - m - n + 1$$

Now a mapping is defined as

$$\varphi : V \to \{1,2,3, \dots, c\} \text{ and } \gamma : E(G) \to \{1\}$$

And we define the labelling of vertices as

$$\varphi(u_\beta^\alpha) = \begin{cases} \frac{(2n-1)(\alpha-1)}{4} & \text{if } \alpha = 4s-3 \\ \left\lfloor \frac{\beta-1}{4} \right\rfloor + \frac{(2n-1)(\alpha-2)}{4} & \text{if } \alpha = 4s-2 \\ n - 1 + \frac{(2n-1)(\alpha-3)}{4} & \text{if } \alpha = 4s-1\,, \beta \in O \\ n + \frac{(2n-1)(\alpha-3)}{4} & \text{if } \alpha = 4s-1\,, \beta \in E \\ \left\lfloor \frac{\beta+2n-2}{4} \right\rfloor + \frac{(2n-1)(\alpha-4)}{4} & \text{if } \alpha = 4s-3 \end{cases}$$

After labelling the vertices we can find the weight of every C_4 denoted by $wt(C_4)$ is,

$$w\left(C_{4,\delta}^{\gamma}\right) = \varphi\left(u_\delta^\gamma\right) + \varphi\left(u_{\delta+1}^\gamma\right) + \varphi\left(u_\delta^{\gamma+1}\right) + \varphi\left(u_{\delta+1}^{\gamma+1}\right)$$

$$\gamma = 1,2,3, \dots ,4m-2;\ \delta = 1,2,3, \dots ,2n+1$$

This shows that

$$w\left(C_{4,\delta}^{\gamma}\right) = \delta + 7 + (2n-1)(\gamma-1)$$

$$\gamma = 1,2,3, \dots ,4m-2;\ \delta = 1,2,3, \dots ,2n+1$$

Thus for the grid graph $G_{qr} = P_q x\, P_r$ having $q = 2m$, $r = 4n - 1$ $\forall\, m \geq 1, n \geq 1$ admits C_4 irregular labelling with $c = 2mn - m - n + 1$.

Theorem: Let Q_n be a connected which is obrained by an edge amalgamtion of $n \geq 2$ C_8-cycles. Then Q_n admits total C_8 irregular labelling with c.

$$c = \left\lceil \frac{n+7}{8} \right\rceil$$

Proof: let G a connected C_8 graph. This graph has $6n+2$ vertices. We denotes the first two vertices by u_0 and u_1 which is equal to 1 and all other vertices is denoted by v_a and $a = \{1,2,3, \dots ,6n\}$.

Lets us define a new labelling for this such that

$$\beta : \ v_a \ \rightarrow \{ 1 , 2 , 3 , \dots , 6n \} \text{ and } \gamma : E(G) \rightarrow \{1\}$$

$$\beta(v_a) = \begin{cases} i & \forall \, v_{6j+1} & 8(i-2)+6 \ \le j \le 8(i-1)+5 \\ i & \forall \, v_{6j+2} & 8(i-2)+5 \ \le j \le 8(i-1)+4 \\ i & \forall \, v_{6j+3} & 8(i-2)+1 \ \le j \le 8(i-1) \\ i & \forall \, v_{6j+4} & 8(i-2)+3 \ \le j \le 8i+2 \\ i & \forall \, v_{6j+5} & 8(i-2)+7 \ \le j \le 8(i-1)+6 \\ i & \forall \, v_{6j+6} & 8(i-1) \ \le j \le 8i-1 \end{cases}$$

And we know that

After labelling the vertices we can find the weight of every C_8 denoted by $w(C_8)$ is,

$$w(C_i) = \beta(v_i) + \beta(v_{i+1}) + \beta(v_{i+2}) + \beta(v_{i+3} + \beta(v_{i+4}) + \beta(v_{i+5}) + \beta(v_{i+6}) + \beta(v_{i+7})$$

This shows that

$$w(C_i) = i + 15 \quad \forall \, i \ge 1$$

$$c = \left\lceil \frac{n-7}{8} \right\rceil$$

Conclusion

In this thesis we have calculated vertex H irregularity strength of *triangular graph and wheel graph* by taking C_3 as a subgraph. We have determined vertex H irregularity strength of *ladder* and *grid graph* taking C_4 as a subgraph. We also found vertex H irregularity strength of connected Q_n graph taking C_8 as a subgraph.

- We have found only C_n irregular strength of some connected graph. We are unable to find these strengths for $H \ncong C_n$. We therefore purpose following open problem for further study.

  Open Problem 1.

  Find vertex T_n irregular strengths of connected graphs, T_n is a tree graph on n-vertices.

  Open Problem 2.

  Find total H irregular strengths of connected graphs, where H is not isomorphic to cycle graph.

Chapter 4

References

[1] A. Ahmad, O. B. S. Al-Mushayt and M. Bača, On edge irregularity strength of graphs, *Appl. Math. Comput*. 243 (2014) 607–610.

[2] M. Aigner and E. Triesch, Irregular assignments of trees and forests, SIAM J. *Discrete Math*. 3 (1990) 439–449.

[3] F. Ashraf, M. Baca, Z. Kimakova, and A. S. Fenovcikova ,On vertex and edge H-irregularity strengths of graphs, *Discrete Mathematics, Algorithms and Applications* ,Vol. 8, No. 4 (2016) 1650070.

[4] M. Anholcer and C. Palmer, Irregular labellings of Circulant graphs, *Discrete Math*. 312 (2012) 3461–3466.

[5] K. Ali, M. Hussain, A. Ahmad and M. Miller, Magic Labellings of type (a,b,c) of families of wheels, *Math. Computer Sci*., 7 (2013) 315-319.

[6] K. Ali, M. Hussain, and Razzaq, Super edge-magic total labeings of a tree, *Util. Math.*, 91 (2013) 355-364.

[7] K. Ali, M. Hussain, H. Shaker, and M. Javaid, Super edge-magic total labelling of subdivided stars, *Ars Combin*., 120 (2015), 161-167.

[8] T. Bohman and D. Kravitz, On the irregularity strength of trees, *J. Graph Theory* 45 (2004) 241–254.

[9] R. Balakrishnan and R. Kumar, Decomposition of complete graphs into isomorphic bipartite subgraphs, *Graphs and Combin*., 10 (1994) 19-25.

[10] M. Baca, J. Jendrol, M. Miller, and J. Ryan, On irregular total labellings, *Discrete Math*., 307 (2007) 1378-1388..

[11] M. Doob, Characterizations of regular magic graphs, *J. Combin*. Theory, Ser. B, 25 (1978) 94-104.

[12] R. J. Faudree and J. Lehel, Bound on the irregularity strength of regular graphs, *Combinatorica* 52 (1987) 247–256.

[13] J.A. Gallian, A dynamic survey of graph labelling, *Electron J. combin.* 18 (2018) #DS6.

[14] J. Gomez and P. Kovar On super vertex-magic total labelling of the disjoint union of k copies of K_n, *Ars Combin*., 113 (2014) 175-192.

[15] I. Gray, New Construction Methods for Vertex-magic Total Labellings of Graphs, *Ph.D. thesis*, University of Newcastle, 2006.

[16] P. Jeyanthi and P. Selvagopal, Supermagic coverings of some simple graphs, *Internat.* J. Math. Combin., 1 (2011) 33-48.

[17] M. Kalkowski, M. Karonski and F. Pfender, A new upper bound for the irregularity strength of graphs, *SIAM J. Discrete Math*. 25(3) (2011) 1319–1321.

[18] J. A. MacDougall, M. Miller, and W. D. Wallis, Vertex-magic total labellings of wheels and related graphs, *Util. Math.*, 62 (2002) 175-183.

[19] T. Nierho and Chartland , A tight bound on the irregularity strength of graphs, *SIAM J. Discrete Math*. 13 (2000) 313–323.

[20] J. Przybylo, Irregularity strength of regular graphs, *Electron. J. Combin*.15 (2008) #R82.

[21] A. Rosa, On certain valuations of the vertices of a graph, *Theory of Graphs (Internat. Symposium, Rome, July 1966), Gordon and Breach, N. Y. and Dunod Paris* (1967) 349-355.

[22] G. Ringel and A. Llado, Another tree conjecture, *Bull. Inst. Combin. Appl*., 18 (1996) 83-85.

[23] A. H. Rokad, 3-Equitable labelling in context of ring sum of graphs Research & Reviews: *Discrete Math. Structures: Open Journal Systems*, 2 no. 3, (2015) 6272- 6275.

[24] J. Sedlacek, Problem 27, In theory and its applications, *Proc.Symp. Smolenice*, 1963, 163-169.