H-Magic Labelings of Graphs under Abelian Groups

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DEDICATION

To

My Parents
&
Family

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ABSTRACT

H-Groupmagic Labelings of Graphs under Abelian Groups

In this thesis, we find the K_2-groupmagic total labeling of unicyclic $C_m \Delta S_{n_j}$ under abelian group $Z_2 \times Z_{(n_j+1)m}$, for all $m \geq 3$ and $1 \leq j \leq m$ and $n_j = n \geq 1$. we also prove that if a graph $G(v,e)$ has super edge magic total labeling, then we can extend this labeling to K_2-groupmagic total labeling under abelian group $\mathbb{Z}_2 \times \mathbb{Z}_v$. We also extend C_3-supermagic labelings to C_3-groupmagic total labeling under abelian group $\mathbb{Z}_3 \times \mathbb{Z}_v$. At the end, we also find C_3-groupmagic total labeling of Anti prism graphs under abelian group $\mathbb{Z}_3 \times \mathbb{Z}_v$.

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Chapter 1

Introduction

”The graph theory started its journey from Königsberg city of Russia around 1736. The people were interested to find the solution of the famous bridge problem of Pregel river which was stated that whether there is a possible way to cross the seven bridges over the river exactly for once and finally reach at the starting point”. Swiss mathematician Leonard Euler solved this mystery after a long time and gave birth to graph theory.
In the following section, we explain some notations, terminologies and basic definitions that will be helpful to understand our studies.

1.1 Elementary Definitions

”A graph $G(V,E)$ consists of a set of vertices or nodes $V(G)$ and set of edges or arcs $E(G)$.” *Order* for a graph G is defined as total number of vertices of G. ”The graphs with order 1 is a *trivial* graphs”. The *size* of a graph is number of edges in it. An edge from a vertex u to itself is a *loop* in a graph G. ”If the end points for two or more than two edges are same, these edges are called *multi-edge/parallel edges*. A graph free of loops and multi-edge is a *simple graph*”. These definition is taken from [27]
“The number of edges around some vertex is called *degree* of that vertex. The vertex is called even or odd if number of edges around that vertex are even or odd respectively.”
”The sum of degrees of all vertices in any graph is always an even number and is double the number of edges, since each edge contribute exactly two degrees to the sum. This statement is called *Hand-shaking lemma*”.
These definition is taken from [27]. A graph H whose nodes and edges both are subsets of nodes and edges respectively, of parent graph G, is a subgraph.
Graph coloring is an assignment to color nodes of a graph so that any two adjacent nodes receive different color. This particular graph coloring is called *vertex coloring*. Similarly, an *edge coloring* is an assignment to color edges so that any two adjacent edges receive the different color.
Two graphs having same number of elements (edges and nodes) and structural similarities are called isomorphic. “*Isomorphism* between graphs G and H is a bijection

$f : V(G) \longrightarrow V(H)$ in such a way that x and y share an edge in G so is $f(x)$ and $f(y)$ in H."

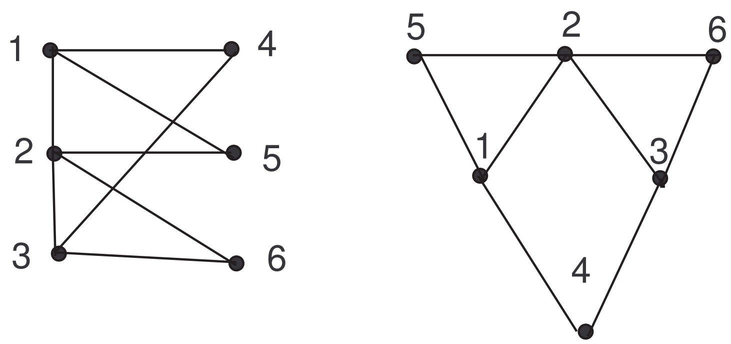


Figure 1.1: Isomorphic graphs

1.1.1 Unicyclic Graph

Let C_m is a cycle has vertices $v_1, v_2, \ldots, v_m$ and S_{n_j}, $j = 1, 2, 3, \ldots, n$, is a star has central vertex w_j and leaves $w_j^j, 1 \leq k \leq n_j$.

"If the star S_{n_j} be adjacent to each vertex v_j, $j = 1, 2, 3, \ldots, n$ by describing v_j and w_j, for $j = 1, 2, 3, \ldots, n$.e attained a unicyclic graph denoted by $C_m \Delta S_{n_j}$.

Let $V(C_m \Delta S_{n_1}) = \{v_j : 1 \leq j \leq m\} \cup \{w_k^j : 1 \leq j \leq m, 1 \leq k \leq n_j\}$is a vertex set and $E(C_m \Delta S_{n_1}) = \{v_j v_{j+1} : 1 \leq j \leq m-1\} \cup \{v_m v_1\} \cup \{v_j w_k^j ; 1 \leq j \leq m, 1 \leq k \leq n_j\}$ is a edge set".

These definition is taken from [4]

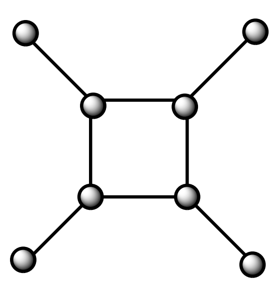


Figure 1.2: Unicyclic graph $C_4 \Delta S_{n_1}$

1.1.2 Generalized Antiprism

" A generalized antiprism A_q^p is a graph obtained by completing the prism graph $C_m \times P_n$ by adding the edges $v_{i,j+1}v_{i+1,j}$ for $1 \leq i \leq m$ and $1 \leq j \leq n-1$".

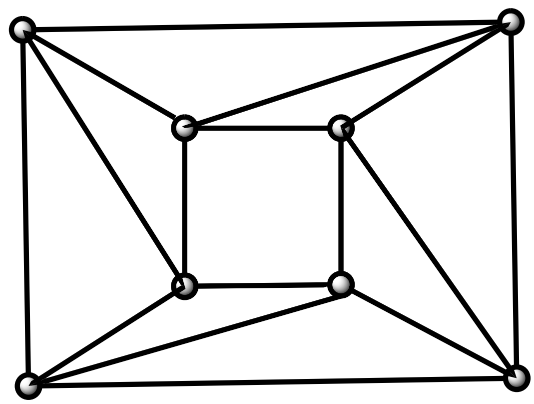


Figure 1.3: Antiprism(A_4^2)

1.2 Application of Graph Theory

Graph theory has application in other fields of mathematics, science and technology. Graphs are used as graph data bases in data base designing [26]. Graph representation with nodes, edges and properties are used in graph database to depict and stock data. The graph anatomy is a vital part in scheming databases, as it has rapid execution using different functions and characteristics of graph anatomy. In Physics [26] real-world structures are modeled into networks and algorithms are defined to compress the information in a network to more simple values (e.g. statistical analysis). Image Analysis [26] is the derivation of information from images. Image analysis is largely executed on digital image processing methods. Graph theoretic approach improves the image processing techniques. For image processing graph search algorithms in segmentation help to find edge boundaries. It helps calculate the picture alignment. The concept of vertex coloring is applied to color map coloring and allocation of frequencies in GSM mobile phone networks. Minimum vertex covers are useful to DNA sequencing and CNS(computer network security) [25]. To determine the synchronization of some network, subgraphs and their complementary subgraphs can be employed. Such graphs especially are chains, cycles, bipartite and product graphs. These can be used to represent network and then synchronization of the network can be characterized by few clear bounds that can be attained for the eigen ratio of the structural matrix of the underlying graph. BIM software is a fantastic tool that can help builders and designers to go through a process of decision making effectively as it is accumulation of the knowledge of mathematical values from the graph theory. With its help the conceptual graph replaces the actual layout of the building and then can be analyzed mathematically. Alternative solution for the designs can be easily assessed.

1.3 Graph Labelings

Graph Labeling has its roots in late 1960s. ”Labeling is a function that allocates the elements of a graph to real numbers, usually positive integers”.

1.3.1 H-Covering

”If each edge of graph G belongs to at least one subgraph H of G, then graph G will possess H-covering”.[4]

1.3.2 H-Magic Labeling

”An H-magic labeling is a total bijective assigning (mapping) of positive numbers from 1 to $v+e$, to the vertex and edge sets of graph G such that for every subgraph D isomorphic to H” has a constant weight j, where the weight of subgraph D is defined as follows: $\sum D = \sum_{v' \in V(D)} \lambda(v') + \sum_{e' \in E(D)} \lambda(e') = j$. In literature, weight j is also called the magic constant. These definition is taken from [27]

1.3.3 Results on H-Magic Labelings of Connected Graphs

Every edge magic graph is H-magic as it is K_2-magic as well. Gutierréz and Lladó after defining H-magic labeling, formulated such labeling for complete bipartite graph $K_{m,n}$ and complete graph K_n. In these cases H was considered star or path [15].
They have proven that "d-regular graph is not $K_{1,h}$-magic for any $1 \leq h \leq d$; complete bipartite graph $K_{n,n}$ is $K_{1,n}$-magic for all n; $K_{n,n}$ is not $K_{1,n}$-supermagic for $n > 1$; for any integer $1 < r < s$, $K_{r,s}$ is $K_{1,h}$-supermagic iff $h = s$."
Authors in [18] have also shown that "path graph P_n is P_h-supermagic for all $2 \leq h \leq n$; K_n is not P_h-magic for any $2 < h \leq n$; C_n is P_h-magic for any $2 \leq h < n$ such that $gcd(n, h(h-1)) = 1$." Lladó et al. [19] have concentrated on *cycle*-magicness of different connected graph. They have also shown C_{2r+k}-magicness of subdivision of wheel graphs $W_n(r,k)$ for any odd $n \neq \frac{2r}{k+1}$ and C_{2r+1}-supermagicness of $W_n(r,k)$ with $k = 1$. They have proven that "the graph which is obtained by joining the endpoints of any number of internally disjoint path of length $P \geq 2$ is C_{2p}-supermagic." Maryati et al. [20] have formulated P_h-supermagic labeling of the graphs obtained after subdivision of star, banana trees and shrubs. Ngurah et al. [22] have constructed all possible C_h-magic labelings of fans and ladder graphs. Ngurah et al. in [22] discussed the C_3-supermagic labeling of fans, $K_{1,n} + K_1$; C_4-supermagic labeling of book graph, C_n-supermagic labeling of chain graphs in which identical blocks are isomorphic to C_n. Their results also include that cycles are P_t-supermagic. Roswitha et al. [3] have proven results on generalize Jahangir graph $J_{k,s}$ to be C_{s+2}-supermagic and $K_{2,n}$ to be C_4-supermagic and W_n to be C_3-supermagic for $n \geq 4$ and even. Roswitha et al. [24] proved that caterpillar, double stars, banana trees and firecrackers are star-supermagic. Gutierréz and Lladó after defining H-magic labeling have shown H-magicness of few connected graphs.

1.3.4 Results on H-Groupmagic Labelings of Graphs

Edelman and Saks [10] discussed the groupmagic labeling of graph in 1979. Gimbel studied vertex,edge labelings of graphs, relationship between vertex and edge labeling of graph over abelian group[14]. Lee et al. [18] examined A-magic graph in which A take as Klei four group. The magic labeling over finite group were studied by Comb [8]. He proved that the star graph have differen edge magic A-labeling. He also proved that any star $K_(1,n)$ for $n \geq 4$ is a vertex A-magic graph, where A is an abelian group of proper order. Recently, Razzaq et al. [23] found the H-groupmagic labelings for fan graph F_n under abelian group $\mathbb{Z}_3 \times \mathbb{Z}_n$ by considering H as K_2 and cycle C_3. For further details on graph labelings, we refer readers to updated survey by Gallian [13].

1.3.5 Application of Graph Labelings

A particular type of graph labeling called radio labeling is used to assign radio channels to radio transmitters. Channels are assigned in a way to avoid disruption of close transmitters. In such labeling locations of the transmitters are being represented by vertices and the labels on the vertices corresponds to the channel or frequencies assigned to stations [6].

Some applications of such graphs in different fields are studied in [21]. In field of electronics ladder networks of resistors are economical to do conversions from digital to analog . One of such devices is weighted ladder and the other one is R/2R ladder. Better choice is R/2R. The reason is its superiority in accuracy and that it is easily manufactured. The function of both devices is conversions from digital voltage information in to analog. In telecommunication systems, a network can be defined by some structure specifically called graph topology. It can be a cycle, star, wheel, fan and moreover a friendship graph. The underlying communication network that is described by some graph consists of nodes and links. In such systems, magic labeling can help detecting route. A magic constant will be defined which is public to network, then router can move along a path that fits the magic constant. This type of routing is usually automatic. For security purposes, reduction of the size of packets and application of labeling scheme help transferring the small information packets quickly and securely as information will be in incomplete chunks. To find a solution of a problem where a system analyst must link its distinct communication ends with unique addresses, edge magic labeling helps creating unique addresses by distinct edge weights. The graph type complete, star, mesh cycle and bus are depicted to be used in computer network system.

Chapter 2

H-groupmagic Labeling over a Finite Abelian Group

In this section we define the vertex and edge set of unicyclic graph $C_m \Delta S_{n_j}$ as follows

$$V(C_m \Delta S_{n_j}) = \{v_j; w_k^j : 1 \leq j \leq m, 1 \leq k \leq n\}$$

$$E(C_m \Delta S_{n_j}) = \{v_j v_{j+1}; w_j w_k^j : 1 \leq j \leq m, 1 \leq k \leq n\}$$

2.0.6 K_2-Groupmagic Labeling of Unicyclic Graph $C_m \Delta S_{n_1}$ for Even Cycle.

Theorem 2.0.1. *For all even $m \geq 3$, the $C_m \Delta S_{n_1}$ is K_2-groupmagic graph over a finite abelian group $A \cong Z_2 \times Z_{2m}$ with groupmagic constant $c = (1, 3m-1)$.*

Proof.

Let $m \geq 3$. We define a labeling $\lambda : V(C_m \Delta S_{n_1}) \to (0,0),(0,1),(0,2),\ldots,(0,2m-1)$ as folllows:

Case 1: $m \equiv 0$ **(**$mod4$**)**

$$\lambda(v_j) = \begin{cases} (0, m-j), & \text{if } 1 \leq j \leq \frac{m}{2} \;\; and \;\; j-odd \\ (0, 2m-j), & \text{if } 1 \leq j \leq \frac{m}{2} \;\; and \;\; j-even \\ (0, m-1-j), & \text{if } \frac{n}{2}+1 \leq j \leq m \;\; and \;\; j \;\; -odd \\ (0, 2m-1-j), & \text{if } 1+\frac{n}{2} \leq j \leq m-1 \;\; and \;\; j \;\; -even \\ (0, 2m-1), & \text{if } j = m \end{cases}$$

$$\lambda(w_j^j) = \begin{cases} (0,1), & \text{if } j=1 \\ (0,m-j), & \text{if } 2 \le j \le \frac{m}{2} \text{ and } j-even \\ (0,2m-j), & \text{if } 2 \le j \le \frac{m}{2}+1 \text{ and } j-even \\ (0,2m+1-j), & \text{if } 2+\frac{n}{2} \le j \le m-1 \text{ and } j-odd \\ (0,m+1-j), & \text{if } \frac{n}{2}+1 \le j \le m-2 \text{ and } j-even \\ (0,m), & \text{if } j=m \end{cases}$$

Case 2: $m = 6, 10, 14, \ldots$

$$\lambda(v_j) = \begin{cases} (0,-1+m), & \text{if } j=1 \\ (0,j-1+m), & \text{if } 2 \le j \le m \text{ and } j-even \\ (0,j-3), & \text{if } 3 \le j \le \frac{m}{2} \text{ and } j-odd \\ (0,j-1), & \text{if } \frac{n}{2}+1 \le j \le m \text{ and } j-odd \end{cases}$$

$$\lambda(w_1^1) = (0,1)$$

$$\lambda(w_m^m) = (0,m)$$

$$\lambda(w_j^j) = \begin{cases} (0, j-1), & \text{if } 2 \leq j \leq \frac{m}{2} \ \ and \ \ j \ \ -even \\ \\ (0, \frac{m}{2}-1), & \text{if } j = \frac{m}{2}+1 \\ \\ (0, -1+j), & \text{if } \frac{n}{2}+2 \leq j \leq m-1 \ \ and \ \ j \ \ -even \\ \\ (0, j-1+m), & \text{if } 3 \leq j \leq m \ \ and \ \ j \ \ -odd \\ \end{cases}$$

.

We can see that the weights of edges are $(0,m),(0,m+1),(0,m+2),\dots,(0,3m-1)$ under λs. Now, by using lemma 1, we have K_2-groupmagic total labeling over a finite abelian group $\mathbb{Z}_2 \times \mathbb{Z}_{2m}$ with weight of every edge is c = $(1,3m-1)$, which completes the proof.

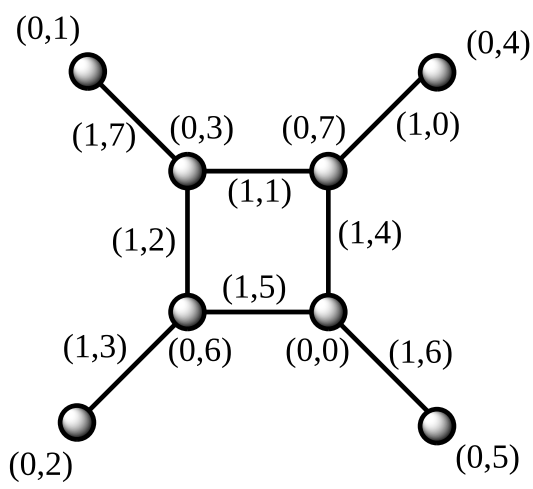


Figure 2.1: $C_4 \Delta S_1$

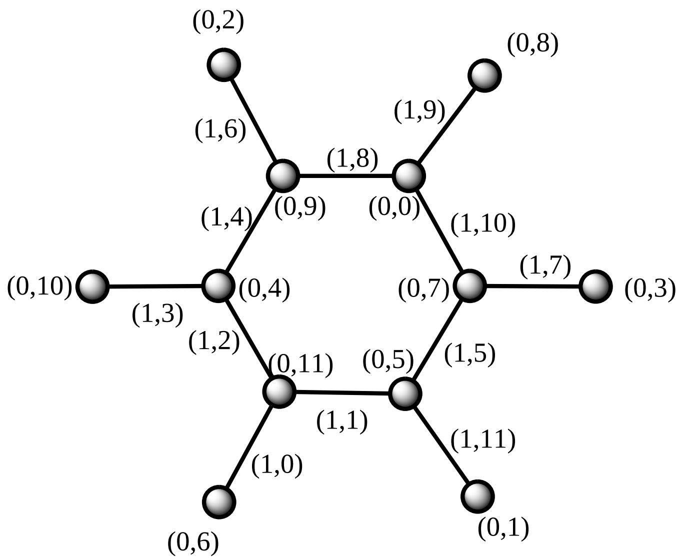


Figure 2.2: $C_6 \Delta S_1$

Theorem 2.0.2. *For all even $m \geq 3$, the $C_m \Delta S_{n_1}$ is K_2-groupmagic graph over a finite abelian group $A \cong Z_2 \times Z_{2m}$ with groupmagic constant $c = (0, 3m-1)$.*

Proof.

Let $m \geq 3$. We define a labeling $\lambda : V(C_m \Delta S_{n_1}) \to (1,0),(1,1),(1,2),\ldots,(1,2m-1)$ as follows:

Case 1: $m \equiv 0$ **(**$mod 4$**)**

$$\lambda(v_j) = \begin{cases} (1, m-j), & \text{if } 1 \leq j \leq \frac{m}{2} \ \ and \ \ j-odd \\ (1, 2m-j), & \text{if } \frac{m}{2} \geq j \geq 1 \ \ and \ \ j-even \\ (1, m-1-j), & \text{if } \frac{n}{2}+1 \leq j \leq m \ \ and \ \ j \ \ -odd \\ (1, 2m-1-j), & \text{if } \frac{n}{2}+1 \leq j \leq m-1 \ \ and \ \ j \ \ -even \\ (1, 2m-1), & \text{if } \ j=m \end{cases}$$

$$\lambda(w_j^j) = \begin{cases} (1,1), & \text{if } \ j=1 \\ (1, m-j), & \text{if } \ 2 \leq j \leq \frac{m}{2} \ \ and \ \ j \ \ -even \\ (1, 2m-j), & \text{if } \ 2 \leq j \leq \frac{m}{2}+1 \ \ and \ \ j-even \\ (1, 2m+1-j), & \text{if } \ 2+\frac{n}{2} \leq j \leq m-1 \ \ and \ \ j \ \ -odd \\ (1, -j+1+m), & \text{if } \ \frac{n}{2}+1 \leq j \leq m-2 \ \ and \ \ j \ \ -even \\ (1, m), & \text{if } \ j=m \end{cases}$$

Case 2: $m = 6, 10, 14, \ldots$

$$\lambda(v_j) = \begin{cases} (1, m-1), & \text{if } j = 1 \\ (1, m-1+j), & \text{if } 2 \leq j \leq m \ \ and \ \ j-even \\ (1, j-3), & \text{if } 3 \leq j \leq \frac{m}{2} \ \ and \ \ j-odd \\ (1, j-1), & \text{if } \frac{n}{2}+1 \leq j \leq m \ \ and \ \ j \ \ -odd \end{cases}$$

$$\lambda(w_j^j) = \begin{cases} (1,1), & \text{if } j = 1 \\ (1, j-1), & \text{if } 2 \leq j \leq \frac{m}{2} \ \ and \ \ j \ \ -even \\ (1, \frac{m}{2}-1), & \text{if } j = \frac{m}{2}+1 \\ (1, j-1), & \text{if } \frac{n}{2}+2 \leq j \leq m-1 \ \ and \ \ j \ \ -even \\ (1, m), & \text{if } j = m \\ (1, j+m-1), & \text{if } 3 \leq j \leq m \ \ and \ \ j \ \ -odd \end{cases}$$

.

We can see that set of weights of edges has elements $(0,m),(0,m+1),(0,m+2),\ldots,(0,3m-1)$ under λ,. Now, by using lemma 1, we have K_2-groupmagic total labeling over a finite abelian group $\mathbb{Z}_2 \times \mathbb{Z}_{2m}$ with weight of every edge is $c = (0,3m-1)$. Which completes the proof.

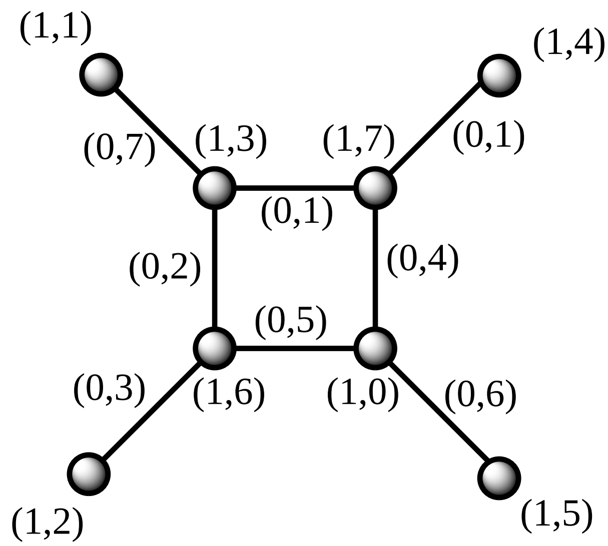


Figure 2.3: $C_4 \Delta S_1$

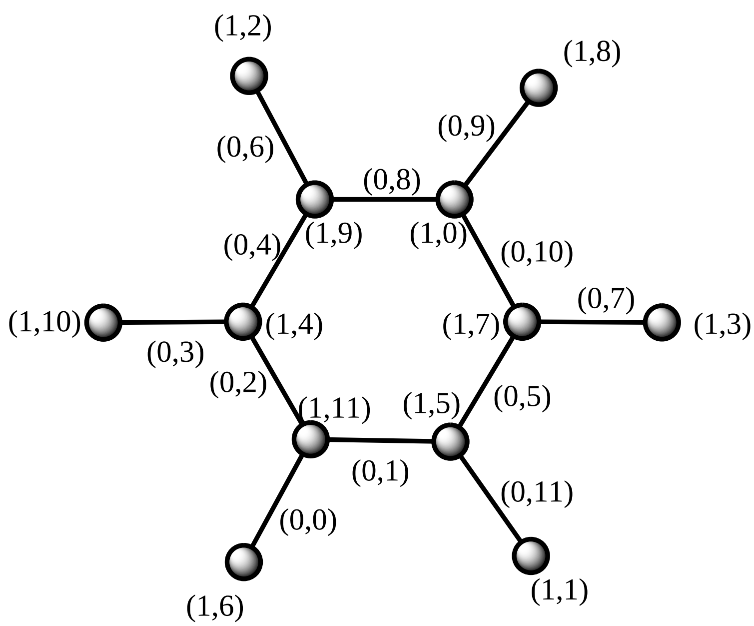


Figure 2.4: $C_6 \Delta S_1$

2.0.7 K_2-Groupmagic Labeling of Unicyclic Graph $C_m \Delta S_{n_j}$ for all Even m and $n_j = n \geq 2$.

Theorem 2.0.3. *For all even $m \geq 3$ and $1 \leq j \leq m$ and $n_j = n \geq 2$, the $C_m \Delta S_{n_j}$ is K_2-groupmagic graph over a finite abelian group $A \cong Z_2 \times Z_{(n_j+1)m}$.*

Proof:

Let $m \geq 3$. We define a labeling $\lambda : V(C_m \Delta S_{n_1}) \to (0,0),(0,1),(0,2),\ldots,(0,(n_j+1)m-1|)$ as follows: **Case 1:** $m \equiv 0$ **(**$mod 4$**)**

$$\lambda(v_j) = \begin{cases} (0,(n+1)(\frac{j}{2})-1), & \text{if } 1 \leq j \leq \frac{m}{2}-1 \ and \ \ j \ \ -even \\ \\ (0,(n+1)(\frac{j}{2})), & \text{if } \frac{m}{2} \leq j \leq m \ and \ \ j \ \ -even \\ \\ (0,(\frac{nj+nm-n}{2})+\lceil\frac{j+m}{2}\rceil), & \text{if } 1 \leq j \leq \frac{m}{2} \ and \ \ j-odd \\ \\ (0,(\frac{nm-n+nj}{2})+\lceil\frac{j+m}{2}\rceil-1), & \text{if } \frac{m}{2}+1 \leq j \leq m \ and \ \ j-odd \\ \end{cases}$$

For i odd

$$\lambda(w_k^j) = \begin{cases} (0,n(\frac{j-1}{2})+\lceil\frac{2k+j}{2}\rceil-2), & \text{if } 1 \leq j \leq \frac{m}{2}, 1 \leq k \leq n \\ \\ (0,n(\frac{j-1}{2})+\lceil\frac{j+2k}{2}\rceil), & \text{if } j = 1+\frac{m}{2}, k=1 \\ \\ (0,n(\frac{j-1}{2})+\lceil\frac{2k+j}{2}\rceil-1), & \text{if } j = \frac{m}{2}+1, 2 \leq k \leq n \\ \\ (0,(\frac{nj-n}{2})+\lceil\frac{2k+j}{2}\rceil-1), & \text{if } \frac{m}{2}+2 \leq j \leq m, 1 \leq k \leq n \\ \end{cases}$$

For i even

$$\lambda(w_k^j)=\begin{cases}(0,(n+1)(\frac{j+m}{2})+k-n), & \text{if } 1\leq j\leq \frac{m}{2}-1, 1\leq k\leq n\\ \\ (0,(n+1)(\frac{j}{2})+1), & \text{if } j=\frac{m}{2}, k=1\\ \\ (0,(1+n)(\frac{m+j}{2})+k-n-1), & \text{if } j=\frac{m}{2}, 2\leq k\leq n\\ \\ (0,(1+n)(\frac{j+m}{2})+k-n-1), & \text{if } \frac{m}{2}+1\leq j\leq m, 1\leq k\leq n\\ \end{cases}$$

We can see that under labeling λ, the weights of edges are $(0,\frac{6m+8}{4}-2),(0,\frac{6m+8}{4}-1),(0,\frac{6m+8}{4}),\ldots,(0,\frac{9m+2}{2}-2)$. Now, by using lemma 1, we have K_2-groupmagic total labeling over a finite abelian group $\mathbb{Z}_2\times\mathbb{Z}_{(n_j+1)m}$ with weight of every edge is $c=(1,\frac{9m+2}{2}-2)$. Which completes the proof.

Case 2: $m\equiv 2$ **(**$mod4$**)**

$$\lambda(v_j)=\begin{cases}(0,(1+n)(\frac{j}{2})-1), & \text{if } 1\leq j\leq \frac{m}{2} \ and \ j \ -even\\ \\ (0,(1+n)(\frac{j}{2})), & \text{if } \frac{m}{2}+1\leq j\leq m \ and \ j \ -even\\ \\ (0,(\frac{nm-n+nj}{2})+\lceil\frac{j+m}{2}\rceil), & \text{if } 1\leq j\leq \frac{m}{2}-1 \ and \ j \ -odd\\ \\ (0,(\frac{nm-n+nj}{2})+\lceil\frac{j+m}{2}\rceil-1), & \text{if } \frac{m}{2}\leq j\leq m \ and \ j \ -odd\\ \end{cases}$$

For j odd

$$\lambda(w_k^j) = \begin{cases} (0, n(\frac{j-1}{2}) + \lceil \frac{j}{2} \rceil - 2 + k), & \text{if } 1 \le j \le \frac{m}{2} - 1, 1 \le k \le n \\ \\ (0, (\frac{nj-n}{2}) + \lceil \frac{j}{2} \rceil + k - 1), & \text{if } \frac{m}{2} \le j \le m, 1 \le k \le n \\ \end{cases}$$

For j even

$$\lambda(v_k^j) = \begin{cases} (0, (n+1)(\frac{m+j}{2}) + k - n), & \text{if } 1 \le j \le \frac{m}{2} - 2, 1 \le k \le n \\ \\ (0, (n+1)(\frac{j+m}{2}) + k - n), & \text{if } j = \frac{m}{2} - 1, 1 \le k \le n - 1 \\ \\ (0, (n+1)(\frac{m+j}{2}) + k + 1 - n), & \text{if } j = \frac{m}{2} - 1, k = n \\ \\ (0, (n+1)(\frac{j-2}{2})), & \text{if } j = \frac{m}{2} + 1, k = 1 \\ \\ (0, (1+n)(\frac{m+j}{2}) + k - n - 1), & \text{if } j = \frac{m}{2} + 1, 2 \le k \le n \\ \\ (0, (1+n)(\frac{m+j}{2}) + k - n - 1), & \text{if } \frac{m}{2} + 2 \le j \le m, 1 \le k \le n \\ \end{cases}$$

We can see that set of the weights of edges are $(0,m),(0,m+1),(0,m+2),\ldots,(0,3m-1)\}$ under λ. Now, by using lemma 1, we have K_2-groupmagic total labeling over a finite abelian group $\mathbb{Z}_2 \times \mathbb{Z}_{(n_j+1)m}$ with weight of every edge is c = $(1, 2mn_j + 2)$. Which completes the proof. □

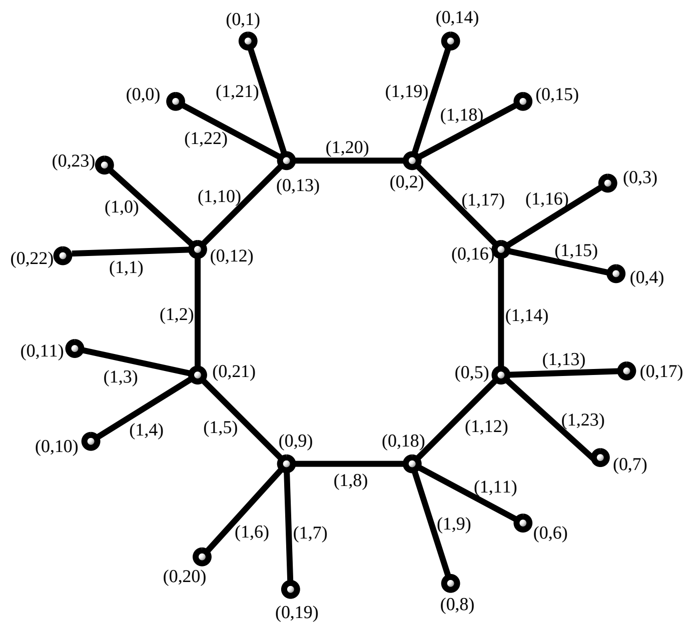


Figure 2.5: $C_8 \Delta S_2$

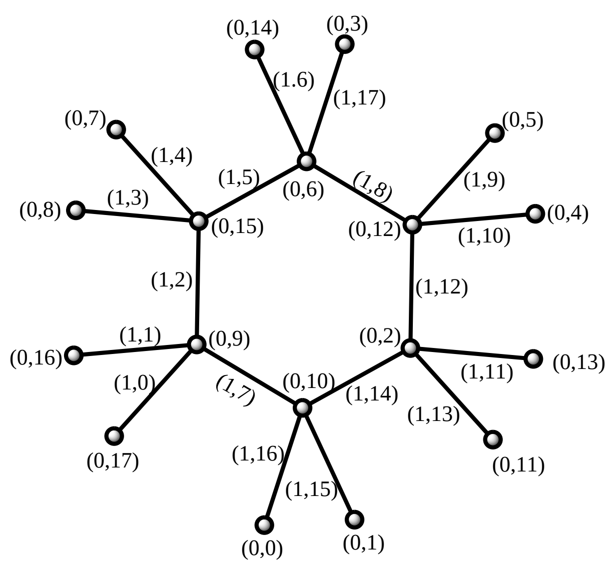


Figure 2.6: $C_6 \Delta S_2$

Theorem 2.0.4. *For all even $m \geq 3$ and $1 \leq j \leq m$ and $n_j = n \geq 2$, the $C_m \Delta S_{n_j}$ is K_2-groupmagic graph over a finite abelian group $A \cong Z_2 \times Z_{(n_j+1)m}$.*

Proof:

Let $m \geq 3$. We define a labeling $\lambda : V(C_m \Delta S_{n_1}) \rightarrow (1,0),(1,1),(1,2),\ldots,(1,(n_j+1)m-1|)$, as follows:

Case 1: $m \equiv 0$ **(**$mod 4$**)**

$$\lambda(v_j) = \begin{cases} (1,(n+1)(\frac{j}{2})-1), & \text{if } 1 \leq j \leq \frac{m}{2}-1 \ \ and \ \ j \ \ -even \\ \\ (1,(n+1)(\frac{j}{2})), & \text{if } \frac{m}{2} \leq j \leq m \ \ and \ \ j \ \ -even \\ \\ (1,(\frac{nm-n+nj}{2})+\lceil\frac{j+m}{2}\rceil), & \text{if } 1 \leq j \leq \frac{m}{2} \ \ and \ \ j-odd \\ \\ (1,(\frac{nm-n+nj}{2})-1+\lceil\frac{m+j}{2}\rceil), & \text{if } \frac{m}{2}+1 \leq j \leq m \ \ and \ \ j-odd \end{cases}$$

For i odd

$$\lambda(w_k^j) = \begin{cases} (1,(\frac{nj-n}{2})+\lceil\frac{2k+j}{2}\rceil-2), & \text{if } 1 \leq j \leq \frac{m}{2}, 1 \leq k \leq n \\ \\ (1,n(\frac{j-1}{2})+\lceil k+\frac{j}{2}\rceil), & \text{if } j = \frac{m}{2}+1, k=1 \\ \\ (1,n(\frac{j-1}{2})-1+\lceil\frac{2k+j}{2}\rceil), & \text{if } j = \frac{m}{2}+1, 2 \leq k \leq n \\ \\ (1,(\frac{nj-n}{2})-1+\lceil\frac{2k+j}{2}\rceil), & \text{if } \frac{m}{2}+2 \leq j \leq m, 1 \leq k \leq n \end{cases}$$

For i even

$$\lambda(w_k^j) = \begin{cases} (1,(n+1)(\frac{m+j}{2})-n+k), & \text{if } 1 \leq j \leq \frac{m}{2}-1, 1 \leq k \leq n \\ (1,(n+1)(\frac{j}{2})+1), & \text{if } j = \frac{m}{2}, k=1 \\ (1,(n+1)(\frac{m+j}{2})+k-n-1), & \text{if } j = \frac{m}{2}, 2 \leq k \leq n \\ (1,(n+1)(\frac{m+j}{2})+k-n-1), & \text{if } \frac{m}{2}+1 \leq j \leq m, 1 \leq k \leq n \end{cases}$$

We can see that under labeling λ, the weights of edges are $(0, \frac{6m+8}{4}-2), (0, \frac{6m+8}{4}-1), (0, \frac{6m+8}{4}), \ldots, (0, \frac{9m+2}{2}-2)$. Now, by using lemma 1, we have K_2-groupmagic total labeling over a finite abelian group $\mathbb{Z}_2 \times \mathbb{Z}_{(n_j+1)m}$ with weight of every edge is $c = (0, \frac{9m+2}{2}-2)$. Which completes the proof.

Case 2: $m \equiv 2$ **(**$mod 4$**)**

$$\lambda(v_j) = \begin{cases} (1,(n+1)(\frac{j}{2})-1), & \text{if } 1 \leq j \leq \frac{m}{2} \text{ and } j - even \\ (1,(n+1)(\frac{j}{2})), & \text{if } \frac{m}{2}+1 \leq j \leq m \text{ and } j - even \\ (1,(\frac{nm-n+nj}{2})+\lceil\frac{m+j}{2}\rceil), & \text{if } 1 \leq j \leq \frac{m}{2}-1 \text{ and } j - odd \\ (1,(\frac{nm-n+nj}{2})-1+\lceil\frac{m+j}{2}\rceil), & \text{if } \frac{m}{2} \leq j \leq m \text{ and } j - odd \end{cases}$$

For j odd

$$\lambda(w_k^j) = \begin{cases} (1, (\frac{nj-n}{2}) + \lceil \frac{j}{2} \rceil + k - 2), & \text{if } 1 \le j \le \frac{m}{2} - 1, 1 \le k \le n \\ \\ (1, n(\frac{j-1}{2}) + \lceil \frac{j}{2} \rceil - 1 + k), & \text{if } \frac{m}{2} \le j \le m, 1 \le k \le n \\ \end{cases}$$

For j even

$$\lambda(v_k^j) = \begin{cases} (1, (n+1)(\frac{m+j}{2}) + k - n), & \text{if } 1 \le j \le \frac{m}{2} - 2, 1 \le k \le n \\ \\ (1, (n+1)(\frac{m+j}{2}) + k - n), & \text{if } j = \frac{m}{2} - 1, 1 \le k \le n - 1 \\ \\ (1, (n+1)(\frac{m+j}{2}) + k + 1 - n), & \text{if } j = \frac{m}{2} - 1, k = n \\ \\ (1, (n+1)(\frac{j-2}{2})), & \text{if } j = \frac{m}{2} + 1, k = 1 \\ \\ (1, (n+1)(\frac{m+j}{2}) + k - 1 - n), & \text{if } j = \frac{m}{2} + 1, 2 \le k \le n \\ \\ (1, (n+1)(\frac{m+j}{2}) + k - 1 - n), & \text{if } \frac{m}{2} + 2 \le j \le m, 1 \le k \le n \\ \end{cases}$$

We can see that set of the weights of edges has elements $(0,m),(0,m+1),(0,m+2),\ldots,(0,3m-1)\}$. Now, by using lemma 1, we have K_2-groupmagic total labeling over a finite abelian group $\mathbb{Z}_2 \times \mathbb{Z}_{(n_j+1)m}$ with weight of every edge is $c = (0, 2mn_j + 2)$. Which completes the proof. $\square$

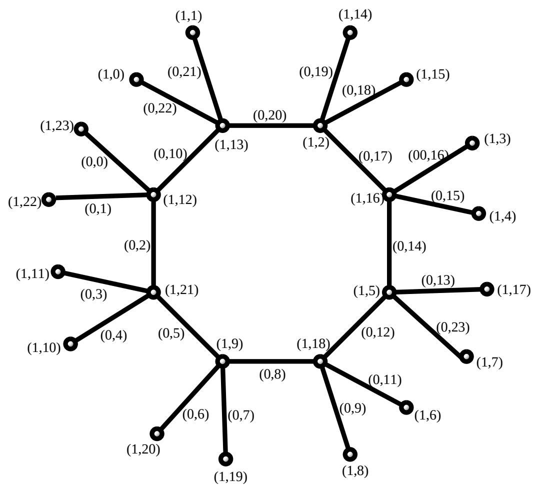


Figure 2.7: $C_8 \Delta S_2$

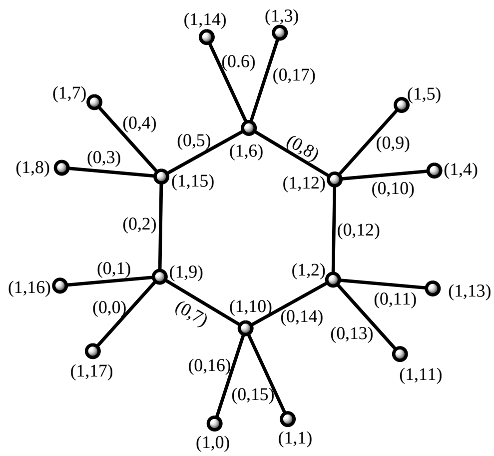


Figure 2.8: $C_6 \Delta S_2$

2.0.8 K_2-Groupmagic Labeling of Unicyclic Graph $C_m \Delta S_{n_j}$ for all Odd m and $n_j = n \geq 2$.

Theorem 2.0.5. *For all odd $m \geq 3$ and $1 \leq j \leq m$, the $C_m \Delta S_{n_j}$ is K_2-groupmagic graph over a finite abelian group $A \cong Z_2 \times Z_{(n_j+1)m}$ with groupmagic constant $c = (1, 3m-2)$.*

Proof.

Let $m \geq 3$. We define a labeling $\lambda : V(C_m \Delta S_{n_j}) \to \{(0,0),(0,1),(0,2),\ldots,(0,(n_j+$

$1)m-1|)\}$, where $j=1,2,3,\ldots,r$ as follows:

Label the vertices as follows.

$$\lambda(v_j)=\begin{cases}(0,\frac{(j-1)(1+n_j)}{2}), & \text{if } j-odd\\ \\ (0,\frac{(m-1+j)(1+n_j)}{2}), & \text{if } j-even\\ \end{cases}$$

Label the leaves verrtices as follows.

$$\lambda(w_k^j)=\begin{cases}(0,\frac{(m+j-2)(n_j+1)}{2}+k), & \text{if } j-odd\\ \\ (0,\frac{(j-2)(n_j+1)}{2}+k), & \text{if } j-even\\ \end{cases}$$

We can see that under labeling λ, the weights of edges are $\{(0,m-1),(0,m),(0,m+1),(0,m+2),\ldots,(0,3m-2)\}$. Now, by using lemma 1, we have K_2-groupmagic total labeling over a finite abelian group $\mathbb{Z}_2\times\mathbb{Z}_{(n_j+1)m}$ with weight of every edge is $c=(1,3m-2)$.

Which completes the proof.

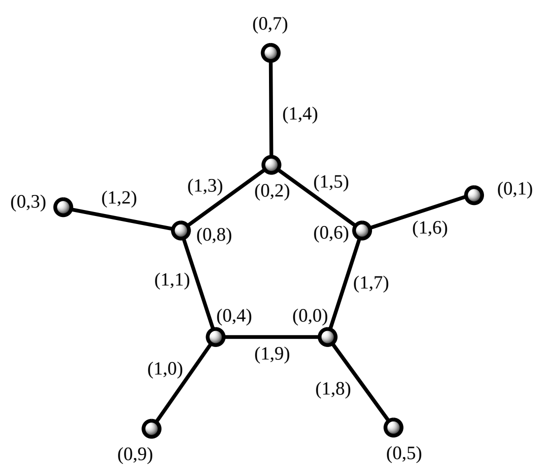


Figure 2.9: $C_5\Delta S_1$

Theorem 2.0.6. *For all odd $m \geq 3$ and $1 \leq j \leq m$, the $C_m \Delta S_{n_j}$ is K_2-groupmagic graph over a finite abelian group $A \cong Z_2 \times Z_{(n_j+1)m}$ with groupmagic constant $c =$* $(0, 3m-2)$.

Proof.

Let$C_m \Delta S_{n_j}$be the unicyclic with V vertex set and E edge set.

Let $m \geq 3$. We define a labeling $\lambda : V(C_m \Delta S_{n_j}) \rightarrow \{(1,0),(1,1),(1,2),\ldots,(1,(n_j+1)m-1|)\}$, where $j = 1,2,3,\ldots,r$ as follows: Label the vertices as follows.

$$\lambda(v_j) = \begin{cases} (1, \frac{(j-1)(r+1)}{2}), & \text{if } \ j-odd \\ \\ (1, \frac{(m-1+j)(1+r)}{2}), & \text{if } \ j-even \\ \end{cases}$$

Label the leaves verrtices as follows.

$$\lambda(v_k^j) = \begin{cases} (1, \frac{(m-2+j)(1+r)}{2} + k), & \text{if } \ j-odd \\ \\ (1, \frac{(j-2)(1+r)}{2} + k), & \text{if } \ j-even \\ \end{cases}$$

We can see that under labeling λ, the weights of edges are $\{(0,m-1),(0,m),(0,m+1),(0,m+2),\ldots,(0,3m-2)\}$. Now, by using lemma 1, we have K_2-groupmagic total labeling over a finite abelian group $\mathbb{Z}_2 \times \mathbb{Z}_{(n_j+1)m}$ with weight of every edge is $c = (0, 3m-2)$.

2.0.9 Lemma

A graph $G(v,e)$ is K_2-groupmagic over finite abelian group $Z_2 \times Z_v$ if and only if there exists a bijective map $\lambda : V(G) \rightarrow \{(a,0),(a,1),(a,2),\ldots,(a,v-1)\}$ under which the set

$Z = \{\lambda(x)+\lambda(y)|xy \in E\} = \{(0,z),(0,z+1),(0,z+2),\ldots,(0,z+e-1)\}$ consists of e order pairs, where $a \in \{0,1\}$. If this set Z exits then λ becomes "K_2-groupmagic total labeling of G" over a finite abelian group A with valence $c = (a+1, v+z-1)$, where $z = min(Z)$ and

$$Z = \{c-(a+1,0), c-(a+1,1), c-(a+1,2), \ldots, c-(a+1,v-1)\}.$$

Proof.

First suppose that such a function λ exists and let $xy \in E(G)$, so that $z = min(Z) = \lambda(x)+\lambda(y)$. Then λ extends to the domain $V(G) \cup E(G)$ in the following manner. Let $\lambda(xy) = c-\lambda(x)-\lambda(y)$ for any edge xy of G. Then
$\lambda(E(G)) = \{(a+1,0),(a+1,1),(a+1,2),\ldots,(a+1,v-1)\}$.
Conversely, if G is K_2-groupmagic over a finite abelian group A with K_2-groupmagic labeling λ over a finite abelian group A of constant c. Then

$$\begin{aligned} Z &= \{c-\lambda(xy)\} \\ &= \{c-(a+1,0), c-(a+1,1), \ldots, c-(a+1,v-1)\}. \end{aligned}$$

□

2.0.10 K_2-Groupmagic Total Labeling over a Finite Abelian Group $A \cong \mathbb{Z}_2 \times \mathbb{Z}_v$ with $|A| = v+e$.

Theorem 2.0.7. *Let $G(v,e)$ be a super edge magic total graph. Then G has a K_2-groupmagic total labeling over a finite abelian group $A \cong \mathbb{Z}_2 \times \mathbb{Z}_v$ with $|A| = v+e$.*

Proof:

Since $G(v,e)$ is a super edge magic total graph. Therefore, there exists a bijective mapping $F : V(G) \cup E(G) \to \{1,2,\ldots,v+e\}$ such that weight of every edge xy under F is a constant c, where $c = F(x)+F(y)+F(xy)$.

Now, we extend this labeling $\digamma$ for K_2-groupmagic total labeling g under finite abelian group $A \cong \mathbb{Z}_2 \times \mathbb{Z}_v$ as follows,

$$\begin{aligned} g(v_i) &= (a, \digamma(v_i) - 1),\ 1 \leq i \leq v. \\ g(e_j) &= (a+1, \digamma(e_j) - v - 1),\ 1 \leq j \leq e. \end{aligned}$$

where, $a \in [0,1]$. We can see that the set Z is of the form $\{(0,z),(0,z+1),(0,z+2),\ldots,(0,z+e-1)\}$ because $\digamma$ is a super edge magic total labeling.

Now, by using lemma 2.4, we have two K_2-groupmagic total labeling over a finite abelian group $\mathbb{Z}_2 \times \mathbb{Z}_v$ by finding the weight of every edge say c' under labeling g as follow:

$$\begin{aligned} c' &= g(x) + g(y) + g(xy) \\ &= (a, \digamma(x) - 1) + (a, \digamma(y) - 1) + (a+1, \digamma(xy) - (v+1)) \\ &= (3a+1, \digamma(x) + \digamma(y) + \digamma(xy) - (v+3)) \\ &= (3a+1, c - v - 3) \end{aligned}$$

We can see that C' is a constant because it is depending on constanta, c and v, therefore G is a K_2-groupmagic graph. $\square$

It has been proved that all odd cycles [11] are super edge magic total graphs. By using above labeling we have following corollary.

Corollary 2.0.8. *All odd cycles C_n are K_2-groupmagic over a finite abelian group $\mathbb{Z}_2 \times \mathbb{Z}_n$.*

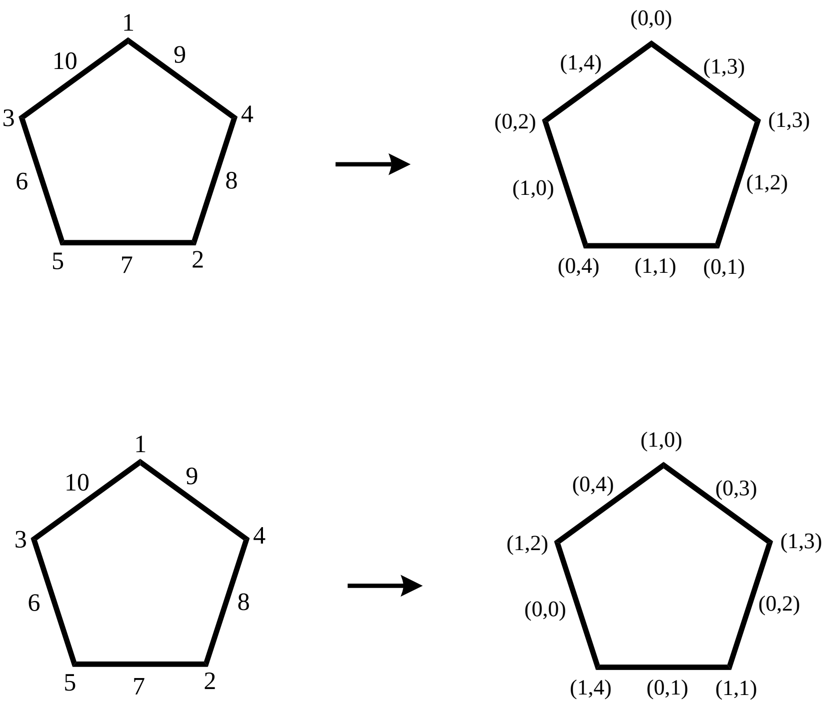


Figure 2.10: C_5

2.0.11 C_3-Groupmagic Total Labeling over a Finite Abelian Group $A \cong \mathbb{Z}_3 \times \mathbb{Z}_v$ with $|A| = v + e$.

Theorem 2.0.9. *Let $G(v,e)$ be a C_3-supermagic total graph. Then G has six C_3-groupmagic total labelings over a finite abelian group $A \cong \mathbb{Z}_3 \times \mathbb{Z}_v$ with $|A| = v + e$ and $e = 2v$.*

Proof:

Since $G(v,e)$ is a C_3-supermagic total graph. Therefore, there exists a bijective mapping $\mathcal{F} : V(G) \cup E(G) \to \{1, 2, \ldots, v+e\}$ such that weight of every subgraph isomorphic to cycle C_3 under $\mathcal{F}$ is a constant k, where $k = \sum\limits_{v \in V(C_3)} \mathcal{F}(v) + \sum\limits_{e \in E(C_3)} \mathcal{F}(e)$.

Now, we extend the labeling $\mathcal{F}$ for C_3-groupmagic labeling. Let g be such C_3-groupmagic labeling and we can define total six labelings under abelian group $A \cong \mathbb{Z}_3 \times \mathbb{Z}_v$. We define such labelings as follows,

$$\begin{aligned}
g_1(v_i) &= (0, F(v_i)-1),\ 1 \le i \le v. \\
g_1(e_j) &= (1, F(e_j)-v-1),\ v+1 \le F(e_j) \le 2v. \\
g_1(e_k) &= (2, F(e_k)-2v-1),\ 2v+1 \le F(e_k) \le 3v.
\end{aligned}$$

Now, we find the weight of every subgraph isomorphic to cycle C_3, say k', under labeling g over a finite abelian group $A \cong \mathbb{Z}_3 \times \mathbb{Z}_v$ as follow:

$$\begin{aligned}
k' &= \sum_{v \in V(C_3)} g_1(v) + \sum_{e \in E(C_3)} g_1(e) \\
&= g_1(x)+g_1(y)+g_1(z)+g_1(xy)+g_1(xz)+g_1(yz) \\
&= (0, F(x)-1)+(0, F(y)-1)+(0, F(z)-1)+(1, F(xy)-(v+1) \\
&+ (1, F(xz)-(v+1))+(2, F(yz)-(2v+1)) \\
&= (4, F(x)+F(y)+F(xy)+F(z)+F(xz)+F(yz)-(4v+6)) \\
&= (1, k-4v-6)
\end{aligned}$$

Where, $4 \equiv 1\ (mod 3)$.

and

$$\begin{aligned}
g_2(v_i) &= (0, F(v_i)-1),\ 1 \le i \le v. \\
g_2(e_j) &= (2, F(e_j)-v-1),\ v+1 \le F(e_j) \le 2v. \\
g_2(e_k) &= (1, F(e_k)-2v-1),\ 2v+1 \le F(e_k) \le 3v.
\end{aligned}$$

Now, we find the weight of every subgraph isomorphic to cycle C_3, say k'', under labeling g over a finite abelian group $A \cong \mathbb{Z}_3 \times \mathbb{Z}_v$ as follow:

$$
\begin{aligned}
k'' &= \sum_{v \in V(C_3)} g_2(v) + \sum_{e \in E(C_3)} g_2(e) \\
&= g_2(x) + g_2(y) + g_2(z) + g_2(xy) + g_2(xz) + g_2(yz) \\
&= (0, F(x) - 1) + (0, F(y) - 1) + (0, F(z) - 1) + (2, F(xy) - (2v + 1) \\
&+ (2, F(xz) - (v + 1)) + (1, F(yz) - (v + 1)) \\
&= (5, F(x) + F(y) + F(z) + F(xy) + F(xz) + F(yz) - (4v + 6)) \\
&= (2, k - 4v - 6)
\end{aligned}
$$

Where, $5 \equiv 2 \ (mod 3)$. and

$$
\begin{aligned}
g_3(v_i) &= (1, F(v_i) - 1),\ 1 \leq i \leq v. \\
g_3(e_j) &= (0, F(e_j) - v - 1),\ v + 1 \leq F(e_j) \leq 2v. \\
g_3(e_k) &= (2, F(e_k) - 2v - 1),\ 2v + 1 \leq F(e_k) \leq 3v.
\end{aligned}
$$

Now, we find the weight of every subgraph isomorphic to cycle C_3, say k''', under labeling g over a finite abelian group $A \cong \mathbb{Z}_3 \times \mathbb{Z}_v$ as follow:

$$
\begin{aligned}
k''' &= \sum_{v \in V(C_3)} g_2(v) + \sum_{e \in E(C_3)} g_2(e) \\
&= g_3(x) + g_3(y) + g_3(z) + g_3(xy) + g_3(xz) + g_3(yz) \\
&= (1, F(x) - 1) + (1, F(y) - 1) + (1, F(z) - 1) + (0, F(xy) - (2v + 1) \\
&+ (0, F(xz) - (v + 1)) + (2, F(yz) - (v + 1)) \\
&= (5, F(x) + F(y) + F(xy) + F(z) + F(xz) + F(yz) - (4v + 6)) \\
&= (2, k - 4v - 6)
\end{aligned}
$$

Where, $5 \equiv 2\ (mod 3)$. and

$$\begin{aligned} g_4(v_i) &= (1, F(v_i) - 1),\ 1 \leq i \leq v. \\ g_4(e_j) &= (2, F(e_j) - v - 1),\ v + 1 \leq F(e_j) \leq 2v. \\ g_4(e_k) &= (0, F(e_k) - 2v - 1),\ 2v + 1 \leq F(e_k) \leq 3v. \end{aligned}$$

Now, we find the weight of every subgraph isomorphic to cycle C_3, say k'''', under labeling g over a finite abelian group $A \cong \mathbb{Z}_3 \times \mathbb{Z}_v$ as follow:

$$\begin{aligned} k'''' &= \sum_{v \in V(C_3)} g_2(v) + \sum_{e \in E(C_3)} g_2(e) \\ &= g_4(x) + g_4(y) + g_4(z) + g_4(xy) + g_4(xz) + g_4(yz) \\ &= (1, F(x) - 1) + (1, F(y) - 1) + (1, F(z) - 1) + (2, F(xy) - (2v + 1) \\ &+ (2, F(xz) - (v + 1)) + (0, F(yz) - (v + 1)) \\ &= (7, F(x) + F(y) + F(xy) + F(z) + F(xz) + F(yz) - (4v + 6)) \\ &= (1, k - 4v - 6) \end{aligned}$$

Where, $7 \equiv 1\ (mod 3)$. .

and

$$\begin{aligned} g_5(v_i) &= (2, F(v_i) - 1),\ 1 \leq i \leq v. \\ g_5(e_j) &= (0, F(e_j) - v - 1),\ v + 1 \leq F(e_j) \leq 2v. \\ g_5(e_k) &= (1, F(e_k) - 2v - 1),\ 2v + 1 \leq F(e_k) \leq 3v. \end{aligned}$$

Now, we find the weight of every subgraph isomorphic to cycle C_3, say k''''', under labeling g over a finite abelian group $A \cong \mathbb{Z}_3 \times \mathbb{Z}_v$ as follow:

$$\begin{aligned}
k''''' &= \sum_{v\in V(C_3)} g_2(v) + \sum_{e\in E(C_3)} g_2(e) \\
&= g_3(x)+g_3(y)+g_3(z)+g_3(xy)+g_3(xz)+g_3(yz) \\
&= (2,F(x)-1)+(2,F(y)-1)+(2,F(z)-1)+(0,F(xy)-(2v+1) \\
&+ (0,F(xz)-(v+1))+(1,F(yz)-(v+1)) \\
&= (7,F(x)+F(y)+F(xy)+F(z)+F(xz)+F(yz)-(4v+6)) \\
&= (1,k-4v-6)
\end{aligned}$$

Where, $7 \equiv 1 \ (mod3)$.and

$$\begin{aligned}
g_6(v_i) &= (2,F(v_i)-1),\ 1 \le i \le v. \\
g_6(e_j) &= (1,F(e_j)-v-1),\ v+1 \le F(e_j) \le 2v. \\
g_6(e_k) &= (0,F(e_k)-2v-1),\ 2v+1 \le F(e_k) \le 3v.
\end{aligned}$$

Now, we find the weight of every subgraph isomorphic to cycle C_3, say k'''''', under labeling g over a finite abelian group $A \cong \mathbb{Z}_3 \times \mathbb{Z}_v$ as follow:

$$\begin{aligned}
k'''''' &= \sum_{v\in V(C_3)} g_2(v) + \sum_{e\in E(C_3)} g_2(e) \\
&= g_3(x)+g_3(y)+g_3(z)+g_3(xy)+g_3(xz)+g_3(yz) \\
&= (2,F(x)-1)+(2,F(y)-1)+(2,F(z)-1)+(1,F(xy)-(2v+1) \\
&+ (1,F(xz)-(v+1))+(0,F(yz)-(v+1)) \\
&= (8,F(x)+\tau(y)+F(xy)+F(z)+F(xy)+F(xz)+F(yz)-(4v+6)) \\
&= (2,k-4v-6)
\end{aligned}$$

Where, $8 \equiv 2 \ (mod3)$. We can see that all the above calculated edge weights are depending on the constant k and v, therefore all the labelings $g_i, 1 \le i \le 6$ are C_3-

groupmagic labelings.

□

Theorem 2.0.10. *An antiprism graph A_q^p is C_3-groupmagic labeling over a finite abelian group $\mathbb{Z}_3 \times \mathbb{Z}_q$ for all $q \geq 3$ and $p = 2$.*

Proof.

"Let A_q^p be the antiprism graph(V,E), V is vertex set and edge set denoted by E. V=$\{V_{s,t} : 1 \leq s \geq q, 1 \leq t \leq p\}$ and E=$\{V_{s,t}V_{s+1,t} : 1 \leq s \leq q, 1 \leq t \leq p\} \cup \{V_{s,t}V_{s,t+1} : 1 \leq s \leq q, 1 \leq t \leq p\} \cup \{V_{s,t+1}V_{s+1,t} : 1 \leq s \leq q, 1 \leq t \leq p\}$". where the suffix $'s'$ is taken modulo $'q'$.

Let $X_{s,t}$ be the 3-cycle $v_{s,t}v_{s+1,t}v_{s,t+1}$ and $X'_{s,t}$ be the 3-cycle $v_{s,t+1}v_{s+1,t+1}v_{s+1,t}$.For$1 \leq s \leq q, 1 \leq t \leq p\}$. where the suffix $'s'$ is taken modulo $'q'$.

Define a total labeling $\lambda : V \cup E \to \{1, (4p-2)q\}$ as follows:

Case 1: q **is odd**

For$1 \leq s \leq q, 1 \leq t \leq p$

$$\lambda(v_{s,t}) = \begin{cases} (0, q(t-1) + \frac{q-s+2}{2} - 1), & \text{if } s-odd, t-odd \\ (0, q(t-1) + \frac{2q-s+2}{2} - 1), & \text{if } s-even, t-odd \\ (0, q(t-1) + \frac{s+1}{2} - 1), & \text{if } s-odd, t-even \\ (0, q(t-1) + \frac{q+s+1}{2} - 1), & \text{if } s-even, t-even \end{cases}$$

For $1 \leq s \leq q, 1 \leq t \leq p$

$$\lambda(v_{s,t}v_{s+1,t}) = \begin{cases} (2, q(p-t) + \frac{s+1}{2} - 1), & \text{if } s-odd, t-odd \\ (2, q(p-t) + \frac{q+s+1}{2} - 1), & \text{if } s-even, t-odd \\ (2, q(p-t) + \frac{q-s}{2} - 1), & \text{if } s-odd, t-even, s \neq q \\ (2, q(p-1) - l), & \text{if } s = q, t-even \\ (2, q(p-t) + \frac{2q-s}{2} - 1), & \text{if } s-even, t-even \end{cases}$$

For $1 \leq s \leq q, 1 \leq t \leq p$

$$\lambda(v_{s,t}v_{s,t+1}) = \begin{cases} (1, q(p-t-1) + \frac{q+s}{2} - 1), & \text{if } s-odd, t-odd \\ (1, q(p-t-1) + \frac{s}{2} - 1), & \text{if } s-even, t-odd \\ (1, q(p-t-1) + \frac{2q-s+1}{2} - 1), & \text{if } s-odd, t-even \\ (1, q(p-t-1) + \frac{q-s+1}{2} - 1), & \text{if } s-even, t-even \end{cases}$$

For $1 \leq s \leq q, 1 \leq t \leq p$

$$\lambda(v_{s,t+1}v_{s+1,t}) = \begin{cases} (1, q(2p-t-2)+\frac{2q-s+1}{2}-1), & \text{if } s-odd, t-odd \\ \\ (1, q(2p-t-2)+\frac{q-s+1}{2}-1), & \text{if } s-even, t-odd \\ \\ (1, q(2p-t-2)+\frac{q+s}{2}-1), & \text{if } s-odd, t-even \\ \\ (1, q(2p-t-2)+\frac{s}{2}-1), & \text{if } s-even, t-even \\ \end{cases}$$

Where the s is taken modulo q

For $1 \leq s \leq q$, $1 \leq t \leq p-1$ where s is odd and t is odd,

$$\sum(B_{s,t}) = (0, q(t-1)+\frac{q-s+2}{2}-1)+(0, q(t-1)+$$

$$+\frac{2q-s+1}{2})-1+(0, qt+\frac{s+1}{2}-1)+$$

$$+(2, q(p-t)+\frac{s+1}{2}-1)+(1, q(2p-t-2)+$$

$$+\frac{2q-s+1}{2}-1)+(1, q(p-t-1)+\frac{q+s}{2}-1)$$

$$\sum(B_{s,t}) = (1, 4pq-4q-3)$$

where s is even and t is odd,

$$\sum(B_{s,t}) = (0, q(t-1)+\frac{2q-s+2}{2}-1)+(0, q(t-1)+$$

$$+\frac{q-s+1}{2}-1)+(0, qt+\frac{q+s+1}{2}-1)+$$

$$+(2, q(p-t)+\frac{q+s+1}{2}-1)+(1, q(2p-t-2)+$$

$$+\frac{q-s+1}{2}-1)+(1,q(p-t-1)+\frac{s}{2}-1)$$

$$\sum(B_{s,t})=(1,4pq-4q-3)$$

where s is odd and t is even,

$$\sum(B_{s,t})=(0,q(t-1)+\frac{s+1}{2}-1)+(0,q(t-1)+$$

$$+\frac{q+s+2}{2}-1)+(0,qt+\frac{q-s+2}{2}-1)+$$

$$+(2,q(p-t)+\frac{q-s}{2}-1)+(1,q(2p-t-2)+$$

$$+\frac{q+s}{2}-1)+(1,q(p-t-1)+\frac{2q-s+1}{2}-1)$$

$$\sum(B_{s,t})=(1,4pq-4q-3)$$

where s is even and t is even,

$$\sum(B_{q,t})=(0,q(t-1)+\frac{q+s+1}{2}-1)+(0,q(t-1)+$$

$$+\frac{s+2}{2}-1)+(0,qt+\frac{2q-s+2}{2}-1)+$$

$$+(2,q(p-t)+\frac{2q-s}{2}-1)+(1,q(2p-t-2)+$$

$$+\frac{q-s+1}{2}-1)+(1,q(p-t-1)+\frac{s}{2}-1)$$

$$\sum(B_{q,t})=(1,4pq-2q-3)$$

Thus, it can be easily verified that $\sum B_{s,t})=(4,4pq-2q-3)$ for the remaining cases of s and t, where $1\leq s\leq q, 1\leq t\leq p-1$.

Next we prove $\sum(C_{s,t})=(1,4pq-4q-3)$, for $1\leq s\leq q, 1\leq t\leq p-1$. where t is odd and s is even and $s \neq q$,

$$\sum(C_{s,t}) = (0, qt + \frac{q+s+1}{2} - 1) + (0, q(t-1) +$$

$$+ \frac{q-s+1}{2} - 1) + (0, qt + \frac{s+2}{2} - 1) +$$

$$+ (2, q(2p-t-2) + \frac{q-s+1}{2} - 1) + (1, q(p-t-1) +$$

$$+ \frac{q+s+1}{2} - 1) + (1, q(p-t-1) + \frac{2q-s}{2} - 1)$$

$$\sum(C_{s,t}) = (1, 4pq - 2q - 3)$$

where t is odd and $s = q$,

$$\sum(C_{q,t}) = (0, qt + \frac{q+1}{2} - 1) + (0, q(t-1) + \frac{q+1}{2} - 1) + (0, qt) +$$

$$+ (1, q(2p-t-2) + \frac{q+1}{2} - 1) + (1, q(p-t-1) + \frac{q+1}{2} - 1) + (2, q(p-t) - 1)$$

$$\sum(C_{q,t}) = (1, 4pq - 4q - 3)$$

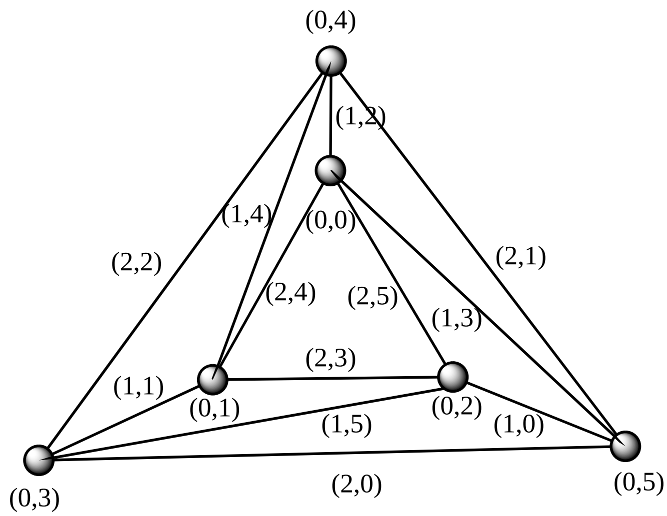
where s is odd and t is odd and $s \neq q$,

$$\sum(C_{s,t}) = (0, qt + \frac{s+1}{2} - 1) + (0, q(t-1) + \frac{2q-s+1}{2} - 1) + (0, qt + \frac{q+s+2}{2} - 1) +$$

$$+ (1, q(2p-t-2) + \frac{2q-s+1}{2} - 1) + (1, q(p-t-1) + \frac{q+1}{2} - 1) + (2, q(p-t-1) + \frac{q-s}{2} - 1)$$

$$\sum(C_{s,t}) = (1, 4pq - 4q - 3)$$

Similarly for the other cases of s and t, $\sum(B_{s,t}) = \sum(C_{s,t}) = (1, 4pq - 4q - 3)$, where $1 \leq s \leq q, 1 \leq t \leq p-1$. Thus we get a C_3-groupmagic labeling of A_q^p with groupmagic constant $c = (1, 4pq - 4q - 3)$. Hence A_q^p is C_3-groupmagic.



Case 2: q**-even**

For$1 \leq s \leq q, 1 \leq t \leq p$

$$\lambda(v_{s,t}) = \begin{cases} (0, q(t-1)+s-1), & \text{if } t-odd \\ (0, qt-s), & \text{if } t-even \end{cases}$$

$$\lambda(v_{s,t}v_{s+1,t}) = \begin{cases} (2, q(2p-t-1)-s-1), & \text{if } t-odd, s \neq q \\ (2, q(2p-t-1)-1), & \text{if } t-odd, s = q \\ (2, q(t-2)+s), & \text{if } t-even, s \neq q \\ (2, q(t-2)) & \text{if } t-even, s = q \end{cases}$$

For$1 \leq s \leq q, 1 \leq t \leq p$

$$\lambda(v_{s,t}v_{s,t+1}) = \left\{ \ (1, q(p-t)-s), \right.$$

For $1 \leq s \leq q, 1 \leq t \leq p$

$$\lambda(v_{s,t+1}v_{s+1,t}) = \left\{ \ (1, q(p-t)+s-1), \right.$$

Where the s is taken modulo q

For $1 \leq s \leq q$, $1 \leq t \leq p-1$, where t is odd and $s \neq q$ is odd,

$$\sum(B_{s,t}) = (0, q(t-1)+s-1) + (0, q(t-1)+s) + (0, qt+q-s) +$$

$$+(2, q(2p-t-1)-s-1) + (1, q(p-t)+s-1) + (1, q(p-t)-s-1)$$

$$\sum(B_{s,t}) = (1, 4pq-4q-3)$$

where j t is odd and $s = q$,

$$\sum(B_{q,t}) = (0, q(t-1)+q-1) + (0, q(t-1)) + (0, q(t+1)-q)$$

$$+(2, q(2p-t-1)-1) + (1, q(p-t)+q-1) + (1, q(p-t)-q)$$

$$\sum(B_{q,t}) = (1, 4pq-4q-3)$$

Similarly for the remaining cases of s and t, it can be easily verified that $\sum(B_{s,t}) = (1, 4pq-4q-3))$ for $1 \leq s \leq q, 1 \leq t \leq p-1$.

Next we prove $\sum(C_{s,t}) = (4, 4pq-4q-3)$, where for $1 \leq s \leq q, 1 \leq t \leq p-1$.

where t is odd and $s \neq q$,

$$\sum(C_{s,t}) = (0, q(t+1)-s) + (0, q(t-1)+s) + (0q(t+1)-s-1) + (1, q(p-t)+s-1) +$$

$$+(1,q(p-t)-s)+(2,q(2p-t-3)+s-1)$$

$$\sum(C_{s,t})=(1,4pq-4q-3)$$

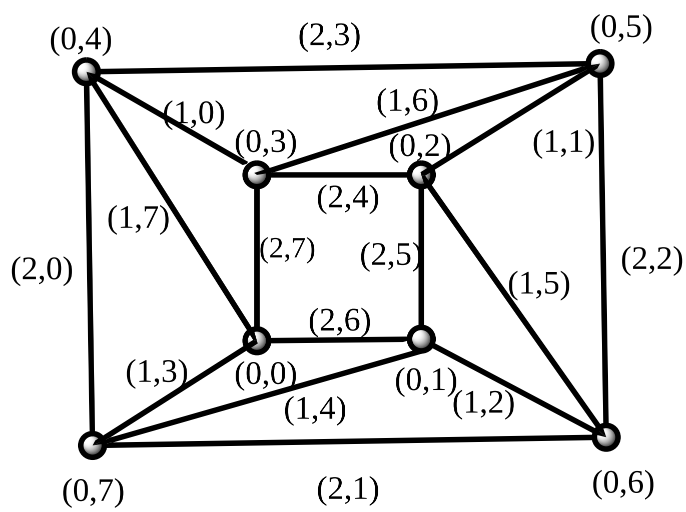
where t is odd and $s=q$,

$$\sum(C_{q,t})=(0,q(t+1)-q)+(0,q(t-1))+(0,q(t+1)-1)$$

$$+(1,q(p-t)+q)+(1,q(p-t)-1)+(2,q(2p-t-3)-1)$$

$$\sum(C_{q,t})=(1,4pq-4q-3)$$

Similarly for the other cases of s and t, $\sum(B_{s,t})=\sum(C_{s,t})=(1,4pq-4q-3$, where $1\leq s\leq q, 1\leq t\leq p-1$. Thus we get a C_3-groupmagic labeling of A_q^p with groupmagic constant $c=(1,4pq-4q-3)$. Hence A_q^p is C_3-groupmagic. $\square$



Chapter 3

Conclusions and Open Problems

3.1 Conclusions

In this thesis we have discussed the K_2-groupmagic labeling of unicyclic graph $(C_m \Delta S_{n_j})$ over a finite abelian group $A \cong Z_2 \times Z_{(n_j+1)m}$. Secondly, we formulated H-groupmagic total labelings of graph $G(v, e)$ over finite abelian group $Z_t \times Z_v$, by considering $H \cong K_2, C_t$ and $e = (t-1)v$. Moreover, we found the C_3-groupmagic labeling for anti prism graph. We are unable to find such labelings when $e \neq (t-1)v$ and propose following open problems:

3.2 Open Problems

Open Problem:1

Find the H-groupmagic total labelings of graph $G(v,e)$ over finite abelian group $Z_t \times Z_v$, by considering $H \cong K_2, C_t$ and $e \neq (t-1)v$.

Open Problem:2

Find the C_3-groupmagic total labelings of antiprism A_q^p if $p \neq 2$ over finite abelian group $Z_3 \times Z_v$.

Chapter 4

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