H-Magic Behaviors of Some Graphs

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DEDICATION

To my parents, brother,

wife, daughter and son,

for their love, support and patience.

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May Allah bless us with bright success in our life and after this life. (Aameen)

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ABSTRACT

H-Magic Behaviors of Some Graphs

A graph $G(V,E)$ has an H-covering if every edge in E belongs to a subgraph of G isomorphic to H. Suppose G admits an H-covering. An H-magic labeling is a mapping λ from $E(G)\cup V(G)$ onto the integers $\{1,2,...,|E(G)\cup V(G)|\}$ with the property that, for every subgraph A of G isomorphic to H, there is a positive integer c such that $\sum A=\sum_{v\in V(A)}\lambda(v)+\sum_{e\in E(A)}\lambda(e)=c$. A graph which possess such type of labeling is known as H-magic graph. Further if in a graph vertices are labeled first with smallest positive numbers, then the graph is called H-supermagic. Moreover a graph is said to be H-(a,d)-anti magic if the magic constant for an arithmetic progression with initial value a and a common difference d. Numerous results on labeling of many families of graphs have been published. In this thesis, research work focuses on to formulate cycle C_3-(a,d) anti-supermagic labeling for the Multi-Wheels graph, supermagic labeling for isomorphic copies with its disjoint union of Multi-Wheels graph and cycle (a,d)-anti-supermagic labeling for Web graph. Also cycle anti-supermagic labeling for isomorphic copies with its disjoint union of ladder and triangular ladder graphs have been formulated.

In addition, investigation of fan, friendship, ladder and wheel line graphs and study of the supermagic and anti-supermagic vertex-edge-face labeling of such graphs and their isomorphic copies have been carried in this thesis. An anti-supermagic labeling of the extension of cycle graphs is also formulated.

Lastly the face supermagic labeling of (1,1,1) type of subdivided triangular ladder graph, subdivided mC_4-snake graph and subdivided kmC_4-triangular snake graph with its (1,1,...,1) and (2,2,...,2) string are also the part of this thesis.

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LIST OF ABBREVIATIONS

$V(G)$	Vertex set of the graph G
$E(G)$	Edge set of the graph G
$F(G)$	Face set of the graph G
mC_n	m copies of C_n graph
ε	Epsilon
ξ	Zeta
η	Eta
λ	Lambda
μ	Mu
∞	Infinity

Chapter 1

Introduction

1.1 Graph theory

The foundation of graph theory was led by a Swiss mathematician Leonhardo Paul Euler when he solved "Koningsberg bridge problem" using the notions of graphs. The aforementioned problems was originated in the city of Koningsberg river named as Pregel River, which had seven bridges. People of the city were curious to know whether there was any way to cross all the bridges once and only once to reach at the initial point or not. Euler provided the solution of the problem by creating first graph of the history. In this way, graph took birth and it became one of the significant mathematical discipline with the passage of time. However, in 1935 D. Konig's [93] provided a systematical and formal study in his book on graph theory. Graph theory is becoming popular in these days, so it is used in the major fields of science, mathematics, technology, computer science, electrical engineering and many more. It plays an important role in various fields. In chemistry, it is used to study molecules and chemical compounds and to model chemical bounds. It is used to reshape samples of new structure in biochemistry and to formulate various algorithms which are used to solve different problems in computer applications. It also provides a way to reduce traffic load on roads. Moreover, it models the networks through which airlines and trains are inter connected with cities so that passengers can move easily through possible trips by airlines and trains. Graph theory helps in formulating a network of species in biology and in arranging a time table for teachers so that there would be no overlapping for any teacher. Graphs are also used in geometry and statistics. By using of graphs one can easily do planning and scheduling his large projects.

Another concept in graph theory is graph labeling that plays an important role in understanding the graph properties, network representation is seen in a circuit design and in a communication network.

The application of graph theory in heterogeneous fields like Astronomy, Biochemistry is enhancing the significance of graph theory and pushing it into the mainstream of mathematics. In physics, a three dimensional complicated atomic structures can be studied with it easily. In statistical physics, graph theory makes it easier to represent the local connections with physical process of its dynamics on such systems. The graphs are also used for the representation of the data of any organization, flow of computation and network communication etc. The story does not end here. Still there are many fields which depend on graph theory to fulfill their demands in an effective way. The use of graph theory has no limitations.

In this section, some basic graph theory definitions, techniques, terminologies and concepts which are used in our thesis are presented. Simple, undirected, planar and finite graphs are considered through out this thesis. The vertices, edges, faces and mapping are represented by $V(G)$, $E(G)$, $F(G)$ and ξ respectively. A graph can be represented in to different ways. However one way to represent a graph is by a diagram. Some examples of simple, undirected, planar and finite graphs are presented. See figures given below,

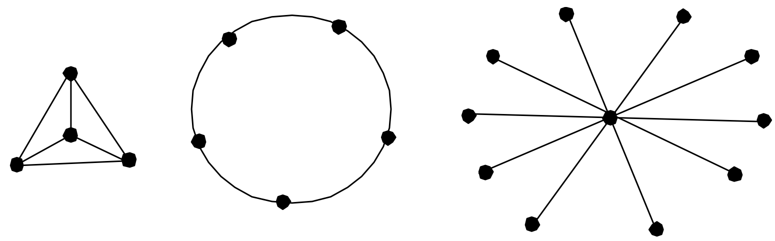

Figure 1.1: Simple, undirected, planar and finite graphs

1.2 Basic terminologies

Most of these basic ideas can be found in [1], [42] and [92].

Definition 1.2.1 "A graph $G(V, E)$ is a mathematical structure consisting of two finite sets V and E. The elements of V are called vertices (or nodes), and the elements of E are called edges. Each edge has a set of one or two vertices associated to it, which are called its endpoints."

Definition 1.2.2 "In a graph the order and size is the total number of vertices and edges respectively."

Definition 1.2.3 "The degree of a vertex v in a graph G, denoted by $\deg(v)$, is the number of edges incident on v plus twice the number of self-loops. The largest degree among the vertices degree in a graph is known as a graph degree."

Definition 1.2.4 "A proper edge is an edge that joins two distinct vertices."

Definition 1.2.5 "A self-loop is an edge that joins a single endpoint to itself."

Definition 1.2.6 "A multi-edge is a collection of two or more edges having identical end points."

Definition 1.2.7 "A simple graph has neither self-loops nor multi-edges."

Definition 1.2.8 "A loopless graph (or multi-graph) may have multi-edges but no self loop."

Definition 1.2.9 "A general graph may have self-loops and/or multi-edges."

Definition 1.2.10 "A null graph is a graph whose vertex and edge-sets are empty."

Definition 1.2.11 "A trivial graph is a graph consisting of one vertex and no edges."

Definition 1.2.12 "A directed graph is a graph each of whose edges is directed."

Definition 1.2.13 "A graph is finite if its vertex set and edge set are finite."

Definition 1.2.14 "A complete graph is a simple graph such that every pair of vertices is joined by an edge. Any complete graph on n vertices is denoted by K_n."

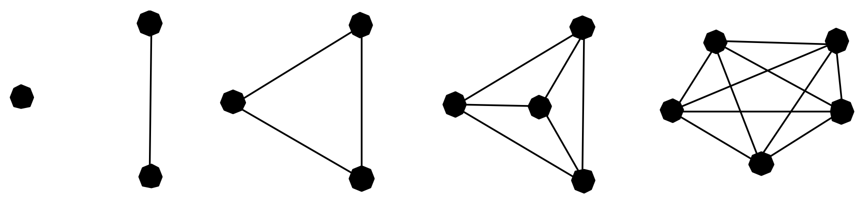

Figure 1.2: First five complete graphs

Definition 1.2.15 "A regular graph is a graph whose all vertices have equal degree. A k-regular graph is a regular graph whose common degree is k."

Definition 1.2.16 "A graph G is bipartite if $V(G)$ is the union of two disjoint (possible empty) independent sets called partite sets of G."

Definition 1.2.17 "A complete bipartite graph is a simple bipartite graph such that every vertex in one of the bipartition subsets is joined to every vertex in the other bipartition subset. Any complete bipartite graph that has m vertices in one of its bipartition subsets and n vertices in other is denoted $K_{m,n}$."

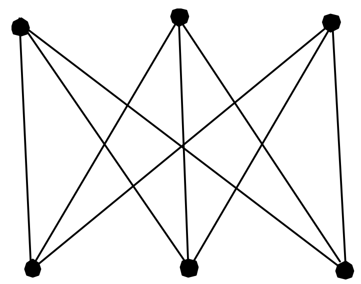

Figure 1.3: A complete bipartite graph

Definition 1.2.18 "A walk is a list $v_0, e_1, v_{1,...}, e_k, v_k$ of vertices and edges such that, for $1 \leq i \leq k$, the edge e_i has endpoints v_{i-1} and v_i."

Definition 1.2.19 "A trail is a walk with no repeated edge."

Definition 1.2.20 "A walk which has same initial and terminal point is called close walk."

Definition 1.2.21 "A path is a simple graph whose vertices can be ordered so that two vertices are adjacent if and only if they are consecutive in the list."

Definition 1.2.22 "A cycle is a graph with an equal number of vertices and edges whose vertices can be placed around a circle so that two vertices are adjacent if and only if they appear consecutively along the circle."

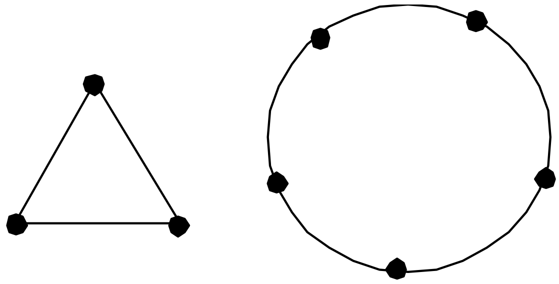

Figure 1.4: Cycle graphs

Definition 1.2.23 "A subgraph of a graph G is a graph H such that $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$ and the assignment of endpoints to edges in H is the same as in G. We then write $H \subseteq G$ and say that G contains H."

Definition 1.2.24 "A graph G is connected if each pair of vertices in G belongs to a path; otherwise, G is disconnected."

Definition 1.2.25 "A graph G is called a planar graph if without any edge crossing we can draw it in a plane."

Definition 1.2.26 "A graph which passes through all vertices of graph is called Hamiltonian graph."

Definition 1.2.27 "An isomorphism from a simple graph G to a simple graph H is a bijection $\xi : V(G) \to V(H)$ such that $uv \in E(G)$ if and only if $\xi(u)\xi(v) \in E(H)$. We say G is isomorphic to H, written $G \cong H$, if there is an isomorphism from G to H."

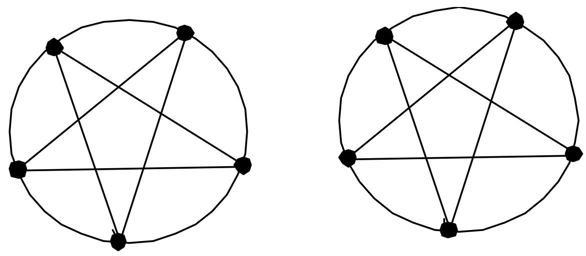

Figure 1.5: An isomorphic graphs

Definition 1.2.28 "A vertex whose degree is 1 is called leaf. *i.e.* $d(x) = 1$."

Definition 1.2.29 "Let G be a connected graph, and uv be any edge such that the deletion of that edge reveals a disconnected graph, then that edge is called bridge."

1.3 Operations on graphs

Definition 1.3.1 "The complement $\bar{G}$ of a simple graph is the simple graph with vertex set $V(G)$ defined by $uv \in E(\bar{G})$ if and only if $uv \notin E(G)$."

Definition 1.3.2 "A cartesian product $G \otimes H$ of graphs $G = (V(G), E(G))$ and $H = (V(H), E(H))$ *is* the graph with vertex set $V(G) \times V(H)$ where vertex (v_1, u_1) is adjacent to vertex (v_2, u_2) whenever $v_1v_2 \in E(G)$ and $u_1 = u_2$, or $v_1 = v_2$ and $u_1u_2 \in E(H)$."

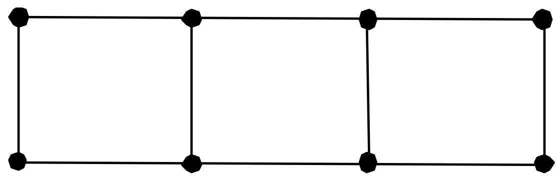

Figure 1.6: A cartesion product of P_2 and P_4 graphs

Definition 1.3.3 "A book graph B_n is the cartesian product $S_{n+1} \times P_2$."

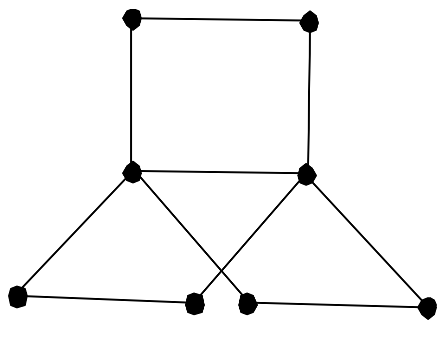

Figure 1.7: A Book graph (B_3)

Definition 1.3.4 "The union of graphs $G_1, G_2, ..., G_k$ written $G_1 \cup G_2 \cup ... \cup G_k$ is the graph with vertex set $\cup_{i=1}^{k} V(G_i)$ and edge set $\cup_{i=1}^{k} E(G_i)$."

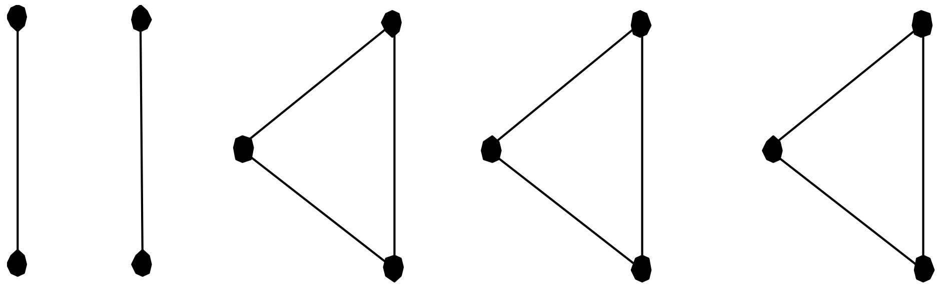

Figure 1.8: Union of two graphs $(2P_2 \cup 3C_3)$

Definition 1.3.5 "If G_1, G_2 be two graphs then sum or join of G_1 and G_2 is denoted by $G_1 + G_2 = G$ with the property that

$$V(G) = V(G_1) \cup V(G_2)$$

$E(G) = E(G_1) \cup E(G_2)$+edges joining all vertices of G_1 with all vertices of G_2."

Definition 1.3.6 "A simple graph G with vertex set $V(G) = \{v_1, v_2, ..., v_n\}$ is an intersection graph if there exists a family of sets $\mathcal{F} = \{S_1, S_2, ..., S_n\}$ such that vertex v_i is adjacent to v_j if and only if $i \neq j$ and $S_i \cap S_j \neq \phi$."

Definition 1.3.7 "The line graph $L(G)$ of a graph G has a vertex of each edge of G, and two vertices in $L(G)$ are adjacent if and only if the corresponding edges in G have a vertex in common. Thus, the line graph $L(G)$ is the intersection graph corresponding to the endpoint sets of the edges of G."

1.4 Common families of graphs

In this part the graphs are taken from [27] and [42].

1.4.1 Wheel graph

"In a graph if we take all vertices of a cycle and connect it to a single vertex it is called a wheel graph. Suppose that $W_n = K_1 + C_n$ be a wheel graph. The even and odd wheel graphs are dependent on the value of n whether it is even or odd respectively."

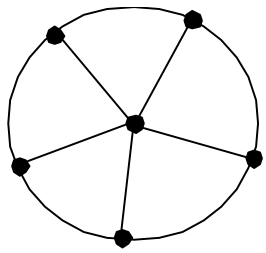

Figure 1.9: A wheel graph(W_5)

1.4.2 Tree graph

"An undirected graph G is known as tree graph if it is connected and acyclic (no cycle). It is denoted by T_n, where $n \geq 1$. A tree T_n, has at-least two leaves."

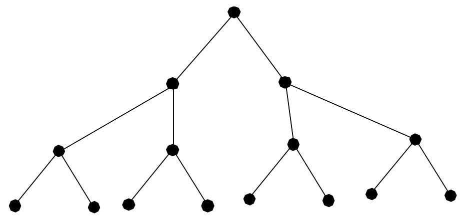

Figure 1.10: A tree graph

1.4.3 Spanning tree

"A graph H is a subgraph of G, will be called spanning tree if

(i) $V(H) = V(G)$ (ii) Tree $\Rightarrow$ connected + acyclic".

1.4.4 Fan graph

"An undirected graph is known as a fan graph if all the vertices of a path P_n joining with a isolated vertex, called the center. In other words a union of K_1 with P_n is known as a fan graph F_n. Here core is known as the vertex which come from K_1 and spokes is the edges incident with the core. Hence if F_n is a fan graph then its number of vertices are $n+1$ and edges are $2n-1$."

Figure given below has shown the fan graph F_7.

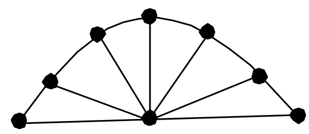

Figure 1.11: A fan graph (F_7)

1.4.5 Ladder graph

"A ladder graph L_n is a undirected planar graph with vertices and edges $2n$ and $3n-2$ respectively. The ladder graph can be obtained as the cartesian product of two path graphs, one of which has only one edge: $L_n = P_n \times P_1$, where $n \geq 2$."

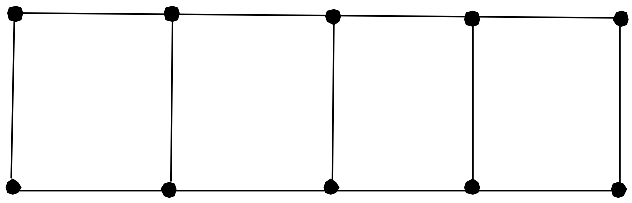

Figure 1.12: A ladder graph(L_5)

1.4.6 Triangular ladder graph

"If we add the edges $u_i v_{i+1}$, where $1 \leq i \leq n-1$, in a ladder graph $L_n = P_n \times P_1$ then the resulting graph is known as a triangular ladder TL_n, where $n > 2$. Such triangular ladder graph has $2n$ vertices and $4n-3$ edges."

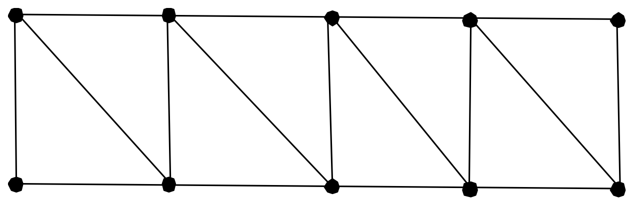

Figure 1.13: A triangular ladder graph(TL_5)

1.4.7 Friendship graph

"An undirected planar graph which is raised by the connection of n copies of the cycle graph C_3 having one vertex common is known as friendship graph F_n, having vertices and edges are $2n+1$ and $3n$ respectively."

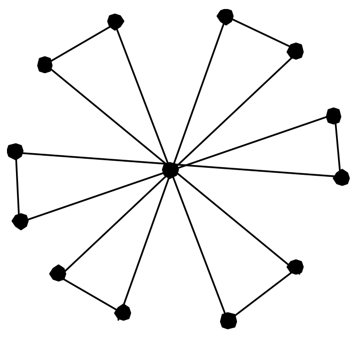

Figure 1.14: A friendship graph

1.4.8 Double-wheel graph

"A double-wheel graph DW_N of size N is isomorphic to $2C_N + K_1$, i.e. it consists of two cycles of size N, where the vertices of the two cycles are all connected to a common hub. On similar lines one can define a Multi-wheels graph mW_N of size N can be composed of $mC_N + K_1$, where $m \geq 2$. i.e. it consists of m-cycles of size N, where the vertices of the m-cycles are all connected to a common hub."

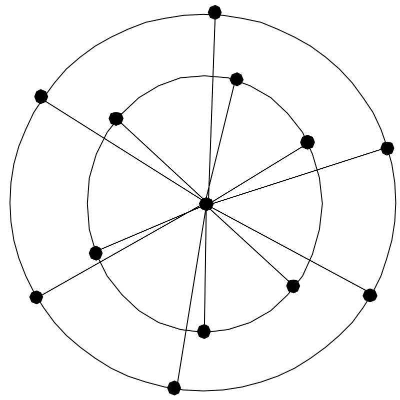

Figure 1.15: A double wheel graph(DW_6)

1.4.9 Thesis outline

Chapter one contains an introduction to the thesis, some basic definitions of graph theory, operation on graphs and some common families of graphs.

Chapter two contains different types of graph labeling such as, vertex labeling, edge labeling, H-magic labeling and magic labeling of (a, b, c) type. An overview of this renowned research area and some known results of graph labeling are presented.

In chapter three, cycle C_3-(a, d) anti-supermagic labeling for the Multi-wheels graph, supermagic labeling for isomorphic copies with its disjoint union of Multi-wheels graph and cycle (a, d)-anti-supermagic labeling for Web graph with some examples are discussed.

In Chapter four, line graphs of some graphs and their labeling have been studied. The aim of this chapter is to first investigate fan, friendship, ladder and wheel line graphs. The supermagic, anti-supermagic over vertex-edge-face labeling of such graphs and their isomorphic copies are formulated along with some examples for the support of under study results. An anti-supermagic labeling of the extension of cycle graphs is also the part of this chapter.

In Chapter five, cycle anti-supermagic labeling for the isomorphic copies with its disjoint union of ladder graphs for different value of d are formulated. Also proof of cycle supermagic and cycle anti-supermagic labeling of the isomorphic copies with its disjoint union of triangular ladder graphs for different value of d are shown.
In Chapter six, face supermagic labeling of $(1, 1, 1)$ type of subdivided triangular ladder graph, subdivided mC_4- snake graph and subdivided kmC_4- triangular snake graph with its $(1, 1, ..., 1)$ and $(2, 2, ..., 2)$ string are established.
Chapter seven, contains summary of all the under study results and also propose some open problems as well.

Chapter 2

Graph labeling

Graph labeling were started in the seventh century, when Chinese mathematicians introduced third order magic square using magic labeling. In magic square M_n, the integers $\{1,\ 2,\ ...,\ n^2\}$ are arranged in a mathematically order such as the total of each column, row, both diagonals are same.

1	15	8	10
12	6	13	3
14	4	11	5
7	9	2	16

Figure 2.1: Magic squar M_4

In the field of graph theory, the graph labeling plays a vital role to easily understand the different problems of the different areas in sciences. Graph labeling is defined as an assignment of real values (usually to the positive integers) to the vertices and edges under certain conditions. Usually the collection of vertices and edges is used as a domain set. Harmonious, cordial, coloring, graceful, edge and vertex magic, H-magic, H-supermagic, H-antimagic and H-anti-supermagic are the different types of graph labeling. Various graph labeling techniques have been developed and are still increasing in number due to their significance. Different techniques are identified with different names. Here we have defined some basic magic labeling techniques which are very helpful to complete the present study.

In 1963, Sedlacek [79] used the idea of Chinese mathematicians in graph theory and opened the new sides regarding the magic labeling. In the mean while Kotzig and Rosa [56] was the men who introduced graph labeling in 1967 and opened a new horizon for mathematicians in this field. This labeling was named by Rosa, as a β-valuation. They defined the labeling as,

Definition 2.0.1 "An injective mapping from the vertices of the graph G to the set of integers $\{0,\ 1,\ 2,\ ...,\ |E(G)|\}$, so that each edge xy is given the label $|\xi(x) - \xi(y)|$, the resulting labels of edges are distinct."

However, Golomb [38] defined the labeling in its own way and named it as graceful labeling.

Definition 2.0.2 "A graph G with q edges is said to be graceful if there is an injection ξ from the vertices of G to the set $\{0, 1, ..., q\}$ such that the labels of the edges are every possible difference of the vertex labels, that is, the set $\{1, 2, ..., q\}$."

Now we leave the said labeling here and concentrate our interest on main labelling which are used in our thesis. See the reference [85].

Definition 2.0.3 "A total labeling of a graph is a graph labeling with the set of vertices and edges as the domain."

Definition 2.0.4 "For any vertex $x \in V$ and edge $xy \in E$ the weights under total labeling are defined as $w(x) = \xi(x)+\sum_{y\in N(x)} \xi(xy)$ and $w(xy) = \xi(x)+\xi(y)+\xi(xy)$ respectively."

Definition 2.0.5 "A vertex magic total labeling of a graph $G = (V, E)$ is a bijection from the set of vertices and edges to the set of numbers defined by $\xi : V \cup E \to \{1, 2, ..., |V| + |E|\}$, so that for every $x \in V,\ w(x) = k$, for some integer k. The constant k is called the magic constant for ξ."

Definition 2.0.6 "A super vertex magic total labeling of a graph $G = (V, E)$ is an vertex magic total labeling ξ that has the property $\xi(V) = \{1, 2, ..., |V|\}$ and $\xi(E) = \{|V| + 1, |V| + 2, ..., |V| + |E|\}$."

Definition 2.0.7 "An edge magic total labeling of a graph $G = (V, E)$ is a bijection from the set of vertices and edges to the set of numbers defined by $\xi : V \cup E \to \{1, 2, ..., |V| + |E|\}$, so that for every $xy \in E,\ w(xy) = k$, for some integer k."

Definition 2.0.8 "A super edge magic total (super edge magic total) labeling of a graph $G = (V, E)$ is an edge magic total labeling that has the property $\lambda(V) = \{1, 2, ..., |V|\}$ and $\lambda(E) = \{|V| + 1, |V| + 2, ..., |V| + |E|\}$."

Definition 2.0.9 "A graph G that admits both vertex magic total and edge magic total labelling is called totally magic."

In this chapter, we also include graph labeling of numerous types, such as vertex labeling, edge labeling, H-magic labeling and magic labeling of type (a, b, c). Further some known results of labeling of each type are listed, which are very useful to build up some new techniques and there by some new results.

2.1 Vertex labeling

Magic labeling is playing an important role in graph theory. In 1963 Sedlacek [79], was introduced magic labeling concept and he defined it as "A function $\xi : E \to R^+ \cup \{0\}$ qualifies for magic labeling if for every $v \in V$, the vertex weight $w(v) = \sum_{e \in V} \xi(e)$ is a constant k, where u belongs to neighbor of v." After its introduction, this topic grabbed the attention of many researchers. Since the introduction of magic labeling, a lot of work has been done on this topic and different types of magic labeling have been introduced. Stewart [83] worked on complete graph K_n, basket graph B_n and fan graph F_n and found the values of $'n'$, for which these graphs are magic. Jezny and Trenkler [52] introduced vertex magic edge and defined it as: "A graph G qualifies for vertex magic iff every edge e of G, $e \in F$ and every pair of edges of G, f is separated by F, where every component is a regular graph of degree 1 or 2." Stewart [82, 83] presented various ideas regarding magic labeling like semi-magic, supermagic and prime-magic labeling. For semi-magic labeling, peoples are not restricted to start the labels from 1. It can be started from any positive integer $'n'$, while in the case of supermagic labeling, the set of edge labels comprises consecutive numbers. For prime labeling, he chooses the prime number for the labeling of edges. During his study on vertex magic edge labeling, Trenkler considered the relationship between edges and vertices and investigated the behavior of a connected graph [89]. In [88], he extended his work to hypergraphs. Sun and Lee presented his findings regarding the product of two vertex magic edge graphs in [84]. vertex antimagic edge labeling has a particular feature which makes it prominent among different types of labeling. In this labeling, the weights of vertex sets form an arithmetic progression with d as a common difference. In [23-25], authors presented some families of graphs which are comfortable for vertex antimagic edge labeling. These families includes paths and trees of odd order, complete graphs, parachutes and Husimi trees. Bača and Millar studied the topic of vertex antimagic edge labeling in detail and presented some important results regarding magic and anti-magic labeling [11-22, 59-63, 68-70]. In [81], distance magic labeling was presented by Simanjuntak et al.. Tezer and Cahit [87] applied the concept of vertex antimagic edge labeling on paths and cycles and presented significant results. In recent years, Nicholas et al. [75] applied this tool on some special trees like caterpillars and spiders and on complete bipartite graphs $k_{n,n+2}$ and $k_{n,n}$. They categorized vertex antimagic edge labeling of unicycle graphs. Bača et al. [12] worked on (a, d)-vertex antimagic total labeling and its basic properties and also made significant contribution by constructing such labeling for some classes of graphs. Dual labeling was also investigated in the same paper. They considered the following graphs P_n, C_n, K_n and $K_{m,n}$ in his paper and discussed its labeling ideas also. Bača et al. [14] formulated vertex anti-magic for convex polytope and some graphs like D_n, Q_n and $P(n, k)$. Bača and his companions did a lot of work on magic and anti magic labeling, which can be seen in the following references [4, 5, 9-19, 68, 69]. MacDougall and his companions presented a new idea (a, d)-vertex antimagic total labeling, where d is taken to be zero. Such type of labeling is called vertex magic total labeling. MacDougall and his companions covered this topic intensively in their manuscripts [$36 - 38, 61, 62, 67$]. McQeillan [69] introduced another variation of (a, d)-vertex antimagic total-labeling on cubic graphs. He ignored the injective property while introducing this variation. Froncek et al. [34] worked on cartesian product of cycles and explored the vertex magic total labeling for them. Kovar extended this work to regular graphs and their copies [54-56]. The factors and compositions for vertex magic total and vertex antimagic total have also studied.

2.2 Edge labeling

In magic labeling, different notions are used for same concepts. It happened because different people worked independently and produced same results. For example, the notions of magic valuation and edge magic labeling were introduced for the same concept by Kotzig et al. [55, 56] and Ringel et al. [76] respectively. Ringel and Laldo [76] presented their study on edge magic total labeling of trees and provided valuable results. Wallis and his companions [91] studied the properties of edge magic total graphs and generated the edge magic total labeling for various graphs including cycles C_n, Sun graphs, stars, complete bipartite graphs and complete graphs up-to K_6. Craft and Tessar [30] and Fukuchi in [36] described a bound on the maximum size and wheels graphs edge magic total respectively. Enomoto and his collaborators [31] in 1998, first time introduced the term of super edge magic labelling. Enomoto et al. [31] presented a valuable study on super edge magic total labeling by providing a necessary condition for the comfortably of graph for super edge magic total labeling. Moreover they claimed that all trees are comfortability for super edge magic total labeling. Their claim is still awaiting for confirmation. Kotzig and Rosa [56] studied caterpillar graph and provided their convertibility for super edge magic total labeling. Fukuchi [35] choose some special type of trees and explained their behavior for super edge magic total labeling. Figueroa-Centeno et al. [33] proved that α-labeling and super edge magic total labeling are the same concepts when these are discussed in the context of tree graphs. In [34], Figueroa and his companions presented their study regarding the possibility of adding an isolated vertex for the conversion of a non-super edge magic total graph into super edge magic total graph. They introduced the notion "super edge deficiency" for the minimum number of additional isolated vertices. In the same paper, they presented their study on cycles C_n, complete bipartite graph $K_{m,n}$ and forests. In recent years, the investigations of super edge magic total labeling of a graph of maximum size are provided. Ivanco and Luckanicova [47] studied some disconnected graphs and explored edge magic total labeling for them. The magic strength of a graph has been discussed in [54]. In his Ph.D. thesis, Muntaner-Batle [73] presented a comprehensive study of edge magic total and super edge magic total labeling. He studied some properties of these labeling and investigated the relationship of edge magic total labeling with some other labeling. He also discussed operations on graphs and presented labeling schemes for some particular graphs. Moreover, he discussed the magic deficiency of edge magic total graphs and explored the magic total graph that contains a large complete graph. Wallis also presented important results on this topic in [90]. Example given below, shows the edge magic of graph with constant $c = 19$.

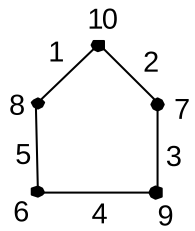

Figure 2.2 : Edge magic graph

2.3 H-magic labeling

Let G be a simple, connected, planar, undirected and finite graph, where the edges and vertices sets are denoted by E and V respectively. An extension of Rosa [56], definition on magic valuation which corresponds the property $H \cong K_2$. The concept of H-magic graph was introduced by Gutierrez along with Llado [43], such that "If H and $G = (V, E)$ be finite and simple graphs with the property that every edge of G belongs to at least one subgraph isomorphic to H. A bijection $\xi : V \cup E \rightarrow \{1, 2, ..., |V| + |E|\}$ is an H-magic labeling of G if there exists a positive integer c (called the magic constant), such that for any subgraph $H'(V', E')$ of G isomorphic to H, the sum $\sum\limits_{v \in V'} \xi(v) + \sum\limits_{e \in E'} \xi(e)$ is equal to magic constant c. This sum is also known as weight of H. A graph is H-magic if it admits an H-magic labeling." Further more, if we labeled the graphs under the property that $\{\xi(v)\}_{v \epsilon V} = \{1, \cdots, |V|\}$, then H-magic labeling becomes H-supermagic labeling.

Different people worked on H-supermagic labeling of graphs and provided so many results in this direction. By using the partitioning sets technique of integers, Llado and Moragas [63] described cycle-magic labeling of wind mill, books, prisms, wheels and subdivided wheels graphs. Some families of graphs like ladders, fans, and books are considered by Ngurah et al. [74] which admit cycle-supermagic labeling. Maryati et al. [66], proved that disjoint union of k-isomorphic copies (k taken as odd) of a graph G which is connected and $|V(G)| + |E(G)|$ is even has G-supermagic. Maryati et al. [32] proved that results are similar for any disconnected graph G if every component contains at least its two vertices. Recently in [77] Rizvi et al. discussed H-supermagic labeling of the disconnected graphs. They formulated cycle-supermagic labeling for the isomorphic copies with its disjoint union of different families of graphs. They also worked on non isomorphic copies with its disjoint union of ladders and fans. Ali et al. [3] determined supermagic labeling for the isomorphic copies with its disjoint union of prism graph and also found out the supermagic labeling for the isomorphic copies with its disjoint union of generalized anti-prism graph. In the mean while Rizvi et al. [78] described the cycle-(super) magic labeling of uniform subdivided graph. In addition, they studied cycle-(super) magic behavior of non-uniform subdivided fans and triangular ladder graphs.

In 1990, Hartsfield and Ringel [44] introduced the notion of antimagic graphs such as "A graph with q edges is called antimagic if its edges can be labeled with $1, 2, 3, ..., q$ such that the sums of the labels of the edges incident to each vertex are distinct." Among the graphs they proved that path with $n \geq 3$, wheels and cycles are antimagic. Bača [6] formulated an antimagic labeling for an antiprism. Koh et al. [53] defined a web graph such that "one obtained by joining the pendent points of a helm to form a cycle and then adding a single pendent edge to each vertex of this outer cycle." Also formulated graceful labeling for such graphs.

A. Semaničová-Feňovčíková et al. [80] investigates the super cycle-antimagic labeling of wheel. Very recently, Bras et al. [27] formulated a graceful labeling for double-wheel graphs. In [51] Selvagopal and Jeyanthi investigated that a k-polygonal snake with length n admit C_k-supermagic also the graphs $C_n \times P_m$, $m \geq 2$, $n \geq 3$, $n \neq 4$ and $P_2 \times P_n$, $P_3 \times P_n$ with $n \geq 2$ admit C_4-supermagic. For further detail see a resent survey of graph labeling and the following research papers [34, 45-47, 65].

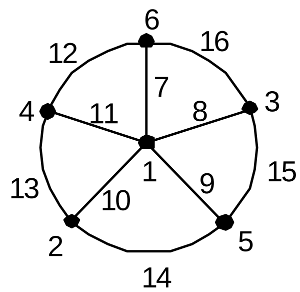

Figure 2.3: C_3-supermagic labeling, where magic constent $c = 41$

2.4 Magic labeling of (a, b, c) type

Labeling is a correspondence between the graph elements to numbers where input is vertex set, face set and edge set and output is a number assigned to that vertex, face or edge. Such type of labeling is known as (a, b, c) type. The total sum of numbers which assigned to a vertex, face and edge is known as weight of the face under this labeling. If all equal sided faces in a graph have the same weight then that labeling is known as magic labeling and if the weights form an arithmetic sequence then labeling is called anti-magic.

Ko-Wei Lih [60] was the first person worked on planer graphs and described the concept of magic labeling, where face magic labeling for friendship graphs, wheels and prisms have been discussed. Magic labeling of M$\ddot{o}$bius ladder, grid graph, planer graphs and prism graph have been given in [8, 9] by Ba$\check{c}$a.
Ali et al. [2], discussed the magic labeling of $(1, 1, 1)$ type for wheel graph and subdivided wheel graph. Hussain and Ali [46, 86] discussed the d-anti-supermagic labeling for kC_n and subdivided ladder graph. Hussain et. al. [45] discussed the magic labeling for $(1, 1, 1)$ type at subdivision of prism graph and showed that it admits face magic labeling of $(1, 1, 1)$ type. For further detail one can see a recent graph labeling survey [37].

The removal of the term face from the above definition provides the definition of labeling of $(1, 1, 0)$ type. The things are more interesting if labels are asinged only to the vertices provides vertex labeling which is also termed as labeling of $(1, 0, 0)$ type. Similarly, $(0, 1, 0)$ and $(0, 0, 1)$ type be asinging labels only to edges and faces respectively. Under this particular labeling, weights are collected by adding the labels of edges, vertices and faces, which surrounds that face. A labeling of graph G is called the d-antimagic labeling if for every number m, the weight form an arithmetic progression such as $W_m = \{a_m, a_m + d, a_m + 2d, ..., , a_m + (f_m - 1)d\}$ for the m-sided face weight, where a_m and d are some integers with $d \geq 0$. Here m-sided faces are denoted by f_m. For different values of m the different sets W_m are exist. However, if all the faces have equal number of sides, then only one arithmetic progression exists which consists on the set of face-weights. There are many common types of graphs which can be mentioned in this content. For example, prism and antiprism does not contain same faces initially, but it can easily be done through simple modification, so that (a, d)-antimagic can be considered on them instead of d-antimagic labeling.

Chapter 3

Multi-wheels are cycle anti-supermagic

A graph $G(V, E)$ has an H-covering if every edge in E belongs to a subgraph of G isomorphic to H. Suppose G admits an H-covering. An H-magic labeling is a total labeling ξ from $E(G) \cup V(G)$ onto the integers $\{1, 2, ..., |E(G) \cup V(G)|\}$ having property that, for every subgraph A of G isomorphic to H there is a positive integer c such that $\sum A = \sum\limits_{v \in V(A)} \xi(v) + \sum\limits_{e \in E(A)} \xi(e) = c$. A graph along with this labeling is known as H-magic. In addition, if $\{\xi(v)\}_{v \epsilon V} = \{1, ..., |V|\}$, then the graph is called H-supermagic. Different people worked on H-supermagic labeling of graphs and provided so many results in this direction. By using the partitioning sets technique of integers, Llado and Moragas [63] described cycle-magic labeling of wind mill, books, prisms, wheels and subdivided wheels graphs. Some families of graphs like ladders, fans, and books are considered by Ngurah et al. [74] which admit cycle-supermagic labeling. Recently in [77] Rizvi et al. discussed H-supermagic labeling of the disconnected graphs. They formulated cycle-supermagic labeling for the isomorphic copies with its disjoint union of different families of graphs. In this chapter, a cycle C_3-(a, d) anti-supermagic labeling for the Multi-wheels graph, supermagic labeling for isomorphic copies with its disjoint union of Multi-wheels graph and cycle (a, d)-anti-supermagic labeling for web graph are formulated.

Theorem 1 *[28] A Multi-wheels graph $G \cong mW_N$ admits C_3-$(a, 1)$ anti-supermagic, where $m \geq 2$ is even and $n \geq 3$.*

Proof.

Let $v = |V(G)| = mn + 1$, and $e = |E(G)| = 2mn$. The edge set and vertex set of graph are defined as,

$E(G) = \{c_0 u_i^j : 1 \leq i \leq n,\ 1 \leq j \leq m\} \cup \{u_i^j u_{i+1}^j : 1 \leq i \leq n-1,\ 1 \leq j \leq m\} \cup \{u_n^j u_1^j : 1 \leq j \leq m\}$.

$V(G) = \{u_i^j : 1 \leq i \leq n,\ 1 \leq j \leq m\} \cup \{c_0\}$.

Now, labeling can be defined as, $\xi : E(G) \cup V(G) \rightarrow \{1, 2, ..., 3mn + 1\}$,for $1 \leq i \leq n, 1 \leq j \leq m$.

$$\begin{aligned}
\xi(c_0) &= mn + 1, \\
\xi(u_i^j) &= m(i-1) + j,
\end{aligned}$$

$$\xi(c_0u_i^j) = 2mn - m(i-1) - (j-2),$$
$$\xi(u_i^ju_{i+1}^j) = 2mn - (i-2) + nj,$$

for $1 \leq i \leq n-1,\ 1 \leq j \leq m$

$$\xi(u_n^ju_1^j) = 2mn - (n-2) + nj,$$

for $1 \leq j \leq m$, where $v = mn+1$ and $e = 2mn$.

Here we can easily calculate the initial value a of every H which is containing in G and isomorphic with C_3, such that $a = \xi(c_0) + \xi(u_i^j) + \xi(u_n^j) + \xi(c_0u_i^j) + \xi(c_0u_n^j) + \xi(u_n^ju_1^j) = 7(mn+1)$ and the common difference $d = 1$. Hence $G \cong mW_n$ is C_3-$(a,1)$ anti-supermagic.

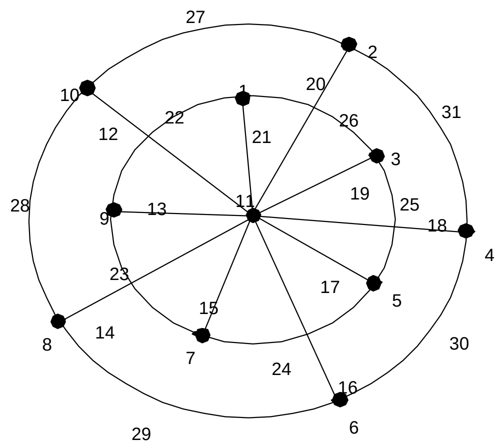

Figure 3.1: $C_3 - (77,1)$ anti-supermagic labeling of $2W_5$.

Theorem 2 *[28] A Multi-wheels graph $G \cong mW_n$ admits C_3-$(a,0)$ anti-supermagic, where even $m \geq 2$ and $n = 3$.*

Proof.

Let $v = |V(G)| = mn+1$, and $e = |E(G)| = 2mn$. The edge set and vertex set of graph are defined as,

$E(G) = \{c_0u_i^j : 1 \leq i \leq n, 1 \leq j \leq m\} \cup \{u_i^ju_{i+1}^j : 1 \leq i \leq n-1, 1 \leq j \leq m\} \cup \{u_n^ju_1^j : 1 \leq j \leq m\}$.

$V(G) = \{u_i^j : 1 \leq i \leq n, 1 \leq j \leq m\} \cup \{c_0\}$.

Let ξ and $s(\xi)$ represent total labeling over G and supermagic sum of every H which is contained in G and isomorphic with C_3 respectively. A supermagic sum is obtained by the formula,

$$s(\xi) = \xi(c_0) + \xi(u_i^j) + \xi(u_{i+1}^j) + \xi(c_0u_i^j) + \xi(c_0u_{i+1}^j) + \xi(u_i^ju_{i+1}^j). \qquad (3.1)$$

For $1 \leq i \leq n$ and $1 \leq j \leq m$.

Furthermore a labeling can be defined as follows:

$$\begin{aligned} \xi(c_0) &= v, \\ \xi(u_i^j) &= m(i-1)+j, \\ \xi(c_0u_i^j) &= 10m - 2mi - 2(j-1), \end{aligned}$$

when $j \geq \frac{m}{2}$

$$\xi(c_0u_i^j) = 10m - mi - 2(j - \frac{m}{2} - 1),$$

when $j > \frac{m}{2}$

$$\xi(u_i^ju_{i+1}^j) = 3(2m+1) + 2m(i-1) + 2(j-1),$$

for $1 \leq i \leq n-1$ and $1 \leq j \leq \frac{m}{2}$

$$\xi(u_3^ju_1^j) = 7m + i - 2(j-1),$$

for $1 \leq j \leq \frac{m}{2}$

$$\xi(u_i^ju_{i+1}^j) = 3(m+1) + 2m(i-1) + 2(j - \frac{m}{2} - 1),$$

for $1 \leq i \leq n-1,\ j \geq \frac{m}{2}+1$

$$\xi(u_3^ju_1^j) = 4m + i - 2(j - \frac{m}{2} - 1),$$

for $j \geq \frac{m}{2}+1$, where $v = mn+1$. In every subgraph C_3 of G, here $s(\xi)$ be a constant sum. Using equation (3.1), we get $s(\xi) = 8mn + 2n$. Hence $G \cong mW_n$ is C_3-$(a,0)$ anti-supermagic.

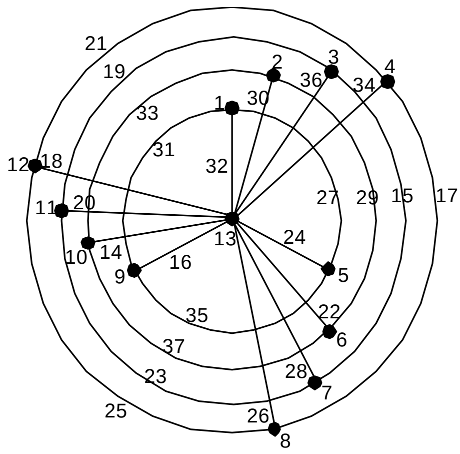

Figure 3.2: $C_3 - (102, 0)$ anti-supermagic labeling of $4W_3$.

3.1 Cycle supermagic labeling for isomorphic copies

In the next theorems, we prove that isomorphic copies with its disjoint union of Multi-wheels graph admit cycle supermagic labeling.

Theorem 3 *[28] An isomorphic copies with its disjoint union of Multi-wheels graph $G \cong mlW_n$ admits C_3-$(a, 0)$ supermagic labeling, where $n = 3,\ m = 2$ and $l \geq 2$.*

Proof.

Let $v = |V(G)| = l(mn + 1)$, and $e = |E(G)| = 2lmn$. The edge set and vertex set of graph are defined as,

$E(G) = \{c_k u^k_{(i,j)} : 1 \leq i \leq n,\ 1 \leq k \leq l,\ j = 1, 2\} \cup \{u^k_{(i,j)} u^k_{(i+1,j)} : 1 \leq i \leq n - 1,\ 1 \leq k \leq l,\ j = 1, 2\} \cup \{u^k_{(n,j)} u^k_{(1,j)}\}$.

$V(G) = \{u^k_{(i,j)} : 1 \leq i \leq n,\ 1 \leq k \leq l,\ \ j = 1, 2\} \cup \{c_k : 1 \leq k \leq l\}$.

Let ξ and $s(\xi)$ represent total labeling over G and supermagic sum of every H which is contained in G and isomorphic with C_3 respectively. A supermagic sum is obtained by the formula,

$$s(\xi) = \xi(c_k) + \xi(u^k_{(i,j)}) + \xi(u^k_{(i+1,j)}) + \xi(c_k u^k_{(i,j)}) + \xi(c_k u^k_{(i+1,j)}) + \xi(u^k_{(i,j)} u^k_{(i+1,j)}). \quad (3.2)$$

For $1 \leq i \leq n,\ 1 \leq k \leq l$ and $\ j = 1, 2$.

Furthermore the total labeling can be defined as follows:

$$
\begin{aligned}
\xi(c_k) &= lmn + k, \\
\xi(u^k_{(i,1)}) &= lm(i-1) + (k-1) + 1, \\
\xi(u^k_{(i,2)}) &= lm(i-1) + (k-1) + (l+1), \\
\xi(c_k u^k_{(i,1)}) &= 8lm - 2ml(i-1) - (k-1), \\
\xi(c_k u^k_{(i,2)}) &= 9lm - 2ml(i-1) - (k-1), \\
\xi(u^k_{(i,1)} u^k_{(i+1,1)}) &= l(8m-1) + 2ml(i-1) - (k-1), \\
\xi(u^k_{(i,2)} u^k_{(i+1,2)}) &= l(5m-1) + 2ml(i-1) - (k-1), \\
\xi(u^k_{(3,1)} u^k_{(1,1)}) &= l(8m+1) - (k-1), \\
\xi(u^k_{(3,2)} u^k_{(1,2)}) &= l(5m+1) - (k-1).
\end{aligned}
$$

In every subgraph C_3 of G, $s(\xi)$ be a constant sum. Using equation (3.2), we have $s(\xi) = 9mnl - 3(l-1)$. Hence an isomorphic copies with its disjoint union of Multi-wheels graph $G \cong mlW_n$ admits C_3-$(a, 0)$ supermagic labeling.

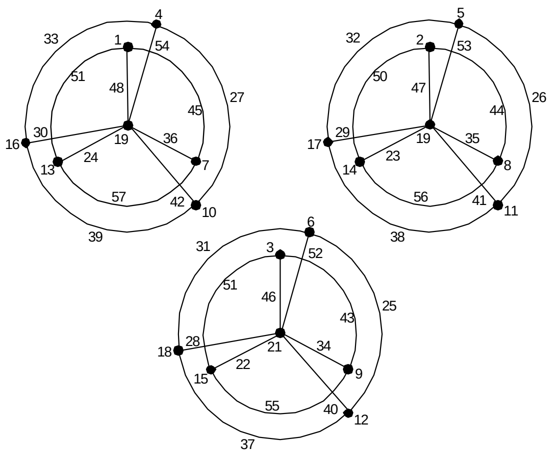

Figure 3.3: $C_3 - (156, 0)$ supermagic labeling of $2(3(W_3))$.

Theorem 4 *[28] An isomorphic copies with its disjoint union of Multi-wheels graph $G \cong mlW_n$ admits C_3-$(a, 0)$ supermagic labeling, where $n = 3$, $m = 4$ and $l \geq 2$.*

Proof.

Let $v = |V(G)| = l(mn + 1)$, and $e = |E(G)| = 2lmn$. The edge set and vertex set of graph are defined as,

$E(G) = \{c_k u^k_{(i,j)} : 1 \leq i \leq n,\ 1 \leq k \leq l,\ 1 \leq j \leq m\} \cup \{u^k_{(i,j)} u^k_{(i+1,j)} : 1 \leq i \leq n-1,\ 1 \leq k \leq l,\ 1 \leq j \leq m\} \cup \{u^k_{(n,j)} u^k_{(1,j)}\}$.
$V(G) = \{u^k_{(i,j)} : 1 \leq i \leq n,\ 1 \leq k \leq l,\ 1 \leq j \leq m\} \cup \{c_k : 1 \leq k \leq l\}$.
Let ξ and $s(\xi)$ be represent total labeling over G and supermagic sum of every H which is contained in G and isomorphic with C_3 respectively.
Furthermore the total labeling can be defined as follows:

$$\begin{aligned}
\xi(c_k) &= lmn + k, \\
\xi(u^k_{(i,j)}) &= lm(i-1) + (k-1) + (lj+1), \\
\xi(c_k u^k_{(i,1)}) &= 8lm - 2ml(i-1) - (k-1), \\
\xi(c_k u^k_{(i,2)}) &= 8lm - 2ml(i-1) - (k-1) - 2l, \\
\xi(c_k u^k_{(i,3)}) &= 9lm - 2ml(i-1) - (k-1), \\
\xi(c_k u^k_{(i,4)}) &= 9lm - 2ml(i-1) - (k-1) - 2l, \\
\xi(u^k_{(i,1)} u^k_{(i+1,1)}) &= l(7m-1) + 2ml(i-1) - (k-1), \\
\xi(u^k_{(i,2)} u^k_{(i+1,2)}) &= l(7m-1) + 2ml(i-1) - (k-1) + 2l, \\
\xi(u^k_{(i,3)} u^k_{(i+1,3)}) &= l(4m-1) + 2ml(i-1) - (k-1), \\
\xi(u^k_{(i,4)} u^k_{(i,4)}) &= l(4m-1) + 2ml(i-1) - (k-1) + 2l, \\
\xi(u^k_{(3,1)} u^k_{(1,1)}) &= l(8m+1) - (k-1) - 2l, \\
\xi(u^k_{(3,2)} u^k_{(1,2)}) &= l(8m+1) - (k-1), \\
\xi(u^k_{(3,3)} u^k_{(1,3)}) &= l(5m+1) - (k-1) - 2l, \\
\xi(u^k_{(3,4)} u^k_{(1,4)}) &= l(5m+1) - (k-1).
\end{aligned}$$

where $1 \leq i \leq n-1,\ 1 \leq k \leq l$ and $1 \leq j \leq m$.
In every subgraph C_3 of G, we have $s(\xi) = 8mnl + 3(l+1)$. Hence the graph $G \cong mlW_n$ is C_3-$(a, 0)$ supermagic. In next theorem, formulation of cycle supermagic labeling of isomorphic copies with its disjoint union of web graphs are presented.

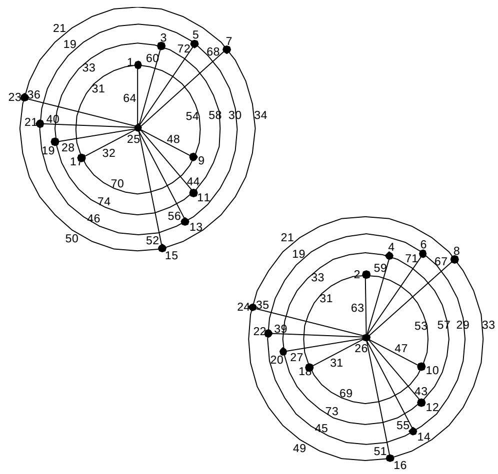

Figure 3.4: $C_3-(201,0)$ supermagic labeling of $4(2(W_3))$.

Theorem 5 *[28] A Multi-wheels graph $G \cong lW(m,n)$ admits multi-cycle-supermagic, where $l \geq 2,\ m=2$ and $n=3$.*

Proof.

Let $v=|V(G)|=l(mn+1)$, and $e=|E(G)|=2lmn$. The edge set and vertex set of graph are defined as,

$E(G)=\{c_kv_i^k : 1\leq i\leq n\}\cup\{v_i^ku_i^k : 1\leq i\leq n\}\cup\{u_i^ku_{i+1}^k : 1\leq i\leq n-1\}\cup\{u_n^ku_1^k\}\cup\{v_i^kv_{i+1}^k : 1\leq i\leq n-1\}\cup\{v_n^kv_1^k\}$.

$V(G)=\{u_i^k : 1\leq i\leq n,\ 1\leq k\leq l\}\cup\{v_i^k : 1\leq i\leq n\}\cup\{c_k\}$.

Now, labeling can be defined as, $\xi : E(G)\cup V(G)\rightarrow\{1,2,...,l(3mn+1)\}$,

For $1\leq i\leq n,\ 1\leq k\leq l$.

$$
\begin{aligned}
\xi(c_k) &= k,\\
\xi(u_i^k) &= li+k,\\
\xi(v_i^k) &= 4l+k+l(i-1),\\
\xi(c_kv_i^k) &= 19l-2m(i-1)-(k-1),\\
\xi(u_i^kv_i^k) &= 18l-2m(i-1)-(k-1),\\
\xi(v_i^kv_{i+1}^k) &= 11l+lm(i-1)-(k-1),\\
\xi(u_i^ku_{i+1}^k) &= 10l+lm(i-1)-(k-1),\\
\xi(u_3^ku_1^k) &= 9l-(k-1),\\
\xi(v_3^kv_1^k) &= 12l-(k-1).
\end{aligned}
$$

In every subgraph C_3 of G, we can easily calculate the value of $\xi(s) = \xi(c_k) + \xi(v_i^k) + \xi(c_k v_i^k) + \xi(v_i^k v_{i+1}^k) + \xi(v_n^k v_1^k) = 19ln + (n-l)$, and also every subgraph C_4 of G, we have $\xi(s) = \xi(c_k) + \xi(u_i^k) + \xi(v_i^k u_i^k) + \xi(u_i^k u_{i+1}^k) + \xi(u_n^k u_1^k) = 23ln - 2(l-2)$. Hence $G \cong W(m,n)$ is multi-cycle-supermagic.

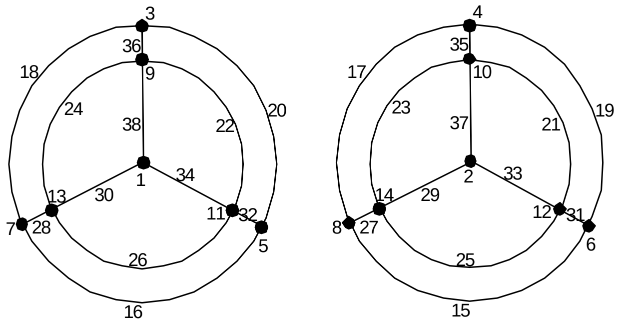

Figure 3.5: A multi-cycle-supermagic labeling of $W(2,3)$.

Our results of this chapter has been published in [28].

Chapter 4

Magic labeling of various line graphs

The aim of this chapter is to investigate fan, friendship, ladder and wheel line graphs and to study the supermagic and anti-supermagic vertex-edge-face labeling of such graphs and their isomorphic copies. Ko-Wei Lih [60] was the first person worked on planer graphs and described the concept of magic labeling, where face magic labeling for friendship graphs, wheels and prisms have been discussed. Magic labeling of Möbius ladder, grid graph, planer graphs and prism graph have been given in [8, 9] by Bača.

Ali et al. [2], discussed the magic labeling of $(1,1,1)$ type for wheel graph and subdivided wheel graph. Hussain and Ali [46, 86] discussed the d-anti-supermagic labeling for kC_n and subdivided ladder graph. Hussain et. al. [45] discussed the magic labeling for $(1,1,1)$ type at subdivision of prism graph and showed that it admits face magic labeling of $(1,1,1)$ type. An anti-supermagic labeling of the extension of cycle graphs is also formulated.

4.1 Line graph of fan and friendship graphs with its labeling

Theorem 6 *[4] A fan graph $G \cong mL(F_n)$, where $m \geq 2$ and $n > 3$ confesses anti-supermagic of* $(1,1,1)$ *type labeling.*

Proof.

Let $v = |V(G)| = 2mn - m$, $e = |E(G)| = 3mn - m$ and $f = |F(G)| = 2mn - m$. Now, labeling can be defined as, $\xi : V(G) \cup F(G) \cup E(G) \rightarrow \{1,\ 2,\ 3,\ ...,\ v+f+e\}$,

$$
\begin{aligned}
\xi(v_i^j) &= mi - m + j, && \text{for } 1 \leq i \leq 2n \\
\xi(v_i^j v_{i+1}^j) &= 4mn - m - mi + 1 - j, && \\
\xi(v_{2i}^j v_{2i-2}^j) &= 6mn - 6m - 2mi + 1 - j, && \text{for } 2 \leq i \leq n \\
\xi(v_{2n-2}^j v_1^j) &= 6mn - 5m + 1 - j, && \\
\xi(v_{2i-1}^j v_{2i+1}^j) &= 6mn - 5m - 2mi + 1 - j, && \text{for } 2 \leq i \leq n \\
\xi(v_{2n-1}^j v_2^j) &= 6mn - 6m + 1 - j, &&
\end{aligned}
$$

$$\xi(f_i^j) = 5mn + nj - n + i, \quad \text{for } 1 \le i \le n$$
$$\xi(f_{int}^j) = 8mn + j,$$
$$\xi(f_{ext}^j) = 8mn + m + j.$$

In $G \cong mL(F_n)$ all 3-sided faces form an arithmetic progression with common difference $d = 1$, hence the said graph admits anti-supermagic and all n-sided faces also have anti-supermagic.

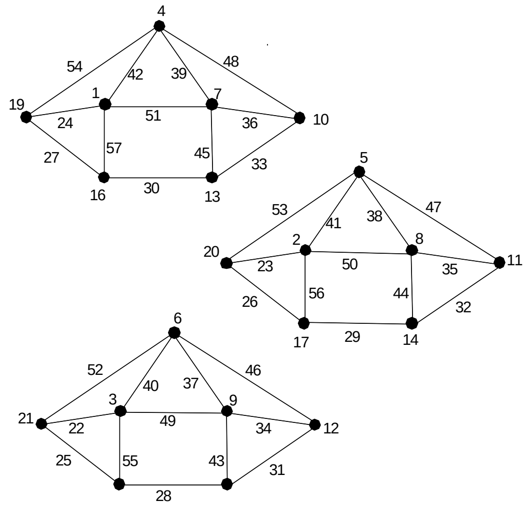

Figure 4.1: In $G \cong 3L(F_3)$ all 3–sided, n-sided and $n+1$ sided form anti supermagic of $(1, 1, 1)$ type labeling.

Theorem 7 *[4] A graph $G \cong mL(F(nC_3))$, where $m \ge 2$ and $n \ge 3$, confesses anti-supermagic of* $(1, 1, 1)$ *type labeling.*

Proof.

Let $v = |V(G)| = 3mn$, $e = |E(G)| = 4mn$ and $f = |F(G)| = 2mn - m$. Now, labeling can be defined as, $\xi : V(G) \cup F(G) \cup E(G) \to \{1,\ 2,\ 3,\ ...,\ v + f + e\}$,

$$\xi(v_i^j) = mi - m + j, \quad \text{for } 1 \le i \le n$$
$$\xi(u_i^j) = 3mn - mi + j, \quad \text{for } 1 \le i \le n$$
$$\xi(v_{2i-1}^j u_i^j) = 5mn - m\left\lfloor \frac{2i-1}{2} \right\rfloor + 1 - j, \quad \text{for } 1 \le i \le n$$

$$
\begin{aligned}
\xi(v_{2i}^j u_i^j) &= 5mn - m\left\lfloor \frac{2i-1}{2} \right\rfloor + 1 - j, \qquad \text{for } 1 \le i \le n \\
\xi(v_{2i-1}^j v_{2i}^j) &= 6mn + m + 1 - j - mi, \\
\xi(v_{2i}^j v_{2i+1}^j) &= 6mn + i + nj - n, \qquad\qquad \text{for } 1 \le i \le n
\end{aligned}
$$

$$
\begin{aligned}
\xi(f_i^j) &= 8mn + m + 1 - j - mi, \qquad\qquad \text{for} 1 \le i \le n \\
\xi(f_{int}^j) &= 8mn + j, \\
\xi(f_{ext}^j) &= 8mn + m + j.
\end{aligned}
$$

In $G \cong mL(F(nC_3))$ all 3-sided faces forms an arithmetic progression with initial value $25mn + 4$ and the common difference $d = 1$. All $2n$-sided faces admit anti-supermagic and also all $3n$-sided faces have anti-supermagic, hence the following graph admits anti-supermagic.

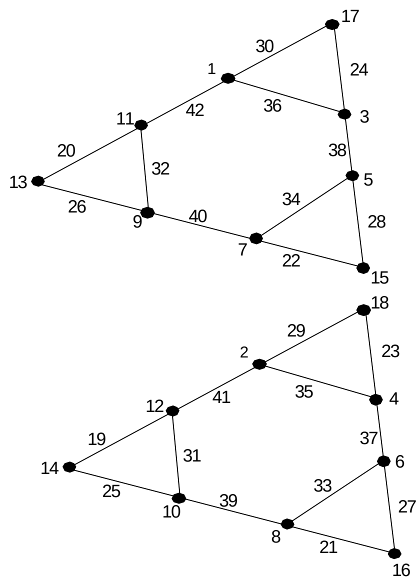

Figure 4.2: In $G \cong 2L(3C_3)$ all 3–sided, 6-sided and 9-sided form anti supermagic of $(1, 1, 1)$ type labeling.

Theorem 8 *[4] A graph $G \cong mL(F(nC_3))$, where $m \ge 2$ and $n \ge 3$, confesses supermagic of $(1, 1, 1)$ type labeling.*

Proof.

Let $v = |V(G)| = 3mn$, $e = |E(G)| = 4mn$ and $f = |F(G)| = 2mn - m$. Now, labeling can be defined as, $\xi : V(G) \cup F(G) \cup E(G) \to \{1,\ 2,\ 3,\ ...,\ v + f + e\}$, where $1 \leq j \leq m$

$$\begin{aligned}
\xi(v_i^j) &= mi - m + j, && \text{for } 1 \leq i \leq 2n \\
\xi(u_i^j) &= 3mn - mi + j, && \text{for } 1 \leq i \leq n \\
\xi(v_{2i-1}^j u_i^j) &= 5mn - m\left\lfloor \frac{2i-1}{2} \right\rfloor + 1 - j, && \text{for } 1 \leq i \leq n \\
\xi(v_{2i}^j u_i^j) &= 5mn - m\left\lfloor \frac{2i-1}{2} \right\rfloor + 1 - j, && \text{for } 1 \leq i \leq n \\
\xi(v_i^j v_{2i}^j) &= 7mn - 1 - 2(j-1) - m(i-1), && \text{where } i \text{ is odd} \\
\xi(v_{2i-2}^j v_{2i-1}^j) &= 6mn + m - m(i-1) - 2(j-1), && \text{where } i \text{ is odd} \\
\xi(v_{2n}^j v_1^j) &= 7mn - 2(j-1), \\
\xi(f_i^j) &= 7mn + 2i + (j-1), \\
\xi(f_{int}^j) &= 7mn + 1 + mn(j-1), \\
\xi(f_{ext}^j) &= 8mn + m + j.
\end{aligned}$$

Now one can easily see that weights of 3-sided faces with this labeling is constant. $\xi(s) = 27nm + m$. Hence the graph admits supermagic labeling, all $2n$-sided faces admit anti-supermagic and also all $3n$-sided faces have anti-supermagic, hence the following graph admits supermagic.

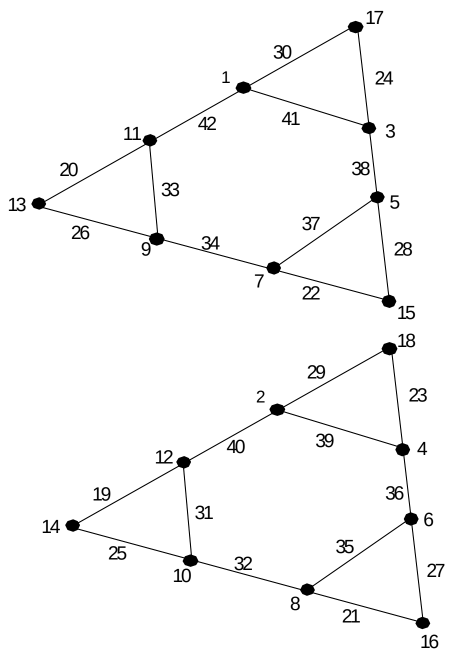

Figure 4.3: In $G \cong 2L(3C_3)$ all 3–sided have supermagic of $(1,1,1)$ type and 6-sided and 9-sided form anti supermagic of $(1,1,1)$ type labeling.

Theorem 9 *[4] A graph $G \cong mL(F(nC_3))$, where $m \geq 2$ and $n \geq 3$, confesses supermagic of $(1,1,1)$ type labeling.*

Proof.

Let $v = |V(G)| = 3mn$, $e = |E(G)| = 4mn$ and $f = |F(G)| = 2mn - m$. Now, labeling can be defined as, $\xi : V(G) \cup F(G) \cup E(G) \rightarrow \{1, 2, 3, ..., v + f + e\}$, where $1 \leq j \leq m$

$$\begin{aligned}
\xi(v_i^j) &= mi - m + j, && \text{for } 1 \leq i \leq n \\
\xi(u_i^j) &= 3mn - mi + j, && \text{for } 1 \leq i \leq n \\
\xi(v_{2i-1}^j u_i^j) &= 5mn - m\left\lfloor \frac{2i-1}{2} \right\rfloor + 1 - j, && \text{for } 1 \leq i \leq n \\
\xi(v_{2i}^j u_i^j) &= 5mn - m\left\lfloor \frac{2i-1}{2} \right\rfloor + 1 - j, && \text{for } 1 \leq i \leq n \\
\xi(v_{2i-1}^j v_{2i}^j) &= 7mn + 1 - j - mi + m, && \\
\xi(v_{2i}^j v_{2i+1}^j) &= 5mn + i + nj - n, && \text{for } 1 \leq i \leq n,
\end{aligned}$$

$$\begin{aligned}
\xi(f_i^j) &= 7mn + i + nj - n, && \text{for } 1 \leq i \leq n, \\
\xi(f_{int}^j) &= 8mn + j, && \\
\xi(f_{ext}^j) &= 8mn + m + j. &&
\end{aligned}$$

In $G \cong mL(F(nC_3))$ all 3-sided faces forms an arithmetic progression with initial value $27mn - m$ and the common difference $d = 1$. All $2n$-sided faces admit anti-supermagic and also all $3n$-sided faces have anti-supermagic, hence the following graph admits anti-supermagic.

4.2 Line graph of ladder and wheel graphs with its labeling

Theorem 10 *[4] A graph $G \cong L(L_n)$, where $n \geq 3$, confesses anti-supermagic of $(1,1,1)$ type labeling.*

Proof.

Let $v = |V(G)| = 3n + 1$, $e = |E(G)| = 6n - 2$ and $f = |F(G)| = 3n - 1$.

The edge set and vertex set of graph are defined as, $E(G) = \{u_i u_{i+1} : 1 \leq i \leq n-1\} \cup \{v_i v_{i+1} : 1 \leq i \leq n-1\} \cup \{u_i w_{i+1} : 1 \leq i \leq n\} \cup \{w_i u_i : 1 \leq i \leq n\} \cup \{v_i w_{i+1} : 1 \leq i \leq n\} \cup \{w_i v_i : 1 \leq i \leq n\}$.
$V(G) = \{u_i : 1 \leq i \leq n\} \cup \{v_i : 1 \leq i \leq n\} \cup \{w_i : 1 \leq i \leq n+1\}$;
Now, labeling can be defined as, $\xi : V(G) \cup F(G) \cup E(G) \to \{1,\ 2,\ 3,\ ...,\ v+f+e\}$,

$$\xi(u_i) = \begin{cases} 2i-1, & \text{when } i = odd, \\ \\ 2i, & \text{when } i = even, \end{cases}$$

$$\xi(v_i) = \begin{cases} 2i, & \text{when } i = odd, \\ \\ 2i-1, & \text{when } i = even, \end{cases}$$

$$\begin{aligned}
\xi(w_i) &= 2n+i, && \text{where } 1 \leq i \leq n+1 \\
\xi(u_i u_{i+1}) &= 9n-2i, && \text{where } 1 \leq i \leq n-1 \\
\xi(v_i v_{i+1}) &= 9n+1-2i, && \text{where } 1 \leq i \leq n-1 \\
\xi(w_i u_i) &= 7n+4-4i, && \text{where } 1 \leq i \leq n \\
\xi(u_i w_{i+1}) &= 7n+2-4i, && \text{where } 1 \leq i \leq n \\
\xi(w_i v_i) &= 7n+5-4i, && \text{where } 1 \leq i \leq n \\
\xi(v_i w_{i+1}) &= 7n+3-4i, && \text{where } 1 \leq i \leq n \\
\xi(f(u_i)) &= 9n+3i-3, && \text{where } 1 \leq i \leq n-1 \\
\xi(f(v_i)) &= 9n+3i-1, && \text{where } 1 \leq i \leq n-1 \\
\xi(f(w_i)) &= 12n+1-3i, && \text{where } 1 \leq i \leq n-1 \\
\xi(f_{external}) &= 12n-3.
\end{aligned}$$

In $G \cong L(L_n)$ all 3-sided faces forms an arithmetic progression with initial value $32n-27$ and the common difference $d = 1$ and also all the weights of the interior 4-sided faces under the labeling, have initial value of a for every subgraph H of G that is isomorphic to C_4 with common difference d. Hence the following graph admit anti-supermagic labeling. The example given below shows a C_3-anti-supermagic and C_4-anti-supermagic of $L(L_5)$:

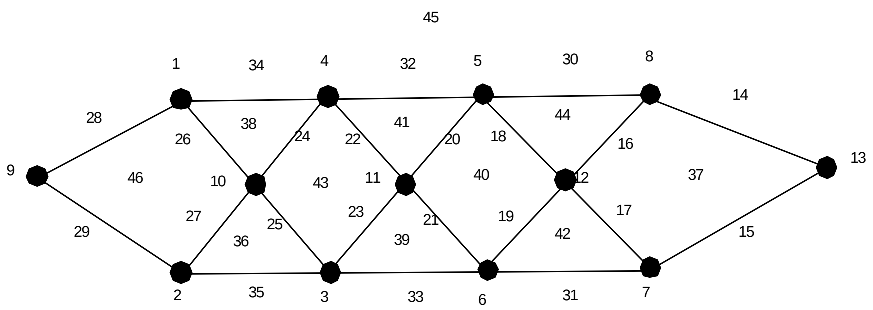

Figure 4.4: C_3-anti supermagic and C_4-anti supermagic of $L(L_5)$.

Theorem 11 *[4] A graph $G \cong L(W_n)$, where $n \geq 3$, confesses supermagic of* $(1,1,1)$ *type labeling.*

Proof.

Here $v = |V(G)| = 2n$, $e = |E(G)| = 4n$ and $f = |F(G)| = 2n+2$. Now, labeling can be defined as, $\xi : V(G) \cup F(G) \cup E(G) \rightarrow \{1,\ 2,\ 3,\ ...,\ v+f+e\}$,

$$\begin{aligned}
\xi(v_i) &= i, && \text{for } 1 \leq i \leq 2n \\
\xi(v_i v_{i+1}) &= 4n+1-i, && \text{for } 1 \leq i \leq 2n \\
\xi(v_{2i} v_{2i-2}) &= 6n+1-i, && \text{for } 2 \leq i \leq n \\
\xi(v_{2n} v_2) &= 6n, && \\
\xi(v_{2i-1} v_{2i-3}) &= 5n+1-i, && \text{for } 2 \leq i \leq n \\
\xi(v_{2n-1} v_1) &= 5n, &&
\end{aligned}$$

$$\xi(f_i) = \begin{cases} 6n+i, & 1 \leq i \leq n, \\ \\ 6n+1+i, & n+1 \leq i \leq 2n, \end{cases}$$

$$\begin{aligned}
\xi(f_{int}) &= 7n+1, \\
\xi(f_{ext}) &= 8n+2.
\end{aligned}$$

Here by above labelling all 3-sided faces, have the same value $21n+4$ and all n-sided faces have supermagic. Hence the graph admit supermagic labeling.

Theorem 12 *[4] A graph $G \cong mL(W_n)$, where $m \geq 2$ and $n \geq 3$, confesses anti-supermagic of* $(1,1,1)$ *type labeling.*

Proof.

Let $v = |V(G)| = 2mn$, $e = |E(G)| = 4mn$ and $f = |F(G)| = 2mn$. Now, labeling can be defined as, $\xi : V(G) \cup F(G) \cup E(G) \to \{1,\ 2,\ 3,\ ...,\ v + f + e\}$,

$$\begin{aligned}
\xi(v_i^j) &= mi - m + j, && \text{for } 1 \le i \le 2n \\
\xi(v_i^j v_{i+1}^j) &= 4mn - mi + m + 1 - j, && \text{for } 1 \le i \le 2n \\
\xi(v_{2i}^j v_{2i-2}^j) &= 6mn - 2mi + m + 1 - j, && \text{for } 2 \le i \le n \\
\xi(v_{2n}^j v_2^j) &= 6mn - m + 1 - j, && \\
\xi(v_{2i-1}^j v_{2i-3}^j) &= 6mn - 2mi + 2m + 1 - j, && \text{for } 2 \le i \le n \\
\xi(v_{2n-1}^j v_1^j) &= 6mn + 1 - j, && \\
\xi(f_i^j) &= 6mn + 2nj - 2n + i, && \text{for } 1 \le i \le 2n \\
\xi(f_{int}^j) &= 8mn + j, && \\
\xi(f_{ext}^j) &= 8mn + m + j. &&
\end{aligned}$$

In $G \cong mL(W_n)$ all 3-sided faces forms an arithmetic progression with initial value $20mn + 4$ and the common difference $d = 1$ and all n-sided faces have also anti-supermagic. Hence the above graph admits anti-supermagic labelling.

Theorem 13 *[4] For every $m \ge 1$, $n \ge 5$, where n is odd, the graph $G \cong C_n^{m*}$ is C_3-anti-supermagic with distance $d = 1$.*

Proof.

Let $v = |V(G)| = n(m + 1)$, and $e = |E(G)| = n(2m + 1)$. Denote the edge and vertex sets as follows:

$E(G) = \{u_i u_{i+1} : 1 \le i \le n - 1\} \cup \{u_i v_i^j : 1 \le i \le n, 1 \le j \le m\} \cup \{u_{i+1} v_i^j : 1 \le i \le n - 1, 1 \le j \le m\} \cup \{u_1 v_n^j : 1 \le j \le m\}$.

$V(G) = \{u_i : 1 \le i \le n\} \cup \{v_i^j : 1 \le i \le n, 1 \le j \le m\}$;

Now, labeling can be defined as, $\xi : E(G) \cup V(G) \to \{1, 2, ..., (3mn + 2n)\}$, For $1 \leq i \leq n, 1 \leq j \leq m. \xi(u_i) = \begin{cases} (\frac{i+1}{2}), & i = odd, \\ \\ (\frac{n-1}{2}) + \lceil \frac{i+1}{2} \rceil, & i = even, \end{cases}$

$$\begin{aligned} \xi(u_i u_{i+1}) &= 3mn + 2n - i, \\ \xi(u_n u_1) &= 3mn + 2n, \\ \xi(v_i^j) &= nj + i, \\ \xi(u_i v_i^j) &= mn + nj + i, \\ \xi(u_{i+1} v_i^j) &= 3mn + 2n + 2 - i - nj, \\ \xi(u_1 v_n^j) &= 3mn + 1 + n - nj. \end{aligned}$$

Now one can easily verify that $\sum C_3^{(i,j)} = \xi(u_i) + \xi(u_{i+1}) + \xi(v_i^j) + \xi(u_i v_i^j) + \xi(u_{i+1} v_i^j) + \xi(u_i u_{i+1}) = 7mn + 6n + 1$. Hence C_n^{m*} is C_3-anti-supermagic with distance $d = 1$.

Our results of this chapter has been published in [4].

Chapter 5

On cycle anti-supermagic labeling of ladder and triangular ladder graphs

Ali et al. [3] determined supermagic labeling for the isomorphic copies with its disjoint union of prism graph and also found out the supermagic labeling for the isomorphic copies with its disjoint union of generalized anti-prism graph. In the mean while Rizvi et al. [78] described the cycle-(super) magic labeling of uniform subdivided graph. In addition, they studied cycle-(super) magic behavior of non-uniform subdivided fans and triangular ladder graphs. In this chapter, labeling of type cycle anti-supermagic for the isomorphic copies with its disjoint union of ladder graphs for different value of d are presented. Also showed that the isomorphic copies with its disjoint union of triangular ladder graphs admit labeling of type cycle supermagic as well as labeling of type cycle anti-supermagic for the isomorphic copies with its disjoint union of triangular ladder graphs for different value of d are exist.

Theorem 14 *[29] For every $m \geq 1$, $n \geq 2$, the graph $G \cong mL_n$ is C_4-anti-supermagic with distance $d = 2$ and $a = 18mn - 5m + 5$.*

Proof.
Let $v = |V(G)| = 2mn$, and $e = |E(G)| = m(3n-2)$. The edge set and vertex set of graph are defined as, $V(G) = \{u_i^j : 1 \leq i \leq n, 1 \leq j \leq m\} \cup \{v_i^j : 1 \leq i \leq n, 1 \leq j \leq m\}$;
$E(G) = \{u_i^j v_i^j : 1 \leq i \leq n, 1 \leq j \leq m\} \cup \{u_i^j u_{i+1}^j : 1 \leq i \leq n-1, 1 \leq j \leq m\} \cup \{v_i^j v_{i+1}^j : 1 \leq i \leq n-1, 1 \leq j \leq m\}$.
Now, labeling can be defined as, $\xi : V(G) \cup E(G) \rightarrow \{1, 2, ..., m(5n-2)\}$,
For $1 \leq i \leq n, 1 \leq i \leq m$.

$$\begin{aligned}
\xi(u_i^j) &= 2m(i-1) + j, \\
\xi(v_i^j) &= m(2i-1) + j, \\
\xi(u_i^j v_i^j) &= v + e - m(i-1) - (j-1),
\end{aligned}$$

where $v = 2mn$ and $e = m(3n-2)$

$$\begin{aligned}
\xi(u_i^j u_{i+1}^j) &= v + e - mn - 2m(i-1) - (j-1), \\
\xi(v_i^j v_{i+1}^j) &= v + e - mn - 2mi + j.
\end{aligned}$$

Here one can easily calculate the value of a of subcycle such that $a = \sum C_4^{(i,j)} = \xi(u_i^j) + \xi(u_{i+1}^j) + \xi(v_i^j) + \xi(v_{i+1}^j) + \xi(u_i^j v_i^j) + \xi(u_i^j u_{i+1}^j) + \xi(v_i^j v_{i+1}^j) + \xi(u_{i+1}^j v_{i+1}^j) = 18mn - 5m + 5$. Hence mL_n has C_4-anti-supermagic labeling with distance $d = 2$.

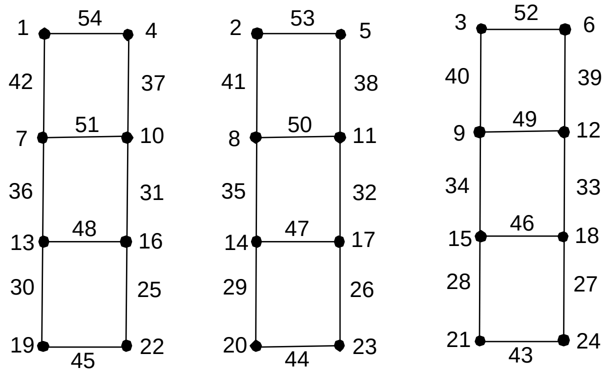

Figure 5.1: A $C_4 - (206, 2)$ anti-supermagic labeling of $3L_4$.

Theorem 15 *[29] For every $m \geq 1$, $n \geq 2$, the graph $G \cong mL_n$ is C_4-anti-supermagic with distance $d = 6$ and $a = 16mn - 3m + 7$.*

Proof.

Let $v = |V(G)| = 2mn$, and $e = |E(G)| = m(3n-2)$. The edge set and vertex set of graph are defined as, $V(G) = \{u_i^j : 1 \leq i \leq n, 1 \leq j \leq m\} \cup \{v_i^j : 1 \leq i \leq n, 1 \leq j \leq m\}$;

$E(G) = \{u_i^j v_i^j : 1 \leq i \leq n, 1 \leq j \leq m\} \cup \{u_i^j u_{i+1}^j : 1 \leq i \leq n-1, 1 \leq j \leq m\} \cup \{v_i^j v_{i+1}^j : 1 \leq i \leq n-1, 1 \leq j \leq m\}$.

Now, labeling can be defined as, $\xi : V(G) \cup E(G) \rightarrow \{1, 2, ..., m(5n-2)\}$,

For $1 \leq i \leq n$ and $1 \leq i \leq m$.

$$\begin{aligned}
\xi(u_i^j) &= 2m(i-1) + j, \\
\xi(v_i^j) &= m(2i-1) + j, \\
\xi(u_i^j v_i^j) &= v + e - m(i-1) - (j-1),
\end{aligned}$$

where $v = 2mn$ and $e = m(3n-2)$

$$\begin{aligned}
\xi(u_i^j u_{i+1}^j) &= v + e - 2mn + m + i + (n-1)(j-1), \\
\xi(v_i^j v_{i+1}^j) &= v + e - 2mn - (m-2) - (i-1)(n-1)(j-1).
\end{aligned}$$

We can easily calculate the value of a of subcycle such that $a = \sum C_4^{(i,j)} = \xi(u_i^j) + \xi(u_{i+1}^j) + \xi(v_i^j) + \xi(v_{i+1}^j) + \xi(u_i^j v_i^j) + \xi(u_i^j u_{i+1}^j) + \xi(v_i^j v_{i+1}^j) + \xi(u_{i+1}^j v_{i+1}^j) = 16mn - 3m + 7$. Hence mL_n is C_4-anti-supermagic with distance $d = 6$.

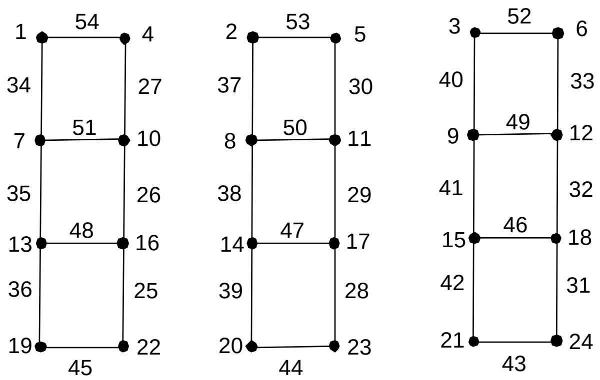

Figure 5.2: A $C_4 - (190, 6)$ anti-supermagic labeling of $3L_4$.

In the next two theorems, we prove that isomorphic copies with its disjoint union of triangular ladder graphs, admit cycle supermagic labeling and cycle anti-supermagic labeling for different values of d.

Theorem 16 *[29] The graph $G \cong mTL_n$ admits C_4-supermagic labeling for any m, n.*

Proof.

Let $v = |V(G)|$ and $e = |E(G)|$. Then $v = 2mn$, $e = m(4n-3)$. The edge set and vertex set of graph are defined as, $V(G) = \{u_i^j : 1 \leq i \leq n, 1 \leq j \leq m\} \cup \{v_i^j : 1 \leq i \leq n, 1 \leq j \leq m\}$,

$E(G) = \{u_i^j v_i^j : 1 \leq i \leq n, 1 \leq j \leq m\} \cup \{u_i^j u_{i+1}^j : 1 \leq i \leq n-1, 1 \leq j \leq m\} \cup \{v_i^j v_{i+1}^j : 1 \leq i \leq n-1, 1 \leq j \leq m\} \cup \{u_{i+1}^j v_i^j : 1 \leq i \leq n-1, 1 \leq j \leq m\}$.

Now, labeling can be defined as, $\xi : V(G) \cup E(G) \rightarrow \{1, 2, ..., 6mn - 3m\}$,

For $1 \leq i \leq n, 1 \leq j \leq m$. $1 \leq i \leq n, 1 \leq j \leq m$

$$\begin{aligned}
\xi(u_i^j) &= 2m(i-1) + j, \\
\xi(v_i^j) &= m(2i-1) + j, \\
\xi(u_i^j v_i^j) &= m(6n-3) - 2m(i-1) - (j-1), \\
\xi(u_{i+1}^j v_i^j) &= m(6n-4) - 2m(i-2) - (j-1), \\
\xi(u_i^j u_{i+1}^j) &= m(4n-2) - 2m(i-1) - (j-1), \\
\xi(v_i^j v_{i+1}^j) &= m(4n-3) - 2m(i-1) - (j-1).
\end{aligned}$$

Here any one can easily calculate the value of a of subcycle such that $a = C_4^{(i,j)} = \xi(u_i^j) + \xi(u_{i+1}^j) + \xi(v_i^j) + \xi(u_i^j v_i^j) + \xi(u_i^j u_{i+1}^j) + \xi(u_{i+1}^j v_i^j) = 20mn - 7m + 4$. Hence mTL_n is C_4-supermagic.

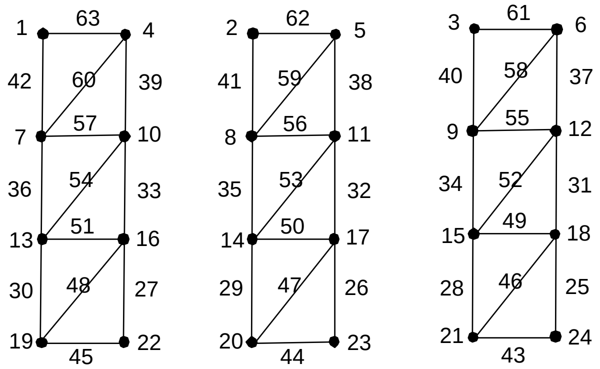

Figure 5.3: A $C_4 - (223, 0)$ supermagic labeling of $3TL_4$.

Theorem 17 *[29] The graph $G \cong mTL_n$ admits C_4-(anti)-supermagic labeling for any m, n.*

Proof.

Let $v = |V(G)|$ and $E = |E(G)|$. Then $v = 2mn$, $e = m(4n - 3)$. The edge set and vertex set of graph are defined as, $V(G) = \{u_i^j : 1 \leq i \leq n, 1 \leq j \leq m\} \cup \{v_i^j : 1 \leq i \leq n, 1 \leq j \leq m\}$,

$E(G) = \{u_i^j v_i^j : 1 \leq i \leq n, 1 \leq j \leq m\} \cup \{u_i^j u_{i+1}^j : 1 \leq i \leq n-1, 1 \leq j \leq m\} \cup \{v_i^j v_{i+1}^j : 1 \leq i \leq n-1, 1 \leq j \leq m\} \cup \{u_{i+1}^j v_i^j : 1 \leq i \leq n-1, 1 \leq j \leq m\}$.

Now, labeling can be defined as, $\xi : V(G) \cup E(G) \to \{1, 2, ..., 6mn - 3m\}$,

For $1 \leq i \leq n, 1 \leq j \leq m$.

$$\begin{aligned}
\xi(u_i^j) &= 2m(i-1) + j, \\
\xi(v_i^j) &= m(2i-1) + j, \\
\xi(u_i^j v_i^j) &= 2m(2n-1) + 1 + 2m(i-1) + (j-1), \\
\xi(u_{i+1}^j v_i^j) &= 2m(2n-1) + 3 + 2m(i-1) + (j-1), \\
\xi(u_i^j u_{i+1}^j) &= m(4n-2) - 2m(i-1) - (j-1), \\
\xi(v_i^j v_{i+1}^j) &= m(4n-3) - 2m(i-1) - (j-1).
\end{aligned}$$

It is easy to verify that for every subcycle $C_4^{(i,j)} : 1 \leq i \leq n$ and $1 \leq j \leq m$, $\xi(u_i^j) + \xi(u_{i+1}^j) + \xi(v_i^j) + \xi(u_i^j v_i^j) + \xi(u_i^j u_{i+1}^j) + \xi(u_{i+1}^j v_i^j) = 16mn - m + 6$. Hence mTL_n is C_4-(anti)-supermagic.

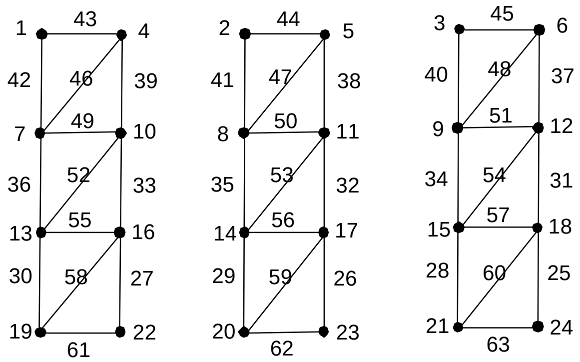

Figure 5.4: A $C_4-(195,0)$ supermagic labeling of $3TL_4$.

Our results of this chapter has been published in [29].

Chapter 6

Super face magic labeling of generalized subdivided graphs

In this chapter, established super face magic labeling of $(1,1,1)$ type of subdivided triangular ladder graph, subdivided mC_4- snake graph and subdivided mC_4- triangular snake graph of $(1,1,...,1)$ and $(2,2,...,2)$ string.

6.1 Face magic labeling of subdivided graphs

In this section the face magic labeling of subdivided triangular ladder are discussed. The faces, edges and vertices of graph of subdivided triangular ladder can be represented in figure as,

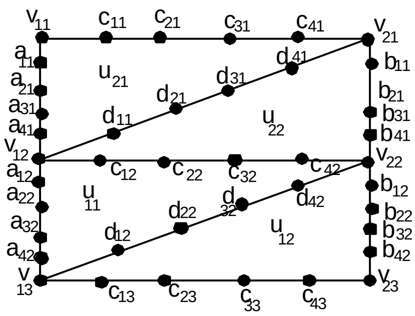

Figure 6.1: For $k = 4$, $H \cong P_3 \times P_2$.

Theorem 18 *Subdivided triangular ladder graph $H \cong P_n \times P_2$, where $n \geq 2$ with k-subdivisions admits super face magic labeling of $(1,1,1)$ type.*

Proof.

If $v = |V(H)|$, $f = |F(H)|$ and $e = |E(H)|$. Then $v = (4k+2)n - 3k$, $e = (4k+1)n - 3(k+1)$ and $f = 2n - 2$.

Now, labeling can be defined as, $\xi : V \cup F \cup E \rightarrow \{1, 2, ..., v + e + f\}$,

where $1 \leq m \leq n$,

$$\xi(v_{q,m}) = \begin{cases} 2m - 1, & for\ q = 1 \\ 2m, & for\ q = 2 \end{cases}$$

$$\xi(c_{q,m}) = \begin{cases} (2q+1)n + 1 - m, & for\ 1 \leq q \leq \lfloor \frac{k+1}{2} \rfloor \\ (2q+1)n + m. & for\ \lfloor \frac{k+1}{2} \rfloor + 1 \leq q \leq k \end{cases}$$

For $1 \leq m \leq n - 1$.

$$\xi(d_{q,m}) = \begin{cases} ((k+3) + 2(q-1))n - m - 2(q-1), & for\ , 1 \leq q \leq \lfloor \frac{k+1}{2} \rfloor \\ ((k+3) + 2(q-1))n + m - 2q + 1, & for\ \lfloor \frac{k+1}{2} \rfloor + 1 \leq q \leq k \end{cases}$$

$$\begin{aligned}
\xi(a_{q,m}) &= (2k+2+2(q-1))n-(k-1+2(q-1))+2(m-1), && 1\le q\le k\\
\xi(b_{q,m}) &= (2k+2+2(q-1))n-(k-2+2(q-1))+2(m-1), && 1\le q\le k\\
\xi(v_{1,m},a_{1,m}) &= v+(3k+3)n-(2k+2)-2(m-1), &&\\
\xi(a_{q,m},a_{q+1,m}) &= v+(3k+3-2q)n-(2k+2-2q)-2(m-1), && 1\le q\le k\\
\xi(a_{k,m},v_{1,m+1}) &= v+(k+3)n-2-2(m-1), &&\\
\xi(v_{2,m},b_{1,m}) &= v+(3k+3)n-(2k+3)-2(m-1), &&\\
\xi(b_{q,m},b_{q+1,m}) &= v+(3k+3-2q)n-(2k+3-2q)-2(m-1), && 1\le q\le k.\\
\xi(b_{k,m},v_{2,m+1}) &= v+(k+3)n-3-2(m-1). &&
\end{aligned}$$

For $1\le m\le n$.

$$\xi(v_{1,m},c_{1,m}) = v+n+1-m,$$

$$\xi(c_{q,k},c_{q+1,k}) = \begin{cases} v+(2q+1)n+1-m, & for\ 1\le q\le \lfloor\frac{k-1}{2}\rfloor\\ v+(2q+1)n+m, & for\ \lfloor\frac{k-1}{2}\rfloor+1\le q\le k-1\end{cases}$$

$$\xi(c_{k,m},v_{2,m}) = \begin{cases} v+kn+m, & for \quad k=odd\\ v+(k+1)n+1-m, & for \quad k=even\end{cases}$$

$$\xi(v_{1,m},d_{1,m-1}) = v+(3k+4)n-(2k+2)-(m-1) \quad for \quad 2\le m\le n.$$

$$\xi(d_{q,m},d_{q+1,m}) = \begin{cases} v+(3k+4+2q)n-(2k+2+2q)-m,\\ for \quad 1\le q\le\lfloor\frac{k-1}{2}\rfloor\\ v+(3k+4+2(q-1))n-(2k+3+2(q-1))+m\\ for \quad \lfloor\frac{k-1}{2}\rfloor+1\le q\le k-1.\end{cases}$$

For $1\le m\le n-1$

$$\xi(d_{k,m}v_{2,m}) = \begin{cases} v+e-(n-1)+m, & for \quad k=odd\\ v+e-m, & for \quad k=even\end{cases}$$

$$\xi(f_{q,m}) = \begin{cases} v+e+2q, & for\ m=1,\ 1\le q\le n-1\\ v+e+2q-1. & for\ m=2,\ 1\le q\le n-1\end{cases}$$

Under this labeling, one can verify that all the weights of the graph $H \cong P_n \times P_2$ with k subdivisions under the labeling, have the same weights. Hence graph admits super face magic labeling of $(1,1,1)$ type.

6.2 Face anti-magic labeling of subdivided snake graph

In this part, let subdivided snake graph mC_4 with $(1,1,...,1)$ and $(2,2,...,2)$ string and formulate its super face anti magic labeling.

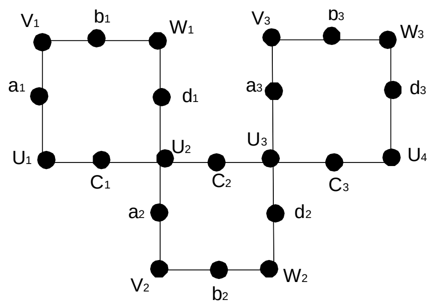

Figure 6.2: $3C_4$ having string $(1,1,...,1)$

Theorem 19 *A subdivided snake graph $H \cong mC_4$ with $(1,1,...,1)$ and $(2,2,...,2)$ string along with its p-subdivisions admits super 1-anti magic of $(1,1,1)$ type labeling.*

Proof.
Consider $v = |V(H)|$, $f = |F(H)|$ and $e = |E(H)|$. Then $v = (4p+3)m+1$, $e = 4m(p+1)$ and $f = m$.
Now, labeling can be defined as, $\xi : V(H) \cup F(H) \cup E(H) \rightarrow \{1,2,...,v+f+e\}$.
Now symbolize the vertices of H of $(1,1,...,1)$ and $(2,2,...,2)$ string as follows:

$$\xi(u_q) = q, \quad 1 \leq q \leq m+1$$

For $1 \leq q \leq m$,

$$\begin{aligned} \xi(w_q) &= 2m+2-q, \\ \xi(u_q) &= 2m+1+q. \end{aligned}$$

We symbolize the partitions of H of string $(1, 1, ..., 1)$ and string $(2, 2, ..., 2)$ as follows:

For $1 \leq q \leq m$, $1 \leq t \leq p$

$$\begin{aligned}
\xi(b_{q,t}) &= 3m + tm + 2 - q, \\
\xi(a_{q,t}) &= 3m + pm + 1 + (t-1)m + q, \\
\xi(d_{q,t}) &= 3m + 2pm + 1 + (t-1)m + q, \\
\xi(c_{q,t}) &= 3m + 4pm + 2 - (t-1)m - q.
\end{aligned}$$

The symbolize for edges of H can be defined as, for string $(2, 2, ..., 2)$ and $1 \leq q \leq m, 1 \leq t \leq p$

$$\begin{aligned}
\xi(v_q b_{q,1}) &= v + (p+1)m - q + 1, \\
\xi(b_{q,t} b_{q,t+1}/u_{q+1}) &= v + (p+1)m - tm - q + 1, \\
\xi(u_q c_{q,1}) &= v + 3(p+1)m + q, \\
\xi(c_{q,t} c_{q,t+1}/w_q) &= v + 3(p+1)m + tm + q, \\
\xi(u_q a_{q,1}) &= v + (p+1)m + q, \\
\xi(a_{q,t} a_{q,t+1}/v_q) &= v + (p+1)m + tm + q, \\
\xi(w_q d_{q,1}) &= v + 3(p+1)m + 1 - q, \\
\xi(d_{q,t} d_{q,t+1}/u_{q+1}) &= v + 3(p+1)m + 1 - tm - q.
\end{aligned}$$

For string $(1, 1, ..., 1)$ and $1 \leq q \leq m, 1 \leq t \leq p$

$$\begin{aligned}
\xi(v_q b_{q,1}) &= v + (p+1)m - q + 1, \\
\xi(b_{q,t} b_{q,t+1}/w_q) &= v + (p+1)m - tm - q + 1, \\
\xi(u_q c_{q,1}) &= v + 3(p+1)m + q, \\
\xi(c_{q,t} c_{q,t+1}/u_{q+1}) &= v + 3(p+1)m + tm + q, \\
\xi(u_q a_{q,1}) &= v + (p+1)m + q,
\end{aligned}$$

$$\begin{aligned}
\xi(a_{q,t}a_{q,t+1}/v_q) &= v+(p+1)m+tm+q, \\
\xi(w_q d_{q,1}) &= v+3(p+1)m+1-q, \\
\xi(d_{q,t}d_{q,t+1}/u_{q+1}) &= v+3(p+1)m+1-tm-q.
\end{aligned}$$

We symbolize the faces of H of $(1,1,...,1)$ and $(2,2,...,2)$ string as follows:

$$\xi(f_q) = v+e+m+1-q,\ 1 \le q \le m.$$

Now it can easily be seen that the weights forms an arithmetic progression of the a subdivided snake graph $H \cong mC_4$ with $(1,1,...,1)$ and $(2,2,...,2)$ string along with its p-subdivisions admits super 1-anti magic of $(1,1,1)$ type under this labeling. Hence the subdivided snake graph have super 1-anti magic of $(1,1,1)$ type labeling.

6.3 Face anti-magic labeling for triangular snake graph

In this part, super face anti-magic labeling for triangular mC_4- snake graph as well as subdivision of triangular mC_4 -snake graph are established.

Triangular mC_4- snake graph can be made by joining vertex u_i with vertex w_i with an edge for string $(1,1,...,1)$ and joining vertex v_i with vertex w_i with an edge for string $(2,2,...,2)$.

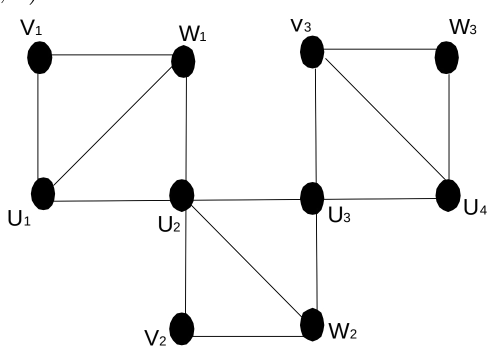

Figure 6.3: Triangular $3C_4$ having string $(1,1,...,1)$

Theorem 20 *A subdivided triangular snake graph $H \cong mC_4$ with $(2,2,...,2)$ string along with its p-subdivisions admits super 1-anti magic of $(1,1,0)$ type labeling.*

Proof.

Here $v = |V(H)|$, $f = |F(H)|$ and $e = |E(H)|$. Then $v = (3+5p)m+1$, $e = 5m(p+1)$ and $f = m$.

Now, labeling can be defined as $\xi : V(H) \cup E(H) \cup F(H) \rightarrow \{1, 2, ..., v+e+f\}$:

$$\xi(u_q) = q, \quad 1 \leq q \leq m+1.$$

For $1 \leq q \leq m$,

$$\begin{aligned} \xi(w_q) &= 2m+2-q, \\ \xi(v_q) &= 2m+1+q, \end{aligned}$$

The subdivisions of H can be symbolize as follows:

For $1 \leq q \leq m, \ 1 \leq t \leq p$,

$$\begin{aligned} \xi(c_{q,t}) &= 3m+tm+2-q, \\ \xi(a_{q,t}) &= 3m+2pm-(t-1)m+2-q, \\ \xi(b_{q,t}) &= 3m+2pm+(t-1)m+1+q, \\ \xi(d_{q,t}) &= 3m+3pm+(t-1)m+1+q, \\ \xi(o_{q,t}) &= 3m+5pm-(t-1)m+2-q. \end{aligned}$$

The edges of H can be symbolize as follows:

$1 \leq q \leq m, 1 \leq t \leq p$

$$\begin{aligned} \xi(w_q o_{q,1}) &= v+q, \\ \xi(o_{q,t} o_{q,t+1}/v_q) &= v+tm+q, \\ \xi(u_{q+1} c_{q,1}) &= v+2(p+1)m+1-q, \\ \xi(c_{q,t} c_{q,t+1}/v_q) &= v+2(p+1)m+1-tm-q, \\ \xi(w_q a_{q,1}) &= v+3(p+1)m+1-q, \\ \xi(a_{q,t} a_{q,t+1}/u_q) &= v+3(p+1)m+1-tm-q, \end{aligned}$$

$$
\begin{aligned}
\xi(u_q b_{q,1}) &= v+3(p+1)m+q, \\
\xi(b_{q,t}b_{q,t+1}/v_q) &= v+3(p+1)m+tm+q, \\
\xi(w_q d_{q,1}) &= v+4(p+1)m+q, \\
\xi(d_{q,t}d_{q,t+1}/u_{q+1}) &= v+4(p+1)m+tm+q.
\end{aligned}
$$

Now it can easily be seen that the weights forms an arithmetic progression of the triangular snake graph of string $(2,2,...,2)$ with p-subdivisions under this labeling. Hence a subdivided triangular snake graph with $(2,2,...,2)$ string forms super 1-anti magic of $(1,1,0)$ type labeling.

Theorem 21 *For all $m \geq 2$, $H \cong mC_4$-triangular snake graph of string $(1,1,...,1)$ with p-subdivisions concedes super 1-anti magic labeling of $(1,1,0)$ type.*

Proof.

Consider $v = |V(H)|$, $f = |F(H)|$ and $e = |E(H)|$. So, we have $v = (3+5p)m+1$, $e = 5m(p+1)$ and $f = m$.

Now, labeling can be defined as, $\xi : V(H) \cup E(H) \cup F(H) \rightarrow \{1,2,...,v+e+f\}$

$$\xi(u_q) = 2m+q, 1 \leq q \leq m+1,$$

where $1 \leq q \leq m$

$$
\begin{aligned}
\xi(w_q) &= 2m+1-q, \\
\xi(v_q) &= q.
\end{aligned}
$$

The subdivisions of H can be symbolize as follows:

For $1 \leq q \leq m,\ 1 \leq t \leq p$,

$$
\begin{aligned}
\xi(o_{q,t}) &= 3m+tm+2-q, \\
\xi(c_{q,t}) &= 3m+2pm-(t-1)m+2-q, \\
\xi(a_{q,t}) &= 3m+2pm+(t-1)m+1+q, \\
\xi(b_{q,t}) &= 3m+4pm-(t-1)m+2-q, \\
\xi(o_{q,t}) &= 3m+4pm+(t-1)m+1+q.
\end{aligned}
$$

The edges of H can be symbolize as follows:

For $1 \leq q \leq m,\ 1 \leq t \leq p,$

$$\begin{aligned}
\xi(u_q o_{q,1}) &= v + q, \\
\xi(o_{q,t} o_{q,t+1}/w_q) &= v + tm + q, \\
\xi(w_{q+1} c_{q,1}) &= v + 2(p+1)m + 1 - q, \\
\xi(c_{q,t} c_{q,t+1}/u_{q+1}) &= v + 2(p+1)m + 1 - tm - q, \\
\xi(v_q a_{q,1}) &= v + 3(p+1)m + 1 - q, \\
\xi(a_{q,t} a_{q,t+1}/u_q) &= v + 3(p+1)m + 1 - tm - q, \\
\xi(v_q b_{q,1}) &= v + 3(p+1)m + q, \\
\xi(b_{q,t} b_{q,t+1}/w_q) &= v + 3(p+1)m + tm + q, \\
\xi(u_q d_{q,1}) &= v + 4(p+1)m + q, \\
\xi(d_{q,t} d_{q,t+1}/u_{q+1}) &= v + 4(p+1)m + tm + q.
\end{aligned}$$

Now it can easily be seen that the weights forms an arithmetic progression of the subdivided triangular snake graph of $(1, 1, ..., 1)$ string along with its p-subdivisions under this labeling. Hence the subdivided triangular snake graph forms super 1-anti magic of $(1, 1, 0)$ type labeling.

Our results of this chapter has been submitted in the journal of "Nanoscience and Nanotechnology Letters".

Chapter 7

Conclusion and open problems

7.1 Conclusion

This dissertation is about the construction of cycle anti-supermagic labeling for the Multi-wheels graph, supermagic labeling for the Multi-wheels graph and for isomorphic copies with its disjoint union of Multi-wheels graph for $m = 2$ and 4 as well as cycle anti-supermagic labeling for Web graphs. The isomorphic copies with its disjoint union of ladders and triangular ladders graphs are studied. Also a cycle supermagic labeling of isomorphic copies with its disjoint union of triangular ladders are formulated. An anti-supermagic labeling of the extension of cycle graphs is also discussed.

Line graphs of different graphs are calculated and then formulated its labelling like as,

(a) isomorphic copies with its disjoint union of line graph of fan graph which admit anti-supermagic of $(1, 1, 1)$ type labeling

(b) isomorphic copies with its disjoint union of line graph of friendship graph satisfying anti-supermagic and supermagic of $(1, 1, 1)$ type labeling

(c) the line graph of ladder graphs of anti-supermagic of $(1, 1, 1)$ type labeling

(d) line graph of wheel graph of supermagic of $(1, 1, 1)$ type labeling

(e) isomorphic copies with its disjoint union of line graph of wheel graphs which admit anti-supermagic of $(1, 1, 1)$ type labeling.

Last but not least, face supermagic of $(1, 1, 1)$ type labeling of subdivided triangular ladder graph, subdivided mC_4- snake graph and subdivided mC_4- triangular snake graph with $(1, 1, ..., 1)$ and $(2, 2, ..., 2)$ string are established. Few examples are presented to support the result.

Some open problems are also proposed in this dissertation.

7.2 Open problems

Open problem 1. An isomorphic copies with its disjoint union of Multi-wheels graph $G \cong mlW_n$ admits C_3-$(a, 0)$ anti-supermagic labeling, where n, m and $l \geq 3$.

Open problem 2. For every m, n and l, the following graphs, $G \cong mL(W_n)$, $G \cong mL(L_n)$,and $G \cong mL(F_n)$ confesses anti-supermagic of $(1, 1, 1)$ type labeling.

Open problem 3. Subdivided triangular ladder and ladder graphs with k subdivisions admits anti super face magic labeling of $(1, 1, 1)$ type for every n.

Open problem 4. Subdivided snake graph with k subdivisions admits d-antisupermagic of $(1, 1, 1)$ type labeling.

Chapter 8

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