Computing Metric Dimension of Certain Families of Toeplitz Graphs

By

Wajiha Nazir

CIIT/SP17-RMT-021/LHR

MS

In

Mathematics

COMSATS University Islamabad

Lahore Campus

Fall, 2018

COMSATS University Islamabad

Computing Metric Dimension of Certain Families of Toeplitz Graphs

A Thesis Presented to

COMSATS University Islamabad

In partial fulfillment

of the requirement for the degree of

MS (Mathematics)

By

Wajiha Nazir

CIIT/SP17-RMT-021/LHR

Fall, 2018

Computing Metric Dimension of Certain Families of Toeplitz Graphs

A Post Graduate Thesis is submitted to the Department of Mathematics as partial fulfillment of the requirement for the award of Degree of MS Mathematics.

Name	Registration Number
Wajiha Nazir	CIIT/SP17-RMT-021/LHR

Supervisor

Dr. Muhammad Faisal Nadeem

Assistant Professor, Department of Mathematics

COMSATS University Islamabad, Lahore Campus

December, 2018

Final Approval

This report titled

Computing Metric Dimension of Certain Families of Toeplitz Graphs

By

Wajiha Nazir

CIIT/SP17-RMT-021/LHR

Has been approved

For the COMSATS University Islamabad, Lahore Campus

External Examiner:______________________________

Dr. Kashif Shafiq

Supervisor: ______________________________

Dr. Faisal Nadeem

Department of Mathematics /CUI, Lahore Campus

HOD:___________________________________

Department of Mathematics, Lahore Campus

Declaration

I Wajiha Nazir, **CIIT/SP17-RMT-021/LHR** hereby declare that I have produced the work presented in this thesis, during the scheduled period of study. I also declare that I have not taken any material from any source except referred to wherever due that amount of plagiarism is within acceptable range. If a violation of HEC rules on research has occurred in this thesis, I shall be liable to punishable action under the plagiarism rules of the HEC.

Date:_______________

____

Wajiha Nazir

CIIT/SP17-RMT-021/LHR

Certificate

It is certified that **<u>Wajiha Nazir (CIIT/SP17-RMT-021/LHR)</u>** has carried out all the work related to this thesis under my supervision at the Department of Mathematics, COMSATS University Islamabad, Lahore Campus, and the work fulfills the requirement for award of MS degree.

Date:__________________

Supervisor:

____

Dr. Faisal Nadeem
Assistant Professor

Head of Department:

____

Dr. Sarfraz Ahmed
Associate Professor
Department of Mathematics

DEDICATION

To

My parents and My family

ACKNOWLEDGEMENT

In the name of Allah, the Most Gracious and Most Merciful, all the praises to Allah for the strengths and His blessing in completing this thesis. I feel much pleasure in expressing my heartiest gratitude to my supervisor Dr. Muhammad Faisal Nadeem for his dynamic supervision and whose inspiring attitude made it very easy to undertake this project. A very special thanks to my respectable teacher Dr. H. M. Afzal Siddiqui, without his effort I was not able to complete this task. It was only through their guidance that my efforts were fruitful. I express my sincerest gratitude to my institution and a special thanks to the Head of Department of Mathematics for providing us with very good research facilities, required for the accomplishment of our goals. My deepest appreciations are extended to my beloved Parents and Family for my success, is really the fruit of their devoted prayers and encouragement. And a special thanks for the person who motivated me to achieve my goals.

Wajiha Nazir
CIIT/SP17-RMT-021/LHR

ABSTRACT

Computing Metric Dimension of Certain Families of Toeplitz Graphs

Let $Q = \{q_1, q_2, \ldots q_k\}$ be an ordered set of vertices of a graph G and a is any vertex in G, then a has representation w.r.t Q, denoted by $r(a|Q)$ is the $k-$tuple which is $(r(a, q_1), r(a, q_2) \ldots r(a, qk))$. If the different vertices of G have the different representation w.r.t Q, then Q is known as a resolving set/locating set. A resolving/locating set having the least count of vertices is basis of G and count of vertices in this basis is called metric dimension of G which is represented as $dim(G)$.
In our work, we study the metric dimension of certain Toeplitz graphs and find out their metric dimension. Firstly, we find the metric dimension of Toeplitz graph $T_n\langle 1, t\rangle$ where $t \geq 2$, which is constant. Secondly, we find the metric dimension of Toeplitz graph $T_n\langle 1, s, t\rangle$, where $t = 3, 4, 5, 6$ and $s = 2$, which is also constant.

TABLE OF CONTENTS

List of Figures

Chapter 1
Introduction and Preliminaries

1 Introduction and Preliminaries

This section is related about the history and some primary concepts related to graphs. Also some basic notations, definitions and terminologies that will be used throughout in this thesis.

1.1 Historical Background of Graph Theory

In 1736, *The Königsberg Bridge Problem* laid the foundation of graph theory. There are seven bridges that connected two sides of Pregel river at Konigsberg, the city in Russia. Euler find out that there is no path that covered each bridge only once. Each land mass is

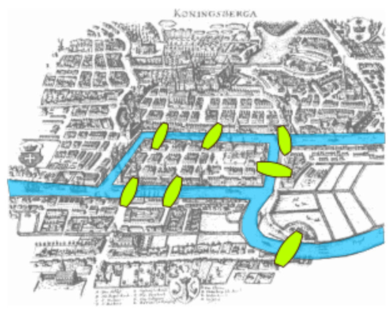

Figure 1: Map of Königsberg

represented by *vertex/node*, and each bridge is represented by an *edge*. The resulting graph from this problem is shown in Fig. 2:

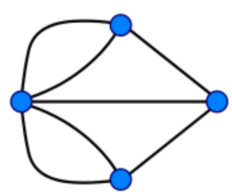

Figure 2: Graph of Königsberg

Graph theory is very useful theory in Mathematics. It is used in the field of Computer Science (Website, Networks, Database and Links), Linguistics (Tree based structures), Chemistry (Crystallography), Physics (Circuit design, Radar, X-Rays), Biology (DNA Structures), Sociology and Economics.

1.2 Basic Definitions

1.2.1 Graph

A *Graph* G is the graphical representation which consists of vertices (nodes or points) which may or may not be connected by the edges (arcs or lines) in different ways. We represent a graph as an ordered pair $G = (V, E)$, where E is the set of edges and V is the set of vertices.

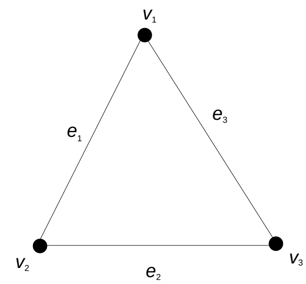

Figure 3: Graph having $V(G) = \{v_1, v_2, v_3\}$ and $E(G) = \{e_1, e_2, e_3\}$

1.2.2 Order of a Graph

The count of vertices of a graph G is said to be the *order* of a graph G and it is denoted by $|V(G)|$.

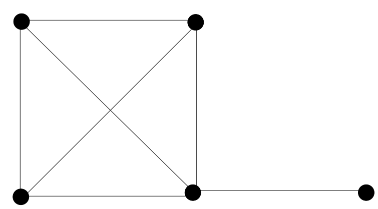

Figure 4: $|V(G)| = 5$

1.2.3 Size of a Graph

The count of edges of a graph G is said to be the *size* of a graph G and it is denoted by $|E(G)|$. In Fig. 4 $|E(G)| = 7$.

1.2.4 Trivial Graph

Graph having a single vertex is known as *trivial graph*, so we can say that minimum vertices of nontrivial graph is 2.

Figure 5: Trivial graph

1.2.5 Parallel Edges

When two or more than two edges are joined with same vertices, such edges are known as *parallel edges*.

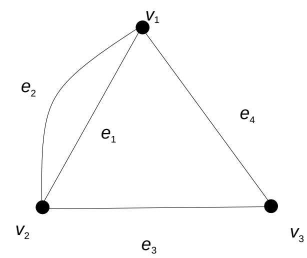

Figure 6: Edge e_1 is parallel to e_2

1.2.6 Loop

When an edge is joined with one vertex, such edge is known as a *loop*.

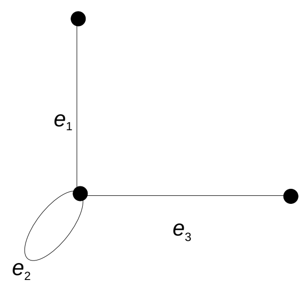

Figure 7: Edge e_2 is a loop

1.2.7 Neighboring Vertices

Two adjacent vertices of a graph G are called the *neighboring vertices* of each other. The set of all neighboring vertices of a graph G with respect to $c \in G$ is denoted by $N_G(c)$.

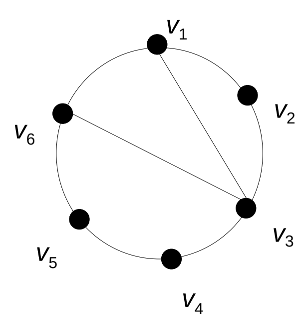

Figure 8: $N_G(v_1) = \{v_2, v_3, v_6\}$

1.2.8 Degree of a Vertex

The count of edges adjacent with a vertex is called the *degree* of that vertex.

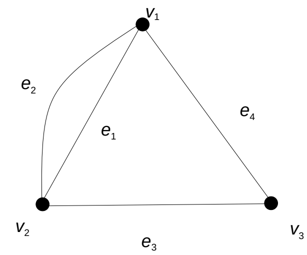

Figure 9: Degree of vertices v_1 and v_2 is 3

The *smallest degree* of a vertex in a graph G is denoted by $\delta(G)$. In Fig. 9 $\delta(G) = 2$. The *greatest degree* of a vertex in a graph G is denoted by $\Delta(G)$. In Fig. 9 $\Delta(G) = 3$. The *average degree* of a graph G is denoted by $d(G)$ that is $d(G) = \frac{\sum_{j=1}^{n} d(v_j)}{n}$, where n is the total number of vertices of a graph G.
We have the following relation for the degree of graph and its order:

$$0 \leq \delta(G) \leq d(t) \leq \Delta(G) \leq n-1$$

where t is any vertex of a graph.

1.2.9 Even and Odd Degree

An isolated vertex has 0 degree and end vertex has 1 degree. A vertex is called an *even* when its degree is even and a vertex is called an *odd* when its degree is odd.

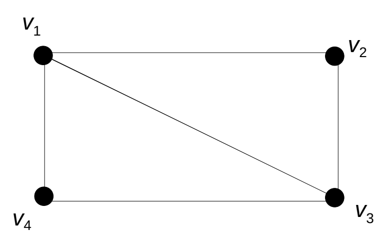

Figure 10: v_1 is odd and v_2 is even

1.2.10 Simple Graph

A graph G is known as *simple graph* if it contains no parallel edges and loops.

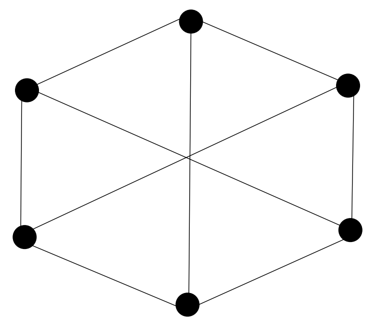

Figure 11: Simple graph

Throughout our work we deal only with simple, connected and undirected graphs.

1.2.11 Walk

A finite alternating sequence of vertices and edges of a graph G in which vertices and edges are repeated is known as a *walk*.

1.2.12 Trail

A walk has no repetition of edges is known as *trail*.

1.2.13 Path

A finite alternating sequence of vertices and edges of a graph G in which no vertices and edges are repeated is known as a *path*, $v_0e_1v_1e_2v_2\cdots v_{n-1}e_nv_n$ is a path, where v_0 and v_n represents starting and ending vertex respectively.

1.2.14 Circuit

A closed walk has no repetition of edges is called a *circuit*.

1.2.15 Cycle

A closed walk has no repetition of vertices is known as a *cycle*. A cycle is represented by C_n.

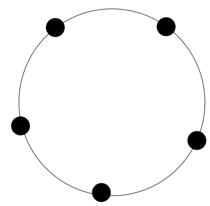

Figure 12: Cycle C_5

1.2.16 Path Graph

A *path graph*, denoted by P_n, is defined as a graph consisting of vertex set $V = \{1, 2, ..., n\}$ and edge set $E = \{v_i, v_{i+1}\}$ where $i = 1, 2, ..., n-1$.

Figure 13: Path graph P_6

1.3 Tree

A connected graph G, which contains no cycle is known as *tree*. If the spanning subgraph is a tree then it is known as *spanning tree*.

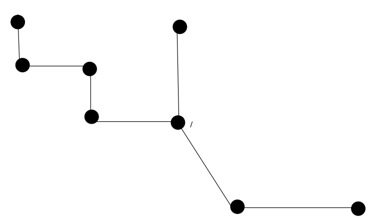

Figure 14: Tree

Theorem 1.1. [6] *"The following statements are equivalent for a graph G of order n.*

(i) *G is a tree.*

(ii) *There is a unique path between every pair of distinct vertices in G.*

(iii) *G is connected and every edge in G is a bridge.*

(iv) *G is connected, and has $n-1$ edges.*

(v) *G is acyclic, and has $n-1$ edges.*

(vi) *G is acyclic, and whenever any two arbitrary nonadjacent vertices in G are joined by an edge, the resulting enlarged graph has a unique cycle".*

Theorem 1.2. [6] *"A graph is connected if and only if it has a spanning tree".*

Theorem 1.3. [3] *"Every nontrivial tree has at least two end-vertices".*

1.4 Connectivity

In this section we will discuss the connectivity of graphs, definitions, some basic terminologies and results.

1.4.1 Connected Graph

A graph G is *connected graph* when it contains a path between any pair of vertices. A connected graph has only one component.

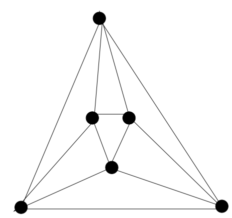

Figure 15: Connected graph

1.4.2 Disconnected Graph

A graph G is *disconnected graph* when it is not connected. A disconnected graph has more than one components.

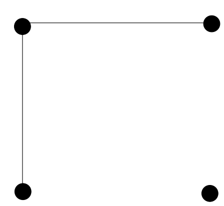

Figure 16: Disconnected graph

1.4.3 Bridge

A single edge which disconnects the graph G into its components is known as *bridge*.

Theorem 1.4. [3] *"An edge e of a graph G is a bridge if and only if e lies on no cycle of G".*

1.4.4 Edge Connectivity

The smallest count of edges in a set, whose elements disconnected the graph is said to be *edge connectivity number* of the graph G and is represented by $\lambda(G)$. This set is *cut set* if it has no proper subset.

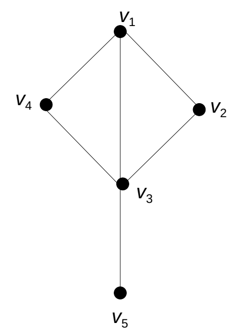

Figure 17: $\lambda(G) = 1$

1.4.5 Separating Set

A set contains those vertices due to which the graph is disconnected is known as *separating set*. In Fig. 17 separating set is $\{v_3\}$.

1.4.6 Connectivity Number

The separating set of smallest size is the *connectivity number* which is denoted $\kappa(G)$ of a graph G. In Fig. 17 $\kappa(G) = 1$

1.4.7 Cut Vertex

A separating set contains a single vertex v is called a *cut vertex*. In Fig. 17 cut vertex is v_3.

Theorem 1.5. [3] *"For any graph G,*

$$\kappa(G) \leq \lambda(G) \leq \delta(G)."$$

Theorem 1.6. [3] *"Let u be a vertex incident with a bridge in a connected graph G. Then u is a cut-vertex of G if and only if $d(u) \geq 2$."*

Theorem 1.7. [3] *"A graph of order at least 3 is nonseparable if and only if every two vertices lie on a common cycle".*

Theorem 1.8. [3] *"If G is a cubic graph, then $\lambda(G) = k(G)$."*

Theorem 1.9. [3] *"Let G be a graph of order n and size $m \geq n-1$, then*

$$\kappa(G) \leq \lfloor \frac{2m}{n} \rfloor."$$

1.5 Distances in Graphs

In this section we deal with the concept of distance in graphs, basic results and terminologies which are very helpful in computing our main results.

1.5.1 Distance

The length of the shortest path between any two vertices such as c and t, where $c,t \in V(G)$ is known as the *distance* in a connected graph G and denoted by $d(c,t)$.
The term distance should satisfy the following conditions in G:

1. "$d(c,t) \geq 0$ for all $c,t \in V(G)$.
2. $d(c,t) = 0$ iff $c = t$.
3. $d(c,t) = d(t,c)\ \forall\ c,t \in V(G)$.
4. $d(c,z) \leq d(c,t) + d(t,z)\ \forall\ c,t,z \in V(G)$."

1.5.2 Eccentricity

The maximum distance from v, to any vertex of G is known as *eccentricity* of v. Thus $e(v) = max\{d(v,c) : c \in V(G)\}$.

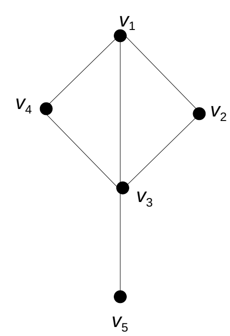

Figure 18: Eccentricities $e(v_1) = 2$, $e(v_2) = 2$, $e(v_3) = 1$, $e(v_4) = 2$, $e(v_5) = 2$

1.5.3 Radius

Any vertex of a graph G has the smallest eccentricity is known as *radius* of graph G. In Fig. 18 radius is 1.

1.5.4 Diameter

Any vertex of a graph G has the greatest eccentricity is known as *diameter* of graph G. In Fig. 18 diameter is 2.

1.5.5 Center

The center has all vertices of G having smallest eccentricity. In Fig. 18 $C(G) = \{v_3\}$.
The following theorem is about the radius, eccentricity and center of a graph G.

Theorem 1.10. [3] *"If G be a nontrivial connected graph, then*

$$rad(G) \leq diam(G) \leq 2rad(G)."$$

1.5.6 Eccentric Graph

A graph G contains all eccentric vertices is known as *eccentric graph*.

Theorem 1.11. [3] *"A nontrivial graph G is an eccentric subgraph of some graph if and only if every vertex of G has eccentricity* 1 *or no vertex of G has eccentricity* 1*."*

Theorem 1.12. [3] *"A unique eccentric vertex graph is self-centered if and only if each vertex of G is eccentric".*

1.6 Matrix Representations of Graph

A graph is usually represented by a diagram of points joined by lines. This representation is not suitable if we want to store a large graph in a computer. It can be done by using matrices.

1.6.1 Adjacency Matrix of a Graph

"For a graph G with m vertices labelled as $\{1, 2, 3, ..., m\}$ its adjacency matrix A is $m \times m$ matrix whose ij^{th} entry is the number of edges joining vertex i and vertex j and 0 otherwise".

1.6.2 Toeplitz Matrices

A $m \times m$ matrix $B = b_{ij}$ is known as Toeplitz Matrix if $b_{ij} = b_{i+1,j+1}$ for each $i, j = 1, ..., m-1$. Toeplitz adjacency matrix is symmetric.

$$\begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

Figure 19: Toeplitz Matrix

1.6.3 Toeplitz Graph

A simple undirected graph T_m is *Toeplitz graph* if matrix $m \times m$ which is $B = b_{ij}$ is the symmetric Toeplitz matrix and for all $i, j = 1, ..., m$ satisfied the following: edge $\{i, j\}$ is in $E(G)$ iff $b_{ij} = b_{ji} = 1$. An $m \times m$ matrix B will be labelled $0, 1, 2, ..., m-1$ which has m distinct diagonals. The main diagonal has $b_{ii} = 0$ for all $i = 1, ..., m$, so Toeplitz graph has no loop. The diagonals $x_1, x_2, ..., x_p$ containing ones $0 < x_1 < x_2 < ... < x_p < m$. The Toeplitz graph $T_m < x_1, x_2, ..., x_p >$ with vertex set $1, 2, 3, ..., m$ has edge $\{i, j\}$, $1 \le i \le j \le m$, occurs if and only if $j - i = t_q$ for some q, $1 \le q \le p$.
Toeplitz graph of the above Toeplitz matrix is given in Fig. 20

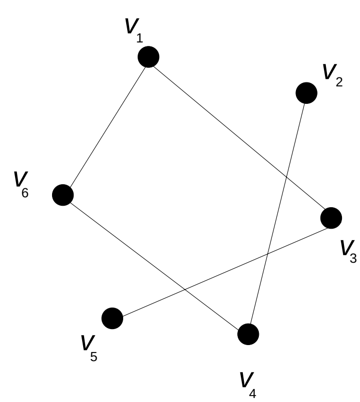

Figure 20: Toeplitz graph $T_6 \langle 2, 5 \rangle$

Important properties of Toeplitz graph are as follows:

Theorem 1.13. [7] *"$T_n \langle t_1, ..., t_k \rangle$ has at least $gcd(t_1, ..., t_k)$ components".*

Theorem 1.14. [7] *"If $gcd(t_1, ..., t_k) > 1$, then $T_n \langle t_1, ..., t_k \rangle$ is disconnected".*

1.6.4 Circulant Graph

The circulant graph was firstly introduced by Wong and Coppersmith [23]. Circulant graphs are the special class of Toeplitz graphs and every circulant graph is Toeplitz graph but the converse is not true. "An undirected *circulant graph*, denoted by $C(n, \pm\{1, 2, ..., j\})$,

$1 \leq j \leq \lfloor \frac{n}{2} \rfloor$, $n \geq 3$ is defined as a graph consisting of vertex set $V = \{0, 1, 2, ..., n-1\}$ and edge set $E = \{(i, j) : |j - i| \equiv s (\mathrm{mod}\, n)\}$, $s \in \{1, 2, ..., j\}$."

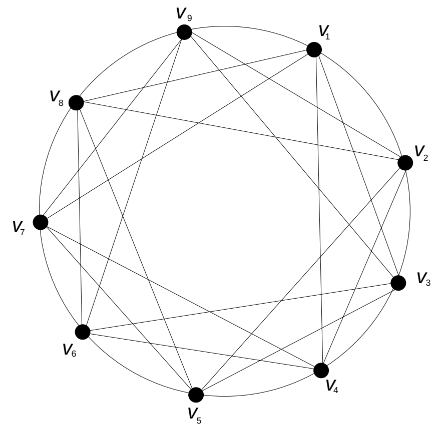

Figure 21: Circulant graph $C(9, \pm\{1, 2, 3\})$

1.6.5 Harary Graph

"*Harary Graph* $H_{r,n}$ is an r-regular graph of order n with $V(H_{r,n}) = \{v_1, v_2, ..., v_n\}$. If r is even then $r = 2p \leq n-1$ for some nonnegative integer $p \leq \frac{n-1}{2}$. If r is odd then $n = 2l$ is even and $r = 2p+1 \leq n-1$ for some nonnegative integer $p \leq \frac{n-2}{2}$."

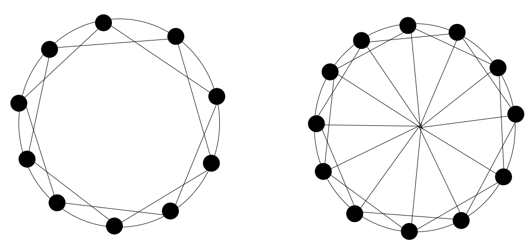

Figure 22: Harary graphs $H_{4,10}$ and $H_{5,12}$

Chapter 2
Literature Review

2 Literature Review

2.1 Applications of Metric Dimension

The word metric dimension was first time introduced by Slater [22] and Harary [9] independently. There are many applications of resolubility in graph theory, for example pattern recognition in pharmaceutical Chemistry [4], in processing of image, in networks, combinatorial optimization, tricky games and tasks on coin-weighing, robot navigation [16], facility location problems and many more. Solutions of many other practical applications are found by the dimension and basis of the graph. For more details see [17, 20, 21, 24, 29, 30].

2.2 Resolving Set

Let $Q = \{q_1, q_2, \ldots, q_k\}$ be an ordered set of vertices of a graph G and a is any vertex in G, then a has representation w.r.t Q, denoted by $r(a|Q)$ is the k-tuple which is $(r(a,q_1), r(a,q_2), \ldots, r(a,q_k))$. If the different vertices of G have the different representation w.r.t Q, then Q is known as a *resolving set/locating set*.

2.3 Metric Dimension

A resolving/locating set having the least count of vertices is *basis* of G and count of vertices in this basis is called *metric dimension* of G which is represented as $dim(G)$.

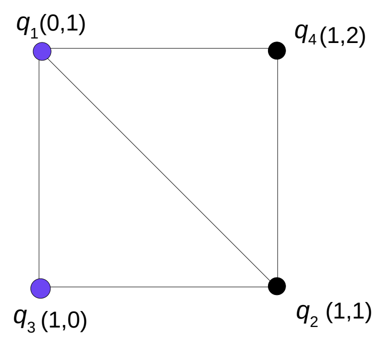

Figure 23: Graph with metric dimension 2

Dimension of graph is 1 if and only if $G = P_n$, by Theorem 2.2. So, dimension of given graph G is not 1 and its dimension is greater than 1. For $Q_1 = \{q_1, q_2\}$, $r(q_3|Q_1) = r(q_4|Q_1) = (1,1)$. So, Q_1 is not a resolving set. For $Q_2 = \{q_1, q_3\}$, $r(q_2|Q_2) = (1,1)$ and $r(q_4|Q_2) = (1,2)$. So, Q_2 is the resolving set. Hence $dim(G) = 2$.
For more concepts see [1, 2, 4, 11, 18, 25, 26].

Lemma 2.1. [28] *"Let Q be a resolving set for a connected graph G and $u, v \in V(G)$. If $d(u,w) = d(v,w)$ for all vertices $w \in V(G) \setminus \{u,v\}$, then $\{u,v\} \cap Q \neq \emptyset$".*

2.4 Nature of Metric Dimension

Let $\mathscr{E}$ be a family of connected graphs defined as $G_m : \mathscr{E} = (G_m)_{m\geq 1}$ which depends on m the order $|V(G)| = \varphi(m)$ and $\lim_{m\to\infty} \varphi(m) = \infty$. Let K, $K > 0$ be any constant, such that $dim(G_m) \leq K$ for each $m \geq 1$, then we say that $\mathscr{E}$ has *bounded metric dimension*; otherwise $\mathscr{E}$ has *unbounded metric dimension*.
If all graphs in $\mathscr{E}$ having the same metric dimension (which is not depending on m), then $\mathscr{E}$ has *constant metric dimension*. The families of graphs having constant metric dimension were discussed in [12, 13, 14, 15].

2.5 Fundamental Results of Metric Dimension

Following are some basic results of dimension of different graphs which will be helpful in generalization of our main results.

Theorem 2.2. [19] "Let G be a simple connected graph of order $n \geq 2$. Then

(a) $dim(G) = 1$ if and only if $G = P_n$.
(b) $dim(G) = n-1$ if and only if $G = K_n$.
(c) $dim(G) = n-2$ if and only if $G = K_{r,s}$, where $r,s \geq 1$ and $n \geq 4$"

Theorem 2.3. [27] *"Let G be a simple connected graph with metric dimension* 2 *and let* $\{v_1, v_2\} \subseteq V(G)$ *be a metric basis in G, then the degree of both* v_1 *and* v_2 *is at most* 3 *and there exists a unique shortest path between* v_1 *and* v_2*."*

Theorem 2.4. [19] "A simple connected graph G with $dim(G) = 2$ can not have following:
- K_5 as a subgraph.
- $K_5 - e$ as a subgraph, where e is an edge.
- $K_{3,3}$ as a subgraph.
- The Peterson graph as a subgraph".

Theorem 2.5. [4] *"For every pair* n,k *of integers with* $1 \leq k \leq n-1$*, there exists a k-dimensional connected graph of order n."*

Theorem 2.6. [10] *"For generalized Peterson graph* $P(n,3)$ *we have*
- $dim(P(n,3)) \leq 3$ for $n \equiv 1 (\text{mod } 6)$ and $n \geq 13$;
- $dim(P(n,3)) \leq 4$ for $n \equiv 0,3,4,5 (\text{mod } 6)$ and $n \geq 17$;
- $dim(P(n,3)) \leq 5$ for $n \equiv 2 (\text{mod } 6)$ and $n \geq 8$."

Theorem 2.7. [10] *"For generalized Peterson graph* $P(n,3)$ *we have* $dim(P(n,3)) = 3$ *for* $n = 6k+1$ *and* $n \geq 25$*."*

An unbounded metric dimension of wheel graph W_n, fan graph f_n and Jahangir graph J_{2n} is given in the following theorem.

Theorem 2.8. [14] *"Let wheel graph W_n , fan graph f_n and Jahangir graph j_{2n}, then*
- $dim(W_n) = \lfloor \frac{2n+2}{5} \rfloor$ for $n \geq 7$;
- $dim(f_n) = \lfloor \frac{2n+2}{5} \rfloor$ for $n \geq 7$;
- $dim(J_{2n}) = \lfloor \frac{2n}{3} \rfloor$ for $n \geq 4$."

Theorem 2.9. [14] *"For every odd positive integer n, $n \geq 5$ then $dim(J_n) = 3$."*

2.5.1 Metric Dimension of Circulant Graph

We study metric dimensions of some graphs such as circulant graph, Harary graph and some results on infinite classes for generalization. Circulant graphs are the special class of Toeplitz graph and every circulant graph is Toeplitz graph but the converse is not true. We also discuss about the nature of metric dimensions of these graphs i.e either it is bounded, unbounded or constant. To find the metric dimension of Toeplitz graphs, some important results which are very useful in our task are metric dimension of circulant graphs which was studied in [8] and some harary graph which was studied in [8].
The following theorem shows that circulant graph has constant and bounded metric dimension.

Theorem 2.10. [8] "Let $G = C(n, \pm\{1,2\})$, then $dim(G) = 3$ when $n \equiv 0,2,3 (\text{mod } 4)$ and $dim(G) \leq 4$ when $n \equiv 1 (\text{mod } 4)$."

Later the result was extended which is given below.

Theorem 2.11. [8] "Let $G = C(n, \pm\{1,2,3\})$, then $dim(G) = 4$ when $n \equiv 2,3,4,5 (\text{mod } 6)$, $dim(G) \leq 5$ when $n \equiv 0 (\text{mod } 6)$ and $dim(G) \leq 6$ when $n \equiv 1 (\text{mod } 6)$."

The general result of the circulant graph is given :

Theorem 2.12. [8] "Let $G = C(n, \pm\{1,2,...,j\})$, $1 < j < \lfloor \frac{n}{2} \rfloor$, $n \geq 3$, then $dim(G) \geq j+1$."

Theorem 2.13. [8] "Let $G = C(n, \pm\{1,2,...,j\})$, then $dim(G) = j+1$ when $n \equiv i (\text{mod } 2j)$, $2 \leq i \leq j+2$ and $dim(G) \leq i-1$ when $n \equiv i (\text{mod } 2j)$, $2+j \leq i \leq 2j+1$."

Theorem 2.14. [8] "Let $G = C(n, \pm\{1,2,...,j,\frac{n}{2}\})$, then $dim(G) = j+2$ when $n \equiv 2j + 2i (\text{mod } 4j)$, $2 \leq i \leq j+1$ and $dim(G) \leq i-1$ when $n \equiv 2j+2i (\text{mod } 4j)$, $2+j \leq i \leq 2j+1$."

2.5.2 Metric Dimension of Harary Graph $H(r,n)$

The following theorem shows that Harary graph has constant and bounded metric dimension.

Theorem 2.15. [15] "Let $H_{4,n}$ be a 4-ragular Harary graph with $n \geq 5$, then $dim(H_{4,n}) = 3$ when $n \equiv 0,2,3 (\text{mod } 4)$ and $dim(H_{4,n}) \leq 4$ other wise."

Theorem 2.16. [8] "Let G be a Harary graph $H_{r,n}$, such that r is odd, n is even and $n \equiv 2j + 2i (\text{mod } 4j)$, $2 \leq i \leq j+1$. Then $dim(G) = j+2$, where $j = \frac{r-1}{2}$."

Theorem 2.17. [8] "Let G be a Harary graph $H_{r,n}$, such that r is odd, n is even and $n \equiv 2j+2i (\mathrm{mod}\, 4j)$, $j+2 \leq i \leq 2j+1$. Then $dim(G) = i-1$, where $j = \frac{r-1}{2}$."

For the detail study of metric dimension of different well known classes of Harary graph and Circulant graph see [5, 8, 12]

Chapter 3
Main Results

3 Metric Dimension of Certain Toeplitz Graph

In this chapter, we discuss about the certain Toeplitz graphs and find out their metric dimension. Firstly, we find the metric dimension of Toeplitz graph $T_n\langle 1,t\rangle$, where $t \geq 2$, which is constant. Secondly, we find the metric dimension of Toeplitz graph $T_n\langle 1,s,t\rangle$, where $t = 3,4,5,6$ and $s = 2$, which is also constant.

3.1 Metric Dimension of Toeplitz Graphs $T_n\langle 1,t\rangle$

In this section we study the Toeplitz graph $T_n\langle 1,t\rangle$, where 1 and t are its generators. We classify the dimension of this graph on the bases of the value of t. If t is even, then dimension is 2. If t is odd, then dimension is 3.

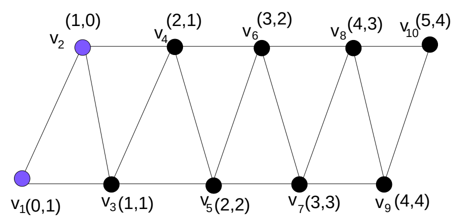

Figure 24: $dim(T_{10}\langle 1,2\rangle) = 2$

Theorem 3.1. *Let $T_n\langle 1,2\rangle$ be the Toeplitz graph. Then $dim(T_n\langle 1,2\rangle) = 2$, where $n \geq 4$.*

Proof. We will show that only two elements in basis set suffice to resolve all the vertices in $V(T_n\langle 1,2\rangle)$. Let $Q = \{v_1, v_2\}$ resolve the vertices of $T_n\langle 1,2\rangle$. Then we have the following two cases:

Case 1: If i is odd, then we have the following representation of v_i where $i \geq 3$ with respect to Q;

$$r(v_i|Q) = \left(\frac{i-1}{2}, \frac{i-1}{2}\right)$$

As, all the representation of different vertices are distinct. So, this shows that $Q = \{v_1, v_2\}$ resolve the vertices of $T_n\langle 1,2\rangle$ which means

$$dim(T_n\langle 1,2\rangle) \leq 2$$

Case 2: If i is even, then we have the following representation of v_i where $i \geq 3$ with respect to Q;

$$r(v_i|Q) = \left(\frac{i}{2}, \frac{i}{2} - 1\right)$$

As, all the representation of different vertices are distinct. So, this shows that $Q = \{v_1, v_2\}$ resolve the vertices of $T_n\langle 1,2\rangle$ which means

$$dim(T_n\langle 1,2\rangle) \leq 2 \tag{1}$$

Conversely:
Now, we show that $dim(T_n\langle 1,2\rangle) \geq 2$. Suppose on contrary that $dim(T_n\langle 1,2\rangle) = 1$.
As, we know that $dim(G) = 1$ if and only if $G = P_n$ by Theorem 2.2. This is not possible for our graph which is Toeplitz graph.
So,

$$dim(T_n\langle 1,2\rangle) \geq 2 \tag{2}$$

Hence from Eq. (1) and (2) we have

$$dim(T_n\langle 1,2\rangle) = 2$$

□

Next theorem is about the metric dimension of Toeplitz graph $T_n\langle 1,t\rangle$ where even $t \geq 4$.

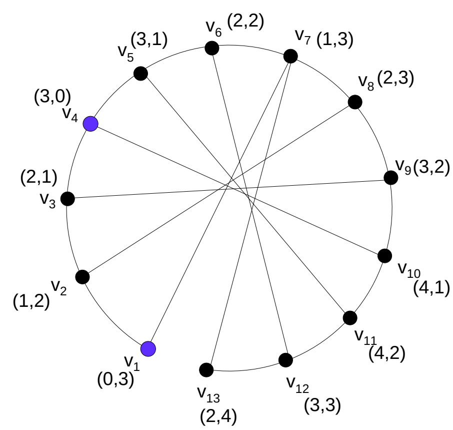

Figure 25: $dim(T_{13}\langle 1,6\rangle) = 2$

Theorem 3.2. *Let $T_n\langle 1,t\rangle$ be the Toeplitz graph with even $t \geq 4$. Then $dim(T_n\langle 1,t\rangle) = 2$, where $n \geq t+2$.*

Proof. We will show that only two elements in basis set suffice to resolve all the vertices in $V(T_n\langle 1,t\rangle)$. Let $Q = \{v_1, v_{\frac{t+2}{2}}\}$ resolve the vertices of $T_n\langle 1,t\rangle$. Then we have the following three cases:
Case 1: If $k \equiv 2,3,...,\frac{t}{2}(\mathrm{mod}\, t)$, then v_k has the following representation with respect to Q;

$$r(v_k|Q) = \left(k - pt + p - 1, \frac{(2p+1)t - 2k}{2} + p + 1\right)$$

where $p = \lfloor \frac{k}{t} \rfloor$. As, all the representation of different vertices are distinct. So, this shows that $Q = \left\{v_1, v_{\frac{t+2}{2}}\right\}$ resolve the vertices of $T_n\langle 1,t\rangle$ which means

$$dim(T_n\langle 1,t\rangle) \leq 2$$

Case 2: If $k \equiv \frac{t}{2}+1 (\bmod t)$, then v_k has the following representation with respect to Q;

$$r(v_k|Q) = \left(k - pt + p - 1, \frac{2k-(2p+1)t}{2} + p - 1\right)$$

where $p = \lfloor \frac{k}{t} \rfloor$. As, all the representation of different vertices are distinct. So, this shows that $Q = \left\{v_1, v_{\frac{t+2}{2}}\right\}$ resolve the vertices of $T_n\langle 1,t\rangle$ which means

$$dim(T_n\langle 1,t\rangle) \leq 2$$

Case 3: If $k \equiv \frac{t}{2}+2, \frac{t}{2}+3, ..., t+1 (\bmod t)$ then v_k has the following representation with respect to Q;

$$r(v_k|Q) = \left(pt - k + p + 1, \frac{2k-(2p-1)t}{2} + p - 2\right)$$

where $p = \lfloor \frac{2k+t}{2t} \rfloor$. As, all the representation of different vertices are distinct. So, this shows that $Q = \left\{v_1, v_{\frac{t+2}{2}}\right\}$ resolve the vertices of $T_n\langle 1,t\rangle$ which means

$$dim(T_n\langle 1,t\rangle) \leq 2 \tag{3}$$

Conversely:
Now, we show that $dim(T_n\langle 1,t\rangle) \geq 2$. Suppose on contrary that $dim(T_n\langle 1,t\rangle) = 1$.
As, we know that $dim(G) = 1$ if and only if $G = P_n$ by Theorem 2.2. This is not possible for our graph which is Toeplitz graph.
So,

$$dim(T_n\langle 1,t\rangle) \geq 2 \tag{4}$$

Hence from Eq. (3) and (4) we have

$$dim(T_n\langle 1,t\rangle) = 2$$

□

Next theorem is about the metric dimension of Toeplitz graph $T_n\langle 1,t\rangle$ when $t = 3$.

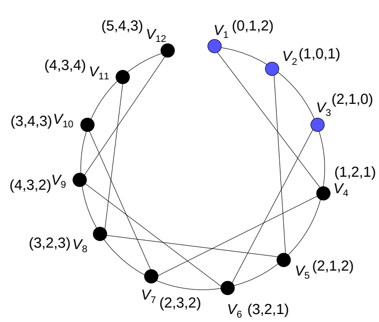

Figure 26: $dim(T_{12}\langle 1,3\rangle) = 3$

Theorem 3.3. *Let $T_n\langle 1,3\rangle$ be the Toeplitz graph. Then $dim(T_n\langle 1,3\rangle) = 3$, where $n \geq 5$.*

Proof. We will show that only three elements in basis set suffice to resolve all the vertices in $V(T_n\langle 1,3\rangle)$. Let $Q = \{v_1, v_2, v_3\}$ resolve the vertices of $T_n\langle 1,3\rangle$. Then we have the following two cases:
Case 1: If $k \equiv 1,2(\text{mod } 3)$, then v_k has the following representation with respect to Q;

$$r(v_k|Q) = (k-2p-1, 4p-k+2, k-2p-1)$$

where $p = \lfloor \frac{k}{3} \rfloor$. As, all the representation of different vertices are distinct. So, this shows that $Q = \{v_1, v_2, v_3\}$ resolve the vertices of $T_n\langle 1,3\rangle$ which means

$$dim(T_n\langle 1,3\rangle) \leq 3$$

Case 2: If $k \equiv 0(\text{mod } 3)$, then v_k has the following representation with respect to Q;

$$r(v_k|Q) = (p+1, p, p-1)$$

where $p = \frac{k}{3}$. As, all the representation of vertices are distinct. So, this shows that $Q = \{v_1, v_2, v_3\}$ resolve the vertices of $T_n\langle 1,3\rangle$ which means

$$dim(T_n\langle 1,3\rangle) \leq 3 \tag{5}$$

Conversely:
Now, we show that $dim(T_n\langle 1,3\rangle) \geq 3$. Suppose on contrary that $dim(T_n\langle 1,3\rangle) = 2$.
There are two possible cases according to n which are given below:
Case 1: When $n = 5, 6$
Following are two possible basis sets by Theorem 2.3.
If $Q_1 = \{v_i, v_{i+1}\}$ where $1 \leq i \leq n-1$, then the following vertices have the same representation;

$$r(v_{i+2}|Q_1) = r(v_{i+4}|Q_1) = (2,1)$$

If $Q_2 = \{v_i, v_a\}$ where $1 \leq i \leq n-1$ and $a \equiv p(\text{mod } 3)$ where $p = 1, 2, ..., n-3$ and $k = \lfloor \frac{n-p}{3} \rfloor$, then the following vertices have the same representation;

$$r(v_{i+2}|Q_2) = r(v_{i+4}|Q_2) = (2,1)$$

Our supposition is wrong, the dimension is not 2 because there exist the same representation.
So,

$$dim(T_n\langle 1,3\rangle) \geq 3$$

Case 2: When $n \geq 7$
Following are two possible basis sets by Theorem 2.3.
If $Q_1 = \{v_i, v_{i+1}\}$ where $i = 1, 2$ and $i = n-2, n-1$, then the following vertices have the same representation;

$$r(v_{i+2}|Q_1) = r(v_{i+4}|Q_1) = (2,1)$$

If $Q_2 = \{v_i, v_a\}$ where $i = 1,2,3$ and $a \equiv p(\bmod 3)$ where $p = 0,1,2$ and $k = \lfloor \frac{n-p}{3} \rfloor$, then the following vertices have the same representation;

$$r(v_{i+2}|Q_2) = r(v_{i+4}|Q_2) = (2,1)$$

Our supposition is wrong, the dimension is not 2 because there exist same representation. So,

$$dim(T_n\langle 1,3\rangle) \geq 3 \tag{6}$$

Hence from Eq. (5) and (6) we have

$$dim(T_n\langle 1,3\rangle) = 3$$

□

Next theorem is about the metric dimension of Toeplitz graph $T_n\langle 1,t\rangle$ when odd $t \geq 5$.

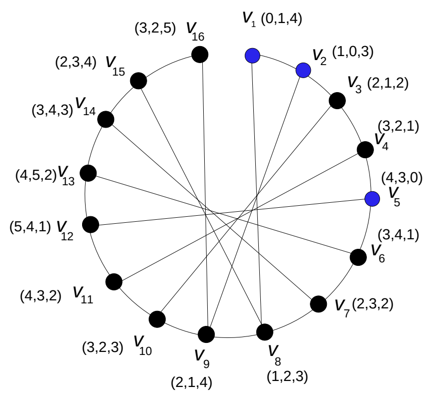

Figure 27: $dim(T_{16}\langle 1,7\rangle) = 3$

Theorem 3.4. *Let $T_n\langle 1,t\rangle$ be the Toeplitz graph with odd $t \geq 5$. Then $dim(T_n\langle 1,t\rangle) = 3$, where $n \geq t+2$.*

Proof. We will show that only three elements in basis set suffice to resolve all the vertices in $V(T_n\langle 1,t\rangle)$. Let $Q = \{v_1, v_2, v_{\frac{t+3}{2}}\}$ resolve the vertices of $T_n\langle 1,t\rangle$. Then we have the following four cases:

Case 1: If $k \equiv 3,4,...,\frac{t+1}{2}(\bmod t)$, then v_k has the following representation with respect to Q;

$$r(v_k|Q) = \left(k - pt + p - 1, k - pt + p - 2, \frac{(2p+1)t - 2k + 2p + 3}{2}\right)$$

where $p = \lfloor \frac{k}{t} \rfloor$. As, all the representation of different vertices are distinct. So, this shows that $Q = \left\{v_1, v_{\frac{t+3}{2}}\right\}$ resolve the vertices of $T_n\langle 1,t\rangle$ which means

$$dim(T_n\langle 1,t\rangle) \leq 3$$

Case 2: If $k \equiv \frac{t+3}{2} (\bmod t)$, then v_k has the following representation with respect to Q;

$$r(v_k|Q) = \left(k - pt + p - 1, k - pt + p - 2, \frac{2k - (2p+1)t + 2p - 3}{2}\right)$$

where $p = \lfloor \frac{k}{t} \rfloor$. As, all the representation of different vertices are distinct. So, this shows that $Q = \left\{v_1, v_{\frac{t+3}{2}}\right\}$ resolve the vertices of $T_n \langle 1, t \rangle$ which means

$$dim(T_n \langle 1, t \rangle) \leq 3$$

Case 3: If $k \equiv \frac{t+5}{2}, \frac{t+7}{2}, ..., t+1 (\bmod t)$, then v_k has the following representation with respect to Q;

$$r(v_k|Q) = \left(pt - k + p + 1, pt - k + p + 2, \frac{2k - (2p-1)t + 2p - 5}{2}\right)$$

where $p = \lfloor \frac{2k+t}{2t} \rfloor$. As, all the representation of different vertices are distinct. So, this shows that $Q = \left\{v_1, v_{\frac{t+3}{2}}\right\}$ resolve the vertices of $T_n \langle 1, t \rangle$ which means

$$dim(T_n \langle 1, t \rangle) \leq 3$$

Case 4: If $k \equiv 2 (\bmod t)$, then v_k has the following representation with respect to Q;

$$r(v_k|Q) = \left(k - pt + p - 1, k - pt + p - 2, \frac{2k - (2p-1)t + 2p - 5}{2}\right)$$

where $p = \lfloor \frac{k}{t} \rfloor$. As, all the representation of different vertices are distinct. So, this shows that $Q = \left\{v_1, v_{\frac{t+3}{2}}\right\}$ resolve the vertices of $T_n \langle 1, t \rangle$ which means

$$dim(T_n \langle 1, t \rangle) \leq 3 \tag{7}$$

Conversely:
Now, we show that $dim(T_n \langle 1, t \rangle) \geq 3$. Suppose on contrary that $dim(T_n \langle 1, t \rangle) = 2$. There are two possible cases according to n which are given below:
Case 1: When $t + 2 \leq n \leq 2t$
Following are three possible basis sets by Theorem 2.3.
If $Q_1 = \{v_i, v_{i+a}\}$ where $1 \leq i \leq n - a$ and $1 \leq a \leq \lfloor \frac{t}{2} \rfloor$, then the following vertices have the same representation;

$$r(v_{i+a+1}|Q_1) = r(v_{t+i+a}|Q_1) = (a+1, 1)$$

If $Q_2 = \{v_i, v_{i+a}\}$ where $i = 1$ and $i = n - a$, and $\frac{t+3}{2} \leq a \leq t - 1$, then the following vertices have the same representation;

$$r(v_{i+a-\frac{t-1}{2}}|Q_2) = r(v_{i+a+\frac{t-1}{2}}|Q_2) = (a - \frac{t-1}{2}, \frac{t-1}{2})$$

If $Q_3=\{v_i,v_a\}$ where $1\leq i\leq n-t$ and $a\equiv p(\bmod t)$ where $p=1,2,...,n-t$ and $k=\lfloor\frac{n-p}{t}\rfloor$, then the following vertices have the same representation;

$$r(v_{i+\frac{t+1}{2}}|Q_3)=r(v_{i+\frac{3t-1}{2}}|Q_3)=(\frac{t+1}{2},k+\frac{t-3}{2})$$

Our supposition is wrong, the dimension is not 2 because there exist same representation. So,

$$dim(T_n\langle 1,t\rangle)\geq 3$$

Case 2: When $n\geq 2t+1$

Following are four possible basis sets by Theorem 2.3.

If $Q_1=\{v_i,v_{i+a}\}$ where $1\leq i\leq t-a$ and $n-(t-a)\leq i\leq n-a$, and $1\leq a\leq\lfloor\frac{t}{2}\rfloor$, then the following vertices have the same representation;

$$r(v_{i+a+1}|Q_1)=r(v_{t+i+a}|Q_1)=(a+1,1)$$

If $Q_2=\{v_i,v_{i+a}\}$ where $i=t-c+1$ with $1\leq c\leq a-1$ and $n=2t+d$ with $1\leq d\leq a-c$, and $2\leq a\leq\lfloor\frac{t}{2}\rfloor$, then the following vertices have the same representation;

$$r(v_{i+a+1}|Q_2)=r(v_{t+i+a}|Q_2)=(a+1,1)$$

If $Q_3=\{v_i,v_{i+a}\}$ where $i=1$ and $i=n-a$, and $\frac{t+3}{2}\leq a\leq t-1$, then the following vertices have the same representation;

$$r(v_{i+a-\frac{t-1}{2}}|Q_3)=r(v_{i+a+\frac{t-1}{2}}|Q_3)=(a-\frac{t-1}{2},\frac{t-1}{2})$$

If $Q_4=\{v_i,v_a\}$ where $1\leq i\leq t$ and $a\equiv p(\bmod t)$ with $p=0,1,2,...,t-1$ and $k=\lfloor\frac{n-p}{t}\rfloor$, then the following vertices have the same representation;

$$r(v_{i+\frac{t+1}{2}}|Q_4)=r(v_{i+\frac{3t-1}{2}}|Q_4)=(\frac{t+1}{2},k+\frac{t-3}{2})$$

Our supposition is wrong, the dimension is not 2 because there exist same representation. So,

$$dim(T_n\langle 1,t\rangle)\geq 3 \tag{8}$$

Hence from Eq. (7) and (8) we have

$$dim(T_n\langle 1,t\rangle)=3$$

□

3.2 Metric Dimension of Toeplitz Graphs $T_n\langle 1,2,t\rangle$

In this section we study the Toeplitz graph $T_n\langle 1,2,t\rangle$, where 1, 2 and t are its generators. We find the metric dimension of Toeplitz graph $T_n\langle 1,2,t\rangle$, where $t=3,4,5,6$, which is 3.

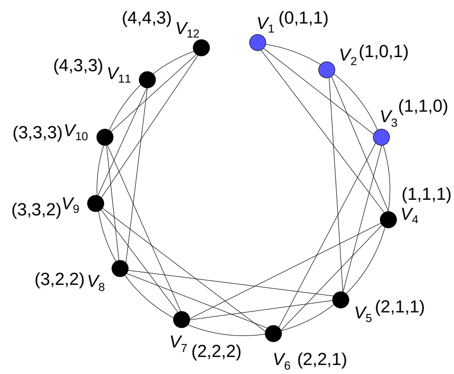

Figure 28: $dim(T_{11}\langle 1,2,3\rangle) = 3$

Theorem 3.5. *Let $T_n\langle 1,2,3\rangle$ be the Toeplitz graph. Then $dim(T_n\langle 1,2,3\rangle) = 3$, where $n \geq 5$.*

Proof. We will show that only three elements in basis set suffice to resolve all the vertices in $V(T_n\langle 1,2,3\rangle)$. Let $Q = \{v_1, v_2, v_3\}$ resolve the vertices of $T_n\langle 1,2,3\rangle$. Then we have the following two cases:
Case 1: If $k \equiv 1,2 (\text{mod } t)$, then v_k has the following representation with respect to Q;

$$r(v_k|Q) = (k - pt + p, p, p)$$

where $p = \lfloor \frac{k}{t} \rfloor$. As, all the representation of different vertices are distinct. So, this shows that $Q = \{v_1, v_2, v_3\}$ resolve the vertices of $T_n\langle 1,2,3\rangle$ which means

$$dim(T_n\langle 1,2,3\rangle) \leq 3$$

Case 2: If $k \equiv 0 (\text{mod } t)$, then v_k has the following representation with respect to Q;

$$r(v_k|Q) = (k - pt + p, k - pt + p, p - 1)$$

where $p = \lfloor \frac{k}{t} \rfloor$. As, all the representation of different vertices are distinct. So, this shows that $Q = \{v_1, v_2, v_3\}$ resolve the vertices of $T_n\langle 1,2,3\rangle$ which means

$$dim(T_n\langle 1,2,3\rangle) \leq 3 \tag{9}$$

Conversely:
Now, we show that $dim(T_n\langle 1,2,3\rangle) \geq 3$. Suppose on contrary that $dim(T_n\langle 1,2,3\rangle) = 2$. There are two possible cases according to n which are given below:
Case 1: For $n \equiv 1 (\text{mod } 3)$ and $n = 3k + 1$ where $k = \lfloor \frac{n}{3} \rfloor$. In $T_n\langle 1,2,3\rangle$ the only possible basis set by Theorem 2.3 is given:
If $Q = \{v_1, v_n\}$, then the following vertices have the same representation;

$$r(v_2|Q) = r(v_3|Q) = (1, k)$$

Our supposition is wrong, the dimension is not 2 because there exist same representation.
So,

$$dim(T_n\langle 1,2,3\rangle) \geq 3$$

Case 2: For the remaining values of n in $T_n\langle 1,2,3\rangle$ the only possible basis set by Theorem 2.3 is $Q = \{v_1, v_n\}$, but v_1 and v_n does not have the unique shortest path. Our supposition is wrong, the dimension is not 2.
So,

$$dim(T_n\langle 1,2,3\rangle) \geq 3 \tag{10}$$

Hence from Eq. (9) and (10) we have

$$dim(T_n\langle 1,2,3\rangle) = 3$$

□

Next theorem is about the metric dimension of Toeplitz graph $T_n\langle 1,s,t\rangle$ when $t = 4$ and $s = 2$.

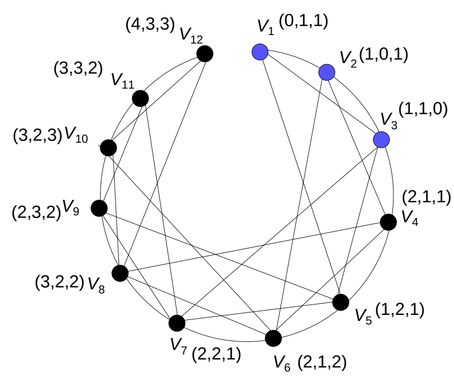

Figure 29: $dim(T_{12}\langle 1,2,4\rangle) = 3$

Theorem 3.6. *Let $T_n\langle 1,2,4\rangle$ be the Toeplitz graph. Then $dim(T_n\langle 1,2,4\rangle) = 3$, where $n \geq 6$.*

Proof. We will show that only three elements in basis set suffice to resolve all the vertices in $V(T_n\langle 1,2,4\rangle)$. Let $Q = \{v_1, v_2, v_3\}$ resolve the vertices of $T_n\langle 1,2,4\rangle$. Then we have the following three cases:
Case 1: If $k \equiv 3(\text{mod } 4)$, then v_k has the following representation with respect to Q;

$$r(v_k|Q) = (k-3p-3, k-3p-3, p)$$

where $p = \lfloor \frac{k}{4} \rfloor$. As, all the representation of different vertices are distinct. So, this shows that $Q = \{v_1, v_2, v_3\}$ resolve the vertices of $T_n\langle 1,2,4\rangle$ which means

$$dim(T_n\langle 1,2,4\rangle) \leq 3$$

Case 2: If $k \equiv 0,1(\mathrm{mod}\ 4)$, then v_k has the following representation with respect to Q;

$$r(v_k|Q) = (5p-k+1, k-3p, p)$$

where $p = \lfloor \frac{2k+4}{8} \rfloor$. As, all the representation of different vertices are distinct. So, this shows that $Q = \{v_1, v_2, v_3\}$ resolve the vertices of $T_n \langle 1,2,4 \rangle$ which means

$$dim(T_n \langle 1,2,4 \rangle) \leq 3$$

Case 3: If $k \equiv 2(\mathrm{mod}\ t)$, then v_k has the following representation with respect to Q;

$$r(v_k|Q) = (p+1, p, p+1)$$

where $p = \lfloor \frac{k}{4} \rfloor$. As, all the representation of different vertices are distinct. So, this shows that $Q = \{v_1, v_2, v_3\}$ resolve the vertices of $T_n \langle 1,2,4 \rangle$ which means

$$dim(T_n \langle 1,2,4 \rangle) \leq 3 \tag{11}$$

Conversely:
Now, we show that $dim(T_n \langle 1,2,4 \rangle) \geq 3$. Suppose on contrary that $dim(T_n \langle 1,2,4 \rangle) = 2$. There are two possible cases according to n which are given below:
Case 1: For $n \equiv 1(\mathrm{mod}\ 4)$ and $n = 4k+1$ where $k = \lfloor \frac{n}{4} \rfloor$. In $T_n \langle 1,2,4 \rangle$ the only possible basis set by Theorem 2.3 is given:
If $Q = \{v_1, v_n\}$, then the following vertices have the same representation;

$$r(v_4|Q) = r(v_6|Q) = (2, k)$$

Our supposition is wrong, the dimension is not 2 because there exist same representation.
So,

$$dim(T_n \langle 1,2,4 \rangle) \geq 3$$

Case 2: For the remaining values of n in $T_n \langle 1,2,4 \rangle$ the only possible basis set by Theorem 2.3 is $Q = \{v_1, v_n\}$, but v_1 and v_n does not have the unique shortest path. Our supposition is wrong, the dimension is not 2.
So,

$$dim(T_n \langle 1,2,4 \rangle) \geq 3 \tag{12}$$

Hence from Eq. (11) and (12) we have

$$dim(T_n \langle 1,2,4 \rangle) = 3$$

□

Next theorem is about the metric dimension of Toeplitz graph $T_n \langle 1,s,t \rangle$ when $t = 5,6$ and $s = 2$.

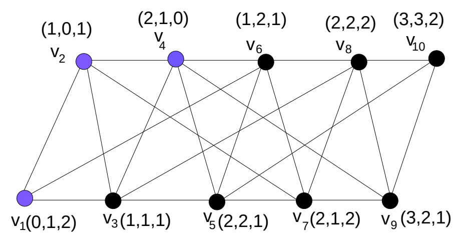

Figure 30: $dim(T_{10}\langle 1,2,5\rangle)=3$

Theorem 3.7. *Let $T_n\langle 1,2,t\rangle$ where $t=5,6$ be the Toeplitz graph. Then $dim(T_n\langle 1,2,t\rangle)=3$, where $n\geq t+3$.*

Proof. We will show that only three elements in basis set suffice to resolve all the vertices in $V(T_n\langle 1,2,t\rangle)$. Let $Q=\{v_1,v_2,v_{t-1}\}$ resolve the vertices of $T_n\langle 1,2,t\rangle$. Then we have the following four cases:

Case 1: If $k\equiv 3,...,t-2(\bmod t)$, then v_k has the following representation with respect to Q;

$$r(v_k|Q)=(k-pt+p-2,p+1,p+1)$$

where $p=\lfloor\frac{k}{t}\rfloor$. As, all the representation of different vertices are distinct. So, this shows that $Q=\{v_1,v_2,v_{t-1}\}$ resolve the vertices of $T_n\langle 1,2,t\rangle$ which means

$$dim(T_n\langle 1,2,t\rangle)\leq 3$$

Case 2: If $k\equiv t-1(\bmod t)$, then v_k has the following representation with respect to Q;

$$r(v_k|Q)=(p+2,k-pt+p-3,p)$$

where $p=\lfloor\frac{k}{t}\rfloor$. As, all the representation of different vertices are distinct. So, this shows that $Q=\{v_1,v_2,v_{t-1}\}$ resolve the vertices of $T_n\langle 1,2,t\rangle$ which means

$$dim(T_n\langle 1,2,t\rangle)\leq 3$$

Case 3: If $k\equiv 0,1(\bmod t)$, then v_k has the following representation with respect to Q;

$$r(v_k|Q)=(pt-k+p+1,p+1,p)$$

where $p=\lfloor\frac{2k+t}{2t}\rfloor$. As, all the representation of different vertices are distinct. So, this shows that $Q=\{v_1,v_2,v_{t-1}\}$ resolve the vertices of $T_n\langle 1,2,t\rangle$ which means

$$dim(T_n\langle 1,2,t\rangle)\leq 3$$

Case 4: If $k\equiv 2(\bmod t)$, then v_k has the following representation with respect to Q;

$$r(v_k|Q)=(p+1,p,p+1)$$

where $p = \lfloor \frac{k}{t} \rfloor$. As, all the representation of different vertices are distinct. So, this shows that $Q = \{v_1, v_2, v_{t-1}\}$ resolve the vertices of $T_n \langle 1,2,t \rangle$ which means

$$dim(T_n \langle 1,2,t \rangle) \leq 3 \tag{13}$$

Conversely:
Now, we show that $dim(T_n \langle 1,2,t \rangle) \geq 3$. Suppose on contrary that $dim(T_n \langle 1,2,t \rangle) = 2$ where $t = 5,6$. There are two possible cases according to n which are given below:
Case 1: For $n \equiv 1(\mathrm{mod}\, t)$ and $n = tk+1$ where $k = \lfloor \frac{n}{t} \rfloor$. In $T_n \langle 1,2,t \rangle$ the only possible basis set by Theorem 2.3 is given:
If $Q = \{v_1, v_n\}$, then the following vertices have the same representation;

$$r(v_2|Q) = r(v_3|Q) = (1, k+1)$$

Our supposition is wrong, the dimension is not 2 because there exist same representation. So,

$$dim(T_n \langle 1,2,t \rangle) \geq 3$$

Case 2: For the remaining values of n in $T_n \langle 1,2,t \rangle$ the only possible basis set by Theorem 2.3 is $Q = \{v_1, v_n\}$, but v_1 and v_n does not have the unique shortest path. Our supposition is wrong, the dimension is not 2.
So,

$$dim(T_n \langle 1,2,t \rangle) \geq 3 \tag{14}$$

Hence from Eq. (13) and (14) we have

$$dim(T_n \langle 1,2,t \rangle) = 3$$

□

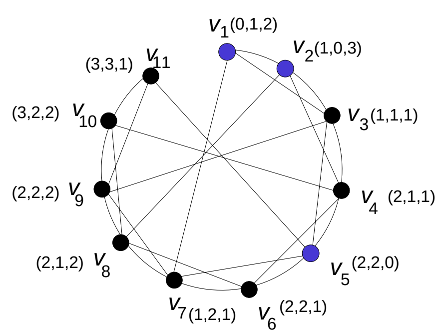

Figure 31: $dim(T_{11} \langle 1,2,6 \rangle) = 3$

Chapter 4
Conclusions and Open Problems

4 Conclusion and Open Problems

We studied the metric dimension of certain families of Toeplitz graphs, $T_n\langle 1,t\rangle$ when t is even and odd and $T_n\langle 1,2,t\rangle$ when $t = 3,4,5,6$. We have also concluded that these graphs have constant metric dimension.

4.1 Open Problems

1. Is $dim(T_n\langle 2,t\rangle)$ for $t \geq 3$ odd constant, bounded or unbounded?
2. Is $dim(T_n\langle 1,2,t\rangle)$ for $t \geq 7$ constant, bounded or unbounded?
3. Is $dim(T_n\langle 1,3,t\rangle)$ for $t \geq 4$ constant, bounded or unbounded?
4. If anyone work on the General result for $dim(T_n\langle t_1,t_2,t_3,...,t_k\rangle)$, then it will be an interesting result.

Chapter 5
References

[1] Ali, M., Ali, G., Imran, M., Baig, A. Q., & Shafiq, M. K. (2012). On the metric dimension of Mobius ladders. *Ars Combinatoria* 105, 403-410.

[2] Cceres, J., Hernando, C., Mora, M., Pelayo, I. M., & Puertas, M. L. (2009). On the metric dimension of infinite graphs. *Electronic Notes in Discrete Mathematics* 35, 15-20.

[3] Chartrand, G. & P. Zhang (2004) *Introduction to Graph Theory*.

[4] Chartrand, G., Eroh, L., Johnson, M. A., & Oellermann, O. R. (2000). Resolvability in graphs and the metric dimension of a graph. *Discrete Appl. Math.* 105(1-3), 99-113.

[5] Hernando, C., Mora, M., Pelayo, I. M., Seara, C., Cceres, J., & Puertas, M. L. (2005). On the metric dimension of some families of graphs. *Electronic Notes in Discrete Mathematics* 22, 129-133.

[6] Diestel, R. (2005). Graph Theory, ser. *Graduate Texts in Mathematics. Springer-Verlag, Heidelberg*, 173.

[7] Van Dal, R., Tijssen, G., Tuza, Z., Van Der Veen, J. A., Zamfirescu, C., & Zamfirescu, T. (1996). Hamiltonian properties of Toeplitz graphs. *Discrete Math.* 159(1-3), 69-81.

[8] Grigorious, C., Manuel, P., Miller, M., Rajan, B., & Stephen, S. (2014). On the metric dimension of circulant and Harary graphs *Appl. Math. Comput.* 248, 4754.

[9] Harary, F., & Melter, R. A. (1976). On the metric dimension of a graph. *Ars Combinatoria* 2, 191-195.

[10] Imran, M., Baig, A. Q., Shafiq, M. K., & Tomescu, I. (2014). On Metric Dimension of Generalized Petersen Graphs $P(n,3)$. *Ars Combinatoria* 117, 113-130.

[11] Imran, M., Baig, A. Q., Bokhary, S. A. U. H., & Javaid, I. (2012). On the metric dimension of circulant graphs. *Appl. Math. Lett.* 25(3), 320-325.

[12] Imran, M., Baig, A. Q., & Ahmad, A. (2012). Families of plane graphs with constant metric dimension. *Utilitas Mathematica* 88, 43-57.

[13] Imran, M., Bokhary, S. A. U. H., & Baig, A. Q. (2010). On families of convex polytopes with constant metric dimension. *Comput. Math. Appl.* 60(9), 2629-2638.

[14] Imran, M., Ahmad, A., & Semanicova-Fenovcikova, A. (2013). On classes of regular graphs with constant metric dimension. *Acta Math. Sci.* 33(1),187-206.

[15] Javaid, I., Rahim, M. T., & Ali, K. (2008). Families of regular graphs with constant metric dimension. *Utilitas Mathematica* 75, 21-34.

[16] Khuller, S., Raghavachari, B., & Rosenfeld, A. (1994). *Localization in graphs*.

[17] Kratica, J., Kovaevi-Vuji, V., angalovi, M., & Stojanovi, M. (2012). Minimal doubly resolving sets and the strong metric dimension of some convex polytopes. *Appl. Math. Comput.* 218(19), 9790-9801.

[18] Melter, R. A., & Tomescu, I. (1984). Metric bases in digital geometry. *Computer Vision, Graphics, and Image Processing*, 25(1), 113-121.

[19] Naeem, R., & Imran, M. (2017). On resolvability and exchange property in antiweb-wheels. *Utilitas Mathematica* 104, 187-200.

[20] Perc, M., Gmez-Gardees, J., Szolnoki, A., Flora, L. M., & Moreno, Y. (2013). Evolutionary dynamics of group interactions on structured populations: a review. *Journal of the royal society interface.* 10(80), 20120997.

[21] Perc, M., & Szolnoki, A. (2010). Coevolutionary gamesa mini review. *BioSystems* 99(2), 109-125.

[22] Slater, P. J. (1975). Leaves of trees. *Congr. Numer.* 14(549-559), 37.

[23] Wong, C. K., & Coppersmith, D. (1974). A combinatorial problem related to multimodule memory organizations. *J. ACM.* 21(3), 392-402.

[24] Szolnoki, A., & Perc, M. (2013). Correlation of positive and negative reciprocity fails to confer an evolutionary advantage: Phase transitions to elementary strategies. *Phys. Rev. X.* 3(4), 041021.

[25] Saputro, S. W., Simanjuntak, R., Uttunggadewa, S., Assiyatun, H., Baskoro, E. T., Salman, A. N. M., & Baca, M. (2013). The metric dimension of the lexicographic product of graphs. *Discrete Math.* 313(9), 1045-1051.

[26] Slater, P. J. (1988). Dominating and reference sets in a graph. *J. Math. Phys. Sci.* 22(4), 445-455.

[27] Sudhakara, G., & Kumar, H. A. (2009). Graphs with metric dimension two-a characterization. *Advances and Applications in Discrete Mathematics* 4(2), 169-186.

[28] Tomescu, I., & Imran, M. (2009). On metric and partition dimensions of some infinite regular graphs. *B. Math. Soc. Sci. Math.*, 4, 461-472.

[29] Yero, I. G., Kuziak, D., & Rodrguez-Velzquez, J. A. (2011). On the metric dimension of corona product graphs. *Comput. Math. Appl.* 61(9), 2793-2798.

[30] Yero, I. G., Jakovac, M., Kuziak, D., & Taranenko, A. (2014). The partition dimension of strong product graphs and Cartesian product graphs. *Discrete Math.*, 331, 43-52.