Convergence Analysis of Hybrid Projection Method for Split Equilibrium Problems in Hilbert Spaces

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Declaration

I Faisal Mumtaz, **CIIT/FA16-RMT-029/LHR** hereby declare that I have produced the work presented in this thesis, during the scheduled period of study. I also declare that I have not taken any material from any source except referred to wherever due, that amount of plagiarism is within acceptable range. If a violation of HEC rules on research has occurred in this thesis, I shall be liable to punishable action under the plagiarism rules of the HEC.

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Certificate

It is certified that Faisal Mumtaz , CIIT/FA16-RMT-029/LHR has carried out all the work related to this thesis under my supervision at the Department of Mathematics, COMSATS University Islamabad, Lahore campus and the work fulfills the requirement for award of MS degree.

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DEDICATION

To

My Parents and family, specially to my elder brothers Muzammil Mumtaz and Moazzam Mumtaz.

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In the name of Allah, the most gracious and the most merciful.
The Prophet (S.A.W) said: “Seeking knowledge is obligatory for every muslim”.
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ABSTRACT

Convergence Analysis of Hybrid Projection Method for Split Equilibrium Problems in Hilbert Spaces

This work is devoted to establish the strong convergence results of an iterative algorithm generated by the shrinking projection method in Hilbert spaces. The proposed approximation sequence is used to find a common element in the set of solutions of a finite family of split equilibrium problems and the set of common fixed points of a finite family of total asymptotically strict pseudo contractions in such setting.

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Chapter 1

Introduction and Basic Definition

This chapter is focused on the presentation of fixed point theory (FPT), split equilibrium theory (SET) and shrinking projection method in the domain of Hilbert spaces. The class of Hilbert spaces has essential geometrical structures crucial for the analysis of problems related to metric fixed point theory (MFPT) and equilibrium problem theory (EPT). In the continuation, different fundamental ideas of FPT and EPT are characterized in such settings.

1.1 Fixed Point Theory

The majority of the phenomena occurring in daily life are generally nonlinear in nature. Subsequently, mathematicians and researchers are constantly in quest for discovering new techniques to handle the nonlinear problems. The most effective techniques to handle with a nonlinear problem is to frame it as an equivalent FPP. The FPT is one of the most amazing system of modern mathematical sciences. It has joined analysis, topology and geometry together. In particular, it has been connected in such areas as arithmetic, economics , game theory, biology, chemistry and physics etc.

FPT is a branch of nonlinear analysis though which the fixed points of any function G can b approximated. i.e,

$$\text{there exists } \xi \in Y \text{ such that } G\xi = \xi,$$

where $Y \neq \emptyset$ is the domain set and G is a self nonlinear map .

FPT has proved itself a useful tool for the analysis of different nonlinear real world problems. The usage of FPT has been expanded widely with the advancement of scientific procedures to construct the fixed points. The notion $G\xi = \xi$ can be simplified into the form $(\mathbb{I} - G)\xi = 0$ which relates the FPP to corresponding problems in theories of nonlinear differential equations and integral equations rising in various fields of study, for example, biology, chemistry, physics, designing, accounting, game theory, software engineering and arithmetic related fields.

In general, FPT is designed for the analysis of the existence or interpolation of associated fixed points for the nonlinear maps under consideration. The existence and/or interpretation of associated fixed points for the nonlinear maps can be utilized to solve problems in different branches of mathematical sciences. We remark that the iterative computation of fixed points has inherent importance in various branches of applied mathematics and suitable for analysis of fixed points in such cases where the derivation of constrains is necessary and it is tough to extract those preconditions for existence.

FPT is classified in following parts

(i) Topological FPT

(ii) Discrete FPT

(iii) Metric FPT

One of the main motivation for the study of MFPT is to analyze the existence as well as uniqueness of the solution associated with the differential equations. Famous mathematicians of nineteenth century to be specific, Picard, Liouville, Peano, Lipschitz, and particularly, Cauchy contributed significantly to the theory of diffeential equations via MFPT. MFPT believes to be originated from the well-known Banach Constriction Principle (BCP) essentially due to Stephan Banach [2].

Theorem 1.1.1 (BCP)**.** *Consider a complete metric space Y and a selfmap $G : Y \longrightarrow Y$ satisfying*

$$d(G\xi, G\eta) \leq l \quad d(\xi, \eta), \quad for\ all \quad \xi, \eta \in Y,$$

while $1 > l > 0$. Then $\exists$ a unique fixed point t for G such that the sequence $x_n = G(x_{n-1})$ converges to that unique fixed point t.

BCP has a significant function in approximation of nonlinear equations. BCP is a bond for the presence of fixed point as well as the uniqueness which can be estimated by utilizing progressive estimation.

The theory of nonexpansive maps was initiated autonomously by F. E. Browder [4], D. Gӧhde [29] and W. A. Kirk [39] in 1965. In nonlinear functional analysis the MFPT is an operational and considerational field of research. The theory of nonexpansive maps has crucial applications. Likewise, we can estimate the solutions of monotone operators, convex minimization problem and variational inequality problem (VIP) of a functions utilizing nonexpansive maps. For a comprehensive discussion of the FPT of nonexpansive maps, see [25, 40, 41].

A class of nonexpansive maps known as asymptotically nonexpansive maps is considered generally in several spaces. In any case, the analysis of approximating the fixed points utilizing the iterative algorithms for asymptotically nonexpansive maps and its distinctive suppositions have been ultimately improving in the general configurations of Hilbert spaces. The iterative algorithms can be used as a fundamental instruments for solving FPP of asymptotically nonexpansive maps and its particular theories.

1.2 Some Concepts Related to MFPT

This part of thesis is concentrated to describe some central thoughts and ideas related to MFPT required in the continuation:

Definition 1.2.1. *"Suppose an arbitrary space H and an arbitrary sequence $\{y_a\} \in H$. if $\{y_a\} \in H$ is cauchy then the space H is known as a complete space. The cauchy criterion is defined as, given $\epsilon_{a,b} > 0$ there exists M such that if $a, b > M$ then:*

$$|y_a - y_b| < \epsilon_{a,b}\,;\, y_a, y_b \in H. \tag{1.1}$$

Any arbitrary sequence is said to be Cauchy if the terms of the sequence in the long run all turn out to be relatively near. "

Let $\mathbb{H}$ to be a Hilbert space (real) and $\emptyset \neq Y \subset \mathbb{H}$ to be a closed as well as convex subset. The set containing fixed points is demonstrated as $\mathbb{F}(G)$: $\{x \in Y : Gx = x\}$ under the action of a map G.

We now characterize a few mappings defined below:

Let $\emptyset \neq Y \subset \mathbb{H}$ be a convex as well as closed subset. Let $a, b \in Y$. and $G : Y \to Y$ be a mapping:

Definition 1.2.2 ([2]). *Contraction is a mapping $G : Y \to Y$ such that:*

$$\|Ga - Gb\| \leq \beta \|a - b\| \ ; \ \beta \in (0, 1) \tag{1.2}$$

Definition 1.2.3 ([4, 29, 39]). *The nonexpansive map is defined as the mapping G : $Y \to Y$ such that*

$$\|Ga - Gb\| \leq \beta \|a - b\| \ ; \ \beta \in (0, 1)$$

if $\beta = 1$.

$$\|Ga - Gb\| \leq \|a - b\| \tag{1.3}$$

Definition 1.2.4 ([25]). *The asymptotically nonexpansive map is the mapping G : $Y \to Y$ if: $\exists$ a sequence $\{y_k\} \subset [1, \infty)$ with $\lim\limits_{k\to\infty} y_k = 1$ such that*

$$\left\|G^k a - G^k b\right\| \leq y_k \|a - b\|, \ \forall \ k \geq 1. \tag{1.4}$$

Definition 1.2.5 ([57]). *The generalized asymptotically non-expansive map is the mapping $G : Y \to Y$ the nonnegative real sequences $\{\lambda_k\}_{k=1}^{\infty}$, $\{\mu_k\}_{k=1}^{\infty}$ be given provided that*

$$\left\|G^k a - G^k b\right\| \leq \|a - b\| + \lambda_k \|a - b\| + \mu_k, \ \forall \ k \geq 1. \tag{1.5}$$

Definition 1.2.6 ([8]). *"The mapping $G : Y \to Y$ satisfying:*

$$\limsup_{k\to\infty} \sup_{a,b\in G} \left(\left\|G^k a - G^k b\right\| - \|a - b\|\right) \leq 0. \tag{1.6}$$

where G is uniformly continuous. Then the mapping is known as asymptotically non-expansive mapping in the intermediate sense. "

Definition 1.2.7 ([1]). *"A total asymptotically nonexpansive is a mapping $G : Y \to Y$ and the nonnegative real sequences $\{\lambda_k\}_{k=1}^{\infty}$, $\{\mu_k\}_{k=1}^{\infty}$ be given with $\lim_{k\to\infty} \lambda_k = 0 = \lim_{k\to\infty} \mu_k$ and*

$$\psi : \mathcal{R}^+ \cup \{0\} \to \mathcal{R}^+ \cup \{0\}$$

with

$$\psi(0) = 0$$

satisfying that

$$\|G^k a - G^k b\| \leq \|a - b\| + \lambda_k \psi(\|a - b\|) + \mu_k. \tag{1.7}$$

where ψ is a strictly increasing continuous function."

Remark 1.2.8.

- *If ψ is identity function then total asymptotically nonexpansive mapping becomes generalized asymptotically nonexpansive mappings.*
- *If ψ is identity function and $\mu_k = 0$ then total asymptotically nonexpansive mapping becomes asymptotically nonexpansive mappings.*
- *If $\mu_k = 0 = \lambda_k$ then we have nonexpansive mappings.*

A nonexpansive map acquires a fixed point for a convex subset of a Banach space (uniformly convex) which is closed as well as bounded. This result was unveild by Browder [4] and Göhde [29] in 1965, autonomously. As a consequence this conclusion was showed by Kirk [39] with marginally weaker conditions. In 1969 Goebel [24] gave another verification which is more elementary geometric in nature. For nonliner mappings, in accordance with the classifications of nonexpansive maps, Goebel and Kirk [26] initiated the class of asymptotically nonexpansive maps in 1972, after this in 2006 the class of total asymptotically nonexpansive maps was initiated by Alber et al. [1]. This class of nonexpansive maps is much broader and had brought together a wide range of definitions of mappings related to asymptotically nonexpansive mappings, Afterwards, in 2007, Shahzad and Zegeye [57] unveiled another class namely generalized asymptotically nonexpansive mapping.

Let $a, b \in Y.$ and $G : Y \to Y$ be a mapping:

Definition 1.2.9. *Lipschitzian is the mapping $G : Y \to Y$ if*

$$\left\|G^k a - G^k b\right\| \leq \Theta \left\|a - b\right\| \textit{for some } \Theta > 0. \tag{1.8}$$

Definition 1.2.10. *A firmly nonexpansive mapping is $G : Y \to Y$ if*

$$\left\|Ga - Gb\right\|^2 + \left\|(\mathbb{I} - G)a - (\mathbb{I} - G)b\right\|^2 \leq \left\|a - b\right\|^2. \tag{1.9}$$

where $\mathbb{I}$ is an identity map.

Definition 1.2.11 ([5])**.** *"The mapping $G : Y \to Y$ such that:*

$$\left\|Ga - Gb\right\|^2 \leq \left\|a - b\right\|^2 + \left\|(\mathbb{I} - G)a - (\mathbb{I} - G)b\right\|^2. \tag{1.10}$$

is called pseudo-contraction. "

Definition 1.2.12 ([5])**.** *"The mapping $G : Y \to Y$ if there exists $m \in [0, 1)$ such that*

$$\|Ga - Gb\|^2 \leq \|a - b\|^2 + m\|(\mathbb{I} - G)a - (\mathbb{I} - G)b\|^2. \tag{1.11}$$

is called strict pseudo contraction."

Definition 1.2.13 ([61])**.** *"The mapping $G : Y \to Y$ if there exist a constant $m \in [0, 1)$ and a sequence $\{\lambda_k\}_{k=1}^{\infty}$ such that*

$$\|G^k a - G^k b\|^2 \leq \lambda_k \|a - b\|^2 + m\|(\mathbb{I} - G^k)a - (\mathbb{I} - G^k)b)\|^2. \tag{1.12}$$

is called asymptotically strict pseudo contraction."

Definition 1.2.14 ([63])**.** *"The mapping $G : Y \to Y$ if there exist a constant $m \in [0, 1)$ and nonnegative real sequences $\{\lambda_k\}_{k=1}^{\infty}$, $\{\mu_k\}_{k=1}^{\infty}$ with $\lim_{k\to\infty} \lambda_k = 0 = \lim_{k\to\infty} \mu_k$ and*

$$\psi : \mathcal{R}^+ \cup \{0\} \to \mathcal{R}^+ \cup \{0\}$$

with

$$\psi(0) = 0 \tag{1.13}$$

satisfying:

$$\|G^k a - G^k b\|^2 \leq \|a-b\|^2 + m\|(\mathbb{I} - G^k)a - (\mathbb{I} - G^k)b)\|^2 + \lambda_k \psi(\|a-b\|) + \mu_k \tag{1.14}$$

for all $a, b \in G$. Where ψ is a strictly increasing continuous function. Then the mapping is called total asymptotically strictly pseudo contraction."

Remark 1.2.15. *It can be seen from above definition that the nonexpansive maps are infact quasi nonexpansive similarly asymptotically nonexpansive maps are infact asymptotically semi quasi nonexpansive. In addition, an asymptotically nonexpansive mapping is consistently Θ-Lipschitzian. In general the reverse formulation is not verified.*

1.3 Equilibrium Problem Theory

In this section, we characterize some essential ideas and thoughts of Equilibrium Problem Theory required in the continuation:

Definition 1.3.1. *"If any vector space $D(G)$ is the domain and $R(G)$ the range is a subset of a vector space over the same field, provided that for a scalar λ and for all $\xi, \eta \in D(G)$*

$$\begin{aligned} G(\xi + \eta) &= G\xi + G\eta \\ G(\lambda\xi) &= \lambda G\xi \end{aligned}$$

then an operator G is called linear."

Definition 1.3.2. *"Let any arbitrary space Y and a linear operator G defined as*

$$G : D(G) \longrightarrow Y$$

where $D(G)$ is a subset of Y. If there is a real number $\mathcal{L}$ such that for all $x \in D(\mathrm{T})$.

$$\|Gx\| \leq \mathcal{L}\,\|x\| \tag{1.15}$$

Then the operator G is said to be BLO."

Definition 1.3.3. *"A Hilbert space $\mathbb{H}$ and a closed as well as convex subset $Y \neq \emptyset$. Let the set of real numbers $\mathbb{R}$ and a functional $\mho$ defined as $\mho : Y \times Y \to \mathbb{R}$. The equilibrium problem (EP) is the evaluation of $\hat{\xi} \in Y$ such that:*

$$\mho(\hat{\xi}, \xi) \geq 0,\ \forall\, \xi \in Y. \tag{1.16}$$

The solutions set of EP is elaborated as $EP(\mho)$. "

Definition 1.3.4 ([48])**.** *"Let the arbitrary sets $Y \neq \emptyset$ $Z \neq \emptyset$ which is a subset of $\mathbb{H}_1$, and $\mathbb{H}_2$ respectively. Let a BLO $A : \mathbb{H}_1 \to \mathbb{H}_2$ be given and two bifunctions $\mho$ and $\eth$ defined as*

$$\mho : Y \times Y \to \mathbb{R}$$

and

$$\eth : Z \times Z \to \mathbb{R}.$$

Then the SEP is to find:

$$\hat{\xi} \in Y \quad \textit{such that}\ \mho(\hat{\xi}, \xi) \geq 0,\ \forall\, \xi \in Y. \tag{1.17}$$

and

$$\eta = A\hat{\xi} \in Z \quad \textit{such that}\ \eth\left(\hat{\xi}, \eta\right) \geq 0. \tag{1.18}$$

for all $\eta \in Z$. while the solution set of SEP is defined as:

$$\Omega = \{\hat{\xi} \in EP(\mho); A\hat{\xi} \in EP(\eth)\}$$

Recently in 2011, Moudafi [48] initiated the split equilibrium problem."

Remark 1.3.5. *It can be easily seen that inequality* (1.16) *represents the classical EP [18] and the set of solution is described as* $EP(f)$*. Moreover, the inequalities* (1.17) *and* (1.18) *consisted of a couple of EPs and are focused to evaluate the solution of* (1.17) *i.e* $\hat{\xi}$ *such that its image* $\eta = A\hat{\xi}$ *also solves further* (1.18) *under action of a bounded linear operator (BLO).*

EPT characterize a extensive area research in mathematical sciences as various problems in operational research, game theory, physics, financial mathematics, optimization, economics and mechanics can be expressed as an equilibrium problem (EP). EP involves variational inequalities problem (VIP), optimization problems and FPP. The evolution of fruitful and executable iterative schemes are fascinating and significant in the theory of variational inequalities and EPs. In past years, some iterative schemes have been investigated to approximate the EP and VIP in Banach spaces and Hilbert spaces. See, for example [9, 13, 22, 23].

1.4 Iterative Algorithms for Nonlinear Maps

Since the nature of FPT is arithmetic. So one can easily characterize the iterative algorithms to estimate the fixed points of different nonlinear problems. One can locate an extensive variety of iterative algorithms in the current writing. We now characterize some iterative plans in the accompanying continuation:

Definition 1.4.1. *"Let* $G : Y \to Y$ *be a mapping and a sequence* $\{x_k\} \in Y$*. Suppose an initial guess is* $x_0 \in Y$ *then:*

$$x_{k+1} = G^k x \tag{1.19}$$

is called Picard's Iterative scheme."

This iterative scheme is quite straightforward and frequently implemented for the approximation of desired solution in various problems from different areas of mathematics especially in fixed point theory. In 1955, Krasnoselskii [42] showed that if G is a nonexpansive mapping and has a unique fixed point then the Picard's iterative scheme is not convergent to the unique fixed point of G.

Definition 1.4.2 ([43]). *"A mapping $G : Y \to Y$. Then a sequence $\{y_{k+1}\}$ defined by :*

$$y_{k+1} = \beta_k y_k + (1 - \beta_k)\, G y_k. \tag{1.20}$$

is said to be Mann's Iterative scheme."

Mann's iterative scheme [43] has been widely utilized and explored in documentation for instance observe [51, 12, 17]. In [10], D. Borwein and J. M. Borwein showed that Mann iteration is not convergent for a $\mathcal{L}$-Lipschitz map (not pseudo contractive), and posses a unique fixed point. While Hicks and Kubicek [32], studied another case for the Mann iteration is not convegent under the action of a discontinuous pseudo constriction having a unique fixed point.

Definition 1.4.3 ([34]). *"In 1974 Ishikawa introduced this scheme. A mapping $G : Y \to Y$. A sequence $\{z_k\}$ defined by*

$$\begin{aligned} x_{k+1} &= (1 - \gamma_k)\, G x_k + \gamma_k z_k \\ z_k &= (1 - \gamma_k)\, G x_k + \gamma_k x_k. \end{aligned} \tag{1.21}$$

is said to be Ishikawa's Iterative scheme."

Remark 1.4.4. *From above definitions, it is clear that if in Ishikawa iteration, $y_0 = z_o$ and $\gamma_k = 0$, $\forall\, n \geq 0$, then we have the Mann iteration and if $\beta_k = 1$, $\forall\, n \geq 0$, and $y_0 = x_o$,then we have Picard iteration. The relation between the convergence of these iterative scheme had been dicussed by many authors.*

Definition 1.4.5 ([52]). *"In 1991 Schu studied the Mann iteration and inintiated the modified Mann iterative scheme. A mapping $G : Y \to Y$. A sequence $\{x_{k+1}\}$ defined by*

$$x_{k+1} = (1 - \beta_k) x_k + \beta_k G^k x_k. \tag{1.22}$$

is said to be Ishikawa's Iterative scheme."

By this iterative scheme, he proved many theorems for convergence of asymptotically non-expansive mappings in a Hilbert space when $0 < \eta \leq \beta_k \leq 1-\eta < 1$ for some $\eta \in (0,1)$ and $\sum\limits_{k=0}^{\infty} \theta_k < +\infty$.

Definition 1.4.6. *"Let Y be a nonempty closed convex subset of a Hilbert space $\mathbb{H}$. A mapping $P_Y : \mathbb{H} \to Y$. If for each $x \in \mathbb{H}_1$, there exists a unique nearest point of Y, denoted by $P_Y x$, such that*

$$\|x - P_Y x\| \leq \|x - y\| \textit{ for all } y \in Y.$$

is known as a metric projection of $\mathbb{H}$ onto Y."

Moreover, P_Y satisfies the characteristics of nonexpansive maps in a Hilbert space and

$$\langle x - P_Y x, P_Y x - y\rangle \geq 0 \text{ for all } x, y \in Y.$$

Remark 1.4.7. *It is remarked that P_Y is firmly nonexpansive mapping from $\mathbb{H}_1$ onto Y, i.e,*

$$\|P_Y x - P_Y y\|^2 \leq \langle x - y, P_Y x - P_Y y \rangle, \textit{ for all } x, y \in Y.$$

Definition 1.4.8 ([53])**.** *"Let $\mathbb{H}_k$ and $\mathbb{W}_k$ be two half-spaces of a Hilbert space $\mathbb{H}$. Then the Iterative Scheme:*

$$\begin{aligned} v_k &\in A y_k\,, \\ \epsilon_k &= v_k + \mu_k (y_k - x_k)\,; \\ \|\epsilon_k\| &\leq \alpha\ \max\{\|v_k\|, \mu_k \|y_k - x_k\|\}\,; \\ \mathbb{H}_k &= \{z \in \mathbb{H}; \langle z - y_k, v_k\rangle \leq 0\}\,; \\ \mathbb{W}_k &= \{z \in \mathbb{H}; \langle x_k - z, x_0 - v_k\rangle \leq 0\}\,; \\ x_{k+1} &= P_{\mathbb{H}_k \cap \mathbb{W}_k} x_0. \end{aligned} \tag{1.23}$$

is called Solodov and Svaiter [53] Hybrid Projection Method scheme."

Haugazeau[31] added the hybrid projection method scheme in the field of mathematical analysis in 1968. Later on, Solodov and Svaiter [53] used maximal monotone operators approximant and initiated a strong convergence result in 2000. Indeed, Solodov and Svaiter utilized projection method and classical proximal point scheme and initiated strong convergence in the settings of intersection of two half-spaces $\mathbb{H}_k$ and $\mathbb{W}_k$ containing the solution set.

Definition 1.4.9 ([49]). *"Let $\mathbb{H}_k$ and $\mathbb{W}_k$ be two half-spaces of a Hilbert space $\mathbb{H}$. Then the following Iterative Scheme:*

$$\begin{aligned}
\textit{chose arbitrarily}, x_0 &\in Y \\
y_k &= \beta_k x_k + (1-\beta_k)Gx_k, \\
\mathbb{C}_k &= \{z \in Y : \|y_k - z\| \leq \|x_k - z\|\}, \\
\mathbb{Q}_k &= z \in Z : \langle x_k - z, x - x_k \rangle \geq 0, \\
x_{k+1} &= P_{\mathbb{C}_k \cap \mathbb{Q}_k} x_0. \qquad (1.24)
\end{aligned}$$

is said to be $\mathbb{CQ}$-Hybrid Method Iterative scheme."

Nakajo and Takahashi [49] presented the $\mathbb{CQ}$-Hybrid method using nonexpansive maps motivated by the fundamental work of half spaces by Solodov and Svaiter [53] in Hilbert spaces. They demonstrated that $\mathbb{CQ}$-Hybrid method is strongly convergent to $P_{F(G)}x$ under few suitable conditions. $\mathbb{CQ}$-Hybrid Method Iterative scheme (1.26) is additionally recognized as the Mann $\mathbb{CQ}$-method in light of the fact that, at each progression, the Mann conspire is utilized to assemble the set $\mathbb{C}_k$ and $\mathbb{Q}_k$ which are in turm used to set up the next iteration of x_{k+1} and henceforth the name.

Definition 1.4.10 ([20]). *"Let $\{\mathbb{Y}_{k+1}\}$ be a closed and convex subset of $\mathbb{H}$, then the* *following iterative scheme:*

$$\begin{aligned} x_0 &\in Y \textit{ chosen arbitrarily where } Y_0 = Y, \\ y_k &= \beta_k x_k + (1-\beta_k)Gx_k, \\ \mathbb{Y}_{k+1} &= \{z \in \mathbb{Y}_k : \|z - y_k\| \leq \|z - x_k\|\}, \\ x_{k+1} &= P_{\mathbb{Y}_{k+1}} x_0. \end{aligned} \tag{1.25}$$

is called often called Shrinking Projection Method or Relaxed $\mathbb{CQ}$*-Hybrid Method Iterative scheme by shrinking effect."*

Later on , Takahashi et al. [50] initiated the hybrid scheme to establish a strong convergence result utilizing the shrinking effect in 2008, and in 2011 Dong et al.[20], introduced a shrinking projection method like (1.25) under the action of nonexpansive mappings in a Hilbert space. Indeed, they initiated a strong convergence result by using shrinking effect of one of the half-spaces as characterized by (1.24), in particular $\mathbb{Q}_k$, see too [36].

1.5 Motivation and Problem Statement

In this section, we'll discuss the Motivation of our work and in the end we'll eleborate the statement of our Problem.

Motivation:

In the literature, results on split equilibrium problems are commonly in Hilbert spaces and are depended on a premier facts of the operator norm. Consequently , It will be crucial to establish the iterative schemes for solving split equilibrium problems which do not require premier facts of operator norm. Furthermore, there is also the need to obtain iterative approximation of split equilibrium problem or any of its generalizations in Hilbert spaces.

Motivated by the previously mentioned outcomes and the continuing research towards this path, we aim to utilize a hybrid shrinking projection algorithm to locate a common component in the set of solutions of a finite family of SEP with the set of common fixed points of a finite family of total asymptotically strict pseudo contractions in Hilbert spaces. These outcomes can be seen as a generalization of different existing outcomes in the present writing.

Problem Statement:

The main motive of this thesis is focused to initiate the study of iterative algorithms to approximate the collective element in the set of solutions for the class of FP together with the fixed points of a family of a nonlinear mapping using hybrid shrinking projection in Hilbert spaces. Our conclusions can be seen as the generalization and advancement of particular current conclusions in the literature. The main motive behind this work is enhancement in the field of MFPT of nonlinear maps, for example, the class of nonexpansive maps, the class of asymptotically nonexpansive maps as well as the class of total asymptotically nonexpansive maps by means of numerous levels of the iterative calculations in the settings of uniformly convex Hilbert spaces.

1.6 Objectives of the Study

The fundamental motive of this study is to state and prove an iterative algorithm approximant to initiate a strong convergence results for a common solution of a FPP, SEP the settings of real Hilbert spaces utilizing Hybrid projection method scheme and the Cesàro Mean Iterative scheme. and to broaden ongoing research on SEP in the settings of Hilbert spaces.

1.7 Arrangements of the Thesis

Rest of the thesis is sorted out as following:

In Chapter 2, we will exhibit the strong convergence results using iterative algorithms formulated by the shrinking projection method for the SEP and FPP in Hilbert spaces.

In Chapter 3, we will exhibit the strong convergence results using iterative algorithms formulated by the shrinking projection method utilizing the Cesáro mean algorithm for the SEP and FPP in Hilbert spaces.

Chapter 2

Convergence Analysis of Shrinking Projection Method

This chapter is focused to initiate the strong convergence results of an iterative algorithm generated by the shrinking projection method in Hilbert spaces. The proposed approximant sequence is used to find a common element in the set of solutions of a finite family of SEP and the set of common fixed points of a finite family of total asymptotically strict pseudo contractions in such setting.

EPT gives a broader way to deal with the variety of scientific issues. The theory of the class of split variational inequality problems (SVIP) was iniatiated by Censor et al. [16] in 2012, whereas Moudafi [48] generalized the concept of SVIP to that of split monotone variational inclusions (SMVIP). The SEP is a special case of SMVIP. The SMVIP have now been considered and effectively utilized in medical field, for example [14, 15].

In 2013, Chang et al. [21] studied the split feasibility problem for a total asymptotically strict pseudo contraction in Hilbert spaces. In 2015, Ma and Wang [44] solved the the split common FPP under the action of a total asymptotically strict pseudo contractions to initiate the strong convergence results Hilbert space. Quite recently, numerous methods have been proposed and analyzed in [37, 38, 56] for the SEP.

Inspired and motivated by the on going research in this field described above, we aim to study SEP under the action of shrinking projection method in the settings of Hilbert spaces.

2.1 Preliminaries

In this section, we recall some lemmas and conditions required in the sequel.

The following lemmas are the collection of some well-known equations in the settings of a real Hilbert space.

Lemma 2.1.1. *"Let $\mathbb{H}_1$ be a real Hilbert space, then:*

$$(i)\ \|\xi-\eta\|^2 = \|\xi x\|^2 - \|\eta\|^2 - 2\langle \xi-\eta, \eta\rangle \text{ for all } \xi,\eta \in \mathbb{H}_1$$
$$(ii)\ \|\xi+\eta\|^2 \le \|\xi\|^2 + 2\langle \xi-\eta, \eta\rangle \text{ for all } \xi,\eta \in \mathbb{H}_1$$
$$(iii)\ \|\delta\xi + (1-\delta)\eta\|^2 = \delta\|\xi\|^2 + (1-\delta)\|\eta\|^2 - \delta(1-\delta)\|\xi-\eta\|^2$$

for all $\xi,\eta \in \mathbb{H}_1$ and $\alpha \in \mathbb{R}$."

Condition 2.1.2 ([9, 18])**.** *"Let $\mho : Y \times Y \to \mathbb{R}$ be a bifunction satisfying the following conditions:*

A1. $\mho(\xi,\xi) = 0$ for all $\xi \in Y$;

A2. $\mho$ is monotone, that is, $\mho(\xi,\eta) + \mho(\eta,\xi) \le 0$ for all $\xi,\eta \in Y$;

A3. $\mho$ is upper hemicontinuous, that is, for each $\xi,\eta,\zeta \in Y$,

$$\lim_{a\to 0} \mho(a\zeta + (1-a)\xi, \eta) \le \mho(x,y);$$

A4. for each $\xi \in Y$, the function $\eta \mapsto \mho(\xi,\eta)$ is convex and lower semi-continuous."

Lemma 2.1.3. *[19]. "Let Y be a closed convex subset of a real Hilbert space $\mathbb{H}_1$ and let $\mho : Y \times Y \to \mathbb{R}$ be a bifunction satisfying Condition 2.1.2 For $r > 0$ and $\xi \in \mathbb{H}_1$, there exists $\zeta \in Y$ such that*

$$\mho(\zeta,\eta) + \frac{1}{r}\langle \eta-\zeta, \zeta-\xi\rangle \ge 0, \text{ for all } \eta \in Y.$$

Moreover, define a mapping $\mathcal{T}_r^{\mho} : \mathbb{H}_1 \to Y$ by

$$\mathcal{T}_r^{\mho}(\xi) = \left\{ \zeta \in Y : \mho(\zeta,\eta) + \frac{1}{r}\langle \eta-\zeta, \zeta-\xi\rangle \ge 0, \text{ for all } \eta \in Y \right\},$$

for all $\xi \in \mathbb{H}_1$. Then, the following hold:

(i) $\mathcal{T}_r^{\mho}$ is single-valued;

(ii) $\mathcal{T}_r^{\mho}$ is firmly nonexpansive, i.e., for every $\xi,\eta \in \mathbb{H}$,

$$\left\|\mathcal{T}_r^{\mho}\xi - \mathcal{T}_r^{\mho}\eta\right\|^2 \le \left\langle \mathcal{T}_r^{\mho}\xi - \mathcal{T}_r^{\mho}\eta, \xi-\eta \right\rangle$$

(iii) $\mho(\mathcal{T}_r^{\mho}) = EP(\mho)$;

(iv) $EP(\mho)$ is closed and convex."

Remark 2.1.4. *"It is remarked that if $\eth : Y \times Y \to \mathbb{R}$ is a bifunction satisfying Condition 2.1.2, then for $s > 0$ and $\beta \in \mathbb{H}_2$ we can define a mapping:*

$$\mathcal{T}_s^{\eth}(\beta) = \left\{ \alpha \in Y : \eth(\alpha, e) + \frac{1}{s}\langle e - \alpha, \alpha - \beta \rangle \geq 0, \text{ for all } e \in Y \right\},$$

which is, nonempty, single-valued and firmly nonexpansive. Moreover, $EP(\eth)$ is closed and convex, and $F(\mathcal{T}_s^{\eth}) = EP(\eth)$."

Lemma 2.1.5. *[47]."Let $G : Y \to Y$ be a mapping. If $\mathbb{F}(G) \neq \emptyset$, then for each $p \in \mathbb{F}(G)$ and for each $x \in Y$, the following equivalent inequalities hold:*

$$\left\|G^k\xi - p\right\|^2 \leq \|\xi - p\|^2 + \| \left\|\xi - G^k\xi\right\|^2 + \lambda_k\xi(\|\xi - p\|) + \mu_k, \tag{2.1}$$

$$\left\langle \xi - G^k\xi, \xi - p \right\rangle \geq \frac{1 - \|}{2} \left\|\xi - G^k\xi\right\|^2 - \frac{\lambda_k}{2}\xi(\|\xi - p\|) - \frac{\mu_k}{2}, \tag{2.2}$$

$$\left\langle \xi - G^k\xi, p - G^k\xi \right\rangle \leq \frac{1 + \|}{2} \left\|\xi - G^k\xi\right\|^2 + \frac{\lambda_k}{2}\xi(\|\xi - p\|) + \frac{\mu_k}{2}. \tag{2.3}$$

where G is a $(\mathcal{K}, \{\lambda_k\}, \{\mu_k\}, \xi)$-total asymptotically strictly pseudo contraction. "

Lemma 2.1.6. *[47]. "Let Y be a nonempty subset of a real Hilbert space $\mathbb{H}_1$ and let $G : Y \to Y$ be a uniformly $\mathcal{L}$-Lipschitzian and $(\mathcal{K}, \{\lambda_k\}, \{\mu_k\}, \xi)$-total asymptotically strictly pseudo contraction, then G is demiclosed at origin. That is, if for any sequence $\{\xi_k\}$ in Y with $\xi_k \rightharpoonup \xi$ and $\|\xi_k - G\xi_k\| \to 0$, we have $\xi = G\xi$. "*

2.2 Main Result

We now prove our main result of this section.

Theorem 2.2.1. *Let $\mathbb{H}_1$ and $\mathbb{H}_2$ be two real Hilbert spaces and let $Y \subseteq \mathbb{H}_1$ and $Z \subseteq \mathbb{H}_2$ be nonempty closed convex subsets of Hilbert spaces $\mathbb{H}_1$ and $\mathbb{H}_2$, respectively. Let $\mho_i : Y \times Y \to \mathbb{R}$ and $\eth_i : Z \times Z \to \mathbb{R}$ be two finite families of bifunctions satisfying Condition 2.1.2 such that $\eth_i$ be upper semicontinuous for each $i \in \{1, 2, 3, \cdots, K\}$. Let $G_i : Y \to Y$ be a finite family of uniformly $\mathcal{L}$-Lipschitzian and continuous total asymptotically strict pseudo contractions and let $A_i : \mathbb{H}_1 \to \mathbb{H}_2$ be a finite family of bounded* *linear operators for each* $i \in \{1,2,3,\cdots,K\}$*. Suppose that* $\mathbb{F} := \left[\bigcap_{i=1}^{K} \mathbb{F}(G_i)\right] \cap \Omega \neq \emptyset$*, where* $\Omega = \left\{ z \in : z \in \bigcap_{i=1}^{K} EP(\mho_i) \text{ and } A_i z \in EP(\eth_i) \text{ for } 1 \leq i \leq K \right\}$*. Let* $\{x_k\}$ *be a sequence generated by:*

$$\begin{aligned}
&x_1 \in Y_1 = Y,\\
&u_k = \mathcal{T}_{r_k}^{\mho_k}\left(x_k - \gamma A_{(k \mod K)}^{*}\left(\mathbb{I} - \mathcal{T}_{s_k}^{\eth_k}\right) A_{(k \mod K)} x_n\right),\\
&y_k = \alpha_k u_k + (1-\alpha_k)\, G_{(k \mod K)}^{k} u_k,\\
&Y_{k+1} = \left\{ z \in \mathbb{H}_1 : \|y_k - z\|^2 \leq \|x_k - z\|^2 + \theta_k \right\},\\
&x_{k+1} = P_{C_{k+1}} x_1, \quad k \geq 1,
\end{aligned} \tag{2.4}$$

where $\theta_k = (1-\alpha_k)\{\lambda_k \xi_k(M_k) + \lambda_k M_k^* D_k + \mu_k\}$ *with* $D_k = \sup\{\|x_k - p\| : p \in \mathbb{F}\}$*. Let* $\{r_k\}, \{s_k\}$ *be two positive real sequences and let* $\{\alpha_k\}$ *be in* $(0,1)$*. Assume that if the following set of conditions holds:*

(C1): $0 \leq \mathcal{K} < \alpha_k < 1$ *and* $\gamma \in \left(0, \frac{1}{L}\right)$ *where* $L = \max\{L_i\}_{i=1}^{K}$ *and* L_i *denotes the spectral radius of* $A_i^* A_i$ *such that* A_i^* *is the adjoint of* A_i *for each* $i \in \{1,2,3,\cdots,K\}$*;*

(C2): $\liminf_{k\to\infty} r_k > 0$ *and* $\liminf_{k\to\infty} s_k > 0$*;*

(C3): $\sum_{k=1}^{\infty} \lambda_k < \infty$ *and* $\sum_{k=1}^{\infty} \mu_k < \infty$*;*

(C4): there exist constants $M_i,\ M_i^* > 0$ *such that* $\xi_i(\lambda_i) \leq M_i^* \lambda_i$ *for all* $\lambda_i \geq M_i, i = 1,2,3,\cdots,K$*, then the sequence* $\{x_k\}$ *generated by (2.4) converges strongly to* $P_{\mathbb{F}} x_1$*.*

Proof : For the sake of simplicity, we define $A_k = A_{(k \mod K)}$ and $G_k = G_{(k \mod K)}$ for all $k \geq 1$. We start our proof to establish that the sequence $\{x_k\}$ defined in (2.4) is well defined. In order to prove this assertion, we first show by mathematical induction that $\mathbb{F} \subset Y_k$ for all $k \geq 1$. Obviously, $\mathbb{F} \subset Y_1 = Y$.

Now, assume that $\mathbb{F} \subset Y_i$ for some $i \geq 1$. Then it follows from (2.4) that

$$\begin{aligned}
\|u_i - p\|^2 &= \left\|\mathcal{T}_{r_i}^{\mho_i}\left(x_i - \gamma A_i^*\left(\mathbb{I} - \mathcal{T}_{s_i}^{\eth_i}\right)A_i x_i\right) - \mathcal{T}_{r_i}^{\mho_i} p\right\|^2 \\
&\leq \left\|x_i - \gamma A_i^*\left(\mathbb{I} - \mathcal{T}_{s_i}^{\eth_i}\right)A_i x_i - p\right\|^2 \\
&\leq \|x_i - p\|^2 + \gamma^2\left\|A_i^*\left(\mathbb{I} - \mathcal{T}_{s_i}^{\eth_i}\right)A_i x_i\right\|^2 + 2\gamma\left\langle p - x_i, A_i^*\left(\mathbb{I} - \mathcal{T}_{s_i}^{\eth_i}\right)A_i x_i\right\rangle \\
&\leq \|x_i - p\|^2 + \gamma^2\left\langle A_i x_i - \mathcal{T}_{s_i}^{\eth_i} A_i x_i, A_i A_i^*\left(\mathbb{I} - \mathcal{T}_{s_i}^{\eth_i}\right)A_i x_i\right\rangle \\
&\quad + 2\gamma\left\langle p - x_i, A_i^*\left(\mathbb{I} - \mathcal{T}_{s_i}^{\eth_i}\right)A_i x_i\right\rangle \\
&\leq \|x_i - p\|^2 + L\gamma^2\left\langle A_i x_i - \mathcal{T}_{s_i}^{\eth_i} A_i x_i, A_i x_i - \mathcal{T}_{s_i}^{\eth_i} A_i x_i\right\rangle \\
&\quad + 2\gamma\left\langle p - x_i, A_i^*\left(\mathbb{I} - \mathcal{T}_{s_i}^{\eth_i}\right)A_i x_i\right\rangle \\
&= \|x_i - p\|^2 + L\gamma^2\left\|A_i x_i - \mathcal{T}_{s_i}^{\eth_i} A_i x_i\right\|^2 + 2\gamma\left\langle p - x_i, A_i^*\left(\mathbb{I} - \mathcal{T}_{s_i}^{\eth_i}\right)A_i x_i\right\rangle .
\end{aligned} \tag{2.5}$$

Now letting $\Lambda = 2\gamma\left\langle p - x_i, A_i^*\left(\mathbb{I} - \mathcal{T}_{s_i}^{\eth_i}\right)A_i x_n\right\rangle$, we have

$$\begin{aligned}
\Lambda &= 2\gamma\left\langle p - x_i, A_i^*\left(\mathbb{I} - \mathcal{T}_{s_i}^{\eth_i}\right)A_i x_i\right\rangle \\
&= 2\gamma\left\langle A_i(p - x_i), A_i x_i - \mathcal{T}_{s_i}^{\eth_i} A_i x_i\right\rangle \\
&= 2\gamma\left\langle A_i(p - x_i) + \left(A_i x_i - \mathcal{T}_{s_i}^{\eth_i} A_i x_i\right) - \left(A_i x_i - \mathcal{T}_{s_i}^{\eth_i} A_i x_i\right), A_i x_i - \mathcal{T}_{s_i}^{\eth_i} A_i x_i\right\rangle \\
&= 2\gamma\left\{\left\langle A_i p - \mathcal{T}_{s_i}^{\eth_i} A_i x_i, A_i x_i - \mathcal{T}_{s_i}^{\eth_i} A_i x_i\right\rangle - \left\|A_i x_i - \mathcal{T}_{s_i}^{\eth_i} A_i x_i\right\|^2\right\} \\
&\leq 2\gamma\left\{\frac{1}{2}\left\|A_i x_i - \mathcal{T}_{s_i}^{\eth_i} A_i x_i\right\|^2 - \left\|A_i x_i - \mathcal{T}_{s_i}^{\eth_i} A_i x_i\right\|^2\right\} \\
&= -\gamma\left\|A_i x_i - \mathcal{T}_{s_i}^{\eth_i} A_i x_i\right\|^2 .
\end{aligned}$$

Using the above simplification of Λ in (2.5), we get

$$\|u_i - p\|^2 \leq \|x_i - p\|^2 + \gamma(L\gamma - 1)\left\|A_i x_i - \mathcal{T}_{s_{n,i}}^{\eth_i} A_i x_i\right\|^2 . \tag{2.6}$$

Since $\gamma \in \left(0, \frac{1}{L}\right)$ by condition (2.1.2) (C1), the above estimate then yields

$$\|u_i - p\|^2 \leq \|x_i - p\|^2 . \tag{2.7}$$

Making use of (2.7), we have the following estimate:

$$
\begin{aligned}
\|y_i - p\|^2 &= \left\|\alpha_i u_i + (1-\alpha_i) G_i^i u_i - p\right\|^2 \\
&= \alpha_i \|u_i - p\|^2 + (1-\alpha_i) \left\|G_i^i u_i - p\right\|^2 - \alpha_i (1-\alpha_i) \left\|u_i - G_i^i u_i\right\|^2 \\
&\leq \alpha_i \|u_i - p\|^2 + (1-\alpha_i) \left\{\|u_i - p\|^2 + \mathcal{K} \left\|u_i - G_i^i u_i\right\|^2 + \lambda_i \xi_i(\|u_i - p\|) + \mu_i\right\} \\
&\quad -\alpha_i (1-\alpha_i) \left\|u_i - G_i^i u_i\right\|^2 \\
&\leq \|u_i - p\|^2 + (1-\alpha_i) \left\{\mathcal{K} \left\|u_i - G_i^i u_i\right\|^2 + \lambda_i \xi_i (M_i) + \lambda_i M_i^* \|u_i - p\|^2) + \mu_i\right\} \\
&\quad -\alpha_i (1-\alpha_i) \left\|u_i - G_i^i u_i\right\|^2 \\
&\leq \|x_i - p\|^2 - (1-\alpha_i)(\alpha_i - \mathcal{K}) \left\|u_i - G_i^i u_i\right\|^2 \\
&\quad + (1-\alpha_i) \left\{\lambda_i \xi_i (M_i) + \lambda_i M_i^* \|x_i - p\|^2 + \mu_i\right\}. \qquad (2.8)
\end{aligned}
$$

Since $\alpha_i - \mathcal{K} \geq 0$ by condition 2.1.2 (C1), so (2.8) implies that

$$\|y_i - p\|^2 \leq \|x_i - p\|^2 + \theta_i, \qquad (2.9)$$

where $\theta_i = (1-\alpha_i)\{\lambda_i \xi_i(M_i) + \lambda_i M_i^* D_i + \mu_i\}$ and $D_i = \sup\left\{\|x_i - p\|^2 : p \in \mathbb{F}\right\}$.
It now follows from the estimate (2.9) that $p \in Y_{i+1}$. Hence, $\mathbb{F} \subset Y_k$ for all $k \geq 1$.
Next, we show that the set Y_k is closed and convex for all $k \geq 1$. Since

$$\left\{z \in \mathbb{H}_1 : \|y_k - z\|^2 \leq \|x_k - z\|^2\right\} = \left\{z \in \mathbb{H}_1 : \|y_k\|^2 - \|x_k\|^2 \leq 2\langle y_k - x_k, z\rangle + \theta_k\right\},$$

it is closed and convex; hence the sequence $\{x_k\}$ defined in (2.4) is well-defined.
Next, from $x_k = P_{Y_k} x_1$, we get

$$0 \leq \langle x_k - x_1, x^* - x_k\rangle,$$

for all $x^* \in Y_k$. Since $\mathbb{F} \neq \emptyset$, then for any $p \in \mathbb{F}$, we get

$$
\begin{aligned}
0 &\leq \langle x_k - x_1, p - x_k\rangle \\
&= \langle x_k - x_1, p + x_1 - x_1 - x_k\rangle \\
&= \langle x_k - x_1, x_1 - x_k\rangle + \langle x_k - x_1, p - x_1\rangle \\
&= -\|x_k - x_1\|^2 + \|x_k - x_1\| \|p - x_1\|.
\end{aligned}
$$

That is,

$$\|x_n - x_1\| \leq \|p - x_1\|\,, \text{ for all } p \in \mathbb{F} \text{ and } k \geq 1.$$

Hence, the sequence $\{x_k\}$ is bounded, so are $\{u_k\}$ and $\{y_k\}$. Moreover, from $x_k = P_{Y_k}x_1$ and $x_{k+1} = P_{Y_{k+1}}x_1 \in Y_{k+1} \subset Y_k$, we have

$$0 \leq \langle x_k - x_1, x_{k+1} - x_k \rangle\,.$$

Similarly, we get the following relation:

$$\|x_k - x_1\| \leq \|x_{k+1} - x_1\|\,, \text{ for all } k \geq 1.$$

From the above assertions, we conclude that the sequence $\{\|x_k - x_1\|\}$ is bounded and nondecreasing, therefore, we have

$$\lim_{k\to\infty} \|x_k - x_1\| \text{ exists.} \tag{2.10}$$

Further observe that

$$\begin{aligned}
\|x_{k+1} - x_k\|^2 &= \|x_{k+1} - x_1 + x_1 - x_k\|^2 \\
&= \|x_{k+1} - x_1\|^2 + \|x_k - x_1\|^2 - 2\langle x_k - x_1, x_{k+1} - x_1 \rangle \\
&= \|x_{k+1} - x_1\|^2 + \|x_k - x_1\|^2 - 2\langle x_k - x_1, x_{k+1} - x_n + x_k - x_1 \rangle \\
&= \|x_{k+1} - x_1\|^2 - \|x_k - x_1\|^2 - 2\langle x_k - x_1, x_{k+1} - x_k \rangle \\
&\leq \|x_{k+1} - x_1\|^2 - \|x_k - x_1\|^2\,.
\end{aligned}$$

From (2.10), we obtain that $\|x_{k+1} - x_1\|^2 - \|x_k - x_1\|^2 \to 0$ as $k \to \infty$, hence the sequence $\{\|x_k - x_1\|\}$ is Cauchy. That is

$$\lim_{k\to\infty} \|x_{k+1} - x_k\| = 0. \tag{2.11}$$

Since $x_{k+1} \in Y_{k+1}$, which implies that $\|y_k - x_{k+1}\| \leq \|x_k - x_{k+1}\| + \theta_{k,i}$. From (2.11), we conclude that

$$\lim_{k\to\infty} \|y_k - x_{k+1}\| = 0, \text{ for all } k \geq 1. \tag{2.12}$$

Now using (2.11), (2.12) and the following triangular inequality, we get

$$\|y_k - x_k\| \le \|y_k - x_{k+1}\| + \|x_{k+1} - x_k\| \to 0 \tag{2.13}$$

as $k \to \infty$.

Consider from (2.4), (2.6) and (2.9), we get

$$\begin{aligned}
\gamma(1-\gamma L)\left\|A_k x_n - \mathcal{T}_{s_n}^{\eth_n} A_k x_k\right\|^2 &\le \|x_k - p\|^2 - \|u_k - p\|^2 \\
&\le \|x_k - p\|^2 - \|y_k - p\|^2 + \theta_k \\
&\le (\|x_k - p\| + \|y_k - p\|)\|x_k - y_k\| + \theta_k.
\end{aligned}$$

Utilizing the fact that $\gamma(1-\gamma L) > 0$ and the estimate (2.13), we have

$$\lim_{k\to\infty} \left\|A_k x_k - \mathcal{T}_{s_k}^{g_k} A_k x_k\right\|^2 = 0 \text{ for all } k \ge 1. \tag{2.14}$$

For any $p \in \mathbb{F}$ and firm nonexpansiveness of $\mathcal{T}_{r_k}^{f_k}$, we have

$$\begin{aligned}
\|u_k - p\|^2 &= \left\|\mathcal{T}_{r_k}^{\mho_k}\left(x_k - \gamma A_k^*\left(\mathbb{I} - \mathbb{T}_{s_k}^{\eth_k}\right)A_k x_k\right) - \mathcal{T}_{r_k}^{\mho_k} p\right\|^2 \\
&\le \left\langle u_k - p, x_k - \gamma A_k^*\left(\mathbb{I} - \mathcal{T}_{s_k}^{\eth_k}\right)A_k x_k - p\right\rangle \\
&= \frac{1}{2}\{\|u_k - p\|^2 + \left\|x_k - \gamma A_k^*\left(\mathbb{I} - \mathcal{T}_{s_k}^{\eth_k}\right)A_k x_k - p\right\|^2 \\
&\quad - \left\|u_k - x_k - \gamma A_k^*\left(\mathbb{I} - \mathcal{T}_{s_k}^{\eth_k}\right)A_k x_k\right\|^2\} \\
&\le \frac{1}{2}\left\{\|u_k - p\|^2 + \|x_k - p\|^2 - \left\|u_k - x_k - \gamma A_k^*\left(\mathbb{I} - \mathcal{T}_{s_k}^{\eth_k}\right)A_k x_k\right\|^2\right\} \\
&= \frac{1}{2}\{\|u_k - p\|^2 + \|x_k - p\|^2 - (\|u_k - x_k\|^2 + \gamma^2\left\|A_k^*\left(\mathbb{I} - \mathcal{T}_{s_k}^{\eth_k}\right)A_k x_k\right\|^2 \\
&\quad -2\gamma\left\langle u_k - x_k, A_k^*\left(\mathbb{I} - \mathcal{T}_{s_k}^{\eth_k}\right)A_k x_k\right\rangle)\} \\
&\le \|x_k - p\|^2 - \|u_k - x_k\|^2 + 2\gamma\|A(u_k - x_k)\|\left\|A_k x_k - \mathcal{T}_{s_k}^{\eth_k} A_k x_k\right\|.
\end{aligned} \tag{2.15}$$

Using the following estimate:

$$\|y_n - p\|^2 \le \alpha_n\|x_k - p\|^2 + (1-\alpha_k)\|u_k - p\|^2 + \theta_k.$$

in (2.15) and re-arranging the terms, we get

$$\begin{aligned}
(1-\alpha_k)\|u_k - x_k\|^2 &\le (\|x_k - p\| + \|y_k - p\|)\|x_k - y_k\| \\
&\quad +2\gamma\|A(u_k - x_k)\|\left\|A_k x_k - \mathcal{T}_{s_k}^{\eth_k} A_k x_k\right\| + \theta_k.
\end{aligned}$$

Letting $k \to \infty$ and utilizing (2.13) and (2.14), we have

$$\lim_{k\to\infty} \|u_k - x_k\| = 0 \text{ for all } k \geq 1. \tag{2.16}$$

Moreover, from (2.13) and (2.16), we obtain

$$\|y_k - u_k\| \leq \|y_k - x_k\| + \|x_k - u_k\| \to 0 \tag{2.17}$$

when $k \to \infty$. Observe that $\|y_k - u_k\| = (1-\alpha_k)\left\|G_k^k u_k - u_k\right\|$. Then it follows from condition (C1) and (2.17) that

$$\lim_{k\to\infty} \left\|G_k^k u_k - u_k\right\| = 0 \text{ for all } k \geq 1. \tag{2.18}$$

Since

$$\left\|G_k^k u_k - x_k\right\| \leq \left\|G_k^k u_k - u_k\right\| + \|x_k - u_k\|.$$

Therefore from (2.16) and (2.18), we get

$$\lim_{k\to\infty} \left\|G_k^k u_k - x_k\right\| = 0 \text{ for all } k \geq 1. \tag{2.19}$$

On a similar reasoning, we also obtain

$$\lim_{k\to\infty} \left\|G_k^k u_k - y_k\right\| = 0 \text{ for all } k \geq 1.$$

Observe that each G_k is uniformly $\mathcal{L}$-Lipschitzian, therefore, we have

$$\begin{aligned}
\left\|G_k^k x_k - x_k\right\| &\leq \left\|G_k^k x_k - G_k^k u_k\right\| + \left\|G_k^k u_k - x_k\right\| \\
&\leq \mathcal{L}\|x_k - u_k\| + \left\|G_k^k u_k - x_k\right\|.
\end{aligned}$$

Now, using (2.16), (2.19) and the above estimate, we get

$$\lim_{k\to\infty} \left\|G_k^k x_k - x_k\right\| = 0 \text{ for all } k \geq 1. \tag{2.20}$$

Moreover, utilizing the uniform continuity of G_k and (2.20) the following estimate:

$$\|x_k - G_k x_k\| \leq \left\|x_k - G_k^k x_k\right\| + \left\|G_k^k x_k - G_k x_k\right\|$$

implies that

$$\lim_{k\to\infty} \|x_k - G_k x_k\| = 0 \text{ for all } k \geq 1. \tag{2.21}$$

Similarly, we also have that

$$\lim_{k\to\infty} \|u_k - G_k u_k\| = 0 \text{ for all } k \geq 1. \tag{2.22}$$

Now, we show that $\omega(x_k) \subset \mathbb{F}$, where $\omega(x_k)$ is the set of all weak ω-limits of $\{x_k\}$. Since $\{x_k\}$ is bounded, therefore $\omega(x_k) \neq \emptyset$. Let $q \in \omega(x_k)$, then there exists a subsequence $\{x_{Kk+i}\}$ of $\{x_k\}$ such that $x_{Kk+i} \rightharpoonup q$. Using the fact that $S_{Kk+i} = G_i$ for all $k \geq 1$ and the semiclosed principle (Lemma 2.1.3) for each G_i, we have that $x \in \mathbb{F}(G_i)$ for each $1 \leq i \leq K$. Next, we show that $q \in \Omega$, i.e., $q \in \bigcap_{i=1}^{K} EP(\mho_i)$ and $A_i q \in EP(\eth_i)$ for each $1 \leq i \leq K$. In order to show that $q \in \bigcap_{i=1}^{K} EP(\mho_i)$, that is, $q \in EP(\mho_i)$ for each $1 \leq i \leq K$, we define subsequence $\{k_j\}$ of index $\{k\}$ such that $k_j = Kj + i$ for all $n \geq 1$. As a consequence, we can write $\mho_{k_j} = \mho_i$ for $1 \leq i \leq K$. From $u_{k_j} = \mathcal{T}_{r_{k_j}}^{\mho_i}\left(\mathbb{I} - \gamma A_{k_j}^{*}\left(\mathbb{I} - \mathcal{T}_{s_{k_j}}^{\eth_{k_j}}\right) A_{k_j}\right) x_{k_j}$ for all $k \geq 1$, we have

$$\mho_i(u_{k_j}, y) + \frac{1}{r_{k_j}}\left\langle y - u_{k_j}, u_{k_j} - x_{k_j} - \gamma A_{k_j}^{*}\left(\mathbb{I} - \mathcal{T}_{s_{k_j}}^{\eth_{k_j}}\right) A_{k_j} x_{k_j}\right\rangle \geq 0, \quad \text{for all } y \in Y.$$

This implies that

$$\mho_i(u_{k_j}, y) + \frac{1}{r_{k_j}}\left\langle y - u_{k_j}, u_{k_j} - x_{k_j}\right\rangle - \frac{1}{r_{k_j}}\left\langle y - u_{k_j}, \gamma A_{k_j}^{*}\left(\mathbb{I} - \mathcal{T}_{s_{k_j}}^{\eth_{k_j}}\right) A_{k_j} x_{k_j}\right\rangle \geq 0$$

From (A2), we have

$$\frac{1}{r_{k_j}}\left\langle y - u_{k_j}, u_{k_j} - x_{k_j}\right\rangle - \frac{1}{r_{k_j}}\left\langle y - u_{k_j}, \gamma A_{k_j}^{*}\left(\mathbb{I} - \mathcal{T}_{s_{k_j}}^{\eth_{k_j}}\right) A_{k_j} x_{k_j}\right\rangle \geq \mho_i(y, u_{k_j}),$$

for all $y \in Y$. Since $\liminf_{j\to\infty} r_{k_i} > 0$ (by (C2)), therefore it follows from (2.14) and (2.16) that

$$\mho_i(y, q) \leq 0, \quad \text{for all } y \in Y \text{ and for } 1 \leq i \leq K.$$

Let $y_t = ty + (1-t)q$ for some $0 < t < 1$ and $y \in Y$. Since $q \in Y$, this implies that $y_t \in Y$. Using (A1) and (A4) from Condition 2.1.2, the following estimate:

$$0 = \mho_i(y_t, y_t) \leq t\mho_i(y_t, y) + (1-t)\mho_i(y_t, q) \leq t\mho_i(y_t, y),$$

implies that

$$\mho_i(y_t, y) \geq 0, \text{ for } 1 \leq i \leq K.$$

Letting $t \to 0$, we have $\mho_i(q, y) \geq 0$ for all $y \in Y$. Thus, $q \in EP(\mho_i)$ for $1 \leq i \leq K$. That is, $q \in \bigcap_{i=1}^{K} EP(\mathbb{F}_i)$. Reasoning as above, we show that $A_i q \in EP(\eth_i)$ for each $1 \leq i \leq K$. Since $x_{k_l} \longrightarrow q$ and A_{k_l} is a bounded linear operator, therefore $A_{k_l} x_{k_l} \longrightarrow A_{k_l} q$. Hence, it follows from (2.14) that

$$\mathcal{T}_{s_{k_l}}^{\eth_{k_l}} A_{k_l} x_{k_l} \longrightarrow A_{k_l} q \quad \text{as} \quad l \to \infty.$$

Now, from Lemma 2.1.5, we have

$$\eth_i \left(\mathcal{T}_{s_{k_l}}^{\eth_{k_l}} A_{k_l} x_{k_l}, z \right) + \frac{1}{s_{k_l}} \left\langle z - \mathcal{T}_{s_{k_l}}^{\eth_{k_l}} A_{k_l} x_{k_l}, \mathcal{T}_{s_{k_l}}^{\eth_{k_l}} A_{k_l} x_{k_l} - A_{k_l} x_{k_l} \right\rangle \geq 0, \text{ for all } z \in Z.$$

Since $\eth_i$ is upper hemicontinuous in the first argument for each $1 \leq i \leq K$, therefore taking $\limsup$ on both sides of the above estimate as $l \to \infty$ and utilizing (C2) and (2.14), we get

$$\eth_i \left(A_{k_l} x, z \right) \geq 0, \text{ for all } z \in Z \text{ and for each } 1 \leq i \leq K.$$

Hence $A_i q \in EP(\eth_i)$ for each $1 \leq i \leq K$ and consequently $q \in \mathbb{F}$. It remains to show that $x_k \to q = P_{\mathbb{F}} x_1$. Let $x = P_{\mathbb{F}} x_1$, then from $\|x_k - x_1\| \leq \|x - x_1\|$, therefore, we have

$$\begin{aligned} \|x - x_1\| &\leq \|q - x_1\| \\ &\leq \liminf_{j \to \infty} \left\|x_{k_j} - x_1\right\| \\ &\leq \limsup_{j \to \infty} \left\|x_{k_j} - x_1\right\| \\ &\leq \|x - x_1\|. \end{aligned}$$

This implies that

$$\lim_{j \to \infty} \left\|x_{k_j} - x_1\right\| = \|q - x_1\|.$$

Hence $x_{k_j} \to q = P_{\mathbb{F}} x_1$. From the arbitrariness of the subsequence $\{x_{k_j}\}$ of $\{x_k\}$, we conclude that $x_k \to x$ as $k \to \infty$. It is easy to see that $y_{k,i} \to x$ and $u_{k,i} \to x$.

This completes the proof. $\square$

Corollary 2.2.2. *Let $\mathbb{H}_1$ and $\mathbb{H}_2$ be two real Hilbert spaces and let $Y \subseteq \mathbb{H}_1$ and $Z \subseteq \mathbb{H}_2$ be nonempty closed convex subsets of Hilbert spaces $\mathbb{H}_1$ and $\mathbb{H}_2$, respectively. Let $\mho_i : Y \times Y \to \mathbb{R}$ and $\eth_i : Z \times Z \to \mathbb{R}$ be two finite families of bifunctions satisfying Condition 2.1.2 such that $\eth_i$ be upper semicontinuous for each $i \in \{1, 2, 3, \cdots, K\}$. Let $G_i : Y \to Y$ be a finite family nonexpansive mappings and let $A_i : \mathbb{H}_1 \to \mathbb{H}_2$ be a finite family of bounded linear operators for each $i \in \{1, 2, 3, \cdots, K\}$. Suppose that $\mathbb{F} := \left[\bigcap_{i=1}^{K} \mathbb{F}(G_i)\right] \cap \Omega \neq \emptyset$, where $\Omega = \left\{z \in Y : z \in \bigcap_{i=1}^{K} EP(\mho_i) \text{ and } A_i z \in EP(\eth_i) \text{ for } 1 \leq i \leq K\right\}$. Let $\{x_k\}$ be a sequence generated by:*

$$\begin{aligned}
&x_1 \in Y_1 = Y,\\
&u_k = \mathcal{T}_{r_k}^{f_k}\left(x_k - \gamma A^*_{(k \mod K)}\left(\mathbb{I} - \mathcal{T}_{s_k}^{\eth_k}\right) A_{(k \mod K)} x_k\right),\\
&y_k = \alpha_k u_k + (1 - \alpha_k)\, G^k_{(k \mod K)} u_k,\\
&Y_{k+1} = \left\{z \in \mathbb{H}_1 \colon \|y_k - z\|^2 \leq \|x_k - z\|^2 + \mathcal{L}_K\right\},\\
&x_{k+1} = P_{Y_{k+1}} x_1, \quad k \geq 1,
\end{aligned} \tag{2.23}$$

where $\theta_k = (1-\alpha_k)\left\{\lambda_k \xi_k(M_k) + \lambda_k M_k^* D_k + \mu_k\right\}$ with $D_k = \sup\left\{\|x_k - p\| : p \in \mathbb{F}\right\}$. Let $\{r_k\}, \{s_k\}$ be two positive real sequences and let $\{\alpha_k\}$ be in $(0, 1)$. Assume that if the following set of conditions holds:

(C1): $0 \leq \mathcal{K} < a \leq \alpha_k \leq b < 1$ and $\gamma \in \left(0, \frac{1}{L}\right)$ where $L = \max\{L_1, L_2, \cdots, L_K\}$ and L_i is the spectral radius of the operator $A_i^* A_i$ and A_i^* is the adjoint of A_i for each $i \in \{1, 2, 3, \cdots, K\}$;

(C2): $\liminf_{k\to\infty} r_k > 0$ and $\liminf_{k\to\infty} s_k > 0$;

(C3): $\sum_{k=1}^{\infty} \lambda_k < \infty$ and $\sum_{k=1}^{\infty} \mu_k < \infty$;

(C4): there exist constants $M_i,\ M_i^* > 0$ such that $\xi_i(\lambda_i) \leq M_i^* \lambda_i$ for all $\lambda_i \geq M_i, i = 1, 2, 3, \cdots, K$, then the sequence $\{x_k\}$ generated by (2.23) converges strongly to $P_{\mathbb{F}} x_1$.

Theorem 2.2.3. *Let $\mathbb{H}_1$ and $\mathbb{H}_2$ be two real Hilbert spaces and let $Y \subseteq \mathbb{H}_1$ and $Z \subseteq \mathbb{H}_2$ be nonempty closed convex subsets of Hilbert spaces $\mathbb{H}_1$ and $\mathbb{H}_2$, respectively. Let*

$\mho_i : Y \times Y \to \mathbb{R}$ and $\eth_i : Z \times Z \to \mathbb{R}$ be two finite families of bifunctions satisfying Condition 2.1.2 such that $\eth_i$ be upper semicontinuous for each $i \in \{1, 2, 3, \cdots, K\}$. Let $G_i : Y \to Y$ be a finite family of uniformly $\mathcal{L}$-Lipschitzian and continuous total asymptotically strict pseudo contractions and let $A_i : \mathbb{H}_1 \to \mathbb{H}_2$ be a finite family of bounded linear operators for each $i \in \{1, 2, 3, \cdots, K\}$. Suppose that $\mathbb{F} := \left[\bigcap_{i=1}^{K} \mathbb{F}(G_i)\right] \cap \Omega \neq \emptyset$, where $\Omega = \left\{z \in Y : z \in \bigcap_{i=1}^{K} EP(\mho_i) \text{ and } A_i z \in EP(\eth_i) \text{ for } 1 \leq i \leq K\right\}$. Let $\{x_k\}$ be a sequence generated by:

$$\begin{aligned}
&x_1 \in Y_1 = Y, \\
&u_k = \mathcal{T}_{r_k}^{\mho_k}\left(x_k - \gamma A^*_{(k \mod K)}\left(\mathbb{I} - \mathcal{T}_{s_k}^{\eth_k}\right) A_{(k \mod K)} x_k\right), \\
&y_k = \alpha_k u_k + (1 - \alpha_k)\, G^k_{(k \mod K)} u_k, \\
&Y_{k+1} = \left\{z \in \mathbb{H}_1 \colon \|y_k - z\|^2 \leq \|x_k - z\|^2 + \mathcal{L}_k\right\}, \\
&x_{k+1} = P_{Y_{k+1}} x_1, \quad k \geq 1,
\end{aligned} \tag{2.24}$$

where $\theta_k = (1-\alpha_k)\{\lambda_k \xi_k(M_k) + \lambda_k M_k^* D_k + \mu_k\}$ with $D_k = \sup\{\|x_k - p\| : p \in \mathbb{F}\}$. Let $\{r_k\}, \{s_k\}$ be two positive real sequences and let $\{\alpha_k\}$ be in $(0, 1)$. Assume that if the following set of conditions holds:

(C1): $0 \leq \mathcal{K} < a \leq \alpha_k \leq b < 1$ and $\gamma \in \left(0, \frac{1}{L}\right)$ where $L = \max\{L_1, L_2, \cdots, L_K\}$ and L_i is the spectral radius of the operator $A_i^* A_i$ and A_i^* is the adjoint of A_i for each $i \in \{1, 2, 3, \cdots, K\}$;

(C2): $\liminf_{k\to\infty} r_k > 0$ and $\liminf_{k\to\infty} s_k > 0$;

(C3): $\sum_{k=1}^{\infty} \lambda_k < \infty$ and $\sum_{k=1}^{\infty} \mu_k < \infty$;

(C4): there exist constants $M_i,\ M_i^* > 0$ such that $\xi_i(\lambda_i) \leq M_i^* \lambda_i$ for all $\lambda_i \geq M_i, i = 1, 2, 3, \cdots, K$, then the sequence $\{x_k\}$ generated by (2.24) converges strongly to $P_{\mathbb{F}} x_1$.

Proof: Set $\mathbb{H}_1 = \mathbb{H}_2,\ Y = Z$ and $A = \mathbb{I}$(the identity mapping) then the desired result then follows from Theorem 2.2.1 immediately.

Chapter 3

Convergence Analysis of Cesáro Mean Algorithm

In this chapter, we study the convergence characteristics of an iterative algorithm, which is a combination of the shrinking projection method and the Cesáro means method, in Hilbert spaces. The proposed algorithm provides an optimal solution in the intersection of the solutions sets of the split equilibrium problem, the variational inequality problem and the fixed point problem associated with the class of total asymptotically nonexpansive mappings in such setting. We aim to establish strong convergence results of the proposed algorithm by employing suitable conditions on the control sequences of parameters. As a consequence, these results can be considered as an improvement and generalization of various results in this field of study.

Theorem 3.0.4. *In 1975, Baillon [3] established the first result of nonlinear ergodic theorem in Hilbert spaces. In fact, he proved that for a nonexpansive self-mapping $\mathcal{T}$ defined on a bounded closed convex subset Y of a Hilbert space $\mathbb{H}$, then for every $x \in Y$, the Cesáro means*

$$\frac{x + \mathcal{T}x + \cdots + \mathcal{T}^k x}{k+1}$$

converges weakly to a fixed point of $\mathcal{T}$ provided that $\mathbb{F}(\mathcal{T})$ is nonempty.

In 1979, Bruck [6] explained the attributes of the famous iterative scheme namely Cesáro mean using non-expansive mapping in the settings of uniformly convex Banach space, see also [7].

Shimizu and Takahashi [58] studied Cesàro mean iteration and initiated a strong convergence result under the action of asymptotically nonexpansive mappings in Hilbert spaces. Hirano-Takahashi [33] continued Baillon's result using asymptotically non-expansive mappings. A few writers have scrutinizeed different techniques using cesàro mean approximant for nonlinear mappings. For instance, [45, 46, 54, 55, 58, 59, 60, 62] and the references cited therein. It is remarked that the nonlinear averages $\frac{1}{k+1}\sum_{i=0}^{k} \mathcal{T}^i x$ exhibit only weak convergence characteristics. In many situations, the strong convergence of an iterative algorithm involving a nonlinear mapping is much more desirable than the weak convergence. In 1967, Halpern [30] introduced the following strongly convergent iterative algorithm:

$$P_0^{\alpha}x = x, \quad P_{k+1}^{\alpha}x = \alpha_{k+1}x + (1-\alpha_{k+1})\,\mathcal{T}P_k^{\alpha}x,$$

where $\alpha = \{\alpha_k\}_{k=1}^{\infty}$ is a sequence in $[0,1]$. Note that if $\mathcal{T}$ is positively homogeneous (i.e., $\mathcal{T}(tx) = t\mathcal{T}x$ for any $t \geq 0$ and $x \in Y$) and $\alpha_k = \frac{1}{k+1}$ then, we have, $P_k^{\alpha}x = \frac{1}{k+1}G_kx$, where $G_0x = x$ and $G_{k+1}x = x + \mathcal{T}(G_kx)$. Thus, for this special choice of $\{\alpha_k\}_{k=1}^{\infty}$, $P_k^{\alpha}x$ is a nonlinear generalization of the Cesáro means method. In 1968, Haugazeau [31] introduced the idea of hybrid projection algorithm or outer-approximation methods in Hilbert spaces. The hybrid projection method has been modified in different ways to ensure the strong convergence of an algorithm.

The main motive of this work is to establish convergence results of the proposed algorithm by employing suitable conditions on the control sequences of parameters. The proposed algorithm strongly converges to a common element of the set of solutions of the SEP, and the set of common fixed points of a finite family of total asymptotically nonexpansive mappings. As a consequence, these results can be considered as an improvement and generalization of various results in this field of study.

3.1 Main Result

We now prove our main result of this section.

Theorem 3.1.1. *Let Y, Z be two nonempty closed convex subsets of two real Hilbert spaces $\mathbb{H}_1$ and $\mathbb{H}_2$, respectively. Let $\{\mho_i\}_{i=1}^{K} : Y \times Y \to \mathbb{R}$ and $\{\eth_i\}_{i=1}^{K} : Z \times Z \to \mathbb{R}$ be two finite families of bifunctions satisfying Condition (2.1.2) such that each $\eth_i$ is upper* *semicontinuous for each $i \in \{1,2,3,\cdots,K\}$. Let $\{A_i\}_{i=1}^{K} : \mathbb{H}_1 \to \mathbb{H}_2$ be a finite family of bounded linear operators and let $\{G_i\}_{i=1}^{K} : Y \to Y$ be a finite family of uniformly $\mathcal{L}$-Lipschitzian and continuous total asymptotically nonexpansive mappings satisfying the condition that*

$$\lim_{k\to\infty} \sup_{x\in \acute{Y}} \left\|G_i^{k+1}x - G_i^k x\right\| = 0,\ 1 \le i \le K$$

for any bounded subset $\acute{Y}$ of Y. Assume that the solution set $\mathbb{F} := \left[\bigcap_{i=1}^{K} \mathbb{F}(G_i)\right] \cap \Omega \neq \emptyset$, where $\Omega = \left\{z \in Y : z \in \bigcap_{i=1}^{K} EP(\mho_i) \text{ and } A_i z \in EP(\eth_i) \text{ for } 1 \le i \le K\right\}$. Let $\{x_k\}$ be a sequence generated by:

$$\begin{aligned}
&x_1 \in Y_1 = Y,\\
&u_k = \mathcal{T}_{r_k}^{\mho_k}\left(x_k - \gamma A^*_{(k \mod K)}\left(\mathbb{I} - \mathcal{T}_{s_k}^{\eth_k}\right) A_{(k \mod K)} x_k\right),\\
&y_k = \alpha_k u_k + (1-\alpha_k)\tfrac{1}{K}\sum_{i=1}^{K} G_i^k u_k,\\
&Y_{k+1} = \left\{z \in \mathbb{H}_1 \colon \|y_k - z\|^2 \le \|x_k - z\|^2 + \theta_k\right\},\\
&x_{k+1} = P_{Y_{k+1}} x_1,\ \ k \ge 1,
\end{aligned} \tag{3.1}$$

where $\theta_k = (1-\alpha_k)\{\lambda_k \xi_k(M_k) + \lambda_k M_k^* D_k + \mu_k\}$ with $D_k = \sup\{\|x_k - p\| : p \in \mathbb{F}\}$. Let $\{r_k\}, \{s_k\}$ be two positive real sequences and let $\{\alpha_k\}$ be in $(0,1)$ such that $\alpha_k \le a$. Assume that if the following set of conditions holds:

(C1): $\gamma \in \left(0, \frac{1}{L}\right)$ where $L = \max\{L_1, L_2, \cdots, L_K\}$ and L_i is the spectral radius of the operator $A_i^* A_i$ where A_i^* is the adjoint of A_i for each $i \in \{1,2,3,\cdots,K\}$;

(C2): $\liminf\limits_{n\to\infty} r_k > 0$ and $\liminf\limits_{k\to\infty} s_k > 0$;

(C3): $\sum\limits_{k=1}^{\infty} \lambda_k < \infty$ and $\sum\limits_{k=1}^{\infty} \mu_k < \infty$;

(C4): there exist constants $M_i,\ M_i^* > 0$ such that $\xi_i(\lambda_i) \le M_i^* \lambda_i$ for all $\lambda_i \ge M_i, i = 1,2,3,\cdots,K$, then the sequence $\{x_k\}$ generated by (3.1) converges strongly to $P_{\mathbb{F}} x_1$.

Proof. For the sake of simplicity, we divide the proof into five steps and set $A_k = A_{(k \mod K)}$ for all $k \ge 1$.

Step 1. The sequence $\{x_k\}$ is well defined.

Proof of Step 1. We first show by mathematical induction that $\mathbb{F} \subset Y_k$ for all $k \ge 1$.

It is obvious from the assumption that $\mathbb{F} \subset Y_1 = Y$. Let $\mathbb{F} \subset Y_i$ for some $i \geq 1$. We show that $\mathbb{F} \subset Y_{i+1}$ for some $i \geq 1$. It follows from (3.1) that

$$\begin{aligned}
\|u_i - p\|^2 &= \left\|\mathcal{T}_{r_i}^{\mho_i}\left(x_i - \gamma A_i^*\left(\mathbb{I} - \mathcal{T}_{s_i}^{\eth_i}\right)A_i x_i\right) - \mathcal{T}_{r_i}^{\mho_i} p\right\|^2 \\
&\leq \left\|x_i - \gamma A_i^*\left(\mathbb{I} - \mathcal{T}_{s_i}^{\eth_i}\right)A_i x_i - p\right\|^2 \\
&\leq \|x_i - p\|^2 + \gamma^2\left\|A_i^*\left(\mathbb{I} - \mathcal{T}_{s_i}^{\eth_i}\right)A_i x_i\right\|^2 + 2\gamma\left\langle p - x_i, A_i^*\left(\mathbb{I} - \mathcal{T}_{s_i}^{\eth_i}\right)A_i x_i\right\rangle \\
&\leq \|x_i - p\|^2 + \gamma^2\left\langle A_i x_i - \mathcal{T}_{s_i}^{\eth_i} A_i x_i, A_i A_i^*\left(\mathbb{I} - \mathcal{T}_{s_i}^{\eth_i}\right)A_i x_i\right\rangle \\
&\quad + 2\gamma\left\langle p - x_i, A_i^*\left(\mathbb{I} - \mathcal{T}_{s_i}^{\eth_i}\right)A_i x_i\right\rangle \\
&\leq \|x_i - p\|^2 + L\gamma^2\left\langle A_i x_i - \mathcal{T}_{s_i}^{\eth_i} A_i x_i, A_i x_i - \mathcal{T}_{s_i}^{\eth_i} A_i x_i\right\rangle \\
&\quad + 2\gamma\left\langle p - x_i, A_i^*\left(\mathbb{I} - \mathcal{T}_{s_i}^{g_i}\right)A_i x_i\right\rangle \\
&= \|x_i - p\|^2 + L\gamma^2\left\|A_i x_i - \mathcal{T}_{s_i}^{\eth_i} A_i x_i\right\|^2 + 2\gamma\left\langle p - x_i, A_i^*\left(\mathbb{I} - \mathcal{T}_{s_i}^{\eth_i}\right)A_i x_i\right\rangle.
\end{aligned} \tag{3.2}$$

Denote $\Lambda = 2\gamma\left\langle p - x_i, A_i^*\left(\mathbb{I} - \mathcal{T}_{s_i}^{\eth_i}\right)A_i x_k\right\rangle$, we have

$$\begin{aligned}
\Lambda &= 2\gamma\left\langle p - x_i, A_i^*\left(\mathbb{I} - \mathcal{T}_{s_i}^{g_i}\right)A_i x_i\right\rangle \\
&= 2\gamma\left\langle A_i(p - x_i), A_i x_i - \mathcal{T}_{s_i}^{\eth_i} A_i x_i\right\rangle \\
&= 2\gamma\left\langle A_i(p - x_i) + \left(A_i x_i - \mathcal{T}_{s_i}^{\eth_i} A_i x_i\right) - \left(A_i x_i - \mathcal{T}_{s_i}^{\eth_i} A_i x_i\right), A_i x_i - \mathcal{T}_{s_i}^{\eth_i} A_i x_i\right\rangle \\
&= 2\gamma\left\{\left\langle A_i p - \mathcal{T}_{s_i}^{\eth_i} A_i x_i, A_i x_i - \mathcal{T}_{s_i}^{g_i} A_i x_i\right\rangle - \left\|A_i x_i - \mathcal{T}_{s_i}^{\eth_i} A_i x_i\right\|^2\right\} \\
&\leq 2\gamma\left\{\frac{1}{2}\left\|A_i x_i - \mathbb{T}_{s_i}^{g_i} A_i x_i\right\|^2 - \left\|A_i x_i - \mathcal{T}_{s_i}^{g_i} A_i x_i\right\|^2\right\} \\
&= -\gamma\left\|A_i x_i - \mathcal{T}_{s_i}^{g_i} A_i x_i\right\|^2.
\end{aligned}$$

Substituting the above simplified value of Λ in (3.2), we have

$$\|u_i - p\|^2 \leq \|x_i - p\|^2 + \gamma(L\gamma - 1)\left\|A_i x_i - \mathcal{T}_{s_{n,i}}^{\eth_i} A_i x_i\right\|^2. \tag{3.3}$$

From the definition of γ and conditions (C1), we obtain

$$\|u_i - p\|^2 \leq \|x_i - p\|^2. \tag{3.4}$$

Let $G^k = \frac{1}{K}\sum_{i=1}^{K} G_i^k$, it then follows that

$$\begin{aligned}
\left\|G^k x - G^k y\right\| &= \left\|\frac{1}{K}\sum_{i=1}^{K} G_i^k x - \frac{1}{K}\sum_{i=1}^{K} G_i^k y\right\| \\
&\leq \frac{1}{K}\sum_{i=1}^{K}(\|x-y\| + \lambda_k \xi_k(\|x-y\|) + \mu_k) \\
&\leq \|x-y\| + \lambda_k \xi_k(\|x-y\|) + \mu_k, \text{ for all } x, y \in Y. \qquad (3.5)
\end{aligned}$$

Now, for any $p \in \mathbb{F}$, we have $G^k p = \frac{1}{K}\sum_{i=1}^{K} G_i^k p = p$. It follows from (3.2) and (3.5), that

$$\begin{aligned}
\|y_i - p\| &= \left\|\alpha_i u_i + (1-\alpha_i) G^i u_i - p\right\| \\
&= \alpha_i \|u_i - p\| + (1-\alpha_i)\left\|G^i u_i - p\right\| \\
&\leq \alpha_i \|u_i - p\| + (1-\alpha_i)\{\|u_i - p\| + \lambda_i \xi_i(\|u_i - p\|) + \mu_i\} \\
&\leq \|u_i - p\| + (1-\alpha_i)\{\lambda_i \xi_i(M_i) + \lambda_i M_i^* \|u_i - p\|) + \mu_i\} \\
&\leq \|x_i - p\| + (1-\alpha_i)\left\{\lambda_i \xi_i(M_i) + \lambda_i M_i^* \|x_i - p\|^2 + \mu_i\right\} \\
&\leq \|x_i - p\| + \theta_i, \qquad (3.6)
\end{aligned}$$

where $\theta_i = (1-\alpha_i)[(\lambda_i \xi_i(M_i) + \lambda_i D_i M_i^* + \mu_i)]$ with $D_i = \sup\{\|x_i - p\| : p \in \mathbb{F}\}$. The estimate (3.6) implies that $p \in Y_{i+1}$ and hence $\mathbb{F} \subset Y_k$ for all $k \geq 1$. Since

$$\left\{z \in \mathbb{H}_1 : \|y_k - z\|^2 \leq \|x_k - z\|^2 + \theta_k\right\} = \left\{z \in \mathbb{H}_1 : \|y_k\|^2 - \|x_k\|^2 \leq 2\langle y_k - x_k, z\rangle + \theta_k\right\},$$

it is closed and convex, therefore, the sequence $\{x_k\}$ is well-defined.

Step 2. The sequence $\{\|x_k - x_1\|\}$ is Cauchy.

Proof of Step 2. As proved in Theorem 2.2.1, the sequence $\{\|x_k - x_1\|\}$ is bounded and nondecreasing, therefore, we have

$$\lim_{k\to\infty} \|x_k - x_1\| \text{ exists.} \qquad (3.7)$$

Moreover, it now follows from the estimate (3.7) that the sequence $\{\|x_k - x_1\|\}$ is Cauchy. That is

$$\lim_{k\to\infty} \|x_{k+1} - x_k\| = 0. \qquad (3.8)$$

Step 3. Show that:

(i): $\lim_{k\to\infty}\|y_k - x_{n+1}\| = \lim_{k\to\infty}\|y_k - x_k\| = 0,$

(ii): $\lim_{k\to\infty}\|u_k - x_n\| = 0,$

(iii): $\lim_{k\to\infty}\|y_k - u_k\| = 0,$

(iv): $\lim_{k\to\infty}\|G_k u_k - u_k\| = 0.$

Proof of Step 3. By $x_{k+1} \in Y_{k+1}$, we have $\|y_k - x_{k+1}\| \leq \|x_k - x_{k+1}\| + \theta_k$. Using (3.8), we have

$$\lim_{k\to\infty}\|y_k - x_{k+1}\| = 0, \text{ for all } k \geq 1. \tag{3.9}$$

Since $\|y_k - x_k\| \leq \|y_k - x_{k+1}\| + \|x_{k+1} - x_k\|$, therefore using (3.8)-(3.9), we obtain

$$\lim_{k\to\infty}\|y_k - x_k\| = 0, \text{ for all } k \geq 1. \tag{3.10}$$

as $k \to \infty$.

Altogether, we deduce from (3.1), (3.3) and (3.6) that

$$\begin{aligned}
\gamma(1-\gamma L)\left\|A_k x_k - \mathcal{T}_{s_k}^{\eth_k} A_k x_k\right\|^2 &\leq \|x_k - p\|^2 - \|u_k - p\|^2 \\
&\leq \|x_k - p\|^2 - \|y_k - p\|^2 + \theta_k \\
&\leq (\|x_k - p\| + \|y_k - p\|)\,\|x_k - y_k\| + \theta_k.
\end{aligned}$$

From $\gamma(1 - Ly) > 0$ and (3.10), we obtain

$$\lim_{k\to\infty}\left\|A_k x_k - \mathcal{T}_{s_k}^{\eth_k} A_k x_k\right\|^2 = 0 \text{ for all } k \geq 1. \tag{3.11}$$

Next, we show that $\|u_k - x_k\| \to 0$ as $k \to \infty$. Since $p \in \mathbb{F}$, we have

$$\begin{aligned}
\|u_k - p\|^2 &= \left\|\mathcal{T}_{r_k}^{\mho_k}\left(x_k - \gamma A_k^*\left(\mathbb{I} - \mathcal{T}_{s_k}^{\eth_k}\right)A_k x_k\right) - \mathcal{T}_{r_k}^{\mho_k} p\right\|^2 \\
&\leq \left\langle u_k - p, x_k - \gamma A_k^*\left(\mathbb{I} - \mathcal{T}_{s_k}^{\eth_k}\right)A_k x_k - p\right\rangle \\
&= \frac{1}{2}\{\|u_k - p\|^2 + \left\|x_k - \gamma A_k^*\left(\mathbb{I} - \mathcal{T}_{s_k}^{\eth_k}\right)A_k x_n - p\right\|^2 \\
&\quad - \left\|u_k - x_k + \gamma A_k^*\left(\mathbb{I} - \mathcal{T}_{s_k}^{\eth_k}\right)A_k x_k\right\|^2\} \\
&\leq \frac{1}{2}\left\{\|u_k - p\|^2 + \|x_k - p\|^2 - \left\|u_k - x_k + \gamma A_k^*\left(\mathbb{I} - \mathcal{T}_{s_k}^{\eth_k}\right)A_k x_k\right\|^2\right\} \\
&= \frac{1}{2}\{\|u_k - p\|^2 + \|x_k - p\|^2 - (\|u_k - x_k\|^2 + \gamma^2\left\|A_k^*\left(\mathbb{I} - \mathcal{T}_{s_k}^{\eth_k}\right)A_k x_k\right\|^2 \\
&\quad - 2\gamma \rangle u_k - x_k, A_k^*\left(\mathbb{I} - \mathcal{T}_{s_k}^{\eth_k}\right)A_k x_k\rangle)\} \\
&\leq \|x_k - p\|^2 - \|u_k - x_k\|^2 + 2\gamma\|A(u_k - x_k)\|\left\|A_k x_k - \mathcal{T}_{s_k}^{\eth_k} A_k x_k\right\| \qquad (3.12)
\end{aligned}$$

Consider the following variant of (3.6) together with (3.4):

$$\|y_k - p\|^2 \leq \alpha_n\|x_k - p\|^2 + (1 - \alpha_k)\|u_k - p\|^2 + \theta_k. \qquad (3.13)$$

Altogether, it follows from (3.12) and (3.13) that

$$\begin{aligned}
(1 - \alpha_k)\|u_k - x_k\|^2 &\leq (\|x_k - p\| + \|y_k - p\|)\|x_k - y_k\| \\
&\quad + 2\gamma\|A(u_k - x_k)\|\left\|A_k x_k - \mathcal{T}_{s_k}^{\eth_k} A_k x_k\right\| + \theta_k.
\end{aligned}$$

From (3.10) and (3.11), the above estimate implies that

$$\lim_{k\to\infty}\|u_k - x_k\| = 0 \text{ for all } k \geq 1. \qquad (3.14)$$

Utilizing (3.10) and (3.13), we get

$$\begin{aligned}
\|y_k - u_k\| &\leq \|y_k - x_k\| + \|x_k - u_k\| \\
&\to 0 \text{ as } k \to \infty. \qquad (3.15)
\end{aligned}$$

Since $\|y_k - u_k\| = (1 - \alpha_k)\left\|G^k u_k - u_k\right\|$ and $\alpha_k \leq a < 1$, then from (3.15) that

$$\lim_{k\to\infty}\left\|G^k u_k - u_k\right\| = 0 \text{ for all } k \geq 1. \qquad (3.16)$$

From (3.14) and (3.16), we obtain

$$\begin{aligned}\left\|G^k u_k - x_k\right\| &\leq \left\|G^k u_k - u_k\right\| + \|x_k - u_k\| \\ &\to 0 \text{ as } k \to \infty. \end{aligned} \tag{3.17}$$

Reasoning as above, we obtain

$$\lim_{k\to\infty} \left\|G^k u_k - y_k\right\| = 0 \text{ for all } k \geq 1. \tag{3.18}$$

Now utilizing the uniform $\mathcal{L}$-Lipschitz property of G^k, together with the estimates (3.14) and (3.17), we get

$$\begin{aligned}\left\|G^k x_k - x_k\right\| &\leq \left\|G^k x_k - G^k u_k\right\| + \left\|G^k u_k - x_k\right\| \\ &\leq \mathcal{L}\,\|x_k - u_k\| + \left\|G^k u_k - x_k\right\| \\ &\to 0 \text{ as } k \to \infty. \end{aligned} \tag{3.19}$$

Note that $\frac{1}{K}\left\|G_i^k u_k - u_k\right\|^2 \leq \frac{1}{K}\sum_{i=1}^{K}\left\|G_i^k u_k - u_k\right\|^2$, therefore using (3.16), we have

$$\lim_{k\to\infty}\left\|G_i^k u_k - u_k\right\| = 0 \text{ for each } i = 1, 2, \cdots, K.$$

Similarly, we also have that

$$\lim_{k\to\infty}\left\|G_i^k x_k - x_k\right\| = 0 \text{ for each } i = 1, 2, \cdots, K.$$

Moreover, utilizing the uniform continuity of G_i and (3.20), we get

$$\begin{aligned}\|x_k - G_i x_k\| &\leq \left\|x_k - G_i^k x_k\right\| + \left\|G_i^k x_k - G_i x_k\right\| \\ &\to 0 \text{ as } k \to \infty. \end{aligned}$$

Similarly, we also have that

$$\lim_{k\to\infty}\|u_k - G_k u_k\| = 0 \text{ for all } k \geq 1.$$

As proved earlier in Theorem 2.2.1, the set $\omega(x_k) \subset \mathbb{F}$, where $\omega(x_k)$ is the set of all weak ω-limits of $\{x_k\}$. One can use the fact that $G_{Kk+i} = G_i$ for all $k \geq 1$ is demiclosed so that $x \in \mathbb{F}(G_i)$ for each $1 \leq i \leq K$. Similarly, we can also show that $q \in \Omega$, i.e., $q \in \bigcap_{i=1}^{K} EP(\mho_i)$ and $A_i q \in EP(\eth_i)$ for each $1 \leq i \leq K$. Finally, we conclude that $x_k \to x$, $y_{k,i} \to x$ and $u_{k,i} \to x$ as $k \to \infty$. This completes the proof. $\square$

3.2 Consequently Results

In this section, we deduce some results from Theorem (3.1.1). An immediate consequence of Theorem 3.1.1 is to establish the same result for a class of non-expansive mappings.

Corollary 3.2.1. *Let $\mathbb{H}_1$ and $\mathbb{H}_2$ be two real Hilbert spaces and let $Y \subseteq \mathbb{H}_1$ and $Z \subseteq \mathbb{H}_2$ be nonempty closed convex subsets of Hilbert spaces $\mathbb{H}_1$ and $\mathbb{H}_2$, respectively. Let $\mho_i : Y \times Y \to \mathbb{R}$ and $\eth_i : Z \times Z \to \mathbb{R}$ be two finite families of bifunctions satisfying conditions (A1)-(A4) such that each $\eth_i$ is upper semicontinuous for each $i \in \{1, 2, 3, \cdots, K\}$. Let $G : Y \to Y$ be a nonexpansive mappings and let $A_i : \mathbb{H}_1 \to \mathbb{H}_2$ be a finite family of bounded linear operators for each $i \in \{1, 2, 3, \cdots, K\}$. Suppose that $\mathbb{F} := \mathbb{F}(G) \cap \Omega \neq \emptyset$, where $\Omega = \left\{ z \in Y : z \in \bigcap_{i=1}^{K} EP(\mho_i) \text{ and } A_i z \in EP(\eth_i) \text{ for } 1 \leq i \leq K \right\}$. Let $\{x_k\}$ be a sequence generated by:*

$$\begin{aligned}
&x_1 \in Y_1 = Y, \\
&u_k = \mathcal{T}_{r_k}^{\mho_k} \left(x_k - \gamma A^*_{(k \mod K)} \left(\mathbb{I} - \mathcal{T}_{s_k}^{\eth_k} \right) A_{(k \mod K)} x_k \right), \\
&y_k = \alpha_k u_k + (1 - \alpha_k) \tfrac{1}{K} \sum_{i=1}^{K} G^i u_k, \\
&Y_{k+1} = \left\{ z \in \mathbb{H}_1 \colon \|y_k - z\|^2 \leq \|x_k - z\|^2 + \theta_k \right\}, \\
&x_{k+1} = P_{Y_{k+1}} x_1, \quad k \geq 1,
\end{aligned} \tag{3.20}$$

where $\theta_k = (1-\alpha_k) \{\lambda_k \xi_k(M_k) + \lambda_k M_k^* D_k + \mu_k\}$ with $D_k = \sup\{\|x_k - p\| : p \in \mathbb{F}\}$. Let $\{r_k\}, \{s_k\}$ be two positive real sequences and let $\{\alpha_k\}$ be in $(0, 1)$. Assume that if the following set of conditions holds:

(C1): $0 \leq \mathcal{K} < a \leq \alpha_k \leq b < 1$ *and* $\gamma \in \left(0, \frac{1}{L}\right)$ *where* $L = \max\{L_1, L_2, \cdots, L_K\}$ *and* L_i *is the spectral radius of the operator* $A_i^* A_i$ *and* A_i^* *is the adjoint of* A_i *for each* $i \in \{1, 2, 3, \cdots, K\}$;

(C2): $\liminf\limits_{k\to\infty} r_k > 0$ *and* $\liminf\limits_{k\to\infty} s_k > 0$;

(C3): $\sum\limits_{k=1}^{\infty} \lambda_k < \infty$ *and* $\sum\limits_{k=1}^{\infty} \mu_k < \infty$;

(C4): there exist constants $M_i,\ M_i^* > 0$ *such that* $\xi_i(\lambda_i) \leq M_i^* \lambda_i$ *for all* $\lambda_i \geq M_i, i = 1, 2, 3, \cdots, K$, *then the sequence* $\{x_k\}$ *generated by (3.2.1) converges strongly to* $P_{\mathbb{F}} x_1$.

Corollary 3.2.2. *Let* $\mathbb{H}_1$ *and* $\mathbb{H}_2$ *be two real Hilbert spaces and let* $Y \subseteq \mathbb{H}_1$ *and* $\mathbb{Q} \subseteq \mathbb{H}_2$ *be nonempty closed convex subsets of Hilbert spaces* $\mathbb{H}_1$ *and* $\mathbb{H}_2$*, respectively. Let* $\mho_i : Y \times Y \to \mathbb{R}$ *and* $\eth_i : Z \times Z \to \mathbb{R}$ *be two finite families of bifunctions satisfying condition 2.1.2 (A1)-(A4) such that each* $\eth_i$ *is upper semicontinuous for each* $i \in \{1, 2, 3, \cdots, K\}$. *Let* $G : Y \to Y$ *be a nonexpansive mappings and let* $A_i : \mathbb{H}_1 \to \mathbb{H}_2$ *be a finite family of bounded linear operators for each* $i \in \{1, 2, 3, \cdots, K\}$. *Suppose that* $\mathbb{F} := \mathbb{F}(G) \cap \Omega \neq \emptyset$*, where* $\Omega = \left\{ z \in Y : z \in \bigcap_{i=1}^{K} EP(\mho_i) \text{ and } A_i z \in EP(\eth_i) \text{ for } 1 \leq i \leq K \right\}$. *Let* $\{x_k\}$ *be a sequence generated by:*

$$
\begin{aligned}
& x_1 \in Y_1 = y, \\
& u_k = \mathcal{T}_{r_k}^{\mho_k}\left(x_k - \gamma A^*_{(k \mod K)}\left(\mathbb{I} - \mathcal{T}_{s_k}^{\eth_k}\right) A_{(k \mod K)} x_k\right), \\
& y_k = \alpha_k u_k + (1 - \alpha_k) G u_k, \\
& Y_{k+1} = \left\{ z \in \mathbb{H}_1 \colon \|y_k - z\|^2 \leq \|x_k - z\|^2 + \theta_k \right\}, \\
& x_{k+1} = P_{Y_{k+1}} x_1, \quad k \geq 1,
\end{aligned}
\tag{3.21}
$$

where $\theta_k = (1 - \alpha_k)\{\lambda_k \xi_n(M_k) + \lambda_k M_k^* D_k + \mu_k\}$ *with* $D_k = \sup\{\|x_k - p\| : p \in \mathbb{F}\}$. *Let* $\{r_k\}, \{s_k\}$ *be two positive real sequences and let* $\{\alpha_k\}$ *be in* $(0, 1)$. *Assume that if the following set of conditions holds:*

(C1): $0 \leq \mathcal{K} < a \leq \alpha_k \leq b < 1$ *and* $\gamma \in \left(0, \frac{1}{L}\right)$ *where* $L = \max\{L_1, L_2, \cdots, L_K\}$ *and* L_i *is the spectral radius of the operator* $A_i^* A_i$ *and* A_i^* *is the adjoint of* A_i *for each* $i \in \{1, 2, 3, \cdots, K\}$;

(C2): $\liminf\limits_{k\to\infty} r_k > 0$ *and* $\liminf\limits_{k\to\infty} s_k > 0$;

(C3): $\sum_{k=1}^{\infty} \lambda_k < \infty$ *and* $\sum_{k=1}^{\infty} \mu_k < \infty$;
(C4): there exist constants $M_i,\ M_i^* > 0$ *such that* $\xi_i(\lambda_i) \leq M_i^* \lambda_i$ *for all* $\lambda_i \geq M_i, i = 1, 2, 3, \cdots, K$, *then the sequence* $\{x_k\}$ *generated by (3.2.1) converges strongly to* $P_{\mathbb{F}} x_1$.

Proof. Set $G^i = G$, then the desired result then follows from Corollary 3.2.1 immediately. □

The following result suggest an iterative construction for the common solution of classical equilibrium problem together with the fixed point problem.

Corollary 3.2.3. *Let* $\mathbb{H}_1$ *and* $\mathbb{H}_2$ *be two real Hilbert spaces and let* $Y \subseteq \mathbb{H}_1$ *and* $Z \subseteq \mathbb{H}_2$ *be nonempty closed convex subsets of Hilbert spaces* $\mathbb{H}_1$ *and* $\mathbb{H}_2$*, respectively. Let* $\mho_i : Y \times Y \to \mathbb{R}$ *and* $\eth_i : Z \times Z \to \mathbb{R}$ *be two finite families of bifunctions satisfying condition 2.1.2 (A1)-(A4) such that each* $\eth_i$ *is upper semicontinuous for each* $i \in \{1, 2, 3, \cdots, K\}$*. Let* $G_i : Y \to Y$ *be a finite family of uniformly* $\mathcal{L}$*-Lipschitzian and continuous total asymptotically strict pseudo contractions and let* $A_i : \mathbb{H}_1 \to \mathbb{H}_2$ *be a finite family of bounded linear operators for each* $i \in \{1, 2, 3, \cdots, K\}$*. Suppose that* $\mathbb{F} := \left[\bigcap_{i=1}^K \mathbb{F}(G_i)\right] \cap \Omega \neq \emptyset$*, where* $\Omega = \left\{z \in C : z \in \bigcap_{i=1}^K EP(\mho_i) \text{ and } A_i z \in EP(\eth_i) \text{ for } 1 \leq i \leq K\right\}$.
Let $\{x_k\}$ *be a sequence generated by:*

$$\begin{aligned}
&x_1 \in Y_1 = Y, \\
&u_k = \mathcal{T}_{r_k}^{\mho_k}\left(x_k - \gamma A^*_{(k \mod K)}\left(\mathbb{I} - \mathcal{T}_{s_k}^{\eth_k}\right) A_{(k \mod K)} x_k\right), \\
&y_k = \alpha_k u_k + (1-\alpha_k)\frac{1}{K}\sum_{i=1}^{K} G_i^k u_k, \\
&C_{k+1} = \left\{z \in \mathbb{H}_1 \colon \|y_k - z\|^2 \leq \|x_k - z\|^2 + \theta_k\right\}, \\
&x_{k+1} = P_{Y_{k+1}} x_1, \quad k \geq 1,
\end{aligned} \tag{3.22}$$

where $\theta_k = (1-\alpha_k)\{\lambda_k \xi_k(M_k) + \lambda_k M_k^* D_k + \mu_k\}$ *with* $D_k = \sup\{\|x_k - p\| : p \in \mathbb{F}\}$. *Let* $\{r_k\}, \{s_k\}$ *be two positive real sequences and let* $\{\alpha_k\}$ *be in* $(0,1)$*. Assume that if the following set of conditions holds:*
(C1): $0 \leq \mathcal{K} < a \leq \alpha_k \leq b < 1$ *and* $\gamma \in \left(0, \frac{1}{L}\right)$ *where* $L = \max\{L_1, L_2, \cdots, L_K\}$ *and* L_i *is the spectral radius of the operator* $A_i^* A_i$ *and* A_i^* *is the adjoint of* A_i *for each* $i \in \{1, 2, 3, \cdots, K\}$;

(C2): $\liminf_{k\to\infty} r_k > 0$ *and* $\liminf_{k\to\infty} s_k > 0$;

(C3): $\sum_{k=1}^{\infty} \lambda_k < \infty$ *and* $\sum_{k=1}^{\infty} \mu_k < \infty$;

(C4): there exist constants $M_i,\ M_i^* > 0$ *such that* $\xi_i(\lambda_i) \le M_i^* \lambda_i$ *for all* $\lambda_i \ge M_i, i = 1, 2, 3, \cdots, K$, *then the sequence* $\{x_k\}$ *generated by (3.21) converges strongly to* $P_{\mathbb{F}} x_1$.

Proof. Set $\mathbb{H}_1 = \mathbb{H}_2,\ C = Q$ and $A = \mathbb{I}$(the identity mapping) then the desired result then follows from Theorem (3.1.1) immediately. $\square$

Corollary 3.2.4. *Let* $\mathbb{H}_1$ *and* $\mathbb{H}_2$ *be two real Hilbert spaces and let* $Y \subseteq \mathbb{H}_1$ *and* $Z \subseteq \mathbb{H}_2$ *be nonempty closed convex subsets of Hilbert spaces* $\mathbb{H}_1$ *and* $\mathbb{H}_2$*, respectively. Let* $\mho_i : Y \times Y \to \mathbb{R}$ *and* $\eth_i : Z \times Z \to \mathbb{R}$ *be two finite families of bifunctions satisfying condition 2.1.2 (A1)-(A4) such that each* $\eth_i$ *is upper semicontinuous for each* $i \in \{1, 2, 3, \cdots, K\}$. *Let* $G_i : Y \to Y$ *be a finite family of uniformly* $\mathcal{L}$*-Lipschitzian and continuous total asymptotically strict pseudo contractions and let* $A_i : \mathbb{H}_1 \to \mathbb{H}_2$ *be a finite family of bounded linear operators for each* $i \in \{1, 2, 3, \cdots, K\}$. *Suppose that* $\mathbb{F} := \left[\bigcap_{i=1}^K \mathbb{F}(G_i)\right] \cap \Omega \neq \emptyset$, *where* $\Omega = \left\{z \in Y : z \in \bigcap_{i=1}^K EP(\mho_i) \text{ and } A_i z \in EP(\eth_i) \text{ for } 1 \le i \le K\right\}$.
Let $\{x_k\}$ *be a sequence generated by:*

$$
\begin{aligned}
&x_1 \in Y_1 = Y,\\
&u_k = \mathcal{T}_{r_k}^{\mho_k}\left(x_k - \gamma A^*_{(k \mod K)}\left(\mathbb{I} - \mathcal{T}_{s_k}^{g_k}\right) A_{(k \mod K)} x_k\right),\\
&y_k = \alpha_k u_k + (1-\alpha_k)\tfrac{1}{K}\sum_{i=1}^{K} G^i u_k,\\
&Y_{k+1} = \left\{z \in \mathbb{H}_1 \colon \|y_k - z\|^2 \le \|x_k - z\|^2 + \theta_k\right\},\\
&x_{k+1} = P_{Y_{k+1}} x_1, \quad k \ge 1,
\end{aligned}
\tag{3.23}
$$

where $\theta_k = (1-\alpha_k)\{\lambda_k \xi_k(M_k) + \lambda_k M_k^* D_k + \mu_k\}$ *with* $D_k = \sup\{\|x_k - p\| : p \in \mathbb{F}\}$. *Let* $\{r_k\}, \{s_k\}$ *be two positive real sequences and let* $\{\alpha_k\}$ *be in* $(0,1)$. *Assume that if the following set of conditions holds:*

(C1): $0 \le \mathbb{K} < a \le \alpha_k \le b < 1$ *and* $\gamma \in \left(0, \frac{1}{L}\right)$ *where* $L = \max\{L_1, L_2, \cdots, L_K\}$ *and* L_i *is the spectral radius of the operator* $A_i^* A_i$ *and* A_i^* *is the adjoint of* A_i *for each* $i \in \{1, 2, 3, \cdots, K\}$;

(C2): $\liminf_{k\to\infty} r_k > 0$ *and* $\liminf_{k\to\infty} s_k > 0$;

(C3): $\sum_{k=1}^{\infty} \lambda_k < \infty$ *and* $\sum_{k=1}^{\infty} \mu_k < \infty$;

(C4): there exist constants $M_i,\ M_i^* > 0$ *such that* $\xi_i(\lambda_i) \leq M_i^* \lambda_i$ *for all* $\lambda_i \geq M_i, i = 1, 2, 3, \cdots, K$, *then the sequence* $\{x_k\}$ *generated by (3.21) converges strongly to* $P_{\mathbb{F}} x_1$.

Proof. *Set* $G^i = G$, *then the desired result then follows from Corollary (3.2.3) immediately.* $\square$

Chapter 4

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