Convergence Analysis of Proximal Point Algorithm in Geodesic Spaces

By

Wakeel Ahmed

CIIT/FA17-RMT-047/LHR

MS Thesis

In

Mathematics

COMSATS University Islamabad
Lahore Campus

Spring, 2019

COMSATS University Islamabad, Lahore Campus

Convergence Analysis of Proximal Point Algorithm in Geodesic Spaces

A Report Presented to

COMSATS University Islamabad, Lahore Campus

In partial fulfillment

of the requirement for the degree of

MS (Mathematics)

By

Wakeel Ahmed

CIIT/FA17-RMT-047/LHR

Spring, 2019

Convergence Analysis of Proximal Point Algorithm in Geodesic Spaces

A Post Graduate Thesis submitted to the Department of Mathematics as partial fulfillment of the requirement for the award of Degree of M.S (Master of Science in Mathematics).

Name	Registration Number
Wakeel Ahmed	CIIT/FA17-RMT-047/LHR

Supervisor

Dr. M. Aqeel Ahmad Khan
Assistant Professor,
Department of Mathematics
Lahore Campus.
COMSATS University Islamabad
Lahore Campus.
June, 2019

Convergence Analysis of Proximal Point Algorithm in Geodesic Spaces

Final Approval

This report titled

By

Wakeel Ahmed

CIIT/FA17-RMT-047/LHR

Has been approved

For the COMSATS University Islamabad, Lahore Campus

External Examiner:_____________________________

Dr. Imran Javaid
Bahauddin Zakariya University, Multan

Supervisor: _____________________________
Dr. M. Aqeel Ahmad Khan
Department of Mathematics , (CUI) Lahore Campus

HoD:___________________________________
Dr. Sarfraz Ahmad
Department of Mathematics,(CUI) Lahore Campus

Declaration

I **Wakeel Ahmed** with registration number **CIIT/FA17-RMT-047/LHR** hereby declare that I have produced the work presented in this thesis, during the scheduled period of study. I also declare that I have not taken any material from any source except referred to wherever due, that amount of plagiarism is within acceptable range. If a violation of HEC rules on research has occurred in this thesis, I shall be liable to punishable action under the plagiarism rules of the HEC.

.

Date:_______________

Wakeel Ahmed

CIIT/FA17-RMT-047/LHR

Certificate

It is certified that **Wakeel Ahmed** with **CIIT/FA17-RMT-047/LHR** has carried out all the work related to this thesis under my supervision at the Department of Mathematics, COMSATS University Islamabad, Lahore campus and the work fulfills the requirement for award of MS degree

Date:_________________

Supervisor:

____

Dr. M. Aqeel Ahmad Khan
Assistant Professor

Head of Department:

____

Dr. Sarfraz Ahmed
Associate Professor
Department of Mathematics

DEDICATED TO

My Parents, specially my elder brothers
Shakeel Ahmed and Late Ashfaq Ahmed.

Acknowledgements

First and foremost, I would like to thank **Allah** Almighty most beneficent and merciful for giving me opportunity, courage, ability and determination to complete my thesis. With the blessings of **Allah** Almighty I am able to achieve my goals. My significant appreciation to **Hazrat Muhammad (PBUH)** Who is everlastingly a track of direction and learning for mankind in general.
I would like to express my true gratitude to my profoundly learned, kind, generous and commited supervisor **Dr. M. Aqeel Ahmad Khan** who did not discriminate about day or night to help me in my research work. I am also thankful for his convenient and vigilant analyze to revamp the aspect of the thesis. His direction, kind supervision and productive recommendations empowered me to finish this work. Also I am thankful to Dr. Sarfraz Ahmad, Head of Department of Mathematics, for his direction amid course work and supportive state of mind to explore work and giving an environment to study and research in the department.

This acknowledgement will remain incomplete if I do not say a few words about my family for their steady care, tolerance, love, support, and prayers which will always remain behind me **In Sha Allah**. I would like to express my sincere gratitude towards my parents specially for my elder brothers **Shakeel Ahmed** and **Late Ashfaq Ahmed** for their help, advices, kind support and encouragement to complete my thesis.

I would also like to express my sincere gratitude to my teachers **Mr. Muhammad Saqib Khan** and **Mr. Iftikhar Ahmed**. Their advices and courage always help me. I am really very thankful to both of them.

Lahore **Wakeel Ahmed**
Spring, 2019 **CIIT/FA17-RMT-047/LHR**

Abstract

In this work, we aim to devise an implicit proximal algorithm to find an approximate common solution associated with the convex optimization problem and the fixed point problem of non-expansive mappings in Hadamard spaces. The convergence analysis of the proposed algorithm guarantees the strong convergence characteristics as well as $\Delta -$convergence characteristics under suitable set of control conditions.

Moreover, we extended our results from the case of single mapping to the case of family of mappings.

TABLE OF CONTENTS

Abbreviations

PPA = Proximal point algorithm
MMO = Maximal monotone operator
CCS = Closed convex subset
HS = Hilbert space
PC = Proper convex
SC = Strong convergence
BS = Banach space
FPP = Fixed point problem
FPT = Fixed point theory
FP = Fixed point
BCP = Banach contraction principle
MS = Metric space
GS = Geodesic space
LSC = Lower semi-continuous
ΔC = $\Delta-$convergent
CA = Convergence analysis

Chapter 1
Introduction and Basic Concepts

1.1 Convex Optimization and Algorithms

Optimization can be regarded as one of the basic tools in all areas of applied mathematics having applications in the field of engineering, medicine, economics, management, industry and other related areas of sciences. Moreover, optimization is considered as a major motivation and driving force for the development of differential and integral calculus. Consequently, this subject is getting attention of researchers and it is not only expanding in all directions of sciences but also serving as an interdisciplinary bridge between various related areas of sciences. Iterative methods or optimization algorithms are ubiquitous in the theory of optimization which can be used to solve problems in optimization for their feasible solution. A general optimization problem is stated as follows: suppose that we are given with a non-empty subset C of $\mathbb{R}$ and $g : C \longrightarrow \mathbb{R}$ as a given function. Recall that the optimization problem for g is to compute a feasible solution $x_0 \in C$ such that:

- $g(x_0) \leq g(x)$ for all $x \in C$ defines the minimization problem
- $g(x_0) \geq g(x)$ for all $x \in C$ defines the maximization problem

Generally, it is difficult to locate the exact solution of the above mentioned optimization problems. As a result, the research community is trying their level best to formulate algorithms and their consequent convergence analysis for the solution of an approximate optimization problem.

The most accurate and effective tool to resolve or approximating the general optimization problem is a PPP, which is prominent in convex optimization theory. In 1970, Martinet [36] proposed and analyzed the PPA as a tool to resolve the convex optimization issue $argmin_{(\bar{x} \in G)} h(\bar{x})$, where G satisfies the axioms of being CCS of HS H and h is $\mathbb{R} \cup \infty$-valued PC as well as LSC function defined on G.

Later on, Rockafellar [40] generalized the results of PPA to MMO's, which is the present form of PPA today and studied the convergence analysis of the PPA in HS. Rockafellar generalized the algorithm proposed by Martinet in the following manner:

$$y_{\theta+1} = (I + q_\theta B)^{-1} y_\theta + e_\theta, \;\; \forall\, \theta \; \geq 0,$$

for general MMO $B : D(B) \subset H \to H$, where the sequence $\{q_\theta\} \geq 0$, $\{e_\theta\}$ is sequence of computing errors and $x_o \in H$ is initial point. Rockafellar called above algorithm PPA as Moreau called in his paper $\bar{p} = (I + cB)^{-1}$ the proximal mapping related with cB.

This PPA is defined as

$$y_{\theta+1} = argmin_{\bar{y}\in\mathbb{H}} \left[g(\bar{y}) + \frac{1}{2\lambda_n} \| \; \bar{y} - \bar{y}_\theta \; \|^2 \right], \tag{1.1}$$

where $\lambda_\theta \geq 0$ for all $\theta \in \mathbb{N}$. The above definitions can also be written by using the resolvent J_f of f :

$$y_{\theta+1} = J_{\lambda_n f} \bar{y}_\theta, \text{ for all } \theta \in \mathbb{N},$$

where $J_f y = \arg\min_{\bar{y}\in H}\{f(\bar{y}) + \frac{1}{2}\|\bar{y} - y\|^2\}$.

The PPA does not have SC characteristics in general; see for example [7, 8, 23]. Since the introduction of PPA, numerous papers have been appeared to study the convergence analysis of PPA in linear setting of different spaces, such as HS as well as Banach space (BS), see for example [17, 25, 26, 35, 38, 39, 20]. It is worth revealing that the classical PPA can be modified in numerous behaviors. In 2013 [3], PPA proposed in Hadamard space (X, d) :

$$y_{\theta+1} = argmin_{\bar{y}\in\mathbb{H}} \left[g(\bar{y}) + \frac{1}{2\lambda_\theta} d(\bar{y}, \bar{y}_\theta)^2 \right], \tag{1.2}$$

where $\lambda_\theta \geq 0$ for all $\theta \in \mathbb{N}$. Furthermore, Chang et al. [13] introduced modification of PPA in Hadamard spaces, explaining minimization issues. They demonstrated the sequence produced by algorithm, SC to minimizer of convex detached functions.

1.2 Fixed Point Theory

Most of the phenomena happening in day by day life are commonly nonlinear in nature. Accordingly, mathematicians and scientists are continually in journey for finding new strategies to deal with the nonlinear problems. One of the appropriate way to deal with nonlinear problem is consider as to frame them and study them within the setting of Fixed Point Theory (FPT). FPT is a standout among most astonishing arrangement of present day mathematical sciences. In particular, it has been associated in such regions as biology, arithmetic, game theory, economics and physical sciences and so on. FPT is a part of nonlinear branch of mathematics however which the FP of any function ϖ can b approximated. i.e,

$$\exists\, x_i \in G \text{ such that } \varpi(x_i) = x_i,$$

where G is Domain set and ϖ is a non-linear map. FPT has substantiated itself a valuable apparatus for the investigation of various nonlinear Problems. The use of FPT has been extended broadly day by day and research community is trying to develop new methods to deal with non-linear problems as well as to develop the FPT and investigating the further importance of FPT .

We can expressed $\varpi(x_i)$ into $(\mathbb{I} - \varpi)x_i$, which relates the FPP to comparing issues in analysis of nonlinear differential equations and integral equations ascending in different fields of study, for instance, science, chemistry, physics, accounting, game hypothesis and programming building related fields.

Generally, FPT is intended for the investigation of the presence of related FP for the nonlinear maps under controlled conditions. The presence as well as understanding of related FP's for the nonlinear maps can be used to take care of issues in various parts of mathematical sciences. We comment that the iterative calculation of FP's has innate significance in different parts of applied sciences and appropriate for investigation of FP in such situations where the determination of constrains is important and it is hard to deal with those preconditions for presence.

FPT is arranged in following parts
(i)Topological FPT
(ii)Discrete FPT
(iii)Metric FPT (MFPT)

One of the principle inspiration for the investigation of MFPT is to find the presence just as uniqueness of the arrangement related with the system of equations. MFPT is a subdivision of FP theory which gives a schematic domain to the development of FP of different nonlinear mapping in various spaces. It is observed that the iterative domain of FP is the main tool to study such generalized classes of nonlinear mappings. There are many motivations to translate and generalize the linear version of the results concerning PPA to the corresponding nonlinear version in CAT(0). One of the main motivations is that a lot of non-convex optimization problems become convex through the introduction of an adequate metric defined on the CAT(0).

MFPT accepts to be started from the understood Banach Constriction Principle. In 1922, Stefan Banach [4] set up an amazing FP hypothesis called "Banach Contraction Principle" (BCP). It is the most broadly connected FP result in numerous parts of arithmetic since it needs format of complete MS with contraction condition.

Theorem 1.2.1 (BCP). *"Assume T is a complete MS and itself map $f : T \longrightarrow T$ satisfying*

$$d(Tx_i, Tz_i) \leq h \quad d(x_i, z_i), \quad for\ all \quad x_i, z_i \in T,$$

while $1 > h > 0$. Then there exists a unique FP t for η such that $x_n = \eta(x_{n-1})$ is the *sequence convergent to particular FP t. "*

BCP has a significant function in estimate of nonlinear conditions. It is a bond for the nearness of FP just as the uniqueness which can be assessed by using significant progressive calculation.

The hypothesis of non-expansive maps was started self-sufficiently in 1965 [11, 22, 27]. In nonlinear useful investigation the MFPT is an operational and considerational field of research. The hypothesis of non-expansive maps has pivotal applications. In like manner, we can evaluate the arrangements of monotone operators, raised minimizational problems and variational inequality of a function using non-expansive maps. For a far reaching talk of the FPT of non-expansive maps, see [21, 31, 29]. A class of non-linear maps known as asymptotically non-linear maps is considered for the most part in a few spaces. Regardless, the investigation of approximating the fixed focuses using the iterative calculations for asymptotically non-expansive maps and its particular suppositions have been eventually improving in the general arrangements of HS's. The iterative calculations can be utilized as a crucial instruments for understanding FPP of asymptotically non-expansive maps and its specific speculations.

1.3 Definitions

Definition 1.3.1. *[5]"If a graph of monotone operator $S(T) = \{(z,x) \in H * H : x \in T(z)\}$ is not properly contained in a graph of other monotone operator then it is said to be MMO."*

Definition 1.3.2. *[46]"The resolvent J_f of f is described as:*
$J_f y = \arg\min_{x\in H}\{f(x) + \frac{\|x-y\|^2}{2}\}$ *for all $x \in H$*
if f be a proper LSC convex function of HS H into $]-\infty,\infty]$. The generalized resolvent J_f in a Hadamard space X is described by
$J_f y = \arg\min_{x\in H}\{f(x) + \frac{d(x,y)^2}{2}\}$ *for all $x \in H$. "*

Definition 1.3.3. *"Let a function $f : S \to W$ is said to be proper function if the inverse image of every compact set in W is compact in S and (S, τ_S) (W, τ_W) are two topological spaces."*

Definition 1.3.4. *[16]"A FP or invariant point is a component of the function's domain which is self mapped by the function.*
Let T be a self mapping of Z then an element $z \in Z$ is said to be FP of T if $T(z) = z$."

As an example, let a function g is a self mapping of real numbers $\mathbb{R}$ and defined as $g(\varphi) = \varphi^2 - 3\varphi + 4$, then the FP of function g is $\varphi = 2$ as we have $g(2) = 2$.

Definition 1.3.5. *[44]"A self mapping T of HS H is said to be non-expansive if"*

$$\parallel T\bar{m} - T\bar{n} \parallel \leq \parallel \bar{m} - \bar{n} \parallel \quad \textit{for each } m, n \in H.$$

Definition 1.3.6. *[41]"Let a MS (X, d), a geodesic is a mapping g of $[0, l]$ into X where $l = d(x, y)$ such that:*

$$d(g(k), g(\bar{k})) = |k - \bar{k}| \ \textit{for all } k, \bar{k} \in [0, l]$$

and

$$g(0) = y, \ g(l) = z,$$

the images $y, \ z$ of g is the set $[y, z]$, called the geodesic segment between y and z. "

Definition 1.3.7. *[43]"Let X be a complete $CAT(0)$ space and $\{x_n\}$ a bounded sequence in X. For $x \in X$, write*

$$r(x, \{x_n\}) = \limsup_{n \to \infty} d(x, x_n).$$

The asymptotic radius r($\{x_n\}$) is given by

$$r(\{x_n\}) = \inf\{r(x, \{x_n\}) : x \in X\},$$

and the asymptotic center A($\{x_n\}$) of $\{x_n\}$ is defined as:

$$A(\{x_n\}) = \{x \in X : r(x, \{x_n\}) = r(\{x_n\})\}.$$

It is known that in a $CAT(0)$ space, A($\{x_n\}$) consists of exactly one point. "

Definition 1.3.8. *[41]"In a geodesic space (GS) (X, d) a geodesic triangle $\triangle(y_1, y_2, y_3)$ consists of the vertices of triangle in X and a geodesic segment between the edges of triangle."*

Definition 1.3.9. *[41]"In Euclidean plane $(\mathbb{R}^2)$ a comparison triangle for $\triangle(y_1, y_2, y_3)$ in a MS (X, d) is a triangle $\bar{\triangle}(\bar{y}_1, \bar{y}_2, \bar{y}_3)$ such that:*

$$d_{\mathbb{R}^2}(\bar{y}_m, \bar{y_n}) = d(y_m, y_n)$$

for $m, n \in \{1, 2, 3\}$ "

Definition 1.3.10. *"If the values of a function f near c are close to $f(c)$ or greater than $f(c)$, then f is said to be LSC at $c \in$ Dom$f$. Ceiling function is a LSC function which is defined by $h(a) = \lceil a \rceil$.*
Let a function ξ of real numbers $\mathbb{R}$ into $\mathbb{R}$ is LSC function which is described by $\xi(z) = 0,\ \ z$ is rational and $\xi(z) = 1,\ \ z$ is irrational, for all $z \in \mathbb{Q}$."

Definition 1.3.11. *[48]"In a norm space Y a sequence $\{y_n\}$ is said to be SC if a component $y \in Y$ such that:*

$$\lim_{n\to\infty} \| y_n - y \| = 0$$

For example, a sequence $\{\frac{1}{m}\}$ SC to zero."

Definition 1.3.12. *[6]"If an asymptotic centre $\mathcal{A}$ $\{y_{n_j}\}$=$\{q\}$ for every subsequence $\{y_{n_j}\}$ of $\{y_n\}$, then a sequence $\{y_n\}$ is said to be $\Delta-$convergent (ΔC) to an element $q \in Y$. "*

Definition 1.3.13. *[10]"If $d(x_n, p) \leq d(x_{n-1}, p)$ for all $p \in C$ and for all $n \geq 1$, then a sequence $\{x_n\}$ in a GS (X, d) is said to be Fejer monotone regarding $C \subset X$. "*

Definition 1.3.14. *[12] "T is said to be semi-compact, if for any bounded sequence x_n in D such that $\parallel x_n - Tx_n \parallel \rightarrow 0(n \rightarrow \infty)$, then there exists a subsequence $x_{n_i} \subset x_n$ such that $x_{n_i} \rightarrow x^* \in D$."*

Definition 1.3.15. *[12] "T is said to be asymptotically non-expansive , if there exists a sequence $\{h_n\} \subset [1, \infty) with \lim_{n\rightarrow\infty} h_n = 1$ such that,"*

$$\parallel T^n(x) - T^n(y) \parallel \leq h_n \parallel x - y \parallel \ \forall\, x, y \in D,\ n \geq 1.$$

1.4 Motivation and Problem Statement

The primary thought process of this thesis is engaged to start the investigation of iterative algorithm to surmised the familiar component in solution set for FP class along with the FP of a group of nonlinear mappings utilizing PPA joined with Maan iteration in Hadamard spaces. Our outcomes is the generalization of some specific current conclusions in the theory of non-linear mapping in GS's. The primary motive behind this work is to contributing in this area of research through establishing the convergence results for non-expansive mapping, finite as well as countable infinite family of non-expansive mapping in Hadamard spaces.

1.5 Objectives of the Study

The essential thought process of this examination is to state and demonstrate an iterative calculation approximation to established a SC and also ΔC for a proposed implicate iterative scheme in Complete $CAT(0)$ spaces.

1.6 Thesis Arrangement

We arrange rest of the thesis as following:
In Chapter 2, we show convergence analysis of proposed algorithmic plan define implicitly for non-expansive mapping in complete GS's.
In chapeter 3, we discuss further results related to SC as well as ΔC for two different types of non-expansive mapping which is Finite Family and Infinite Family in the setting of Hadmarad spaces for the Implicit type Mann iteration collaborate with PPA.

Chapter 2
Convergence Analysis of a Non-expansive Mapping

In this chapter, we are describing convergence analysis(CA) of our proposed implicit iterative algorithm for the non-expansive mapping and we successively obtained results related to SC as well as ΔC for our proposed algorithm, under some appropriate controlled conditions within $CAT(0)$ space.

It is of prominent concern to interpreting the linear form of a familiar problem to its corresponding nonlinear form. In different methods of science mostly problems are nonlinear in description. Without linear system, analysis of various problems in spaces has its own significance in applied and pure sciences. To propose an convex arrangement on MS, numerous experiments have been formed Takahashi [42] studied the properties of convex MS and introduced view of convexity. For non-expansive mapping they defined FP theorems and examined in certain MS. Corresponding convex structures feasible in $CAT(0)$ spaces. We deal with $CAT(0)$ spaces throughout this article. Some of the interesting established results exist in $CAT(0)$ can be studied from [15, 18, 33].
In FPT it follow that for different classes of maps in various spaces the iterative development of FP's is vigorously investigated and proposed. By utilizing distinctive iterative procedures on various spaces by estimating FP's of nonlinear mappings has stayed at the core of FPT [32]. The improved estimate of FP's is present in implicit algorithm than explicit algorithm.

2.1 Preliminaries

We review a few lemmas and some basic concepts in this section which will be utilized in the main result.

Let (X, d) fulfills the axioms of MS and $x, y \in X$ such that $d(x, y) = l$. A mapping $\zeta : [0, l] \to X$ defines a geodesic path between these two points in such a way that $\zeta(0) = x,\ \zeta(l) = y$ and

$$d(\zeta(a), \zeta(b) = |a - b| \quad \forall a, b \in [0, l]$$

. This also means that the mapping ζ is an isometry. We denote the geodesic segment between $x, y \in X$ by $[x, y]$. Suppose we have geodesic space triangle $\triangle(x_1, x_2, x_3)$ and its comparison Euclidean space triangle $\overline{\triangle}(\bar{x_1}, \bar{x_2}, \bar{x_3})$, then the following expression

$$d(y, x) \leq d_{\mathbb{R}^2}(\overline{y}, \overline{x})$$

is called the $CAT(0)$ space inequality. Geometrically, it implies that the geodesic space triangle is as thin as its comparison Euclidean space triangle. Moreover, a GS becomes the $CAT(0)$ space if it satisfies the following inequality:

$$d^2((1-\alpha)a \oplus \alpha b, c) \leq (1-\alpha)d^2(a, c) + \alpha d^2(b, c) - \alpha(1-\alpha)d^2(a, b), \tag{2.1}$$

for all $a, b, c \in X$ and $\alpha \in [0, 1]$. Also note that

$$d((1-\alpha)a \oplus \alpha b, c) \leq (1-\alpha)d(a, c) + \alpha d(b, c). \tag{2.2}$$

Some example of $CAT(0)$ are:

(i) Hyperbolic spaces

(ii) HS's

(iii) Euclidean spaces $\mathbb{R}^n$,

(iv) $\mathbb{R}-$Trees

In a MS Lim [34] studied the ΔC technique and Panyanak and Dhompongsa [19] proposed the analogue of ΔC in Hadamard spaces. In this paper, for general Hadamard spaces we explored the ΔC. To follow this, we have the following approaches.

Remark: "The uniformaly convex BS's and also $CAT(0)$ spaces appreciate the property which is, bounded sequences have particular asymptotic centers regarding to CCS."

Lemma 2.1.1. *[24]"The resolvent J_λ of f is non-expansive for any $\lambda > 0$, if $f : X \to (-\infty, \infty]$ is LSC and PC function of a complete $CAT(0)$ space (X, d)."*

Lemma 2.1.2. *[1]"Following statement hold for $\lambda > 0$ and for all $z, y \in X$,*

$$f(y) \geq \frac{1}{2\lambda} d^2(J_\lambda z, y) - \frac{1}{2\lambda} d^2(z, y) + \frac{1}{2\lambda} d^2(z, J_\lambda z) + f(J_\lambda z),$$

where function $f : X \to (-\infty, \infty]$ be LCS and PC map in Hadamard spaces."

Lemma 2.1.3. *[28]"If $\sum_{n=1}^{\infty} \alpha_n < \infty$ and let two non-negative sequences $\{\beta_n\}$ and $\{\alpha_n\}$, which satisfy the successive condition:*

$$\beta_{n+1} \leq (1 + \alpha_n)\beta_n \;\; \forall\, n \geq n_0 \, \textit{for some } n_0 \geq 1,$$

then $\lim_{n\to\infty} \beta_n$ exists."

Lemma 2.1.4 (Resolvent Identity). *[24, 37]"The following statement hold*

$$J_\lambda x = J_\nu\left(\frac{\lambda - \nu}{\lambda} J_\lambda x \oplus \frac{\nu x}{\lambda}\right)$$

for all $x \in X$ and $\lambda > \nu > 0$, in Hadamard spaces where $f : X \to (-\infty, \infty]$ satisfy the axioms of PC and also LSC function."

Lemma 2.1.5. *[30]"In Hadamard spaces, always has a ΔC subsequences for every bounded sequence."*

Lemma 2.1.6. *[19]"A sequence $\{x_n\}$ has the asymptotic center with in C, if it is a bounded sequence itself, where C be a CCS of Hadamard spaces."*

Lemma 2.1.7. *[13]"Let a nonempty CCS C of a complete MS (X, d) and the sequence $\{x_n\}$ be a Fejer monotone w.r.t. C. Then $\{x_n\}$ converges to some point $p \in C$ if and only if $\lim_{n\to\infty} d(X_n, C) = 0$."*

Lemma 2.1.8. *[2]"Let family of non-expansive mapping $\{T_n\}$ satisfies $AKTT$ condition. Then, for every $z \in C, T_n z$ SC to point in C, where T is expressed as*

$$Tz = \lim_{n\to\infty} T_n z \ \ \forall z \in C.$$

Then for every bounded subset B of C, $\lim_{n\to\infty} \sup_{z\in B} \parallel Tz - T_n z \parallel = 0$. "

2.2 Main Result

We demonstrate the fundamental results for this section below as.

Lemma 2.2.1. *Let C be a non-empty CCS in a complete $CAT(0)$ space (X, d) and let $f : X \to (-\infty, \infty]$ be PC and LSC function and let non-expansive mapping T on C such that $\Omega = F(T) \cap argmin_{z\in C} f(z) \neq \emptyset$. Suppose sequence λ_θ such that $\lambda_\theta \geq \lambda > 0$ for some λ and for all $\theta \in \mathbb{N}$. Assume that $y_1 \in C$ chosen arbitrarily and y_θ is generated in the following manner:*

$$\begin{aligned} z_\theta &= argmin_{z\in C}[f(z) + \frac{1}{2\lambda_\theta} d^2(z, y_{\theta+1})] \\ y_{\theta+1} &= \gamma_\theta y_\theta \oplus (1-\gamma_\theta) T z_\theta. \end{aligned} \tag{2.3}$$

for each $\theta \in \mathbb{N}$, where $\gamma_\theta \subset [\delta, 1-\delta]$ for some $\delta \in (0, \frac{1}{2})$, then

$$\lim_{\theta\to 0} d(y_\theta, Ty_\theta) = 0.$$

Proof. Firstly, we show that $\{y_\theta\}$ generated by (2.3) is well-defined for every $\theta \geq 1$. Let us describe a map

$$G_\theta : X \to X,$$

such that

$$G_\theta(y) = \gamma_\theta y_\theta \oplus (1-\gamma_\theta) T(J_{\lambda_\theta} y),$$

we have

$$
\begin{aligned}
d(G_\theta(y), G_\theta(z)) &= d(\gamma_\theta y_\theta \oplus (1-\gamma_\theta)T(J_{\lambda_\theta}y), \gamma_\theta z_\theta \oplus (1-\gamma_\theta)T(J_{\lambda_\theta}z) \\
&\leq (1-\gamma_\theta)(J_{\lambda_\theta}y, J_{\lambda_\theta}z),
\end{aligned}
$$

as J_{λ_θ} is non-expansive

$$
d(G_\theta(y), G_\theta(z)) \leq (1-\gamma_\theta)d(y,z),
$$

where $\gamma_\theta \subset [\delta, 1-\delta]$ for some $\delta \in (0, \frac{1}{2})$ also G is well-defined. Now we show $\{y_\theta\}$ is the bounded. Since

$$
\begin{aligned}
d(y_{\theta+1}, \bar{y}) &\leq \gamma_\theta d(y_\theta, \bar{y}) + (1-\gamma_\theta)d(Tz_\theta, \bar{y}) \\
&\leq \gamma_\theta d(y_\theta, \bar{y}) + (1-\gamma_\theta)d(z_\theta, \bar{y}) \\
&= \gamma_\theta d(y_\theta, \bar{y}) + (1-\gamma_\theta)d(J_{\lambda_\theta}y_{\theta+1}, \bar{y}) \\
&\leq \gamma_\theta d(y_\theta, \bar{y}) + (1-\gamma_\theta)d(y_{\theta+1}, \bar{y}) \\
&\leq d(y_\theta, \bar{y}). \qquad (2.4)
\end{aligned}
$$

By applying Induction, we examine

$$
d(y_{\theta+1}, \bar{y}) \leq d(y_1, \bar{y}), \qquad \text{for all} \quad \theta \geq 1
$$

which implies sequence $\{y_\theta\}$ is bounded and so $\{Ty_\theta)\}$, $\{z_\theta\}$, $\{Tz_\theta\}$.
Now, we prove $d(y_\theta, Ty_\theta) = 0$ as $\theta \to \infty$, so therefore

$$
\begin{aligned}
d(y_\theta, Ty_\theta) &\leq d(y_\theta, y_{\theta+1}) + d(Tz_\theta, Ty_\theta) + d(y_{\theta+1}, Tz_\theta) \\
&\leq d(y_\theta, y_{\theta+1}) + \gamma_\theta d(y_\theta, Tz_\theta) + (1-\gamma_\theta)d(Tz_\theta, Ty_\theta). \qquad (2.5)
\end{aligned}
$$

Using (2.1)

$$
\begin{aligned}
d^2(y_{\theta+1}, \bar{y}) &\leq \gamma_\theta d^2(y_\theta, \bar{y}) + (1-\gamma_\theta)d^2(Tz_\theta, \bar{y}) - \gamma_\theta(1-\gamma_\theta)d^2(y_\theta, Tz_\theta) \\
&\leq \gamma_\theta d^2(y_\theta, \bar{y}) + (1-\gamma_\theta)d^2(z_\theta, \bar{y}) - \gamma_\theta(1-\gamma_\theta)d^2(y_\theta, Tz_\theta) \\
&\leq \gamma_\theta d^2(y_\theta, \bar{y}) + (1-\gamma_\theta)d^2(y_{\theta+1}, \bar{y}) - \gamma_\theta(1-\gamma_\theta)d^2(y_\theta, Tz_\theta),
\end{aligned}
$$

now by utilizing (2.4), we get

$$d^2(y_{\theta+1}, \bar{y}) \ \leq \ d^2(y_\theta, \bar{y}) - \gamma_\theta(1-\gamma_\theta)d^2(y_\theta, Tz_\theta),$$

assume that $\sigma_\theta \ = \ d(y_\theta, Tz_\theta)$, which implies

$$2\delta^3\sigma_\theta^2 \ \leq \ d^2(y_\theta, \bar{y}) - d^2(y_{\theta+1}, \bar{y}),$$

for some $m \geq 1$

$$2\delta^3 \sum_{\theta=1}^{m} \sigma_\theta^2 \ \leq \ d^2(y_o, \bar{y}) - d^2(y^{m+1}, \bar{y}) \leq d^2(y_o, \bar{y}),$$

when $m \to \infty$

$$\sum_{\theta=1}^{\infty} \sigma_\theta^2 \ < \ \infty,$$

which means

$$\lim_{\theta\to\infty} \sigma_\theta \ = \ \lim_{\theta\to\infty} d(y_\theta, Tz_\theta) \ = \ 0.$$

As we have

$$d(y_{\theta+1}, y_\theta) \leq \ (1-\gamma_\theta)d(Tz_\theta, y_\theta) + \gamma_\theta d(y_\theta, y_\theta),$$

when $\theta \to \infty$, we have $d(Tz_\theta, y_\theta) = 0$, implies $d(y_{\theta+1}, y_\theta) = 0$.

By Lemma 2.1.2

$$\frac{1}{2\lambda_\theta}d^2(z_\theta, \bar{y}) + \frac{1}{2\lambda_\theta}d^2(y_\theta, z_\theta) - \frac{1}{2\lambda_\theta}d^2(y_\theta, \bar{y}) \leq f(z) - f(J_\lambda y),$$

such as $f(z) \leq f(J_\lambda y)$, it follows that

$$d^2(z_\theta, \bar{y}) \leq d^2(y_\theta, \bar{y}) - d^2(y_\theta, z_\theta).$$

As we know that

$$d^2(y_{\theta+1}, \bar{y}) = d^2(\gamma_\theta y_\theta \oplus (1-\gamma_\theta)Tz_\theta, \bar{y}).$$

By (2.1)

$$\begin{aligned} d^2(y_{\theta+1},\bar{y}) &\le \gamma_\theta d^2(y_\theta,\bar{y})+(1-\gamma_\theta)d^2(Tz_\theta,\bar{y})-\gamma_\theta(1-\gamma_\theta)d^2(y_\theta,Tz_\theta)\\ &\le \gamma_\theta d^2(y_\theta,\bar{y})+(1-\gamma_\theta)d^2(z_\theta,\bar{y})-\gamma_\theta(1-\gamma_\theta)d^2(y_\theta,Tz_\theta)\\ &\le (1-\gamma_\theta)[d^2(y_\theta,\bar{y})-d^2(y_\theta,z_\theta)]+\gamma_\theta[d^2(y_\theta,\bar{y})-(1-\gamma_\theta)d^2(y_\theta,Tz_\theta)]\\ &\le (1-\gamma_\theta)d^2(y_\theta,\bar{y})+(\gamma_\theta-1)d^2(y_\theta,z_\theta)+\gamma_\theta d^2(y_\theta,\bar{y})-\gamma_\theta(1-\gamma_\theta)d^2(y_\theta,Tz_\theta)\\ &\le d^2(y_\theta,\bar{y})-(1-\gamma_\theta)d^2(y_\theta,z_\theta)-\gamma_\theta(1-\gamma_\theta)d^2(y_\theta,Tz_\theta), \end{aligned}$$

from above estimate, we calculate that

$$(1-\gamma_\theta)d^2(y_\theta,z_\theta) \;\le\; d(y_{\theta+1},y_\theta)[d(y_\theta,\bar{y})+d(y_{\theta+1},\bar{y})]-\gamma_\theta(1-\gamma_\theta)d^2(y_\theta,Tz_\theta),$$

which implies $(1-\gamma_\theta)d^2(y_\theta,z_\theta)=0$, when $\theta\to\infty$.

Now from (2.5)

$$d(y_\theta,Ty_\theta)\le d(y_\theta,y_{\theta+1})+\gamma_\theta d(y_\theta,Tz_\theta)+(1-\gamma_\theta)d(z_\theta,y_\theta),$$

we have $d(y_\theta,y_{\theta+1}),d(y_\theta,Tz_\theta)\ and\ d(y_\theta,z_\theta)=0$, as $\theta\to\infty$, implies $\lim_{\theta\to\infty}d(y_\theta,Ty_\theta)=0$.

Lemma 2.2.2. *Let $T,\ C,\ \Omega$ and γ_θ be as given in Lemma 2.2.1. Starting from $y_1\in C$ an arbitrary point, the sequence $\{y_\theta\}$ defined in (2.3) is Fejer monotone w.r.t Ω. Moreover, $\{y_\theta\}$ SC to an element in Ω if and only if $\lim_{\theta\to\infty}d(y_\theta,\Omega)=0$.*

Proof. We show that regarding to Ω, $\{y_\theta\}$ is a Fejer monotone and let $\bar{p}\in\Omega$.

From (2.3)

$$\begin{aligned} d(y_{\theta+1},\bar{p}) &= d(\gamma_\theta y_{\theta-1}\oplus(1-\gamma_\theta)Tz_\theta)\\ &\le \gamma_\theta d(y_\theta,\bar{p})+(1-\gamma_\theta)d(y_{\theta+1},\bar{p})\\ &\le d(y_\theta,\bar{p}), \end{aligned} \tag{2.6}$$

which also implies

$$d(y_{\theta+1},\Omega)\le d(y_\theta,\Omega). \tag{2.7}$$

From (2.6) and (2.7) we conclude that $\{y_\theta\}$ is Fejer monotone. Next we establish $\{y_\theta\}$ SC to an element in Ω if and only if $\lim_{\theta\to\infty} d(y_\theta, \Omega) = 0$.
$\lim_{\theta\to\infty} d(y_\theta, \bar{p}) = 0$ if $y_\theta \to \bar{p} \in \Omega$, since $0 \leq d(y_\theta, \Omega) \leq d(y_\theta, \bar{p})$ implies $\liminf_{\theta\to\infty} d(y_\theta, \Omega) = 0$.
Contrary, assume $\lim_{\theta\to\infty} d(y_\theta, \bar{p}) = 0$, after implementing Lemma 2.1.4 and 2.1.5, $\lim_{\theta\to\infty} d(y_\theta, \Omega) = 0$. By supposition $\liminf_{\theta\to\infty} d(y_\theta, \Omega) = 0$, we observe $\lim_{\theta\to\infty} d(y_\theta, \Omega) = 0$. Next we show the sequence $\{y_\theta\}$ is Cauchy.
Since from (2.6)

$$d(y_{\theta+1}, \bar{p}) \leq d(y_\theta, \bar{p}),$$

from (2.4), we have

$$d(y_{\theta+1}, \bar{p}) \leq d(y_1, \bar{p}),$$

for $m \in \mathbb{N}$, we have

$$d(y_{\theta+m}, \bar{p}) \leq d(y_\theta, \bar{p}), \tag{2.8}$$

as $\lim_{\theta\to\infty} d(y_\theta, \Omega) = 0$, for any $\varepsilon > 0 \ \exists\, \theta_o \in \mathbb{N}$ such that $d(y_\theta, \Omega) < \frac{\varepsilon}{2}$, we can find that $\bar{p} \in \Omega$ such that $d(y_\theta, \bar{p}) \leq \frac{\varepsilon}{2}$.

$$d(y_{\theta+m}, y_\theta) \leq d(y_{\theta+m}, \bar{p}) + d(y_\theta, \bar{p}) \text{ for all } \theta \geq \theta_o, \ \ m \geq 1$$

from (3.7)

$$\begin{aligned} &\leq d(y_\theta, \bar{p}) + d(y_\theta, \bar{p}) \\ &\leq 2d(y_\theta, \bar{p}) \\ &< 2d(y_{\theta_o}, \bar{p}) \\ &< \varepsilon, \end{aligned} \tag{2.9}$$

which ensure that the sequence $\{y_\theta\}$ is Cauchy, assume $\lim_{\theta\to\infty} y_\theta = z$. As C is closed, implies $z \in C$. Now we show $z \in \Omega$. The following two inequalities

$$\begin{aligned} d(z,\bar{p}) &\leq d(z,y_\theta) + d(y_\theta,\bar{p}) \text{ for all } \bar{p} \in \Omega,\ \theta \leq 1 \\ d(z,y_\theta) &\leq d(z,\bar{p}) + d(y_\theta,\bar{p}) \text{ for all } \bar{p} \in \Omega,\ \theta \leq 1 \end{aligned} \tag{2.10}$$

given that

$$|d(z,\Omega) - d(y_\theta,\Omega)| \leq d(z,y_\theta), \quad \theta \geq 1$$

as $\lim_{\theta\to\infty} y_\theta = z$ and $\lim_{\theta\to\infty} d(y_\theta,\Omega) = 0$, we conclude $z \in \Omega$.

From above Lemmas, we can conclude the following results.

This completes the proof.

Theorem 2.2.3. Let $T,\ C,\Omega$ and γ_θ be as given in Lemma 2.2.1 and suppose the sequence $\{y_\theta\}$ defined in (2.3) SC to an element in Ω if and only if there exist a subsequence $\{y_{\theta_j}\}$ of $\{y_\theta\}$ which converges to $p \in \Omega$.

Theorem 2.2.4. Let $T,\ C$ and Ω be as given in Lemma 2.2.1 and let a sequence $\{y_\theta\}$ is defined implicity in (2.3) is Fejer monotone and SC to an element in Ω.

Proof. By following Lemma 2.2.1, $\lim_{\theta\to\infty} d(y_\theta, Ty_\theta) = 0$, furthermore from Lemma 2.2.2, we observe that, $\lim_{\theta\to\infty} d(y_\theta,\Omega) = 0$. Therefore, $\{y_\theta\}$ SC to $p \in \Omega$. This completes the proof.

Theorem 2.2.5. Let $T,\ C$ and Ω be as given in Lemma 2.2.1. Then the sequence $\{y_\theta\}$ defined implicity in (2.3), ΔC to an element in Ω.

Proof. From Lemma 2.2.1, $\{y_\theta\}$ is a bounded sequence and also from Lemma 2.1.5 and 2.1.6 and also recalling the definition of ΔC we observe that the sequence $\{y_\theta\}$ has a unique asymptotic center, which is A($\{y_\theta\}$)=$\{y_\theta\}$.

For a sequence $\{y_\theta\}$ suppose there exist any subsequence $\{\bar{y}_\theta\}$ such that A($\{\bar{y}_\theta\}$)=$\{\bar{y}\}$. From Lemma 2.2.1 we conclude that $d(\bar{y}_\theta, T_\theta \bar{y}_\theta) = 0,$ as $\lim_{\theta\to\infty}$ for every $\theta \geq \mathbb{N}$.

Let $\bar{y} \in \Omega$ and by Lemma 2.2.1, we observe that $\lim_{\theta\to\infty} d(y_\theta, \bar{y})$ exists. Assume that $y \neq \bar{y}$. By particular asymptotic center

$$\begin{aligned}\limsup_{\theta\to\infty} d(\bar{y}_\theta, \bar{y}) &< \limsup_{\theta\to\infty} d(\bar{y}_\theta, y)\\ &\leq \lim_{\theta\to\infty} supd(y_\theta, y)\\ &< \lim_{\theta\to\infty} supd(y_\theta, \bar{y})\\ &= \lim_{\theta\to\infty} supd(\bar{y}_\theta, \bar{y}),\end{aligned}$$

which shows contradiction. Thus, $y = \bar{y}, \quad (\bar{y}_\theta)$ is any randomly chosen subsequence of $\{y_\theta\}$, so A($\{\bar{y}_\theta\}$)=$\{\bar{y}\}$ for all subsequences $\{\bar{y}_\theta\}$ of $\{y_\theta\}$. This proves the sequence $\{y_\theta\}$ is ΔC to $\bar{y} \in \Omega$.

This completes the proof.

Chapter 3
Convergence Analysis for Family of Non-expansive Mapping

This chapter is focused on CA of finite family of non-expansive mapping and countable infinite family, the purpose of this work is to established the results related to SC as well as ΔC.

3.1 Finite Family

Xu and Ori [45] proved in HS, an implicit iteration procedure converges to a common FP of finite family of non-expansive mapping. They established successive procedure for finite family $\{T_\theta : \theta \in I\}$, I=$\{1, 2, ..., \Theta\}$, taking a sequence $\{\gamma_\theta\}$ in interval $(0, 1)$ and $y_0 \in C$ is an initial point:

$$\begin{aligned} y_1 &= \gamma_1 y_0 + (1-\gamma_1)T_1 y_1, \\ y_2 &= \gamma_2 y_0 + (1-\gamma_2)T_2 y_2, \\ &. \\ &. \\ &. \\ y_\Theta &= \gamma_\Theta y_{\Theta-1} + (1-\gamma_\Theta)T_\Theta y_\Theta, \\ y_{\Theta+1} &= \gamma_{\Theta+1} y_\Theta + (1-\gamma_{\Theta+1})T_1 y_{\Theta+1}, \\ &. \\ &. \\ &. \end{aligned}$$

and its compact structure can be written as

$$y_\theta = \gamma_n y_{\theta-1} + (1-\gamma_\theta)T_\theta y_\theta, \tag{3.1}$$

for all $\theta \geq 1, T_\theta = T_{\theta(\mod \Theta)}$ and $\mod \Theta$ takes values in I

Zhou and Chang [47] presented work in BS's, they discussed implicit iteration method for weak convergence as well as SC to common FP for asymptotically and finite family of non-expansive mapping. Below given the result they proved:

Theorem.[45]"Let a uniformly convex BS E, a nonempty CCS $C \subset E$, $\{T_1, T_2, ..., T_\Theta\}$: $C \to C$ be Θ semi-compact non-expansive mappings with $F = \cap_{\theta=1}^{\Theta} F(T_\theta) \neq \emptyset$ and $\{\alpha_\theta\}$ in interval $[0,1]$ satisfying $b < \alpha_\theta < c$, where b, c are constants $b, c \in (0,1)$. Then implicit iterative sequence $\{x_\theta\}$ described by (3.1), weakly converges to a common FP $\{T_1, T_2, ..., T_\Theta\}$."

Motivated by the work which is described above, we obtain convergence results of the prepared schematic algorithm under some suitable control condition on control sequences of parameters. The proposed algorithm SC to common component of solution set of the minimizer and familiar set of FP's of Finite family of non-expansive mapping. As a consequence, we contributed in this area of research as a modification and generalization of convergence results suited to our proposed implicit type algorithm.

3.2 Main Result

In this section we are dealing with finite family of non-expansive mapping in a complete $CAT(0)$ space for the generalized proposed implicit iterative scheme (3.2) describe below and we establish SC as well as ΔC.

Lemma 3.2.1. *Let C be a nonempty CCS of a complete $CAT(0)$ space (X, d). Let $\{T_\theta\}_{\theta=1}^{\Theta}$ be a finite family of non-expansive mappings of C into itself and let $\{f_\theta\}_{\theta=1}^{\Theta}$ be a finite family of PC and LSC functions of C into $(-\infty, \infty]$ in a complete $CAT(0)$ spaces (X, d). Assume that $\Omega = \cap_{\theta=1}^{\Theta} \arg\min_{\nu \in C} f_\theta(\nu) \cap_{\theta=1}^{\Theta} F(T_\theta)$ is nonempty. For $y_1 \in C$,* *let a sequence $\{y_\theta\}$ in C defined as follows:*

$$\begin{aligned} z_\theta &= \arg\min_{\nu \in C}\left[f_\theta(\nu) + \frac{1}{\lambda_\theta} d(\nu, y_{\theta+1})^2\right], \\ y_{\theta+1} &= \gamma_n y_\theta \oplus (1-\gamma_\theta) T_n z_\theta. \end{aligned} \tag{3.2}$$

Let $\gamma_\theta \subset [\delta, 1-\delta]$ for some $\delta \in (0, \frac{1}{2})$, then $\lim_{\theta\to\infty} d(y_\theta, q^*)$ exists for each $q^* \in \Omega$.

Proof. Suppose for each $q^* \in \Omega$ and suppose $q^* = T_n q^*$ for all $\theta = 1, 2, ..., \Theta$ and $f_\theta(q^*) \le f_\theta(z)$ for all $z \in C$. Therefore

$$f_\theta(q^*) + \frac{d(q^*, q^*)^2}{2\lambda_\theta} \le f_\theta(z) + \frac{d(z, q^*)^2}{2\lambda_\theta} \text{ for all } z \in C, \;\; \theta = 1, 2, ..., \Theta.$$

Hence we have $q^* = J_{\lambda_\theta} q^*$ for all $\theta \ge 1$. As $z_\theta = J_{\lambda_\theta} y_{\theta+1}$ for all $\theta \ge 1$, $\theta = 1, 2, ..., \Theta$. We have

$$d(z_\theta, q^*) = d(J_{\lambda_\theta} y_{\theta+1}, J_{\lambda_\theta} q^*) \le d(y_{\theta+1}, q^*). \tag{3.3}$$

It follows from convexity

$$\begin{aligned} d(y_{\theta+1}, q^*) &= d(\gamma_\theta y_\theta \oplus (1-\gamma_\theta) T_\theta z_\theta, q^*) \\ &\le \gamma_\theta d(y_\theta, q^*) + (1-\gamma_\theta) d(T_\theta z_\theta, q^*). \end{aligned}$$

By utilizing (3.2)

$$\begin{aligned} d(y_{\theta+1}, q^*) &\le \gamma_\theta d(y_\theta, q^*) + d(z_\theta, q^*) - \gamma_\theta d(z_\theta, q^*) \\ &\le d(y_\theta, q^*). \end{aligned} \tag{3.4}$$

Now, from (3.4) $\lim_{\theta\to\infty} d(y_\theta, q^*)$ exists.

This completes the proof

Lemma 3.2.2. *Let $\{T_\theta\}_{\theta=1}^{\Theta}$, $\{f_\theta\}_{\theta=1}^{\Theta}$, C and $\{y_\theta\}$ be as given in Lemma 3.2.1. Assume that $\Omega = \cap_{\theta=1}^{\Theta} \arg\min_{\nu\in C} f_\theta(\nu) \cap_{\theta=1}^{\Theta} F(T_\theta)$ is nonempty, then*

$$(i) \lim_{\theta\to 0} d(y_\theta, T_l y_\theta) = 0. \qquad l < \mathbb{N}$$

$$(ii) \lim_{\theta\to\infty} d(y_\theta, J_{\lambda_\theta} y_\theta) = 0. \textit{ for all } \theta \in I$$

Proof. Firstly, we show $\{y_\theta\}$ generated by (3.2) is well-define for each $\theta \geq 1$. Let define a map

$$T_\theta : X \to X,$$

such that

$$T_\theta(y) = \alpha_\theta y_\theta \oplus (1-\gamma_\theta) T_\theta(J_{\lambda_\theta} y),$$

$$\begin{aligned} d(T_\theta(y), T_\theta(z)) &= d(\gamma_\theta y_\theta \oplus (1-\gamma_\theta) T_\theta(J_{\lambda_\theta} y) \gamma_\theta y_\theta \oplus (1-\gamma_\theta) T_\theta(J_{\lambda_\theta} z)) \\ &\leq (1-\gamma_\theta)(J_{\lambda_\theta} y, J_{\lambda_\theta} z), \end{aligned}$$

as we know $d(J_{\lambda_\theta} y, J_{\lambda_\theta} z) \leq d(y,z)$, so therefore

$$d(T_\theta(y), T_\theta(z)) \leq (1-\gamma_\theta) d(y,z),$$

where $\gamma_\theta \subset [\delta, 1-\delta]$ for some $\delta \in (0, \frac{1}{2})$ and T_θ is well-define. Next we show that $\{y_\theta\}$ is bounded.

Suppose $q^* \in \Omega$ and $J_{\lambda_\theta} y_{\theta+1} \ = \ z_\theta$, we have

$$\begin{aligned} d(y_{\theta+1}, q^*) &\leq \gamma_\theta d(y_\theta, q^*) + (1-\gamma_\theta) d(T_\theta z_\theta, q^*), \\ &\leq \gamma_\theta d(y_\theta, q^*) + (1-\gamma_\theta) d(z_\theta, q^*), \\ &\leq \gamma_\theta d(y_\theta, q^*) + (1-\gamma_\theta) d(J_{\lambda_\theta} y_{\theta+1}, q^*), \end{aligned}$$

utilizing (3.3)

$$\begin{aligned} &\leq \gamma_\theta d(y_\theta, q^*) + (1-\gamma_\theta) d(y_{\theta+1}, q^*), \\ &\leq d(y_\theta, q^*). \end{aligned} \qquad (3.5)$$

By applying Induction

$$d(y_{\theta+1}, q^*) \leq d(y_1, q^*), \text{ for all } \theta \in \mathbb{N},$$

which ensure us that $\{y_\theta\}$ is bounded and also $\{T_\theta y_\theta\}$, $\{z_\theta\}$, $\{T_\theta z_\theta\}$.

Now we prove that $d(y_\theta, T_\theta y_\theta) = 0$ as $\theta \to \infty$.

we have,

$$\begin{aligned} d(y_\theta, T_\theta y_\theta) &\leq d(y_{\theta+1}, T_\theta z_\theta) + d(T_\theta z_\theta, T_\theta y_\theta) + d(y_{\theta+1}, y_\theta) \\ &\leq d(\gamma_\theta y_\theta \oplus (1-\gamma_\theta) T_\theta z_\theta, T_\theta z_\theta) + d(y_{\theta+1}, y_\theta) + d(T_\theta z_\theta, T_\theta y_\theta) \\ &\leq d(y_\theta, y_{\theta+1}) + \gamma_\theta d(y_\theta, T_\theta y_\theta) + (1-\gamma_\theta) d(T_\theta z_\theta, T_\theta y_\theta). \end{aligned} \tag{3.6}$$

Using (2.1)

$$\begin{aligned} d^2(y_{\theta+1}, q^*) &\leq \gamma_\theta d^2(y_\theta, q^*) + (1-\gamma_\theta) d^2(T_\theta z_\theta, q^*) - \gamma_\theta(1-\gamma_\theta) d^2(y_\theta, T_\theta z_\theta) \\ &\leq \gamma_\theta d^2(y_\theta, q^*) + (1-\gamma_\theta) d^2(z_\theta, q^*) - \gamma_\theta(1-\gamma_\theta) d^2(y_\theta, T_\theta z_\theta) \\ &\leq \gamma_\theta d^2(y_\theta, q^*) + (1-\gamma_\theta) d^2(y_{\theta+1}, q^*) - \gamma_\theta(1-\gamma_\theta) d^2(y_\theta, T_\theta z_\theta), \end{aligned}$$

now utilizing (3.5)

$$d^2(y_{\theta+1}, q^*) \leq d^2(y_\theta, q^*) - \gamma_\theta(1-\gamma_\theta) d^2(y_\theta, T_\theta z_\theta),$$

assume that $\sigma_\theta = d(y_\theta, T_\theta z_\theta)$, which implies that

$$2\delta^3 \sigma_\theta^2 \leq d^2(y_\theta, q^*) - d^2(y_{\theta+1}, q^*),$$

for some $m \geq 1$

$$2\delta^3 \sum_{\theta=1}^{m} \sigma_\theta^2 \leq d^2(y_o, q^*) - d^2(y_{m+1}, q^*) \leq d^2(y_o, q^*),$$

when $m \to \infty$

$$\sum_{\theta=1}^{\infty} \sigma_\theta^2 < \infty,$$

which means

$$\lim_{\theta\to\infty} \sigma_\theta = \lim_{\theta\to\infty} d(y_\theta, T_\theta z_\theta) = 0, \tag{3.7}$$

We know that

$$d(y_{\theta+1}, y_\theta) \leq \gamma_\theta d(y_\theta, y_\theta) + (1-\gamma_\theta) d(T_\theta z_\theta, y_\theta),$$

when $\theta \to \infty$, $\;d(T_\theta z_\theta, y_\theta) = 0$, as

$$d(y_{\theta+1}, y_\theta) = 0, \tag{3.8}$$

which also implies

$$\limsup_{\theta\to\infty} d(y_{\theta+1}, y_\theta) = 0. \tag{3.9}$$

Clearly, we know

$$d(y_{\theta+l}, y_\theta) \leq d(y_{\theta+1}, y_\theta) + d(y_{\theta+1}, y_{\theta+2}) + ... + d(y_{\theta+l-1}, y_{\theta+l}),$$

from (3.9)

$$\limsup_{\theta\to\infty} d(y_{\theta+l}, y_\theta) = 0, \tag{3.10}$$

for $l < \Theta$.

By Lemma 2.2

$$\frac{1}{2\lambda_\theta} d^2(z_\theta, q^*) - \frac{1}{2\lambda_\theta} d^2(y_\theta, q^*) + \frac{1}{2\lambda_\theta} d^2(y_\theta, z_\theta) \leq f(z) - f(J_\lambda y),$$

such as $f(z) \leq f(J_\lambda y)$, it follows that

$$d^2(z_\theta, q^*) \leq d^2(y_\theta, q^*) - d^2(y_\theta, z_\theta).$$

We have

$$d^2(y_{\theta+1}, q^*) = d^2(\gamma_\theta y_\theta \oplus (1-\gamma_\theta) T_\theta z_\theta, q^*).$$

By (2.1)

$$\begin{aligned}
d^2(y_{\theta+1}, q^*) &\leq \gamma_\theta d^2(y_\theta, q^*) + (1-\gamma_\theta) d^2(T_\theta z_\theta, q^*) - \gamma_\theta(1-\gamma_\theta) d^2(y_\theta, T_\theta z_\theta) \\
&\leq \gamma_\theta d^2(y_\theta, q^*) + (1-\gamma_\theta) d^2(z_\theta, q^*) - \gamma_\theta(1-\gamma_\theta) d^2(y_\theta, T_\theta z_\theta) \\
&\leq (1-\gamma_\theta)[d^2(y_\theta, q^*) - d^2(y_\theta, z_\theta)] + \gamma_\theta[d^2(y_\theta, q^*) - (1-\gamma_\theta) d^2(y_\theta, T_\theta z_\theta)] \\
&\leq d^2(y_\theta, q^*) - (1-\gamma_\theta) d^2(y_\theta, z_\theta) - \gamma_\theta(1-\gamma_\theta) d^2(y_\theta, T_\theta z_\theta).
\end{aligned}$$

From above estimate, we have

$$(1-\gamma_\theta)d^2(y_\theta, z_\theta) \leq d(y_{\theta+1}, y_\theta)[d(y_\theta, q^*) + d(y_{\theta+1}, q^*)] - \gamma_\theta(1-\gamma_\theta)d^2(y_\theta, T_\theta z_\theta).$$

By utilizing (3.7) and (3.8)

$$(1-\gamma_\theta)d^2(y_\theta, z_\theta) = 0, \text{when } \theta \to \infty. \tag{3.11}$$

Now from (3.6)

$$d(y_\theta, T_\theta y_\theta) \leq d(y_\theta, y_{\theta+1}) + \gamma_\theta d(y_\theta, T_\theta z_\theta) + (1-\gamma_\theta)d(z_\theta, y_\theta),$$

as $\theta \to \infty$, from (3.8) and (3.12)

$$d(y_\theta, T_\theta z_\theta) = 0,$$

$$\Rightarrow \lim_{\theta\to\infty} d(y_\theta, T_\theta y_\theta) = 0.$$

Further more for every $l \in I$

$$\begin{aligned} d(y_\theta, T_{\theta+l} y_\theta) &\leq d(y_\theta, y_{\theta+l}) + d(y_{\theta+l}, T_{\theta+l} y_{\theta+l}) + d(T_{\theta+l} y_{\theta+1}, T_{\theta+l} y_\theta) \\ &\leq 2d(y_{\theta+l}, y_\theta) + d(y_{\theta+l}, T_{\theta+l} y_{\theta+l}), \end{aligned}$$

from above estimation we observe that $d(y_\theta, T_{\theta+l} y_\theta) = 0$, as $\theta \to \infty$ for each $l \in I$, $\{d(y_\theta, T_l y_\theta)\}$ is subsequence of $\cup_{t=1}^{\Theta} \{d(y_\theta, T_{\theta+l} y_\theta)\}$ and $\lim_{\theta\to\infty} d(y_\theta, T_{\theta+l} y_\theta) = 0$ for every $l \in I$

$$\lim_{\theta\to\infty} d(y_\theta, T_l y_\theta) = 0.$$

This completes the proof of part (i).

(ii) As $\lambda_\theta \geq \lambda \geq 0$, by Lemma 2.4 and non-expensivety of J_λ, we have

$$
\begin{aligned}
d(y_\theta, J_\lambda y_\theta) &\leq d(y_\theta, z_\theta) + d(z_\theta, J_\lambda y_\theta) \\
&= d(y_\theta, z_\theta) + d(J_\lambda y_\theta, J_\lambda y_\theta)) \\
&= d\Big(y_\theta, z_\theta) + d\Big(J_\lambda\Big(\frac{\lambda_\theta - \lambda}{\lambda_\theta} J_\lambda y_\theta \oplus \frac{\lambda_\theta}{\lambda}\Big) y_\theta\Big), J_\lambda y_\theta\Big) \\
&\leq d(y_\theta, z_\theta) + \Big(1 - \frac{\lambda}{\lambda_\theta}\Big)(y_\theta, J_\lambda y_\theta)) \\
&= \Big(2 - \frac{\lambda}{\lambda_\theta}\Big) d(y_\theta, z_\theta),
\end{aligned}
$$

as above we already calculate $\lim_{\theta\to\infty} d(y_\theta, z_\theta) \ = \ 0$. So therefore

$$\lim_{\theta\to\infty} d(y_\theta, J_\lambda x_\theta) \ = \ 0.$$

Finally we obtain our desire results. This completes the proof of part (ii) and of Lemma 3.2.2.

Theorem 3.2.3. Let $\{T_\theta\}_{\theta=1}^{\Theta}$, $\{f_\theta\}_{\theta=1}^{\Theta}$, C and $\{y_\theta\}$ be as given in Lemma 3.2.1, then sequence $\{y_\theta\}$ defined in (3.2), ΔC to an element in Ω.

Proof. From Lemma 3.2.1, $\{y_\theta\}$ is a bounded sequence and also from Lemma 2.1.5 and 2.1.6 and also recalling the definition of ΔC we observe that, there is unique asymptotic center of $\{y_\theta\}$, that is A($\{y_\theta\}$)=$\{y_\theta\}$.

Let $\{\mu_\theta\}$ be subsequence of a sequence $\{y_\theta\}$ such that A($\{\mu_\theta\}$)=$\{\mu\}$. From above Lemma 3.2.2, we conclude $d(\mu_\theta, T_l\mu_\theta) = 0 \lim_{\theta\to\infty}$ for every $l \in I$.

Let $\mu \in \Omega$ and from Lemma 3.2.1, $\lim_{\theta\to\infty} d(y_\theta, \mu)$ exists. Suppose $y \neq \mu$, from the asymptotic centre uniqueness

$$
\begin{aligned}
\limsup_{\theta\to\infty} d(\mu_\theta, \mu) &< \limsup_{\theta\to\infty} d(\mu_\theta, x) \\
&\leq \lim_{\theta\to\infty} supd(y_\theta, y) \\
&< \lim_{\theta\to\infty} supd(y_\theta, \mu) \\
&= \lim_{\theta\to\infty} supd(\mu_\theta, \mu),
\end{aligned}
$$

which shows contradiction. Thus, $y = \mu$, since (μ_θ) is an arbitrary subsequence of $\{y_\theta\}$, therefore A($\{\mu_\theta\}$) = $\{\mu\}$ for every subsequences $\{\mu_\theta\}$ of $\{y_\theta\}$. This proves that $\{y_\theta\}$ ΔC to a point $\mu \in \Omega$. This completes the proof.

Theorem 3.2.4. Let $\{T_\theta\}_{\theta=1}^{\Theta}$, $\{f_\theta\}_{\theta=1}^{\Theta}$, C and $\{y_\theta\}$ be as given in Lemma 3.2.1, then the sequence $\{y_\theta\}$ defined in (3.2) SC to an element in $p \in \Omega$ if and only if $\liminf_{\theta\to\infty} d(y_\theta, \Omega) = 0$, where $d(y_\theta, \Omega) = \inf\{d(y_\theta, z) : z \in \Omega\}$.

Proof. The sequence $\{y_\theta\}$ is Fejer monotone W.r.t. $\bar{q} \in \Omega$ follows from (3.4) and also by Lemma 3.2.1 we observe that, $\lim_{\theta\to\infty} d(y_\theta, q^*)$ exists. By applying Lemma 2.1.3 to (3.4), $\lim_{\theta\to\infty} d(y_\theta, \Omega)$ exists. Conversely, $\liminf_{\theta\to\infty} d(y_\theta, \Omega) = 0$.

Let for $\varepsilon > 0 \ \exists\ m_0 \in \mathbb{N}$ such that

$$d(y_\theta, \Omega) < \frac{\varepsilon}{4}, \quad forall\ \theta \geq m_0.$$

In particular $\inf\{d(y_{m_0}, \bar{p}) \ : \ q^* \in \Omega\} < \frac{\varepsilon}{4}$, therefore there must be $q^* \in \Omega$ exists such that

$$d(q^*, y_{m_0}) < \frac{\varepsilon}{2},$$

for all $m,\ \theta \geq m_0$, we have that

$$\begin{aligned} d(y_{\theta+m}, y_\theta) &\leq d(y_{\theta+m}, q^*) + d(q^*, y_\theta) \\ &\leq 2d(y_{m_0}, q^*), \\ &< \varepsilon. \end{aligned}$$

Thus, the sequence $\{y_\theta\}$ is Cauchy in C. Since C is assumed to be closed subset of a complete $CAT(0)$ space, therefore we conclude that the sequence $\{y_\theta\}$ converges to some q^* in C. Since Ω is closed and $\lim_{\theta\to\infty} d(y_\theta, \Omega) = 0$. This completes the proof.

Lemma 3.2.3. *[9]"Let C be as given in Lemma 3.2.1 and let the sequence $\{y_\theta\}$ be Fejer monotone W.r.t. C . Then $\{y_\theta\}$ converges to some $q^* \in \Omega$ if and only if* $\lim_{\theta\to\infty} d(y_\theta, C) = 0$. "

Theorem 3.2.5. Let $\{T_\theta\}_{\theta=1}^{\Theta}$, $\{f_\theta\}_{\theta=1}^{\Theta}$, C and $\{y_\theta\}$ be as given in Lemma 3.2.1. Suppose that $\{T_\theta : \theta \in I\}$ is semi-compact then the sequence $\{y_\theta\}$ SC to an element in $q^* \in \Omega$.

Proof. Using Lemma 3.2.2 and the action proof in Theorem 3.2.4. Assume that for some $\theta \in I$ we have T_l which is semi-compact. From Lemma 3.2.2 it follows that

$$\lim_{\theta\to\infty} d(T_l x_\theta, y_\theta) = 0.$$

From semi-compactness of mapping T_l there exist a subsequence $\{y_{\theta_j}\}$ of the sequence $\{y_\theta\}$ such that

$$\lim_{j\to\infty} y_{\theta_j} = q^* \in C \text{ and } \lim_{\theta\to\infty} d(T_l y_{\theta_j}, y_{\theta_j}) = 0.$$

Now, by Lemma 3.2.3 we obtain that $\lim_{\theta\to\infty} d(y_{\theta_j}, T_l y_{\theta_j}) = 0$ for all $l \in J$. This implies $d(q^*, T_l q^*) = 0$ for all $l \in J$.
Thus, $q^* \in \Omega$ and also from Theorem 3.4, $\lim_{\theta\to\infty} d(y_\theta, \Omega) = 0$ and by Lemma 3.2.3 $\{y_\theta\}$ SC to an element in $q^* \in \Omega$. This completes the proof.

Theorem 3.2.6. Let $\{T_\theta\}_{\theta=1}^{\Theta}$, $\{f_\theta\}_{\theta=1}^{\Theta}$, C and $\{y_\theta\}$ be as given in Lemma 3.2.1 and suppose there exist a nondecreasing function $g : [0,\infty] \to [0,\infty)$ with $g(0) = 0$ and $g(r) > 0$ for all $r > 0$, such that

$$g(d(y,\Omega)) \leq d(y, T_\theta y)$$

or

$$g(d(y,\Omega)) \leq d(y, J_\lambda^{(\theta)} y)$$

for some $\theta \in I$ and for all $y \in C$, then the sequence $\{y_\theta\}$ defined in (3.2) SC to an element in $q^* \in \Omega$.

Proof. It follows from Lemma 3.2.1 that, $\lim_{\theta\to\infty} d(y_\theta, \Omega)$ exists and Lemma 3.2.2 implies that, $\lim_{\theta\to\infty} d(y_\theta, T_l y_\theta) = 0$ for all $l \in I$. Since g is nondecreasing with $g(0) = 0$, it follows that $\lim_{\theta\to\infty} d(y_\theta, \Omega) = 0$. Thus, Lemma 3.2.2 implies $\{y_\theta\}$ SC to an element in q^* in Ω. This completes the proof.

We acquire few outcomes from theorem 3.2.4, 3.2.5 and 3.2.6. An prompt result of Theorem 3.2.4 and 3.2.6 is to set up a similar result for class of non-expansive mappings.

Corollary 3.2.4. *[45] "Let H be a HS and the sequence $\{y_\theta\}$ generated by (3.12) is SC to an element in Ω if and only if $\{y_\theta\}$ is regular asymptotic, i.e.* $\lim_{\theta\to\infty} \| y_\theta - y_{\theta+1} \| = 0$.

$$y_\theta = z_\theta u + (1 - z_\theta)T_\theta y_\theta. \tag{3.12}$$

where $u \in C, z_\theta \in (0, 1)$."

Corollary 3.2.5. *[14] "Let E be a uniformly convex BS and C be a nonempty CCS of E. Let $\{T_\theta : \theta \in I\}$ be θ semi-compact non-expansive self mappings of C with $F = \cap_{\theta=1}^{\Theta} F(T_\theta) = \emptyset$, (here $F(T_\theta)$ denotes the set of FP's of T_θ). Suppose that $x_o \in C$ and $\{x_\theta\} \subset (b, c)$ for some $b, c \in (0, 1)$. Then the sequence $\{x_\theta\}$ defined by the implicit iteration process is given as*

$$y_\theta = \alpha_\theta y_{\theta-1} + (1 - \alpha_\theta)T_\theta y_\theta, \ \textit{for all } \theta \geq 1,$$

where $T_\theta = T_{\theta(mod\Theta)}$, (here the $mod\Theta$ function takes the value in I), SC to an element in Ω. "

Corollary 3.2.6. *[14] "Let E be a uniformly convex BS and C be a nonempty CCS of E. Let $\{T_\theta : \theta \in I\}$ be Θ non-expansive self mappings of C with $F = \cap_{\theta=1}^{\Theta} F(T_\theta) = \emptyset$. Suppose that one of the mappings in $\{T_\theta : \theta \in I\}$ is semi-compact and let $\{\alpha_\theta\}_{\theta\geq1} \subset [\delta, 1 - \delta]$ for some $\delta \in (0, 1)$. From arbitrary $x_o \in C$, define the sequence $\{x_\theta\}$ by*

$$y_\theta = \alpha_\theta y_{\theta-1} + (1 - \alpha_\theta)T_\theta y_\theta, \ \textit{for all } \theta \geq 1,$$

where $T_\theta = T_{\theta(mod\Theta)}$, (here the $mod\Theta$ function takes the value in I).

Then the sequence $\{x_\theta\}$ SC to a common FP of the mappings $\{T_\theta : \theta \in I\}$. "

3.3 Infinite Family

The main motive of this work is to form convergence of proposed algorithm by employing suitable conditions on control sequences of parameters. The proposed algorithm strongly converges to common component of solution set of minimizer and the common set of FP's of a countable infinite family of total non-expansive mapping. As a consequence, these results can be considered as an improvement and generalization of various results in this field of study.

In 2007, Aoyama et al. [2] firstly proposed the successive condition: countably infinite family of self mapping $\{T_\theta\}$ from C and C is nonempty subset of Hadamard space (X, d), then $\{T_\theta\}$ satisfies $AKTT$ condition, where the alphabets A, K, T, T represents Aoyama, Kimura, Takahashi and Toyoda, if

$$\sum_{\theta=1}^{\infty} \sup_{z \in D} \{d(T_{\theta+1}z, T_\theta z)\} < \infty,$$

for every bounded subset D of C. We can define a self mapping $T : C \to C$ by $Ty = \lim_{\theta \to \infty} T_\theta y$ for all $y \in C$, if $\{T_\theta\}$ satisfies $AKTT$ condition and C is closed subset of X. For this case, additionally we state ($\{T_\theta\}$,T) fulfills the $AKTT$ condition.

3.4 Main Result

We discuss in this section infinite family of non-expansive mappings in a complete $CAT(0)$ spaces for the generalized proposed implicit iterative scheme (3.2) and we establish SC as well as ΔC.

Theorem 3.4.1: Let f be a proper LSC convex function of C into $(-\infty, \infty]$ in a Complete $CAT(0)$ space (X, d) such that $F = \cap_{\theta=1}^{\infty} F(T_\theta) \neq \emptyset$ and let C be a nonempty CCS of a Complete $CAT(0)$ space and let $\{T_\theta\}_{\theta=1}^{\infty}$ be a family of non-expansive self mapping of C. Assume that $\Omega = \arg\min_{\mu\in C} f(\mu) \cap_{\theta=1}^{\infty} F(T_\theta)$ is nonempty and for $y_1 \in C$, let $\{y_\theta\}$ be a sequence in C defined by

$$\begin{aligned} z_\theta &= \arg\min_{\mu\in C}\left[f(\mu) + \frac{1}{2\lambda}d(\mu, y_{\theta+1})^2\right], \\ y_{\theta+1} &= \gamma_\theta y_\theta \oplus (1-\gamma_\theta)T_\theta z_\theta. \text{ for all } \theta \geq 1 \end{aligned} \tag{3.13}$$

where $\{\gamma_\theta\}$ is a subset of $[\delta, 1-\delta]$ for some $\delta \in (0, \frac{1}{2})$ and if $\{T_\theta\}_{\theta=1}^{\infty}$ satisfies AKTT-condition then the following conditions hold:

(1) $\lim_{\theta\to\infty} d(y_\theta, p^*)$ exists for all $p^* \in F$ and hence, the sequence $\{y_\theta\}$ is bounded.

(2)$\lim_{\theta\to\infty} d(y_{\theta+1}, T_{\theta+1}y_{\theta+1}) = 0$.

Proof.

(1) Let $p^* \in \Omega$ and by convexity and definition of $\{y_\theta\}$, we have

$$\begin{aligned} d(y_{\theta+1}, p^*) &= d(\gamma_\theta y_\theta \oplus (1-\gamma_\theta)T_\theta z_\theta, p^*) \\ &\leq \gamma_\theta d(y_\theta, p^*) + (1-\gamma_\theta)d(T_\theta z_\theta, p^*) \\ &\leq \gamma_\theta d(y_\theta, p^*) + (1-\gamma_\theta)d(z_\theta, p^*) \\ &\leq \gamma_\theta d(y_\theta, p^*) + (1-\gamma_\theta)d(y_{\theta+1}, p^*) \\ &\leq d(y_\theta, p^*). \end{aligned} \tag{3.14}$$

By induction we observe that

$$d(y_{\theta+1}, p^*) \leq d(y_1, p^*),$$

from (3.14) we prove that $\{y_\theta\}$ is bounded and so $\lim_{\theta\to\infty} d(y_\theta, p^*)$ exists. This completes the proof of part (1).

(2) Next we show $\lim_{\theta\to\infty} d(y_{\theta+1}, T_{\theta+1}y_{\theta+1}) = 0$, such that

$$\begin{aligned} d(y_{\theta+1}, T_{\theta+1}y_{\theta+1}) &\leq d(y_{\theta+1}, y_\theta) + d(y_\theta, T_{\theta+1}y_{\theta+1}) \\ &\leq d(y_{\theta+1}, y_\theta) + d(T_{\theta+1}y_{\theta+1}, T_\theta y_{\theta+1}) + d(y_\theta, T_\theta y_{\theta+1}), \end{aligned}$$

as

$$d(y_\theta, T_\theta y_{\theta+1}) \leq d(y_\theta, y_{\theta+1}) + d(y_{\theta+1}, T_\theta y_{\theta+1}),$$

and

$$d(y_{\theta+1}, T_\theta y_{\theta+1}) \leq d(y_{\theta+1}, y_\theta) + d(y_\theta, T_\theta, z_\theta) + d(T_\theta z_\theta, T_\theta y_\theta),$$

which implies

$$\begin{aligned} d(y_{\theta+1}, T_{\theta+1}y_{\theta+1}) &\leq 3d(y_\theta, y_{\theta+1}) + d(T_{\theta+1}y_{\theta+1}, T_\theta y_{\theta+1}) + d(y_\theta, T_\theta, z_\theta) + d(T_\theta z_\theta, T_\theta y_\theta) \\ &\leq 3d(y_\theta, y_{\theta+1}) + d(T_{\theta+1}y_{\theta+1}, T_\theta y_{\theta+1}) + d(y_\theta, T_\theta z_\theta) + d(z_\theta, y_\theta) \end{aligned} \quad (3.15)$$

Using (2.1)

$$\begin{aligned} d^2(y_{\theta+1}, p^*) &\leq \gamma_\theta d^2(y_\theta, p^*) + (1-\gamma_\theta)d^2(T_\theta z_\theta, p^*) - \gamma_\theta(1-\gamma_\theta)d^2(y_\theta, T_\theta z_\theta) \\ &\leq \gamma_\theta d^2(y_\theta, p^*) + (1-\gamma_\theta)d^2(z_\theta, p^*) - \gamma_\theta(1-\gamma_\theta)d^2(y_\theta, T_\theta z_\theta) \\ &\leq \gamma_\theta d^2(y_\theta, p^*) + (1-\gamma_\theta)d^2(y_{\theta+1}, p^*) - \gamma_\theta(1-\gamma_\theta)d^2(y_\theta, T_\theta z_\theta), \end{aligned}$$

now by utilizing (3.14)

$$d^2(y_{\theta+1}, p^*) \leq d^2(y_\theta, p^*) - \gamma_\theta(1-\gamma_\theta)d^2(y_\theta, T_\theta z_\theta),$$

assume that $\sigma_\theta = d(y_\theta, T_\theta z_\theta)$, which implies that

$$2\delta^3\sigma_\theta^2 \leq d^2(y_\theta, p^*) - d^2(y_{\theta+1}, p^*),$$

for some $m \geq 1$, we have

$$2\delta^3 \sum_{\theta=1}^{m} \sigma_\theta^2 \leq d^2(y_o, p^*) - d^2(y_{m+1}, p^*) \leq d^2(y_o, p^*),$$

when $m \to \infty$

$$\sum_{\theta=1}^{\infty} \sigma_\theta^2 < \infty,$$

which means

$$\lim_{\theta \to \infty} \sigma_\theta = \lim_{\theta \to \infty} d(y_\theta, T_\theta z_\theta) = 0,$$

As we know that

$$d(y_{\theta+1}, y_\theta) \leq \gamma_\theta d(y_\theta, y_\theta) + (1 - \gamma_\theta) d(T_\theta z_\theta, y_\theta),$$

$$\text{when } \theta \to \infty, \text{we have } d(T_\theta z_\theta, y_\theta) = 0 \text{ and also } d(y_{\theta+1}, y_\theta) = 0, \tag{3.16}$$

therefore, which also implies that

$$\limsup_{\theta \to \infty} d(y_{\theta+1}, y_\theta) = 0. \tag{3.17}$$

By Lemma 2.2

$$\frac{1}{2\lambda_\theta} d^2(z_\theta, p^*) - \frac{1}{2\lambda_\theta} d^2(y_\theta, p^*) + \frac{1}{2\lambda_\theta} d^2(y_\theta, z_\theta) \leq f(z) - f(J_\lambda y),$$

such as $f(z) \leq f(J_\lambda y)$, it follows that

$$d^2(z_\theta, p^*) \leq d^2(y_\theta, p^*) - d^2(y_\theta, z_\theta).$$

Since

$$d^2(y_{\theta+1}, p^*) = d^2(\gamma_\theta y_\theta \oplus (1 - \gamma_\theta) T_\theta z_\theta, p^*).$$

By (2.1)

$$\begin{aligned}
d^2(y_{\theta+1}, p^*) &\leq \gamma_\theta d^2(y_\theta, p^*) + (1 - \gamma_\theta) d^2(T_\theta z_\theta, p^*) - \gamma_\theta (1 - \gamma_\theta) d^2(y_\theta, T_\theta z_\theta) \\
&\leq \gamma_\theta d^2(y_\theta, p^*) + (1 - \gamma_\theta) d^2(z_\theta, p^*) - \gamma_\theta (1 - \gamma_\theta) d^2(y_\theta, T_\theta z_\theta) \\
&\leq (1 - \gamma_\theta)[d^2(y_\theta, p^*) - d^2(y_\theta, z_\theta)] + \gamma_\theta [d^2(y_\theta, p^*) - (1 - \gamma_\theta) d^2(y_\theta, T_\theta z_\theta)] \\
&\leq (1 - \gamma_\theta) d^2(y_\theta, p^*) + (\gamma_\theta - 1) d^2(y_\theta, z_\theta) + \gamma_\theta d^2(y_\theta, p^*) - \gamma_\theta (1 - \gamma_\theta) d^2(y_\theta, T_\theta z_\theta) \\
&\leq d^2(y_\theta, p^*) - (1 - \gamma_\theta) d^2(y_\theta, z_\theta) - \gamma_\theta (1 - \gamma_\theta) d^2(y_\theta, T_\theta z_\theta),
\end{aligned}$$

from above calculation, we say that

$$\begin{aligned}(1-\gamma_\theta)d^2(y_\theta,z_\theta) &\leq |d(y_\theta,p^*)-d(y_{\theta+1},p^*)|[d(y_\theta,p^*)+d(y_{\theta+1},p^*)-\gamma_\theta(1-\gamma_\theta)d^2(y_\theta,T_\theta z_\theta)]\\ &\leq |-d(y_{\theta+1},y_\theta)|[d(y_\theta,p^*)+d(y_{\theta+1},p^*)-\alpha_\theta(1-\gamma_\theta)d^2(y_\theta,T_\theta z_\theta)]\\ &\leq d(y_{\theta+1},y_\theta)[d(y_\theta,p^*)+d(y_{\theta+1},p^*)]-\gamma_\theta(1-\gamma_\theta)d^2(y_\theta,T_\theta z_\theta),\end{aligned}$$

as $\gamma\in[\delta,1-\delta],$ so therefor when $\theta\to\infty,$ then

$$d^2(y_\theta,z_\theta) \quad = \quad 0. \tag{3.18}$$

from (3.16), (3.18) and through $AKTT$ condition, we observe that $d(y_{\theta+1},T_{\theta+1}y_{\theta+1})=0,$ as $\theta\to\infty$. This completes the proof of part (2) and of Theorem 3.4.1.

Theorem 3.4.2. Let $\{T_\theta\}_{\theta=1}^{\infty},$ C and F be as given in Theorem 3.4.1. Then the sequence $\{y_\theta\}$ defined implicity in (3.13), ΔC to an element in Ω.

Proof. It follows from theorem 3.4.1, the sequence $\{y_\theta\}$ is bounded and also from Lemma 2.1.5 and 2.1.6 and also recalling the definition of ΔC, we observe the sequence $\{y_\theta\}$ has a unique asymptotic center, that is A($\{y_\theta\}$)=$\{y_\theta\}$.

$\{\mu_\theta\}$ is subsequence of sequence $\{y_\theta\}$ such that A($\{\mu_\theta\}$)=$\{\mu\}$. Then by theorem 3.4.1, $\lim_{\theta\to\infty} d(\mu_\theta,T_\theta\mu_\theta)=0$

Let $\mu\in\Omega$ and from theorem 3.4.1, we have $\lim_{\theta\to\infty} d(y_\theta,\mu)$ exists. Suppose $y\neq\mu$. By uniqueness of asymptotic center, we have

$$\begin{aligned}\limsup_{\theta\to\infty} d(\mu_\theta,\mu) &< \limsup_{\theta\to\infty} d(\mu_\theta,y)\\ &\leq \lim_{\theta\to\infty} supd(y_\theta,y)\\ &< \lim_{\theta\to\infty} supd(y_\theta,\mu)\\ &= \lim_{\theta\to\infty} supd(\mu_\theta,\mu),\end{aligned}$$

which shows contradiction. Hence, $y = \mu$, since (μ_θ) is an arbitrary subsequence of the sequence $\{y_\theta\}$, so A($\{\mu_\theta\}$)=$\{\mu\}$ for all subsequences $\{\mu_\theta\}$ of $\{y_\theta\}$. This proves $\{y_\theta\}$ ΔC to $\mu \in \Omega$. This completes the proof.

Theorem 3.4.3: Let $\{T_\theta\}_{\theta=1}^{\infty}$, C and $\{y_\theta\}$ be as given in Theorem 3.4.1, then the sequence $\{y_\theta\}$ defined in (3.2) is Fejer monotone with respect to Ω. Moreover, $\{y_\theta\}$ SC to an element in $q^* \in \Omega$ if and only if $\lim_{\theta\to\infty} d(y_\theta, \Omega) = 0$ where $d(y_\theta, \Omega) = \inf\{d(y, z) : z \in \Omega\}$.

Proof. From (3.15), the sequence $\{y_\theta\}$ is Fejer monotone W.r.t. $p^* \in \Omega$ and also from Theorem 3.4.1, we obtain $\lim_{\theta\to\infty} d(y_\theta, p^*)$ exists. By applying Lemma 2.3 to (3.14), $\lim_{\theta\to\infty} d(y_\theta, \Omega)$ exists. The need is self-evident and then we just demonstrate sufficiency. Suppose $\liminf_{\theta\to\infty} d(y_\theta, \Omega) = 0$, there exist $m_0 \in \not\leq$ for any $\varepsilon > 0$ such that

$$d(y_\theta, \Omega) < \frac{\varepsilon}{4}, \quad forall\ \theta \geq m_0.$$

Specifically $\inf\{d(y_{m_0}, q^*) : q^* \in \Omega\}< \frac{\varepsilon}{4}$, therefore there must be $p^* \in \Omega$ exists such that

$$d(y_{m_0}, p^*) < \frac{\varepsilon}{2}.$$

Then for all $m,\ \theta \geq m_0$, we have

$$\begin{aligned} d(y_{\theta+m}, y_\theta) &\leq d(y_{\theta+m}, p^*) + d(p^*, y_\theta) \\ &\leq 2d(y_{m_0}, p^*), \\ &< \varepsilon. \end{aligned}$$

Thus, the sequence $\{y_\theta\}$ in C be Cauchy, this implies the sequence $\{y_\theta\}$ converges to some point q^* in C. Since Ω is closed and $\lim_{\theta\to\infty} d(y_\theta, \Omega) = 0$.
This completes the proof.

Chapter 4
References

Bibliography

[1] L. Ambrosio, N. Gigli and G. Savare, Gradient flows in metric spaces and in the space of probability measures, 2nd ed. Lectures in Mathematics ETH Zurich, Brikhauser Verlag, Basel, 2008.

[2] K. Aoyama, Y. Kimura, W. Takahashi and M. Toyoda, Approximation of common fixed points of a countable family of nonexpansive mappings in a Banach space, Nonlinear Anal., 67(2007), 2350-2360.

[3] M. Bacak, The proximal point algorithm in metric spaces, Isr. J. Math. **194**(2013),689-701.

[4] S. Banach, *Sur les opération dans les ensembles abstracts et leur applications aux équations intégrals*, Fund. Math. J., **3**(1922),133-181.

[5] V. Barbu, Maximal Monotone Operators in Banach Spaces, Springer, (2009).

[6] M. Basarir and A. Sahin, On the Strong and Δ-Convergence of S-Iteration Process for Generalized Nonexpansive Mappings on $CAT(0)$ Space, Thai. J. Maths, 12(2014),549-559.

[7] H.H. Bauschke, E. Matouskova and S. Reich, Projection and proximal point methods, convergence results and counter examples, Nonlinear Anal., 56(2004),715-738.

[8] H.H. Bauschke, J.V. Burke, F.R. Deutsch, H.S. Hunadal and J.D. Vanderwerff, A new proximal point iteration that converges weakly but not in norm, Proc. Amer. Math. Soc., 133(2005),1829-1835.

[9] H.H. Bauschke and P.L. Combettes, Convex analysis and monotone operator theory in Hilbert spaces, Springer-Verlag, New York (2011).

[10] L. Berg, W. Engel and G. Pazderski, Basic properties of fejer monotone sequesnces, Rostok. Math. Kolloq, 49(1995),57-74.

[11] F.E. Browder, *Nonexpansive nonlinear operators in a Banach space*, Proc. Natl. Acad. Sci. U.S.A., **54**(1965),1041-1044.

[12] S.S. Chang and K.K. Tan, On the convergence of implicit iteration process with error for a finite family of asymptotically nonexpansive mappings, J. Math. Anal. Appl. 313(2006),273Ű283

[13] S.S. Chang, L. Wang, C.F. Wen, and J.Q. Zhang, The modified proximal point algorithm in Hadamard spaces, J. Ine. Appl., 2018(2018),124.

[14] C.E. Chidumea and N. Shahzad, Strong convergence of an implicit iteration process for a finite family of nonexpansive mappings, J. Nonlinear Analysis, 62(2005),1149 Ű 1156.

[15] Y.J. Cho, L. Ciric and S. Wang, Convergence theorems for nonexpansive semigroups in CAT(0) spaces, Nonlinear Anal., 74(2011),6050-6059.

[16] P. Cholamjiak, Y.J. Cho and S. Suantai ,Strong convergence theorems for a sequence of nonexpansive mappings with gauge functions, An. St. Univ. Ovidius Constanta, 21(2013),183.

[17] P.L. Combettes and J.C. Pesquet, Proximal splitting methods in signal processing, In. H.H. Bauschke, R. Burachik, P.L. Combettes, V. Elser, D.R. Luke and H. Wolkowicz, (eds.) Fixed point Algorithm for inverse problems in science and Engineering, pp.185-212 springer, New York(2011).

[18] A. Cuntavepanit and B. Panyanak, Strong convergence of modified Halpern iterations in CAT(0) spaces, Fixed point theory Appl. Art., ID 869458(2011),11.

[19] S. Dhompongsa and B. Panyanak, On Δ-convergence theorems in CAT(0) spaces, Comput. Math. Appl., 56(2008),2572-2579.

[20] S. Dhompongsa, W.A. Kirk and B. Panyanak, Nonexpansive set-valued mappings in metric and Banach spaces, J. Nonlinear Convex Anal., 8(2007),35Ű45.

[21] K. Goebel and W.A. Kirk, *Topics in metric fixed point theory.*, Cambridge studies in advanced mathematics 28, Cambridge University Press, Cambridge, (1990).

[22] D. Göhde, *Zum Prinzip der Kontraktiven Abbildung,* Math. Nachr., **30** (1965),251-258.

[23] O. Guler, On the convergence of the proximal point algorithm for convex minimization, SIAM J. Control Optim., 29(1991),403-419.

[24] J. Jost, Convex functionals and generalized harmonic maps into spaces of nonpositive curvature, Comment. Math. Helv., 70(1995),659-673.

[25] S. Kamimura and W. Takahashi, Strong convergence of Approximating solutions of maximal monotone operators in Hilbert spaces, J. Approx. Theory, 106(2000),226-240.

[26] S. Kamimura and W. Takahashi, Strong convergence of approximal-type algorithm in a Banach space, SIAM J. Optim., 13(2003),938-945.

[27] W.A. Kirk, *A fixed point theorem for mappings which do not increase distances*, Amer. Math. Monthly, **72**(1965),1004-1006.

[28] W.A. Kirk, Geodesic geometry and Fixed point theory, Seminar of Mathematical Analysis (Malaga/Seville,2002/2003), 195-225, Univ. Sevilla Secr. Pub., (2003).

[29] W.A. Kirk and M.A. Khamsi, *An introduction to metric spaces and fixed point theory*, J. Math. Anal. Appl., **341**(2008),1-11.

[30] W.A. Kirk and B. Panyanak, A concept of convergence in geodesic spaces, Nonlinear Anal., 68(2008),3689Ű3696.

[31] W.A. Kirk, B. Sims, *Handbook of metric fixed point theory*, Kluwer, Dordrecht-Boston-London, (2001).

[32] S.H. Khan, Fixed point approximation of nonexpansive mappings on a nonlinear domain, Abs. App. Anal., 2014(2014),5.

[33] W. Laowang and B. Pankyanak, Storng and Δ-convergence theorems for multivalued mappings in CAT(0) spaces, J. Ineq. Appl., 2009(2009),16.

[34] T.C. Lim, Remarks on some Fixed point theorems, Proc. Am. Math. Soc., 60(1976),179-182.

[35] G. Marino and H.K. Xu, Convergence of generalized proximal point algorithm, Commun. Pure Appl. Anal, 3(2004),791-808.

[36] B. Martinet, Regularization dinequationvariationelles per approximation successive, Rev. Fr. Inform. Rech. Oper., 4(1970),154-158.

[37] U.F. Mayer, Gradient flows on nonpositively curved metric spaces and harmonic maps, Commun. Anal. Geo., 6(1998),199-253.

[38] S. Reich and S. Sabach, Two strong convergence theorems for a proximal method in reflexive Banach spaces, Numer. Funct. Anal. Optim., 31(2010),22-44.

[39] S. Reich and Z. Salinas, Weak convergence of infinite products of operators in Hadamard spaces, Rend. Circ. Mat. Palermo., 65(2016),55-71.

[40] R.T. Rockafellar, Monotone Operators and the proximal point algorithm, SIAM J. Control Optim., 14(1976),877-898.

[41] K. Sombut, N. Pakkaranang and P. Saipara, Modified proximal point algorithms for solving fxed point problem and convex minimization problem in non-positive curvature metric spaces, Thai. J. Maths, (2018),1-16.

[42] W. Takahashi, A convexity in metric space and nonexpansive mappings, Math Sem. Rep., 22(1970),142-149.

[43] I. Uddin, J.J. Nieto and J. Ali, One-Step Iteration Scheme for Multivalued Nonexpansive Mappings in $CAT(0)$ Spaces, Mediterr. J. Math.

[44] B. XU, Fixed point iterations for asymptotically nonexpansive mappings in Banach spaces, J. Maths. Anal. Appli. 267(2002),444-453.

[45] H.K. Xu and R.G. Ori, An implicit iteration process for nonexpansive mapping, Num. Funct. Anal. Optim., 22(5-6),767-773.

[46] Y. Yao and N. Shahzad, Strong convergence of a proximal point algorithm with general errors, Optim. Li., 6(2012),621-628.

[47] Y.Y. Zhou and S.S. Chang, Convergence of implicit iterative process for a finite family of asymptotically nonexpansive mappings in Banach spaces, Numer. Funct. Anal. Optim., 23(2002),911-921.

[48] E. Kreyszig, Introductory functional analysis with application.