

On Algebraic and Combinatorial properties of the Poset $C_2 \times C_2 \times C_j$



By

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CIIT/FA17-RMT-023/LHR

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A Post Graduate Thesis submitted to the Department of Mathematics as partial fulfillment of the requirement for the award of Degree of M.S in Mathematics.

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Declaration

I Husna Muzaffar (CIIT/ FA17-RMT-023/LHR) hereby declare that I have produced the work presented in this thesis, during the scheduled period of study. I also declare that I have not taken any material from any source except referred to wherever due that amount of plagiarism is within acceptable range. If a violation of HEC rules on research has occurred in this thesis. I shall be liable to punishable action under the plagiarism rules of the HEC.

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It is certified that Husna Muzaffar (CIIT/ FA17-RMT-023/LHR) has carried out all the work related to this thesis under my supervision at the Department of Mathematics, COMSATS University Islamabad, Lahore Campus and the work fulfills the requirement for award of MS degree.

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DEDICATION

To
My Parents

ACKNOWLEDGEMENTS

First of all, I would like to thank Almighty Allah Who is most Gracious and Merciful. Peace be upon The Holy prophet who exhort his followers to seek knowledge.

I would like to thank Professor Dr. Sarfraz Ahmad, my supervisor for his many valuable suggestions and constant support. His excellent knowledge, motivation and help combined to make my research experience worthwhile.

Of course at this stage I cannot forget my parents, my brother and sisters. Their prayers have lightened up my spirit to finish this thesis.

Husna Muzaffar (CIIT/FA17-RMT-023/LHR)

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Abstract

In this research we investigated different algebraic invariants using combinatorial techniques. In particular we computed Krull dimension, minimal generating set of the ideal associated to the $C_2 \times C_2 \times C_j$ for $j \geq 2$ using Stanley Reisner simplicial complex. Secondly we computed metric dimension of the line graph of Honeycomb network.

Chapter 1

Introduction

A binary relation R on a set X is a partial order if and only if it is reflexive, antisymmetric and transitive. For example $\leq$ is a partial ordering relation on integers, $/$ is partial ordering for +ve integers. Let P be a set, P with a partial ordering relation $\leq$ is called a poset. The partial ordering $\leq$ is a relation between the elements of the poset. As if x and y contained in P then $x \leq y$ means, x is related to y . We say in a Poset two elements x, y to be comparable if any of the following holds $x \leq y$ or $y \leq x$. If no two elements can be related they are called incomparable. Let C_i be the chain, $1 < 2 < \dots < i$. We consider the poset $P = C_2 \times C_2 \times C_j$ for $i, j, k > 0$. This is the generalization of the poset $P = C_i \times C_j$ studied in [1]. For example for $i = 2 = k$, $j = 3$, the $P = C_2 \times C_2 \times C_j$ is shown in Figure 1. Let K be a field, then $K[x_1, \dots, x_n]$ is called the ring (polynomial) in n number of variables. $K[x_1, \dots, x_n]$ is a collection of all polynomials in x_i variables with coefficient from K . The product $x^a = x_1^{a_1}, \dots, x_n^{a_n}$; $\alpha_i \leq 1$ is said to be a monomial. Every polynomial is a monomial but reverse statement does not holds. For example $x_1^3 x_2^2$ and $x_1^2 x_3^3$ are monomials but $x_1^2 x_3^3$ is not a monomial. An ideal generated by monomials is called a monomial ideal. A monomial $x^a = x_1^{a_1}, \dots, x_n^{a_n}$ is a "square free" monomial if $\alpha_i \leq 1$. An ideal whose generating set

contains elements "square free" is known as square free monomial ideal. To each poset P one can relate a square free monomial ideal I_p in this way. Let $C : i_1 < i_2 < \dots < i_r$ be a maximal chain, then we define a monomial $x_c = x_{i_1}x_{i_2}\dots x_{i_r}$ corresponding to C . Then I_p is generated by $(x_{c_1}, x_{c_2}, \dots, x_{c_r})$ where $C_1, C_2, \dots, C_r$ is the set of all maximal chain. Thus I_p is a monomial ideal and called chain ideal associated to the poset P . It has been widely studied in the literature For more details see[10]. An ideal I is said to be "maximal ideal" of a ring R , if there is no any other ideal which one can find among I and R . A subset A of a poset P is a chain blocker if it satisfies the minimality condition(inclusion) which consists of atleast one element from each maximal chain. The set of chain blockers has been studied for $P = C_a \times C_b$. Results such as for $A = (1, 1), (2, 1), \dots, (a, 1), (a, 2), \dots, (a, b)$ and let $B \subseteq C_2 \times C_2$ be a chain blocker. Then $A \cap B$ contains one and only one element and another one is for $1 \leq m \leq n$ and B be a chain blocker in $\subseteq C_2 \times C$ such that $(*)\|B\| = m, (m, 1) \in B$ and for all $(\acute{m}, \acute{n}) \in B$ we have $(\acute{m} + \acute{n} \leq a + 1$. If $(\acute{m}, \acute{n}) \in B$ for some $1 < \acute{m}$ then exactly one of $(\acute{m} - 1, 2), \dots, (\acute{m} - 1, \acute{n} - 1)$ lies in B are figured out. For the poset $P = C_2 \times C_2 \times C_j$ shown in Figure 1, the I_p is generally by following square free monomials, $x_1 = x_11x_121x_131x_132x_232$, $x_1 = x_11x_112x_122x_222x_232$, $x_1 = x_11x_121x_131x_231x_232$, $x_1 = x_11x_112x_212x_222x_232$, $x_1 = x_11x_121x_122x_132x_232$, $x_1 = x_11x_211x_221x_231x_232$, $x_1 = x_11x_121x_122x_222x_232$, $x_1 = x_11x_211x_221x_222x_232$, $x_1 = x_11x_121x_221x_222x_232$, $x_1 = x_11x_211x_212x_222x_232$, $x_1 = x_11x_112x_122x_132x_232$, $x_1 = x_11x_121x_221x_231x_232$ An ideal $P \subset R$ is said to be a prime ideal [4] if for the condition $x, y \in R$ and $xy \in P$, then either $x \in P$ or $y \in P$. For example the prime ideals of Z are $2Z$ and $3Z$. If we take maximum lenght of all chains(prime ideals) then we call it as "Krull dimension" [5] of ring R (a commutative

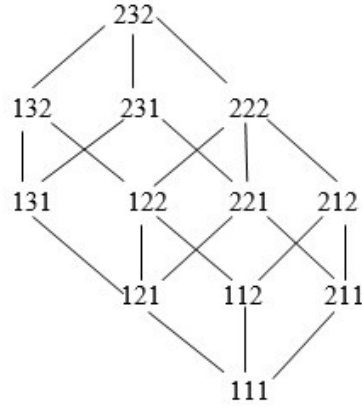


Figure 1.1: poset

one). It is named after mathematician Wolfgang Krull. For example $\dim(K) = 0$, and $\dim(K[x_1, \dots, x_n]) = n$. An ideal I of a ring R is known as "primary ideal" if when ever we observe the statements $\rho \neq R$ as well as $pq \in \rho$, we must have $p \in \rho$ or $q^n \in \rho$. The primary decomposition is decomposing the ideal in the $\cap$ of primary ideals. That is $I = \phi_1 \cap \phi_2 \cap \dots \cap \phi_s$ where ϕ_i are primary ideals. For example primary decomposition [6] of ideal $I = \langle x, yz \rangle$ is $\langle x, y \rangle \cap \langle x, z \rangle$. A finitely generated algebra is an algebra A over a field k where there is finite set of elements $x_1 \cdots x_n$ of A in a way that when ever we have all elements of A we can write in the form of $x_1 \cdots x_n$ by taking there multiples from k . Polynomials rings and their quotients by graded ideals are graded algebras. So if S is finitely generated graded commutative algebra over field K , the Hilbert function $HF_s : n \mapsto \dim_k S_n$ such that the range has n on the number of elements in the basis of the vector space S_n taken over field " K ". The Hilbert series is $HS_s(t) = \sum HF_s(n)t^n$. For example if we take a field K such that $S = K[x]$ be a polynomial ring and $\deg x = 1$. We define "Hilbert series" of the polynomial or quotient ring is $HS_s(t) = 1/(1-t)$. For more detail see [5].

Secondly, we will consider a graph G which do not contain disconnected components over the set of vertices V . The length $d(u, v)$ is set to be shortest way between u and v for any two vertices in G . Let a set $S = q_1, q_2, \dots, q_t$ be a subset of v . The set $r(u/S) = d(u, q_1), d(u, q_2), \dots, d(u, q_t)$ is called resolving set [3], if each vertex in graph has different distance from the vertices of S . That is $d(u, q) \neq d(v, q)$. The resolving set in this way obtained with minimum vertices is said to be basis(metric). A detailed literature has been discussed in [2,8,7] related to metric basis. The number of elements of a basis is called the dimension(metric) of G [11]. The metric dimension of G is denoted by $dim(G)$. The main idea to study properties of the graph in term of dimension (metric) is given by Slater. A line graph consists of all the vertices of each edge of G . We say that if two vertices have a vertex in between them we said those vertices to be adjacent. Line graph of some graph can be written as $L(G)$. Subdivision of graph can be performed on vertices or edges both. Subdivision of edge is basically the splitting an edge into two edges by adding a vertex on edge. A broad explanation of subdivision of graphs is explained in [9]. There are numerous patterns or designs which build honeycomb network. $(HC)(1)$ is a hexagonal honeycomb network, by adding six hexagons to the exterior edges of $HC(1)$ we will obtain $HC(2)$. To get $HC(n)$ continue this process. In this thesis chapter number 2 deals with results of dimension(metric) while in chapter number 3 we have results related to combinatorial and algebraic properties related to the poset $P = C_2 \times C_2 \times C_j$ to generalize the results given in [1] for the poset $P = C_a \times C_b$.

Chapter 2

Basic Definition and Notations

This chapter contains all basic stuff used through out the contents of the thesis. We start with following basic definition.

Definition 2.0.1. *An object containing two sets namely set of vertex V and set of edges E , is called a graph G .*

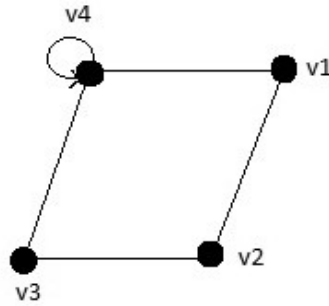


Figure 2.1: Graph

Definition 2.0.2. *The number of vertices and number of edges in a graph is said to be the "order of a graph" and the size of a graph.*

Definition 2.0.3. *An edge whose initial and ending points are same is known as loop.*

Definition 2.0.4. *The "degree of a vertex" represented as $\deg(V)$ is the number of edges containing that vertex.*

Definition 2.0.5. *A graph that does not have any loop or multi edges, is called simple graph.*

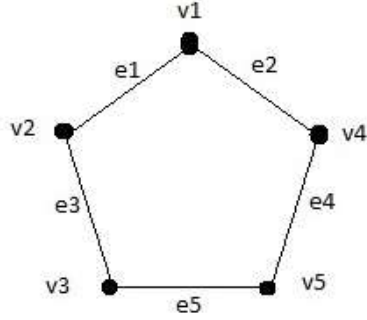


Figure 2.2: Order and size of G is 5,5.

Definition 2.0.6. The graph $\bar{G}$ is called as to be the complement of some graph G , if the edges in G are not the edges in $\bar{G}$. And it is written by $\bar{G}$.

Definition 2.0.7. A graph containing the condition that there are more than one edges between two adjacent vertices, is called multiple graph.

Definition 2.0.8. A graph G is called as to be a complete graph if we take all combinations of cardinality 2 in the set of edges of the graph. It is written by K_n , where n shows the cardinality of the set vertices.

Definition 2.0.9. If the outdegree and indegree of all the vertices in some graph G is denoted by same number, then G is said to be a "regular graph".

Definition 2.0.10. A graph in which edges are arcs, or edges have some arrows in or out, is called "directed graph" otherwise it is "undirected graph".

Definition 2.0.11. If there is some graph G , then its sub-graph G' will consist of the subset of set of vertices and edges of G . And G in that case will be called super-graph.

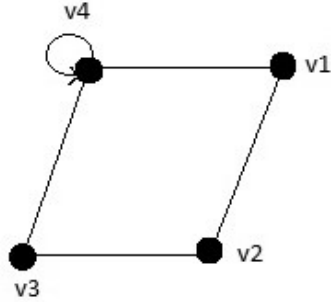


Figure 2.3: vertex with degree

Definition 2.0.12. *A walk in a graph between two vertices is a collection of vertices and edges in the form of sets such that an edge say e_1 joins vertex $v_{(i-1)}$ and v_i . The length of walk is the cardinality of the set of edges in it.*

Definition 2.0.13. *A trail is a walk with no repeated edges.*

Definition 2.0.14. *A path is a walk in which we do not have no vertices which are repeated. All the vertices in a path should be distinct.*

Definition 2.0.15. *A "closed walk" is a walk that has the starting and ending vertex to be same vertex.*

Definition 2.0.16. *If there is a path between every pair of vertices, then this graph is said to be a connected graph.*

Definition 2.0.17. *A cycle is a closed walk with no repeated vertices and edges.*

Definition 2.0.18. *The distance between two vertices v_1 and v_2 in a graph G , is the smallest path which connects them.*

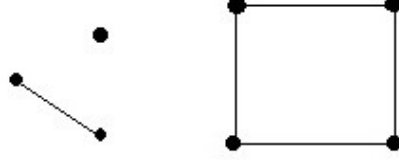


Figure 2.4: Graph

Definition 2.0.19. The length $d(u, v)$ be the path between u and v with minimum number of edges and for any two vertices in G . Let a set $S = \{q_1, q_2, \dots, q_t\}$ be a subset of V . The set $r(u/s) = \{d(u, q_1), d(u, q_2), \dots, d(u, q_t)\}$ is called resolving set if each vertex in graph has different distance from the vertices of S . That is $d(u, q) \neq d(v, q)$.

Definition 2.0.20. The "resolving set" with most less vertices is called basis(metric).

Definition 2.0.21. The "number of elements" of a basis is called the dimension(metric). The dimension(metric) of G is denoted by $\dim(G)$.

Definition 2.0.22. Subdivision of graph can be performed on vertices or edges both. Subdivision of edge is basically the splitting an edge into two edges by adding a vertex on edge.

Definition 2.0.23. There are numerous patterns or designs which build honeycomb network. $(HC)(1)$ is a hexagonal honeycomb network, by adding six hexagons to the exterior edges of $(HC)(1)$ we will obtain $(HC)(2)$. To get $(HC)(n)$ continue this process.

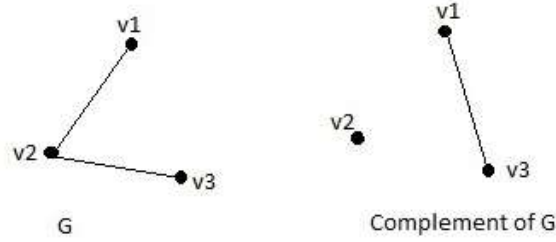


Figure 2.5: "Complement of graph G "

Definition 2.0.24. For $G \simeq LSHC(n)$; $n = 1$, then G has metric dimension 2.

Definition 2.0.25. $LSHC(1)$ is a cycle having 12 vertices. As paths have metric dimension 1 and cycles have metric dimension 2, $LSHC(1)$ has metric dimension 2.

Definition 2.0.26. A line graph of a simple graph G is a graph which is obtained by interchanging the edges into vertices. And two vertices in line graph will be connected if there is a common vertex between two edges in simple graph. Line graph is denoted by $L(G)$.

Definition 2.0.27. A binary relation R on a set X is a partial order if and only if it is refl., antisym. and tran. As an example $\leq$ is said to be a partial ordering relation on integers, $|$ is partial ordering for +ve integers.

Definition 2.0.28. A partial ordering relation is said to be a total ordering relation if it satisfies three of the properties of partial order relation and one additional property that is comparability. For example $\leq$ is also a total ordering relation on integers.

Definition 2.0.29. Let P be a set, P with a partial ordering relation R is called poset. The partial ordering $\leq$ is a relation between the elements of poset.

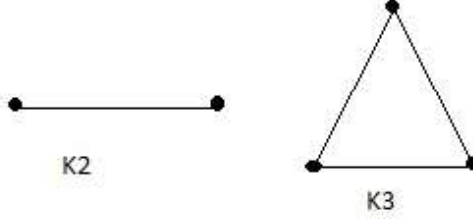


Figure 2.6: Complete graph

Definition 2.0.30. Any two elements a and b in some poset are said to be comparable if $a \leq b$ or $b \leq a$.

Definition 2.0.31. If no two elements can be related in some poset then they are called incomparable.

Definition 2.0.32. If P is a poset, then the subset A of P , in which every element is comparable, then the subset A is said to be the chain. And if it contains all the possible comparable elements then it is called maximal chain.

Definition 2.0.33. A subset A of some poset is said to be anti-chain if no elements in it are comparable to each other.

Definition 2.0.34. An ideal P is called as a "maximal ideal" of a ring R , if there "we do not find any ideal between I and R ".

Definition 2.0.35. A set is said to be a simplicial complex if it satisfies the following properties (a) For some simplicial complex A , every face form A should be in A . (b) intersection (which do not have any element) of any two simplices from A , should have a face of both simplices.

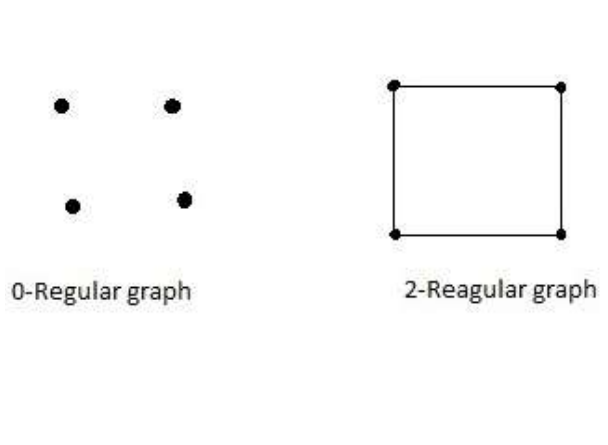


Figure 2.7: Regular graph

Definition 2.0.36. Let K be a field, then $K[x_1, \dots, x_n]$ is said to be ring (polynomial) in n number of variables. $K[x_1, \dots, x_n]$ is a collection of all polynomials in x_i variables with coefficients from K .

Definition 2.0.37. The product $x^a = x_1^{a_1}, \dots, x_n^{a_n}$, $\alpha_i \geq 0$ is said to be a monomial. "Every polynomial is a monomial but converse is not true". For example $x_1^3 x_2^2$ and $x_1^2 x_3^3$ are monomials but $x_1^2 x_3^3$ is not a monomial.

Definition 2.0.38. An ideal I generated by the edges of a graph is called to be edge ideal.

Definition 2.0.39. An ideal I generated by the maximum paths of a graph is said to be path ideal.

Definition 2.0.40. An ideal generated by monomials is called monomial ideal.

Definition 2.0.41. A monomial $x^a = x_1^{a_1}, \dots, x_n^{a_n}$, $\alpha_i \leq 1$. An ideal is generated by "square free monomials" is called "square free monomial ideal".

Definition 2.0.42. A subset A of a poset P is a chain blocker if it is inclusionwise minimal set which consists of atleast one element from each maximal chain.

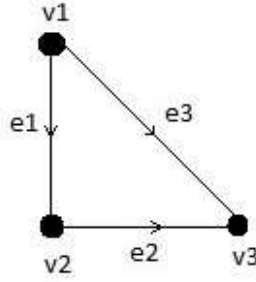


Figure 2.8: Directed Graph

Definition 2.0.43. An ideal $\phi \subset S$ is called to be a prime ideal if for any $p, q \in S$, if $pq \in \phi$, then $p \in \phi$ or $q \in \phi$. For example the prime ideals of Z are $2Z$ and $3Z$.

Definition 2.0.44. "The supremum of lengths of all the chains of prime ideals is called krull dimension. It is named after the mathematician Wolfgang Krull. For example dimension of a field K is 0. And krull dimension of $K[x_1, \dots, x_n]$ is n ".

Definition 2.0.45. An ideal I of a ring R is said to be a primary ideal if $I \neq R$ and $ab \in I$, either $a \in I$ or $b^n \in I$. Every prime ideal is primary for $n = 1$.

Definition 2.0.46. "The primary decomposition is decomposing the ideal into the intersection of primary ideals. That is $I = Q_1 \cap Q_2 \cap \dots \cap Q_i$, where Q_i are the primary ideals. For example primary decomposition of the ideal $I = \langle x, yz \rangle$ is $\langle x, y \rangle$ and $\langle y, z \rangle$ ".

Definition 2.0.47. A finitely generated algebra is an algebra A over a field K , where there is a finite number of elements $x_1 \dots x_n$ such that every element of A can be expressed as a polynomial in $x_1 \dots x_n$ with coefficients in K .

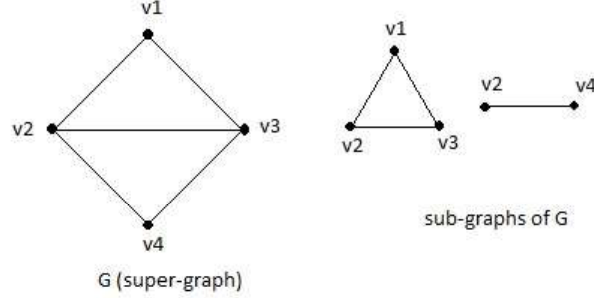


Figure 2.9: Sub-graph and Super-graph

Definition 2.0.48. *Polynomials rings and their quotients by graded ideals are graded algebras. So if S is finitely generated graded commutative algebra over field K , the Hilbert function $HF_s : n \rightarrow \dim_k S_n$ maps the integer n onto the dimension of the K -vector space S_n . The Hilbert series is $HS_s(t) = HF_s(n)t^n$. For example let K be a field and $S = K[x]$ be a polynomial ring. And $\deg x = 1$. Then the Hilbert series of the polynomial ring is $HS_s(t) = 1/(1 - t)$.*

Definition 2.0.49. *Hilbert basis theorem states that if G is a generating set of ideal I in some polynomial ring R , then I is finitely generated.*

Definition 2.0.50. *Dickson's lemma states that if G is a generating set for some monomial ideal I in polynomial ring R , then I is finitely generated.*

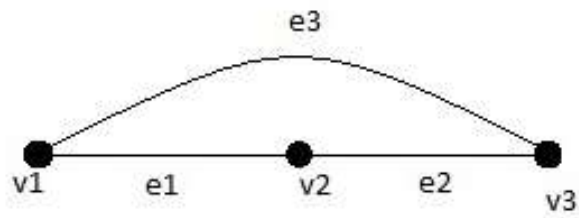


Figure 2.10: walk

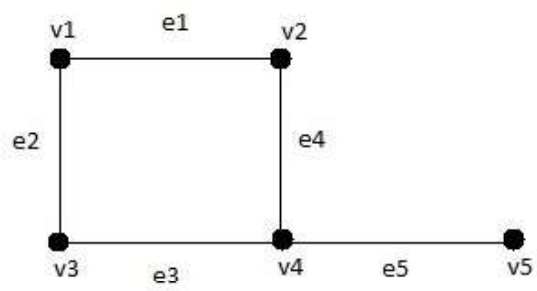


Figure 2.11: trail

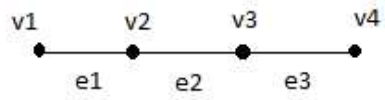


Figure 2.12: path

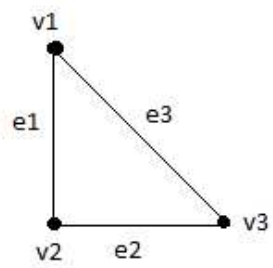


Figure 2.13: cycle

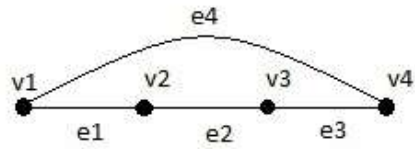


Figure 2.14: $d(v_1, v_2)$

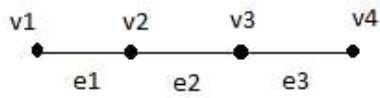


Figure 2.15: Resolving set

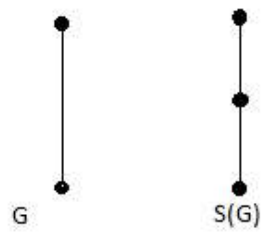


Figure 2.16: sub division

Chapter 3

Metric Dimension of Line graph of sub-divided Honey comb network

3.1 Examples

In this chapter we studied metric dimension of the honey comb network. Following relations are computed to find the metric dimension of the honey comb network. Throughout the chapter, the graph G denote the honey comb network on the vertex set V . Let $LSHC(6)$ be the graph which has the vertex set $V(LSHC(6)) = v_{i,j} \cup u_{i,j} \cup c_{i,j} \cup d_{i,j}$. The values for the central vertices $C_{i,j}$ are given by the following relations. For 1st component:

$$C_{ij} = \begin{cases} (i-2)+4j, & \text{if } n \geq 2 \text{ and } 2 \leq i \leq 3; \\ (i-4)+4j+4, & \text{if } n \geq 2 \text{ and } 4 \leq i \leq 5; \\ (i-6)+4j+8, & \text{if } n \geq 2 \text{ and } 6 \leq i \leq 7; \\ (i-8)+4j+12, & \text{if } n \geq 2 \text{ and } 8 \leq i \leq 9; \\ (i-10)+4j+16, & \text{if } n \geq 2 \text{ and } 10 \leq i \leq 11. \end{cases}$$

2nd component

$$C_{ij} = \begin{cases} 8(n-1)-4(j-1)-(i-2), & \text{if } n \geq 2 \text{ and } 2 \leq i \leq 3; \\ 8(n-1)-4(j-1)-i, & \text{if } n \geq 2 \text{ and } 4 \leq i \leq 5; \\ 8(n-1)-4(j-1)-(i+2), & \text{if } n \geq 2 \text{ and } 6 \leq i \leq 7; \\ 8(n-1)-4(j-1)-(i+4), & \text{if } n \geq 2 \text{ and } 8 \leq i \leq 9; \\ 8(n-1)-4(j-1)-(i+4), & \text{if } n \geq 2 \text{ and } 10 \leq i \leq 11. \end{cases}$$

3rd component

$$C_{ij} = \begin{cases} 8n-4j+(i-2), & \text{if } n \geq 2 \text{ and } 2 \leq i \leq 3; \\ 8n-4j+(i-4), & \text{if } n \geq 3 \text{ and } 4 \leq i \leq 5; \\ 8n-4j+(i-6), & \text{if } n \geq 4 \text{ and } 6 \leq i \leq 7; \\ 8n-4j+(i-8), & \text{if } n \geq 5 \text{ and } 8 \leq i \leq 9; \\ 8n-4j+(i-10), & \text{if } n \geq 6 \text{ and } 10 \leq i \leq 11. \end{cases}$$

1st component

$$D_{ij} = \begin{cases} (i-2)+3+4(j-1), & \text{if } n \geq 2 \text{ and } 2 \leq i \leq 3; \\ (i-2)+7+4(j-1), & \text{if } n \geq 3 \text{ and } 4 \leq i \leq 5; \\ (i-2)+11+4(j-1), & \text{if } n \geq 4 \text{ and } 6 \leq i \leq 7; \\ (i-2)+15+4(j-1), & \text{if } n \geq 5 \text{ and } 8 \leq i \leq 9; \\ (i-2)+19+4(j-1), & \text{if } n \geq 6 \text{ and } 10 \leq i \leq 11. \end{cases}$$

2nd component

$$D_{ij} = \begin{cases} 8n-4j-(i-1), & \text{if } n \geq 2 \text{ and } i = 2; \\ 8n-4j-(i-3), & \text{if } n \geq 2 \text{ and } i = 3; \\ 8n-4j-(i-3), & \text{if } n \geq 3 \text{ and } i = 4; \\ 8n-4j-(i-1), & \text{if } n \geq 3 \text{ and } i = 5; \\ 8n-4j-(i-5), & \text{if } n \geq 4 \text{ and } i = 6; \\ 8n-4j-(i-7), & \text{if } n \geq 4 \text{ and } i = 7; \\ 8n-4j-(i-7), & \text{if } n \geq 5 \text{ and } i = 8; \\ 8n-4j-(i-9), & \text{if } n \geq 5 \text{ and } i = 9; \\ 8n-4j-(i-9), & \text{if } n \geq 6 \text{ and } i = 10; \\ 8n-4j-(i-11), & \text{if } n \geq 6 \text{ and } i = 11. \end{cases}$$

3rd component

$$D_{ij} = \begin{cases} 8(n-1)-4(j-1)+1-(i-2), & \text{if } n \geq 2 \text{ and } 2 \leq i \leq 3; \\ 8(n-1)-4(j-1)-(i-1), & \text{if } n \geq 3 \text{ and } 4 \leq i \leq 5; \\ 8(n-1)-4(j-1)-(i-6), & \text{if } n \geq 4 \text{ and } 6 \leq i \leq 7; \\ 8(n-2)-4(j-1)-3-(i-8), & \text{if } n \geq 5 \text{ and } 8 \leq i \leq 9; \\ 8(n-3)-4(j-1)+1-(i-10), & \text{if } n \geq 6 \text{ and } 10 \leq i \leq 11. \end{cases}$$

Special cases for C_{ij} 2nd component

$$C_{ij} = \begin{cases} 2n-(i-2)-(j-2), & \text{if } n = 2 \text{ for } C_{2,2}, C_{3,2}, C_{2,3}, C_{3,3}; \\ 4(n-1)-(i-2)-1, & \text{if } n \geq 3 \text{ for } \forall j \text{ in } 2 \leq i \leq 3; \\ 4(n-1)-(i-4)-4(j-2), & \text{if } n = 3, j = 3 \text{ and } 4 \leq i \leq 5; \\ n-(i-4), & \text{if } n = 3 \text{ for } C_{4,4}, C_{5,4}; \\ 4(n-2)-(i-4)-1, & \text{if } n \geq 4 \forall j \text{ in } 4 \leq i \leq 5; \\ n-(i-5), & \text{if } n = 4 \text{ for } C_{6,5}, C_{7,5}; \\ 4(n-3)-(i-6)-1, & \text{if } n \geq \forall j \text{ in } 6 \leq i \leq 7; \\ n-(i-6), & \text{if } n = 5 \text{ for } C_{9,6}, C_{8,6}; \\ 4(n-4)-(i-8)-1, & \text{if } n \geq 4 \text{ and } 8 \leq i \leq 9; \\ n-(i-7), & \text{if } n = 6 \text{ for } \forall j \text{ in } 10 \leq i \leq 11. \end{cases}$$

special cases for C_{ij} 3rd Component

$$C_{ij} = \begin{cases} 4n+(i-1), & \text{if } n \geq 2 \text{ and for last } j = n \text{ in } 2 \leq i \leq 3; \\ 4n+5+(i-4), & \text{if } n \geq 3 \text{ and for last } j = n \text{ in } 4 \leq i \leq 5; \\ 4n+9+(i-6), & \text{if } n \geq 4 \text{ and for last } j = n \text{ in } 6 \leq i \leq 7; \\ 4n+13+(i-13), & \text{if } n \geq 5 \text{ and for last } j = n \text{ in } 8 \leq i \leq 9; \\ 4n+17+(i-10), & \text{if } n \geq 6 \text{ and for last } j = n \text{ in } 10 \leq i \leq 11. \end{cases}$$

Special cases for D_{ij} 2nd Component

$$D_{ij} = \begin{cases} 4n+i, & \text{if } n \geq 2 \text{ and for last } j = n \text{ in } 2 \leq i \leq 3; \\ 4n+(i+2), & \text{if } n \geq 3 \text{ and for last } j = n \text{ in } 4 \leq i \leq 5; \end{cases}$$

C_{ij} for $i=1$ 1st,2nd and 3rd component

$$C_{ij} = \begin{cases} 6i+4(j-2), & \text{if } \forall n \text{ and } i = 1; \\ 8(n-1)-i-4(j-2), 4n-2 & \text{if } \forall n \text{ and first } j = n, \text{for last } j = n \text{ in } i = 1; \\ 8(n-1)+i-4(j-2), 4n-2 & \text{if } \forall n \text{ and first } j = n, \text{for last } j = n \text{ in } i = 1. \end{cases}$$

Special cases for D_{ij} 1st,2nd and 3rd component for $i = 1$

$$D_{ij} = \begin{cases} 5+4(j-2), & \text{if } \forall n \text{ and } i = 1; \\ 8(n-1)-4(j-2), 4n-2 & \text{if } \forall n \text{ and first } j = n, \text{for last } j = n \text{ in } i = 1; \\ 8(n-1)-4(j-2), 4n-3 & \text{if } \forall n \text{ and first } j = n, \text{for last } j = n \text{ in } i = 1. \end{cases}$$

V_{ij} 1st, 2nd and 3rd component for $i = 6$

$$V_{ij} = \begin{cases} 3n+3+j, & \text{if } n = 6 \text{ and } \forall jini = 6; \\ 3n+4-j, & \text{if } n = 6 \text{ and } \forall jini = 6; \\ 6n+10-(j-1), & \text{if } n = 6 \text{ and } 1 \leq j \leq 4ini = 6; \end{cases}$$

Special cases for V_{ij} 3rd component for $i = 6$

$$V_{ij} = \begin{cases} 43, & \text{if } n = 6 \text{ for } j = 5, 8, 9, 12, 13, 16, 17, 20, 21, 24 \text{ in } i = 6; \\ 44, & \text{if } n = 6 \text{ for } j = 6, 7, 10, 11, 14, 15, 18, 19, 22, 23 \text{ in } i = 6; \end{cases}$$

U_{ij} 1st, 2nd and 3rd component for $i = 6$

$$U_{ij} = \begin{cases} 3n+2+j, & \text{if } n = 6 \text{ and } \forall jini = 6; \\ 6n+8+j, & \text{if } n = 6 \text{ and } 1 \leq j \leq 2ini = 6; \\ 3n+6-j, & \text{if } n = 6 \text{ and } \forall jini = 6; \end{cases}$$

Special cases for U_{ij} 2nd component for $i = 6$

$$U_{ij} = \begin{cases} 44, & \text{if } n = 6 \text{ for } j = 4, 5, 8, 9, 12, 13, 16, 17, 20, 21, 24 \text{ in } i = 6; \\ 45, & \text{if } n = 6 \text{ for } j = 3, 6, 7, 10, 11, 14, 15, 18, 19, 22, 23 \text{ in } i = 6; \end{cases}$$

V_{ij} 1st component

$$V_{ij} = \begin{cases} 4i+j-3, & \text{if } \forall n, i \text{ and } j; \end{cases}$$

Special cases for V_{ij} 2nd component for $i = 1$

$$D_{ij} = \begin{cases} 8(n-1)+2i+1-j, & \text{if } n \geq 2 \text{ and } j = 4n - 2 \text{ for } i = 1; \end{cases}$$

Special cases for V_{ij} 2nd component

$$V_{ij} = \begin{cases} 4n-3+(j-7), & \text{if } n = 2 \text{ for } V_{1,7}, V_{1,8}; \\ 4n-3+(j-11), & \text{if } n = 3 \text{ for } V_{1,11}, V_{1,12}; \\ 4n-3+(j-15), & \text{if } n = 4 \text{ for } V_{1,15}, V_{1,16}; \end{cases}$$

Special cases for V_{ij} 2nd Component

$$V_{ij} = \begin{cases} 5, & \text{if } n = 2 \text{ for } V_{1,12}, V_{1,9}; \\ 9, & \text{if } n = 3 \text{ for } j = 13, 16, 17, 20 \text{ and } i = 1; \\ 13, & \text{if } n = 4 \text{ for } j = 17, 20, 21, 24, 25, 28 \text{ and } i = 1; \\ 17, & \text{if } n = 5 \text{ for } j = 21, 24, 25, 28, 29, 32, 33, 36 \text{ and } i = 1; \\ 21, & \text{if } n = 6 \text{ for } j = 25, 28, 29, 32, 33, 36, 37, 40, 41, 44 \text{ and } i = 1; \\ 4, & \text{if } n = 2 \text{ for } j = 10, 11 \text{ and } i = 1; \\ 8, & \text{if } n = 3 \text{ for } j = 14, 15, 18, 19 \text{ and } i = 1; \\ 12, & \text{if } n = 4 \text{ for } j = 18, 19, 22, 23, 26, 27 \text{ and } i = 1; \\ 16, & \text{if } n = 5 \text{ for } j = 22, 23, 26, 27, 30, 31, 34, 35 \text{ and } i = 1; \\ 20, & \text{if } n = 6 \text{ for } j = 26, 27, 30, 31, 34, 35, 38, 39, 42, 43 \text{ and } i = 1; \end{cases}$$

V_{ij} 3rd component $i = 1$

$$V_{ij} = \begin{cases} 8n-3i-(j-1), & \text{if } i = 1, j = 4n - 4, n \geq 2; \end{cases}$$

Special cases for V_{ij} 3rd component $i = 1$

$$V_{ij} = \begin{cases} 4n+2i-(j-5), & \text{if } n = 2 \text{ for } j = 5, 6, 7, 8; \\ 4n+2i-(j-9), & \text{if } n = 3 \text{ for } j = 9, 10, 11, 12; \\ 4n+2i-(j-13), & \text{if } n = 4 \text{ for } j = 13, 14, 15, 16; \\ 4n+2i-(j-17), & \text{if } n = 5 \text{ for } j = 17, 18, 19, 20; \\ 4n+2i-(j-21), & \text{if } n = 6 \text{ for } j = 21, 22, 23, 24; \end{cases}$$

Special cases for V_{ij} 3rd Component for $i = 1$

$$V_{ij} = \begin{cases} 7, & \text{if } n = 2 \text{ for } j = 9, 12; \\ 11, & \text{if } n = 3 \text{ for } j = 13, 16, 17, 20 \text{ and } i = 1; \\ 15, & \text{if } n = 4 \text{ for } j = 17, 20, 21, 24, 25, 28 \text{ and } i = 1; \\ 19, & \text{if } n = 5 \text{ for } j = 21, 24, 25, 28, 29, 32, 33, 36 \text{ and } i = 1; \\ 23, & \text{if } n = 6 \text{ for } j = 25, 28, 29, 32, 33, 36, 37, 40, 41, 44 \text{ and } i = 1; \\ 8, & \text{if } n = 2 \text{ for } j = 10, 11 \text{ and } i = 1; \\ 12, & \text{if } n = 3 \text{ for } j = 14, 15, 18, 19 \text{ and } i = 1; \\ 16, & \text{if } n = 4 \text{ for } j = 18, 19, 22, 23, 26, 27 \text{ and } i = 1; \\ 20, & \text{if } n = 5 \text{ for } j = 22, 23, 26, 27, 30, 31, 34, 35 \text{ and } i = 1; \\ 24, & \text{if } n = 6 \text{ for } j = 26, 27, 30, 31, 34, 35, 38, 39, 42, 43 \text{ and } i = 1; \end{cases}$$

3.2 Conjecture

By above calculations we state the following conjecture at the end of this chapter.

Conjecture 3.2.1. *For $G \cong LSHC(n)$; $n \geq 2$ then G has metric dimension 3.*

Chapter 4

Some Algebraic properties of Poset $P = C_2 \times C_2 \times C_j$

A poset P is a **chain** if every pair of element can be comparable. A maximal chain in a poset can be defined as in which we cant add any other element between the pair of two elements and a chain is said to be a maximum chain in any poset if that chain contains more points than any other chain in that poset.

Let K be field and $R = K[x_1, \dots, x_n]$ be the polynomial ring in n variables. An ideal I in R is said to be polynomial ideal if $G(I)$ contains only square free monomials. There are many ways to associate a square free monomial I to a poset P .

Definition 4.0.2. For a finite partially ordered set P we consider the set $M(P)$ of its maximal chains. We take an algebraic approach to $M(P)$ considering the polynomial ring $S = k[x_p | p \in P]$. We identify every maximal chain $c : p_0 < \dots < p_r$ with the monomials $x(c) := x_{p_0} \dots x_{p_r}$. We write I_P for the ideal generated by the $x(c)$ for all $c \in M(P)$.

Definition 4.0.3. For a finite partially ordered set P we consider the set $C(P)$ of its maximal chains of cover relations. We take an algebraic approach to $C(P)$ considering the polynomial ring $S = k[x_{p < q} | p < q \text{ is a cover relation}]$. We identify every maximal chain $c : p_0 < \dots < p_r$ of cover relations with the monomials

$y(c) := y_{p_0 < p_1} \cdots x_{p_{r-1} < p_r}$. We write J_P for the ideal generated by the $y(c)$ for all $c \in C(P)$.

Remark 4.0.4. Let $P = C_2 \times C_2 \times C_j$ be a poset. Then number of cover relations in P is given by

$$8j - 10.$$

This is true because, let $\mathcal{H}_P$ be the Hasse diagram of P which is in the form of vertical cubes as shown in Figure. Now each relation is corresponding to an edge in the cube. Since there are 12 edge in the base cube, and each addition cube give rise to number be edges by 8. Since there are total $j - 1$ cubes, we have total number of edges is given by $12 + (j - 2)8$. Now removing first and last element we remove 6 edges. Thus subtracting 6 we get the required result.

Lemma 4.0.5. Let $P = C_2 \times C_2 \times C_j$ and J_P be the ideal as defined above. Then for each $u \in G(J_P)$,

$$\deg(u) = j - 1.$$

Proof. Clearly by structure of P , the length of all maximum chains $c : p_0 < \cdots < p_r$ of cover relations in P is same. As per Definition 4.0.3, it is sufficient to show that this same length of is given by $j - 1$. Let $\mathcal{H}_P$ be the Hasse diagram of P which is in 3 dimensional cubical shape. Then each path starting from $(1, 1, 1)$ and ending at $(2, 2, j)$ has 1 horizontal (say in x -direction), 1 horizontal (say in y -direction) and $j - 1$ vertical (say in z -direction) steps. So ignoring first and last element there are exactly $j - 1$ steps for each maximal chain of cover relations, as required. $\square$

Lemma 4.0.6. Let $P = C_2 \times C_2 \times C_j$ and J_P be the ideal as defined above. Then

$$|G(J_P)| = j(j + 1).$$

Proof. Be Definition 4.0.3, we need to show number of all maximum chains $c : p_0 < \cdots < p_r$ of cover relations in P is $j(j + 1)$. Let $f(j)$ be number of all maximal chains of cover relations in $P = C_2 \times C_2 \times C_j$ and $\mathcal{H}_P$ be the Hasse diagram of P with labeling of the vertices is shown in the Figure. Each maximal chain has initial element $(2, 1, 1)$, $(1, 2, 1)$ or $(1, 1, 2)$. It is clear number of maximal chain starting from $(2, 1, 1)$ and ending at $(2, 2, j - 1)$ or $(2, 1, j)$ is given by j . Similarly number of maximal chain starting from $(1, 2, 1)$ and ending at $(2, 2, j - 1)$ or $(1, 2, j)$ is given by j . Lastly number of maximal chains starting from $(1, 1, 2)$ is equal to $f(j - 1)$. Thus

$$f(j) = 2j + f(j - 1).$$

Solving this recursive relation, we get the required result. $\square$

Theorem 4.0.7. *Let $P = C_2 \times C_2 \times C_j$ and J_P be the ideal as defined above. Then*

$$\dim(J_P) = 8j - 6$$

for $j \geq 4$.

Proof. Since J_P be a square free monomial ideal. So let Δ_{J_P} be the simplicial complex associated to J_P . Now by definition of the Stanley Reisner Ideal, we need to show that the cardinality of maximal facet in Δ_{J_P} is $8j - 6$. Since by Remark 4.0.9, number of cover relations in the Poset P is given by $8j - 10$. Thus $\dim(J_P) > 8j - 10$. Now to find the cardinality of maximal facet in Δ_{J_P} it is sufficient to find a maximum subset A of cover relations of P such that non of the maximal chains of cover relations completely belongs to A . Now each maximal chain must contains exactly one element from the set of edges $e_1, \dots, e_4$ as shown in the Figure. Thus deleting all 4 edges from the set of all cover relations of P is one of the required subsets. Thus its cardinality is given by $8j - 6$ as required. $\square$

Remark 4.0.8. *In the above theorem, for $j = 2, 3$, the dimension of J_P is 0 and 8 respectively.*

Remark 4.0.9. *Let $P = C_2 \times C_2 \times C_j$ be a poset. Then number of maximal chains in P is given by*

$$8j - 10.$$

This is true because, let $\mathcal{H}_P$ be the Hasse diagram of P which is in the form of vertical cubes as shown in Figure. Now each maximal chain is corresponding to a vertex in the cube. Since there are 8 vertices in the base cube, and each addition cube give rise to number of vertices by 4. Since there are total $j - 1$ cubes, we have total number of vertices is given by $8 + (j - 2)4$. Now removing first and last element we remove 6 vertices. Thus subtracting 6 we get the required result.

Lemma 4.0.10. *Let $P = C_2 \times C_2 \times C_j$ and I_P be the ideal as defined above. Then for each $u \in G(J_P)$,*

$$\deg(u) = j.$$

Proof. Clearly by structure of P , the length of all maximal chains $c : p_0 < \dots < p_r$ of vertices in P is same. As per Definition 4.0.2, it is sufficient to show that this same length of maximal chains is given by j . Let $\mathcal{H}_P$ be the Hasse diagram of P which is in 3 dimensional cubical shape. Then each chain starting from $(1, 1, 1)$ and ending at $(2, 2, j)$ has 1 horizontal (say in x -direction), 1 horizontal (say in y -direction) and j vertical (say in z -direction) steps. So ignoring first and last element there are exactly j steps for each maximal chain of cover relations, as required. $\square$

Lemma 4.0.11. *Let $P = C_2 \times C_2 \times C_j$ and I_P be the ideal as defined above. Then*

$$|G(I_P)| = j(j+1).$$

Proof. By Definition 4.0.2, we need to show number of all maximal chains $c : p_0 < \dots < p_r$ in P is $j(j+1)$. Let $f(j)$ be number of all maximal chains in $P = C_2 \times C_2 \times C_j$ and $\mathcal{H}_P$ be the Hasse diagram of P with labeling of the vertices is shown in the Figure. Each maximal chain has initial element $(2, 1, 1)$, $(1, 2, 1)$ or $(1, 1, 2)$. It is clear number of maximal chains starting from $(2, 1, 1)$ and ending at $(2, 2, j-1)$ or $(2, 1, j)$ is given by j . Similarly number of maximal chain starting from $(1, 2, 1)$ and ending at $(2, 2, j-1)$ or $(1, 2, j)$ is given by j . Lastly number of maximal chains starting from $(1, 1, 2)$ is equal to $f(j-1)$. Thus

$$f(j) = 2j + f(j-1).$$

Solving this recursive relation, we get the required result. $\square$

Theorem 4.0.12. *Let $P = C_2 \times C_2 \times C_j$ and I_P be the ideal as defined above. Then*

$$\dim(I_P) = 4(j-1)$$

for $j \geq 4$.

Proof. Since I_P be a square free monomial ideal. So let Δ_{I_P} be the simplicial complex associated to I_P . Now by definition of the Stanley Reisner Ideal, we need to show that the cardinality of maximal facet in Δ_{I_P} is $4(j-1)$. Since by Remark 4.0.9, number of maximal chains in the Poset P is given by $8j-10$. Thus $\dim(I_P) > 8j-10$. Now to find the cardinality of maximal facet in Δ_{I_P} it is sufficient to find a maximum subset A of maximal chains of P such that non of the maximal chains completely belongs to A . Now each maximal chain must contains exactly one element from the set of vertices $v_1, \dots, v_4$ as shown in the Figure. Thus deleting all 4 vertices from the set of all maximal chains of P is one of the required subsets. Thus its cardinality is given by $8j-6$ as required. $\square$

Remark 4.0.13. *In the above theorem, for $j = 2, 3$, the dimension of I_P is 3 and 7 respectively.*

Chapter 5

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