Interaction of Multi Nanoparticles in Flow of Hybrid Nanofluid over Disk

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Declaration

I Imran Latif Bhatti registration number **CIIT/FA16-RMT-006/LHR** hereby declare that I have produced the work presented in this thesis, during the scheduled period of study. I also declare that I have not taken any material from any source except referred to wherever due that amount of plagiarism is within acceptable range. If a violation of HEC rules on research has occurred in this thesis, I shall be liable to punishable action under the plagiarism rules of the HEC.

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It is certified that Imran Latif Bhatti **(CIIT/FA16-RMT-006/LHR)**

has carried out all the work related to this report under my supervision at the Department of Mathematics, COMSATS University Islamabad, Lahore Campus, and the work fulfills the requirement for award of MS degree.

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DEDICATION

To

My Family

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Firstly, I am thankful to Almighty Allah, who created us, taught us everything we did not know, provided us with balance, health, knowledge and intelligence to explore his world. I offer salutation upon the Holy Prophet Hazrat Muhammad (PBUH), who has lightened the life of all mankind with his guidance. He is a source of knowledge and blessings for the entire creations. His teachings make us to ponder and to explore this world with direction of Islam.

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ABSTRACT

Interaction of Multi Nanoparticles in Flow of Hybrid Nanofluid over Disk

Nanofluid is an important chemical fluid that is used in industries due to high heat transfer rates. Lately, a new kind of nanofluid is called hybrid nanofluid is used for further enhancement in heat transfer rate and is produced by two or more dissimilar nanomaterials [1]. Hybrid nanofluid show amazing thermo-physical properties that are not existed in the convective heat transfer fluids (water, oil etc.) and the nanofluids containing single nanomaterial. The main purpose of hybrid nanofluids is to achieve great enhancement in thermal or rheological properties. Moreover, the hybrid nanofluid is expected to achieve more thermal conductivity as compared to a single material nanofluids.

In a continuation of achieving better thermal conductivity and thermal performs through single nano-material, many studies have been done [2-4]. In an extension of nanofluid research, few papers have recently discussed the topic on thermal performance of hybrid nanofluids [5-7] and found a deficiency in agreement among their results. In addition, there is deficiency of theoretical models that can guess the exact thermal performance as well as fluid flow characteristic of these fluids. Thus, it is needed to be focused on theoretical modeling of hybrid nanofluid flow along with effective physical properties to measure better understanding of its heat and mass flow the behavior.

In this study, to analyses theoretically heat and mass flow behavior, the physical problems will be modeled through equations of continuity, momentum and energy which base on different conservation laws of physics. For physical properties, we will product mathematical co-relational models by different theories. So, by solving these equations with effective physical properties, we will obtain results for velocity and temperature profiles. These results will be used also in further physical quantities of engineering. In addition, disk shaped bodies are often used in automobile and chemical industries etc. For examples, it is used for rotating heat exchangers, bio-fluids production applications and in tubular membrane system etc. So, it is also important to discuss about configuration of geometry when such fluid flow over it.

Chapter 1

Introduction

1.1 Hybrid nanofluid

The hybrid nanofluid is new type of nanofluid and its performance evaluation is still in development phase. The term hybrid can be considered as different materials which are a combination of physical and chemical properties to form a homogeneous phase. Generally, there are three types of the hybrid materials such as metal nano-composites, ceramic nano-composites and polymer matrix nano-composites [8]. Hybrid nano-materials show amazing physicochemical and thermo-physical properties that are not existing in the base fluid as well as in single material nanofluid. The main purpose of hybrid nanofluids is to achieve great enhancement in thermal or rheological properties as compared to a single type of nanofluid and base fluid. It is used for better performance than single martial nanofluid in similar applications of nanofluids. After discovery of nanofluids, many researches associated with various nanofluid's applications have been investigated. Some of the applications contain; electronic cooling, car radiators, heat pipes, nuclear plant, coolant in welding and machining, heat exchanger [9-18] etc.

1.2 Literature Review

Since the past decade, studies on the nanofluid have been augmented rapidly. In the face of some discrepancy in the conveyed results and deficiency in sympathetic of heat transfer in nanofluid, it is considered as an auspicious convective heat transfer fluid. In the continuance of nanofluids investigation, recently the researchers have tried to utilize the hybrid nanofluids. To expand the thermal characteristics of oil, recently, Choi et al. [19] prepared oil base Al_2O_3 / AIN -hybrid nanofluid. By this hybrid nanofluid, they found 8% improvement of thermal conduction and overall the value of heat transfer coefficient is improvement 20% at 0.5% hybrid nanoparticles volume fraction. The performance in heat transfer in water base hybrid nanofluid in tubular heat exchanger is investigated experimentally by Madhesh and Kalaiselvam [20] through titania-copper hybrid nanomaterial. Their results illustrate up to 30.4% improvement in heat transfer coefficient at 0.7% concentrations of hybrid nanocomposite. A fully developed flow of water base $\left(Cu - Al_2O_3\right)$-hybrid nanofluid along convective heat transfer behavior in heated circular tube were examined by Suresh et al. [21]. The experimental results of this showed a 6.09% enhanced in Nusselt number via Al_2O_3 nanoparticles at

0.1% concentration when related to water. On the other hand, the value of Nusselt number is improved 13.56% significantly than water by composition of copper and alumina nanoparticles. This means that heat transfer rate in single material nanofluid can be improved via hybrid nanoparticles. In another study, Suresh et al. [22] studied the heat transfer behavior in turbulent flow of water base $\left(Al_2O_3 - Cu\right)$-hybrid nanofluid and found 8.02% enhancement in heat transfer rate as compared to water. Selvakumar and Suresh [23] measured the impact of $Al_2O_3 - Cu$ hybrid material on heat transfer performance of deionized water in an electronic sink hybrid nanofluid at 0.1% nanoparticle volume fraction. They found 24.35% incensement in convective heat transfer rate through this hybrid nanomaterial.

In terms of heat transfer performance, hybrid nanofluids have shown interesting behavior. As a new kind of nanofluid, some challenges are existing in development like deficiency of agreement among results which are obtained by diverse researchers and lack of theoretical models that can guess the exact behavior of these fluids. Thus, it is needed to be focus on theoretical modeling of flow along with physical properties for better understanding the flow characteristics and heat transfer performance of the hybrid fluid.

1.3 Physical properties of nanofluid and hybrid nanofluid

1.3.1 Density

The co-relation of density for base fluid and nanomaterial is based on the physical principle of the mixture rule. For nanofluid and hybrid nanofluid, it can be represented as

Nanofluid:

$$\rho_{nf} = \left(\frac{M}{V}\right)_{nf} = \frac{M_f + M_{np}}{V_f + V_{np}} = \frac{\rho_f V_f + \rho_{np} V_{np}}{V_f + V_{np}}, \tag{1.1}$$

$$\rho_{nf} = 1 - \left(\frac{V_{np}}{V_f + V_{np}}\right)\rho_f + \left(\frac{V_{np}}{V_f + V_{np}}\right)\rho_{np}, \tag{1.2}$$

$$\rho_{nf} = \left(1 - \phi\right)\rho_f + \left(\phi\right)\rho_{np}, \tag{1.3}$$

Where V is volume, ϕ is the volume fraction of nanoparticles, M is the mass, ρ_{np} is density of nanofluid and ρ_f is density of base fluid. The subscripts f and np refer to the fluid and nanoparticle.

Hybrid nanofluid:

$$\rho_{hnf}=\left(\frac{M}{V}\right)_{hnf}=\frac{M_f+M_{np1}+M_{np2}}{V_f+V_{np1}+V_{np2}}=\frac{\rho_f V_f+\rho_{np1}V_{np1}+\rho_{np2}V_{np2}}{V_f+V_{np1}+V_{np2}}, \tag{1.4}$$

$$\rho_{hnf}=1-\left(\frac{V_{np1}+V_{np2}}{V_f+V_{np1}+V_{np2}}\right)\rho_f+\left(\frac{V_{np1}}{V_f+V_{np1}+V_{np2}}\right)\rho_{np1}+\left(\frac{V_{np2}}{V_f+V_{np1}+V_{np2}}\right)\rho_{np2}, \tag{1.5}$$

$$\rho_{hnf}=\left(1-\phi_{np1}-\phi_{np2}\right)\rho_f+\phi_{np1}\rho_{np1}+\phi_{np2}\rho_{np2}, \tag{1.6}$$

Here $np1$ and $np2$ subscripts respectively refer to the first and second nanoparticles while ϕ_{np1} and ϕ_{np2} are the volume fractions of first and second nanoparticle.

1.3.2 Heat capacity

The co-relations of specific heat for mixture of base fluid and nanomaterial are represented as

Nanofluid:

$$\left(\rho C_p\right)_{nf}=\rho_{nf}\frac{\left(MC_p\right)_f\Delta T+\left(MC_p\right)_{np}\Delta T}{M_f\Delta T+\left(M\right)_{np}\Delta T}, \tag{1.7}$$

$$\left(\rho C_p\right)_{nf}=\rho_{nf}\frac{\left(MC_p\right)_f+\left(MC_p\right)_{np}}{M_f+M_{np}}, \tag{1.8}$$

$$\left(\rho C_p\right)_{nf}=\rho_{nf}\frac{\left(\rho VC_p\right)_f+\left(\rho VC_p\right)_{np}}{\frac{(M_f+M_{np})}{\left(V_f+V_{np}\right)}\times\left(V_f+V_{np}\right)}, \tag{1.9}$$

$$\left(\rho C_p\right)_{nf}=\phi\left(\rho C_p\right)_{np}+\left(1-\phi\right)\left(\rho C_p\right)_f, \tag{1.10}$$

Where $\left(C_p\right)_f$ and $\left(C_p\right)_{np}$ are heat capacities of the base fluid and nanomaterial respectively.

Hybrid nanofluid:

$$\left(\rho C_p\right)_{hnf} = \rho_{hnf}\,\frac{\left(MC_p\right)_f \Delta T + \left(MC_p\right)_{np1} \Delta T + \left(MC_p\right)_{np2} \Delta T}{M_f \Delta T + \left(MC_p\right)_{np1} \Delta T + \left(MC_p\right)_{np2} \Delta T}, \tag{1.11}$$

$$\left(\rho C_p\right)_{hnf} = \rho_{hnf}\,\frac{\left(MC_p\right)_f + \left(MC_p\right)_{np1} + \left(MC_p\right)_{np2}}{M_f + \left(MC_p\right)_{np1} + \left(MC_p\right)_{np2}}, \tag{1.12}$$

$$\left(\rho C_p\right)_{hnf} = \rho_{hnf}\,\frac{\left(\rho VC_p\right)_f + \left(\rho VC_p\right)_{np1} + \left(\rho VC_p\right)_{np2}}{\dfrac{(M_f + M_{np1} + M_{np2})}{\left(V_f + V_{np1} + V_{np2}\right)} \times \left(V_f + V_{np1} + V_{np2}\right)}, \tag{1.13}$$

$$\left(\rho C_p\right)_{hnf} = \phi_{np1}\left(\rho C_p\right)_{np1} + \phi_{np2}\left(\rho C_p\right)_{np2} + \left(1 - \phi_{np1} - \phi_{np2}\right)\left(\rho C_p\right)_f \tag{1.14}$$

1.3.3 Thermal expansion coefficient

The co-relations of the thermal expansion coefficient for base fluid and nanomaterial can be estimated utilizing the volume fraction of the nanoparticles as

Nanofluid:

The thermal expansion coefficient for nanofluid can be given as

$$\left(\beta\right)_{nf} = \frac{\phi\left(\rho\beta\right)_{np} + \left(1-\phi\right)\left(\rho\beta\right)_f}{\rho_{nf}}, \tag{1.15}$$

Where β_{np} and β_f are the thermal expansion coefficient the nanoparticles and the base fluid, respectively.

Hybrid nanofluid:

$$\left(\rho\beta\right)_{hnf} = \left(1 - \phi_{np1} - \phi_{np2}\right)\left(\rho\beta\right)_f + \phi_{np1}\left(\rho\beta\right)_{np1} + \phi_{np2}\left(\rho\beta\right)_{np2}, \tag{1.16}$$

Where β_{np1} and β_{np2} the thermal expansion coefficient of the first and second nanoparticles.

1.3.4 Viscosity

Nanofluid:

Brinkman's viscosity model for nanofluid is given as [24]

$$\mu_{nf} = \frac{\mu_f}{(1-\phi)^{2.5}}, \tag{1.17}$$

Hybrid nanofluid:

Krieger-Dougherty [25] formulated the viscosity for higher particle volume fraction is given as,

$$\mu_{hnf} = \frac{\mu_f}{\left(1-\phi_{np1}-\phi_{np2}\right)^{2.5}} \tag{1.18}$$

1.3.5 Thermal conductivity of hybrid nanofluid:

The co-relation of the thermal conductivity for base fluid and nanomaterial can be written as

Nanofluid:

It was the Maxwell [26] who firstly developed the model for the thermal conductivity of nanofluid as follow

$$\frac{k_{nf}}{k_f} = \frac{k_{np}+2k_f+2\phi\left(k_{np}-k_f\right)}{k_{np}+2k_f-\phi\left(k_{np}-k_f\right)}, \tag{1.19}$$

where k_f and k_{np} are the thermal conductivities of base fluid and nanomaterial.

Hybrid nanofluid:

The thermal conductivity of hybrid nanofluid can be calculated through modified Maxwell model [26] can be written as,

$$k_{hnf} = \frac{\dfrac{\phi_{np1}k_{np1}+\phi_{np2}k_{np2}}{\phi_{np1}+\phi_{np2}}+2k_f+2\left(\phi_{np1}k_{np1}+\phi_{np2}k_{np2}\right)-2\left(\phi_{np1}+\phi_{np2}\right)k_f}{\dfrac{\phi_{np1}k_{np1}+\phi_{np2}k_{np2}}{\phi_{np1}+\phi_{np2}}+2k_f-\left(\phi_{np1}k_{np1}+\phi_{np2}k_{np2}\right)-\left(\phi_{np1}+\phi_{np2}\right)k_f}, \tag{1.20}$$

here k_{np1} and k_{np2} are the thermal conductivity's of first and second nanoparticles.

1.4 Governing equations

The governing equations depend upon the following universal laws of conservation.

1.4.1 Continuity equation

1.4.1.1 Continuity equation for compressible fluids

The fluid flow based on law of conservation of mass is known as equation of continuity and defined as

$$\frac{\partial \rho}{\partial t}+\nabla\cdot\left(\rho\mathbf{V}\right)=0 \tag{1.21}$$

Where **V** is velocity of fluid.

1.4.1.2 Continuity equation for incompressible fluid

A flow of fluid having constant density is called incompressible fluid. Which is given as

$$\frac{\partial \rho}{\partial t}=0, \tag{1.22}$$

which reduces Eq. (1.21) to

$$\nabla\cdot\mathbf{V}=0, \tag{1.23}$$

Cartesian coordinate system:

The Eq. (1.23) in Cartesian coordinate system as follow

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0, \tag{1.24}$$

where $u,\ v,\ w$ represent $x,\ y,\ z$ the components of the velocity vector.

Cylindrical coordinate system:

The Eq. (1.23) in cylindrical coordinate system as follow

Put $x=r\cos\theta$, $\mathrm{y}=r\cos\theta$ and $z=z$ in Eq. (1.23) we get

$$\frac{1}{r}\frac{\partial}{\partial r}\left(rv_r\right)+\frac{1}{r}\frac{\partial v_\theta}{\partial \theta}+\frac{\partial v_z}{\partial z}=0 \tag{1.25}$$

where v_r , v_θ and v_z are the velocity components in directions of r , θ and z respectively.

Spherical coordinate system:

The Eq. (1.23) in spherical coordinate system as follow

Putting $x = r\cos\theta\sin\phi$, $y = r\cos\phi\sin\theta$ and $z = \cos\phi$ in Eq. (1.23)

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 v_r\right)+\frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta v_\theta\right)+\frac{1}{r\sin\theta}\frac{\partial v_\varphi}{\partial\varphi}=0 \tag{1.26}$$

1.4.2 Conservation of moment

1.4.2.1 Conservation of momentum for compressible fluid

The equation Conservation of momentum is also called as the Navier-Stokes equation. It has different forms and one of is

$$\rho\left(\frac{\partial\mathbf{V}}{\partial t}+\left(\mathbf{V}\cdot\nabla\right)\mathbf{V}\right)=\boldsymbol{F}+\nabla\cdot\boldsymbol{\tau} \tag{1.27}$$

The terms in parentheses of left hand side are called local and convective acceleration respectively. The first term on the R.H.S of Eq. (1.27) represents the body force $\boldsymbol{F}$ such as gravity, electric, and magnetic force. The second term on the R.H.S Eq. (1.27) is the surface forces (such as the pressure forces due to hydrostatic pressure and shear stresses due to viscous effects) per unit volume. These forces can apply on the fluid element by the external stresses. The stresses consist of shearing stresses and normal stresses and are represented by the components of the stress tensor. Different types of fluids are defined by different types of stress tensor, i.e,

Newtonian fluid:

The stress tensor for Newtonian fluid is defined as

$$\boldsymbol{\tau}=2\mu\boldsymbol{S}_0+3\zeta\boldsymbol{C}, \tag{1.28}$$

where $\boldsymbol{S}_0$ is pure shear flow and can be written as

$$\boldsymbol{S}_0=\boldsymbol{S}-\boldsymbol{C}, \tag{1.29}$$

where $\boldsymbol{S}$ and $\boldsymbol{C}$ are the strain rate stress tensor and compression flow respectively and given as

$$\boldsymbol{S}=\frac{1}{2}\left[\nabla\mathbf{V}+\left(\nabla\mathbf{V}\right)^T\right], \tag{1.30}$$

$$\boldsymbol{C} = \frac{1}{3}(\nabla \cdot \mathbf{V})\mathbf{I}, \tag{1.31}$$

Putting Eq. (1.30) and (1.31) in (1.29) and we get

$$\boldsymbol{S}_0 = \frac{1}{2}\left[\nabla\mathbf{V} + (\nabla\mathbf{V})^T\right] - \frac{1}{3}(\nabla \cdot \mathbf{V})\mathbf{I} \tag{1.32}$$

Putting Eq. (1.31) and (1.32) in (1.28) and we get

$$\boldsymbol{\tau} = \mu\left[\nabla\mathbf{V} + (\nabla\mathbf{V})^T\right] + \left(\zeta - \frac{2\mu}{3}\right)(\nabla \cdot \mathbf{V})\mathbf{I} \tag{1.33}$$

Putting Eq. (1.33) in (1.27) we get

$$\rho\left(\frac{\partial \mathbf{V}}{\partial t} + (\mathbf{V}\cdot\nabla)\mathbf{V}\right) = \boldsymbol{F} - \nabla p + \mu\nabla^2\mathbf{V} + \frac{\mu}{3}\nabla(\nabla\cdot\mathbf{V}) + \boldsymbol{F} \tag{1.34}$$

or Eq. (1.34) can be written as

$$\rho\frac{D\mathbf{V}}{Dt} = -\nabla p + \mu\nabla^2\mathbf{V} + \frac{\mu}{3}\nabla(\nabla\cdot\mathbf{V}) + \boldsymbol{F} \tag{1.35}$$

1.4.2.2 Conservation of momentum for incompressible fluid

The equation (1.34) for incompressible fluid can be written as

$$\rho\left(\frac{\partial \mathbf{V}}{\partial t} + (\mathbf{V}\cdot\nabla)\mathbf{V}\right) = -\nabla p + \mu\nabla^2\mathbf{V} + \boldsymbol{F} \tag{1.36}$$

Cartesian coordinate system:

The Eq. (1.36) for Cartesian coordinate system can be written as

$$\rho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}\right) = -\frac{\partial p}{\partial x} + \mu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 w}{\partial z^2}\right) + F_x\ , \tag{1.37}$$

$$\rho\left(\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z}\right) = -\frac{\partial p}{\partial y} + \mu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2}\right) + F_y\ , \tag{1.38}$$

$$\rho\left(\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z}\right) = -\frac{\partial p}{\partial z} + \mu\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2}\right) + F_z\ , \tag{1.39}$$

Here u, v and w are the velocities in x, y and z directions respectively.

Cylindrical coordinate system:

The Eq. (1.36) in cylindrical coordinate system

$$\rho\left(\frac{\partial v_r}{\partial t}+v_r\frac{\partial v_r}{\partial r}+\frac{v_\theta}{r}\frac{\partial v_r}{\partial \theta}+v_z\frac{\partial v_r}{\partial z}-\frac{v_\theta{}^2}{r}\right)=$$

$$-\frac{\partial p}{\partial r}\;+\mu\left[\frac{\partial^2 v_r}{\partial r^2}+\frac{\partial}{\partial r}\left(\frac{v_r}{r}\right)+\frac{1}{r^2}\frac{\partial^2 v_r}{\partial \theta^2}+\frac{\partial^2 v_r}{\partial z^2}-\frac{2}{r^2}\frac{\partial v_\theta}{\partial \theta}\right]+F_r\;, \tag{1.40}$$

$$\rho\left(\frac{\partial v_\theta}{\partial t}+v_r\frac{\partial v_\theta}{\partial r}+\frac{v_\theta}{r}\frac{\partial v_\theta}{\partial \theta}+v_z\frac{\partial v_\theta}{\partial z}+\frac{v_r v_\theta}{r}\right)=$$

$$-\frac{1}{r}\frac{\partial p}{\partial \theta}+\mu\left[\frac{\partial^2 v_\theta}{\partial r^2}+\frac{\partial}{\partial r}\left(\frac{v_\theta}{r}\right)+\frac{1}{r^2}\frac{\partial^2 v_\theta}{\partial \theta^2}+\frac{\partial^2 v_\theta}{\partial z^2}-\frac{2}{r^2}\frac{\partial v_r}{\partial \theta}\right]+F_\theta\;, \tag{1.41}$$

$$\rho\left(\frac{\partial v_z}{\partial t}+v_r\frac{\partial v_z}{\partial r}+\frac{v_\theta}{r}\frac{\partial v_z}{\partial \theta}+v_z\frac{\partial v_z}{\partial z}\right)=$$

$$-\frac{\partial p}{\partial z}+\mu\left[\frac{\partial^2 v_z}{\partial r^2}+\frac{1}{r}\frac{\partial v_z}{\partial r}+\frac{1}{r^2}\frac{\partial^2 v_z}{\partial \theta^2}+\frac{\partial^2 v_z}{\partial z^2}\right]+F_z\;, \tag{1.42}$$

where v_r, v_θ and v_z be the velocity components in r, θ and z directions respectively.

Spherical coordinate system:

The Eq. (1.36) in spherical coordinate system

$$\rho\left(\frac{\partial v_r}{\partial t}+v_r\frac{\partial v_r}{\partial r}+\frac{v_\theta}{r}\frac{\partial v_r}{\partial \theta}+\frac{v_\varphi}{r\sin\theta}\frac{\partial v_r}{\partial \varphi}-\frac{v_\theta{}^2+v_\varphi{}^2}{r}\right)=-\frac{\partial p}{\partial r}$$

$$+\mu\left[\begin{array}{l}\dfrac{1}{r^2}\dfrac{\partial}{\partial r}\left(r^2\dfrac{\partial v_r}{\partial r}\right)+\dfrac{1}{r^2\sin\theta}\dfrac{\partial}{\partial \theta}\left(\sin\theta\dfrac{\partial v_r}{\partial \theta}\right)+\dfrac{1}{r^2\sin^2\theta}\dfrac{\partial^2 v_r}{\partial \varphi^2}\\ -\dfrac{2v_r}{r}-\dfrac{2}{r^2}\dfrac{\partial v_\theta}{\partial \theta}-\dfrac{2\cot\theta}{r^2}v_\theta-\dfrac{2}{r^2\sin\theta}\dfrac{\partial v_\varphi}{\partial \varphi}\end{array}\right]+F_r\,, \tag{1.43}$$

$$\rho\left(\frac{\partial v_\theta}{\partial t}+v_r\frac{\partial v_\theta}{\partial r}+\frac{v_\theta}{r}\frac{\partial v_\theta}{\partial \theta}+\frac{v_\varphi}{r\sin\theta}\frac{\partial v_\theta}{\partial \varphi}+\frac{v_\theta v_r}{r}-\frac{2\cot\theta}{r}v_\varphi{}^2\right)=-\frac{1}{r}\frac{\partial p}{\partial \theta}$$

$$+\mu\left[\begin{array}{l}\dfrac{1}{r^2}\dfrac{\partial}{\partial r}\left(r^2\dfrac{\partial v_\theta}{\partial r}\right)+\dfrac{1}{r^2\sin\theta}\dfrac{\partial}{\partial \theta}\left(\sin\theta\dfrac{\partial v_\theta}{\partial \theta}\right)+\dfrac{1}{r^2\sin^2\theta}\dfrac{\partial^2 v_\theta}{\partial \varphi^2}\\ +\dfrac{2}{r^2}\dfrac{\partial v_r}{\partial \theta}-\dfrac{2\cos\theta}{r^2\sin^2\theta}\dfrac{\partial v_\varphi}{\partial \varphi}-\dfrac{v_\theta}{r^2\sin^2\theta}\end{array}\right]+F_\theta\,, \tag{1.44}$$

$$\rho\left(\frac{\partial v_\varphi}{\partial t}+v_r\frac{\partial v_\varphi}{\partial r}+\frac{v_\theta}{r}\frac{\partial v_\varphi}{\partial \theta}+\frac{v_\varphi}{r\sin\theta}\frac{\partial v_\varphi}{\partial \varphi}+\frac{v_\varphi v_r}{r}+\frac{\cot\theta}{r}v_\theta v_\varphi\right)=-\frac{1}{r\sin\theta}\frac{\partial p}{\partial \varphi}$$

$$+\mu\left[\begin{array}{l}\dfrac{1}{r^2}\dfrac{\partial}{\partial r}\left(r^2\dfrac{\partial v_\varphi}{\partial r}\right)+\dfrac{1}{r^2\sin\theta}\dfrac{\partial}{\partial \theta}\left(\sin\theta\dfrac{\partial v_\varphi}{\partial \theta}\right)+\dfrac{1}{r^2\sin^2\theta}\dfrac{\partial^2 v_\varphi}{\partial \varphi^2}\\ +\dfrac{2\cos\theta}{r^2\sin^2\theta}\dfrac{\partial v_\theta}{\partial \varphi}+\dfrac{2}{r^2\sin^2\theta}\dfrac{\partial v_r}{\partial \varphi}-\dfrac{v_\varphi}{r^2\sin^2\theta}\end{array}\right]+F_\varphi\,, \tag{1.45}$$

1.4.3 Conservation of energy

Energy equation can be written in many different ways, such as the one given below

$$\rho C_p\left(\frac{\partial T}{\partial t}+\left(T.\nabla\right)T\right)=k\nabla^2 T+\mu\Phi \tag{1.46}$$

Where L.H.S of Eq. (1.46) represents the energy by mass. The first term of R.H.S represent the energy by heat conduction and second term demonstrate the energy by work.

Also Φ the viscous dissipation function can be written as

$$\Phi=\mu\left[tr\left(\boldsymbol{S}\right)^2\right] \tag{1.47}$$

Viscous dissipation is always positive because all terms are quadratic.

Cartesian system:

The energy Eq. (1.46) in Cartesian form can be written as

$$\rho C_p\left(\frac{\partial T}{\partial t}+u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y}+w\frac{\partial T}{\partial z}\right)=k\left(\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}+\frac{\partial^2 T}{\partial z^2}\right)+\Phi \tag{1.48}$$

For incompressible viscous fluid, the dissipation function has the form

$$\Phi=\mu\begin{bmatrix}2\left(\frac{\partial u}{\partial x}\right)^2+2\left(\frac{\partial v}{\partial y}\right)^2+2\left(\frac{\partial w}{\partial z}\right)^2+2\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)^2\\ +2\left(\frac{\partial w}{\partial y}+\frac{\partial v}{\partial z}\right)^2+2\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\right)^2-\frac{2}{3}\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}\right)^2\end{bmatrix} \tag{1.49}$$

Cylindrical coordinate system:

The energy equation in cylindrical form can be written as

$$\rho C_p\left(\frac{\partial T}{\partial t}+v_r\frac{\partial T}{\partial r}+\frac{v_\theta}{r}\frac{\partial T}{\partial \theta}+v_z\frac{\partial T}{\partial z}\right)=k\left[\frac{\partial^2 T}{\partial r^2}+\frac{1}{r}\frac{\partial T}{\partial r}+\frac{1}{r^2}\frac{\partial^2 T}{\partial \theta^2}+\frac{\partial^2 T}{\partial z^2}\right]+\Phi\ , \tag{1.50}$$

The work dissipation in cylindrical form can be given as

$$\Phi=\mu\begin{bmatrix}2\left\{\left(\frac{\partial v_r}{\partial r}\right)^2+\left(\frac{1}{r}\frac{\partial v_\theta}{\partial \theta}+\frac{v_r}{r}\right)^2+\left(\frac{\partial v_z}{\partial z}\right)^2\right\}+\left(\frac{1}{r}\frac{\partial v_z}{\partial \theta}+\frac{\partial v_\theta}{\partial z}\right)^2\\ +\left(\frac{\partial v_r}{\partial z}+\frac{\partial v_z}{\partial r}\right)^2+\left(\frac{1}{r}\frac{\partial v_r}{\partial \theta}+\frac{\partial v_\theta}{\partial r}-\frac{v_\theta}{r}\right)^2\end{bmatrix} \tag{1.51}$$

$$-\frac{2}{3}\mu\left[\frac{\partial v_r}{\partial r}+\frac{1}{r}\frac{\partial v_\theta}{\partial \theta}+\frac{v_r}{r}+\frac{\partial v_z}{\partial z}\right]^2$$

Spherical coordinate system:

The energy equation in cylindrical form can be written as

$$\rho C_p\left(\frac{\partial T}{\partial t}+v_r\frac{\partial T}{\partial r}+\frac{v_\theta}{r}\frac{\partial T}{\partial \theta}+\frac{v_\varphi}{r\sin\theta}\frac{\partial T}{\partial \varphi}\right)= \\ +k\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial T}{\partial r}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial T}{\partial \theta}\right)+\frac{1}{r^2\sin^2\theta}\frac{\partial^2 T}{\partial \varphi^2}\right]+\Phi, \tag{1.52}$$

The work dissipation in cylindrical form can be given as

$$\Phi=\mu\left[\begin{array}{c}\left(\frac{\partial v_r}{\partial r}\right)^2+\left(\frac{1}{r}\frac{\partial v_\theta}{\partial \theta}+\frac{v_r}{r}\right)^2+\left(\frac{1}{r\sin\theta}\frac{\partial v_\varphi}{\partial \varphi}+\frac{v_r}{r}+\frac{v_\theta\cot\theta}{r}\right)^2 \\ +\frac{1}{2}\left(r\frac{\partial}{\partial r}\left(\frac{v_\theta}{r}\right)+\frac{1}{r}\frac{\partial v_r}{\partial \theta}\right)^2+\frac{1}{2}\left(r\frac{\partial}{\partial \varphi}\left(\frac{v_\varphi}{r}\right)+\frac{1}{r\sin\theta}\frac{\partial v_r}{\partial \varphi}\right)^2 \\ +\frac{1}{2}\left(\frac{\sin\theta}{r}\frac{\partial}{\partial \theta}\left(\frac{v_\varphi}{\sin\theta}\right)+\frac{1}{r\sin\theta}\frac{\partial v_\varphi}{\partial \varphi}\right)^2\end{array}\right] \\ +\left(\zeta-\frac{2}{3}\mu\right)\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 v_r\right)+\frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}\left(v_\theta\sin\theta\right)+\frac{1}{r\sin\theta}\frac{\partial v_\varphi}{\partial \theta}\right]^2 \tag{1.53}$$

1.5 Significance of governing parameters

1.5.1 Velocity and thermal boundary layers

The region of the flow above the geometry in which the effects of the viscous shearing forces caused by fluid viscosity are felt is called the velocity boundary layer.

The flow region over the geometry in which the temperature variation in the direction normal to the surface is significant is the thermal boundary layer.

1.5.2 Boundary layer thickness

The boundary layer thickness, d, is typically defined as the distance normal from the surface to free stream region where velocity u become equal to free stream velocity $0.99u_o$.

The thickness of the thermal boundary layer at any location along the surface is defined as the distance from the surface at which the temperature difference $T-T_s$ equals free stream temperature $0.99(T_\infty-T_s)$.

1.5.3 Displacement thickness

It is defined as the distance by which the external potential flow is displaced outwards due to the decrease in velocity in the boundary layer. The displacement thickness is calculated as

$$\delta^* = \int_0^\infty \left(1 - \frac{u}{u_\infty}\right) dy. \tag{1.54}$$

1.5.4 Heat Transfer Coefficient

The convective heat transfer is the phenomenon between a solid surface and a fluid flowing passing over the surface. The heat transfer rate can be written in form of

$$Q_z = hA\left(T_w - T_\infty\right), \tag{1.55}$$

Where A is area of plate and where h is the convection heat transfer coefficient of the flow. The heat transfer at the surface by conduction is

$$Q_z = -k_{hnf} A \left.\frac{\partial\left(T_w - T_\infty\right)}{\partial y}\right|_{z=0}. \tag{1.56}$$

Equating Eq. (1.56) and (1.57) we get, thus

$$hA\left(T_w - T_\infty\right) = -k_{hnf} A \left.\frac{\partial\left(T_w - T_\infty\right)}{\partial y}\right|_{z=0}. \tag{1.57}$$

Rearranging

$$h = \frac{-k_{hnf} \left.\dfrac{\partial\left(T - T_\infty\right)}{\partial y}\right|_{z=0}}{\left(T_w - T_\infty\right)}. \tag{1.58}$$

1.5.5 Wall shear stress

A fluid moving along solid boundary will incur a shear stress on that boundary. The shear stress at wall or skin friction coefficient for a Newtonian nano-fluid can be written as

$$\tau_w = \mu_{nf} \left.\frac{\partial u}{\partial y}\right|_{y=0}. \tag{1.59}$$

Chapter 2

Interaction of Multi Nanoparticles in Flow of Hybrid Nanofluid over Disk

In this study, mathematical investigation on the effects of nanoparticles interaction on the heat and mass flow of hybrid nanofluid over disk is presented. The hybrid nanofluid is prepared by suspending of silver and copper nanoparticles in water. The results are obtained in the form velocity and temperature at different nanoparticles concentrations and further used in to calculate the engineering parameters such as skin friction coefficient, Nusselt numbers, boundary layer displacement thickness, and discussed with nanofluids which synthesized by silver/water and copper/water. It is concluded that velocity and temperature profiles as well as engineering parameters can be improved or impaired by suspension of together diverse nanoparticles but depend on material and concentration.

2.1 Mathematical formulation of the problem

2.1.1 Flow modeling

Consider the axially symmetric laminar flow of an incompressible hybrid nanofluid over a stretchable rotating disk. Disk is stretching proportional to the radius r with velocity $\alpha_v \Omega_v r/1-\beta_v t$ and has rotation velocity $\left(\Omega_v/1-\beta_v t\right)$ which varying with time. The geometry of the physical problem is shown in Fig. 2.1.

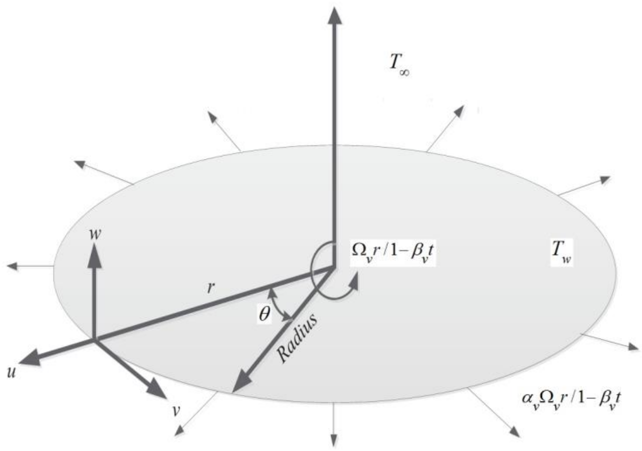

Fig. 2.1: Geometry of the problem.

The basic governing equations containing continuity, momentum and energy in vector form are

$$\nabla.\mathbf{V}=0, \tag{2.1}$$

$$\rho_{hnf}\frac{D\mathbf{V}}{Dt}=-\nabla p+\mu_{hnf}\nabla^2\mathbf{V}, \tag{2.2}$$

$$\left(\rho C_p\right)_{hnf}\frac{DT}{Dt}=k_{hnf}\nabla^2 T. \tag{2.3}$$

Where, $\frac{D}{Dt}=\frac{\partial}{\partial t}+\mathbf{V}.\nabla$, $\mathbf{V}=\left(v_r,v_\theta,v_z\right)$ is velocity vector and T is temperature.

Under boundary layer approximations, these equations in cylindrical components form can be written as

$$\frac{\partial v_r}{\partial r}+\frac{v_r}{r}+\frac{\partial v_z}{\partial z}=0, \tag{2.4}$$

$$\rho_{hnf}\left[\frac{\partial v_r}{\partial t}+v_r\frac{\partial v_r}{\partial r}+v_z\frac{\partial v_r}{\partial z}-\frac{{v_\theta}^2}{r}\right]=-\frac{\partial p}{\partial r}+\left[\frac{\partial^2 v_r}{\partial r^2}+\frac{\partial}{\partial r}\left(\frac{v_r}{r}\right)+\frac{\partial^2 v_r}{\partial z^2}\right], \tag{2.5}$$

$$\rho_{hnf}\left[\frac{\partial v_\theta}{\partial t}+v_r\frac{\partial v_\theta}{\partial r}+v_z\frac{\partial v_\theta}{\partial z}+\frac{v_r v_\theta}{r}\right]=\mu_{hnf}\left[\frac{\partial^2 v_\theta}{\partial r^2}+\frac{\partial}{\partial r}\left(\frac{v_\theta}{r}\right)+\frac{\partial^2 v_\theta}{\partial z^2}\right], \tag{2.6}$$

$$\rho_{hnf}\left[\frac{\partial v_\theta}{\partial t}+v_r\frac{\partial v_\theta}{\partial r}+v_z\frac{\partial v_\theta}{\partial z}+\frac{v_r v_\theta}{r}\right]=\mu_{hnf}\left[\frac{\partial^2 v_\theta}{\partial r^2}+\frac{\partial}{\partial r}\left(\frac{v_\theta}{r}\right)+\frac{\partial^2 v_\theta}{\partial z^2}\right], \tag{2.7}$$

$$\left(\rho C_p\right)_{hnf}\left[\frac{\partial T}{\partial t}+v_r\frac{\partial T}{\partial r}+v_z\frac{\partial T}{\partial z}\right]=k_{hnf}\left[\frac{\partial^2 T}{\partial r^2}+\frac{1}{r}\frac{\partial T}{\partial r}+\frac{\partial^2 T}{\partial z^2}\right] \tag{2.8}$$

along the following boundaries conditions

$$\left.\begin{array}{l} at\ \ z=0;\quad v_r\left(r,z,\mathrm{t}\right)=\dfrac{\alpha_v\,\Omega_v r}{1-\beta_v t},\ v_\theta\left(r,z,\mathrm{t}\right)=\dfrac{\Omega_v r}{1-\beta_v t},\ v_z=0,\ T\left(r,z,\mathrm{t}\right)=T_w \\ at\ \ z\to\infty;\quad v_r\left(r,z,\mathrm{t}\right)\to 0,\ v_\theta\left(r,z,\mathrm{t}\right)\to 0,\ T\left(r,z,\mathrm{t}\right)\to T_\infty \end{array}\right\}. \tag{2.9}$$

Now, introduce the following similarity transformation

$$\left.\begin{array}{l} \eta=\sqrt{\dfrac{\Omega_v}{\nu_f}}\dfrac{z}{\sqrt{1-\beta_v t}},\ \mathrm{v}_r\left(r,z,t\right)=\dfrac{\Omega_v\, r}{1-\beta_v t}F\left(\eta\right),\ v_\theta\left(r,z,t\right)=\dfrac{\Omega_v\, r}{1-\beta_v t}G\left(\eta\right), \\ \mathrm{v}_z\left(r,z,t\right)=\sqrt{\dfrac{\Omega_v\nu_f}{1-\beta_v t}}E\left(\eta\right),\quad \theta\left(\eta\right)=\dfrac{T-T_\infty}{T_w-T_\infty},\ p=-\dfrac{\Omega_v\mu_f}{1-\beta_v t}\mathrm{P}\left(\eta\right) \end{array}\right\}. \tag{2.10}$$

By using Eq. (2.10) into Eq. (2.4) to (2.9), get the dimensionless nonlinear system of ordinary differential equations as

$$2F(\eta)+E'(\eta)=0, \tag{2.11}$$

$$\frac{\rho_{hnf}}{\rho_f}\left[F^2-G^2+EF'+S\left(F+\frac{\eta}{2}F'\right)\right]=\frac{\mu_{hnf}}{\mu_f}F'', \tag{2.12}$$

$$\frac{\rho_{hnf}}{\rho_f}\left[EG'+2FG+S\left(G+\frac{\eta}{2}G'\right)\right]=\frac{\mu_{hnf}}{\mu_f}G'', \tag{2.13}$$

$$\frac{\rho_{hnf}}{\rho_f}\left[EE'+\frac{S}{2}\left(E+\eta E'\right)\right]=-\frac{dP}{d\eta}+\frac{\mu_{hnf}}{\mu_f}E'', \tag{2.14}$$

$$\frac{\left(\rho C_p\right)_{hnf}}{\left(\rho C_p\right)_f}\Pr\left[E\theta'+S\frac{\eta}{2}\theta'\right]=\frac{k_{hnf}}{k_f}\theta'' \tag{2.15}$$

and corresponding boundary conditions are

$$\left.\begin{array}{c} F(0)=\alpha_v,\quad G(0)=1,\quad E(0)=0,\quad \theta(0)=1, \\ F(\infty)=0,\quad G(\infty)=0,\quad \theta(\infty)=0 \end{array}\right\}. \tag{2.16}$$

Here $\Pr=\dfrac{\mu_f C_{pf}}{k_f}$ is Prandtl number and $S=\dfrac{\beta_v}{\Omega_v}$ is the unsteadiness parameter.

2.1.2 Modeling of physical properties for hybrid nanofluid

In the Eqs. (2.12) to (2.15), the effective density ρ_{hnf}, heat capacity $\left(C_p\right)_{hnf}$, thermal conductivity k_{hnf} and viscosity μ_{hnf} for the hybrid-nanofluid are defined as

$$\rho_{hnf}=\phi_{np1}\rho_{np1}+\phi_{np2}\rho_{np2}+\left(1-\phi_{np1}-\phi_{np2}\right)\rho_f \tag{2.17}$$

$$\left(C_p\right)_{hnf}=\frac{\phi_{np1}\left(\rho C_p\right)_{np1}+\phi_{np2}\left(\rho C_p\right)_{np2}+\left(1-\phi_{np1}-\phi_{np2}\right)\left(\rho C_p\right)_f}{\rho_{nf}} \tag{2.18}$$

$$k_{hnf}=\frac{\dfrac{\phi_{np1}k_{np1}+\phi_{np2}k_{np2}}{\phi_{np1}+\phi_{np2}}+2k_f+2\left(\phi_{np1}k_{np1}+\phi_{np2}k_{np2}\right)-2\left(\phi_{np1}+\phi_{np2}\right)k_f}{\dfrac{\phi_{np1}k_{np1}+\phi_{np2}k_{np2}}{\phi_{np1}+\phi_{np2}}+2k_f-\left(\phi_{np1}k_{np1}+\phi_{np2}k_{np2}\right)-\left(\phi_{np1}+\phi_{np2}\right)k_f}, \tag{2.19}$$

$$\mu_{hnf}=\frac{\mu_f}{\left(1-\phi_{np1}-\phi_{np2}\right)^{2.5}} \tag{2.20}$$

In above, $np1$ and $np2$ are represented to copper and silver nanoparticles. The subscripts hnp and f show the hybrid nanofluid and base fluid. The values of these physical properties are estimated through Eqs. (2.17) to (2.20) for nanofluid and hybrid nanofluid and summarized in Table 2.1.

Table 2.1: Thermodynamics properties of Base fluid and Hybrid Nanofluid

Physical properties	Base fluid	Nanofluids				Hybrid Nanofluid	
	Water	Cu/water		Ag/water		Cu - Ag/water $(50:50 \text{ vol.\%})$	
		$\phi = 5\%$	$\phi = 10\%$	$\phi = 5\%$	$\phi = 10\%$	$\phi = 5\%$	$\phi = 10\%$
k (W/m.K)	0.613	1.15732	1.33209	1.15736	1.33217	1.15734	1.33213
ρ (kg/m^3)	997.1	1395.25	1793.39	1471.75	1946.39	1433.50	1869.89
$\mu \times 10^{-3}$ (kg/m.s)	0.894	1.01632	1.16341	1.01632	1.16341	1.01632	1.16341
C_p (J/kg.K)	4179	3989.35	3799.70	3981.70	3784.40	3985.52	3792.05
$\beta \times 10^{-5}$ (1/K)	21	20.03	19.06	20.045	19.09	20.0375	19.075

2.1.3 Parameters of physical interest

Skin friction coefficient:

The skin friction coefficient C_f is a dimensionless wall shear stress which is defined as

$$C_f = \frac{\sqrt{\left(\tau_{wr}\big|_{z=0}\right)^2 + \left(\tau_{w\theta}\big|_{z=0}\right)^2}}{\rho_f \left(\dfrac{\alpha_v \Omega r}{1-\beta t}\right)}, \tag{2.21}$$

where τ_{wr} and $\tau_{w\theta}$ are the radial and the transversal skin fraction or shear stress and given as

$$\tau_{wr} = \mu_{hnf}\left(\frac{\partial v_r}{\partial z} + \frac{\partial v_z}{\partial \theta}\right),\ \tau_{w\theta} = \mu_{hnf}\left(\frac{\partial v_\theta}{\partial z} + \frac{1}{r}\frac{\partial v_z}{\partial \theta}\right), \tag{2.22}$$

After substituting Eq. (2.10) in Eq. (2.21), we have

$$\mathrm{Re}^{\frac{1}{2}} C_f = \frac{\mu_{hnf}}{\mu_f}\sqrt{F'(0)^2 + G'(0)^2}\,, \tag{2.23}$$

Here $\mathrm{Re} = \Omega_v r^2 / v_f (1-\beta t)$ is the rotational Reynolds number

Nusselt number:

Nusselt number (Nu) is dimensionless quantity which indicates the magnitude of convectional heat transfer and introduced by

$$Nu = \frac{h\, r}{k_f} \tag{2.24}$$

where h is convection heat transfer coefficient and has relation

$$h = \frac{k_{hnf} \left. \frac{\partial}{\partial z}\left(T - T_\infty\right)\right|_{z=0}}{\left(T_s - T_\infty\right)} \tag{2.25}$$

After substituting Eq. (2.10) in Eq. (2.24), we have

$$\mathrm{Re}^{-\frac{1}{2}}\, Nu = -\frac{k_{hnf}}{k_f}\theta'(0) \tag{2.26}$$

Boundary layer displacement thickness:

It is defined as the distance by which the external potential flow is displaced outwards due to the decrease in velocity in the boundary layer and its mathematical relation is

$$d = \frac{1-\beta t}{\Omega r}\int_{z=0}^{\infty} v_\theta dz = r\,\mathrm{Re}^{-1/2}\int_{\eta=0}^{\infty} G(\eta)d\eta. \tag{2.27}$$

2.2 Solution of the problem

In this section, we will provide the analytic and uniformly valid solutions by using BVPh 2 Package. The BVPh 2.0 requirements to put the governing equations along with corresponding boundary conditions, initial guess and auxiliary linear operators. Thus, taken the auxiliary linear operators corresponding to the Eqs. (2.11) to (2.15) as:

$$\left.\begin{array}{ll} £_1(H)=\dfrac{dE}{d\eta}, & £_2(F)=\dfrac{d^2F}{d\eta^2}+\dfrac{dF}{d\eta}, \\ £_3(G)=\dfrac{d^2G}{d\eta^2}+\dfrac{dG}{d\eta}\ , & £_4(\theta)=\dfrac{d^2\theta}{d\eta^2}+4\dfrac{d\theta}{d\eta}. \end{array}\right\} \tag{2.28}$$

Initial guess with respect to boundary conditions in Eq. (2.16) choose as:

$$\left.\begin{array}{ll} E_0(\eta)=0, & \mathrm{F}_0(\eta)=\alpha e^{-\eta}, \\ G_0(\eta)=e^{-\eta}\ , & \theta_0(\eta)=e^{-4\eta}. \end{array}\right\} \tag{2.29}$$

Finally, the solutions of velocities and temperature distributions can be exposed explicitly by means of an infinite series of the following form

$$\left.\begin{array}{l} E(\eta)=\sum\limits_{m=0}^{\infty}E_{i,m}(\eta),\ \ F(\eta)=\sum\limits_{m=0}^{\infty}F_{i,m}(\eta), \\ G(\eta)=\sum\limits_{m=0}^{\infty}G_{i,m}(\eta),\ \ \theta(\eta)=\sum\limits_{m=0}^{\infty}\theta_{i,m}(\eta) \end{array}\right\} \tag{2.30}$$

where $E_{i,m}(\eta)$, $F_{i,m}(\eta)$, $G_{i,m}(\eta)$ and $\theta_{i,m}(\eta)$ are governed by high-order deformation equations.

Further, the results for velocity components and temperature up to first iteration are as follow:

$$\begin{aligned} F=&\begin{pmatrix} \dfrac{1}{2}+\dfrac{83}{400}\left(\dfrac{\mu_{hnf}}{\mu_f}+\dfrac{3}{2}\phi g\right)+\dfrac{83}{800}\left(\dfrac{\rho_{hnf}}{\rho_f}\right)- \\ \dfrac{83S}{1600}\left(\dfrac{\rho_{hnf}}{\rho_f}\right)-\dfrac{249}{800}\left(\dfrac{\mu_{hnf}}{\mu_f}\right) \end{pmatrix}e^{-z} \\ &+\begin{pmatrix} \dfrac{-83}{400}\left(\dfrac{\mu_{hnf}}{\mu_f}\right)+\dfrac{83S}{1600}\left(\dfrac{\rho_{hnf}}{\rho_f}\right)- \\ \dfrac{83S}{800}\left(\dfrac{\rho_{hnf}}{\rho_f}\right)z-\dfrac{249}{800}\left(\dfrac{\mu_{hnf}}{\mu_f}\right) \end{pmatrix}e^{-2z}\ \ -\left(\dfrac{83}{800}\left(\dfrac{\rho_{hnf}}{\rho_f}\right)\right)e^{-3z}, \end{aligned} \tag{2.31}$$

$$G=\begin{pmatrix}1+\frac{43}{200}\left(\frac{\mu_{hnf}}{\mu_f}\right)-\frac{43}{600}\left(\frac{\rho_{hnf}}{\rho_f}\right)- \\ \frac{43S}{800}\left(\frac{\rho_{hnf}}{\rho_f}\right)+\frac{192}{400}\left(\frac{\mu_{hnf}}{\mu_f}\right)\end{pmatrix}e^{-z}$$

$$+\begin{pmatrix}-\frac{43}{200}\left(\frac{\mu_{hnf}}{\mu_f}\right)+\frac{43S}{800}\left(\frac{\rho_{hnf}}{\rho_f}\right)- \\ \frac{43S}{400}\left(\frac{\rho_{hnf}}{\rho_f}\right)z-\frac{192}{400}\left(\frac{\mu_{hnf}}{\mu_f}\right)\end{pmatrix}e^{-2z}-\left(\frac{43}{600}\left(\frac{\rho_{hnf}}{\rho_f}\right)\right)e^{-3z}, \tag{2.32}$$

$$E=e^{-2z}\left(-\frac{73}{100}+\frac{73}{100}\right), \tag{2.33}$$

$$T=\left(-\frac{53}{75}\left(\frac{k_{hnf}}{k_f}\right)-\frac{53}{225}\Pr S-\frac{53}{300}\Pr S\left(\frac{\left(\rho C_p\right)_{hnf}}{\left(\rho C_p\right)_f}\right)z\right)e^{-3z},$$

$$+\left(1+\frac{53}{75}\left(\frac{k_{hnf}}{k_f}\right)+\frac{53}{225}\Pr S\left(\frac{\left(\rho C_p\right)_{hnf}}{\left(\rho C_p\right)_f}\right)\right)e^{-2z}. \tag{2.34}$$

Chapter 3

Results and discussion

In this section, the heat and mass flow characteristics of hybrid nanofluid which synthesized by (silver-copper (50:50 vol. %)/water is analysis and its results are compared with single nanomaterial contained nanofluids that synthesized via copper/water and silver/water. To see the behavior of nanoparticles concentration on heat and mass flow characteristics, other parameters are taken fix such as such $\alpha_v = 0.5$ and $S = -0.5$.

Figs. 2.2 to 2.4 represent the radial, azimuthal and axial velocity profiles of hybrid nanofluid with various nanoparticles volume concentrations. These figure show that velocity profiles of hybrid nanofluid in all direction are decreased by nanoparticles volume concentrations. This behavior can be explained in enhancing the resistance to shearing flows, where adjacent parallel layers passage to each other. Moreover, the velocities of this fluid are more influenced at low nanoparticles concentration as relate to high concentration. Figs. 2.5 characterizes the temperature profile of hybrid nanofluid at different nanoparticles volume fraction. The results illustrate that the temperature profile is increased by increasing the nanoparticle volume fraction. This behavior is observed by modify fluid properties like viscosity, capacity of heat storage and thermal conductivity.

Figs. 2.6 to 2.8 show a comparison of velocity profiles of hybrid nanofluid with both nanofluids at 2% nanoparticles concentration. It is perceived that velocity profile of hybrid nanofluid in all direction is greater than and less than to velocity profiles of copper/water and silver/water nanofluids respectively. This midst behavior of hybrid nanofluid is noted due to equal distribution of silver and copper nanoparticles. It is concluded that velocity profile of nanofluid can be increased or decreased through distribution of diverse nanoparticles. In Fig. 2.9, the (silver-copper)/water hybrid nanofluid is evaluated with silver/water and steel/water nanofluids at 2% nanoparticles concentration. In Fig. 2.9, it is noticed that the temperature profile of hybrid nanofluid is heightened as compare to silver/water and condensed than copper/water nanofluid. In other sense, temperature profile of silver/water nanofluid is improve by dispersion of copper nanoparticles in it and reduces in case of copper/water nanofluid by mixing of silver.

Second set of results in terms of Nusselt number, skin friction coefficient and boundary layer displacement thickness is accomplished in tabular form. Table 2 shows the results of skin friction coefficient for hybrid nanofluid at different nanoparticles concentration and its results are compare with both nanofluids. It is seen that the values of skin fraction coefficient for hybrid nanofluid as well as for nanofluids are increased by increasing of nanoparticles volume fraction. Moreover, the values of skin fraction coefficient illustrations that silver/water nanofluid exert a maximum shear stress at wall as compare to copper/water nanofluid. This shear stress is reduced through dispersing of copper nanoparticles in this fluid. So, with help of hybrid nanomaterial, the exertion of this interaction force between the liquid and surface can be modified. The results of Nusselt number for hybrid nanofluid and nanofluids are dispatched in table 3 and evaluated by comparison. The values of Nusselt number are improved for all proposed fluids by nanoparticle concentration. The values of Nusselt number for silver/water nanofluid show a minimum convection heat transfer rate as compare to other fluids. This minutest heat transfer performance of silver/water nanofluid is augmented by dispersion of hybrid composition of nanoparticles. On other side, heat transfer performance of copper/water is maximum as relate to other, but its performance decreased when silver particles are added in it. Thus, hybrid nanoparticles composition can improve or decay the heat transfer characteristics of fluids. It depends on material and its concentrations. In table 4, the results of boundary layer displacement thickness for single and hybrid material contained nanofluids are displayed. The results show the discrepancy of mass flow in boundary layer with associating of free stream layers. It is found that mass flow in boundary layer region is reduced as compare to inviscid flow region when nanoparticles concentration is increased. The maximum reduction is found in case of copper/water nanofluid due to its lowest velocity and can be amended by hybrid nanoparticles composition as seen in table 4.

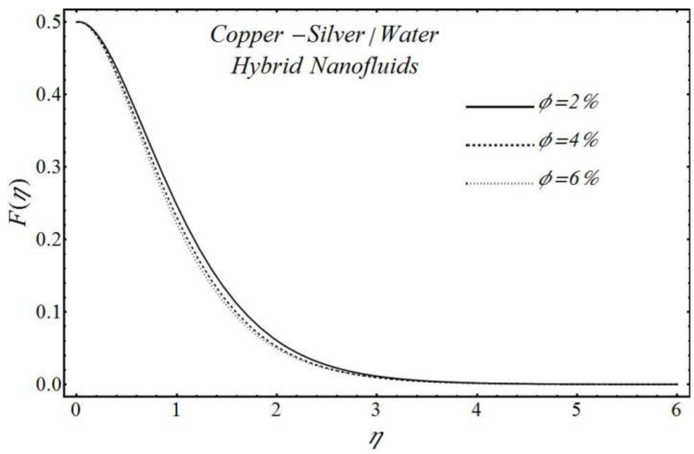

Fig. 2.2: Radial velocity profile of hybrid nanofluid at different nanoparticle concentrations.

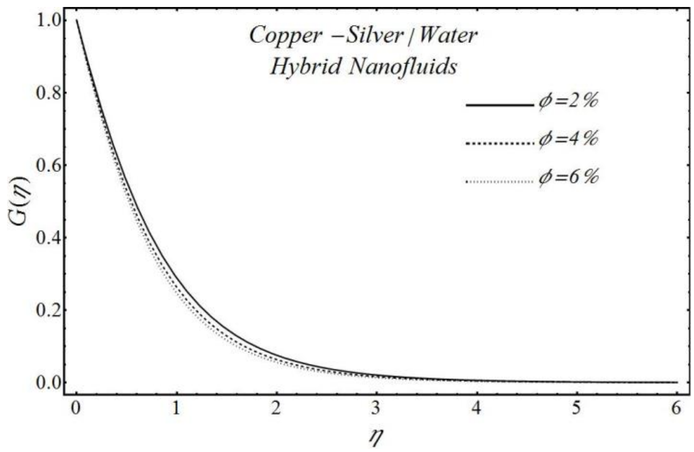

Fig. 2.3: Azimuthal velocity profile of hybrid nanofluid at different nanoparticle concentrations.

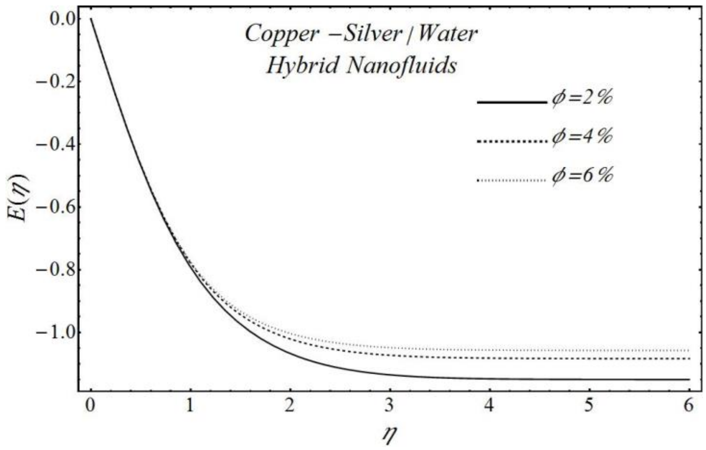

Fig. 2.4: Axial velocity profile of hybrid nanofluid at different nanoparticle concentrations.

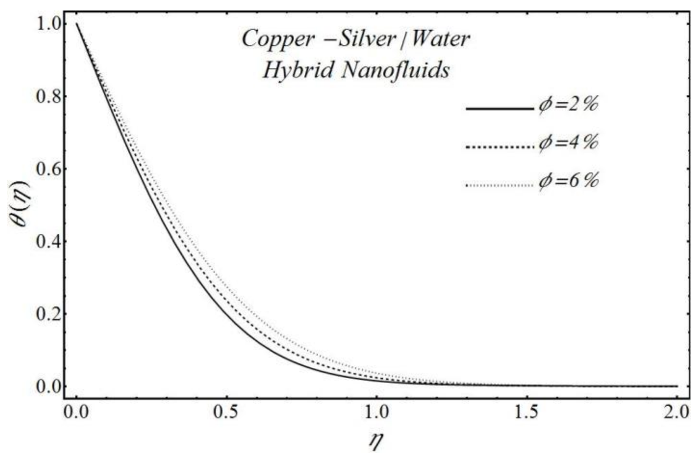

Fig. 2.5: Temperature profile of hybrid nanofluid at different nanoparticle concentrations.

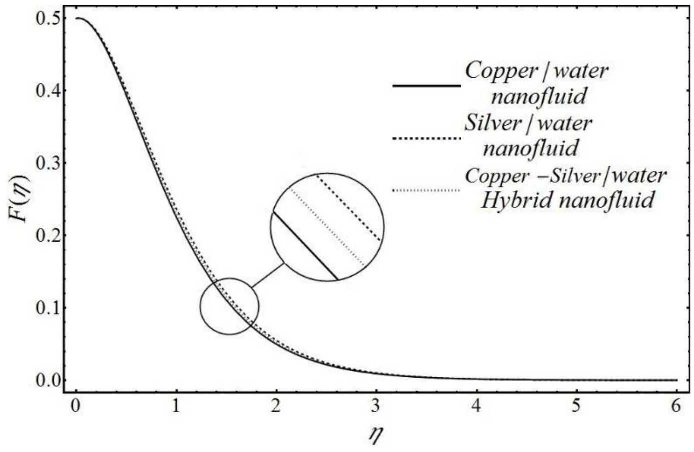

Fig. 2.6: Comparison of radial velocity profile of silver-copper/water hybrid nanofluid with copper/water and silver/water nanofluids.

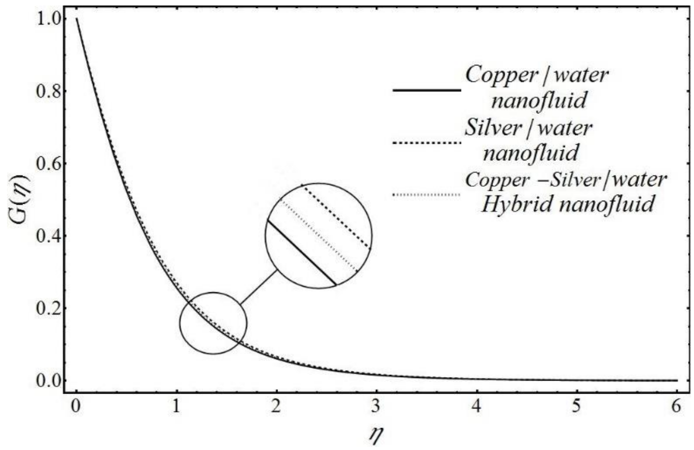

Fig. 2.7: Comparison of azimuthal velocity profile of silver-copper/water hybrid nanofluid with copper/water and silver/water nanofluids.

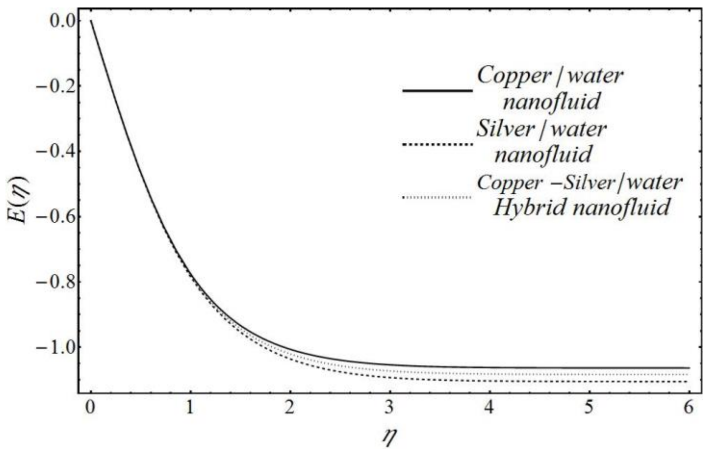

Fig. 2.8: Comparison of axial velocity profile of silver-copper/water hybrid nanofluid with copper/water and silver/water nanofluids.

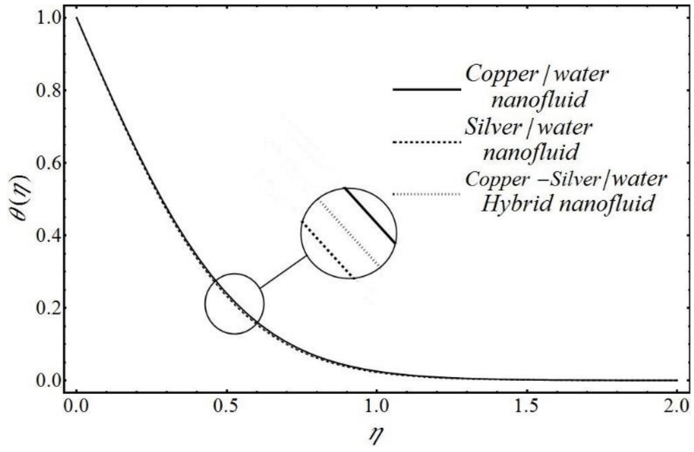

Fig. 2.9: Comparison of temperature profile of silver-copper/water hybrid nanofluid with copper/water and silver/water nanofluids.

Table 2.2: Results of skin fraction coefficient at different nanoparticles concentrations for copper/water and silver/water nanofluids and copper- silver/water hybrid nanofluid.

Nanofluids			Hybrid Nanofluids			
Particle Concentration	*Copper*	*Silver*	Particle Concentration	*Copper – Silver*	Particle Concentration	*Copper - Silver*
2%	1.05859	1.07325	1% – 1%	1.06605	1.5% – 0.5%	1.06237
4%	1.16021	1.18688	2% – 2%	1.17363	3.0% – 1.0%	1.16694
6%	1.26204	1.30083	3% – 3%	1.28158	4.5% – 1.5%	1.27185

Table 2.3: Results of Nusselt Number at different nanoparticles concentrations for copper/water and silver/water nanofluids and copper- Silver/water hybrid nanofluid.

Nanofluids			Hybrid Nanofluids			
Particle Concentration	*Copper*	*Silver*	Particle Concentration	*Copper – Silver*	Particle Concentration	*Copper – Silver*
2%	2.34836	2.34101	1% – 1%	2.34463	1.5% – 0.5%	2.34648
4%	2.40687	2.39238	2% – 2%	2.39964	3.0% – 1.0%	2.40326
6%	2.46603	2.44335	3% – 3%	2.45473	4.5% – 1.5%	2.46039

Table 2.4: Results of boundary layer displacement thickness at different nanoparticles concentrations for copper/water and silver/water nanofluids and copper- Silver/water hybrid nanofluid.

Nanofluids			Hybrid Nanofluids			
Particle Concentration	*Copper*	*Silver*	Particle Concentration	*Copper – Silver*	Particle Concentration	*Copper – Silver*
2%	0.83482	0.81873	1% – 1%	0.82627	1.5% – 0.5%	0.83038
4%	0.79212	0.77331	2% – 2%	0.78236	3.0% – 1.0%	0.78715
6%	0.76534	0.74341	3% – 3%	0.75396	4.5% – 1.5%	0.75954

Chapter 4

Conclusion

In this study, heat and mass flow along with convective heat transfer characteristics of Copper-Silver/water hybrid-nanofluid over disk are investigated. Characteristics of Copper-Silver/water hybrid-nanofluid are also compare with single material nanofluids. In this investigation, it is realized that hybrid nanofluid can be used to improve heat and mass flow as well as heat transfer rate. It is seen that Copper/water nanofluid has least velocity and temperature profiles and improve though hybrid nanofluid with composition of Silver particles. The value of shear stress at wall is found maximum in case of Silver/water nanofluid and reduced through hybrid nanoparticles. On the other hand, through mix particles heat transfer rate can be improved as seen in case of Silver/water nanofluid. Priority research topics aiming engineering applications should be related to the optimized selection of various combinations of nanocomposite materials, thermophysical properties and heat transfer.

Chapter 5

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