Computing Metric Dimension of M-Level Wheel Related Graphs

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I Ayesha Razzaq (CIIT/SP16-RMT-010/LHR) hereby declare that I have produced the work presented in this thesis, during the scheduled period of study. I also declare that I have not taken any material from any source except referred to wherever due that amount of plagiarism is within acceptable range. If a violation of HEC rules on research has occurred in this thesis. I shall be liable to punishable action under the plagiarism rules of the HEC.

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Certificate

It is certified that Ayesha Razzaq (CIIT/SP16-RMT-010/LHR) has carried out all the work related to this thesis under my supervision at the department of mathematics, COMSATS Institute of Information Technology, Lahore and the work fulfills the requirement for award of MS degree.

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DEDICATION

My loving parents, teachers and my siblings. Who always encouraged me to work hard and guided me towards the right way and destination.

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ABSTRACT

An ordered subset $Z = \{z_1, z_2, \dots, z_k\} \subseteq V(G)$ of nodes of G is called a resolving set or locating set for G if each node is individually determined by its code of distances to the nodes in Z. A resolving set of minimum number of nodes is called a basis for G and this cardinality is the metric dimension or location number of G, represented as $\beta(G)$.

In this thesis, we compute the metric dimension of some wheel related graphs, such as m-level gear graph, m-level antiweb-wheel graph, m-level antiweb-gear graph and an infinite class of convex polytopes denoted by $J_{2n,m}$, $AWW_{n,m}$, $AWJ_{2n,m}$ and D_n^m, respectively. We prove that the metric dimension of aforementioned wheel related graphs and a family of convex polytope is not bounded. The unbounded metric dimension of convex polytope also furnish a negative answer to the problem given in [19]:

Open Problem: “Is it the case that the graph of every convex polytope has constant metric dimension?”

It is natural to ask for characterization of graphs with unbounded metric dimension.

TABLE OF CONTENTS

Contents

1 Introduction and Preliminaries **1**

1.1 Historical Background of Graphs 1
1.2 Basic Definitions 2
1.3 Graph Operations 3
1.4 Connectivity 4
1.5 Planarity 5
1.6 Distances in Graphs 6
1.7 Certain Graphs Families 7

2 Metric Dimension of Some Graphs **11**

2.1 Resolving Sets and Metric Dimension 11
2.2 Some Fundamental Results on Metric Dimension 13
2.3 Some Motivational Results on Metric Dimension 15
2.3.1 Metric Dimension of a Wheel Graph 15
2.3.2 Metric Dimension of a Jhangir/Gear Graph 16
2.3.3 Metric Dimension of An Antiweb Wheel Graph 17
2.3.4 Metric Dimension of An Antiweb Gear Graph 18
2.3.5 Metric Dimension of Convex Polytope 19

3 Vertex Resolvability of Some Generalized Wheel related Graphs **22**

3.1 Metric Dimension of m-level Antiweb Wheel Graphs 22
3.2 Metric Dimension of m-level Gear Graph 26
3.3 Metric Dimension of m-level Antiweb Gear Graphs 29
3.4 Metric Dimension of m-level Convex Polytopes $\mathbb{D}_n^m$ 32

4 Conclusion And Open Problems **38**

5 References **40**

LIST OF FIGURES

List of Figures

1 Map of Königsberg 1
2 Graph of Königsberg 1
3 A graph G and its subgraphs 3
4 Graph of houses and utilities problem 5
5 A planar graph G 6
6 Complete graph K_5 8
7 Bipartite graphs 8
8 A network diagram 12
9 A Wheel graph W_{12} 15
10 $W_{12,m}$ 16
11 A gear graph J_{12} 17
12 An antiweb-wheel graph AWW_{12} 18
13 AWJ_{12} 19
14 Convex polytop $\mathbb{D}_n$ 20
15 An antiweb-wheel graph AWW_{12} 23
16 m-level gear graph $J_{12,m}$ 27
17 An antiweb-gear graph AWJ_{12} 29
18 The convex polytope graph $\mathbb{D}_n^m$ 32

Chapter 1
Introduction and Preliminaries

1 Introduction and Preliminaries

In this chapter, we discuss the brief history of graphs, some basic concepts related to graphs. Also some basic notions, definitions and terminologies that will be used throughout in this thesis.

1.1 Historical Background of Graphs

The foundation of graph theory was given by the most familiar and oldest problem *The Königsberg Bridge Problem*. In 1736, Euler has given the solution of this Königsberg bridge problem, which was the first theorem in the history of graph theory and the first true proof in mathematics. In Russia the city of Königsberg lies on the both sides of the Pregel River, and this river included two huge islands connected with each other, by seven bridges. The problem was to make a walk throughout the city that wouldn't cross each of those bridges more than once. A map of Königsberg city at time, which shows the actual location of the seven bridges and the river Pregel is given below:

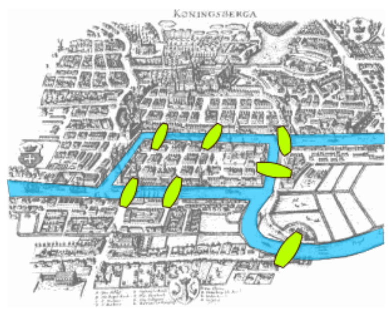


Figure 1: Map of Königsberg

First, Euler thought that the most important thing of a route is the sequence of bridges, crossing each other instead, the choice of route passing by each land mass. This idea helped him to present the problem in graphical structure which laid the foundations of graph theory. In graph theory, by the definition of nodes and edges, he replaced every land mass with *vertex/node*, and each bridge with an *edge*, showing that which pair of nodes (land masses) are connected with each other by the edge (bridge). He named this graphical structure as a *graph*. The first graph of graph theory he formed was:

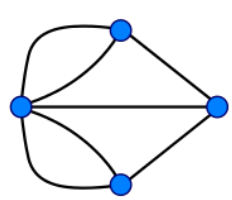


Figure 2: Graph of Königsberg

1.2 Basic Definitions

A *Graph G* is the graphical representation which consists of vertices or nodes or points which may or may not be connected by the edges, arcs or lines in different ways. A graph is represented as an ordered pair $G = (\mathbb{V}, \mathbb{E})$, where $\mathbb{E}$ represents the set of edges and $\mathbb{V}$ represents the set of nodes.

In a graph, number of its nodes is termed as *order* and the number of its edges is *size*. These two terms are expressed as, $|\mathbb{V}(G)|$ and $|\mathbb{E}(G)|$, respectively. Two graphs are said to be equal if they have same order and same size, i.e., $|\mathbb{V}(K_1)| = |\mathbb{V}(K)|$ and $|\mathbb{E}(K_1)| = |\mathbb{E}(K)|$.

A graph, that contains an isolated node only and no edge, is known as *trivial graph*, its order is always one. When two or more edges are incident with a common pair of nodes, such edges are referred as *parallel edges*. A node adjacent with itself is referred as *loop*. A *simple graph* contains no parallel edges and loops. If there exist a connection between any two nodes c and t of a graph, then these nodes are said to be *connected*. On this basis we divide the graphs into two types, connected and disconnected graphs. If there is a connection between every pair of nodes, then a graph is said to be a *connected graph* ;else, a graph is *disconnected graph*. All through our work we discuss, simple graphs and connected graphs only.

Let c and t be two different nodes of a graph then, c is the *neighbor* of t when, both are connected and adjacent to each other . In a graph, the set of all adjacent nodes of c is known as neighbourhood of c in G represented by $N_G(c)$ and . In a connected graph, *degree* of an any node c is, the number of edges incident with c. The minimum degree of a node is known as *minimum degree* of a graph, that is given by $\delta(G)$; the maximum degree of a node in a graph is *maximum degree* of G, that is given by $\Delta(G)$ and the *average degree* of a graph, is sum of all the degrees of nodes in graph is divided by the order of a graph, given by $d(G)$ and represented as $d(G) = \frac{\sum_{j=1}^{n} d(c_j)}{n}$, where n represents the order of a graph. The following inequality shows the relation for the maximum degree, minimun degree and average degree of a graph G with its order:

$$0 \leq \delta(G) \leq d(c) \leq \Delta(G) \leq n - 1.$$

A node is said to be an even node when its degree is even and it is said to be odd node when its degree is odd. The *complement* of a graph is a graph, which contains the vertex set same as the graph, whereas edge set contains those edges which are not present in the edge set of graph.

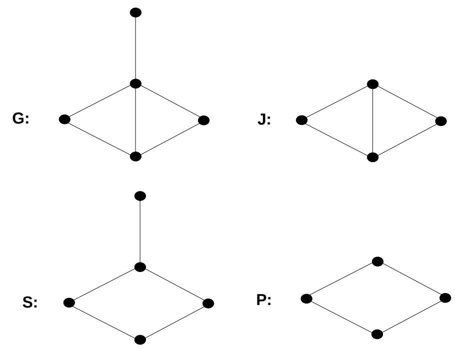


Figure 3: A graph G and its subgraphs

A graph $P = (\mathbb{V}_1, \mathbb{E}_1)$ is a *subgraph* of a graph $G = (\mathbb{V}, \mathbb{E})$, and $\mathbb{V}_1 \subseteq \mathbb{V}$ and $\mathbb{E}_1 \subseteq \mathbb{E}$, where $\mathbb{V}$, $\mathbb{V}_{\nVdash}$ are the sets of nodes of a graph and subgraph, and $\mathbb{E}$, $\mathbb{E}_{\nVdash}$ are sets of edges . As shown in Figure. 3, J, S and P are subgraphs of the graph G. Let J be the subgraph of G and c, t be any two nodes of J then J is said to be an *induced subgraph* iff, ct is an edge of J also ct is an edge of G. In Fig. 3 the graph J is an induced subgraph of the graph G. A *spanning sub graph*, is a subgraph of G which contains the same set of nodes as in G. In Fig. 3 the graph S is a spanning subgraph of G.
Let us consider two graphs $G = (\mathbb{V}, \mathbb{E})$ and $G_1 = (\mathbb{V}_1, \mathbb{E}_1)$, and if there are bijective mappings between the vertices and edges of G and G_1 as, $\eta : \mathbb{V}(G) \to \mathbb{V}_1(G_1)$ and $\sigma : \mathbb{E}(G) \to \mathbb{E}_1(G_1)$ such that the node $\eta(c)$ and an edge $\sigma(f)$ are incident in G_1 iff c and f are incident in G, so the graphs G and G_1 are said to be *isomorphic graphs*. And the mappings (η, σ) are defined as isomorphism from G to G_1 and represented as $G \simeq G_1$.

Let c and k be the vertices of a connected graph, then a *walk* from c to k is a sequence of nodes and edges, i.e $c_0e_1c_1e_2c_2\cdots c_{n1}e_nc_n$, where c_i's and e_i's represents the nodes and edges, respectively and $c_0 = c$ and $c_n = k$ represents initial and final node, respectively. A walk with no repeated edge is known as *trail*. A closed trail is known as *circuit*. A walk from c to k with no repeated edge and vertex is known as *path*. The *length* of a walk or path is defined as number of edges in it. Furthermore, P_n is a path, with length $n-1$. A circuit in which no node is repeated is termed as *cycle*. The number of edges in a cycle is *length* of a cycle. In a simple connected graph G, a cycle consists of t nodes is known as t-cycle. A cycle is an *even cycle* if number of nodes are even, and if the number of nodes are odd then it is said to be an *odd cycle*.

1.3 Graph Operations

Let $H_1 = (\mathbb{V}_1, \mathbb{E}_1)$ and $H_2 = (\mathbb{V}_2, \mathbb{E}_2)$ be two connected graphs, then:
• The *union* of graphs H_1 and H_2 is given by $H_1 \cup H_2$ whose set of nodes is $\mathbb{V}_1 \cup \mathbb{V}_2$ and edge set is $\mathbb{E}_1 \cup \mathbb{E}_2$.
• The *intersection* of graphs H_1 and H_2 is given by $H_1 \cap H_2$ whose set of nodes is $(\mathbb{V}_1 \cap \mathbb{V}_2$ and edge set is $\mathbb{E}_1 \cap \mathbb{E}_2)$.
• $H_1 + H_2$ represents the *join* of two graphs, whose vertex set is $\mathbb{V}_1(H_1) \cup \mathbb{V}_2(H_2)$ and the edge set is $\mathbb{E}_1(H_1) \cup \mathbb{E}_2(H_2) \cup \{ct : c \in \mathbb{V}_1(H_1), t \in \mathbb{V}_2(H_2)\}$.

• $H_1 \square H_2$ represents the *Cartesian product* of two graphs, with set of nodes as, $\mathbb{V}_1(H_1) \times \mathbb{V}_2(H_2)$; and the nodes (c_1, c_2) and (t_1, t_2) are neighboring nodes in $H_1 \square H_2$ iff either,

1. $c_1 = t_1$ and c_2 is connected with t_2 in H_2, or
2. $c_2 = t_2$ and c_1 is connected with t_1 in H_1.

The *Grid graphs*$(G_{m,n})$ defined as $G_{m,n} = P_m \square P_n$, *Ladder Graphs*(L_m), defined as $L_m = P_m \square P_2$, *Prism Graphs*(Y_m), defined as $Y_m = C_m \square P_2$ are some exmaples of Cartesian product. The Cartesian product of two graphs is connected iff both graphs are connected.
• The *direct product* of the graphs H_1 and H_2 is a graph $H_1 \times H_2$ having set of nodes,
$\mathbb{V}(H_1 \times H_2) = \mathbb{V}(H_1) \times \mathbb{V}(H_2) = \{(u_1, u_2) |\ u_1 \in H_1 \text{ and } u_2 \in H_2\}$
and edge set
$\mathbb{E}(H_1 \times H_2) = \{(u_1, u_2)(w_1, w_2) |\ u_1 w_1 \in \mathbb{E}(H_1) \text{ and } u_2 w_2 \in \mathbb{E}(H_2)\}$
The direct product $H_1 \times H_2$ of graphs is connected iff H_1 and H_2 are connected and atleast H_1 or H_2 is non-bipartite.
• $H_1 \boxtimes H_2$ represents the *strong product* of two graphs, with set of nodes $\mathbb{V}(H_1 \boxtimes H_2) = \mathbb{V}(H_1) \times \mathbb{V}(H_2)$ and two nodes (u_1, u_2) and (w_1, w_2) are adjacent in $H_1 \boxtimes H_2$ iff either,

1. $u_1 = w_1$ and $u_2 w_2 \in \mathbb{E}(H_2)$, or
2. $u_1 w_1 \in \mathbb{E}(H_1)$ and $u_2 = w_2$, or
3. $u_1 w_1 \in \mathbb{E}(H_1)$ and $u_2 w_2 \in \mathbb{E}(H_2)$.

Furthermore, $H_1 \square H_2$ and $H_1 \times H_2$ are subgraphs of $H_1 \boxtimes H_2$. $H_1 \boxtimes H_2$ is connected iff H_1 and H_2 are connected.

1.4 Connectivity

In this section, we will discuss the concept of connectivity of graphs, definitions, some basic terminologies and fundamental results. Let c and t be two nodes of a connected graph, if there is a connection between them, then these two nodes are said to be *connected*. If a graph consists of only one component then it is said to be *connected*. Let $\mathbb{E}_1$ be the set of edges in a connected graph and , if we delete all the edges of $\mathbb{E}_1$ from G then we get a graph $G - \mathbb{E}_1$. If $G - \mathbb{E}_1$ consists of more than one component then, the set $\mathbb{E}_1$ is known as *disconnecting set*. A *bridge* is an edge, if we delete that edge from a graph, it disconnects the graph G into components.

A set which disconnects the graph, consists of p-edges then graph is p-edge connected. An *edge connectivity number* is minimum cardinality of disconnecting set and is denoted by $\phi(G)$. A disconnecting set, which has no proper subset then it is known as *cut set*.
Similarly, we can define the idea of vertex disconnecting set, let A be the set of nodes in a graph, If we delete all the nodes of A and edges incident to these nodes from the graph, then we get a graph $G - A$. If A consists of a single vertex c, then we have $G - c$. If $G - A$ has more than one component, then the set A is known as *separating set*. A *cut vertex* is

a vertex, such that by deleting that vertex, the graph is disconnected. A set which disconnects the graph G with minimum size is known as *vertex connectivity number* of G and is represented by $\psi(G)$.

1.5 Planarity

In this section, we discuss the concept of planarity, some definitions and results related to planarity which are used to compute metric dimension of graphs related to planarity. The concept of planarity is explained by a very common problem. Let there be three houses in which three different facilities such as electricity, water, and gas is supplied by electricity lines, water mains and gas pipe lines respectively. The problem is to supply all three facilities to the houses, such that none of the lines cross each other. This problem is known as *"Three Houses and Three Utilities Problem"*. This problem can be graphically represented as in Figure 4, which is basically, $K_{3,3}$.

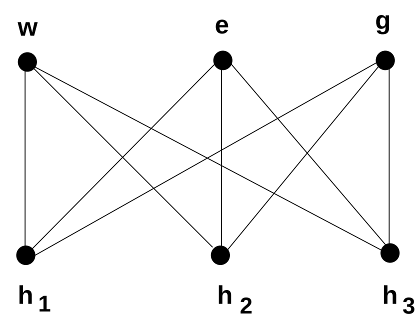


Figure 4: Graph of houses and utilities problem

The planar graphs and planarity of graphs are very helpful to provide the solutions of such problems.
A graph is reffered as a *planar graph* if no edge in a graph cross any other edge as shown in Figure 5, otherwise; graph is *non-planar*.

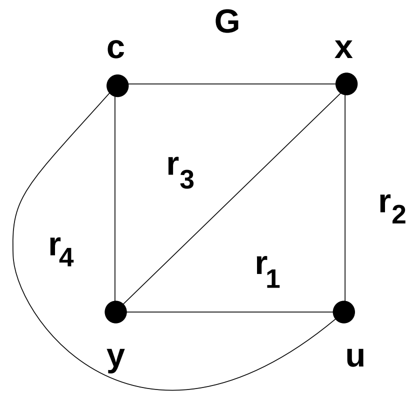


Figure 5: A planar graph G

If there is no edge crossing in any connected graph G, then such graph is referred as *plane graph*. The distinct connected components of a plane graph are called *regions* of a graph. A plane graph G is given in the Figure 5, in which the graph is divided into four regions such as r_1, r_4, r_3 (internal regions) and R_2 (external region). The components are also known as *faces* of a graph. Let the number of nodes, edges and the faces are represented by k, l, m , respectively. Then the relation between these three terms is given by formula, $k-l+m=2$, which has been proved by *Leonhard Euler*. This property of a planar graph given in the following theorem is *Euler Identity*.

Theorem 1.1. *(The Euler Identity)* [10] *"Let G be a connected graph of order k, size l and having m faces, then* $k-l+m=2$*".*

Theorem 1.2. [10] *Let there be a planar graph having size l and order k where,* $k \geq 3$ *and , then* $l \leq 3k-6$.

Theorem 1.2 furnishes a sufficient condition for a planar graph and if a graph not satisfies this condition, it would be a non-planar graph. The converse of Theorem 1.2 is given in the following result:
Let the order of a graph is atleast 3 and size is l, if $l > 3k-6$*, then G is nonplanar.*
If a single edge is added between any two non adjacent nodes of a planar graph we get a non planar graph, such a planar graph is known as *maximal planar*. A graph is a *subdivision* of a graph, if an edge is divided into two edges by inserting a node on an edge.

1.6 Distances in Graphs

In this section, we shall discuss the concepts of distance in graphs, basic results and terminologies which are very helpful in computing our main results. Also we discuss some practical applications of distance in our daily life problems and other scientific fields.

If we want to build a city. We have to place emergency facilities and other services where one can approach in minimum time furthermore, we can not have all facilities at one place. So our aim is to provide such a plan/design that all possible emergency situations are simultaneously, approachable in minimum time.

The length of the shortest path between any two nodes c and t of a graph is known as *distance*, where $c,t \in \mathbb{V}(G)$ and it can be represented as $d(c,t)$. The term distance should satisfy the following conditions:

1. $d(c,t) \geq 0 \; \forall \; c,t \in \mathbb{V}(G)$.
2. $d(c,t) = 0$ iff $c = t$.
3. $d(c,t) = d(t,c) \; \forall \; c,t \in \mathbb{V}(G)$.
4. $d(c,k) \leq d(c,t) + d(t,k) \; \forall \; c,t,k \in \mathbb{V}(G)$.

Since distance d satisfies the properties given in $1-4$, so $(\mathbb{V}(G), d)$ is known as *metric space*.
In a connected graph, the longest distance to a node t, from any node is known as *eccentricity* of a node, it is denoted by $e(t)$. So it is represented as, $e(t) = max\{d(t,c) : t,c \in \mathbb{V}(G)\}$.
A node having the minimum eccentricity, among all the nodes of a graph is known as *radius* and is given by $rad(G)$, where the *diameter* is maximum eccentricity of a node among all the nodes of a graph and is given by $diam(G)$.
If $e(t) = rad(G)$, then a node t is said to be *central node* of G and all the central vertices forms a subgraph $H(G)$ which is termed as, the *center* of a graph, and is given by $Cen(G)$. A graph is *self-centered*, when all the nodes of a graph G are central nodes. If $e(t) = diam(G)$, then a node $t \in \mathbb{V}(G)$ is known as *peripheral vertex* and all the peripheral vertices forms subgraph of $I(G)$, that is known as *periphery* of a graph and is represented as, $Per(G)$.
The relation between diameter and radius of the graph is given below:

Theorem 1.3. [10] *"Let G be a connected graph of order n, if $n \geq 2$ then,*

$$rad(G) \leq diam(G) \leq 2rad(G)."$$

All nodes at maximum distances from t, forms a set this set is known as *eccentric nodes*. An *eccentric graph* is a graph in which, every node of a graph is an eccentric. The subgraph of an eccentric graph is known as *The eccentric subgraph*.

1.7 Certain Graphs Families

In this section, we discuss some well known graphs, their definitions, basic terminologies and useful results related to them.

If there is an edge between each pair of vertices in a simple, connected graph, then graph is known as *complete graph* represented as K_n, where n is order of a graph.

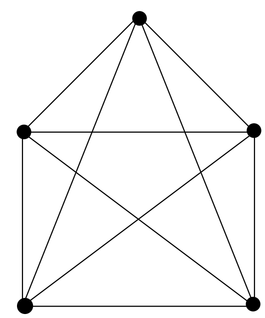


Figure 6: Complete graph K_5

The graph K_1 is a *trivial graph* which contains an isolated node only. The size of a complete graph K_n is given as $\binom{n}{2}$=$\frac{n(n-1)}{2}$.

Theorem 1.4. [10] *Every subgraph of a complete graph K_n, has order at most n.*

Let a graph be a nontrivial and connected, and if the set of nodes can be split into two disjoint sets V_1 and V_2 and every edge of G joins the vertices of V_1 with vertices of V_2, then a graph is called a *bipartite graph*, and the sets V_1 and V_2 are called *partite sets*. It is denoted by $G(V_1, V_2, \mathbb{E})$. The bipartite graph in which the every node of V_1 is connected with every node in V_2 is known as *complete bipartite graph* and is represented as $K_{c,t}$, where c and t denotes the vertices of partite sets V_1 and V_2, respectively.

Bipartite graphs are given as follows:

Figure 7: Bipartite graphs

A graph will be necessarily a bipartite if it has the following property:

Theorem 1.5. [10] *A graph is bipartite iff it contains no odd cycle.*

Trees have many applications in diverse fields, which includes computer science, study of chemical structures, the study of communication networks, electric circuits, road networks and many more. To get the minimum cost for constructing different networks, spanning trees are used. To construct the shortest routes in a city, the concept of spanning trees is used.
A graph which contains no cycle is referred as *cycle free graph* (*acyclic graph*). A *tree* is cycle free connected graph. A subgraph is referred as *spanning tree* if it is a tree subgraph of any graph. *Leaves* are nodes of degree 1 of a tree.
This theorem gives some facts about a tree.

Theorem 1.6. [14] *"Let G be a connected graph of order n, then then statements given below are equivalent:*

(i) *G is a tree.*

(ii) *Between each pair of different nodes in G there exist a unique path.*

(iii) *Every edge in G is a bridge.*

(iv) *G is a cycle free graph, and has* $n-1$ *edges.*

(v) *G is a cycle free graph, and has a unique cycle whenever any two nonadjacent nodes in G are coonected by an edge."*

Theorem 1.7. [14] *A graph is connected iff it has a spanning tree.*

Theorem 1.8. [10] *There exist at least two end-vertices in every nontrivial tree.*

Chapter 2
Metric Dimension of Some Graphs

2 Metric Dimension of Some Graphs

In this chapter, we will review the definition of resolving sets, basis sets, some fundamental results and bounds on metric dimension of graphs. We discuss some of the practical applications of these parameters in our daily life. We also study metric dimensions of some wheel related graphs such as m-level wheel graph, antiweb-wheel graph, antiweb-gear graph and some results on infinite classes of convex polytopes which are the motivational results for our work. We also discuss about the nature of metric dimensions of these graphs i.e, either it is bounded, unbounded or constant.

2.1 Resolving Sets and Metric Dimension

In the history of Graph theory, Slater introduced the concept of metric dimension for the first time in 1975 [41] and later on, it was discussed in 1976 by Melter and Harary [17], and then it has been widely studied. Metric dimension is widely used in the field of graph theory, robot navigation [31], chemistry [11], networking, combinatorial optimization, facility location problems and many more. It is given with details in [30, 47, 48]. The metric dimension of graphs with its various different applications are given in [37, 38, 44]. In chemistry a very common problem is to give different chemical compounds, mathematical representations for their structures, such that different compounds have different representations. The structure of a chemical compound can be given by a graph whose vertices and edge represent atoms and bonds, respectively which is given in [11]. So, by the concept of metric dimension this problem can be easily solved.
One of its common application is *Path Optimization*. In an airline agency, they needed to design such a system of flights that provides maximum facilities to the customers. If the airline agency had the routes of planes that are flying between 20 cities. The system should consider the three important and necessary conditions: lowest cost rates, minimum number of stops and earliest arrival time. The graphical representation is given by, the cities represented by nodes and routes of flights between cities are given by edges. This relatively simple problem has a very straightforward solution. All the problems of an airline can be simply solved by using parameters of metric dimension like making the most efficient less time taking routes and maps, scheduling the flights and many more.

Another most common application of metric dimension is *Network Analysis*. Computer networks can be easily represented by a graph, different systems and machines of a network are represented by nodes and the connection between them is given by edges. A simple network diagram is given below for understanding:

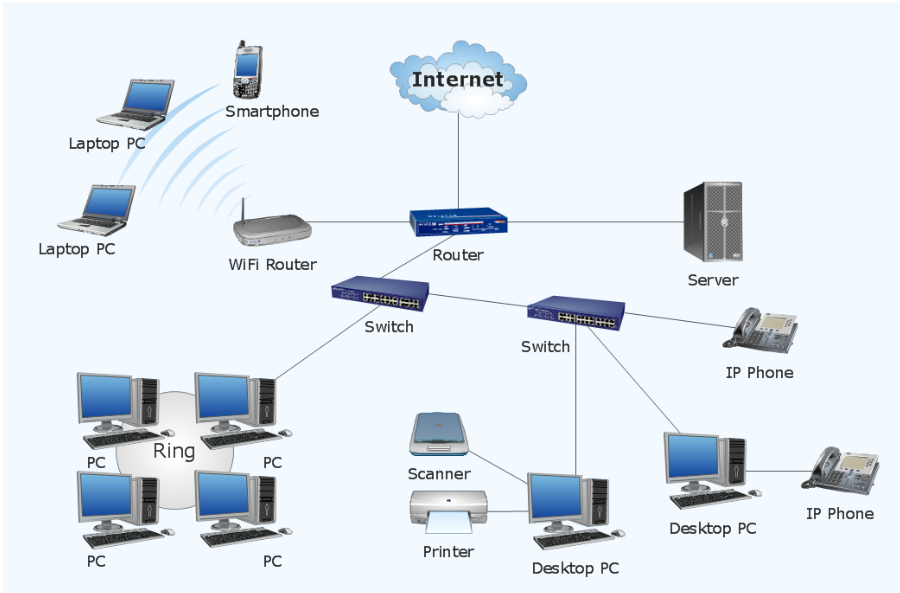
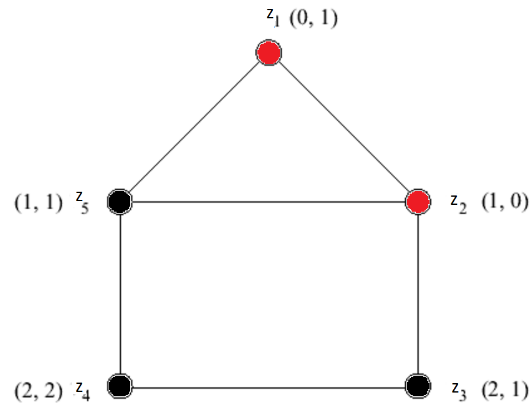


Figure 8: A network diagram

It is easy to see that nodes of a graph represents all the physical objects within a network such as computers, routers, switches, bridges, hubs etc. and the connections such as wires or hardwired are given by edges. Many problems related to networks can be easily solved by using metric dimension such as, how long will it take to transfer the information from one system to the next?

In a connected graph, the length of smallest path, that connects c and k, where $c,k \in \mathbb{V}(G)$ is known as *distance*, represented as $d(c,k)$ and the largest path between c and k is known as *diameter*. Let $Z = \{z_1, z_2 \cdots, z_k\}$ be a subgraph of G and $z_i \in \mathbb{V}(G)$. The (metric) code $d(z|Z)$ of vertex z according to Z is given as $d(z,z_1), d(z,z_2) \cdots, d(z,z_k)$. If each vertex of G is having different code with respect to Z, then we say Z is a *resolving set/locating set* of G. A resolving/locating set, which have the least number of nodes is a *basis* set for a graph and cardinality of this resolving basis set is termed *metric dimension*, which is denoted as $\beta(G)$ or $dim(G)$. The concept of basis sets, resolving sets and metric dimension are also given in [1, 9, 11, 13, 20, 21, 22, 24, 25, 29, 34, 36, 40, 42].
Let a connected graph has an ordered subset of nodes as, $Z = \{z_1, z_2, \cdots, z_k\}$. Then, Z is said to be resolving set if $r(c|Z) \neq r(k|Z)$ for distinct nodes $c,k \in \mathbb{V}(G) \setminus Z$.
For example, consider a graph:



The set $Z = \{z_2, z_3\}$ is not a resolving set of G because the $r(z_1|Z) = (1,2) = r(z_5|Z)$. If we take the set $Z_1 = \{z_1, z_2, z_4\}$, then Z_1 is resolving set with codes representations as $r(z_1|Z_1) = (0,1,2)$, $r(z_2|Z_1) = (1,0,2)$, $r(z_3|Z_1) = (2,1,2)$, $r(z_4|Z_1) = (2,2,2)$, and $r(z_5|Z_1) = (1,1,1)$. But, Z_1 is not a minimum resolving set because $Z_2 = \{z_1, z_2\}$ is also a resolving set which contains only two nodes. Therefore, from the definition of metric dimension, Z_2 is a resolving set of a graph and the dimension is $\beta(G) = 2$.

To find the vertices of a basis set we use the following result:

Lemma 2.1. [45] *"Let $G = (\mathbb{V}, \mathbb{E})$ be a nontrivial connected graph, W is a basis set and $c, k \in \mathbb{V}(G)$. If $d(c,w) = d(k,w)\ \forall\ w \in \mathbb{V}(G) \setminus \{c,k\}$, then $\{c,k\} \cap W \neq \emptyset$".*

Let the family of connected graphs $G = (\mathbb{V}, \mathbb{E})$ is given by $\digamma$, which is defined as $O_n : \digamma = (O_n)_{n \geq 1}$ which depends on n the order $|\mathbb{V}(G)| = \lambda(n)$ and $\lim\limits_{n \to \infty} \lambda(n) = \infty$. Let K be any constant and $K \geq 0$, then family of the connected graphs O_n have *bounded metric dimension* if $\dim(O_n) \leq K$ for each $n \geq 1$; otherwise $\digamma$ does not have bounded metric dimension.
And if $\dim(O_n) = K$ for each $n \geq 1$, then $\digamma$ has constant metric dimension .
For detailed study about constant metric dimension one can see ([18, 19, 23, 27]). The several classes of convex polytopes and their metric dimensions has been given in [22, 23, 25].
Our work is to the compute the basis set with minimum cardinality of some generalized wheel related graphs, such as m-level gear graph, m-level convex polytopes $\mathbb{D}_n$, m-level antiweb-wheel graphs and m-level antiweb-gear graphs denoted by $J_{2n,m}$, $\mathbb{D}_n^m$, $AWW_{n,m}$ and $AWJ_{2n,m}$, respectively. The unbounded metric dimension of convex polytope also negates the problem given in [19].

2.2 Some Fundamental Results on Metric Dimension

Some properties of the graphs related to the metric basis are given below:

Theorem 2.2. *[35]* Let G be a simple connected graph, having order n, where $n \geq 2$. Then:

(a) $\beta(G) = 1$ iff graph is a path graph P_n.

(b) $\beta(G) = n-1$ iff graph is a complete graph K_n .
(c) for $n \geq 4$, $\beta(G) = n-2$ iff graph is a complete bipartite graph $K_{c,u}$, where $c,u \geq 1$, or $G = K_c + \overline{K_u}$, where $c \geq 1$, $u \geq 2$ or $G = K_c + (K_1 \cup K_u)$, where $c,u \geq 1$.

The graph with metric dimension 2 must satisfies the conditions given below:

Theorem 2.3. [43] *Let a graph be a nontrivial with basis set $\{t_1,t_2\} \subseteq \mathbb{V}(G)$, then both the vertices t_1 and t_2 are having degree at most 3 and there is a unique path between t_1 and t_2.*

The following theorem gives a list of forbidden subgraphs for a graph with metric dimension 2.

Theorem 2.4. *[35]* A graph having metric dimension $\beta(G) = 2$ cannot have following graphs as subgraphs:
- A complete graph K_5
- A complete graph with one edge deleted, $K_5 - t$, where t is an edge.
- A complete bipartite graph $K_{3,3}$
- The Petersen graph $P(5,2)$.

In a *tree graph*, a node with degree 3 is known as a major vertex. Let t be the major vertex of tree. The number of terminal nodes of a major vertex is known as the *terminal degree* denoted by $ter(t)$. A major vertex t with positive terminal degree is known as *exterior major vertex* of T. If we add up the terminal degrees, then we have $\sum_{t \in V(T)} ter(t) = \mu(t)$, and cardinality of exterior major vertices of T is given by $ex(T)$. The metric dimension of T, using $\mu(T)$ and $ex(T)$ is given in the following theorem:

Theorem 2.5. [15] *"Let T be a tree which is not a path, then*

$$\beta(T) = \mu(T) - ex(T)\text{"}$$

Theorem 2.6. [12] *"Let T be a tree of order 3 and s be an edge in its complement $\bar{T}$. Then*

$$\beta(T) - 2 \leq \beta(T+s) \leq \beta(T) + 1.$$

Furthermore, there is a tree T_j and an edge s_j in $\bar{T}_j$, where $-2 \leq j \leq 1$, such that

$$\beta(T_j + s_j) = \beta(T_j) + j\text{"}$$

The following results gives some bounds of metric dimension:

Theorem 2.7. [13] *In a nontrivial graph, d is diameter and n is order of a graph, then*

$$dim(G) \leq n - d$$

Theorem 2.8. [13] *Let $\Delta(G)$ be the maximum degree of a graph, then we have:*

$$dim(G) \geq \lceil log_3(\Delta(G)+1) \rceil$$

2.3 Some Motivational Results on Metric Dimension

To find the metric dimension of m-level wheel related graphs some important results which are very useful in our task are metric dimension of wheel graph W_n which was studied in [1] and some wheel related graphs such as Jahangir graph J_{2n} which was studied in [18], anti-web wheel graph AWW_n which was studied in [15], and anti-web Jahangir graph AWJ_{2n} which was studied in [17]. The most important result which we will use in our work is metric dimension of m-level wheel graph $W_{n,m}$, which was given by Siddiqui and Imran in 2014 [39], this result help us to generalize the metric dimension of wheel related graph at m-level to study their metric dimension. Here are some results which we will use for generalization:

2.3.1 Metric Dimension of a Wheel Graph

A *wheel graph* given by W_n can be defined as, join of two graphs, as $W_n \cong C_n + K_1$, where $C_n : c_1, c_2, \cdots, c_n, c_1$ are vertices of C_n where $n \geq 3$. W_n has order $n+1$ and size $2n$.
A wheel graph W_{12} is shown in the Figure 9:

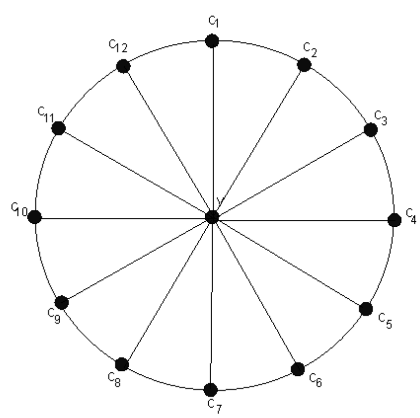


Figure 9: A Wheel graph W_{12}

It has been proved in the following result that the metric dimension of a wheel is not bounded .

Theorem 2.9. *[8]* $\beta(W_n) = \lfloor \frac{2n+2}{5} \rfloor$ for $n \geq 7$

A *double-wheel graph* $W_{n,2}$, can be obtained as, it is join of $2C_n$ and K_1 i.e, $W_{n,2} \cong 2C_n + K_1$. It can be explained as, if two copies of cycle C_n are joined by a single vertex such that every vertex of each cycle is connected with the central vertex. Proceeding in the same manner upto m-level, Imran and Siddiqui in 2014 [39] defined m-level wheel graph $W_{n,m}$ as, $W_{n,m} \cong mC_n + K_1$. $W_{12,m}$ is shown in the Figure 10

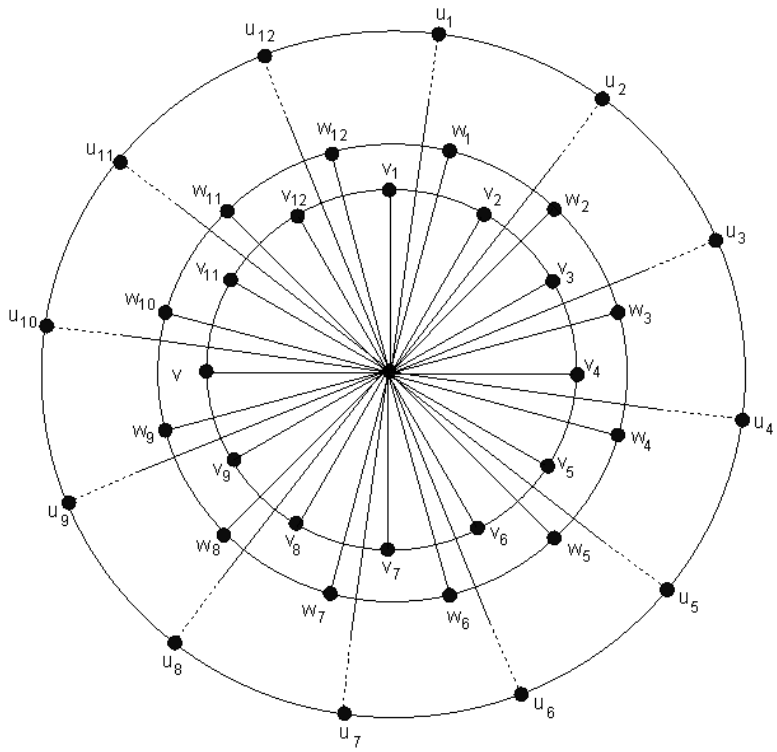


Figure 10: $W_{12,m}$

In 2014, Imran and Siddiqui used the result of metric dimension of W_n given in Theorem 2.9 which was given by Buckowski [8], to compute the metric dimension of double wheel graph $W_{n,2}$ and m-level wheel graph $W_{n,m}$. In the following theorems, the results of metric dimension of $W_{n,2}$ and $W_{n,m}$ are given, which shows the metric dimension of these graphs is unbounded.

Theorem 2.10. *[39] "For a double wheel $n \geq 7$, $dim(W_{n,2}) = dim(W_{n,1}) + \lfloor \frac{2n+4}{5} \rfloor$".*

Theorem 2.11. *[39] "For m-level wheel with $n \geq 7$ and $m \geq 3$, $dim(W_{n,m}) = dim(W_{n,1}) + (m-1)\lfloor \frac{2n+4}{5} \rfloor$".*

2.3.2 Metric Dimension of a Jhangir/Gear Graph

A *Jhangir/Gear Graph*, denoted by J_{2n} was defined by Tomescu and Javed in 2007 [45] as, A gear graph is join of two graphs C_{2n} and K_1, i.e $J_{2n} \cong C_{2n} + K_1$ whose alternate n spokes are deleted, where $C_{2n} : c_1, c_2, \cdots, c_{2n}, c_1$ are vertices of C_{2n} where $n \geq 3$ and it is a cycle of length $2n$. J_{2n} has order $2n+1$ and size $3n$. It can also be obtained from the wheel graph W_{2n}, whose alternate n spokes are deleted . The vertices of J_{2n} are of two kinds, nodes with degree 2 and nodes with degree 3. The nodes with degree 2 are minor vertices and nodes with degree 3 are major vertices.
A gear graph J_{12} is shown in the Figure 11:

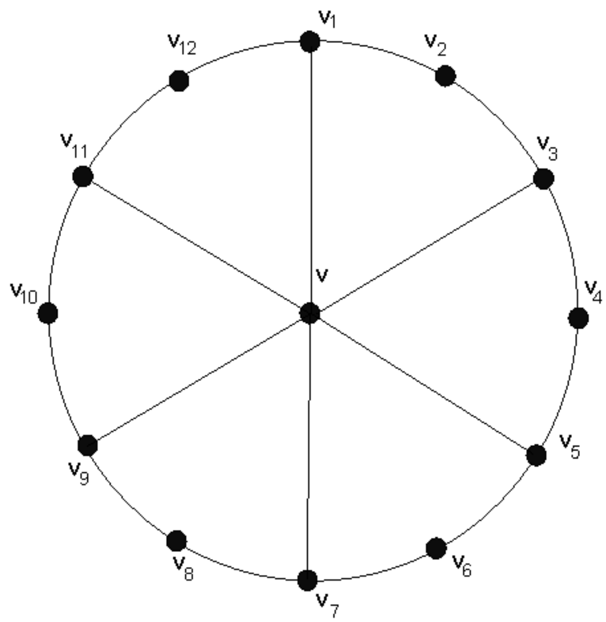


Figure 11: A gear graph J_{12}

The result of metric dimension for a gear graph J_{2n} is given in the following theorem which shows, the metric dimension is unbounded:

Theorem 2.12. *[45]* $\beta(J_{2n}) = \lfloor \frac{2n}{3} \rfloor$ for $n \geq 6$

We use this result as motivation for the generalization of the metric dimension of gear graphs upto m-level. This result forms the basis for the computation of metric dimension of m-level gear graph.

2.3.3 Metric Dimension of An Antiweb Wheel Graph

In 2013, Naeem and Imran [35] defined an *antiweb wheel graph* as, it can be obtained by join of square cycle C_n^2 and K_1 i.e, $AWW_n \cong C_n^2 + K_1$. The set of nodes we have $\mathbb{V}(AWW_n) = \mathbb{V}(W_n)$ and for edge set $\mathbb{E}(AWW_n) = \mathbb{E}(W_n) \cup \{c_j c_{j+2} : 0 \leq j \leq n\}$. The number of nodes in AWW_n is $|\mathbb{V}(AWW_n)| = |\mathbb{V}(W_n)| = n+1$ and number of edges is $|\mathbb{E}(AWW_n)| = 3n$. v is the hub vertex of AWW_n and all other nodes of cycle(s) are known as rim vertices given as $c_1, c_2, \cdots, c_n$. An antiweb-wheel graph AWW_{12} is shown in the Figure 12

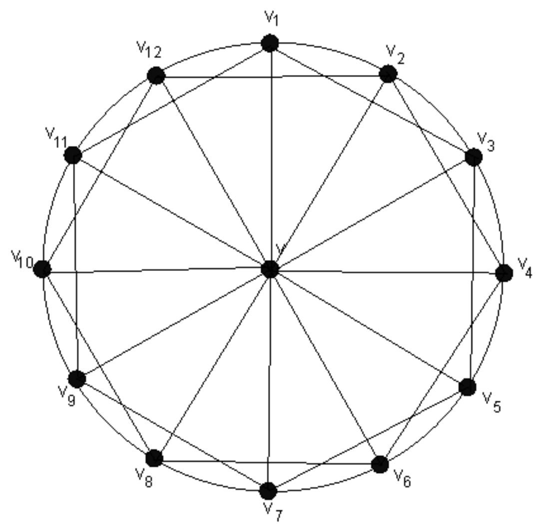


Figure 12: An antiweb-wheel graph AWW_{12}

In the following theorem it has been proved that metric dimension of antiweb-wheel graph AWW_n is unbounded:

Theorem 2.13. *[35]*

$$\beta(AWW_n) = \begin{cases} \lceil \frac{n+1}{3} \rceil & : if\ n\ is\ odd; for n \geq 8 \\ \lceil \frac{n}{3} \rceil & : otherwise. \end{cases}$$

We use this result as motivation for the generalization of the metric dimension of antiweb-wheel graphs upto m-level.

2.3.4 Metric Dimension of An Antiweb Gear Graph

In 2014, Siddiqui and Imran [39] defined an *antiweb-gear graph* as, it can be constructed from gear graph J_{2n} by replacing the cycle C_n with a square cycle C_{2n}^2 and is denoted by AWJ_{2n}. The vertex set is given by $\mathbb{V}(AWJ_{2n}) = \mathbb{V}(J_{2n})$ and edge set is $\mathbb{E}(AWJ_{2n}) = \mathbb{E}(J_{2n}) \cup \{c_i c_{i+2} : 0 \leq i \leq n\}$, the indices are being considered as modulo n. An antiweb-gear graph AWJ_{12} is shown in the Figure 13

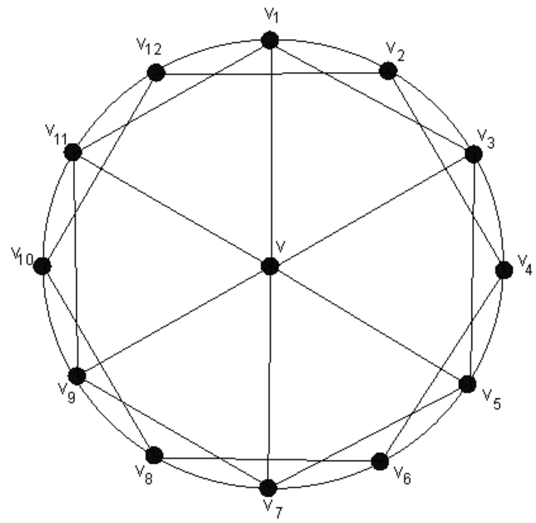


Figure 13: AWJ_{12}

In the following theorem metric dimension of antiweb-gear graphs AWJ_{2n} is given which shows that it has unbounded metric dimension.

Theorem 2.14. *[39]* $\beta(AWJ_{2n}) = \lceil \frac{n+1}{3} \rceil$ for every $n \geq 15$

We use this result as motivation for the generalization of the metric dimension of antiweb-gear graphs upto *m* level.

2.3.5 Metric Dimension of Convex Polytope

Let $\mathscr{D}_n$ is the *prism* graph and $L = \{1, 2, \cdots, n\}$ and $M = \{1, 2\}$ be sets of indices. The prism graph $\mathscr{D}_n$, is the Cartesian product of path graph P_2 and a cycle graph C_n i.e $P_2 \square C_n$, where $(n \geq 3)$. The set of the nodes of $\mathscr{D}_n$ is $\mathbb{V}(\mathscr{D}_n) = \{c_{u,v} : u \in L \text{ and } v \in M\}$ and set of edges is $\mathbb{E}(\mathscr{D}_n) = \{c_{u,v}c_{u,v+1} : u \in L \text{ and } v \in M\} \cup \{c_{1,v}c_{2,v} : v \in M\}$. Also, $c_{u,n+1} = c_{u,1}$ $c_{u,n+2} = c_{u,2}$ for $u \in L$.
The region set $\mathbb{R}(\mathscr{D}_n)$ contains n 4-sided regions and two n-sided regions (interior and exterior). Only one vertex $y(z)$ is added in the interior (exterior) n-sided regions of $\mathscr{D}_n$. Suppose the graph $\mathbb{D}_n$ where n is even, and $n \geq 4$, with set of nodes $\mathbb{V}(\mathbb{D}_n) = \mathbb{V}(\mathscr{D}_n) \cup \{y, z\}$ and the set of edges is given as $\mathbb{E}(\mathbb{D}_n) = \mathbb{E}(\mathscr{D}_n) \cup \{c_{1,2k-1}y : k = 1, 2, \cdots, \frac{n}{2}\} \cup \{c_{2,2k}z : k = 1, 2, \cdots, \frac{n}{2}\}$. The $\mathbb{D}_n$, where n is order of a graph and n is atleast 4, is defined in [3], with number of vertices $|\mathbb{V}(\mathbb{D}_n)| = 2n + 2$, number of edges $|\mathbb{E}(\mathbb{D}_n)| = 4n$ and number of faces $|\mathbb{R}(\mathbb{D}_n)| = 2n$. It can be shown as:

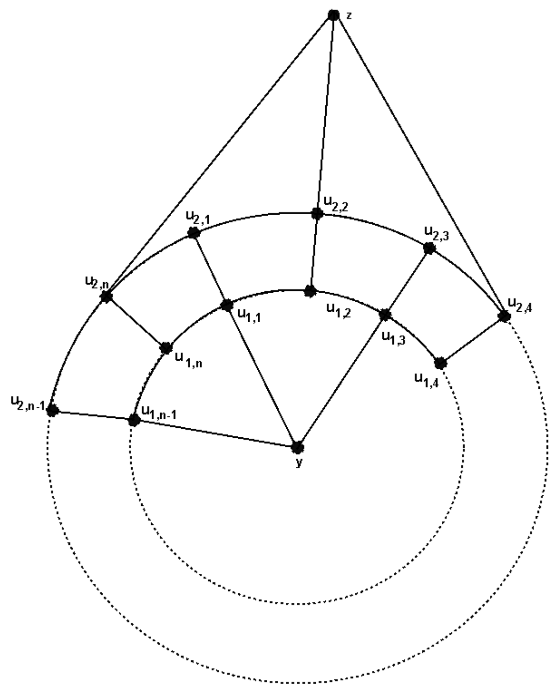


Figure 14: Convex polytop $\mathbb{D}_n$

In the folllowing theorem the metric dimension of $\mathbb{D}_n$ is given, which shows it is unbounded:

Theorem 2.15. *[39]* $\beta(\mathbb{D}_n) = \frac{n}{2}$, For every even integer $n \geq 8$

We used this result as motivation for the generalization of metric dimension of m-level $\mathbb{D}_n$.

Chapter 3
Vertex Resolvability of Some Generalized Wheel Related Graphs

3 Vertex Resolvability of Some Generalized Wheel related Graphs

In this section, the metric dimension of diverse classes of wheel related graphs, m-level gear graph, m-level antiweb-wheel graph, m-level antiweb-gear graph, denoted by $J_{2n,m}$, $AWW_{n,m}$, $AWJ_{2n,m}$, respectively have been studied and shown that the dimension of these families of graphs is not bounded . We also consider the family of convex polytopes, denoted as $\mathbb{D}_n^m$ and prove that the dimension of m-level convex polytope increases as we increase the order, which negates the problem given in [19]:
Open Problem: "Is it the case that the graph of every convex polytope has constant metric dimension?"

3.1 Metric Dimension of m-level Antiweb Wheel Graphs

In the following section, we will compute the metric dimension of m-level antiweb-wheel graph by the result of metric dimension of antiweb-wheel graph given in [35].

"An *antiweb wheel graph* can be obtained by join of square cycle C_n^2 and K_1 i.e, $AWW_n \cong C_n^2 + K_1$. The set of nodes is, $\mathbb{V}(AWW_n) = \mathbb{V}(W_n)$ and edge set $\mathbb{E}(AWW_n) = \mathbb{E}(W_n) \cup \{c_i c_{i+2} : 0 \leq i \leq n\}$. The order of AWW_n is $|\mathbb{V}(AWW_n)| = |\mathbb{V}(W_n)| = n+1$ and the size is $|\mathbb{E}(AWW_n)| = 3n$. Where v is the hub vertex of AWW_n and all other vertices of cycles of AWW_n are called rim vertices given by $c_1, c_2, \cdots, c_n$". In [35], it was proved that

$$\beta(AWW_n) = \begin{cases} \lceil \frac{n+1}{3} \rceil & : if\ n\ is\ odd; for n \geq 8 \\ \lceil \frac{n}{3} \rceil & : otherwise. \end{cases}$$

A *double antiweb wheel graph* can be defined as $AWW_{n,2} \cong 2C_n^2 + K_1$, where $c_1, c_2, \cdots, c_n$ are vertices of $C_{n,1}^2$ and $z_1, z_2, \cdots, z_n$ are vertices of $C_{n,2}^2$ and inductively we can construct an m-level antiweb wheel graph, which is represented as $AWW_{n,m} \cong mC_n^2 + K_1$. Where $C_{n,1}^2, C_{n,2}^2, \cdots, C_{n,m}^2$ represents the cycles of m-level of $AWW_{n,m}$ respectively as shown in the Figure 15.

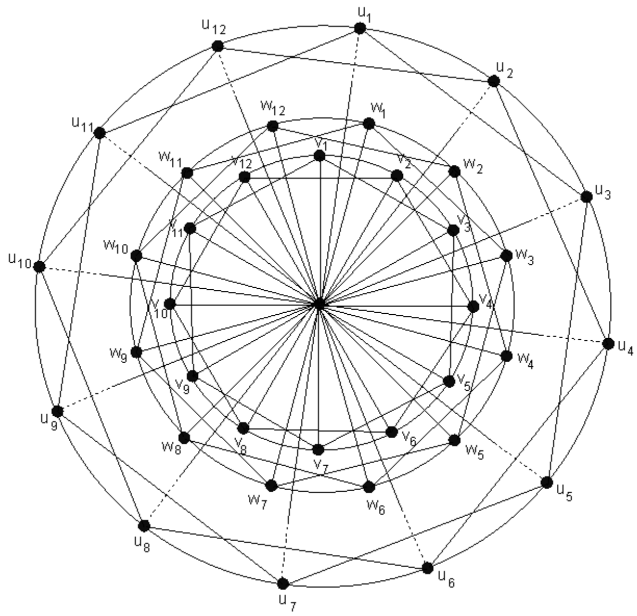


Figure 15: An antiweb-wheel graph AWW_{12}

Now to find the metric dimension of $AWW_{n,2}$ we use the concepts of gap and size of gaps as used by [35] for the metric dimension of $AWW_{n,1}$, Let C_n be a cycle with n vertices. The vertices are given as $c_1, c_2, \cdots, c_n$. Let the vertices $c_{k+1}, c_{k+2}, \cdots, c_{l-1}$ are vertices of gap formed by c_k and c_l where, $1 \leq k \leq l \leq n$. The size of gap is $k-l-1$. The gaps G_λ and G_μ are neighboring gaps where $|\lambda - \mu| = 1$ which is given in [45]. Let $\mathbb{B}$ be the basis of $AWW_{n,1}$. Then the gap conditions given in lemma 3.23 - lemma 3.8 must be satisfied by $\mathbb{B}$.

Lemma 3.1. *[35] At most five vertices are contained in every gap of $\mathbb{B}$.*

Lemma 3.2. *[35] There is atmost one gap of either five or four vertices.*

Lemma 3.3. *[35] In $\mathbb{B}$, a gap which is having four or five vertices, have neighboring gaps that contains at most three vertices and other one is empty.*

Lemma 3.4. *[35] If $\mathbb{B}$ consists of a gap with three vertices, then its neighboring gaps consists of atmost two vertices.*

Lemma 3.5. *[35] If $\mathbb{B}$ contains two vertices, then neighboring gaps consists of at most three vertices and other one is empty.*

Lemma 3.6. *[35] "If $\mathbb{B}$ contains a gap of exactly one vertex then the neighboring gaps contain at most three vertices".*

Lemma 3.7. *[35] "In $\mathbb{B}$ if a gap is empty, then one of its neighboring gap contains atmost five vertices then the other one contains atmost two vertices".*

Lemma 3.8. *[35] In $\mathbb{B}$, for $n = 2k+1$ there exist atleast one empty gap, where $k \geq 4$.*

All the above conditions must be satisfied to get basis $\mathbb{B}$ for $AWW_{n,2}$.

Now we find the dimension of double antiweb wheel graph $AWW_{n,2}$.

In double antiweb wheel graph $AWW_{n,2}$, where $n \geq 8$ the hub vertex v does not belong to any basis as proved in [45]. The basis vertices are rim vertices of $W_{n,1}$ and $W_{n,2}$ in $AWW_{n,2}$. We can construct $\mathbb{B}$ basis set for $AWW_{n,2}$, by using the gap conditions given in lemmas (3.23-3.8).

In the following theorem, we provide the metric dimension of double antiweb wheel graph $AWW_{n,2}$ where $n \geq 8$. This result furnish the base for finding metric dimension of m-level antiweb wheel graph.

Theorem 3.9. $\beta(AWW_{n,2}) = 2\beta(AWW_{n,1})$, *for* $n \geq 8$

Proof. To prove this result we use the method of double inequalities. The double antiweb wheel graph is given by $AWW_{n,2} \cong (2C_n^2 + K_1)$, its order is $2n+1$. We start our proof with: $\beta(AWW_{n,2}) \leq 2\beta(AWW_{n,1})$ i.e,

$$\beta(AWW_{n,2}) \leq 2\begin{cases} \lceil \frac{n+1}{3} \rceil & : if\ n\ is\ odd; \\ \lceil \frac{n}{3} \rceil & : otherwise. \end{cases}$$

To prove this we form a resolving set with $2\lceil \frac{n}{3} \rceil$ nodes, where n is even and with $\lceil \frac{n+1}{3} \rceil$ nodes when n is odd. We have the following cases to consider:

(i) If $n \equiv 0\ (mod\ 6)$, where $n = 6q$, for $q \geq 2$, and $2(\lceil \frac{n}{3} \rceil) = 2q$. In this case $\mathbb{B} = \{c_{6i+1}, c_{6i+3}, z_{6j+1}, z_{6j+3} : 0 \leq i, j \leq q-1\}$ consists of $2q$ nodes and satisfies the conditions given in lemmas (3.23-3.8), So in this case $\mathbb{B}$ is resolving set for $AWW_{n,2}$.

(ii) If $n \equiv 1\ (mod\ 6)$, where $n = 6q+1$, for $q \geq 2$, and $2(\lceil \frac{n+1}{3} \rceil) = 2q+1$. In this case $\mathbb{B} = \{c_1, z_1\} \cup \{c_{6i+3}, c_{6i+7}, z_{6j+3}, z_{6j+7} : 0 \leq i, j \leq q-2\} \cup \{c_{6q-2}, c_{6q-1}, z_{6q-2}, z_{6q-1}\}$ contains $2q+1$ nodes and satisfies the conditions given in lemmas (3.23-3.8), So for this case $\mathbb{B}$ is resolving set for $AWW_{n,2}$.

(iii) If $n \equiv 2\ (mod\ 6)$, where $n = 6q+2$, for $k \geq 1$, and $2(\lceil \frac{n}{3} \rceil) = 2q+1$. In this case $\mathbb{B} = \{c_1, z_1\} \cup \{c_{6i+3}, c_{6i+5}, z_{6j+3}, z_{6j+5} : 0 \leq i, j \leq q-1\}$ consists of $2q+1$ nodes and satisfies the conditions given in lemmas (3.23-3.8), So in this case $\mathbb{B}$ is the resolving set for $AWW_{n,2}$.

(iv) If $n \equiv 3\ (mod\ 6)$, where $n = 6q+3$, for $q \geq 1$, and $2(\lceil \frac{n+1}{3} \rceil) = 2q+2$. In this case $\mathbb{B} = \{c_1, c_3, z_1, z_3\} \cup \{c_{6i+5}, c_{6i+9}, z_{6j+5}, z_{6j+9} : 0 \leq i, j \leq k-2\} \cup \{c_{6q}, c_{6q+1}, z_{6q}, z_{6q+1}\}$ consists of $2k+1$ nodes and satisfies the conditions given in lemmas (3.23-3.8), So for this case $\mathbb{B}$ is resolving set for $AWW_{n,2}$.

(v) If $n \equiv 4\ (mod\ 6)$, where $n = 6q+4$, for $k \geq 1$, and $2(\lceil \frac{n}{3} \rceil) = 2q+2$. In this case $\mathbb{B} = \{c_1, c_3, z_1, z_3\} \cup \{c_{6i+5}, c_{6i+7}, z_{6j+5}, z_{6j+7} : 0 \leq i, j \leq q-1\}$ consists of $2q+2$ nodes and satisfies the conditions given in lemmas (3.23-3.8), So for this case $\mathbb{B}$ is resolving set for $AWW_{n,2}$.

(vi) If $n \equiv 5\ (mod\ 6)$, where, $n = 6q+5$, for $k \geq 1$, and $2(\lceil \frac{n+1}{3} \rceil) = 2q+2$. In this case $\mathbb{B} = \{c_{6i+1}, c_{6i+5}, z_{6j+1}, z_{6j+5} : 0 \leq i, j \leq k-1\} \cup \{c_{6q+2}, c_{6q+3}, z_{6q+2}, z_{6q+3}\}$ contains $2q+2$ nodes and satisfies the conditions given in lemmas (3.23-3.8), So for this case $\mathbb{B}$ is resolving set for $AWW_{n,2}$.

Hence, we have:

$$\beta(AWW_{n,2}) \leq 2 \begin{cases} \lceil \frac{n+1}{3} \rceil & : if\ n\ is\ odd; \\ \lceil \frac{n}{3} \rceil & : otherwise. \end{cases}$$

Now we have to show:

$$\beta(AWW_{n,2}) \geq 2 \begin{cases} \lceil \frac{n+1}{3} \rceil & : if\ n\ is\ odd; \\ \lceil \frac{n}{3} \rceil & : otherwise. \end{cases}$$

To prove this let $\mathbb{B}$ be the metric basis of $AWW_{n,2}$. We now consider following three cases:
• When $|\mathbb{B}| \equiv 0\ (mod\ 3)$, we have $|\mathbb{B}| = 6l$ for $l \geq 1$. From the conditions given in lemmas (3.23-3.8), $\mathbb{B}$ contains atmost one gap of four or five vertices. $\mathbb{B}$ consists of atmost $l+1$ gaps of three vertices and $\mathbb{B}$ contains atmost $l-4$ gaps of two vertices. Atmost $l+2$ gaps consists a single vertex. Moreover, from these $l+2$ gaps exactly two are empty. So the number of nodes in the gaps of $\mathbb{B}$ is $6l$ or $6l-1$ in $AWW_{n,1}$ and $12l$ or $12l-2$ in $AWW_{n,2}$. Thus, we have $(2n-6l) \leq 12l$, which proves that $|\mathbb{B}| = 6l \geq 2\lceil \frac{n}{3} \rceil$ or $|\mathbb{B}| \geq 2\beta(AWW_{n,1})$.
• When $|\mathbb{B}| \equiv 1\ (mod\ 3)$, we have $|\mathbb{B}| = 6l+2$ for $l \geq 1$. From the conditions given in lemmas (3.23-3.8), $\mathbb{B}$ contains atmost one gap of four or five vertices. $\mathbb{B}$ contains atmost $l+1$ gaps of three vertices and $\mathbb{B}$ contains atmost $l-3$ gaps of two vertices. $\mathbb{B}$ contains atmost $l+2$ gaps of one vertex. Moreover, from these $l+2$ gaps exactly two are empty. Thus total nodes lying in the gaps of $\mathbb{B}$ are atmost $6l+2$ or $6l+1$ in $AWW_{n,1}$ and $12l+4$ or $12l+2$ in $AWW_{n,2}$. Thus, we have $(2n-6l-2) \leq 12l+4$, which proves that $|\mathbb{B}| = 6l+2 \geq 2\lceil \frac{n}{3} \rceil$ or $|\mathbb{B}| \geq 2\beta(AWW_{n,1})$ and if we have $(2n-6l-2) \leq 12l+2$ which proves that $|\mathbb{B}| = 6l+2 \geq 2\lceil \frac{n+1}{3} \rceil$ or $|\mathbb{B}| \geq 2\beta(AWW_{n,1})$.
• When $|\mathbb{B}| \equiv 2\ (mod\ 3)$, we have $|\mathbb{B}| = 6l+4$ for $l \geq 1$. From the conditions given in lemmas (3.23-3.8), $\mathbb{B}$ contains atmost one gap of four or five nodes. $\mathbb{B}$ consists of atmost $l+2$ gaps of three vertices and $\mathbb{B}$ consists of atmost $l-4$ gaps of two vertices. $\mathbb{B}$ consists of atmost $l+3$ gaps of one vertex. Moreover, from these $l+3$ gaps exactly two are empty. Therefore, the number of nodes in gaps of $\mathbb{B}$ is atmost $6l+4$ or $6l+3$ in $AWW_{n,1}$ and $12l+8$ or $12l+6$ in $AWW_{n,2}$. Thus, we have $(2n-6l-4) \leq 12l+8$, which proves $|\mathbb{B}| = 6l+4 \geq 2\lceil \frac{n}{3} \rceil$ or $|\mathbb{B}| \geq 2\beta(AWW_{n,1})$ and if we have $(2n-6l-4) \leq 12l+6$ which proves that $|\mathbb{B}| = 6l+4 \geq 2\lceil \frac{n+1}{3} \rceil$ or $|\mathbb{B}| \geq 2\beta(AWW_{n,1})$.
So we have

$$\beta(AWW_{n,2}) \geq 2 \begin{cases} \lceil \frac{n+1}{3} \rceil & : if\ n\ is\ odd; \\ \lceil \frac{n}{3} \rceil & : otherwise. \end{cases}$$

and hence,

$$\beta(AWW_{n,2}) = 2 \begin{cases} \lceil \frac{n+1}{3} \rceil & : if\ n\ is\ odd; \\ \lceil \frac{n}{3} \rceil & : otherwise. \end{cases}$$

□

Now we generalize the result for m-level antiweb wheel given by $AWW_{n,m}$. For this purpose, we use the method of induction hypothesis on the levels of antiweb wheel graph.

Theorem 3.10. *For $n \geq 8$ and $m \geq 3$, we have $\beta(AWW_{n,m}) = m\beta(AWW_{n,1})$*

Proof. We prove this result by mathematical induction on levels of antiweb wheel graph. For $m = 1$, we have

$$\beta(AWW_{n,1}) = \begin{cases} \lceil \frac{n+1}{3} \rceil & : if\ n\ is\ odd; \\ \lceil \frac{n}{3} \rceil & : otherwise. \end{cases}$$

When $m = 2$, then

$$\beta(AWW_{n,2}) = 2\begin{cases} \lceil \frac{n+1}{3} \rceil & : if\ n\ is\ odd; \\ \lceil \frac{n}{3} \rceil & : otherwise. \end{cases}$$

by Theorem 3.10. Now let the result is valid in case of $m = k$, i.e.,

$$\beta(AWW_{n,k}) = k\begin{cases} \lceil \frac{n+1}{3} \rceil & : if\ n\ is\ odd; \\ \lceil \frac{n}{3} \rceil & : otherwise. \end{cases} \tag{1}$$

We prove that, the result is valid in case of $m = k+1$.
Let $\beta(AWW_{n,k+1}) = \beta(AWW_{n,k}) + \beta(AWW_{n,1}) + 1$, then (1) yields
$\beta(AWW_{n,k+1}) = (k\beta(J_{2n,1}) + \beta(AWW_{n,1})) = (k+1)\beta(AWW_{n,1})$. Therefore the result is valid $\forall\ m \geq 3$. □

3.2 Metric Dimension of m-level Gear Graph

In the following section, we will compute the metric dimension of m-level gear graph by using the result of metric dimension of gear graph given in [45]

A gear graph, $J_{2n,1}$ is defined as $J_{2n,1} \cong C_{2n,1} + K_1$, where $C_{2n,1} : c_2, c_4, \cdots, c_{2n}, c_2$ for $n \geq 2$ is a cycle of length $2n$. $J_{2n,1}$ with order $2n+1$ and size $3n$. It can also be constructed from wheel graph W_{2n}, if alternate n spokes are deleted. The nodes of J_{2n} are of two kinds, nodes with degree 2 and nodes with degree 3. The nodes with degree 2 will be considered as minor vertices and nodes with degree 3 will be considered as major vertices.
It is given in [45], that $dim(J_{2n,1}) = \lfloor \frac{2n}{3} \rfloor$ for $n \geq 6$.

A gap formed by neighboring nodes c_i and c_j is known as $\lambda - \mu$ gap, where $\lambda \leq \mu$ such that $d(c_i) = \lambda$ and $d(c_j) = \mu$. So there are three types of gaps: 2-2, 2-3, 3-3 that satisfies the properties which are given in lemma 3.23- lemma 3.26 .
Let $\mathbb{B}$ be a basis set of $J_{2n,1}$ then:

Lemma 3.11. *[45] $\mathbb{B}$ consists of at most 5,4 and 3 nodes in every 2-2, 2-3 and 3-3 gaps (major gaps) respectively.*

Lemma 3.12. *[45] There is at most one major gap in $\mathbb{B}$.*

Lemma 3.13. *[45] Any two neighboring gaps of $\mathbb{B}$ contains at most six nodes, if one of them is major gap.*

Lemma 3.14. *[45] At most six nodes are present in any two minor neighboring gaps of $\mathbb{B}$.*

Definition 1. A double-gear graph $J_{2n,2}$ can be defined as, $J_{2n,2} \cong C_{2n,2} + K_1$, whose alternate n spokes are deleted, where $C_{2n,1} : c_2, c_4, \cdots, c_{2n}, c_2$ and $C_{2n,2} : z_2, z_4, \cdots, z_{2n}, z_2$ are vertices of cycles. Similarly, an m-level gear graph $J_{2n,m}$ can be defined as, $J_{2n,m} \cong C_{2n,m} + K_1$, whose alternate n spokes are deleted, where $C_{2n,1} : v_2, v_4, \cdots, v_{2n}, c_2$, $C_{2n,2} : w_2, w_4, \cdots, w_{2n}, z_2 \cdots$, $C_{2n,m} : u_2, u_4, \cdots, u_{2n}, c_2$ are vertices of cycles.

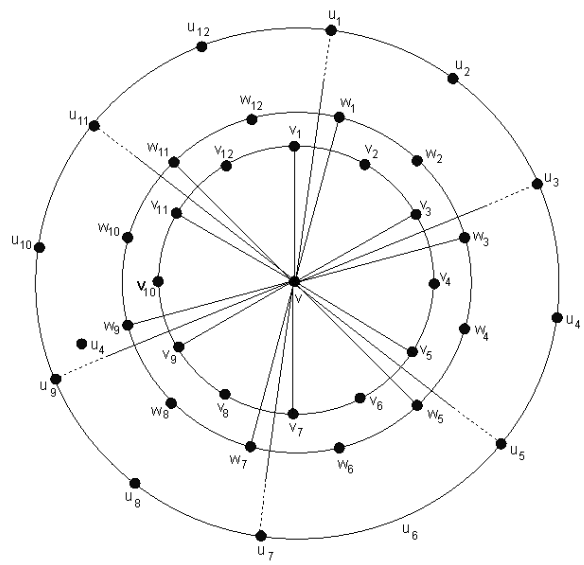


Figure 16: m-level gear graph $J_{12,m}$

Let $C_{2n,1}, \cdots, C_{2n,m}$ represents the cycles of $W_{2n,m}$, as given in Figure. 16 We have to find the metric dimensions of $J_{2n,2}, \cdots, J_{2n,m}$. First we compute the metric dimension of $J_{2n,2}$.
Let $J_{2n,2}$ where $n \geq 6$, as proved in [45] the basis set $\mathbb{B}$ does not contain the central vertex v, it contains only the rim vertices of cycles of $J_{2n,2}$. Let $\mathbb{B}$ be the basis set of $J_{2n,2}$, then $\mathbb{B}$ contains vertices from $C_{2n,1}$ and $C_{2n,2}$. To construct the basis set, we use the conditions, given below:

(a) *Every gap of $\mathbb{B}$ formed by the vertices of $C_{2n,1}$ satisfies conditions given in lemmas (*3.23-3.26*) presented for $J_{2n,1}$.*

(b) *Every set of vertices of $\mathbb{B}$ of $J_{2n,2}$ must contain one more vertex from $C_{2n,2}$. Otherwise, any two vertices one from first level and other from second level will have same code representation, i.e $r(c_{i+4}|\mathbb{B}) = r(z_{i+6}|\mathbb{B}) : 1 \leq i \leq 2n$, which furnish a contradiction.*

Thus, a set $\mathbb{B}$ satisfying the gap conditions given in (a)-(b) is a basis set for $J_{2n,2}$.
For finding the metric dimension of double gear graph $J_{2n,2}$ where $n \geq 6$, we have the following result.

Theorem 3.15. $\beta(J_{2n,2}) = 2\beta(J_{2n,1})+1$ *for* $n \geq 6$.

Proof. Let $J_{2n,1} \cong W_{2n,1}$ (with alternate n spokes deleted), v is the hub vertex of $J_{2n,2}$ and $C_{2n,1} : c_2, \cdots, c_{2n}, c_2$ and $C_{2n,2} : z_2, \cdots, z_{2n}, z_2$ are inner and outer cycles of $J_{2n,2}$. If we construct a resolving set of $J_{2n,2}$ with $2\beta(J_{2n,1})+1$ nodes, we have to show that $\beta(J_{2n,2}) \leq 2\beta(J_{2n,1})+1$. We consider the following cases where n is equivalent to the residue class modulo 3.

(i) If $n \equiv 0\ (mod\ 3)$, then we take $2n = 3k$, such that k is even and $k \geq 4$, and $2\beta(J_{2n,1})+1 = 2k+1$ or $\lfloor \frac{2n}{3} \rfloor = 2k+1$. In this case $\mathbb{B} = \{c_{6i+2}, c_{6i+4}, z_{6j+2}, z_{6j+4} : 0 \leq i, j \leq \frac{k}{2} - 1\} \cup \{c_2, c_{2n}, z_2, z_{2n}\} \cup \{z_4\}$.

(ii) If $n \equiv 1\ (mod\ 3)$, then we take $2n = 3k+2$, such that k is even and $k \geq 4$, and $2\beta(J_{2n,1})+1 = 2k+1$ or $\lfloor \frac{2n}{3} \rfloor = 2k+1$. In this case $\mathbb{B} = \{c_{6i+2}, c_{6i+4}, z_{6j+2}, z_{6j+4} : 0 \leq i, j \leq \frac{k}{2} - 1\} \cup \{c_2, c_{2n}, z_2, z_{2n}\} \cup \{z_4\}$.

(iii) If $n \equiv 2\ (mod\ 3)$, then we take $2n = 3k+1$, such that k is odd and $k \geq 5$, and $2\beta(J_{2n,1})+1 = 2k+1$ or $\lfloor \frac{2n}{3} \rfloor = 2k+1$. In this case $\mathbb{B} = \{c_{6i+2}, c_{6i+4}, z_{6j+2}, z_{6j+4} : 0 \leq i, j \leq \frac{k-1}{2}\} \cup \{c_2, z_2,\} \cup \{z_4\}$.

Hence, from above discussion we have $\beta(J_{2n,2}) \leq 2\beta(J_{2n,1})+1$.
Now we have to show that $\beta(J_{2n,2}) \geq 2\beta(J_{2n,1})+1$. Let $\mathbb{B}$ be the basis of $J_{2n,2}$ and $|\mathbb{B}| = 2r+1$, such that $\mathbb{B}$ forms $2r+1$ gaps on $C_{2n,1}$ and $C_{2n,2}$ denoted by $G_1, G_2, \cdots, G_{2r+1}$ where G_i and G_{i+1} for every $1 \leq i \leq 2r$ also, G_1 and G_r are neighboring gaps.

By using gap conditions given in lemma 3.23 - lemma 3.26 at most one of the gap say G_1 is major gap and we have $|G_1|+|G_2| \leq 6$, $|G_r|+|G_1| \leq 6$ and $|G_i|+|G_{i+1} \leq 4$ for every $i = 2, \cdots, 2r$. By summing up these inequalities, we have:

$$2(4n-2r-1) = 2\sum_{i=1}^{2r+1} \leq 8r+8)$$

where $r \geq \frac{4n-5}{6}$. Since r is integer so for each $n \equiv 0, 1, 2\ (mod\ 3)$ we obtain $r \geq \lfloor \frac{4n}{6} \rfloor$

or

$$2r+1 \geq 2(\lfloor \frac{2n}{3} \rfloor)+1$$

or

$$|\mathbb{B}| \geq 2(\lfloor \frac{2n}{3} \rfloor)+1$$

$\square$

Now we compute the result for m-level gear graph given as $J_{2n,m}$. For this purpose, we use method of mathematical induction.

Theorem 3.16. *For every integer $n \geq 6$ and $m \geq 3$, We have $dim(J_{2n,m}) = m\beta(J_{2n,1}) + (m-1)$.*

Proof. We use method of induction hypothesis on levels of gear graph to prove this result, where levels are represented by m. If $m = 1$, then we have, $\beta(J_{2n,1}) = \lfloor \frac{2n}{3} \rfloor$ is given in [45]. If $m = 2$, then we have, $\beta(J_{2n,2}) = 2(\lfloor \frac{2n}{3} \rfloor)$ by Theorem 3.16. Let the result is valid in case of $m = k$, i.e.,

$$\beta(J_{2n,k}) = k\beta(J_{2n,1}) + (k-1). \tag{2}$$

We now prove that the result is valid in case of $m = k+1$. Let $\beta(J_{2n,k+1}) = \beta(J_{2n,k}) + \beta(J_{2n,1}) + 1$, now from the equation (2), we have $\beta(J_{2n,k+1}) = (k\beta(J_{2n,1}) + (k-1) + \beta(J_{2n,1}) + 1) = (k+1)\beta(J_{2n,1}) + k$. Therefore, for all $m \geq 3$, the result is true. □

3.3 Metric Dimension of m-level Antiweb Gear Graphs

In this section, we will compute the metric dimension of m-level antiweb-gear graph by using the result of metric dimension of antiweb-gear graph given in [39].

The *gear graph* denoted by J_{2n}. The *antiweb-gear graph* $AWJ_{2n,1}$ is obtained from gear graph J_{2n} by replacing C_{2n} with C_{2n}^2. $C_{2n,1} : c_2, c_4, \cdots, c_{2n}, c_2$ and $C_{2n,2} : z_2, z_4, \cdots, z_{2n}, z_2$ are vertices of cycles. The set of nodes is given by $\mathbb{V}(AWJ_{2n}) = \mathbb{V}(J_{2n})$ and edge set is given by $\mathbb{E}(AWJ_{2n}) = \mathbb{E}(J_{2n}) \cup \{c_i c_{i+2} : 0 \leq i \leq n\}$. First we find the metric dimension of $AWJ_{2n,2}$, then by applying induction on levels of antiweb-gear graph we will get metric dimension of $AWJ_{2n,m}$.
It is given that, if $n \geq 15$, then $\beta(AWJ_{2n,1}) = \lceil \frac{n+1}{3} \rceil$ [39].

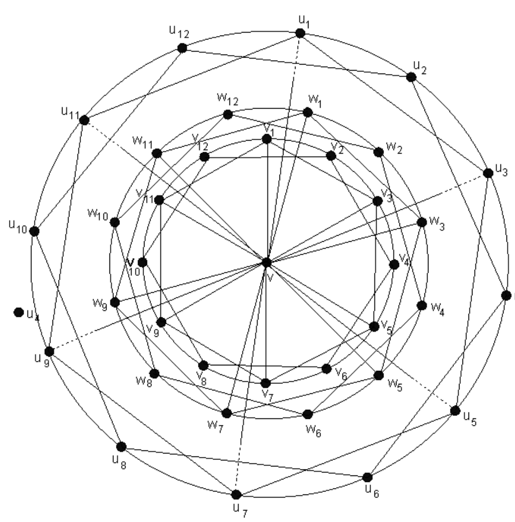


Figure 17: An antiweb-gear graph AWJ_{12}

To find the metric dimension of $AWJ_{2n,2}$, first we discuss the gap conditions and size of gaps as used in [39]. A gap determined by neighboring nodes c_i and c_j is known as $\lambda - \mu$

gap with $\lambda \le \mu$ when $deg(c_i) = \lambda$ and $deg(c_j) = \mu$ or when $deg(c_i) = \mu$ and $deg(c_j) = \lambda$. In Antiweb wheel graphs there are three types of gaps, i.e. $4-4$, $5-4$ and $5-5$ gaps. Following lemmas gives gap conditions for $AWJ_{2n,1}$ proved in [39]:

Lemma 3.17. *[39] Let $\mathbb{B}$ be the basis set then number of vertices are 9, 8 and 7 in every $4-4$, $5-4$ and $5-5$ gaps of $\mathbb{B}$.*

Lemma 3.18. *[39] There is atmost one major gap $4-4$, $5-4$ in the basis $\mathbb{B}$ set.*

Lemma 3.19. *[39] Any two neighbouring gaps, of $\mathbb{B}$ one of them is major may contain together atmost 12 vertices.*

Lemma 3.20. *[39] Any two minor neighbouring gaps of $\mathbb{B}$ contains together atmost 10 vertices.*

Now to compute the basis and metric dimension for $AWJ_{2n,2}$ we have following gap conditions:

(a) Every gap of $\mathbb{B}$ for the vertices of $C^2_{2n,1}$ in $AWJ_{2n,2}$ must satisfy the conditions given in 3.23-3.26 which were presented for $AWJ_{2n,1}$ in [39].

(b) Every set of vertices of B for $AWJ_{2n,2}$ must contain one more vertex, otherwise, any two vertices one from first level and other from second level will have same code representation, i.e $r(c_5|\mathbb{B}) = r(z_5|\mathbb{B})$, which is a contradiction.

Thus every set $\mathbb{B}$ satisfies the conditions given in (a) and (b) is a metric generator for $AWJ_{2n,2}$. The dimension of double antiweb gear graph has been given in the following theorem. This result provide us base for generalizing the result for m-level antiweb gear graph.

Theorem 3.21. *If $n \ge 15$, then we have $\beta(AWJ_{2n,2}) = 2\beta(AWJ_{2n,1}) + 1$.*

Proof. Let $J_{2n,1} \cong W_{2n,1}$, whose n alternating spokes are deleted. Similarly we can construct antiweb gear graph as $AWJ_{2n,2} \cong AWW_{2n,2}$, with n alternate spokes deleted of antiweb wheel graph. We use the method of double inequalities to prove this result. First we prove that $\beta(AWJ_{2n,2}) \le 2\beta(AWJ_{2n,1}) + 1$, if we construct a resolving set for $AWJ_{2n,2}$ with $2\beta AWJ_{2n,1} + 1$ vertices, for this purpose the following cases are needed to be addressed:

(i) If $n \equiv 0(mod\ 6)$, we have $2n = 6q$, for $q \ge 6$ and $2\lceil \frac{n+1}{3} \rceil + 1 = 2q+3$. In this case, $\mathbb{B} = \{c_1, c_{10}, x_1, x_{10}\} \cup \{c_{12i+14}, c_{12i+18}, x_{12j+14}, x_{12j+18} : 0 \le i, j \le \frac{q-4}{2}\} \cup \{c_{2n-2}, x_{2n-2}\}$.

(ii) If $n \equiv 1(mod\ 6)$, we have $2n = 6q+2$, for $q \ge 6$ and $2\lceil \frac{n+1}{3} \rceil + 1 = 2q+3$. In this case, $\mathbb{B} = \{c_1, c_{10}, x_1, x_{10}\} \cup \{c_{12i+14}, c_{12i+18}, x_{12j+14}, x_{2j+18} : 0 \le i, j \le \frac{q-4}{2}\} \cup \{c_{2n-2}, x_{2n-2}\}$.

(iii) If $n \equiv 2(mod\ 6)$, we have $2n = 6q+4$, for $q \ge 6$ and $2\lceil \frac{n+1}{3} \rceil + 1 = 2q+3$. In this case, $\mathbb{B} = \{c_1, c_{10}, x_1, x_{10}\} \cup \{c_{12i+14}, c_{12i+18}, x_{12j+14}, x_{12j+18} : 0 \le i, j \le \frac{q-4}{2}\} \cup \{c_{2n-2}, x_{2n-2}\}$.

(iv) If $n \equiv 3(mod\ 6)$, we have $2n = 6q$, for $q \ge 5$ and $2\lceil \frac{n+1}{3} \rceil + 1 = 2q+3$. In this case, $\mathbb{B} = \{c_1, c_{10}, x_1, x_{10}\} \cup \{c_{12i+14}, c_{12i+18}, x_{12j+14}, x_{12j+18}, : 0 \le i, j \le \frac{q-5}{2}\} \cup \{c_{2n-4}, c_{2n}, x_{2n-4}, x_{2n}\}$.

(v) If $n \equiv 4(mod\ 6)$, we have $2n = 6k+2$, for $q \ge 5$ and $\lceil \frac{n+1}{3} \rceil = 2q+3$. In this case,

$\mathbb{B} = \{c_1, c_{10}, x_1, x_{10}\} \cup \{c_{12i+14}, c_{12i+18}, x_{12j+14}, x_{12j+18} : 0 \leq i, j \leq \frac{q-3}{2}\}$

(vi) If $n \equiv 5(mod\ 6)$, we have $2n = 6q+4$, for $q \geq 5$ and $\lceil \frac{n+1}{3} \rceil = 2q+3$. We define $\mathbb{B} = \{c_1, c_{10}, x_1, x_{10}\} \cup \{c_{12i+14}, c_{12i+18}, x_{12j+14}, x_{12j+18} : 0 \leq i, j \leq \frac{q-3}{2}\}$.

so we have,
$\beta(AWJ_{2n,2}) \leq 2\beta(AWJ_{2n,1}) + 1$. Now we have to show,
$\beta(AWJ_{2n,2}) \geq 2\beta(AWJ_{2n,1}) + 1$

To prove that $\beta(AWJ_{2n,2}) \geq 2\lceil \frac{n+1}{3} \rceil + 1$, let $\mathbb{B}$ be a basis of $AWJ_{2n,2}$ and $|B| = 2l+1$. Then $\mathbb{B}$ forms $2l+1$ gaps on $C_{2n,2}$, $g_1, \cdots, g_{2l+1}$ such that g_j and g_{j+1} are neighboring gaps for every $1 \leq j \leq 2l$, and also g_1 and g_{2l+1} are neighboring gaps. By using the gap conditions given in lemmas 3.23-3.26, we have

$$| g_1 | + | g_2 | \leq 11;$$

$$| g_2 | + | g_3 | \leq 6;$$

$$| g_{2l} | + | g_{2l+1} | \leq 9;$$

$$| g_{2l+1} | + | g_1 | \leq 12$$

and

$$| g_j | + | g_{j+1} | \leq 10,$$

for every $j = 3, \cdots, 2l-1$. By summing up these inequalities, we have

$$2(4n-2l-1) = 2\sum_{j=1}^{l} | g_j | \leq 10(2l+1) - 4.$$

Hence, $l \geq 2\lceil \frac{n+1}{3} \rceil + 1$. Where l is an integer, and for each $2n \equiv 0, 2, 4(mod\ 6)$, we have $l \geq 2\lceil \frac{n+1}{3} \rceil + 1$, which completes the proof.
Finally we have $\beta(AWJ_{2n,2}) = 2\lceil \frac{n+1}{3} \rceil + 1$ □

Now we extend our result to m-level antiweb gear graphs $AWJ_{2n,m}$. For this purpose, we use the induction hypothesis on the levels of antiweb gear graph to prove the result.

Theorem 3.22. *For $n \geq 15$ and $m \geq 3$, we have $\beta(AWJ_{2n,m}) = m\beta(AWJ_{2n,1}) + (m-1)$*

Proof. Let the levels of $AWJ_{2n,m}$ be given by m. For $m = 1$, we have $\beta(AWJ_{2n,1}) = \lceil \frac{n+1}{3} \rceil$ which is given in [39]. For $m = 2$, we have $\beta(AWJ_{2n,2}) = 2(\lceil \frac{n+1}{3} \rceil) + 1$ by Theorem 3.22. Now consider the result is valid in case of $m = k$, i.e.,

$$\beta(AWJ_{2n,k}) = k\beta(AWJ_{2n,1}) + (k-1). \tag{3}$$

Now we show that ir is valid for $m = k+1$.
Let $\beta(AWJ_{2n,k+1}) = \beta(AWJ_{2n,k}) + \beta(AWJ_{2n,1}) + 1$, then equation (3) yields
$\beta(AWJ_{2n,k+1}) = (k\beta(AWJ_{2n,1}) + (k-1) + \beta(AWJ_{2n,1}) + 1) = (k+1)\beta(AWJ_{2n,1}) + k$. Hence the assertion is valid $\forall\ m \geq 3$. □

3.4 Metric Dimension of m-level Convex Polytopes $\mathbb{D}_n^m$

The metric dimension of convex polytope $\mathbb{D}_n^m$ will be discussed in this section. In fact, we generalize the results given in [39] about the metric dimension of $\mathbb{D}_n$.

We can define m-level convex polytope graphs $\mathbb{D}_n^m$ as, Let $L = \{1,2,\cdots,n\}$ and $M = \{1,2,\cdots,m,m+1\}$ be index set and $\mathbb{D}_n^m$ be the prism graph. The prism $\mathbb{D}_n^m$, $(n \geq 3)$, can be obtained by taking m copies of a prism and alternate vertices of outer most cycle are attached to an exterior vertex z and alternate vertices of inner most cycle are attached with a central vertex y, as shown in the Figure 18. The set of nodes of $\mathbb{D}_n^m$ is given by $\mathbb{V}(\mathbb{D}_n^m) = \{u_{j,i} : j \in M \text{ and } i \in L\}$ and edge set given by $\mathbb{E}(\mathbb{D}_n^m) = \{u_{j,i}u_{j,i+1} : j \in M \text{ and } i \in L\} \cup \{u_{1,i}u_{2,i} : i \in L\}$. We have $u_{j,n+1} = u_{j,1}$ $u_{j,n+2} = u_{j,2}$ for $j \in M$.

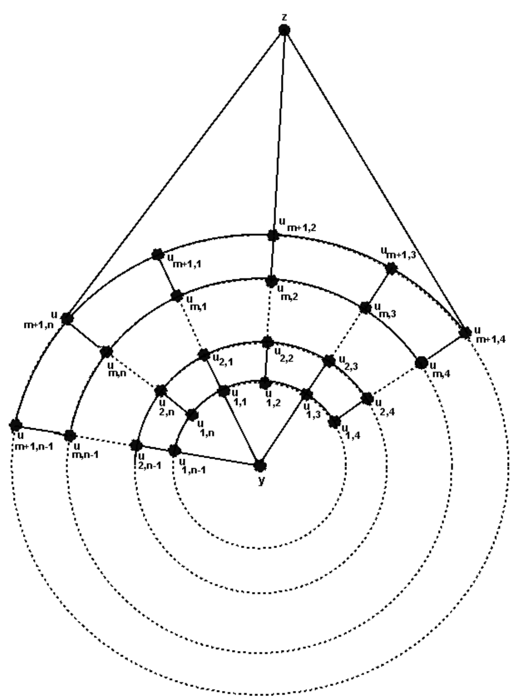


Figure 18: The convex polytope graph $\mathbb{D}_n^m$

For the metric dimension of $\mathbb{D}_n^m$, the basis vertices will be taken from rim vertices of $\mathbb{D}_n^m$.

We use following gap conditions of $\mathbb{D}_n$ as discussed in [39] : A gap formed by neighboring nodes $u_{j,i}$ and $u_{j,k}$ is known as $\lambda - \mu$ gap with $\lambda \leq \mu$ where $deg(u_{j,i}) = \lambda$ and $deg(u_{j,k}) = \mu$ or $deg(u_{j,i}) = \mu$ and $deg(u_{j,k}) = \lambda$. Hence we have three types of gaps i.e. $3-3$, $4-3$ and $4-4$ gaps in $\mathbb{D}_n$, which is given in [39]. Following lemmas (3.23-3.26), which we use as necessary conditions for finding bases:

Lemma 3.23. *[39] Let $\mathbb{B}$ be a basis, then atmost* 5, 4 *and* 3 *vertices are contained in every* $3-3$, $3-4$ *and* $4-4$ *gaps, respectively.*

The $3-3$, $3-4$ and $4-4$ gaps, with 5, 4 and 3 vertices, respectively are known as *major gaps*, and all other gaps are known as *minor gaps*.

Lemma 3.24. *[39] At most one major $3-3$, $3-4$ or $4-4$ gap is contained in $\mathbb{B}$ basis.*

Lemma 3.25. *[39] Any two neighboring gaps of $\mathbb{B}$, one of them is a major gap, contains together at most six vertices.*

Lemma 3.26. *[39] Any two minor neighboring gaps of $\mathbb{B}$ contains at most four vertices.*

To generate the basis set of $\mathbb{D}_n$, we use the following claims given in [39].
(i) The node $u_{2,2} = u_{2,q+1} \in \mathbb{B}$. Else, there are nodes such that $u_{1,3} = u_{1,q+2}$, $u_{1,q+1}, u_{1,q+n-1}, u_{1,q+7}$ on cycle $C_{n,1}$ with $u_{1,q+1}, u_{1,q+n-1}, u_{1,q+7} \in \mathbb{B}$, which gives, $r(u_{2,q+1}|\mathbb{B}) = r(u_{1,q+2}|\mathbb{B})$.

(ii) $\mathbb{B}$ must contain a $4-4$ gap of size three. Else, there exist a $4-4$ gap in $\mathbb{B}$ on $C_{n,2}$ that contains five nodes, namely $u_{2,3} = u_{2,q+2}, u_{2,q+3}, u_{2,q+4}, u_{2,q+5}, u_{2,q+6}$ with $u_{1,q+1}, u_{1,q+7}, u_{1,q+9}, u_{2,q+1}, u_{2,}$
$\mathbb{B}$, which gives, $r(u_{2,q+3}|\mathbb{B}) = r(u_{2,q+4}|\mathbb{B}) = (3,3,\cdots,3,2,2)$.

(iii) "$\mathbb{B}$ consists, one of the neighboring $4-4$ gap that is having at most seven nodes and the other one is having at most five nodes. Else, $\mathbb{B}$ contains $4-4$ neighboring gaps on $C_{n,2}$ that contains seven and nine nodes. Consider a path: $u_{2,3}, \overset{*}{u}_{2,4}, u_{2,5}, \overset{*}{u}_{2,6}, u_{2,7}, \overset{*}{u}_{2,8}$, $u_{2,9}, \overset{*}{u}_{2,10}, u_{2,11}, \overset{*}{u}_{2,12}, u_{2,13}$ which consists of eleven vertices with $u_{1,2}, u_{1,8}, u_{1,10}, u_{1,14}, \overset{*}{u}_{2,2}, \overset{*}{u}_{2,6}, \overset{*}{u}_{2,14} \in \mathbb{B}$, which gives, $r(u_{2,4}|\mathbb{B}) = r(u_{2,12}|\mathbb{B})$, which is a contradiction. Now consider,: $u_{2,n-7}$, $\overset{*}{u}_{2,n-6}$,
$u_{2,n-5}, \overset{*}{u}_{2,n-4}, u_{2,n-3}, \overset{*}{u}_{2,n-2}, u_{2,n-1}, \overset{*}{u}_{2,n}, u_{2,1}, \overset{*}{u}_{2,22}, u_{2,3}, \overset{*}{u}_{2,4}, u_{2,5}$ with thirteen nodes, such that $u_{1,2}, u_{1,8}, u_{1,n}, u_{1,n-2}, u_{1,n-4}, u_{1,n-6}, u_{1,n-8}, \overset{*}{u}_{2,2}, \overset{*}{u}_{2,6}, \overset{*}{u}_{2,n-8} \in \mathbb{B}$, which gives, $r(u_{2,4}|\mathbb{B}) = r(u_{2,n-6}|\mathbb{B})$, which is a contradiction."

(iv) There are seven nodes in at most one $4-4$ gap of $\mathbb{B}$.

• "In $\mathbb{B}$, let there be a gap of size 3 which contains a node t and $t = u_{2,1}, \overset{*}{u}_{2,n}$ or $u_{2,n-1}$ with $u_{1,2}, u_{1,8}$,
$u_{1,n-4}, u_{1,n}, \overset{*}{u}_{2,2}, \overset{*}{u}_{2,6}, \overset{*}{u}_{2,n-2} \in \mathbb{B}$, then
$r(y|\mathbb{B}) = r(2,4,\cdots,4,2,\overline{1,3,\cdots,3})$, $r(t|\mathbb{B}) = r(3,3,\cdots,3,1,\overline{2,\cdots,2})$ and
$r(t|\mathbb{B}) = r(4,4,\cdots,4,2,\overline{3,\cdots,3,1})$, respectively. Where overlined codes are the codes of $\mathbb{B}$ on $C_{n,2}$. By conditions given in lemmas (3.23-3.26) and by claims $(1-4)$, there is no other node of $\mathbb{D}_n$ that satisfy this property."

• "In $\mathbb{B}$, let there be a gap of size 3 which contains a node t on $C_{n,2}$ and $t = u_{2,3}, \overset{*}{u}_{2,4}$ or $u_{2,5}$ with $u_{1,2}, u_{1,8}$,
$u_{1,10}, u_{1,n}, \overset{*}{u}_{2,2}, \overset{*}{u}_{2,6}, \overset{*}{u}_{2,12} \in \mathbb{B}$, then $r(t|\mathbb{B}) = r(2,4,\cdots,4,\overline{1,3,\cdots,3})$,
$r(t|\mathbb{B}) = r(3,3,\cdots,3,\overline{2,\cdots,2})$ and
$r(t|\mathbb{B}) = r(4,4,\cdots,4,\overline{3,1,3,\cdots,3})$, respectively. Where overlined codes are the nodes of $\mathbb{B}$ on $C_{n,2}$. By conditions given in lemmas (3.23-3.26) and by claims $(1-4)$, there exist no

other node of $\mathbb{D}_n$ that satisfy this property."

• "In $\mathbb{B}$, let there be a gap of size 5 which contains a node t on $C_{n,2}$ and $t = u_{2,7}, \overset{*}{u}_{2,8}, u_{2,9}, \overset{*}{u}_{2,10}$ or $u_{2,11}$ with $u_{1,2}, u_{1,8}, u_{1,10}, u_{1,14}, \overset{*}{u}_{2,2}, \overset{*}{u}_{2,6}, \overset{*}{u}_{2,12} \in \mathbb{B}$, then $r(t|\mathbb{B}) = r(4,2,\cdots,4,\overline{3,1,3,\cdots,3})$, $r(t|\mathbb{B}) = r(3,1,3,\cdots,3,\overline{2,\cdots,2})$, $r(t|\mathbb{B}) = r(4,2,2,4,\cdots,4,\overline{3,\cdots,3})$, $r(t|\mathbb{B}) = r(3,3,1,3,\cdots,3,\overline{2,\cdots,2})$ and $r(t|\mathbb{B}) = r(4,4,2,3,4\cdots,4,\overline{3,3,1,3,\cdots,3})$, respectively. Where overlined codes are nodes of $\mathbb{B}$ on $C_{n,2}$. By conditions given in lemmas (3.23-3.26) and by claims $(1-4)$, there is no other node of $\mathbb{D}_n$ satisfy this property."

• "In $\mathbb{B}$, let there be a gap of size 5 which contains a node t on $C_{n,2}$ and $t = u_{2,1}, \overset{*}{u}_{2,n}, u_{2,n-1}, \overset{*}{u}_{2,n-2}, u_{2,n-3}, \overset{*}{u}_{2,n-4}$ or $u_{2,n-5}$, with $u_{1,2}, u_{1,8}, u_{1,n}, u_{1,n-2}, u_{1,n-4}, u_{1,n-8}\overset{*}{u}_{2,2}, \overset{*}{u}_{2,n-6}, \overset{*}{u}_{2,n-12} \in \mathbb{B}$, then $r(t|\mathbb{B}) = r(2,4,\cdots,4,2,\overline{1,3,\cdots,3})$, $r(t|\mathbb{B}) = r(3,3,\cdots,3,1,\overline{2,\cdots,2})$, $r(t|\mathbb{B}) = r(4,4,\cdots,4,2,2,\overline{3,\cdots,3})$, $r(t|\mathbb{B}) = r(3,3,\cdots,3,1,3,\overline{2,\cdots,2})$, $r(t|\mathbb{B}) = r(4,4,\cdots,4,2,2,4,\overline{3,\cdots,3})$, $r(t|\mathbb{B}) = r(3,3,\cdots,3,1,3,3,\overline{2,\cdots,2})$ and $r(t|\mathbb{B}) = r(4,4\cdots,4,2,4,4,\overline{3,3,\cdots,3,1})$, respectively. Where overlined codes are nodes of $\mathbb{B}$ on $C_{n,2}$. By gap conditions given in lemmas (3.23-3.26) and by claims $(1-4)$, there is no other vertex of $\mathbb{D}_n$ that satisfy this property."

The metric dimension of $\mathbb{D}_n$ is given as:

Theorem 3.27. *[39] If n is even and $n \geq 8$, then $\beta(\mathbb{D}_n) = \frac{n}{2}$.*

A set $\mathbb{B}$ that satisfy all the gap conditions and claims would be the basis set for $\mathbb{D}_n$ and if a set B satisfying all the gap conditions given in lemmas (3.23-3.26) and all the claims mentioned above in [39] also the following condition would be the basis set for $\mathbb{D}_n^m$

• At second level one vertex will be added in basis set B, otherwise any two vertices one from first level and other from second level will have same code representation, i.e $r(u_{1,6}|\mathbb{B}) = r(u_{2,7}|\mathbb{B})$, which is a contradiction.

In the following theorem, we compute the dimension of $\mathbb{D}_n^2$ for n even and $n \geq 8$. This result will provide us a base to generalize the metric dimension for m-level convex polytope.

Theorem 3.28. *Let $\mathbb{D}_n^2$ be a convex polytope, then $\beta(\mathbb{D}_n^2) = \frac{n}{2}+1$, $\forall$ even $n \geq 8$.*

Proof. Let $\mathbb{D}_n^2$ be a graph, where n is even and $n \geq 14$, we use the method of double inequality to prove this result. First we prove that $\beta(\mathbb{D}_n^2) \leq \frac{n}{2}+1$ by making a basis set $\mathbb{B}$ in $\mathbb{D}_n^2$ with $\frac{n}{2}+1$ nodes. Let, $n = 2t$, then we have following cases are need to be addressed:

(i) When $t \equiv 0(mod\ 3)$ then we have $2t = 3q$, such that q is even, $q \geq 4$ and $t = \frac{3q}{2}$. For this case, $\mathbb{B} = \{u_{1,2}, u_{1,n}\} \cup \{u_{1,6i+2}, u_{1,6i+4} : 1 \leq i \leq \frac{q}{2}-1\} \cup \{u_{2,10i+2}, u_{2,12i+6} : 0 \leq i \leq \frac{q-4}{4}$ for t even$\} \cup [\{u_{2,2}\} \cup \{u_{2,12i+6}, u_{2,12i+12} : 0 \leq i \leq \frac{q-6}{6}\}$ for t odd$]$.

(ii) When $t \equiv 1(mod\ 3)$ then we have $2t = 3q+2$, such that q is even, $q \geq 4$ and $t = \frac{3q+2}{2}$. For this case, $\mathbb{B} = \{u_{1,2}, u_{1,n}\} \cup \{u_{1,6i+2}, u_{1,6i+4} : 1 \leq i \leq \frac{q}{2}-1\} \cup \{x_{2,10i+2}, x_{2,12i+6} : 0 \leq$

$i \leq \frac{q-2}{4}$ for t even$\} \cup [\{u_{2,2}\} \cup \{u_{2,12i+6}, u_{2,12i+12} : 0 \leq i \leq \frac{q-4}{4}\}$ for t odd$]$.

(iii) When $t \equiv 2(mod\ 3)$ then we have $2t = 3q+1$, such that q is odd, $q \geq 5$ and $t = \frac{3q+1}{2}$. For this case, $\mathbb{B} = \{u_{1,2}\} \cup \{u_{1,6i+2}, u_{1,6i+4} : 1 \leq i \leq \frac{q-1}{2}\} \cup \{u_{2,10i+2}, u_{2,12i+6} : 0 \leq i \leq \frac{q-3}{4}$ for t odd$\} \cup [\{u_{2,2}\} \cup \{u_{2,12i+6}, u_{2,12i+12} : 0 \leq i \leq \frac{q-5}{4}\}$ for t even$]$.

Now to show that $\beta(\mathbb{D}_n^2) \geq \frac{n}{2}+1$, let $\mathbb{B}$ be a basis of $\mathbb{D}_n^2$ and $\mid B \mid = t+s+1$, $(t, s \geq 1)$, where t and s are the number of nodes of $\mathbb{B}$ which lies on $C_{n,1}$ and $C_{n,2}$, respectively. Such that $\mathbb{B}$ forms l gaps on $C_{n,1}$, namely $z_1, \cdots, z_l$ such that z_j and z_{j+1} are neighboring gaps for every $1 \leq j \leq t-1$, and also z_1 and z_l are neighboring gaps. Atmost one of the gaps is major, z_1 given in lemma 3.24, . By lemma 3.25 , and 3.26 , we have,

$$\mid z_1 \mid + \mid z_2 \mid \leq 6;$$

$$\mid z_{t-1} \mid + \mid z_t \mid \leq 6;$$

and

$$\mid z_j \mid + \mid z_{j+1} \mid \leq 4,$$

for every $j = 2, \cdots, t-2$. Where $\mathbb{B}$ forms s gaps on $C_{n,2}$. By the conditions given in claims $1-4$, at most $s-1$ gaps consists of five nodes on $C_{n,2}$. If we sum up all these conditions of the gaps on $C_{n,2}$, we have,

$$2(n-t)+n-s = 2\sum_{j=1}^{t} \mid z_j \mid +n-r \leq 4t+4+5(s-1).$$

$$\Rightarrow 3n+1 \leq 6t+6s$$

$$\Rightarrow \frac{n}{2} \leq \frac{n}{2}+\frac{1}{6} \leq t+s$$

$$\Rightarrow \frac{n}{2}+1 \leq t+s+1$$

for $\mathbb{D}_n^2$ Therefore we have, $t+s+1 \geq \frac{n}{2}+1$. Since $t+s$ is an integer, for each $m \equiv 0,1,2(mod\ 3)$, hence we have $t+s+1 \geq \frac{n}{2}+1$. □

This result can be generalized to m-level convex polytope represented as $\mathbb{D}_n^m$. For this purpose, we apply induction hypothesis on the levels of prism graph to prove the result.

Theorem 3.29. *For $n \geq 8$ and $m \geq 3$, we have $\beta(\mathbb{D}_n^m) = \beta(\mathbb{D}_n^1) + (m-1)$*

Proof. We prove our result by induction hypothesis on levels of prism graph, where the levels of prism graph are given by m. For $m = 1$, we have $\beta(\mathbb{D}_n^1) = \frac{n}{2}$ which is given in [39]. For $m = 2$, we have, $\beta(\mathbb{D}_n^2) = \frac{n}{2}+1)$ by Theorem 3.29. Now let the assertion is valid in case of $m = k$, i.e.,

$$\beta(\mathbb{D}_n^k) = \beta(\mathbb{D}_n^1) + (k-1). \tag{4}$$

We show that the assertion is valid in case of $m = k+1$. Let $\beta(\mathbb{D}^(k+1)_n) = \beta(J_{2n,k})+1$, then equation (4) yields $\beta(J_{2n,k+1}) = (\beta(\mathbb{D}_n^1)+(k-1)+1) = \beta(\mathbb{D}_n^1)+k$. Hence the result is valid $\forall\ m \geq 3$. □

The above result for $\mathbb{D}_n^m$ shows that it has unbounded metric dimension. The unbounded metric dimension of convex polytope also furnish a negative answer to the problem given in [19].

Chapter 4
Conclusions And Open Problems

4 Conclusion And Open Problems

In this thesis, we dealt with metric dimension of certain wheel related graphs, namely m-level gear graph, m-level antiweb-wheel graph, m-level antiweb-gear graph and an infinite class of convex polytopes denoted by $J_{2n,m}$, $AWW_{n,m}$, $AWJ_{2n,m}$ and $\mathbb{D}_n^m$, respectively. We have also proved that metric dimension of family of wheel related graphs is not bounded. Moreover, In the final part of this thesis we examine the metric dimension of convex polytope and it turned out to be unbounded, which also furnish a negative narrative to the problem presented in [19].

The readers are invited to classify the diverse families with respect to nature of the resolving set having least number of elements which is a core problem in this area. So, the readers are invited to consider the following open problems for their future study:

Open Problems:

1. Is it the case that m-level convex polytope generated by prism/anti-prism have constant metric dimension?
2. Can we find some other generalizations of convex polytopes having unbounded metric dimension?
3. "Is it the case that the metric dimension of convex polytope $\mathbb{Q}_n^m$ is given by the following formula

$$dim(\mathbb{Q}_n^m) = \begin{cases} \lfloor \frac{3n}{5} \rfloor & n \equiv 0 (mod\, 5), \\ \\ \lfloor \frac{3n-3}{5} \rfloor & ; otherwise \end{cases}$$

for all $n \geq 7$ and $m \geq 4$" ?

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5 References

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