Chapter 1
Introduction and Background

1 Introduction and background

1.1 Human nervous system

There are many parts in human body that reacts of produced stimuli by nervous system. The human nervous system is the body's conductive system that consist of billion of neurons to send nerve impulses throughout whole body. These nerve cells are spreaded in whole body to accomplish the all body functions. The word stimuli used for this change that may be visual, auditory, electrical, chemical or mechanical. [44]
There are two types of human nervous system i.e central nervous system(CNS) that consist of brain and spinal cord and peripheral nervous system(PNS) contains nerves that transmit nerve impulse to or from the nervous system. [43][45] These two systems collaborate and collect information from inside and outside of the environment. As a result all parts communicate. Here i will discuss the most important part of central nervous system i.e the Brain

1.1.1 Human Brain

The human brain is the most complex organ of human nervous system. It is present in head and covered with cranium and meninges. It is similar to the brain of mammals but cerebral cortex is the largest and well developed in humans. [48] It contains many different portions which are highly important in their respective functions. The human brain is divided into three parts which are the following [41]

- **Forebrain**
- **Midbrain**
- **Hindbrain**

1- The Forebrain

The forebrain contains cerebrum which surrounds mid brain and brain stem. It is grayish in colour because the nerves present in this area are short of insulation due to which other parts appears white. It is densely packed with neurons. It is larger in size and convulated. [37] [55]
Cerebrum is the outer surface that covers human brain. It contains two hemisphere, one is right hemisphere which controls left side of body and other is left hemisphere which controls our right side of body. The outer surface of cortex is about 2 to 4 millimeters thick and it is surrounding round about 1.5 square meter surface area. Its function is thought, perception, awareness, realization and remembrance. [37] [38] [55]

2- The Midbrain

The mid brain is present in between fore brain and hind brain. It is composed of brain stem which is base of the brain connected to spinal cord. It plays important role in communication from brain to all other parts of body. It controls breathing, temperature, blood pressure, listening and vision.[36] [38]

3- The Hindbrain

The hind brain is the last portion of brain . It consist of medulla oblongata,pons and cerebellum(controls short term memory). Medulla works as involuntary function. The pons is the connection between cortex and cerebellum.Pons help in communication to body parts .Though its smaller in size but without pons, all body movements would not be possible.[34] [35]
The cerebellum is the smaller structure (little brain) at the back of the brain. it is about ten percent of total weight. It receives sensory information through spinal cord and then regulate these movements. It is responsible for balance and coordination. cause in damage to cerebellum body losses association and become unstable. [52] although brain controls movements . It controls our thoughts, plans, actions even our body temperature and breathing. its the major part which take part in decision making . In fact we can say all body decisions are made by this part. [64]

The cerebral cortex is composed of four lobes that are given as

- **Temporal lobe**
- **Occipital lobe**
- **Frontal lobe**
- **Perital lobe**

Each lobe has its own functions and responsible for it. [54][56]

- **Temporal lobe**

The temporal lobe is present behind the left side of all lobes and in bottom middle part of cortex. Its work is hearing, listening and understanding sounds having different pitches. A person without temporal lobe is unable to understand a meaningful talk. So due to this lobe we are able to listen, understand and talk about everything.[71]

- **Occipital lobe**

The occipital lobe that is present in the back of head. It helps in understanding and vision. Similar to the temporal lobe,this makes sense of visual information.[51]

- **Frontal lobe**

The Frontal lobe is more developed and larger part in humans that is located in the front of brain. It controls a series of functions like speak, memory, decision making etc. Its function is short term memory, motivation, planning etc. It is the last lobe to mature thats why children do not effectively take decisions. The frontal lobe is the most vulnerable due to which a person loses his personality and faces difficulty in taking decisions. [53]

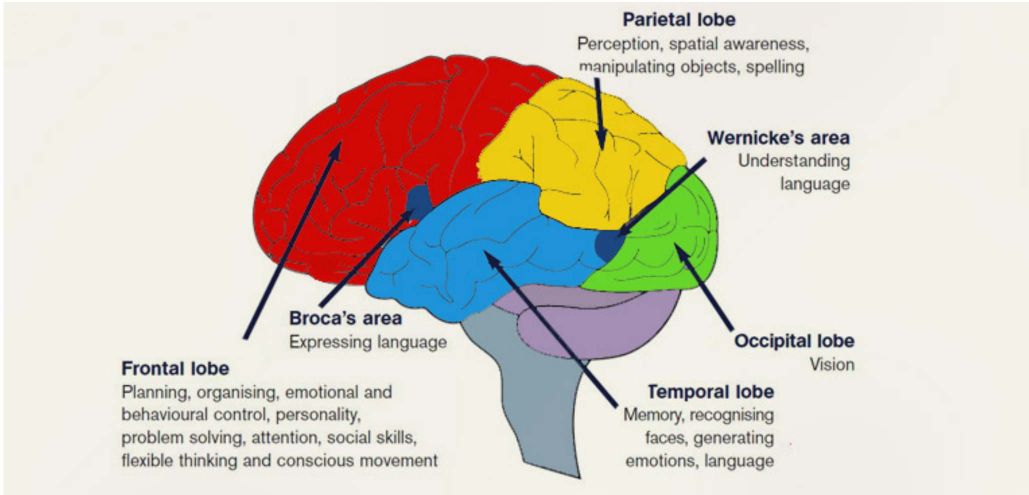

Figure 1: Human Brain with different lobes [47]

- **Perital lobe**

The last one is Perietal lobe is the main part of cerebrum present in middle of all lobes. It is a part of cortex due to which it has many responsibilities. It convey message to other parts within seconds. It is responsible for many functions such as perceptions, taste, touch or feel etc. When this lobe damaged,it losses sense of touch. [50]

1.1.2 Brain Cells

The human brain is main coordinator of body and the structural and functional unit of brain is neuron (nerve cells). The basic structure of neuron is shown in Figure 2. A human brain 1 consist of billions of neurons which communicate throughout the whole body. There are three types of neuron. Sensory neurons recieve the stimuli and send signals to brain through spinal cord. [40] Neurons are the basic unit of brain in human nervous system. All nerve cells are of three kinds, one of them carry the message of brain to all other parts of body(motor nerves)and others carry nerve impulse to the brain when they recieve the stimuli called afferent neurons(sensory neurons) [43][45] A neuron structure and presynaptic terminal is shown as in Fig 2.

It consist of several parts i.e dendrites, soma, axon, synapse and myelin. Dendrites are the input terminals that receives signals from other neurons through its branch like structure and axon is the long tube like output terminal of cell that convey information to other parts of brain. The soma is the main cell body. It is also called processing unit. [49]
Synapse is a small gap contained three parts. The first one is presynaptic terminal which contains neurotransmitter, that produces energy in the cell and cell bodies called organelles. The second part is postsyanaptic terminal that contains receptors for neurotransmitter. The

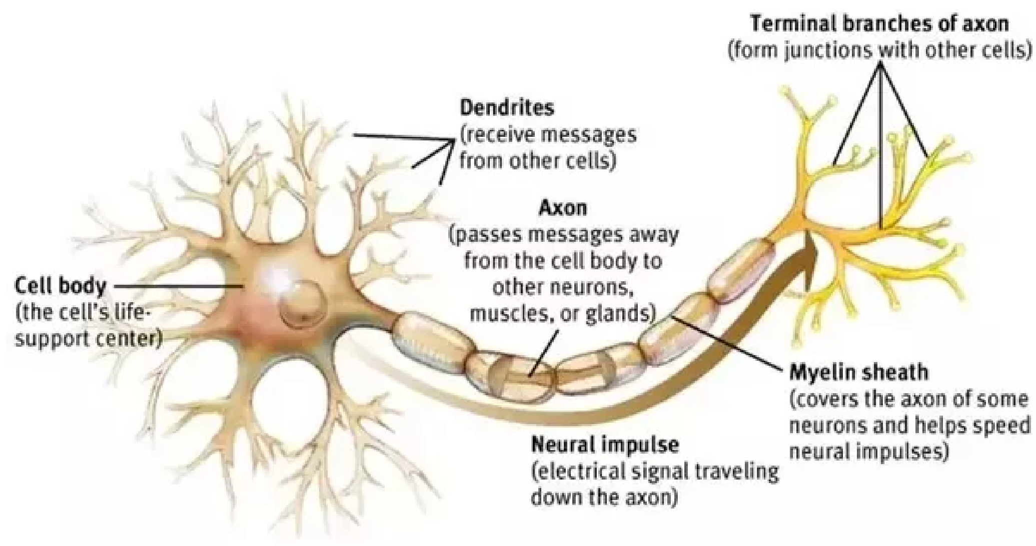

Figure 2: Neurone Structure

last one is the space between presynaptic and postsyanaptic terminal. Thus the presynaptic terminal of one neuron is connected to postsyanaptic terminal of other neuron through this synaptic gap. [33]

Myelin is the necessary coating surrounded axon and helps neuron in conveying signals quickly through long distances. [49]

Dendrites catch input waves from other cells and then convey these to soma. Soma interpret this information till the potential approaches to a certain threshold value. Then neurons emits an electrical spark which can be a series of length from a centimeter to about one meter called action potential. This action potential then fastly transmitted to thousand of other neurons through axon. In fact action potential is the way of communication between thousand of neurons. [42] [64]

Neurons convey message to other part of body through junctions called synapses. Synapses have two forms i.e excitatory and inhibitory. Excitatory synapses brings positive action in whole body while inhibitory decrease the firing action and cause negative action in cells. Excitatory neurons are make up of round about 80% of our settlement and try to bring the affected neurons closed to the action potential while the remaining 20% are inhibitory and they take away neurons from action potential. [40]

Message is carried from one part of brain to another part through neurons,mathematical models is a source to describe it by various methods. These models are distinguished by their scope, simplicity and the amount of biophysical details incorporated in the description. These modeling approaches are divided into three main categories: compartmental modeling, simplified spiking models and firing rate models. For an overview of different modeling approaches, see [3, 4, 5, 6, 8, 9, 10].

The firing rate models have the only level of detail, where the firing action-potential probability is made.Firing existing rate models contains sets of differential equations, e.g. for synaptic drive models [57, 58, 59] the following general form describes the differential equations as follows:

$$\tau_j \frac{du_j}{dt} = -u_j + P_j \left(\sum_j \omega_{jk} u_k \right) \tag{1}$$

Here ω_{jk} is the symbol of synaptic weight function from the presynaptic element k to the post synaptic element j, u_j is the neuron j synaptic drive, τ_j is representing the constant time which is the decay of the synaptic derive u_j follows the potential action in the element j.Firing rate function is represented by $P_j(x)$ changing the synaptic drive into the firing rate function (from element k to element j) [57, 59] .

The model (1) can be derived from the Volterra integral equation model using the linear chain trick with an exponentially decaying temporal kernels which describes the effect neural firing on the existing activity level in the network [3, 5, 13, 14, 58].

In other mathematical biology, rate equation models also occur, for instant gene regulatory network [15, 16, 17, 18, 19, 20]. A model of simple gene regulatory network is given as follows;

$$\frac{dy_i}{dt} = F_i(Z_1, Z_2, ..., Z_n) - G_i(Z_1, Z_2, ..., Z_n) y_i, \quad i = 1, 2, ..., n, \tag{2}$$

where productive functions is F_i and G_i is the rate of relative degradation of gene i, respectively, and y_i is meant gene product concentrations. The function $Z(q_i, \theta_i, y_i)$ is called the activation function, where q_i and θ_i represent the steepness and threshold, respectively. This function expresses the effect of various transcription factors regulating the gene expression and is oftenly approximated by step function Fig. 8.

The model (2) has been studied by approximating the threshold function with the unit step function [15, 16, 21]. In this system right hand side of (2) in this case owns jump discontinuity along with many surfaces in phase space.Singular domain of this dynamical system is an other name of these surfaces. The system (2) is transformed into two differential equations. For the steep activation function the problem left is to prove the convergence of the solution (2). This provides, in particular, a mechanism of combining the limiting solutions in the regular domains together with solutions in the singular domains. In this type of analysis, the key tool in singular perturbation theory and switching dynamical systems [15, 16].

Since the gene regulatory network model (2) and the firing rate model (1) are very much alike with monotonic increasing and bounded activation functions. It is therefore of great interest to investigate the dynamics of model (1) by applying same techniques as used in the gene regulatory networks. This give out as a background and a motivation for the present study. We, therefore apply the theory of switching dynamical systems and singular perturbation analysis in order to find the approximate solutions of the firing rate model in the regime of steep firing rate functions.

1.2 Different Modeling Approaches

Here we will illustrate many mathematical models to identify the transmission of electrical signals which run through neurons. A model is a mathematical state which we have

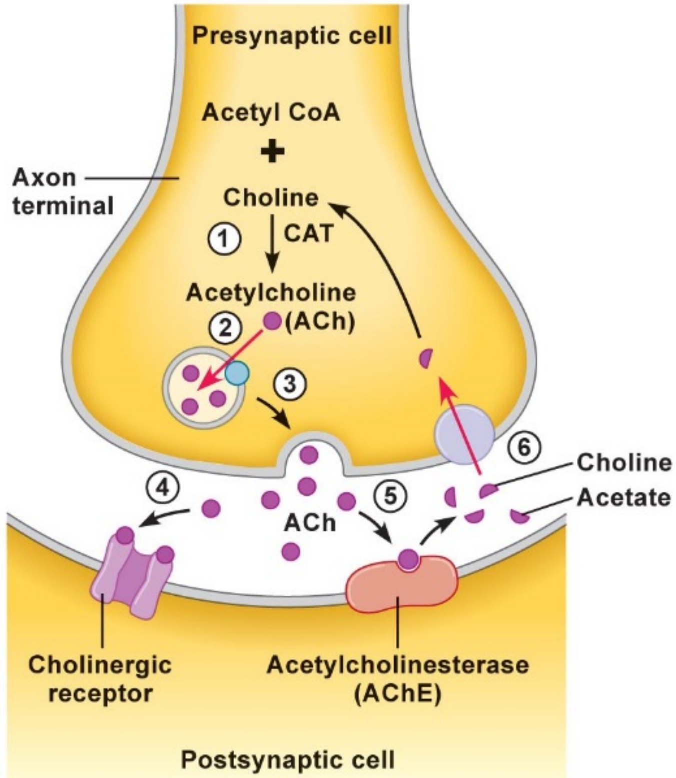

Figure 3: Structure of presynaptic and postsyanaptic terminal[7]

designed to calculate and clarify the basic biological procedures. A mathematical state communicate the signal transformation of neuron. There are many mathematical models to create division due to their size, ease, sum of all biophysical facts which built-in the representation and parameters drawn in. ([66]-[67])

It consist of several parts i.e dendrites, soma, axon, synapse and myelin. Dendrites are the input terminals that receives signals from other neurons through its branch like structure and axon is the long tube like output terminal of cell that convey information to other parts of brain. The soma is the main cell body. It is also called processing unit. [49]

Synapse is a small gap contained three parts and shown in figure 3. The first one is presynaptic terminal which contains neurotransmitter, that produces energy in the cell and cell bodies called organelles. The second part is postsyanaptic terminal that contains receptors for neurotransmitter. The last one is the space between presynaptic and postsyanaptic terminal. Thus the presynaptic terminal of one neuron is connected to postsyanaptic terminal of other neuron through this synaptic gap. [33]

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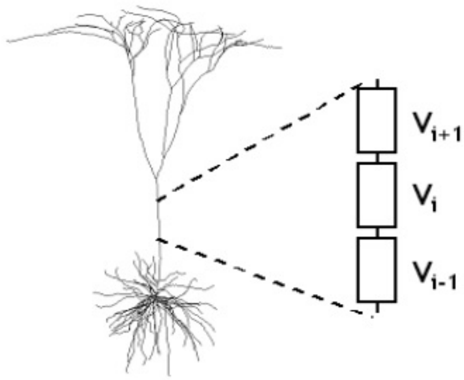

Figure 4: Construction of compartmental model for a neuron [64]

ter called action potential. This action potential then fastly transmitted to thousand of other neurons through axon. Infect action potential is the way of communication between thousand of neurons. [42] [64]
Neurons convey message to other part of body through junctions called synapses. Synapses have two forms i.e excitatory and inhibitory. Excitatory synapses brings positive action in whole body while inhibitory decrease the firing action and cause negative action in cells. Excitatory neurons are make up of round about 80% of our settlement and try to bring the affected neurons closed to the action potential while the remaining 20% are inhibitory and they take away neurons from action potential. [40]
A short report of some familiar mathematical models is given in next sections.

1.2.1 compartmental modeling

Compartmental modeling is often used to illustrate the transformation of materials in biological scheme. It is the scheme which contains various different compartments which the material is transformed in many rules. Here the following (Fig 4). shows construction of compartmental modeling. This modeling linked with its electrical and geometrical properties then its chemical properties. A compartment model contains a number of sections(compartments) containing fine assorted material. These sections then swap this material under certain rules. [66]
So a neuron is separated into various small compartments due to which it comprises a high stage of detail.A single section is so small that the electrical potential is assumed to be same in every section. Membrane potential is the main variable in this case. Mathematical

equation for nth compartments is given from Kirchhoff's law of current as

$$\mathbf{g_{n,n+1}(V_{n+1}-V_n)-g_{n-1,n}(V_n-V_{n-1})=k_n\frac{dV_n}{\mathit{dt}}+\sum_t I_n^t+\sum_j I_n^j} \tag{3}$$

Here the left side of the equation shows the ohmic current among nth and the other two neighboring sections. The first right term gave membrane capacitance although the other term represents the current injected from the pre-synaptic neurons and the last term is due to the leak ion channels. From end to end these channels current depends on the absorption of certain ions in and out side of the membrane. [66] [68]

1.2.2 Point Neuron Models

The compartmental modeling having various disadvantages which is due to use of nunmber of parameters and pricey calculations. This problem is to create difficulty to work out and run these parameters. Then to facing this difficulty, the model is concentrated into a single compartment by removing a number of compartments. In this manner, dendrites and soma deformed into a single point and neuron becomes a one dimensionless component. [63][66]

It is a easy type for repetitive firing in neurons. It is just a point and when threshold is approached it fires spike of constant width and amplitude. [63]

In moving towards this, the potential and membrane current are taken to be similar throughout the point neuron. A conversion of compartmental modeling into point neuron is shown in Fig 5 .

Point neuron model is the simple point model that focus on the sub threshold action and output of neuron. Its well known example is fire models which we have spike is recorded when potential of membrane approaches to threshold value. Each single action potential having almost equal amplitude of same behavior but there is no information of time period of action potential. As a result, there is no need of calculation of time course every time.[64]

The sub-threshold action and output of the neuron are focal point of this model. In a single section model, action potential sub-threshold dynamics of the membrane is given as

$$\mathbf{k\frac{dV}{\mathit{dt}}=-g_L(V-E_L)-\sum_t I^t} \tag{4}$$

Here g_L and E_L are the conductance and resting potential of the neuron, when input current is not there. In this model currents absent between sections of a single neuron, instead we consider only the Ohmic leak current. But after converting compartmental modeling into point neuron, it is still not easy to come across significant result for the full dynamics of system for such simple spiking neurons, or even just the settlement rate dynamics. [66][69]

1.2.3 Firing Rate Models

To generate a huge system of neurons, one of the most common technique is to apply a generalization is known as a firing rate model. Instead of spiking of each neuron, one consider

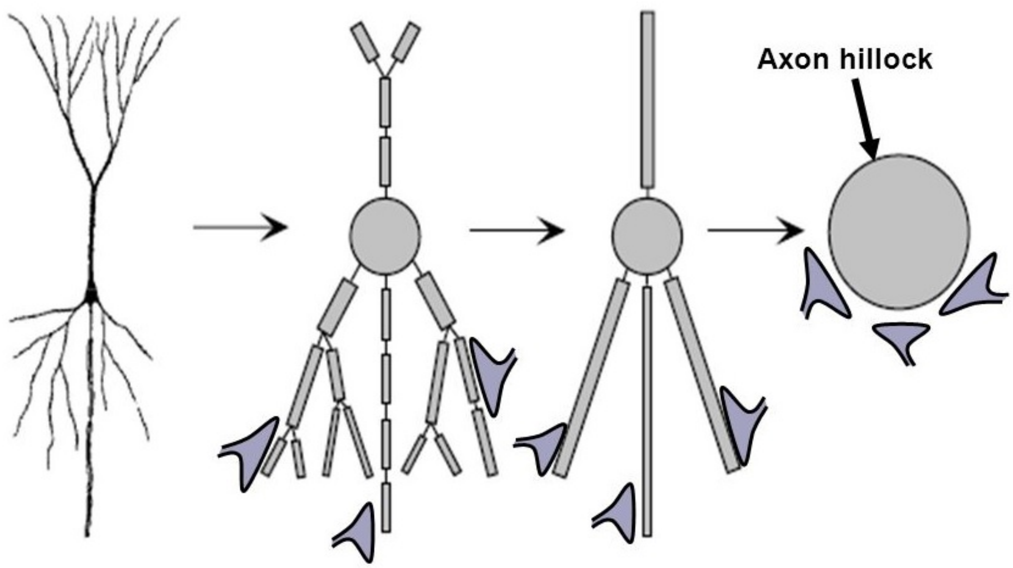

Figure 5: Conversion from compartmental models to point neurons [62]

the spike rates of groups of neurons within the circuit. While these models identify the entire population of neurons then a single neuron, so they called two population models. [70]
A neural tissue has a huge dynamics which often studied by a firing rate models. Firing rate model is in fact explains that how group of neurons give firing response to the input signals and what effect produced to other neurons after this response. Two quantities necessary to as input and output of neuron as shown in Fig 5 [65]
These models are simply to appreciate and compute as they have mathematical formulation of differential and integro-differential developed in applied mathematics. These modeling create the analysis simpler and inherent understanding. Thus these models, instead of modeling each neuron spikes independently, we focus on modeling the opportunity of action potential. The membrane potential is usually transformed to a firing rate by a firing rate function. These models have been frequently utilized in investigative the network properties of strongly interconnected cortical networks, e.g. the two population synaptic drive model and neural field models. [64]

1.2.4 Neural network models

In neural field models the cortical tissue is modeled in a continuous lines or sheets of neurons. In other words, it is the network composed of layers. [61] [64]
Since all parts of body receives messages and connect each other through nerve tissues known as neurons so we extend a space time model for neural network. The easy derivation of neural network is shown in Fig 6
The Fig 6 shows three neurons connected each other in such a way that axon of neuron j is

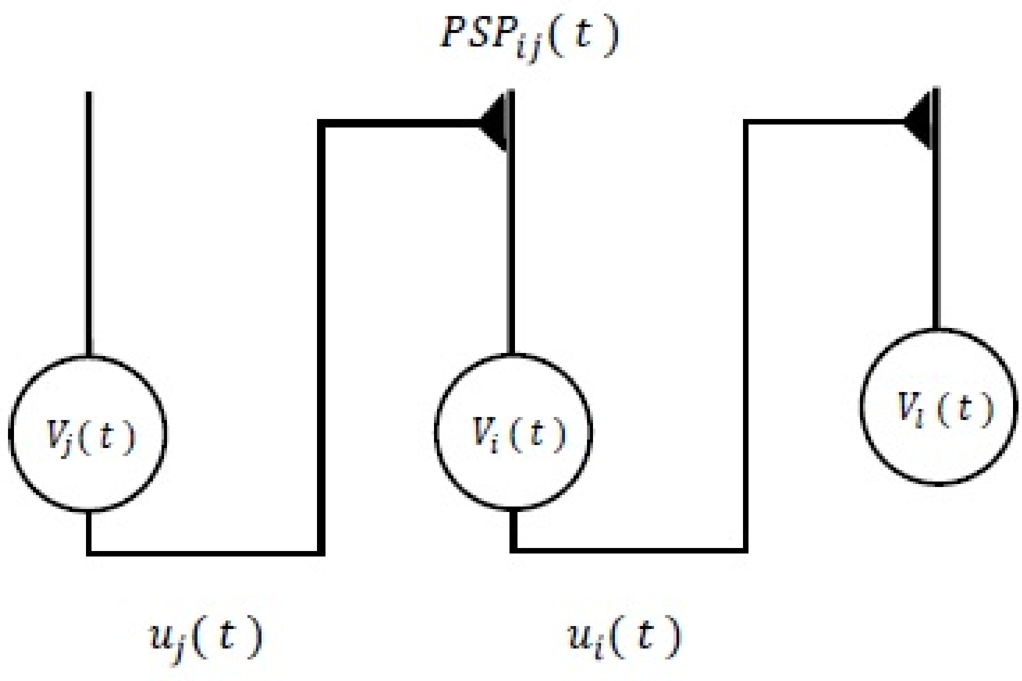

Figure 6: Synaptical interactions between neurons

connected to dendrites of neuron i. Here let u_i identify the firing rate of neuron i at positive time($t > 0$) and this is given as

$$\mathbf{u_i}(\mathbf{t}) = \mathbf{P_i}(\mathbf{V_i}(t)) \tag{5}$$

Where Pi is the firing rate function that changes potential V_i into firing rate u_i at time t. This change can be delayed due to the distance traveled beside axon. This time delay is represented by δ_{ij}.
The amount of all action potentials, approaches at the terminal, form the total potential that is defined as

$$\Omega_{ij} = \sum_k \mathbf{PSP_{ij}}(\mathbf{t}_k) \tag{6}$$

- where t_k represents the time of action potential occurred for k=1,2,3...
- Let S be the time representing total potential approached and t represents the time of action potential fired.

The total potential of neuron(i,j) at time t and $t + \triangle t$ is

$$\Omega_{ij}(t) = \int_0^t \mathbf{PSP_{ij}}(\mathbf{t} - \mathbf{s})\mathbf{u_j}(\mathbf{s} - \delta_{\mathbf{ij}})ds \tag{7}$$

We get the following Volterra integral equation as a result of hitting presynaptic neurons to the postsyanaptic neurons i.

$$\mathbf{V_i}(t) = \sum_j \int_{t_o}^{t} \mathbf{PSP_{ij}}(t-s)\mathbf{P_j}(\mathbf{V_j}(s-\delta_{ij}))ds \tag{8}$$

Hence the final form of action potential formula of neuron i is

$$\mathbf{u_i}(t) = \mathbf{P_i}(\sum_j \int_{t_o}^{t} \mathbf{PSP_{ij}}(t-s)\mathbf{P_j}(\mathbf{V_j}(s-\delta_{ij}))ds) \tag{9}$$

1.2.5 General Synaptic Drive Model

If the shape of the post synaptic potential (PSP) depends only on the nature of the pre-synaptic neuron, the post synaptic potential at the neuron i is the combined effect of the connection strength and the PSP at j-th cell, i.e.,

$$PSP_{ij}(t) = \omega_{ij} P_j(V_j(t)) \tag{10}$$

Further we introduce the time-averaged firing rate of the cell j as

$$U_j(t) = \int_{t_0}^{t} PSP_j(t-s)\, u_j(s) ds \tag{11}$$

Since the PSP is more oftener assumed to be a function of exponentials and its powers. Differentiating (11) yields:

$$L_j U_j(t) = u_j(t) \tag{12}$$

where L_j is linear differential operator. Now by using the equation (5), we end up with

$$L_i U_i(t) = P_i \left(\sum_j \omega_{ij} \int_{t_0}^{t} PSP_j(t-s)\, U_j(s-\delta_{ij}) ds \right) \tag{13}$$

If there is no delay, then the operator L_i is first order linear operator. Hence

$$\tau_i \frac{dU_i}{dt} = -U_i + P_i \left(\sum_j \omega_{ij} U_j \right) \tag{14}$$

where τ_i is the time constant related to synaptic decay and U_i is called the synaptic drive[57] [58][59].

2 Synaptic Drive Model(Two-population)

The model with excitatory and inhibitory activities simplified into two-population synaptic drive model, starting point is given by the Volterra equations

$$v_e = \alpha_e * Z_e(\omega_{ee} v_e - \omega_{ei} v_i) \tag{15a}$$

$$v_i = \alpha_i * Z_i(\omega_{ie} v_e - \omega_{ii} v_i), \tag{15b}$$

where

$$(\alpha_m * Z_m(x))(t) = \int_{-\infty}^{t} \alpha_m(t-s) Z_m(x(s)) ds. \tag{16}$$

Here Z_m for $m = e, i$ is the Hill function is given in 25, the temporal kernels α_m for $m = e, i$ are representing the effect of past neural firing on the present activity levels in the network [23, 24, 57]. These kernels are generally represented by a single time constant. The temporal kernels used in this work are given as:

$$\alpha_i(t) = \frac{1}{\tau} e^{-\frac{t}{\tau}} \ , \ \alpha_e(t) = e^{-t} \tag{17}$$

Here the functions v_m for $(m = e, i)$ in (21) are average activity levels for inhibitory and excitatory populations. The connectivity parameters ω_{mn} $(m = e, i)$ model the connection strength (from n to m cells) in the network. The parameter,τ is the relative inhibition time constant. In this present study, the parameters follows these conditions:

$$0 < \omega_{mn} \leq 1, \ \tau > 0. \tag{18}$$

2.1 Some important definitions

Here we revise some basic definitions which have been encountered in the coming sections.

2.1.1 Dynamical System

The origin of word dynamical system is from Newtonian mechanics. In mathematics, it is a concept where a function explains the time dependence of a point in a geometrical space.

2.1.2 Stationary Points

A point x is said to be stationary if at that point the derivative of given function $f(x)$ becomes disappear as

$$\mathbf{f}'(\mathbf{x}) = \mathbf{0} \tag{19}$$

At this point the function then neither increases nor decreases.

2.1.3 Regular Domain

Any domain which is subset of R^n , where $Z_m = costant = (0,1)$ for $(m = e, i)$ is known as regular domain.
In (Fig 7) there are four regular domains which is define in equ. 29 .

2.1.4 Singular Domain

The compliment of regular domain is known as singular domain and there are five regular domains which are L_e^0 , L_e^1 , L_i^0 , L_i^1 and intersection point are given in (Fig 7)

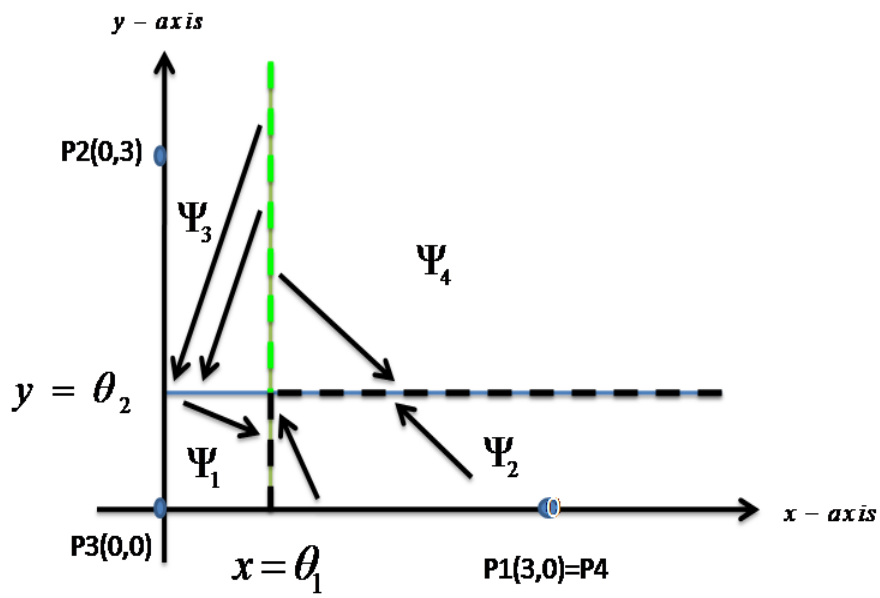

Figure 7: Singular domains $x = \theta_1$, $y = \theta_2$ shows the behavior of black, white and transparent walls respectively, here $\theta_1 = 1$ and $\theta_2 = 1$. Green and black dashes lines show the behavior of white and black walls respectively, while the remaining lines shows the behavior of transparent walls. Ψ_1,Ψ_2,Ψ_3,Ψ_4 are the regular domains.In each regular domain the dashes arrows are represents the integral curve.In (Fig 7) P_1,P_2,P_3,P_4 are the focal points.

2.2 Black wall

Black wall is a singular domain L_m for $(m = e,i)$, which attracts trajectories from both sides of the wall.
In (Fig 7) the black dashes line represent the behavior of black wall.

2.3 White wall

White wall is a singular domain L_m for $(m = e,i)$, which repel trajectories from both sides of the wall.
In (Fig 7) the green dashes line represent the behavior of white walls.

2.4 Transparent wall

Transparent wall is a singular domain L_m for $(m = e, i)$, which trajectories cross the wall. In (Fig 7) those line are show the behavior of transparent walls which are not in dashes .

Now singular domains L_e and L_i are divided into two parts:

2.5 Singular stationary point

The stationary points are said to be singular stationary points (SSP), if it is lying in one of the singular domains.

2.6 Regular stationary point

The stationary points are said to be regular stationary points (RSP), if it is lying in one of the regular domains.

For this, we divide each of the singular domains L_e and L_i further into two parts.
in the next example, we will demonstrate the behavior of singular domains in the phase space of system of two ODEs representing gene regulatory network.

• **Example (Gene regulatory network)**

$$x' = 1 - Z_1 - Z_2 + 2Z_1Z_2 - 1/3x \tag{20a}$$

$$y' = Z_1 - Z_1Z_2 - 1/3y \tag{20b}$$

Here x and y represents two genes with Z_1 and Z_2 corresponding steep activation functions of genes. Corresponding regular domains are given as:
$\Psi_1 = \{(x,y) : x < \theta_1, y < \theta_2\}$ then $Z_1 = 0$ and $Z_2 = 0$
Then 20 gives the values of x' and y' which is follows,
x'= 1-1/3x and y'= -1/3y
The focal point corresponding to these lines is $P_1(3,0)$.

$\Psi_2 = \{(x,y) : x > \theta_1, y < \theta_2\}$ then $Z_1 = 1$ and $Z_2 = 0$
Then 20 gives the values of x' and y' which is follows,
x'= -1/3x and y'= 1-1/3y
The focal point corresponding to these lines is $P_2(0,3)$.

$\Psi_3 = \{(x,y) : x < \theta_1, y > \theta_2\}$ then $Z_1 = 0$ and $Z_2 = 1$
Then 20 gives the values of x' and y' which is follows,
x'= -1/3x and y'= -1/3y
The focal point corresponding to these lines is $P_3(0,0)$.

$\Psi_4 = \{(x,y) : x > \theta_1, y > \theta_2\}$ then $Z_1 = 1$ and $Z_2 = 1$
Then 20 gives the values of x' and y' which is follows,
x'= 1-1/3x and y'= -1/3y
The focal point corresponding to these lines is $P_4(3,0)$.

These focal points are show in (Fig 7).

In (Fig 7) $x = \theta_1$ and $y = \theta_2$ are the regular domains which shows the behavior of white, black and transparent walls and Ψ_1,Ψ_2,Ψ_3,Ψ_4 are the regular domains. In each regular domain the dashes arrow are represents the integral curve. If we are in the region Ψ_1 the solutions are converging into corresponding focal point P_1. After crossing switching domain $x = \theta_1$ solution behavior changes in regular domain Ψ_2. Now integral curve will converge to corresponding point P_2. The integral curve are attracted each other when it crossing the wall and this wall serve as black wall behavior. After crossing switching domain $y = \theta_2$ solution behavior changes in regular domain Ψ_3. Now integral curve will converge to corresponding point P_3. Both side of integral curve are cross the wall then this wall serve as transparent wall behavior. After crossing switching domain $x = \theta_1$ solution behavior changes in regular domain Ψ_4. Now integral curve will converge to corresponding point P_4. In this region, the integral curve are shows the behavior of white wall in that region when the integral curve repel both side of the wall.

Chapter 2
Model

3 Model

The model with excitatory and inhibitory activities simplified into two-population synaptic drive model, starting point is given by the Volterra equations

$$v_e = \alpha_e * Z_e(\omega_{ee} v_e - \omega_{ei} v_i) \tag{21a}$$

$$v_i = \alpha_i * Z_i(\omega_{ie} v_e - \omega_{ii} v_i), \tag{21b}$$

where

$$(\alpha_m * Z_m(x))(t) = \int_{-\infty}^{t} \alpha_m(t-s) Z_m(x(s)) ds. \tag{22}$$

In the network [23, 24, 57] ,temporal kernels are α_m for $m = e, i$ indicating the impact on the present activity levels to past neural firing . These kernels are generally parameterized by a single time constant. In this particular research the temporal kernels are made by means of exponentially decaying functions.

$$\alpha_i(t) = \frac{1}{\tau} \exp(-\frac{t}{\tau}) \ , \ \ \alpha_e(t) = \exp(-t) \tag{23}$$

For inhibitory and excitatory populations, the functions u_m for $m = e, i$ in (21) are the average activity levels.τ is consider as the relative inhibition time that is specialized as the ratio among inhibitory to excitatory time constants. The connectivity parameters ω_{mn} $(m = e, i)$ in the network are modeled by the connection strength (from n to m cells). In this model, the parameters satisfy the following conditions:

$$0 < \omega_{mn} \leq 1, \ \ \tau > 0. \tag{24}$$

The activation function Z_m for $m = e, i$ is made by

$$Z_m(x_m) = H(q_m, \theta_m, x_m) = \begin{cases} 0 & x_m < 0 \\ \frac{x_m^{1/q_m}}{x_m^{1/q_m} + \theta_m^{1/q_m}} & x_m \geq 0, \end{cases} \tag{25}$$

H is used to indicate Hill function. Threshold values are given by the parameter θ_m, In order to measure the steepness of firing rate function q_m for $m = e, i$. In Fig.8 the Hill function is signified for various values of the steepness parameter q for a fixed threshold value $\theta = 0.5$ (for the detailed review see [20, 25]). We suppose that $q_e = q_i = q$ by means of the (linear chain trick). The system of Volterra equations (21) can be converted to a 2D autonomous dynamical system by means of the linear chain trick [22, 26, 57] using exponentially decaying temporal kernels (23):

$$v_e' = -v_e + Z_e(\omega_{ee} v_e - \omega_{ei} v_i) \tag{26a}$$

$$\tau v_i' = -v_i + Z_i(\omega_{ie} v_e - \omega_{ii} v_i). \tag{26b}$$

The following definitions are represents behaviors of singular and regular domains which is define in Fig 9

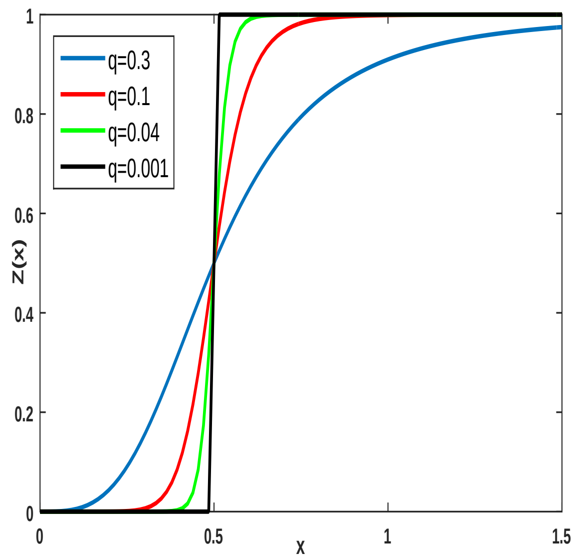

Figure 8: The Hill function for various values of q (steepness parameter) at $\theta = 0.5$.

3.1 Definition

The line segment L_e^0 consists of points satisfying

$$L_e^0 = \{(v_e, v_i) \mid \ \omega_{ee}v_e - \omega_{ei}v_i \ = \theta_e \ , \ \ \omega_{ie}v_e - \omega_{ii}v_i \ < \theta_i\}, \tag{27a}$$

In (Fig 9) the singular domain L_e^0 is white wall with green (dashes) color.
while the line segment L_e^1 satisfy the conditions

$$L_e^1 = \{(v_e, v_i) \mid \omega_{ee}v_e - \omega_{ei}v_i \ = \theta_e \ , \ \ \omega_{ie}v_e - \omega_{ii}v_i \ > \theta_i\}. \tag{27b}$$

In (Fig 9) the singular domain L_e^1 is transparent wall.

3.2 Definition

The line segment L_i^0 consists of points satisfying

$$L_i^0 = \{(v_e, v_i) \mid \ \omega_{ee}v_e - \omega_{ei}v_i \ < \theta_e \ , \ \ \omega_{ie}v_e - \omega_{ii}v_i \ = \theta_i\}, \tag{28a}$$

In (Fig 9) the singular domain L_i^0 is dashes with black color is the black wall.
while line segment L_i^1 satisfy the conditions

$$L_i^1 = \{(v_e, v_i) \mid \omega_{ee}v_e - \omega_{ei}v_i \ > \theta_e \ , \ \ \omega_{ie}v_e - \omega_{ii}v_i \ = \theta_i\}. \tag{28b}$$

In (Fig 9) the singular domain L_i^1 is transparent wall.

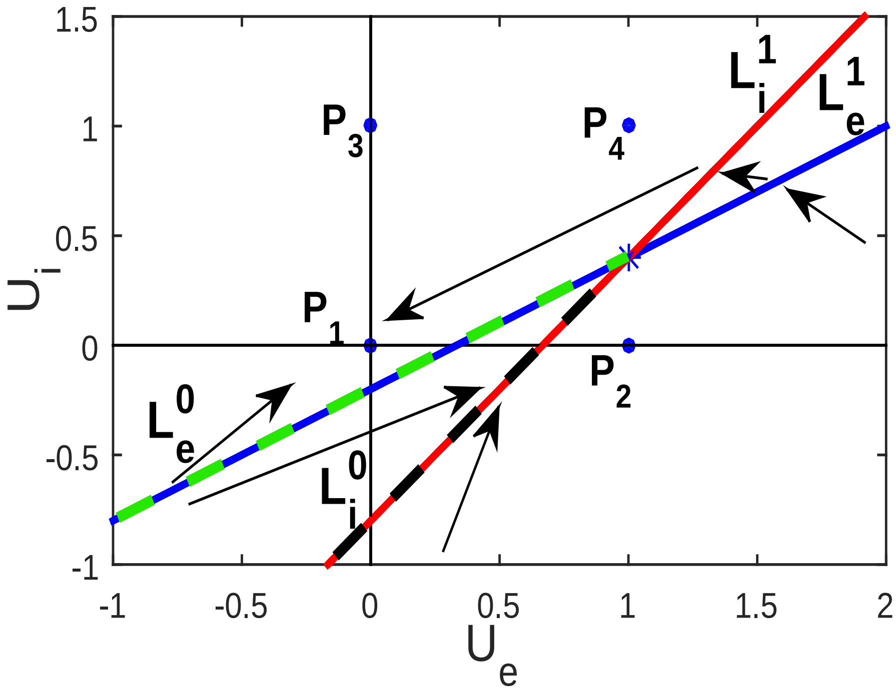

Figure 9: Singular domains L_e^0 , L_e^1(blue) and L_i^0 , L_i^1 (red) as transparent, black and white for the choice of parameter sets given in Table 2. Black and green colors (dashed) are represents the behavior of black and white walls, respectively.The other lines show the behavior of transparent walls. The dashes arrows are represent the integral curves in each regular domain.

This approximation naturally divides the (v_e, v_i) plane in to four disjoint regular domains.

$$\Psi_1 = \{(v_e, v_i) \mid \ \omega_{ee}v_e - \omega_{ei}v_i \ < \theta_e \ , \ \ \omega_{ie}v_e - \omega_{ii}v_i \ < \theta_i\} \tag{29a}$$

$$\Psi_2 = \{(v_e, v_i) \mid \omega_{ee}v_e - \omega_{ei}v_i \ > \theta_e \ , \ \ \omega_{ie}v_e - \omega_{ii}v_i \ < \theta_i\} \tag{29b}$$

$$\Psi_3 = \{(v_e, v_i) \mid \omega_{ee}v_e - \omega_{ei}v_i \ < \theta_e \ , \ \ \omega_{ie}v_e - \omega_{ii}v_i \ > \theta_i\} \tag{29c}$$

$$\Psi_4 = \{(v_e, v_i) \mid \omega_{ee}v_e - \omega_{ei}v_i \ > \theta_e \ , \ \ \omega_{ie}v_e - \omega_{ii}v_i \ > \theta_i\} \tag{29d}$$

We get the following four dynamical systems in these regions:

$$v_e' = -v_e \ \ , \ \ \tau v_i' = -v_i, \ \ (v_e, v_i) \in \Psi_1 \tag{30a}$$

$$v_e' = -v_e + 1 \ , \ \tau v_i' = -v_i, \ (v_e, v_i) \in \Psi_2 \tag{30b}$$

$$v_e' = -v_e \ , \ \tau v_i' = -v_i + 1, \ (v_e, v_i) \in \Psi_3 \tag{30c}$$

$$v_e' = -v_e + 1 \ , \ \tau v_i' = -v_i + 1, \ (v_e, v_i) \in \Psi_4 \tag{30d}$$

The focal points for dynamical system corresponding to (30) are given as

$$P_1 = (0,0) \quad P_2 = (1,0) \ , \ P_3 = (0,1) \ , \ P_4 = (1,1) \tag{31}$$

The singular domains are defined as the lines

$$L_e : \omega_{ee} v_e - \omega_{ei} v_i = \theta_e \tag{32a}$$

$$L_i : \omega_{ie} v_e - \omega_{ii} v_i = \theta_i \tag{32b}$$

We define the connectivity matrix Ψ

$$\Psi = \begin{bmatrix} \omega_{ee} & -\omega_{ei} \\ \omega_{ie} & -\omega_{ii} \end{bmatrix} \tag{33}$$

We assume that Ψ is invertible, i.e. $det(\Psi) \neq 0$. This means that we have a unique crossing point between the lines L_e and L_i.
The vectors

$$\underline{n}_e = \begin{bmatrix} \omega_{ee} \\ -\omega_{ei} \end{bmatrix}, \ \underline{n}_i = \begin{bmatrix} \omega_{ie} \\ -\omega_{ii} \end{bmatrix} \tag{34}$$

are normal vectors to the lines L_e and L_i respectively, whereas threshold values θ_e and θ_i are level curve constants of the lines L_e and L_i respectively. The location of the lines L_e and L_i relative to the focal points P_1, P_2, P_3 and P_4 determine the behavior of the lines, i.e. whether they are transparent, white or black lines.

In the following four figures, the lines L_i and L_e are represented by red and green color lines respectively, direction of integral curves of (30) in each region is represented by black directed curve segments. Focal points defined in (31) are also marked in each regular domain. In all figures we used $\tau = 0.5$

Following are show the behavior of the singular domains L_e and L_i for the parameter sets 1-4 in Table 1:

- In set 1, the parameters both of the singular domains L_e and L_i are shows the behavior of transparent walls in Table 1 and is expressed in (Fig10a).

- In set 2 parameters in line segment L_e^0 with green dashes color is used to show the behavior of white wall, while the lines L_e^1 , L_i^1 , L_i^0 are shows the behavior of transparent walls in Table 1 (Fig. 10b).

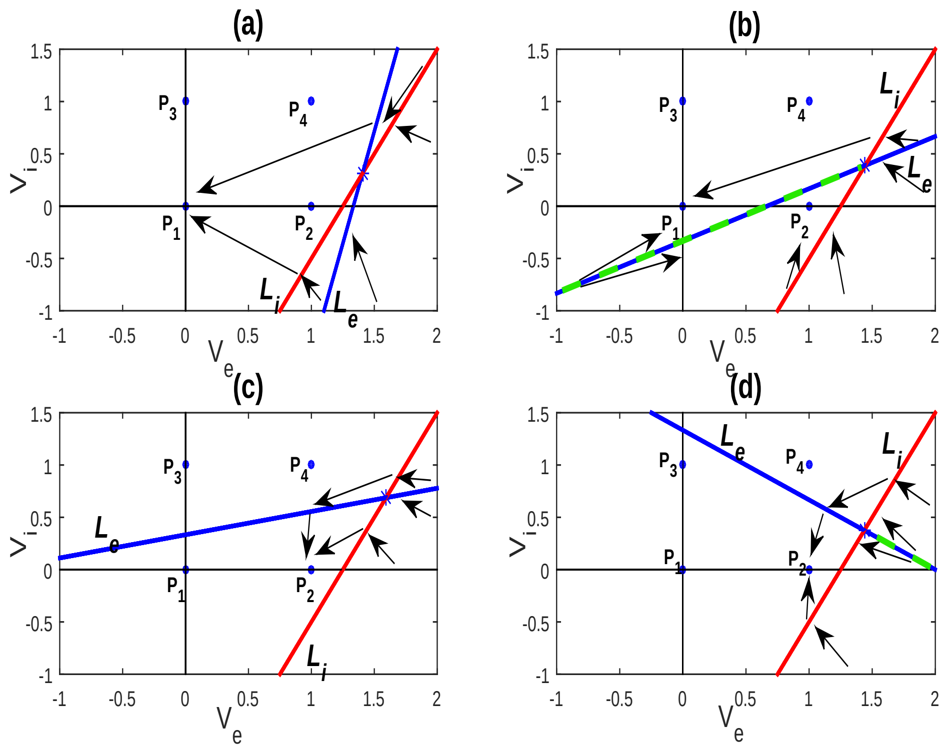

Figure 10: Singular domains L_i (red) and L_e (blue) represents the behavior of white, transparent and black for the parameter sets given in Table 1. Green and black (dashes) lines are shows the behavior of White and black walls respectively. In each regular domain the dashes arrow are represents the integral curve. (a) All the walls for the parameters set 1 are shows the behavior og transparent walls. (b) In parameters set 2 green (dashes) lines are shows the behavior of white wall while the other walls are shows the behavior of transparent wall respectively. (c) All the walls for the parameters set 3 are shows the behavior of transparent walls. (d) In parameters set 4 green (dashes) lines are shows the behavior of white wall while the other walls are shows the behavior of transparent wall respectively, in the Table 1. Decay time constant is assumed to be $\tau = 0.5$.

- In set 3, the parameters both of the singular domains L_e and L_i are shows the behavior of transparent walls in Table 1 and is expressed in (Fig10c).

- In set 4 the parameter with singular domain L_e^1 with green dashes color shows the behavior of white wall, while the remaining line segments L_e^0, L_i^0 and L_i^1 are shows the behavior of transparent walls in Table 1 (Fig. 10d).

The singular domains L_e and L_i are shows the behavior of walls for the parameter sets 1-4 in Table 2:

- Both of the singular domains L_i^1 and L_e^1 are dashes with black and green color are represents black and white walls respectively, while the singular domain L_i^0 and L_e^0

Parameters	ω_{ee}	ω_{ei}	ω_{ie}	ω_{ii}	θ_e	θ_i
Set 1	0.3	0.07	0.08	0.04	0.4	0.1
Set 2	0.3	0.6	0.08	0.04	0.2	0.1
Set 3	0.2	0.9	0.08	0.04	−0.3	0.1
Set 4	0.1	−0.15	0.08	0.04	0.2	0.1

Table 1: Singular domains in Parameter sets for Figure 10 shows the different behaviors. In set 1 parameters are used to shows the behavior of transparent walls. In set 2 parameters are used to shows the behavior of white and transparent walls. In set 3 parameters are used to shows the behavior of transparent walls. In set 4 parameters generate transparent and white walls at the same time.

Parameters	ω_{ee}	ω_{ei}	ω_{ie}	ω_{ii}	θ_e	θ_i
Set 1	0.3	0.03	0.06	0.05	0.4	0.04
Set 2	0.9	0.5	0.06	0.05	0.8	0.04
Set 3	0.3	0.5	0.06	0.05	0.1	0.04
Set 4	0.1	−0.3	0.06	0.05	0.2	0.04

Table 2: Singular domains in Parameter sets for Figure 11 shows the different behaviors. In set 1 parameters are used to shows the behavior of white , black and transparent walls respectively. In set 2 parameters are used to shows the behavior of white, black and transparent walls respectively. In set 3 parameters are used to shows the behavior of white, black and transparent walls respectively. In set 4 parameters generate the behavior of black and transparent walls at the same time.

are transparent walls for the parameter set 1 in Table 2 (Fig. 11a).

- Both of the singular domains L_i^1 and L_e^1 are dashes with black and green color are represents black and white walls respectively, while the singular domain L_i^0 and L_e^0 are transparent walls for the parameter set 2 in Table 2 (Fig. 11b).

- Both of the singular domains L_i^0 and L_e^0 are dashes with black and green color are represents black and white walls respectively, while the singular domain L_i^1 and L_e^1 are transparent walls for the parameter set 3 in Table 2 (Fig. 11b).

- The line segment L_i^0 is dashes with black color is black wall, while the lines L_i^1, L_e^0 , L_e^1 are represents the behavior of transparent walls for the parameter set 4 in Table 2 (Fig. 11d).

The singular domains L_e and L_i are shows the behavior of walls for the parameter sets 1-4 in Table 3:

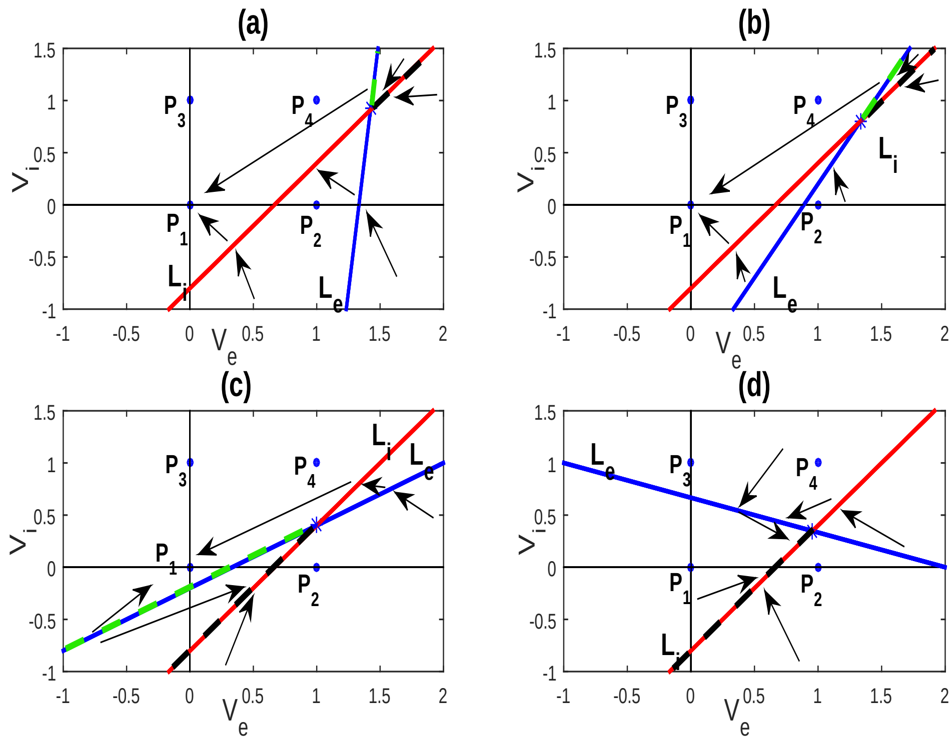

Figure 11: Singular domains L_i (red) and L_e (blue) represents the behavior of white, transparent and black for the parameter sets given in Table 2. Green and black (dashes) lines are shows the behavior of White and black walls respectively. In each regular domain the dashes arrow are represents the integral curve. (a) In parameter set 1 green and black (dashes) lines are shows the behavior of white and black walls respectively, while the other lines represents the behavior of transparent walls. (b) In parameter set 2 green and black (dashes) lines are shows the behavior of white and black walls respectively, while the other lines represents the behavior of transparent walls. (c) In parameter set 3 green and black (dashes) lines are shows the behavior of white and black walls respectively, while the other lines represents the behavior of transparent walls.. (d) In parameters set 4 black (dashes) lines are shows the behavior of black wall while the other walls shows the behavior of transparent wall respectively, in the Table 2. Decay time constant is assumed to be $\tau = 0.5$.

- Both of the singular domains L_i^0 and L_i^1 are dashes with black color are black walls, while the singular domain L_e^0 and L_e^1 are transparent walls for the parameter set 1 in Table 3 (Fig. 12a).

- Both of the singular domains L_i^1 and L_i^0 are dashes with black color are black walls and the singular domain L_e^1 is dashes with green color is white wall, while the singular domain L_e^0 is transparent wall respectively for the parameter set 2 in Table 3 (Fig. 12b).

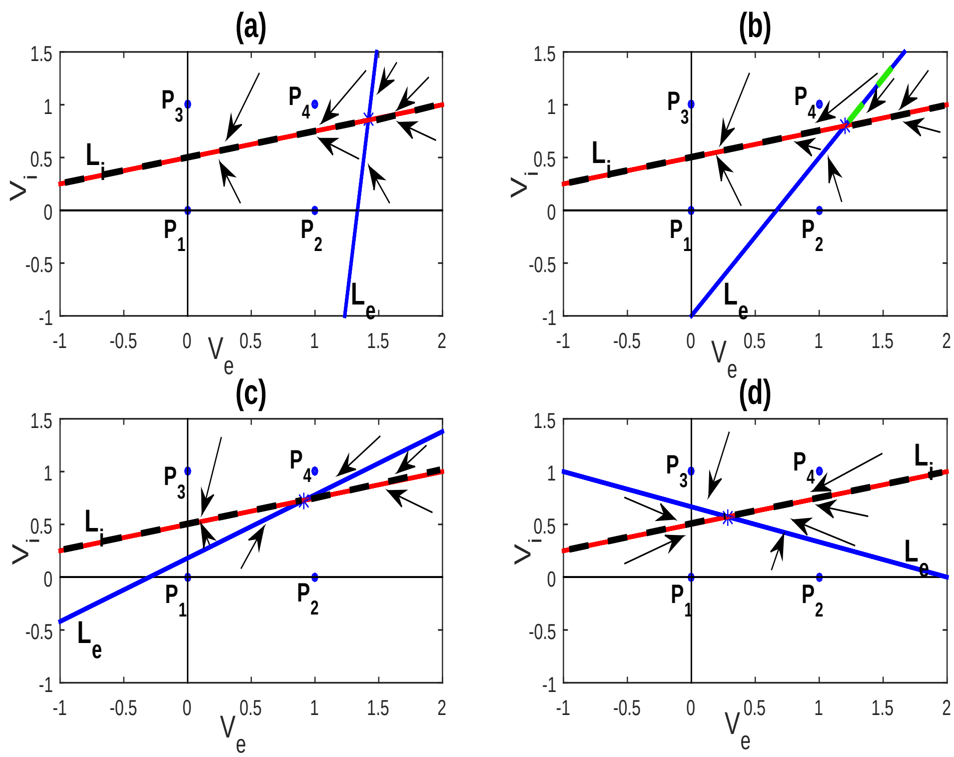

Figure 12: Singular domains L_i (red) and L_e (blue) represents the behavior of white, transparent and black for the parameter sets given in Table 3. Green and black (dashes) lines are shows the behavior of White and black walls respectively. In each regular domain the dashes arrow are represents the integral curve. (a) In parameter set 1 black (dashes) lines are shows the behavior of black walls while the other lines represents the behavior of transparent walls. (b) In parameter set 2 green and black (dashes) lines are shows the behavior of white and black walls respectively, while the other lines represents the behavior of transparent walls. (c) In parameter set 3 black (dashes) lines are shows the behavior of black walls while the other lines represents the behavior of transparent walls. (d) In parameters set 4 black (dashes) lines are shows the behavior of black wall while the other walls shows the behavior of transparent wall respectively, in the Table 3. Decay time constant is assumed to be $\tau = 0.5$.

- Both the singular domains L_i^0 and L_i^1 are dashes with black color are black walls, while the lines L_e^1 and L_e^0 are transparent walls for the parameter set 3 in Table 3 (Fig. 12c).

- Both the singular domains L_i^0 and L_i^1 are dashes with black color are black walls, while the lines L_e^1 and L_e^0 are transparent walls for the parameter set 4 in Table 3 (Fig. 12d).

Parameters	ω_{ee}	ω_{ei}	ω_{ie}	ω_{ii}	θ_e	θ_i
Set 1	0.3	0.03	0.1	0.4	0.4	−0.2
Set 2	0.6	0.4	0.1	0.4	0.4	−0.2
Set 3	0.3	0.5	0.1	0.4	−0.09	−0.2
Set 4	0.1	−0.3	0.1	0.4	0.2	−0.2

Table 3: Singular domains in Parameter sets for Figure 12 shows the different behaviors. In set 1 parameters are used to shows the behavior of black and transparent walls respectively. In set 2 parameters are used to shows the behavior of white, black and transparent walls respectively. In set 3 parameters are used to shows the behavior of black and transparent walls respectively. In set 4 parameters generate the behavior of black and transparent walls at the same time.

Parameters	ω_{ee}	ω_{ei}	ω_{ie}	ω_{ii}	θ_e	θ_i
Set 1	0.4	0.1	−0.4	0.5	0.4	−0.6
Set 2	0.6	0.6	−0.4	0.5	0.2	−0.6
Set 3	0.3	0.9	−0.4	0.5	−0.4	−0.6
Set 4	0.7	−0.09	−0.4	0.5	0.4	−0.6

Table 4: Singular domains in Parameter sets for Figure 13 shows the different behaviors. In set 1 parameters are used to shows the behavior of white and transparent walls respectively. In set 2 parameters are used to shows the behavior of white and transparent walls respectively. In set 3 parameters are used to shows the behavior of white and transparent walls respectively. In set 4 parameters generate the behavior of white and transparent walls at the same time.

The singular domains L_e and L_i are shows the behavior of walls for the parameter sets 1-4 in Table 4:

- The singular domain L_i^1 is white wall with green (dashes) color, while the singular domains L_i^0, L_e^0 and L_e^1 are transparent walls for the parameter set 1 in Table 4 (Fig. 13a).

- Both of the singular domains L_i^1 and L_e^0 are white wall with green (dashes) color, while the singular domain L_e^1 and L_i^0 are transparent walls for the parameter set 2 in Table 4 (Fig. 13b).

- The singular domain L_i^1 is white wall with green (dashes) color, while the singular domains L_i^0, L_e^0 and L_e^1 are transparent walls for the parameter set 3 in Table 4 (Fig. 13c).

- Both of the singular domains L_i^1 and L_e^1 are dashes with green color are white walls, while the singular domain L_e^0 and L_i^0 are transparent walls for the parameter set 4 in

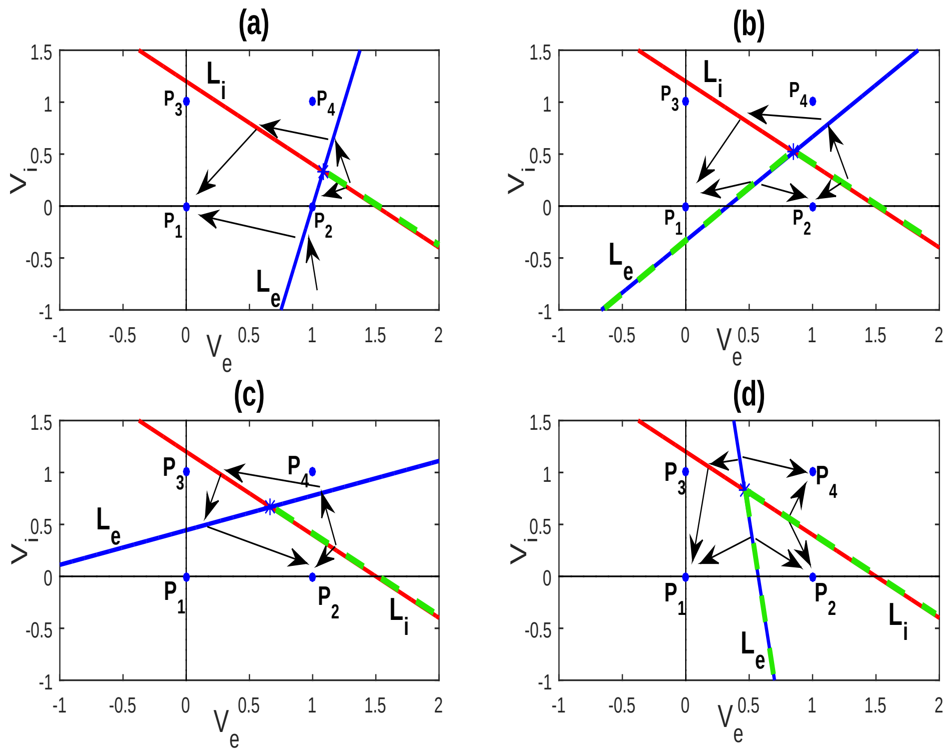

Figure 13: Singular domains L_i (red) and L_e (blue) represents the behavior of white, transparent and black for the parameter sets given in Table 4. Green and black (dashes) lines are shows the behavior of White and black walls respectively. In each regular domain the dashes arrow are represents the integral curve. (a) In parameter set 1 green (dashes) lines are shows the behavior of white walls while the other lines represents the behavior of transparent walls. (b) In parameter set 2 green (dashes) lines are shows the behavior of white walls while the other lines represents the behavior of transparent walls. (c) In parameter set 3 green (dashes) lines are shows the behavior of white walls while the other lines represents the behavior of transparent walls. (d) In parameters set 4 green (dashes) lines are shows the behavior of white wall while the other walls shows the behavior of transparent wall respectively, in the Table 4. Decay time constant is assumed to be $\tau = 0.5$.

Table 4 (Fig. 13d).

The singular domains L_e and L_i are shows the behavior of walls for the parameter sets 1-4 in Table 5:

- The singular domains L_e , L_i both are shows the behavior of transparent walls for the parameter set 1 in Table 5 (Fig. 14a).

- The singular domain L_e^0 is white wall with green (dashes) color, while the singular domain L_e^1, L_i^1 and L_i^0 are transparent walls for the parameter set 2 in Table 5

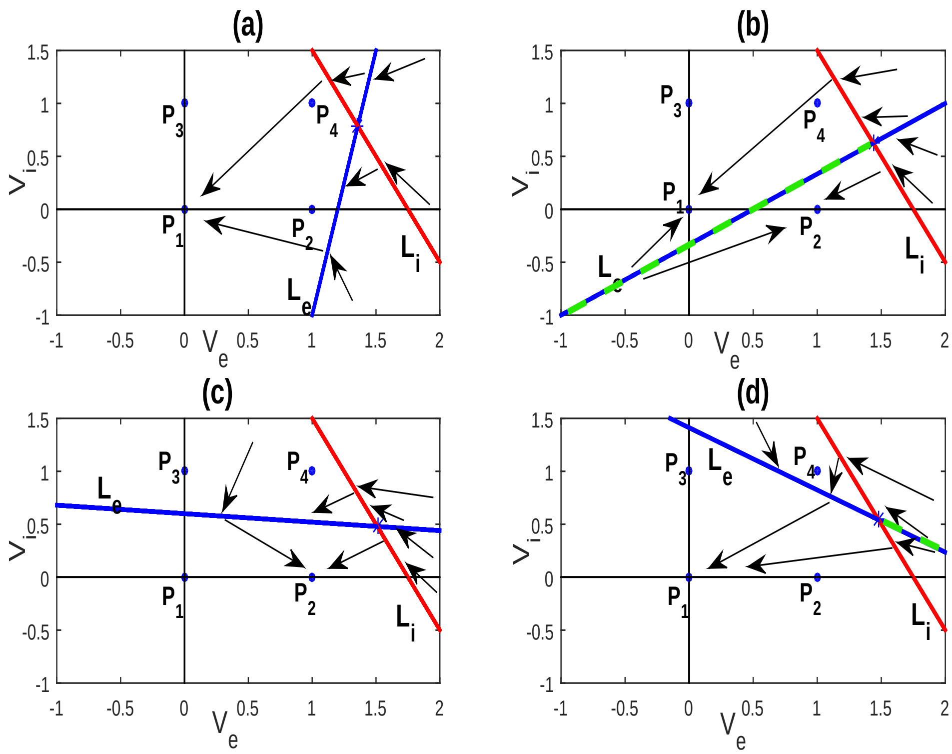

Figure 14: Singular domains L_i (red) and L_e (blue) represents the behavior of white, transparent and black for the parameter sets given in Table 5. Green and black (dashes) lines are shows the behavior of White and black walls respectively. In each regular domain the dashes arrow are represents the integral curve. (a) In parameter set 1 all the lines represents the behavior of transparent walls. (b) In parameter set 2 green (dashes) lines are shows the behavior of white while the other lines represents the behavior of transparent walls. (c) In parameter set 3 all the lines are shows the behavior of transparent walls.. (d) In parameters set 4 green (dashes) lines are shows the behavior of white wall while the other walls shows the behavior of transparent wall respectively, in the Table 5. Decay time constant is assumed to be $\tau = 0.5$.

(Fig. 14b).

- Both of the singular domains L_e and L_i are transparent walls for the parameter set 3 in Table 5 (Fig. 14c).

- The singular domain L_e^1 is white wall with green (dashes) color, while the singular domain L_e^0, L_i^1 and L_i^0 are transparent walls for the parameter set 4 in Table 5 (Fig. 14d).

Parameters	ω_{ee}	ω_{ei}	ω_{ie}	ω_{ii}	θ_e	θ_i
Set 1	0.5	0.1	-0.4	0.2	0.6	-0.7
Set 2	0.4	0.6	-0.4	0.2	0.2	-0.7
Set 3	0.04	-0.5	-0.4	0.2	0.3	-0.7
Set 4	0.1	-0.17	-0.4	0.2	0.24	-0.7

Table 5: Singular domains in Parameter sets for Figure 14 shows the different behaviors. In set 1 parameters are used to shows the behavior of transparent walls. In set 2 parameters are used to shows the behavior of white and transparent walls respectively. In set 3 parameters are used to shows the behavior of transparent walls. In set 4 parameters generate the behavior of white and transparent walls at the same time.

Parameters	ω_{ee}	ω_{ei}	ω_{ie}	ω_{ii}	θ_e	θ_i
Set 1	0.1	-0.08	0.08	0.04	0.2	0.1
Set 2	0.1	-0.08	0.06	0.05	0.2	0.04
Set 3	0.1	-0.08	0.1	0.4	0.2	-0.2
Set 4	0.7	-0.09	-0.4	0.4	0.4	-0.5

Table 6: Singular domains in Parameter sets for Figure 15 shows the different behaviors. In set 1 parameters are used to shows the behavior of transparent walls. In set 2 parameters are used to shows the behavior of white and transparent walls respectively. In set 3 parameters are used to shows the behavior of transparent walls. In set 4 parameters generate the behavior of transparent and white walls at the same time.

The singular domains L_e and L_i are shows the behavior of walls for the parameter sets 1-4 in Table 6:

- The singular domains L_e , L_i are shows the behavior of transparent walls for the parameter set 1 in Table 6 (Fig. 15a).

- The singular domain L_i^0 is white wall with green (dashes) color, while the singular domains L_i^1, L_e^0 and L_e^1 are transparent walls for the parameter set 2 in Table 6 (Fig. 15b).

- The singular domains L_e , L_i are shows the behavior of transparent walls for the parameter set 3 in Table 6 (Fig. 15c).

- The singular domain L_i^1 is white wall with green (dashes) color, while the singular domains L_i^0, L_e^0 and L_e^1 are transparent walls for the parameter set 4 in Table 6 (Fig. 15d).

Following tables shows the behavior of all the above figures.

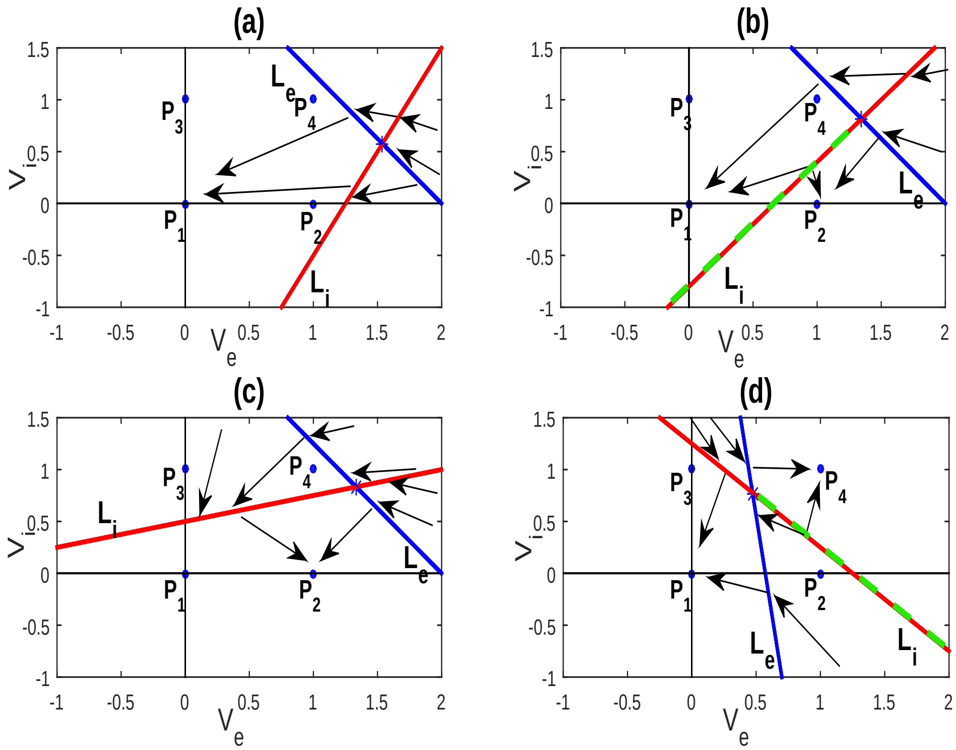

Figure 15: Singular domains L_i (red) and L_e (blue) represents the behavior of white, transparent and black for the parameter sets given in Table 6. Green and black (dashes) lines are shows the behavior of White and black walls respectively. In each regular domain the dashes arrow are represents the integral curve. (a) In parameter set 1 all the lines are shows the behavior of transparent walls. (b) In parameter set 2 green (dashes) lines are shows the behavior of white while the other lines represents the behavior of transparent walls. (c) In parameter set 3 all the lines are shows the behavior of transparent walls.. (d) In parameters set 4 green (dashes) lines are shows the behavior of white wall while the other walls shows the behavior of transparent wall respectively, in the Table 6. Decay time constant is assumed to be $\tau = 0.5$.

Parameters	a	b	c	d
L_e^0	Transparent	White	Transparent	Transparent
L_e^1	Transparent	Transparent	Transparent	White
L_i^0	Transparent	Transparent	Transparent	Transparent
L_i^1	Transparent	Transparent	Transparent	Transparent

Parameters	a	b	c	d
L_e^0	Transparent	Transparent	White	Transparent
L_e^1	White	White	Transparent	Transparent
L_i^0	Transparent	Transparent	Black	Black
L_i^1	Black	Black	Transparent	Transparent

Parameters	a	b	c	d
L_e^0	Transparent	Transparent	Transparent	Transparent
L_e^1	Transparent	White	Transparent	Transparent
L_i^0	Black	Black	Black	Black
L_i^1	Black	Black	Black	Black

Parameters	a	b	c	d
L_e^0	Transparent	White	Transparent	Transparent
L_e^1	Transparent	Transparent	Transparent	White
L_i^0	Transparent	Transparent	Transparent	Transparent
L_i^1	White	Black	White	White

Parameters	a	b	c	d
L_e^0	Transparent	White	Transparent	Transparent
L_e^1	Transparent	Transparent	Transparent	White
L_i^0	Transparent	Transparent	Transparent	Transparent
L_i^1	Transparent	Transparent	Transparent	Transparent

Parameters	a	b	c	d
L_e^0	Transparent	Transparent	Transparent	Transparent
L_e^1	Transparent	Transparent	Transparent	Transparent
L_i^0	Transparent	White	Transparent	Transparent
L_i^1	Transparent	Transparent	Transparent	White

3.3 Wellposedness

With reference if $q > 0$ (steepness parameter) , the function of the vector $\underline{\mathbf{F}}$ is continuously differentiable. Therefore the value of the initial problem for (26) is locally well posed. It will be enough to prove that the vector valued function $\underline{\mathbf{F}}$ is bounded by some linear function, whereas, for the global wellposedness of the problem (26). $\underline{\mathbf{F}}$ is the Hill function for the nonlinear part of the vector function, that is bounded for the initial positive steepness q. Matrix $\mathbf{D}$ have non-negative eigenvalues ($\lambda_1 = -1 < 0$, $\lambda_2 = -\frac{1}{\tau} < 0$) and the nonlinear

part of $\underline{\mathbf{F}}$ is bounded for a finite positive q .
The parameters involved satisfy the following conditions:
The model (26) in the vector form as:

$$\underline{\mathbf{V}}' = \underline{\mathbf{F}}(\underline{\mathbf{V}}), \quad \underline{\mathbf{F}}(\underline{\mathbf{V}}) = -\mathbf{D}\,\underline{\mathbf{V}} + \mathbf{D}\,\underline{\mathbf{G}}(\Psi\underline{\mathbf{V}}), \tag{35}$$

where

$$\mathbf{D} = \begin{bmatrix} 1 & 0 \\ 0 & \tau^{-1} \end{bmatrix}, \ \underline{\mathbf{V}} = \begin{bmatrix} v_e \\ v_i \end{bmatrix}; \ \Psi = \begin{bmatrix} \omega_{ee} & -\omega_{ei} \\ \omega_{ie} & -\omega_{ii} \end{bmatrix} \tag{36}$$

and

$$\underline{\mathbf{G}}(\Psi\underline{\mathbf{V}}) = \begin{bmatrix} Z_e(\omega_{ee}v_e - \omega_{ei}v_i) \\ Z_i(\omega_{ie}v_e - \omega_{ii}v_i). \end{bmatrix} \tag{37}$$

We get,

$$|\mathbf{D}\underline{\mathbf{G}}(\Psi\underline{\mathbf{V}})| \leq ||\mathbf{D}|| \cdot |\underline{\mathbf{G}}(\Psi\underline{\mathbf{V}})| \leq \ c_1, \tag{38}$$

where c_1 is some positive constant and $||.||$ is the matrix norm. Also the function $\underline{\mathbf{F}}$ is bounded.

$$|\mathbf{D}\,\underline{\mathbf{V}}| \ \leq ||\mathbf{D}|| \cdot |\underline{\mathbf{V}}| \leq \ c_2 \cdot |\underline{\mathbf{V}}|, \tag{39}$$

where c_2 is positive constant. By (equ 38) and (equ 39) we get,

$$|\underline{\mathbf{F}}(\underline{\mathbf{V}})| \leq c_1 + c_2 \cdot |\underline{\mathbf{V}}|, \tag{40}$$

This clearly indicates that the vector valued function $\underline{\mathbf{F}}$ is bounded by some linear function $c_1 + c_2 \cdot \underline{\mathbf{V}}$. This shows that our initial value problem (26) is wellposed globally as well.

Chapter 3
Net input variables model

4 Net input variables model

In order to find the general formula for stationary points, we will transform the non delay part of dynamical system(i.e (26) to a new set of variables x and y through following transformation.

$$x = \omega_{ee} v_e - \omega_{ei} v_i \tag{41}$$

$$y = \omega_{ie} v_e - \omega_{ii} v_i \tag{42}$$

Its matrix form is

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \omega_{ee} & -\omega_{ei} \\ \omega_{ie} & -\omega_{ii} \end{bmatrix} \begin{bmatrix} v_e \\ v_i \end{bmatrix} \tag{43}$$

which is

$$\mathbf{X} = \Psi\, \mathbf{V} \tag{44}$$

here X, Ψ and V are

$$\Psi = \begin{bmatrix} \omega_{ee} & -\omega_{ei} \\ \omega_{ie} & -\omega_{ii} \end{bmatrix} \tag{45}$$

$$\mathbf{X} = \begin{bmatrix} x \\ y \end{bmatrix} \tag{46}$$

$$\mathbf{V} = \begin{bmatrix} v_e \\ v_i \end{bmatrix} \tag{47}$$

As non delay part of system (26) is

$$v_e' = -v_e + Z_e(\omega_{ee} v_e - \omega_{ei} v_i) \tag{48a}$$

$$\tau v_i' = -v_i + Z_i(\omega_{ie} v_e - \omega_{ii} v_i) \tag{48b}$$

converting into matrix form

$$\begin{bmatrix} v_e' \\ v_i' \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -\tau^{-1} \end{bmatrix} \begin{bmatrix} v_e \\ v_i \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & \tau^{-1} \end{bmatrix} \begin{bmatrix} Z_e \\ Z_i \end{bmatrix} \tag{49}$$

$$\begin{bmatrix} v_e' \\ v_i' \end{bmatrix} = - \begin{bmatrix} 1 & 0 \\ 0 & \tau^{-1} \end{bmatrix} \begin{bmatrix} v_e \\ v_i \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & \tau^{-1} \end{bmatrix} \begin{bmatrix} Z_e \\ Z_i \end{bmatrix} \tag{50}$$

$$\mathbf{V}' = -\mathbf{C}\,\mathbf{V} + \mathbf{C}\,\mathbf{G}(\mathbf{X}) \tag{51}$$

where

$$\mathbf{V}' = \begin{bmatrix} v_e' \\ v_i' \end{bmatrix} \tag{52}$$

$$\mathbf{C} = \begin{bmatrix} 1 & 0 \\ 0 & \tau^{-1} \end{bmatrix} \tag{53}$$

$$\mathbf{G(x)} = \begin{bmatrix} Z_e \\ Z_i \end{bmatrix} \tag{54}$$

$$\mathbf{V} = \begin{bmatrix} v_e \\ v_i \end{bmatrix} \tag{55}$$

Taking derivative of equation (44)

$$\mathbf{X}' = \Psi\, \mathbf{V}' \tag{56}$$

using value of $\mathbf{V}'$ from (51)

$$\mathbf{X}' = \Psi\, (-\mathbf{C}\, \mathbf{V} + \mathbf{C}\, \mathbf{G}(\mathbf{X})) \tag{57}$$

As by assumption Ψ is invertible, we get

$$\mathbf{X}' = \Psi\, (-\mathbf{C}\, \Psi^{-1}\mathbf{X} + \mathbf{C}\, \mathbf{G}(\mathbf{X})) \tag{58}$$

or

$$\mathbf{X}' = -\Psi\mathbf{C}\Psi^{-1}\mathbf{X} + \Psi\mathbf{C}\mathbf{G}(\mathbf{X}) \tag{59}$$

or

$$\mathbf{X}' = \mathbf{A} \cdot \mathbf{X} + \mathbf{B}(\mathbf{X}) \tag{60}$$

where

$$\mathbf{A} = -\Psi\mathbf{C}\, \Psi^{-1} \tag{61}$$

$$\mathbf{B}(\mathbf{X}) = \Psi\mathbf{C}\mathbf{G}(\mathbf{X}) \tag{62}$$

Since

$$\mathbf{A} = -\Psi\mathbf{C}\, \Psi^{-1} \tag{63}$$

And

$$\Psi = \begin{bmatrix} \omega_{ee} & -\omega_{ei} \\ \omega_{ie} & -\omega_{ii} \end{bmatrix} \tag{64}$$

$$\mathbf{C} = \begin{bmatrix} 1 & 0 \\ 0 & \tau^{-1} \end{bmatrix} \tag{65}$$

So

$$\Psi\mathbf{C} = \begin{bmatrix} \omega_{ee} & -\omega_{ei} \\ \omega_{ie} & -\omega_{ii} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & \tau^{-1} \end{bmatrix} \tag{66}$$

$$\Psi\mathbf{C} = \begin{bmatrix} \omega_{ee} & -\tau^{-1}\omega_{ei} \\ \omega_{ie} & -\tau^{-1}\omega_{ii} \end{bmatrix} \tag{67}$$

$$-\Psi\mathbf{C} = \begin{bmatrix} -\omega_{ee} & \tau^{-1}\omega_{ei} \\ -\omega_{ie} & \tau^{-1}\omega_{ii} \end{bmatrix} \tag{68}$$

As

$$\Psi^{-1} = \frac{Adj\Omega}{det\Omega} \tag{69}$$

And

$$Adj(\Psi) = \begin{bmatrix} -\omega_{ii} & \omega_{ei} \\ -\omega_{ie} & \omega_{ee} \end{bmatrix} \tag{70}$$

$$det(\Psi) = -\omega_{ee}\omega_{ii} + \omega_{ei}\omega_{ie} \tag{71}$$

Using these values in Eq.(63)

$$\mathbf{A} = -\Psi\mathbf{C}\,\Psi^{-1} \tag{72}$$

$$\mathbf{A} = \frac{1}{-\omega_{ee}\omega_{ii} + \omega_{ei}\omega_{ie}} \begin{bmatrix} -\omega_{ee} & \tau^{-1}\omega_{ei} \\ -\omega_{ie} & \tau^{-1}\omega_{ii} \end{bmatrix} \begin{bmatrix} -\omega_{ii} & \omega_{ei} \\ -\omega_{ie} & \omega_{ee} \end{bmatrix} \tag{73}$$

or

$$\mathbf{A} = \frac{1}{d} \begin{bmatrix} \omega_{ee}\omega_{ii} - \tau^{-1}\omega_{ei}\omega_{ie} & -\omega_{ee}\omega_{ei} + \tau^{-1}\omega_{ee}\omega_{ei} \\ \omega_{ii}\omega_{ie} - \tau^{-1}\omega_{ii}\omega_{ie} & -\omega_{ei}\omega_{ie} + \tau^{-1}\omega_{ee}\omega_{ii} \end{bmatrix} \tag{74}$$

Where the parameter **d** is

$$d = \omega_{ei}\omega_{ie} - \omega_{ee}\omega_{ii} \tag{75}$$

or

$$\mathbf{A} = \begin{bmatrix} \frac{1}{\tau d}\left(\tau\omega_{ee}\omega_{ii} - \omega_{ei}\omega_{ie}\right) & \frac{1}{\tau d}\,\omega_{ee}\omega_{ei}(1-\tau) \\ \frac{1}{\tau d}\,\omega_{ii}\omega_{ie}(\tau-1) & \frac{1}{\tau d}\left(-\tau\omega_{ei}\omega_{ie} + \omega_{ee}\omega_{ii}\right) \end{bmatrix} \tag{76}$$

or

$$\mathbf{A} = \begin{bmatrix} \alpha_1 & \beta_1 \\ \alpha_2 & \beta_2 \end{bmatrix} \tag{77}$$

where the coefficients α_1, α_2, β_1 andβ_2 are given as

$$\alpha_1 = \frac{1}{\tau d}\left(\tau\omega_{ee}\omega_{ii} - \omega_{ei}\omega_{ie}\right) \tag{78a}$$

$$\beta_1 = \frac{1}{\tau d}\,\omega_{ee}\omega_{ei}(1-\tau) \tag{79a}$$

$$\alpha_2 = \frac{1}{\tau d}\,\omega_{ii}\omega_{ie}(\tau-1) \tag{80a}$$

$$\beta_2 = \frac{1}{\tau d}\left(-\tau\omega_{ei}\omega_{ie} + \omega_{ee}\omega_{ii}\right) \tag{81a}$$

Now considering Eq.(62). As ΨC is already calculated given in (67) then

$$\mathbf{B}(\mathbf{x}) = \Psi\mathbf{CG(x)} = \begin{bmatrix} \omega_{ee} & -\tau^{-1}\omega_{ei} \\ \omega_{ie} & -\tau^{-1}\omega_{ii} \end{bmatrix} \begin{bmatrix} Z_e(x) \\ Z_i(y) \end{bmatrix} \tag{82}$$

or

$$\mathbf{B}(\mathbf{x}) = \begin{bmatrix} \omega_{ee}Z_e(x) - \tau^{-1}\omega_{ei}Z_i(y) \\ \omega_{ie}Z_e(x) - \tau^{-1}\omega_{ii}Z_i(y) \end{bmatrix} \tag{83}$$

Using these values of Eq.(83) and Eq.(77) in Eq.(60), we get

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \alpha_1 & \beta_1 \\ \alpha_2 & \beta_2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} \omega_{ee}Z_e(x) - \tau^{-1}\omega_{ei}Z_i(y) \\ \omega_{ie}Z_e(x) - \tau^{-1}\omega_{ii}Z_i(y) \end{bmatrix} \tag{84}$$

or

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \alpha_1 x + \beta_1 y \\ \alpha_2 x + \beta_2 y \end{bmatrix} + \begin{bmatrix} \omega_{ee}Z_e(x) - \tau^{-1}\omega_{ei}Z_i(y) \\ \omega_{ie}Z_e(x) - \tau^{-1}\omega_{ii}Z_i(y) \end{bmatrix} \tag{85}$$

From here **x** and **y** components are

$$x' = \alpha_1\, x\; + \beta_1\, y + \omega_{ee}\, Z_e(x) - \frac{1}{\tau}\,\omega_{ei}\, Z_i(y) \tag{86a}$$

$$y' = \alpha_2\, x\; + \beta_2\, y + \omega_{ie}\, Z_e(x) - \frac{1}{\tau}\,\omega_{ii}\, Z_i(y) \tag{86b}$$

For any activation function Z_m for $m = (e,i)$ one can show that the system (86) possess the stationary point (x_0, y_0) given by putting $x' = 0, y' = 0$ in (Equ 86) then we have the following equations become ,

$$0 = \alpha_1\, x_0\; + \beta_1\, y_0 + \omega_{ee}\, Z_e(x) - \frac{1}{\tau}\,\omega_{ei}\, Z_i(y) \tag{87a}$$

$$0 = \alpha_2\, x_0\; + \beta_2\, y_0 + \omega_{ie}\, Z_e(x) - \frac{1}{\tau}\,\omega_{ii}\, Z_i(y) \tag{87b}$$

Now multiply $(equ.87(a))$ by β_2 and $(equ.87(b))$ by β_1 then we have ,

$$0 = \alpha_1\,\beta_2\, x_0\; + \beta_1\,\beta_2\, y_0 + \omega_{ee}\,\beta_2\, Z_e(x) - \frac{1}{\tau}\,\omega_{ei}\,\beta_2\, Z_i(y)$$

$$0 = \alpha_2\,\beta_1\, x_0\; + \beta_1\,\beta_2\, y_0 + \omega_{ie}\,\beta_1\, Z_e(x) - \frac{1}{\tau}\,\omega_{ii}\,\beta_1\, Z_i(y)$$

Now by subtracting these equations we get

$$0 = \alpha_1\,\beta_2\, x_0\; + \omega_{ee}\,\beta_2\, Z_e(x) - \frac{1}{\tau}\,\omega_{ei}\,\beta_2\, Z_i(y)\; - \left(\alpha_2\,\beta_1\, x_0 + \omega_{ie}\,\beta_1\, Z_e(x) - \frac{1}{\tau}\,\omega_{ii}\,\beta_1\, Z_i(y)\right)$$

$$0=\alpha_1\,\beta_2\,x_0\;+\omega_{ee}\,\beta_2\,Z_e(x)-\frac{1}{\tau}\,\omega_{ei}\,\beta_2\,Z_i(y)\;-\alpha_2\,\beta_1\,x_0-\omega_{ie}\,\beta_1\,Z_e(x)+\frac{1}{\tau}\,\omega_{ii}\,\beta_1\,Z_i(y))$$

$$0=(\alpha_1\,\beta_2\;-\alpha_2\,\beta_1)\,x_0\;-(-\omega_{ee}\,\beta_2\;+\omega_{ie}\,\beta_1)\,Z_e(x)-\frac{1}{\tau}\,(-\omega_{ii}\,\beta_1\;+\;\omega_{ei}\,\beta_2)\,Z_i(y)$$

$$(\alpha_1\,\beta_2\;-\alpha_2\,\beta_1)\,x_0\;=(\omega_{ie}\,\beta_1-\omega_{ee}\,\beta_2\;)\,Z_e(x)+\frac{1}{\tau}\,(\omega_{ei}\,\beta_2-\omega_{ii}\,\beta_1\;)\,Z_i(y)$$

So we get the equation of x_0 is

$$x_0=\frac{1}{\alpha_1\beta_2-\alpha_2\beta_1}\;\;[(\omega_{ie}\,\beta_1-\omega_{ee}\,\beta_2\;)Z_e(x)+\frac{1}{\tau}\,(\omega_{ei}\,\beta_2-\omega_{ii}\,\beta_1)Z_i(y)] \tag{89}$$

Now for y_0 we multiply $(equ.87(a))$ by α_2 and $(equ.87(b))$ by α_1 we have ,

$$0=\alpha_1\alpha_2\,x_0\;+\beta_1\alpha_2\,y_0+\omega_{ee}\alpha_2\,Z_e(x)-\frac{1}{\tau}\,\omega_{ei}\alpha_2\,Z_i(y)$$

$$0=\alpha_2\alpha_1\,x_0\;+\beta_2\alpha_1\,y_0+\omega_{ie}\alpha_1\,Z_e(x)-\frac{1}{\tau}\,\omega_{ii}\alpha_1\,Z_i(y)$$

By subtracting these equations, we have

$$0=\beta_1\alpha_2\,y_0+\omega_{ee}\alpha_2\,Z_e(x)-\frac{1}{\tau}\,\omega_{ei}\alpha_2\,Z_i(y)-(\beta_2\alpha_1\,y_0+\omega_{ie}\alpha_1\,Z_e(x)-\frac{1}{\tau}\,\omega_{ii}\alpha_1\,Z_i(y))$$

$$0=\beta_1\alpha_2\,y_0+\omega_{ee}\alpha_2\,Z_e(x)-\frac{1}{\tau}\,\omega_{ei}\alpha_2\,Z_i(y)-\beta_2\alpha_1\,y_0-\omega_{ie}\alpha_1\,Z_e(x)+\frac{1}{\tau}\,\omega_{ii}\alpha_1\,Z_i(y)$$

$$0=\beta_1\alpha_2\,y_0-\beta_2\alpha_1\,y_0+\omega_{ee}\alpha_2\,Z_e(x)-\omega_{ie}\alpha_1\,Z_e(x)+\frac{1}{\tau}\,\omega_{ii}\alpha_1\,Z_i(y)-\frac{1}{\tau}\,\omega_{ei}\alpha_2\,Z_i(y)$$

$$0=(\beta_1\alpha_2\;-\beta_2\alpha_1\;)y_0+(\omega_{ee}\alpha_2-\omega_{ie}\alpha_1)\,Z_e(x)+\frac{1}{\tau}\,(\omega_{ii}\alpha_1-\;\omega_{ei}\alpha_2)\,Z_i(y)$$

$$(\alpha_1\beta_2\;-\alpha_2\beta_1\;)y_0=(\omega_{ee}\alpha_2-\omega_{ie}\alpha_1)\,Z_e(x)+\frac{1}{\tau}\,(\omega_{ii}\alpha_1-\;\omega_{ei}\alpha_2)\,Z_i(y)$$

So we get the equation of y_0

$$y_0=\frac{1}{\alpha_1\beta_2\;-\alpha_2\beta_1}\;\;[(\omega_{ee}\alpha_2-\omega_{ie}\alpha_1)\,Z_e(x)+\frac{1}{\tau}\,(\omega_{ii}\alpha_1-\;\omega_{ei}\alpha_2)\,Z_i(y)] \tag{90}$$

We know that the regular domain Ψ_1 which is

$$\Psi_1=\{(v_e,v_i)\mid\;\omega_{ee}v_e-\omega_{ei}v_i\;<\theta_e\;,\;\;\omega_{ie}v_e-\omega_{ii}v_i\;<\theta_i\}, \tag{91}$$

This equation show that, the focal point for regular domain Ψ_1 is $(0,0)$ because at this point the value of $Z_e(x)$ and $Z_i(y)$ are zero. So point is,

$$P_1(x_1,y_1)=(0,0), \tag{92}$$

Here (x_1,y_1) is the stationary points. Now the regular domain Ψ_2 is,

$$\Psi_2=\{(v_e,v_i)\mid\;\omega_{ee}v_e-\omega_{ei}v_i\;>\theta_e\;,\;\;\omega_{ie}v_e-\omega_{ii}v_i\;<\theta_i\}, \tag{93}$$

This equation shows, the focal point for regular domain Ψ_2 is $(1,0)$ because at this point the value of $Z_e(x)$ is one and the value of $Z_i(y)$ is zero.So the equations (82) and (90) gives the stationary points (x_2, y_2). which is ,

$$x_2 = \frac{1}{\alpha_1\beta_2 - \alpha_2\beta_1} \ (\omega_{ie}\,\beta_1 - \omega_{ee}\,\beta_2\,)$$

$$y_2 = \frac{1}{\alpha_1\beta_2 \ - \alpha_2\beta_1} \ (\omega_{ee}\alpha_2 - \omega_{ie}\alpha_1)$$

The general formula for that focal point of that system is given as,

$$P_2(x_2, y_2) = ((\omega_{ie}\beta_1 - \omega_{ee}\beta_2)/c,\ (\omega_{ee}\alpha_2 - \omega_{ie}\alpha_1)/c), \tag{94}$$

Here $c = \alpha_1\beta_2 \ - \alpha_2\beta_1$. Now the regular domain Ψ_3 is

$$\Psi_3 = \{(v_e, v_i) \mid \ \omega_{ee}v_e - \omega_{ei}v_i \ < \theta_e\ ,\ \ \omega_{ie}v_e - \omega_{ii}v_i \ > \theta_i\}, \tag{95}$$

This equation shows, the focal point for regular domain Ψ_3 is $(0,1)$ because at this point the value of $Z_e(x)$ is zero and the value of $Z_i(y)$ is one.So the equations (82) and (90) gives the stationary points (x_3, y_3). which is ,

$$x_3 = \frac{1}{\alpha_1\beta_2 - \alpha_2\beta_1} \ [\frac{1}{\tau}\,(\omega_{ei}\,\beta_2 - \omega_{ii}\,\beta_1)] \tag{96}$$

$$y_3 = \frac{1}{\alpha_1\beta_2 \ - \alpha_2\beta_1} \ [\frac{1}{\tau}\,(\ \omega_{ee}\alpha_2 - \omega_{ie}\alpha_1)] \tag{97}$$

The general formula for that focal point of that system is given as,

$$P_3(x_3, y_3) = ((\omega_{ei}\beta_2 - \omega_{ii}\beta_1)/c\tau,\ (\omega_{ee}\alpha_2 - \omega_{ie}\alpha_1)/c\tau), \tag{98}$$

Here $c = \alpha_1\beta_2 \ - \alpha_2\beta_1$. Now the regular domain Ψ_4 is

$$\Psi_4 = \{(v_e, v_i) \mid \ \omega_{ee}v_e - \omega_{ei}v_i \ > \theta_e\ ,\ \ \omega_{ie}v_e - \omega_{ii}v_i \ > \theta_i\}, \tag{99}$$

This equation shows, the focal point for regular domain Ψ_4 is $(1,1)$ because at this point the value of $Z_e(x)$ is one and the value of $Z_i(y)$ is also one.So the equations (82) and (90) gives the stationary points (x_4, y_4). which is ,

$$x_4 = \frac{1}{\alpha_1\beta_2 - \alpha_2\beta_1} \ [(\omega_{ie}\,\beta_1 - \omega_{ee}\,\beta_2\,) + \frac{1}{\tau}\,(\omega_{ei}\,\beta_2 - \omega_{ii}\,\beta_1)] \tag{100}$$

$$y_4 = \frac{1}{\alpha_1\beta_2 \ - \alpha_2\beta_1} \ [(\omega_{ee}\alpha_2 - \omega_{ie}\alpha_1)\ \frac{1}{\tau}\,(\omega_{ii}\alpha_1 - \ \omega_{ei}\alpha_2)\,] \tag{101}$$

The general formula for that focal point of the system is given as,

$$P_4(x_4, y_4) = ((x_2 + x_3,\ y_2 + y_3), \tag{102}$$

$$\alpha_1 = \frac{-\omega_{ei}\omega_{ie} + \tau\omega_{ee}\omega_{ii}}{\tau d}, \;\; \beta_1 = \frac{\omega_{ee}\omega_{ei}(1-\tau)}{\tau d}, \tag{103a}$$

$$\alpha_2 = \frac{\omega_{ie}\omega_{ii}(\tau-1)}{\tau d}, \quad \beta_2 = \frac{-\tau\omega_{ei}\omega_{ie} + \omega_{ee}\omega_{ii}}{\tau d}, \tag{103b}$$

$$d = \omega_{ei}\omega_{ie} - \omega_{ee}\omega_{ii}. \tag{103c}$$

The general formulae for the focal points of the system 86 are given as

$$P_1(x_1, y_1) = (0, 0) \tag{104a}$$

$$P_2(x_2, y_2) = ((\omega_{ie}\beta_1 - \omega_{ee}\beta_2)/c, \; (\omega_{ee}\alpha_2 - \omega_{ie}\alpha_1)/c), \tag{104b}$$

$$P_3(x_3, y_3) = ((\omega_{ei}\beta_2 - \omega_{ii}\beta_1)/c\tau, \; (\omega_{ee}\alpha_1 - \omega_{ie}\alpha_2)/c\tau), \tag{104c}$$

$$P_4(x_4, y_4) = ((x_2 + x_3, \; y_2 + y_3), \tag{104d}$$

where $c = \alpha_1\beta_2 - \alpha_2\beta_1$.

Parameters	ω_{ee}	ω_{ei}	ω_{ie}	ω_{ii}	θ_e	θ_i
Set 1	0.3	0.5	0.06	0.05	0.1	0.04
Set 2	0.1	−0.3	0.06	0.05	0.2	0.04
Set 3	0.6	0.2	0.4	1	0.2	−0.2
Set 4	0.1	−0.08	0.08	0.04	0.2	0.1

Table 7: To show various behaviors of singular domains, here we have sets of parameter are given. The parameter in Fig 16, is used to shows the behavior of white and black walls. The Fig 17 produce transparent and black walls. The Fig 18 produces black, white and transparent walls. The Fig 19 show the behavior of transparent walls.

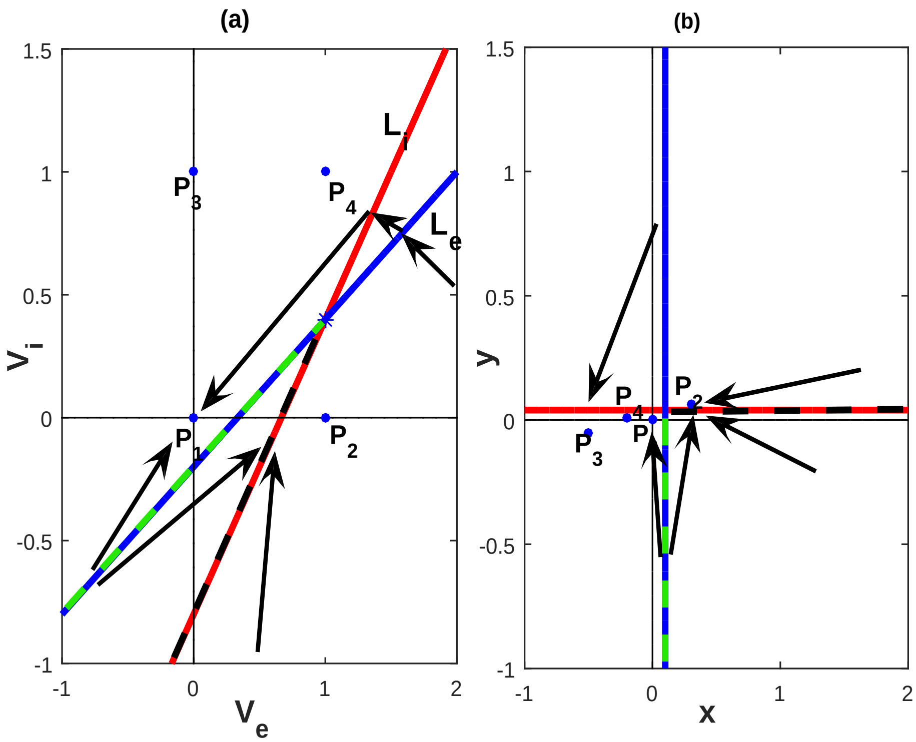

Figure 16: Singular domains L_i (red) and L_e (blue) are represents the behavior of transparent, black and white for the parameter sets given in Table 7. Black and green colors (dashed) lines are shows the behavior of black and white walls respectively. In each regular domain the dashes arrows are represented the integral curve.Now we can convert this integral curve into new variables "x" and "y" .In which represent the white and black wall in the new variables for parameters of set 1 in Table 7. The decay time constant is assumed to be $\tau = 0.5$.

The corresponding focal points in (Fig. 16) are given as ,

$$P_1 = (0,0) \quad P_2 = (-0.3,-0.06) \;\; , \;\; P_3 = (0.5,0.89) \;\; , \;\; P_4 = (0.2,0.83) \tag{105}$$

The parameters $\alpha_1, \beta_1, \alpha_2$ and β_2

$$\alpha_1 = 3, \;\; \beta_1 = -10, \;\; \alpha_2 = 0.2, \;\; \beta_2 = 0 \tag{106}$$

The focal points corresponding to (Fig. 17) are given as ,

$$P_1 = (0,0) \quad P_2 = (-0.007,0.051) \;\; , \;\; P_3 = (0.5,-0.188) \;\; , \;\; P_4 = (0.044,-0.066) \tag{107}$$

The parameters $\alpha_1, \beta_1, \alpha_2$ and β_2

$$\alpha_1 = -1.783, \;\; \beta_1 = -1.304, \;\; \alpha_2 = 0.130, \;\; \beta_2 = -1.217 \tag{108}$$

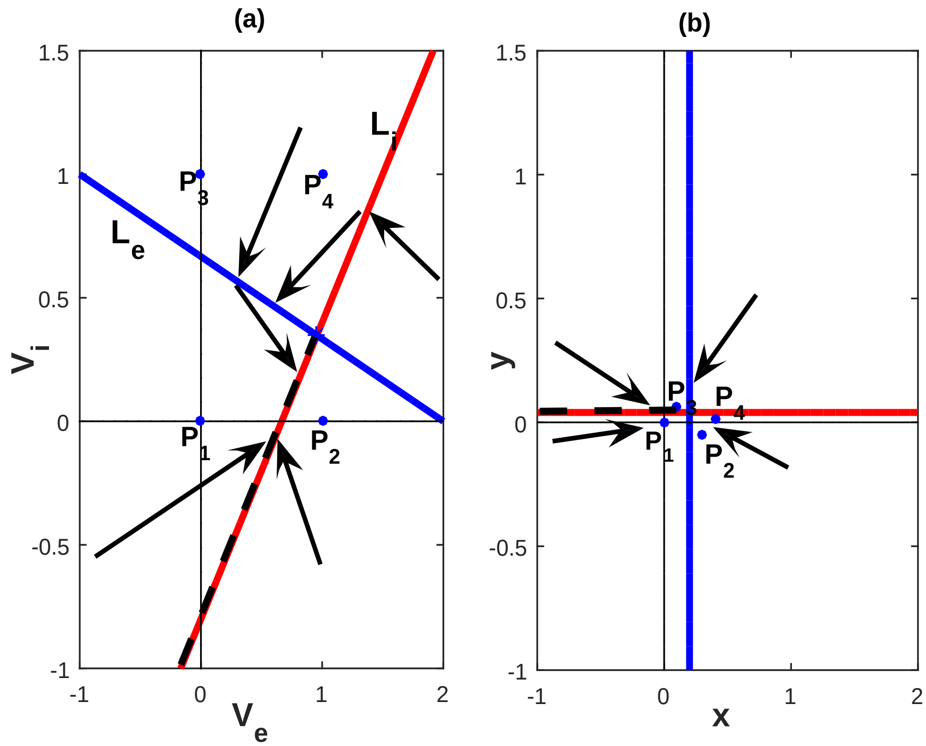

Figure 17: Singular domains L_i (red) and L_e (blue) are represents the behavior of transparent, black and white for the parameter sets given in Table 7. Black and green colors (dashed) lines are shows the behavior of black and white walls respectively. In each regular domain the dashes arrows are represented the integral curve. Now we can convert this integral curve into new variables "x" and "y" .In which represent the black and transparent walls in the new variables for parameters of set 2 in Table 7. The decay time constant is assumed to be $\tau = 0.5$.

The corresponding focal points to (Fig 18) are,

1. $\overline{P}_1(0,0)$ for $\underline{X} \in \overline{\Psi}_1$
2. $\overline{P}_2(0.56, 0.39)$ for $\underline{X} \in \overline{\Psi}_2$
3. $\overline{P}_3(-0.21, -0.98)$ for $\underline{X} \in \overline{\Psi}_3$
4. $\overline{P}_4(0.39, -0.59)$ for $\underline{X} \in \overline{\Psi}_4$

The parameters $\alpha_1, \beta_1, \alpha_2$ and β_2

$$\alpha_1 = -0.85, \ \ \beta_1 = -0.23, \ \ \alpha_2 = 0.77, \ \ \beta_2 = -2.2 \tag{109}$$

The focal points corresponding to (Fig. 19) are given as ,

$$P_1 = (0,0) \quad P_2 = (-0.039, -0.049) \ , \ P_3 = (-0.142, 0.137) \ , \ P_4 = (-0.18, 0.088) \tag{110}$$

The parameters $\alpha_1, \beta_1, \alpha_2$ and β_2

$$\alpha_1 = 1.615, \ \ \beta_1 = 0.769, \ \ \alpha_2 = 0.308, \ \ \beta_2 = 1.385 \tag{111}$$

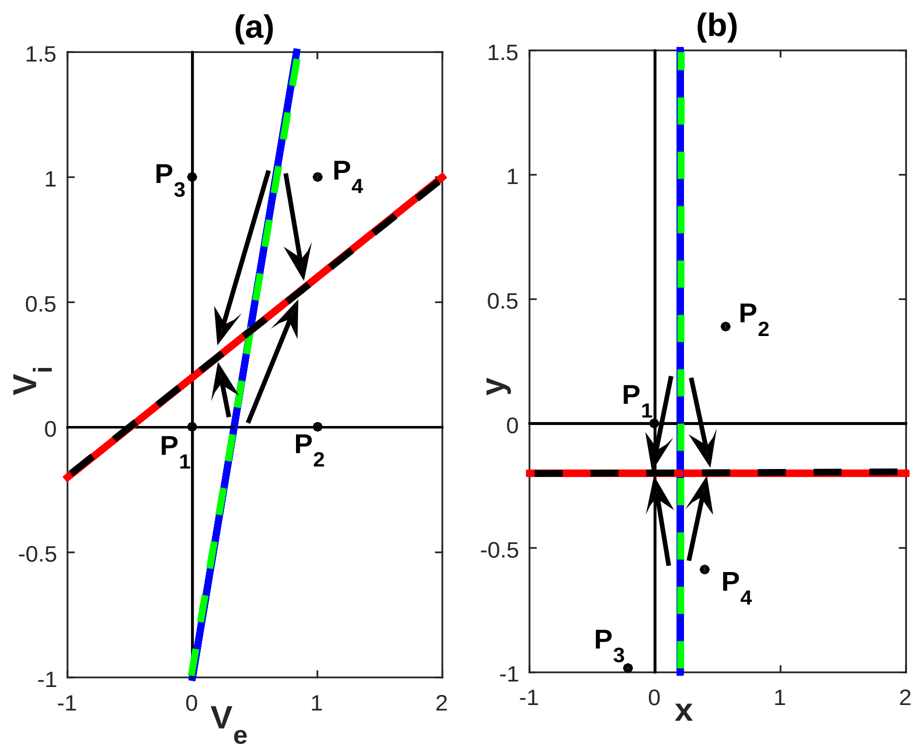

Figure 18: Singular domains L_i (red) and L_e (blue) are represents the behavior of transparent, black and white for the parameter sets given in Table 7. Black and green colors (dashed) lines are shows the behavior of black and white walls respectively. In each regular domain the dashes arrows are represented the integral curve. Now we can convert this integral curve into new variables ”x” and ”y” .In which Fig 18 represent the black, white and transparent walls in the new variables for parameters of set 3 in Table 7. The decay time constant is assumed to be $\tau = 0.5$.

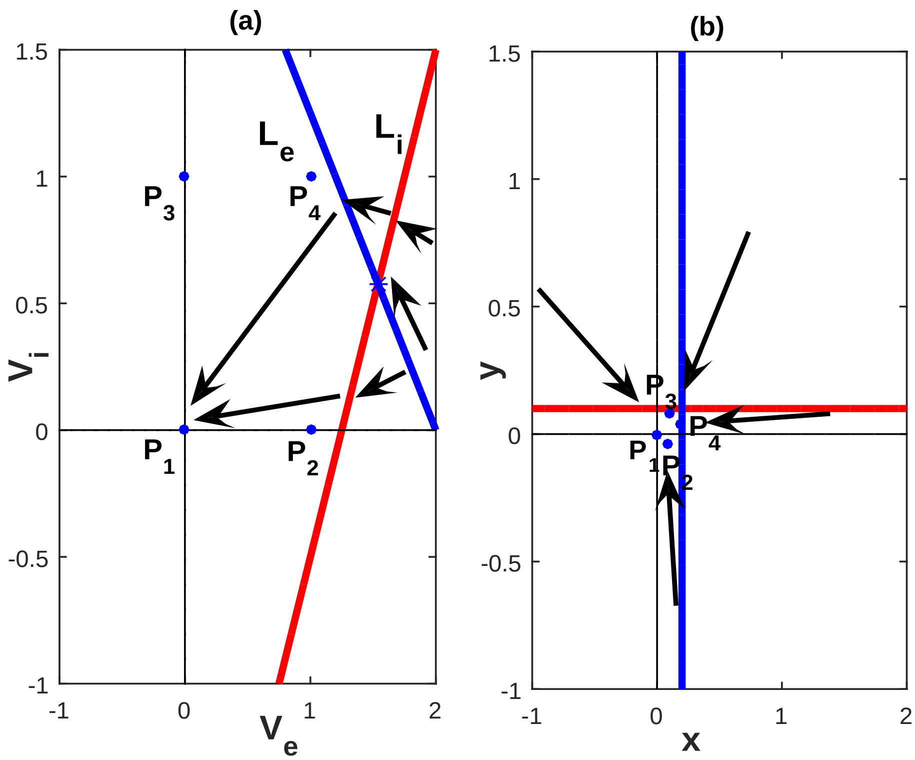

Figure 19: Singular domains L_i (red) and L_e (blue) are represents the behavior of transparent, black and white for the parameter sets given in Table 7. Black and green colors (dashed) lines are shows the behavior of black and white walls respectively. In each regular domain the dashes arrows are represented the integral curve. Now we can convert this integral curve into new variables "x" and "y" .In which Fig 19 represent the transparent wall in the new variables for parameters of set 4 in Table 7. The decay time constant is assumed to be $\tau = 0.5$.

Chapter 4
Dynamics along regular domain and black wall

4.1 Dynamics in the regular domain

The substitution Equ 44 defines a linear transformation

$$L_{\Psi}(\mathbf{V}) = \Psi \cdot \mathbf{V} \tag{112}$$

which maps regular(singular) domains of the **V**-system to the regular(singular) domains of the **X**-system. We introduce the transformed regular domains given in (Equs 29) are ,

$$\overline{\Psi}_1 = L_{\Psi}(\Psi_1) = \{(x,y) \mid x < \theta_e,\ y < \theta_i\} \tag{113a}$$

$$\overline{\Psi}_2 = L_{\Psi}(\Psi_2) = \{(x,y) \mid x > \theta_e,\ y < \theta_i\} \tag{113b}$$

$$\overline{\Psi}_3 = L_{\Psi}(\Psi_3) = \{(x,y) \mid x < \theta_e,\ y > \theta_i\} \tag{113c}$$

$$\overline{\Psi}_4 = L_{\Psi}(\Psi_4) = \{(x,y) \mid x > \theta_e,\ y > \theta_i\} \tag{113d}$$

Moreover, the transformed singular domains are given as

$$\overline{L}_e = L_{\Psi}(L_e) = \{(x,y) \mid x = \theta_e\} \tag{114a}$$

$$\overline{L}_i = L_{\Psi}(L_i) = \{(x,y) \mid y = \theta_i\} \tag{114b}$$

Notice that the eigenvalues of the matrix **A** are given as

$$\lambda_1 = -1\ ,\ \ \lambda_2 = -\tau^{-1} \qquad \tau > 0 \tag{115}$$

Moreover, the matrix **A** is diagonalizable. Eigenvectors $\underline{v}_1, \underline{v}_2$ corresponding to the eigenvalues λ_1 and λ_2 are given as

$$\underline{v}_1 = \begin{bmatrix} -\beta_1 \\ \alpha_1 \end{bmatrix},\ \underline{v}_2 = \begin{bmatrix} -\beta_1 \\ \alpha_2 \end{bmatrix} \tag{116}$$

Here $\beta_1, \alpha_1, \alpha_2$ are constant parameters, which are defined as

$$\alpha_1 = \frac{\tau\ \omega_{ee}\omega_{ii} - \omega_{ei}\omega_{ie} + \tau \cdot d}{\tau \cdot d} \tag{117}$$

$$\alpha_2 = \frac{\tau\omega_{ee}\omega_{ii} - \omega_{ei}\omega_{ie} + d}{\tau \cdot d} \tag{118}$$

$$\beta_1 = \frac{\omega_{ee}\omega_{ei}(1-\tau)}{\tau \cdot d} \tag{119}$$

where $d = det(\Psi)$. When going to the limit $q_m \to 0 (m = e, i)$, The component functions $Z_m(m = e, i)$ of the vector field $\underline{\mathrm{G}}$ is given as

$$Z_e(x) = \begin{cases} 0 & : x < \theta_e \\ 1 & : x > \theta_e \end{cases}\ ,\ Z_i(y) = \begin{cases} 0 & : y < \theta_i \\ 1 & : y > \theta_i \end{cases}$$

Thus, in this case the vector field $\underline{G}$ assumes the values

$$\underline{G}(\underline{X}) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \ \texttt{for} \ \underline{X} \in \overline{\Psi}_1 \tag{120a}$$

$$\underline{G}(\underline{X}) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \ \texttt{for} \ \underline{X} \in \overline{\Psi}_2 \tag{120b}$$

$$\underline{G}(\underline{X}) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \ \texttt{for} \ \underline{X} \in \overline{\Psi}_3 \tag{120c}$$

$$\underline{G}(\underline{X}) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \ \texttt{for} \ \underline{X} \in \overline{\Psi}_4 \tag{120d}$$

Hence we get the linear system for each of the regular domains $\overline{\Psi}_1$, $\overline{\Psi}_2$, $\overline{\Psi}_3$, and $\overline{\Psi}_4$. Each of these systems can be solved by means of standard techniques. Let

$$\underline{X} = \underline{\xi} + \underline{X}_{eq}\,, \quad \underline{\xi} = \begin{bmatrix} \xi_e \\ \xi_i \end{bmatrix} \tag{121}$$

where

$$\underline{X}_{eq} = \underline{\Psi} \cdot \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \ \texttt{for} \ \underline{X} \in \overline{\Psi}_1 \tag{122a}$$

$$\underline{X}_{eq} = \underline{\Psi} \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \omega_{ee} \\ \omega_{ie} \end{bmatrix}, \ \texttt{for} \ \underline{X} \in \overline{\Psi}_2 \tag{122b}$$

$$\underline{X}_{eq} = \underline{\Psi} \cdot \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -\omega_{ei} \\ -\omega_{ii} \end{bmatrix}, \ \texttt{for} \ \underline{X} \in \overline{\Psi}_3 \tag{122c}$$

$$\underline{X}_{eq} = \underline{\Psi} \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} \omega_{ee} - \omega_{ei} \\ \omega_{ie} - \omega_{ii} \end{bmatrix}, \ \texttt{for} \ \underline{X} \in \overline{\Psi}_4 \tag{122d}$$

Then ξ evolves according to

$$\underline{\xi}' = \mathbf{A}\underline{\xi} \tag{123}$$

The solution to the system (123) is given as

$$\xi_e = -c_1\, \beta_1\, e^{-t} - c_2\, \beta_1 e^{\frac{-t}{\tau}} \tag{124}$$

$$\xi_i = c_1\, \alpha_1\, e^{-t} - c_2\, \alpha_2 e^{\frac{-t}{\tau}} \tag{125}$$

Phase portrait of the system is given in the figure.20

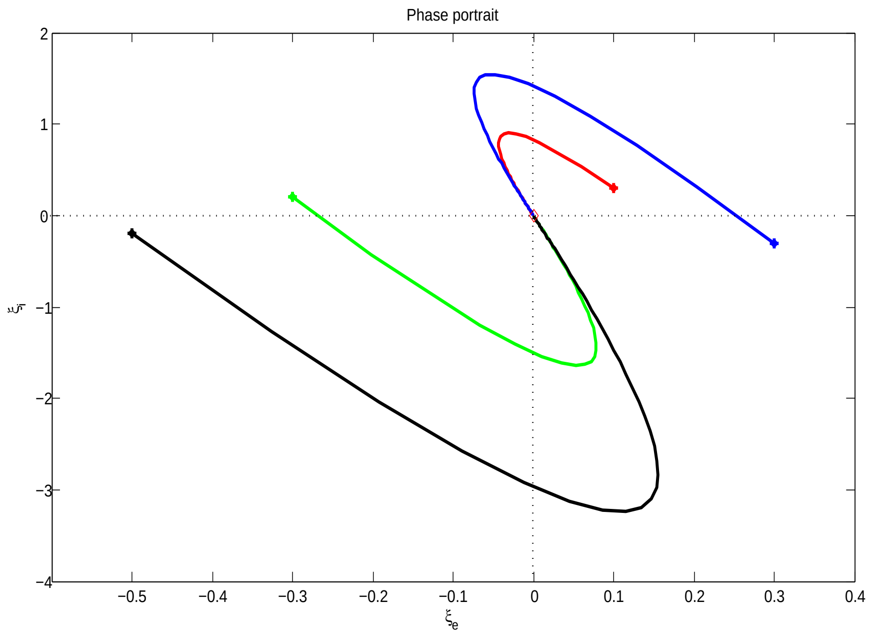

Figure 20: Four different phase curves with different values of the constants (c_1, c_2), but have the same behavior.

5 Dynamical evolution along the black line

It is obvious from the [Fig 16,17,18,19] that the line $y = \theta_i$ is a black line. We now study the dynamics along this line by means of singular perturbation analysis (SPA)[30]. We conventionally write the system (57) in component form as

$$x' = \mu_1\, x + \nu_1\, y + \omega_{ee}\, Z_e(x) - \frac{1}{\tau}\, \omega_{ei}\, Z_i(y) \tag{126a}$$

$$y' = \mu_2\, x + \nu_2\, y + \omega_{ie}\, Z_e(x) - \frac{1}{\tau}\, \omega_{ii}\, Z_i(y) \tag{126b}$$

where the coefficients are defined as where

$$\mu_1 = \frac{-\omega_{ei}\omega_{ie} + \tau\omega_{ee}\omega_{ii}}{\tau d},\; \nu_1 = \frac{\omega_{ee}\omega_{ei}(1-\tau)}{\tau d} \tag{127}$$

$$\mu_2 = \frac{\omega_{ie}\omega_{ii}(\tau - 1)}{\tau d}\; \nu_2 = \frac{-\tau\omega_{ei}\omega_{ie} + \omega_{ee}\omega_{ii}}{\tau d} \tag{128}$$

$$d = \omega_{ei}\omega_{ie} - \omega_{ee}\omega_{ii} \tag{129}$$

We then make the change of variables

$$\begin{bmatrix} x \\ y \end{bmatrix} \longrightarrow \begin{bmatrix} x \\ Z_i \end{bmatrix} \tag{130}$$

where $Z_i = H(y,\theta_i,q_i)$ and H is the Hill function. We readily find that

$$y = H^{-1}(Z_i,\theta_i,q_i) = \theta_i(\frac{Z_i}{1-Z_i})^{q_i} \tag{131}$$

and that the set of equations (126) transforms to

$$x' = \mu_1\, x\, + \nu_1\, H^{-1}(Z_i,\theta_i,q_i) + \omega_{ee}\, Z_e(x) - \frac{1}{\tau}\, \omega_{ei}\, Z_i^{\circ} \tag{132a}$$

$$q_i\, Z_i' = \frac{Z_i(1-Z_i)}{H^{-1}(Z_i,\theta_i,q_i)}\{\mu_2\, x\, + \nu_2\, H^{-1}(Z_i,\theta_i,q_i) + \omega_{ie}\, Z_e(x) - \frac{1}{\tau}\, \omega_{ii}\, Z_i^{\circ}\} \tag{132b}$$

The stretching transformation is,

$$\tilde{\tau} = \frac{t}{q_i} \tag{133}$$

and the system (132) is wriiten in the form:

$$\frac{dx}{d\tilde{\tau}} = q_i\{\mu_1\, x\, + \nu_1\, H^{-1}(Z_i,\theta_i,q_i) + \omega_{ee}\, Z_e(x) - \frac{1}{\tau}\, \omega_{ei}\, Z_i^{\circ}\} \tag{134a}$$

$$\frac{dZ_i}{d\tilde{\tau}} = \frac{Z_i(1-Z_i)}{H^{-1}(Z_i,\theta_i,q_i)}\{\mu_2 x + \nu_2\, H^{-1}(Z_i,\theta_i,q_i) + \omega_{ie} Z_e(x) - \frac{1}{\tau}\omega_{ii} Z_i^{\circ}\} \tag{134b}$$

The SPA now consists of letting $q_m \to 0$ $(m = e,i)$, in this limit we make approximation

$$H^{-1}(Z_i,\theta_i,q_i) = \theta_i(x) \tag{135}$$

Hence we get

$$\frac{dx}{d\tilde{\tau}} = q_i\{\mu_1\, x\, + \nu_1\, \theta_i + \omega_{ee}\, Z_e(x) - \frac{1}{\tau}\, \omega_{ei}\, Z_i^{\circ}\} \tag{136a}$$

$$\frac{dZ_i}{d\tilde{\tau}} = \frac{Z_i(1-Z_i)}{\theta_i}\{\mu_2 x + \nu_2\, \theta_i + \omega_{ie} Z_e(x) - \frac{1}{\tau}\omega_{ii} Z_i^{\circ}\} \tag{136b}$$

The equation (136) is the boundary layer equation. The stationary points for this equation are given as

$$Z_i = 0,\;\; Z_i = 1,\;\; ,Z_i^{\circ} = \frac{\tau(\mu_2\, x\, + \nu_2\, \theta_i + \omega_{ie} Z_e(x))}{\omega_{ii}} \tag{137}$$

For SPA $0 < Z_i^{\circ} < 1$, which imposes restriction on x

$$-(\frac{\theta_i \nu_2 + \omega_{ie} Z_e(x)}{\mu_2}) < x < \frac{\omega_{ii} - \tau(\theta_i \nu_2 + \omega_{ie} Z_e(x))}{\mu_2 \tau} \tag{138}$$

For SPA first, we have to check the conditions of Tikhonov's theorem. Since $\theta_i < 0$ in our case, which implies that Z_i° is not uniformly asymptotic stable. Hence we can not perform SPA.

But if $\theta_i > 0, Z_i^{\circ} \in (0,1)$ and x satisfying the condition (138), then the conditions of Tikhonov's theorem are fulfilled and the dynamics along the black line $x = \theta_i$ is given as

1. When $Z_e(x) = 0$, i.e. $(x < \theta_e)$,

$$x(t) = \frac{\omega_{ei}\, Z_i^{\circ} - \tau \nu_1\, \theta_i}{\tau \mu_1} + c_1\, e^{\mu_1 t} \tag{139}$$

   here c_1 is constant. The equation (139) implies that for $\mu_1 < 0$, the solutions of the system for $Z_e(x) = 0$ will converge to the singular stationary point $P_L(\frac{\omega_{ei}\, Z_i^{\circ} - \tau \nu_1\, \theta_i}{\tau \mu_1}, \theta_i)$.

2. When $Z_e(x) = 1$, i.e. $(x > \theta_e)$, the dynamical evolution equation along $y = \theta_i$ is given as

$$x(t) = \frac{\omega_{ei}\, Z_i^{\circ} - \tau \nu_1\, \theta_i + \omega_{ee}}{\tau \mu_1} + c_2\, e^{\mu_1 t} \tag{140}$$

   where c_2 is constant. The equation (140)shows that if $\mu_1 < 0$, the solutions of the system for $Z_e(x) = 1$ converge to the singular stationary point $P_R(\frac{\omega_{ei}\, Z_i^{\circ} - \tau \nu_1\, \theta_i + \omega_{ee}}{\tau \mu_1}, \theta_i)$

Chapter 5
Conclusions and discussions

6 Conclusion

We have detected examples of black, white and transparent lines in our system (28). We have to explored more than one case of different types of singular domain i.e L_e^0 , L_e^1 ,L_i^0 ,L_i^1 in detail.Transformed these singular domain in new dynamical system and have discussed the stability of singular stationary points.Also discuss the singular domain of each case in detail.
In first chapter we have to discuss synaptic drive model(two-population) with steep firing rate function. Also discuss different models in detail.

Finally we used singular perturbation analysis(SPA) to study the dynamics along black line. It turns out that the conditions of Tikhonov's theorem are fulfilled for $\theta_i > 0$ and certain conditions on variable x. For $\theta_i < 0$, which corresponds to the example we have studied, the stationary point ($\in (0,1)$ of the boundary layer equation turns out to be unstable.

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