COMSATS University Islamabad, Lahore Campus

Creeping Flow of Viscous Fluid through a Porous Permeable Slit with Exponential Reabsorption

By

Sadia Karamat

CIIT/FA17-RMT-048/LHR

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A Post Graduate Thesis submitted to the department of Mathematics as partial fulfillment of the requirement for the award of Degree of MS in Mathematics.

Name	Registeration Number
Sadia Karamat	CIIT/FA17-RMT-048/LHR

Supervisor

Dr. Qurat-ul-Ain Azim

Assistant Professor Department of Mathematics

COMSATS University Islamabad, Lahore Campus.

Certificate of Approval

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Student Name: Sadia Karamat Signature: ________________

External Committee:

External Examiner 1 Name
(Designation & Office Address)

____

____

External Examiner 2 Name
(Designation & Office Address)

____

____

Dr. Qurat-ul-Ain Azim
Supervisor,
Department of Mathematics,
CUI, Lahore Campus

Dr. Sarfraz Ahmad
HOD,
Department of Mathematics,
CUI, Lahore Campus

Prof. Dr. Shamsul Qamar
Chairperson,
Department of Mathematics,
CUI, Lahore Campus

Prof. Dr. Arshad Saleem Bhatti
Dean,
Department of Mathematics,
CUI, Lahore Campus

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Sadia Karamat

CIIT/FA17-RMT-048/LHR

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Sadia Karamat

CIIT/FA17-RMT-048/LHR

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Date:_____________

Supervisor:

____

Dr. Qurat-ul-Ain Azim
Assistant Professor
Department of Mathematics
CUI, Lahore Campus

Head of Department:

____

Dr. Sarfraz Ahmad
Associate Professor
Department of Mathematics
CUI, Lahore Campus

DEDICATION

To

My Loving Parents,

My Brothers, Sisters and Teachers

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All praises to ALLAH Almighty who created the universe and knows whatever is in it hidden or evident, who bestowed upon me the intellectual ability and wisdom to search for it's secrets and whose blessings flourished my thoughts and thrived my ambitions and enabled me to contribute sincerely to the sacred wealth of knowledge. Special praise and regards to his last messenger Muhammad (PBUH) who is a tower of guidance and knowledge for humanity.

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COMSATS University Islamabad, Lahore Campus.

Sadia Karamat

CIIT/FA17-RMT-048/LHR

Abstract

Creeping flow across a porous channels occur in a diverse number of physical phenomena like filtration plants in engineering and industry, glomerular tubule filtration, proximal tubule reabsorption etc. There are a great many studies in literature that address the creeping flow of Newtonian liquid through porous cut, but corresponding studies for flow through porous media are unaccounted for. To address this space in ongoing research literature, we have done to execute the analytical study of creeping flow of viscous liquid through a slit with porous walls where the reabsorption function is exponential. The study is motivated by diseased proximal convoluted tubule in context, where the tubular space acts as a porous medium due to disease. Modeling is the first step of present work which interprets the physical problem into mathematical form. Then the resulting complex differential equations we solved to obtain exact solutions. Physical quantities like components of velocity, pressure distribution, leakage flux and fractional reabsorption are evaluated. Finally we present the graphical analysis of the obtained results.

Contents

Chapter 1

Introduction and Preliminaries

1.1 Fluid Mechanics

The area of Applied Mathematics and Physics that addresses the behavior of fluids (both gases and liquids) that can be in the state of motion or at rest is known as Fluid Mechanics

1.2 Types of Fluid Mechanics

Studies in Fluid Mechanics can be categorized into following three types
1 Fluid Dynamics

2 Fluid Statics

3 Fluid Kinematics

1.2.1 Fluid Dynamics

Within Fluid Mechanics, Fluid dynamics is the subfield that addresses the motion of fluids due to the cause of forces.

1.2.2 Fluid Statics

Fluids in the state of rest are studied under the subfield of Fluid Mechanics.

1.2.3 Fluid Kinematics

When fluid flow is studied without considering the forces involved, such studies are considered to be a part of Fluid Kinematics.

1.3 Fluid

In physics, fluid is a material or substance that frequently changes its shape under when shear stress acts upon it.

1.4 Types of Fluids

There are several types of fluids. These types are discussed as under:

1.4.1 Ideal Fluid

It is defined as a fluid that has uniform density and also lacks viscosity.

1.4.2 Real Fluid

A fluid that exhibits finite level of viscosity is termed as real fluid.

1.4.3 Newtonian Fluid

It is defined as the fluid that follow Newtonian's law of viscosity. Newton's law says that

$$\tau \propto \frac{du}{dy} \tag{1.1}$$

where τ represent the shear stress and $\frac{du}{dy}$ represent the velocity gradient.

Examples: Air, alcohol, water and thin motor oil etc.

1.4.4 Non-Newtonian Fluid

Some fluids not follow the Newtonian's law of viscosity. These fluids are called as non-Newtonian fluids. The law that they follow is given by

$$\tau = k(\frac{du}{dy})^n \tag{1.2}$$

where k represent the index related to consistency and n denote the flow behavior index.

Examples: Gels, honey, blood and toothpaste etc.

1.5 Fluid Flow

It is a branch of fluid mechanics that addresses the fluid dynamics (motion of fluids) the motion of fluids such as liquids and gases is known as fluid flow.

1.6 Types of Flow

Basic kinds of fluid flow:

- Compressible Flow
- Incompressible Flow
- Steady Flow
- Unsteady Flow
- Laminar Flow
- Turbulent Flow
- Rotational and Irrotational Flow

1.6.1 Compressible Flow

Compressible flow is the flow where fluids possess non-constant density.

1.6.2 Incompressible Flow

It is the type of flow where fluids possess constant density.

1.6.3 Steady Flow

The type of low where fluid velocities and all other characteristics that do not change relative to time at every point is regarded as steady flow. In other words, the time derivatives of these quantities are zeros in steady flow. That is

$$\frac{\partial v}{\partial t} = 0, \quad \frac{\partial \rho}{\partial t} = \quad \frac{\partial p}{\partial t} = \text{............} = 0 \text{ and so on.} \tag{1.3}$$

1.6.4 Unsteady Flow

Unsteady flow is a flow in which velocity and other properties of fluids change with time at every point.

$$\frac{\partial v}{\partial t} \neq 0 \text{ etc.} \tag{1.4}$$

1.6.5 Laminar Flow

When fluid particles flow in opposite layers with no disturbance between the layers is known as laminar flow.

1.6.6 Turbulent Flow

Sometimes, fluids flow in an irregular pattern in all directions and in a way that cannot be considered as uniform. Such types of flows are termed as turbulent flows.

1.6.7 Rotational Flow

When fluid particles exhibit a spinning motion about a certain axis, it is regarded as rotational flow

$$\nabla \times \mathbf{V} \neq 0, \tag{1.5}$$

where ∇ is a vector operator called curl and $\mathbf{V}$ represent the velocity.

1.6.8 Irrotational Flow

When no spinning motion abut a particular axis is observed in the flow, such a flow is termed as irrotational flow. Mathematically, it is given by

$$\nabla \times \mathbf{V} = 0, \tag{1.6}$$

1.6.9 Internal Flow

A flow is said to be internal if the fluid particles flows in bounded surface like pipes and open channels.

1.6.10 External Flow

A flow is said to be external if the fluid particles flows over a unbounded surface.

1.7 Stream Function

The stream function is denoted by ψ and is defined for 2-Dimensional and 3-dimensional coordinates of incompressible flow. It is used to plot the streamlines of the fluid flow and also used to find the velocity.

For 2-dimensional flow, the velocity component can be calculated in cartesian form:

$$\psi = \int (udy - vdx) \tag{1.7}$$

where u and v represent components of velocity.

1.8 Reynolds Number

The Reynolds number is define as the inertial forces is proportion to the viscous forces. It is a dimensionless quantity that measure how rough the flow of fluid is.

The Reynolds number is define as:

$$Re = \frac{\rho U L}{\mu} \tag{1.8}$$

where ρ denote the density, U represent the velocity, L is a characteristic linear dimension and μ represent the dynamic viscosity.

- At low Reynolds number, the flow of fluid particles is laminar.
- At high Reynolds number, the flow of fluid particles is turbulent.

1.9 Viscosity

Fluid viscosity is defined as the amount of fluid friction. i.e. The viscosity of honey is greater than the viscosity of water.

1.10 Kinds of Viscosity

Two kinds of viscosity are discusses ad under:

1.10.1 Kinematic Viscosity

This type of viscosity is the ratio of the absolute viscosity to the fluid's density.

Mathematical form of kinematic viscosity is given as:

$$\upsilon = \frac{\mu}{\rho} \tag{1.9}$$

1.10.2 Dynamic Viscosity

It is the measure of internal resistance of the fluid to flow.

The mathematical form of dynamic viscosity is:

$$\mu = \frac{\tau}{\frac{du}{dy}} \tag{1.10}$$

1.11 Continuity Equation

The amount of mass that enters the system should equal the amount of mass that leaves the system plus collection of mass inside the system. It is also known as the law of mass conservation and its mathematical form is called the continuity equation.

The generalized mode of continuity equation is given by

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{V}) = 0 \tag{1.11}$$

There are simplified forms of continuity equation depending on whether the flow is steady or incompressible or both.

1.11.1 Continuity equation in Cartesian Coordinates

The continuity equation in 3-Dimensional form for each and every point in cartesian coordinates is given as:

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho u_x) + \frac{\partial}{\partial y}(\rho u_y) + \frac{\partial}{\partial z}(\rho u_z) = 0 \tag{1.12}$$

Here, u_i where $i = (x, y, z)$ is the velocity component.

1.11.2 Continuity equation in Cylindrical Coordinates

The 3-Dimensional continuity equation for each and every point in cylindrical coordinates is given as:

$$\frac{1}{r}(\frac{\partial}{\partial r}(r v_r)) + \frac{1}{r}(\frac{\partial v_\theta}{\partial \theta}) + \frac{\partial v_z}{\partial z} = 0 \tag{1.13}$$

1.12 Momentum Equation

Momentum equation shows themomentum conservation law and represents the balance of inertial, viscous and body forces on the fluid. The mathematical form in vector notation is given as:

$$\rho \frac{DV}{Dt} = -\nabla p + \rho B + \mu \nabla^2 V \tag{1.14}$$

Where $\rho \frac{DV}{Dt}$ show the total derivative, ∇p represent the pressure gradient, ρB show the body forces and $\mu \nabla^2 V$ represent the diffusion.

1.12.1 Momentum equation in Cartesian Coordinates

The Cartesian coordinate representation of omentum equation is written as:

◇ x-component:

$$\rho(\frac{\partial v_x}{\partial t} + v_x \frac{\partial v_x}{\partial x} + v_y \frac{\partial v_x}{\partial y} + v_z \frac{\partial v_x}{\partial z}) = -\frac{\partial P}{\partial x} + \frac{\partial \tau_{xx}}{\partial t} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z} + \rho g_x \tag{1.15}$$

◇ y-component:

$$\rho(\frac{\partial v_y}{\partial t} + v_x \frac{\partial v_y}{\partial x} + v_y \frac{\partial v_y}{\partial y} + v_z \frac{\partial v_y}{\partial z}) = -\frac{\partial P}{\partial y} + \frac{\partial \tau_{yx}}{\partial t} + \frac{\partial \tau_{yy}}{\partial y} + \frac{\partial \tau_{yz}}{\partial z} + \rho g_y \tag{1.16}$$

◇ z-component:

$$\rho(\frac{\partial v_z}{\partial t} + v_x \frac{\partial v_z}{\partial x} + v_y \frac{\partial v_z}{\partial y} + v_z \frac{\partial v_z}{\partial z}) = -\frac{\partial P}{\partial z} + \frac{\partial \tau_{zx}}{\partial t} + \frac{\partial \tau_{zy}}{\partial y} + \frac{\partial \tau_{zz}}{\partial z} + \rho g_x \tag{1.17}$$

1.12.2 Momentum equation in Cylindrical Coordinates

In cylindrical form, the momentum equation is written as:

◇ r-component:

$$\frac{\partial v_r}{\partial t} + v_r \frac{\partial v_r}{\partial r} + \frac{v_\phi}{r} \frac{\partial v_r}{\partial \phi} + v_z \frac{\partial v_r}{\partial z} - \frac{{u_\phi}^2}{r} = -\frac{1}{\rho} \frac{\partial P}{\partial r} + \frac{1}{r\rho} \frac{\partial (r\tau_{rr})}{\partial r} + \frac{1}{r\rho} \frac{\partial \tau_{\phi r}}{\partial \phi} + \frac{1}{\rho} \frac{\partial \tau_{zr}}{\partial z} - \frac{\tau_{\phi\phi}}{r\rho} + g_r \tag{1.18}$$

$\diamond$ ϕ-component:

$$\frac{\partial v_\phi}{\partial t}+v_r\frac{\partial v_\phi}{\partial r}+\frac{v_\phi}{r}\frac{\partial v_\phi}{\partial \phi}+v_z\frac{\partial v_\phi}{\partial z}+\frac{u_r {u_\phi}^2}{r}=-\frac{1}{r\rho}\frac{\partial P}{\partial \phi}+\frac{1}{r\rho}\frac{\partial \tau_{\phi\phi}}{\partial r}+\frac{1}{r^2\rho}\frac{\partial (r^2\tau_{r\phi})}{\partial r}+\frac{1}{\rho}\frac{\partial \tau_{z\phi}}{\partial z}+g_\phi \tag{1.19}$$

$\diamond$ z-component:

$$\frac{\partial v_z}{\partial t}+v_r\frac{\partial v_z}{\partial r}+\frac{v_\phi}{r}\frac{\partial v_z}{\partial \phi}+v_z\frac{\partial v_z}{\partial z}=-\frac{1}{\rho}\frac{\partial P}{\partial z}+\frac{1}{\rho}\frac{\partial \tau_{zz}}{\partial z}+\frac{1}{r\rho}\frac{\partial \tau_{\phi z}}{\partial \phi}+\frac{1}{r\rho}\frac{\partial (r\tau_{rz})}{\partial r}+g_z \tag{1.20}$$

Chapter 2

Problem Description and Formulation

2.1 Introduction

In the field of engineering, flow in a porous medium has a lot of applicability in dehydration of food, protection of buildings and glomerular tubule filtration etc. One of the most important functions of porous walled channels is in proximal tubule reabsorption and kidney dialysis. Kidney works as a natural filtration plant that is present in the living organisms. Kidneys are like a bean shaped organ. The functions of kidney are excreting waste material from the body and maintaining balance of body fluid. Each kidney consists of millions of small units called nephrons that are same in structure and function. There is about 1 to 1.3 million nephrons in each kidney. Renal corpuscle and renal tubule are main parts of nephron. The main structures of renal corpuscle are Bowman's capsule and glomerules where the filtration of blood starts. The part of nephron that is connected to the Bowman's capsule is called renal tubule. In renal tubule, substances and fluids like water, electrolytes and sugars that are present in the glomerular filtrate are filtered through the tubular wall. The glomerular filtration is 150 to 170 litres per day in which 1.0 to 1.5 litres of waste are urine. When the kidneys catch various diseases, biofouling takes place in which case various substances from the filtrate get stuck on tubular walls. Due to this, the absorption through tubular walls does not remain uniform. Moreover, if proteins and fatty substances get stuck inside the tubular channel, the flow permeability gets restricted and the medium acts as a porous medium.

To study the phenomena of flow through renal tubules effectively, we need a mathematical model and also understand the physiological process of flow in a diseased kidney. Lawrence [23] and Burgen [4] discussed that renal model taking a constant rate of absorption. Kelman [11] suggested that the flow in the proximal tubule changes exponentially. The mathematical model of flow of an incompressible Newtonian fluid passing through renal tubule with first analyzed by Macey [13]. Beavers et al. [2] explained that on the porous boundary, the creep velocity is proportional to the gradient velocity. Kozinski et al [12] reformed Macey's work in tubes and extend it for permeable slits. Marshall [14] studied the flow of Newtonian fluid along the renal tubule with the several physical assumptions. Radhakrishnamacharya [18] investigated the hydrodynamic attitude of creeping flow in renal tubule. Chaturani and Ranganatha [6] examined the creeping flow along a converging tube with porous movable wall. Palatt [17] discussed the hydrodynamic miniature of porous tubule. Muthu [15] [16] analyze the flow problem in a porous medium with varying cross-section. E.FrSmter and K. GeBner [7] discussed the free flow potential portrait of kidney through proximal tubule. Ahmad et al. [1] investigated the flow of viscous fluid along permeable slit with periodic assumption. Haroon et al. [8] investigated the filtrate reabsorption along renal tubule with uniform reabsorption at the walls. White [24], Berman [3] and Terrill [22] discussed the laminar flow of fluid in a uniform permeable medium with different assumptions. Laminar and steady flow of two immiscible fluids in two different channels(porous as well as non-porous) was carried out by Chamkha AJ [5]. Siddiqui et al. [20] [9] [10] [20] studied distinct mathematical study of creeping flow of fluid along permeable slit with different situations.

Most of the work mentioned above address the creeping flow of viscous fluid through porous walled ducts with different biofouling phenomena and reabsorption velocities. But these works do not take into account a diseased tubule with a finite level of permeability . There is no literature available for creeping or Stokes flow of fluid through porous slits with reabsorption that follows exponential form through porous medium. In this dissertation, our aim is to analyze the creeping flow of viscous fluid in a porous slit taking into account biofouling as well as porous medium. We shall model the flow and obtain exact solutions for important physical quantities like velocity profiles, pressure difference, flow rates, wall shear stress, leakage flux, and fractional reabsorption. The results will also be presented graphically.

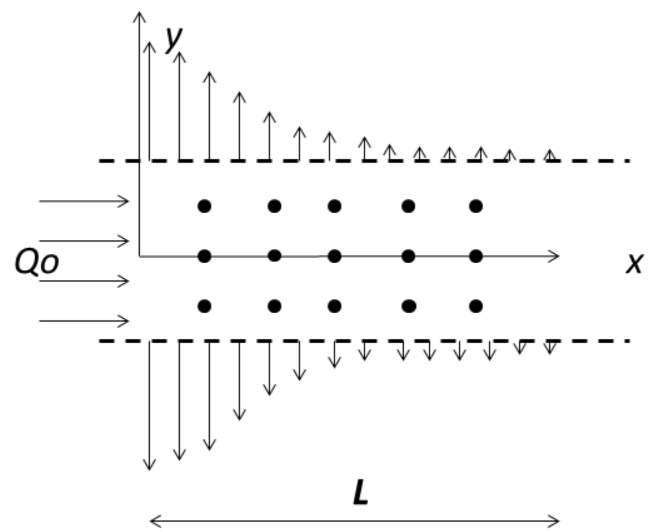

Figure 2.1: Schematic diagram of flow through porous slit

2.2 Problem Formulation

We deal with the a steady, homogeneous, incompressible flow of a Newtonian fluid through a porous medium. We prefer a Rectangular or Cartesian coordinate system (x, y) for characterization of the problem. We assume that the mid line of the slit is adjusted with the x-axis and y-axis is perpendicular to it as shown in figure 2.1. We consider that the fluid is entering through the wall of the slit at $x = 0$ (entrance) with constant volume flow rate Q_0. We also assume that the slit is porous due to reabsorption velocity that changes exponentially throughout the length of the slit. We also consider that the width of the slit is $2H$, the slit length is L and breadth is W. We assume that he fluid is consumed at the walls of the slit (Both upper and lower) at velocity $V_0 e^{-\alpha x}$. We consider flow in two dimesions and the velocity profile is presented as:

$$\mathbf{V} = [u(x, y), v(x, y)], \tag{2.1}$$

In above equation, u is longitudinal velocity and v is transverse velocity. For steady and incompressible flow, due to constant density, the continuity equation is assumes the form:

$$\nabla \cdot \mathbf{V} = 0, \tag{2.2}$$

Using equation (2.1), the continuity equation (2.2) turn into:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \tag{2.3}$$

and for creeping flow through porous medium, the momentum equation is given by:

$$0 = -\frac{\partial p}{\partial x} + \mu \nabla^2 u - \frac{\mu}{k} u, \tag{2.4}$$

$$0 = -\frac{\partial p}{\partial y} + \mu \nabla^2 v - \frac{\mu}{k} v, \tag{2.5}$$

where p represent the pressure, μ is fluid viscosity and k denote the medium permeability.

2.2.1 Boundary Conditions

The boundary conditions associated with the flow under consideration are:
Symmetric condition

$$\frac{\partial u}{\partial y} = 0, \text{ and } v = 0, \text{ at } y = 0, \tag{2.6}$$

No-slip conditions

$$u = 0, \quad v = V_0 e^{-\alpha x}, \text{ at } y = H, \tag{2.7}$$

$v = V_0 e^{-\alpha x}$ is the absorption at wall where α is the biofouling parameter.
Initial flow rate condition

$$Q_0 = 2W \int_0^H u(0, y) dy, \tag{2.8}$$

2.3 Stream Function Formulation of the problem

Here we propose the stream function denoted by $\psi(x, y)$ in terms of the velocity components as:

$$u = \frac{\partial \psi}{\partial y}, \text{ and } v = -\frac{\partial \psi}{\partial x}, \tag{2.9}$$

Now we use equation (2.9) in equations (2.5) and (2.6) then we obtain that form of equations:

$$\frac{\partial p}{\partial x} = \mu \nabla^2 \left(\frac{\partial \psi}{\partial y} \right) - \frac{\mu}{k} (\frac{\partial \psi}{\partial y}), \tag{2.10}$$

$$\frac{\partial p}{\partial y} = -\mu \nabla^2 \left(\frac{\partial \psi}{\partial x} \right) + \frac{\mu}{k} (\frac{\partial \psi}{\partial x}), \tag{2.11}$$

where $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ is a Laplacian operator.
We differentiate equation (2.10) that is related with y and (2.11) that is related with x, to obtain:

$$\frac{\partial^2 p}{\partial y \partial x} = \mu \nabla^2 \left(\frac{\partial^2 \psi}{\partial y^2} \right) - \frac{\mu}{k} (\frac{\partial^2 \psi}{\partial y^2}), \tag{2.12}$$

$$\frac{\partial^2 p}{\partial x \partial y} = -\mu \nabla^2 \left(\frac{\partial^2 \psi}{\partial x^2} \right) + \frac{\mu}{k} (\frac{\partial^2 \psi}{\partial x^2}), \tag{2.13}$$

Now subtracting equations (2.12) and (2.13), we obtain the equation:

$$0 = \mu[\nabla^2 (\frac{\partial^2 \psi}{\partial y^2}) - \frac{1}{k} (\frac{\partial^2 \psi}{\partial y^2}) + \nabla^2 (\frac{\partial^2 \psi}{\partial x^2}) - \frac{1}{k} (\frac{\partial^2 \psi}{\partial x})] \tag{2.14}$$

then further simplifying the equation, we get

$$0 = \nabla^2 (\frac{\partial^2 \psi}{\partial y^2} + \frac{\partial^2 \psi}{\partial x^2}) - \frac{1}{k} (\frac{\partial^2 \psi}{\partial y^2} + \frac{\partial^2 \psi}{\partial x}) \tag{2.15}$$

Here we put $\frac{1}{k} = \beta^2$ in (2.15), we obtain

$$0 = \nabla^2 (\frac{\partial^2 \psi}{\partial y^2} + \frac{\partial^2 \psi}{\partial x^2}) - \beta^2 (\frac{\partial^2 \psi}{\partial y^2} + \frac{\partial^2 \psi}{\partial x}) \tag{2.16}$$

We finally get the equation:

$$\nabla^4\psi - \beta^2\left(\nabla^2\psi\right) = 0 \tag{2.17}$$

where $\nabla^4 = \nabla^2(\nabla^2)$

2.3.1 Boundary Conditions in terms of Stream function

In terms of the stream function $\psi(x, y)$, the boundary conditions (2.6), (2.7) and (2.8) become:

Symmetry condition:

$$\frac{\partial^2\psi}{\partial y^2} = 0, \quad \text{and } \frac{\partial\psi}{\partial x} = 0, \text{ at } y = 0, \tag{2.18}$$

Non-slip conditions:

$$\frac{\partial\psi}{\partial y} = 0, \quad -\frac{\partial\psi}{\partial x} = V_0 e^{-\alpha x}, \text{ at } y = H, \tag{2.19}$$

Initial flow rate conditions:

$$\psi(0, 0) = 0, \tag{2.20}$$

and

$$\psi(0, H) = \frac{Q_0}{2W}. \tag{2.21}$$

Chapter 3

Problem Solution

In this chapter, we will discussed the solutions of the stream function, components of velocity, fractional reabsorption, wall shear stress and leakage flux.

3.1 Method of Solution

3.1.1 Formulation of $\psi(x, y)$ for the Problem

To obtain the definite solutions of the above 2-dimensional boundary value problem, we use inverse method [19],[21] to evaluate the boundary conditions and to find out the stream function $\psi(x, y)$ of the given equation:

$$\psi = V_0 e^{-\alpha x} F(y) + K(y). \tag{3.1}$$

Here, we have used the notations $F(y)$ and $K(y)$ to denote arbitrary unknown functions. Putting equation (3.1) in equation (2.17), we have:

$$\nabla^4 \left(V_0 e^{-\alpha x} F(y) + K(y)\right) - \beta^2 \nabla^2 \left(V_0 e^{-\alpha x} F(y) + K(y)\right) = 0 \tag{3.2}$$

Here we find that

$$\nabla^2(\psi) = \alpha^2 V_0 e^{-\alpha x} F(y) + V_0 e^{-\alpha x} F''(y) + K''(y) \tag{3.3}$$

then further (3.3) takes the form:

$$\nabla^2(\psi) = V_0 e^{-\alpha x} \left(\frac{d^2 F}{dy^2} + \alpha^2 F\right) + \frac{d^2 K}{dy^2} \tag{3.4}$$

This means that

$$\nabla^4(\psi) = \alpha^4 V_0 e^{-\alpha x} F(y) + 2\alpha^2 V_0 e^{-\alpha x} F''(y) + V_0 e^{-\alpha x} F''''(y) + K''''(y) \quad (3.5)$$

then further (3.5) takes the form:

$$\nabla^4(\psi) = V_0 e^{-\alpha x}[\frac{d^4F}{dy^4} + 2\alpha\frac{d^2F}{dy^2} + \alpha F] + \frac{d^4K}{dy^4} \quad (3.6)$$

Now putting equations (3.4) and (3.6) in equation (3.2),we get the equation

$$V_0 e^{-\alpha x}[\frac{d^4F}{dy^4} + 2\alpha^2\frac{d^2F}{dy^2} + \alpha^4 F - \beta^2(\frac{d^2F}{dy^2} + \alpha^2 F)] + \frac{d^4K}{dy^4} - \beta^2\frac{d^2K}{dy^2} = 0 \quad (3.7)$$

which can take the form:

$$V_0 e^{-\alpha x}[\frac{d^4F}{dy^4} + 2\alpha^2\frac{d^2F}{dy^2} + \alpha^4 F - \beta^2(\frac{d^2F}{dy^2} + \alpha^2 F)] = 0 \quad (3.8)$$

Since $V_0 e^{-\alpha x} \neq 0$, therefore

$$\frac{d^4K}{dy^4} - \beta^2\frac{d^2K}{dy^2} = 0, \quad (3.9)$$

and

$$[\frac{d^4F}{dy^4} + 2\alpha^2\frac{d^2F}{dy^2} + \alpha^4 F - \beta^2(\frac{d^2F}{dy^2} + \alpha^2 F)] = 0, \quad (3.10)$$

We now substitute the value of equation (3.1) in the boundary conditions (2.18), (2.19), (2.20) and (2.21).
For $y = 0$ (2.18), we take derivative of the equation (3.1) relative to y to get

$$V_0 e^{-\alpha x} F'(y) + K'(y) = 0 \quad (3.11)$$

We now taking the derivative of the above expression relative to y and further solve the equation to get:

$$V_0 e^{-\alpha x} F'(y) = 0, K'(y) = 0 \quad (3.12)$$

where $V_0 e^{\alpha x} \neq 0$, so we obtain

$$F'(y) = 0, K'(y) = 0 \quad (3.13)$$

At $y = 0$, the equation (3.13) becomes:

$$F'(0) = 0, K'(0) = 0 \tag{3.14}$$

From equation (2.18), we take derivative of the equation (3.1) that is related with x to get:

$$-\alpha V_0 e^{-\alpha x} F(y) = 0 \tag{3.15}$$

we then further solve the above equation and obtain:

$$F(y) = 0 \tag{3.16}$$

At $y = 0$, the equation (3.16) becomes:

$$F(0) = 0 \tag{3.17}$$

From equation (2.19), when we take derivative of the equation (3.1) that is related with y, we get:

$$V_0 e^{-\alpha x} F'(y) + K'(y) = 0 \tag{3.18}$$

then further solve the above equation, we obtain

$$V_0 e^{\alpha x} F'(y) = 0, K'(y) = 0 \tag{3.19}$$

here $V_0 e^{\alpha x} \neq 0$ then the equation (3.18) at $y = H$ becomes:

$$F'(H) = 0, K'(H) = 0 \tag{3.20}$$

From equation (2.19), we take derivative of the equation (3.1) relative to x to get:

$$\alpha V_0 e^{-\alpha x} F(y) = V_0 e^{-\alpha x} \tag{3.21}$$

then further solve the above equation, we obtain:

$$F(y) = \frac{1}{\alpha} \tag{3.22}$$

At $y = H$, the above equation becomes:

$$F(H) = \frac{1}{\alpha} \tag{3.23}$$

From equation (2.20), we put $x = 0$ and $y = 0$ in equation (3.1), we get:

$$V_0 e^{-\alpha 0} F(0) + K(0) = 0 \tag{3.24}$$

then further solve the above equation, we obtain

$$K(0) = 0 \tag{3.25}$$

From equation (2.21), we put $x = 0$ and $y = H$ in equation (3.1), we get:

$$V_0 e^{-\alpha 0} F(H) + K(H) = \frac{Q_0}{2W} \tag{3.26}$$

then further solve the above equation, we obtain

$$K(H) = \frac{Q_0}{2W} - \frac{V_0}{\alpha} \tag{3.27}$$

then further solve the equation (3.27), we get

$$K(H) = \frac{\alpha Q_0 - 2V_0 W}{2\alpha W} \tag{3.28}$$

Now we obtain two subproblems with associated boundary conditions.

F problem:

$$\frac{d^4F}{dy^4} + 2\alpha^2 \frac{d^2F}{dy^2} + \alpha^4 F - \beta^2 \left(\frac{d^2F}{dy^2} + \alpha^2 F\right) = 0, \tag{3.29}$$

F boundary conditions:

$$F(0) = 0, \frac{d^2F(0)}{dy^2} = 0 \tag{3.30}$$

$$F(H) = \frac{1}{\alpha}, \frac{dF(H)}{dy} = 0 \tag{3.31}$$

and

K problem:

$$\frac{d^4K}{dy^4} - \beta^2 \frac{d^2K}{dy^2} = 0, \tag{3.32}$$

K boundary conditions:

$$K(0) = 0, \frac{d^2K(0)}{dy^2} = 0 \tag{3.33}$$

$$K(H) = \frac{\alpha Q_0 - 2V_0 W}{2\alpha W}, \frac{dK(H)}{dy} = 0 \tag{3.34}$$

3.1.2 Solution for ψ involving $F(y)$

Now we first find the value of $F(y)$. We solve the fourth order homogeneous differential equation having constant coefficient or distinct real roots. The auxiliary equation for equation (3.10) takes the form:

$$[\frac{d^4F}{dy^4} + 2\alpha^2\frac{d^2F}{dy^2} + \alpha^4 F - \beta^2(\frac{d^2F}{dy^2} + \alpha^2 F)] = 0, \tag{3.35}$$

$$m^4 + m^2(2\alpha^2 - \beta^2) + (\alpha^4 - \alpha^2\beta^2) = 0 \tag{3.36}$$

The two roots of (3.36) are given by

$$m^2 = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \tag{3.37}$$

Here $a = 1$,$b = 2\alpha^2 - \beta^2$ and $c = \alpha^4 - \alpha^2\beta^2$. Substituting the value of a, b and c in equation (3.37) then we get the value of m:

$$m = \pm\sqrt{\beta^2 - \alpha^2}, m = \pm\alpha\iota \tag{3.38}$$

We obtain the value of $F(y)$ as

$$F(y) = C_1 e^{y\sqrt{\beta^2-\alpha^2}} + C_2 e^{-y\sqrt{\beta^2-\alpha^2}} + e^{\alpha y}[C_3\cos(\alpha y) + C_4\sin(\alpha y)] \tag{3.39}$$

Using BC (3.28) in equation (3.39), we get equation:

$$F(y) = C_4\sin(\alpha y) - C_2 2\sinh(y\sqrt{\beta^2 - \alpha^2}) \tag{3.40}$$

Then using boundary condition (3.30), $\frac{dF(H)}{dy} = 0$ in equation (3.40), we get equation:

$$C_4 = \frac{2C_2\Gamma\cosh(H\Gamma)}{\alpha\cos(\alpha H)} \tag{3.41}$$

Putting the value of C_4 in (3.40), we get

$$F(y) = 2C_2 \frac{[\Gamma \cosh(H\Gamma) \sin(\alpha y) - \alpha \cos(\alpha H) \sinh(y\Gamma)]}{\alpha \cos(\alpha H)} \tag{3.42}$$

Now using $F(H) = \frac{1}{\alpha}$ in (3.42), we obtain

$$C_2 = \frac{\cos(\alpha H)}{2[\Gamma \cosh(H\Gamma) \sin(\alpha H) - \alpha \cos(\alpha H) \sinh(H\Gamma)]} \tag{3.43}$$

substituting the value of C_2 in equation (3.42), now we get

$$F(y) = \frac{[\Gamma \cosh(H\Gamma) \sin(\alpha y) - \alpha \cos(\alpha H) \sinh(y\Gamma)]}{\alpha \Delta_1} \tag{3.44}$$

where

$$\Gamma = \sqrt{\beta^2 - \alpha^2}, \tag{3.45}$$

and

$$\Delta_1 = \Gamma \cosh(H\Gamma) \sin(\alpha H) - \alpha \cos(\alpha H) \sinh(H\Gamma). \tag{3.46}$$

3.1.3 Solution for ψ involving $K(y)$

Now we solve for arbitrary function $K(y)$.

$$\frac{d^4K}{dy^4} - \beta^2 \frac{d^2K}{dy^2} = 0, \tag{3.47}$$

We first integrate the equation (3.47) that is related with y. We get equation:

$$K''' - \beta^2 K' = A \tag{3.48}$$

then again integrate the expression in equation (3.48) relative to y, we have:

$$K'' - \beta^2 K = D_1 y + D_2 \tag{3.49}$$

The Auxiliary equation for (3.49) is given as:

$$m^2 - \beta^2 = 0 \tag{3.50}$$

which gives.

$$m = \pm\beta \tag{3.51}$$

The complementary solution of $K(y)$ is:

$$K_c = A_1 e^{\beta y} + A_2 e^{-\beta y} \tag{3.52}$$

and we assume the particular solution of $K(y)$ is of the form:

$$K_p = A_3 y + A_4 \tag{3.53}$$

We take the derivative of (3.53) relative to y to get:

$$K_p{}' = A_3 \tag{3.54}$$

again differentiate the expression (3.54) relative to y, to get:

$$K_p{}'' = 0 \tag{3.55}$$

Now putting the value of equations (3.53) and (3.55) in equation (3.49), we get equation:

$$K_p = -\frac{D_1}{\beta^2} y - \frac{D_2}{\beta^2} \tag{3.56}$$

To get the complete solution of $K(y)$, we have

$$K(y) = K_c + K_p \tag{3.57}$$

Putting the equations (3.52) and (3.56) in equation (3.57), we get equation of $K(y)$:

$$K = A_1 e^{\beta^2 y} + A_2 e^{-\beta^2 y} - \frac{D_1}{\beta^2} y - \frac{D_2}{\beta^2} \tag{3.58}$$

Now using BC (3.33) in equation (3.58), we get:

$$K = -A_2 (e^{\beta y} - e^{-\beta y}) - \frac{D_1}{\beta^2} y \tag{3.59}$$

then using BC (3.34) in equation (3.59), we get:

$$K(y) = \frac{\alpha Q_0 - 2V_0 W}{2\alpha W \Delta_2} \left[\beta y \cosh(\beta H) - \sinh(\beta y)\right] \tag{3.60}$$

where

$$\Delta_2 = \beta H \cosh(\beta H) - \sinh(\beta H). \tag{3.61}$$

The expression of $\psi(x, y)$ is achieved by substituting expressions (3.44) and (3.60) in equation (3.1) and we achieve equation:

$$\begin{aligned}\psi =& \frac{V_0 e^{-\alpha x}}{\alpha \Delta_1} \left[\Gamma \cosh(H\Gamma) \sin(\alpha y) - \alpha \cos(\alpha H) \sinh(y\Gamma)\right] \\ &+ \frac{\alpha Q_0 - 2V_0 W}{2\alpha W \Delta_2} \left[\beta y \cosh(\beta H) - \sinh(\beta y)\right],\end{aligned} \tag{3.62}$$

3.2 Components of Velocity

The components of velocity are achieved by using equation (2.9):

3.2.1 Solution of Equation involving $u(x, y)$

In order to evaluate longitudinal velocity(u), one can differentiate the equation of $\psi(x, y)$ (3.62) relative to y. We obtain equation of $u(x, y)$:

$$\begin{aligned}u(x, y) =& \frac{V_0 e^{-\alpha x}\Gamma}{\Delta_1} \left[\cosh(H\Gamma) \cos(\alpha y) - \cos(\alpha H) \cosh(y\Gamma)\right] \\ &+ \frac{\alpha Q_0 - 2V_0 W}{2\alpha W \Delta_2} \beta \left[\cosh(\beta H) - \cosh(\beta y)\right]\end{aligned} \tag{3.63}$$

Equations (3.63) and (3.65) provide a complete explanation of the velocity of fluids in each point of the slit. From equation (3.63), we get equation of the maximum $u(x, y)$ that is given as:

$$\begin{aligned}u_{max} =& \frac{V_0 e^{-\alpha x}\Gamma}{\Delta_1} \left[\cosh(H\Gamma) - \cos(\alpha H)\right] \\ &+ \frac{\alpha Q_0 - 2V_0 W}{2\alpha W \Delta_2} \beta \cosh(\beta H)\end{aligned} \tag{3.64}$$

for fixed axial distance x at $y = 0$.

3.2.2 Solution of Equation involving $v(x, y)$

To evaluate transverse velocity(v), one can differentiate the equation of $\psi(x, y)$ (3.63) relative to x. One finds that:

$$v(x,y) = \frac{V_0 e^{-\alpha x}}{\Delta_1} \left[\Gamma \cosh(H\Gamma) \sin(\alpha y) - \alpha \cos(\alpha H) \sinh(y\Gamma)\right] \tag{3.65}$$

Now from equation (3.65), we get the expression for maximum $v(x,y)$ for fixed x at $y = 1$ is given as:

$$v_{max} = V_0 e^{-\alpha x} \tag{3.66}$$

3.3 Axial Volume Flow Rate $Q(x)$

$Q(x)$ can be achieved by accepting formula that is given as:

$$Q(x) = 2W \int_0^H u(x,y)dy \tag{3.67}$$

We first integrate the expression in (3.63) relative to y, to arrive at:

$$\int_0^H u(x,y)dy = \frac{V_0 e^{-\alpha x}}{\alpha} + \frac{\alpha Q_0 - 2V_0 W}{2\alpha W} \tag{3.68}$$

Putting equation (3.68) in expression (3.67), we get

$$Q(x) = Q_0 - \frac{2WV_0}{\alpha}(1 - e^{-\alpha x}) \tag{3.69}$$

3.4 Mass Flow Rate $\bar{q}(x)$

The mass flow rate can be achieved by acceptance of the formula that is given by:

$$\bar{q}(x) = \rho Q(x) \tag{3.70}$$

where ρ is the fluid's density. Substituting the equation (3.69) in equation (3.70), we get equation:

$$\bar{q}(x) = \rho[Q_0 + \frac{2WV_0}{\alpha}(e^{-\alpha x} - 1)] \tag{3.71}$$

We get equation for mass flow rate at the existence of the slit ($x = L$):

$$\bar{q}(L) = \rho[Q_0 + \frac{2WV_0}{\alpha}(e^{-\alpha L} - 1)] \tag{3.72}$$

To obtain the equation for the mass of fluid that is reabsorbed through the walls of slit, then we get

$$\bar{q}(0) - \bar{q}(L) = \frac{2W\rho V_0}{\alpha}(e^{-\alpha L} - 1) \tag{3.73}$$

3.5 Pressure Distribution

To evaluate the distribution for pressure, we make use of the equations (2.12) and (2.13) in equation (3.1). First we take derivative of the equation (3.1) relative to x and get

$$\frac{\partial \psi}{\partial x} = -\alpha V_0 e^{-\alpha x} F(y) \tag{3.74}$$

Then we take derivative of the expression (3.1) relative to y to obtain

$$\frac{\partial \psi}{\partial y} = V_0 e^{-\alpha x} F'(y) + K'(y) \tag{3.75}$$

We find the term ∇^2 of both sides equation (3.74) to get an expression:

$$\nabla^2(\frac{\partial \psi}{\partial x}) = -\alpha^3 V_0 e^{-\alpha x} F(y) - \alpha V_0 e^{-\alpha x} F''(y) \tag{3.76}$$

We find the term ∇^2 of both sides equation (3.75), we get an expression:

$$\nabla^2(\frac{\partial \psi}{\partial y}) = \alpha^2 V_0 e^{-\alpha x} F'(y) + V_0 e^{-\alpha x} F'''(y) + K'''(y) \tag{3.77}$$

Now putting equations (3.74) and (3.76) in equation (2.12), we get an expression:

$$\frac{\partial p}{\partial x} = \mu[V_0 e^{-\alpha x}(F'(\alpha^2 - \beta^2) + F''') + (K''' - \beta^2 K')] \tag{3.78}$$

and putting equations (3.74) and (3.77) in equation (2.13), we get an expression:

$$\frac{\partial p}{\partial y} = \mu[\alpha V_0 e^{-\alpha x}((\alpha^2 - \beta^2)F + \frac{d^2 F}{dy^2})] \tag{3.79}$$

Integrating equation (3.78) that is related with x, we obtain

$$p(x,y)=\mu[\frac{V_0e^{-\alpha x}}{-\alpha}(F'(\alpha^2-\beta^2)+F''')+(K'''-\beta^2K')x]+R(y), \quad (3.80)$$

where $R(y)$ is a function of y arising from integration relative to x. Now we want to determined this function. By differentiating equation (4.2) relative to y, we get

$$\frac{\partial p}{\partial y}=\mu[\frac{V_0e^{-\alpha x}}{-\alpha}(F''(\alpha^2-\beta^2)+F'''')+(K''''-\beta^2K'')x]+R'(y), \quad (3.81)$$

Now comparing equations (4.1) and (4.3), we obtain the value of $R'(y)$ as:

$$\frac{dR}{dy}=0 \quad (3.82)$$

Integrating equation (4.4) relative to y, we get

$$R(y)=D \quad (3.83)$$

where D is constant. Equation (4.2) becomes

$$p(x,y)=\mu[-\frac{V_0e^{-\alpha x}}{\alpha}((\alpha^2-\beta^2)\frac{dF}{dy}+\frac{d^3F}{dy^3})+x(\frac{d^3K}{dy^3}-\beta^2\frac{dK}{dy})]+D, \quad (3.84)$$

Now we take derivative of the equation (3.44) relative to y, we obtain equation:

$$F'(y)=\frac{[\alpha\Gamma\cosh(H\Gamma)\cos(\alpha y)-\alpha\cos(\alpha H)\cosh(y\Gamma)]}{\Delta_1} \quad (3.85)$$

again taking the derivative of the expression in equation (4.8) relative to y, one finds that:

$$F''(y)=\frac{[-\alpha^2\Gamma\cosh(H\Gamma)\sin(\alpha y)-\alpha\Gamma^2\cos(\alpha H)\sinh(y\Gamma)]}{\Delta_1}, \quad (3.86)$$

We again take the derivative of the equation (3.86) relative to y, to get:

$$F'''(y)=\frac{[-\alpha^3\Gamma\cosh(H\Gamma)\cos(\alpha y)-\alpha\Gamma^3\cos(\alpha H)\cosh(y\Gamma)]}{\Delta_1}. \quad (3.87)$$

Now taking derivative of the expression (3.60) relative to y, we arrive at:

$$K'(y) = \frac{\alpha Q_0 - 2V_0W}{2\alpha W\Delta_2}\left[\beta\cosh(\beta H) - \beta\cosh(\beta y)\right], \tag{3.88}$$

and again we taking the derivative of the equation (3.88) relative to y, we get:

$$K''(y) = \frac{\alpha Q_0 - 2V_0W}{2\alpha W\Delta_2}(-\beta^2\sinh(\beta H)). \tag{3.89}$$

We again taking the derivative of the equation (3.89) relative to y to obtain:

$$K'''(y) = \frac{\alpha Q_0 - 2V_0W}{2\alpha W\Delta_2}(-\beta^3\cosh(\beta H)) \tag{3.90}$$

Inserting equations (4.8), (3.86), (3.87), (3.88), (3.89) and (3.90) into equation (4.7), we obtain:

$$\begin{aligned} p(x,y) - p(0,0) =& \mu\frac{V_0\beta^2\Gamma\cosh(H\Gamma)}{\Delta_1}\left[e^{-\alpha x}\cos(\alpha y) - 1\right] \\ &- \mu\frac{\alpha Q_0 - 2V_0W}{2\alpha W\Delta_2}\left[\beta^3\cosh(\beta H)\right]x \end{aligned} \tag{3.91}$$

where $p(0,0)$ is the pressure at the start of slit with coordinated $(x,y) = (0,0)$.

3.5.1 Averages Pressure

The average pressure $\bar{p}(x)$ is determined by using the formula that is given as:

$$\bar{p}(x) = \frac{1}{H}\int_0^H [p(x,y) - p(0,0)]dy. \tag{3.92}$$

Integrating the equation (3.91), we get

$$\begin{aligned} \int_0^H [p(x,y) - p(0,0)]dy =& \mu\frac{V_0\beta^2\Gamma\cosh(H\Gamma)}{\alpha\Delta_1}\left[e^{-\alpha x}\sin(\alpha H) - \alpha H\right] \\ &- \mu\frac{\alpha Q_0 - 2V_0W}{2\alpha W\Delta_2}\left[\beta^3\cosh(\beta H)\right]xH. \end{aligned} \tag{3.93}$$

Now putting the equation (3.93) in the equation (3.92), we obtain

$$\begin{aligned}\bar{p}(x) =& \frac{1}{H}\frac{\mu V_0\beta^2\Gamma\cosh(H\Gamma)}{\alpha\Delta_1}\left[e^{-\alpha x}\sin(\alpha H)-\alpha H\right]\\ &-\mu\frac{\alpha Q_0-2V_0W}{2\alpha W\Delta_2}\left[\beta^3\cosh(\beta H)\right]x.\end{aligned} \tag{3.94}$$

3.5.2 Mean Pressure Drop

The mean pressure drop along the length L of the slit can be determined by using the given formula:

$$\Delta\bar{p}(L) = \bar{p}(0) - \bar{p}(L) \tag{3.95}$$

Putting $x = 0$ in the equation (3.94), we obtain

$$\bar{p}(0) = \mu\frac{V_0\beta^2\Gamma\cosh(H\Gamma)}{\alpha H\Delta_1}[\sin(\alpha H)-\alpha H] \tag{3.96}$$

Then we put $x = L$ in equation (3.94) to obtain

$$\begin{aligned}\bar{p}(L) =& \mu\frac{V_0\beta^2\Gamma\cosh(H\Gamma)}{\alpha H\Delta_1}[e^{-\alpha L}\sin(\alpha H)-\alpha H]\\ &+\mu\frac{\alpha Q_0-2V_0W}{2\alpha W\Delta_2}\left[\beta^3\cosh(\beta H)\right]L.\end{aligned} \tag{3.97}$$

Now substituting equations (3.96) and (3.97) in the equation (3.95), we get

$$\begin{aligned}\bar{p}(L) =& \mu\frac{V_0\beta^2\Gamma\cosh(H\Gamma)}{\alpha H\Delta_1}\sin(\alpha H)[1-e^{-\alpha L}]\\ &+\mu\frac{\alpha Q_0-2V_0W}{2\alpha W\Delta_2}\left[\beta^3\cosh(\beta H)\right]L.\end{aligned} \tag{3.98}$$

3.6 Wall Shear Stress

To find the equation of wall shear stress by using the following formula:

$$\tau_w = -\mu(\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}) \tag{3.99}$$

evaluated at $y = H$. Differentiating the equation (3.63) that is related with y, we get:

$$\begin{aligned} \frac{\partial u}{\partial y} =& \frac{V_0 e^{-\alpha x}\Gamma}{\Delta_1}\left[-\alpha \cosh(H\Gamma)\sin(\alpha y) - \Gamma\cos(\alpha H)\sinh(y\Gamma)\right] \\ &- \frac{\alpha Q_0 - 2V_0 W}{2\alpha W \Delta_2}\left[\beta^2 \sinh(\beta y)\right] \end{aligned} \tag{3.100}$$

We now differentiate the expression in (3.65) relative to x, we obtain equation:

$$\frac{\partial v}{\partial x} = \frac{-V_0 e^{-\alpha x}}{\Delta_1}\left[\Gamma\cosh(H\Gamma)\sin(\alpha y) - \alpha\cos(\alpha H)\sinh(y\Gamma)\right] \tag{3.101}$$

If we substitute equations (3.100) and (3.101) in equation (3.109), we obtain:

$$\begin{aligned} \tau_w|_{y=H} =& \frac{\mu V_0 e^{-\alpha x}}{\Delta_1}\left[2\alpha\Gamma\cosh(H\Gamma)\sin(\alpha H) + \cos(\alpha H)\sinh(H\Gamma)(2\alpha^2 - \beta^2)\right] \\ &- \mu\frac{\alpha Q_0 - 2V_0 W}{2\alpha W \Delta_2}\left[\beta^2\cosh(\beta H)\right] \end{aligned} \tag{3.102}$$

which is the expression for shear stress at the slit wall.

3.7 Fractional Reabsorption

The fractional reabsorption (FA) is achieved by using the given relation:

$$FA = \frac{Q(0) - Q(L)}{Q(0)} \tag{3.103}$$

We substitute $x = 0$ in the equation (3.69), the equation becomes:

$$Q(0) = Q_0 \tag{3.104}$$

Then we substitute $x = L$ in the equation (3.69), the equation becomes:

$$Q(L) = Q_0 - \frac{2WV_0}{\alpha}(1 - e^{-\alpha L}) \tag{3.105}$$

Now putting the equations (3.104) and (3.105) in equation (3.103), we get expression:

$$FA = \frac{2V_0 W}{\alpha}(1 - e^{-\alpha L}). \tag{3.106}$$

3.8 Leakage Flux

The expression for leakage flux can be assessed by acceptance of the given formula:

$$q(x) = -\frac{dQ(x)}{dx} \tag{3.107}$$

We differentiate (3.69)with respect to the variable x to obtain

$$\frac{dQ}{dx} = -2WV_0e^{-\alpha L} \tag{3.108}$$

Now putting eq (3.108) in the expression fiven in (3.107), we find that

$$q(x) = 2WV_0e^{-\alpha x}. \tag{3.109}$$

This shows that the leakage flux varies linearly as V_0 is changed and exponentially with x.

Chapter 4

Graphical Results and Discussion

4.1 Non-Dimensionalisation of Results

In order to draw some meaningful results from expressions obtained us previous chapter, we convert them into dimensionless form. This will also help us in the graphical analysis of the problem. We introduce the following dimensionless variables.

$$x^* = \frac{x}{H}, \quad y^* = \frac{y}{H}, \quad V_0^* = \frac{V_0}{U_0}, \quad \psi^* = \frac{\psi}{U_0 H}, \quad p^* = \frac{pH}{\mu U_0}, \quad \tau^* = \frac{\tau}{\mu U_0},$$

$$Q_0^* = Q_0 U_0 W H, \quad \beta^* = \frac{\beta}{H}, \quad \alpha^* = \frac{\alpha}{H}, \quad u^* = \frac{u}{U_0}, \quad v^* = \frac{v}{U_0}, \quad q^* = \frac{q}{U_0 W}, \quad \text{Ł}^* = \frac{L}{H}.$$

We now using this dimensionless quantities to get the stream function, longitudinal velocity, transverse velocity, axial volume flow rate $Q(x)$, pressure difference, wall shear stress, leakage flux $q(x)$ and fractional reabsorption FA in dimensionless form.

4.1.1 Stream function

$$\begin{aligned}\psi^*(x^*, y^*) =& \frac{V_0^* e^{-\alpha^* x^*}}{\alpha^* \Delta_1^*} \left[\Gamma^* \cosh(\Gamma^*) \sin(\alpha^* y^*) - \alpha^* \cos(\alpha^*) \sinh(y^* \Gamma^*)\right] \\ &+ \frac{\alpha^* Q_0^* - 2V_0^*}{2\alpha^* \Delta_2^*} \left[\beta^* y^* \cosh(\beta^*) - \sinh(\beta^* y^*)\right]\end{aligned} \tag{4.1}$$

4.1.2 Longitudinal Velocity

$$
\begin{aligned}
u^*(x^*, y^*) =& \frac{V_0^* e^{-\alpha^* x^*} \Gamma^*}{\Delta_1^*} \left[\cosh(\Gamma^*) \cos(\alpha^* y^*) - \cos(\alpha^*) \cosh(y^* \Gamma^*)\right] \\
&+ \frac{\alpha^* Q_0^* - 2V_0^*}{2\alpha^* \Delta_2^*} \beta^* \left[\cosh(\beta^*) - \cosh(\beta^* y^*)\right]
\end{aligned} \tag{4.2}
$$

4.1.3 Transverse Velocity

$$
v^*(x^*, y^*) = \frac{V_0^* e^{-\alpha^* x^*}}{\Delta_1^*} \left[\Gamma^* \cosh(\Gamma^*) \sin(\alpha^* y^*) - \alpha^* \cos(\alpha^*) \sinh(y^* \Gamma^*)\right] \tag{4.3}
$$

4.1.4 Pressure Distribution

$$
\begin{aligned}
p^*(x^*, y^*) - p^*(0,0) =& \frac{V_0^* \beta^{2*} \Gamma^* \cosh(\Gamma^*)}{\Delta_1^*} \left[e^{-\alpha^* x^*} \cos(\alpha^* y^*) - 1\right] \\
&- \frac{\alpha^* Q_0^* - 2V_0^*}{2\alpha^* \Delta_2^*} \left[\beta^{3*} \cosh(\beta^*)\right] x^*
\end{aligned} \tag{4.4}
$$

4.1.5 Volume Flow Rate

$$
Q(x) = Q_0^* - \frac{2V_0^*}{\alpha^*}(1 - e^{-\alpha^* x^*}) \tag{4.5}
$$

4.1.6 Wall Shear Stress

$$
\begin{aligned}
\tau_w^*|_{y=H} =& \frac{V_0^* e^{-\alpha^* x^*}}{\Delta_1^*} \left[2\alpha^* \Gamma^* \cosh(\Gamma^*) \sin(\alpha^*) + \cos(\alpha^*) \sinh(\Gamma^*)(2\alpha^{2*} - \beta^{2*})\right] \\
&- \frac{\alpha^* Q_0^* - 2V_0^*}{2\alpha^* \Delta_2^*} \left[\beta^{2*} \cosh(\beta^*)\right]
\end{aligned} \tag{4.6}
$$

4.1.7 Fractional Reabsorption

$$
FA = \frac{2V_0^*}{\alpha^*}(1 - e^{-\alpha^* L^*}) \tag{4.7}
$$

4.1.8 Leakage Flux

$$
q^*(x^*) = 2V_0^* e^{-\alpha^* x^*} \tag{4.8}
$$

4.2 Graphical Results

Figure (4.1) shows the variation of streamlines $\psi(x, y)$ with different values of V_0. We observe that a higher values of V_0, reverse flow situation is seen. This is because excess fluid is absorbed at the walls of the slit. In figure (4.2), we observed the change in longitudinal velocity $u(x, y)$ with different values of V_0. The impact of V_0 is observed at the three different positions across the slit: nearly at the beginning of the slit, in the mid of the slit, then almost at the existence of the slit. We notice that $u(x, y)$ decreases as V_0 increases. In figure (4.3), we also observed the impact of V_0 along the transverse velocity $v(x, y)$ at three different positions across the slit: nearly at the beginning of the slit, in the mid of the slit, then almost at the existence of the slit. We notice that $v(x, y)$ increases as V_0 increases. Figure (4.4) shows how several other quantities are influenced by different values of V_0. In panel (a), $Q(x)$ shows decrease along x. We observe that the volume flow rate is decreases as V_0 increases. In panel (b), we show that pressure difference also undergoes as V_0 is increased. In panel (c), we see that the mass flow rate $\bar{q}(x)$ is increase with decrease in V_0 as expected. In panel (d) of figure (4.4) shows a linear fluctuation of fractional reabsorption FA with V_0. We observe that the FA is inversely proportional to V_0.

Figure (4.5) shows the effect of various values of Q_0 on streamlines. We see that longitudinal flow is increase with the increase of Q_0 while (see for example, panel (b)) with the decrease of Q_0, the fluid flows back. In figure (4.6), we predict that the longitudinal velocity $u(x, y)$ increases with increase in Q_0 values along the x-axis. We know the transverse velocity $v(x, y)$ is not related of Q_0, so here we can not plot it. Figure (4.7) shows how various quantities are influenced by different values of Q_0. Panel (a) represent that the $Q(x)$ increases with increase in the value of Q_0. In panel (b), we show that the pressure difference is directly proportional to Q_0 along x-axis at $y = 0$. In panel (b), we see that the wall shear stress τ also show the direct relation with Q_0. Panel (d) of figure (4.7) shows that the fractional reabsorption FA is decreases rapidly with the increase in Q_0 values.

Figure (4.8) represent the analysis of $\psi(x, y)$ with the effect of porosity medium β. We predict the increase in the longitudinal flow with the increase of β. In figure (4.9), where the longitudinal velocity $u(x, y)$ is plotted at different values of x in the slit against y. We see that longitudinal velocity becomes flatter along $x - axis$ with the increase in β. In figure (4.10), we observe that $v(x, y)$ for different values of β decreases away from $x - axis$.

For high β values, the transverse velocity is decrease in between $y = 0$ and $y = H$. Figure (4.11) show that the absolute value of pressure difference increases when the β values is increases. In figure (4.12), where wall shear stress τ is plotted against x. We observe that the τ value decreases when β value is increases. We know the volume flow rate, mass flow rate and fractional reabsorption are not dependent on β, so here we can not plot them.

Figure (4.13) shows that the effect of different values of α(medium porosity) on streamlines. We observe the reverse flow phenomenon in panel (a) at low values of α. In figure (4.14), we observe that the longitudinal velocity $u(x, y)$ varies with the values of α. In figure (4.15), we observe that the transverse velocity $v(x, y)$ increases as α increases. The panel (a) of figure (4.16) show that the volume flow rate increases when the value of α is increased. In panel (b), we observe that the pressure difference has direct relation with the values of α. In panel (c), we show that the τ increases as α decreases. The panel (d) of figure (4.16) show that the fractional reabsorption decreases when α is increases.

From the above mentioned graphical analysis, we conclude that all the physiological properties are dependent on the flow and physical parameters involved.

4.2.1 Graphs

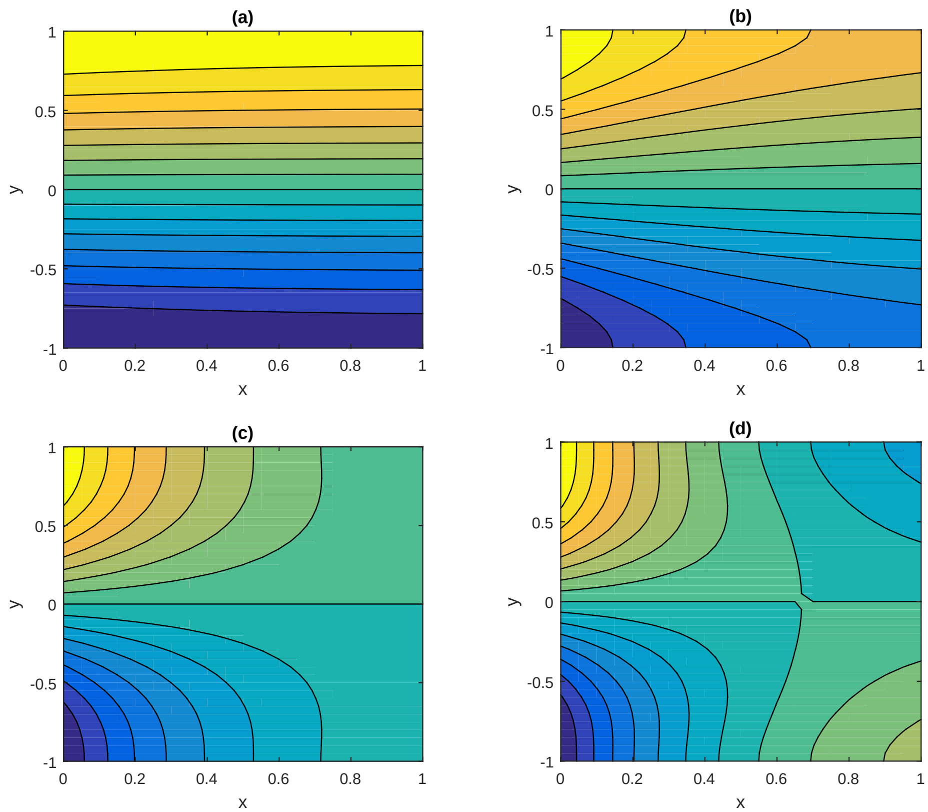

Figure 4.1: Variation in ψ as V_0 changes, when $Q_0 = 2$, $\beta = 3$ and $\alpha = 2$ (a) $V_0 = 0.1$, (b) $V_0 = 1$, (c) $V_0 = 5$, (d) $V_0 = 10$

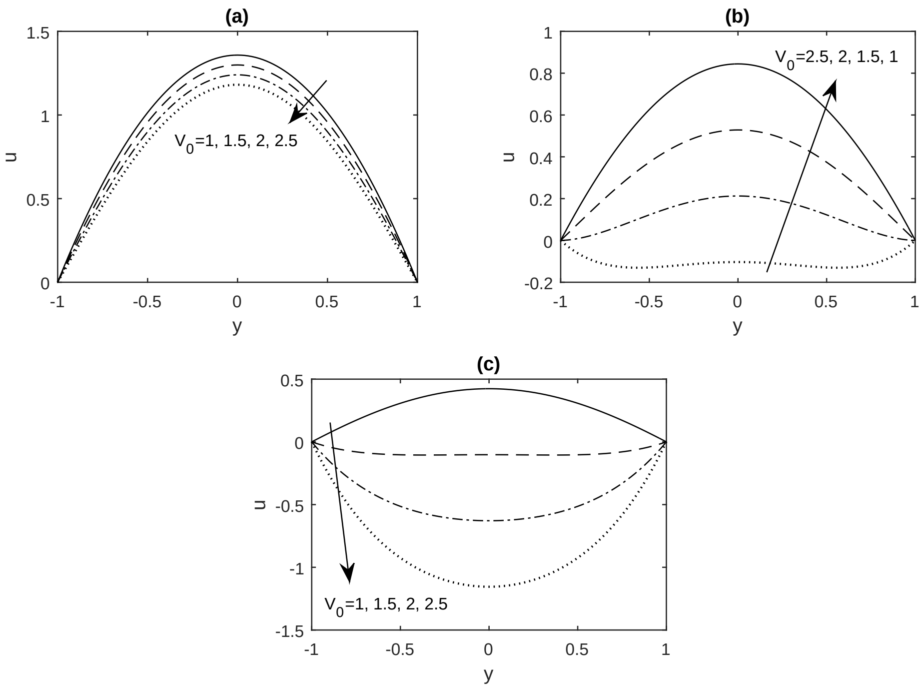

Figure 4.2: Variation in longitudinal velocity as V_0 changes, when $Q_0 = 2$, $\alpha = 0.5$ and $\beta = 1$ near (a) the entrance of slit, $x = 0.1$, (b) the centre of slit, $x = 0.5$, and (c) the exit of slit, $x = 0.9$

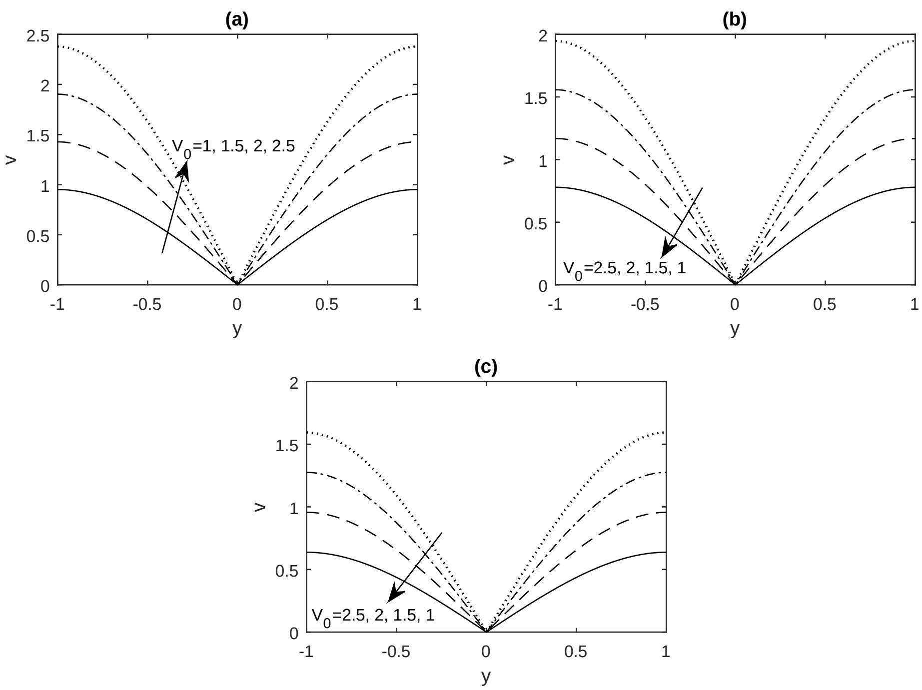

Figure 4.3: Variation in transverse velocity as V_0 changes, when $Q_0 = 2$, $\alpha = 0.5$ and $\beta = 1$ near (a) the entrance of slit, $x = 0.1$, (b) the centre of slit, $x = 0.5$, and (c) the exit of slit, $x = 0.9$

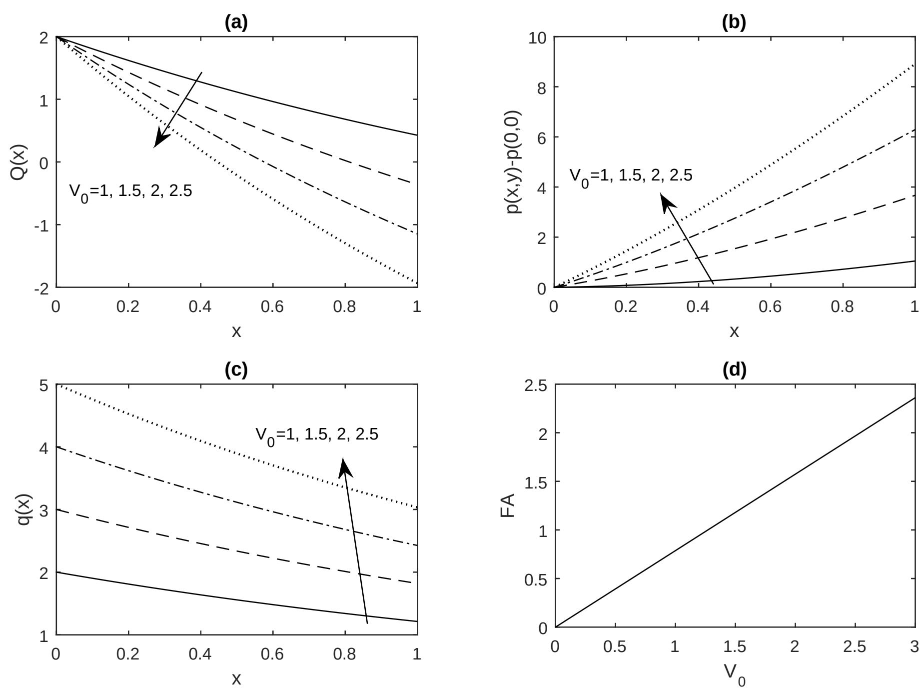

Figure 4.4: Variation in Q(x), pressure, q(x) and FA as V_0 changes, when $Q_0 = 2$, $\alpha = 0.5$, $\beta = 1$, $L = 1$ and $y = 0$

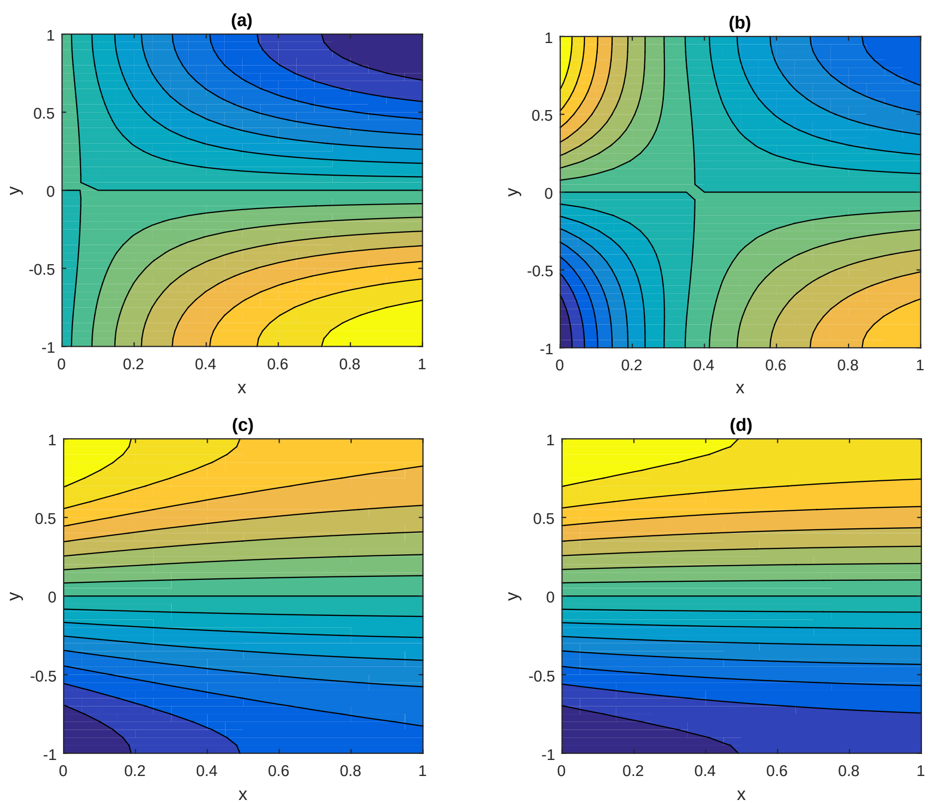

Figure 4.5: Variation in ψ as Q_0 changes, when $V_0 = 2$, $\beta = 1$ and $\alpha = 2$ (a) $Q_0 = 0.1$, (b) $Q_0 = 1$, (c) $Q_0 = 5$, (d) $Q_0 = 10$

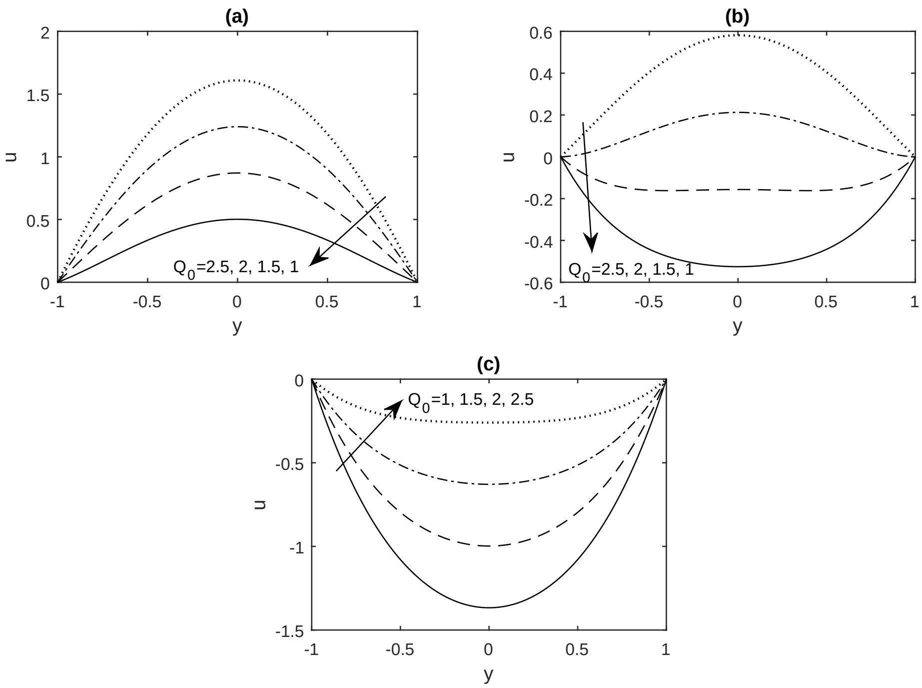

Figure 4.6: Variation in longitudinal velocity as Q_0 changes, when $Q_0 = 2$, $\alpha = 0.5$ and $\beta = 1$ near (a) the entrance of slit, $x = 0.1$, (b) the centre of slit, $x = 0.5$, and (c) the exit of slit, $x = 0.9$

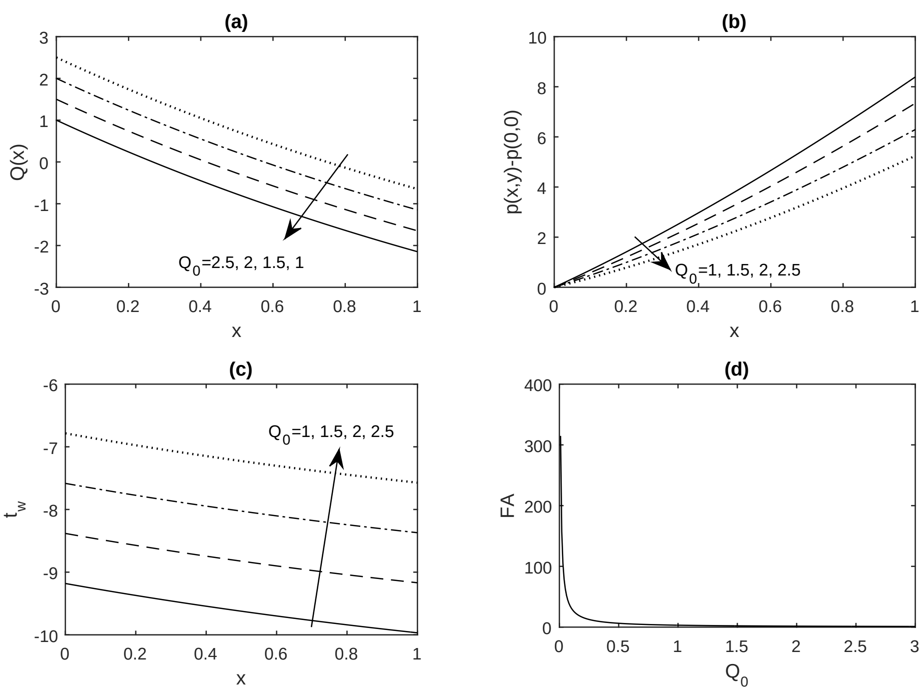

Figure 4.7: Variation in Q(x), pressure, wall shear stress and FA as Q_0 changes, when $V_0 = 2$, $\alpha = 0.5$, $\beta = 1$, $L = 1$ and $y = 0$

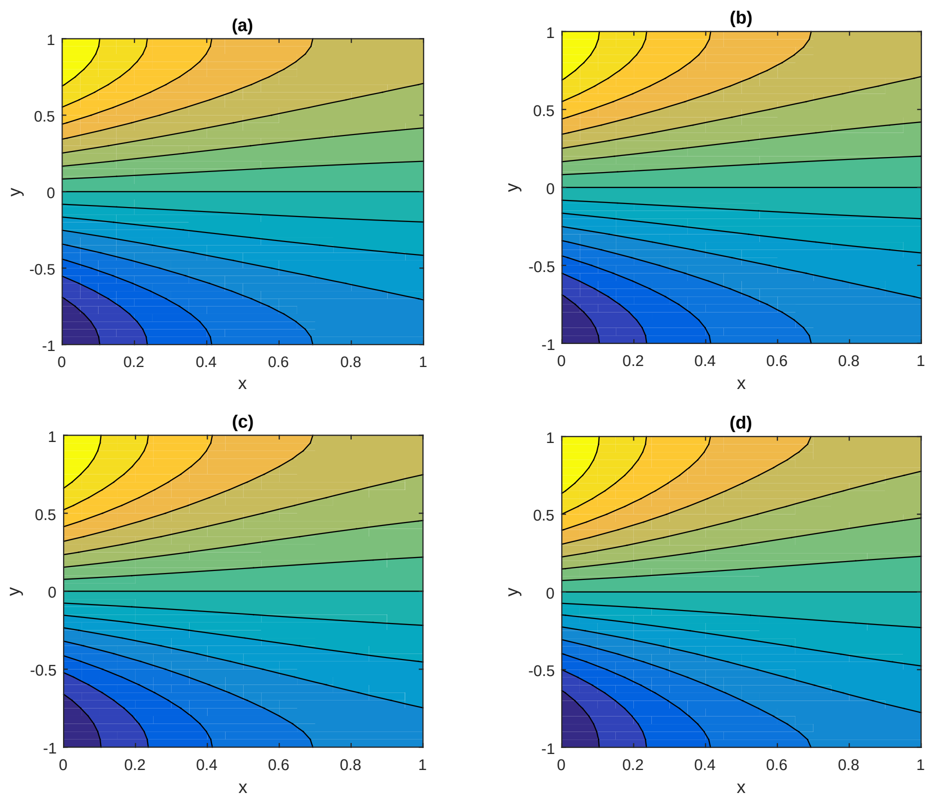

Figure 4.8: Variation in ψ as β changes, when $V_0 = 2$, $Q_0 = 3$ and $\alpha = 2$ (a) $\beta = 0.1$, (b) $\beta = 1$, (c) $\beta = 5$, (d) $\beta = 10$

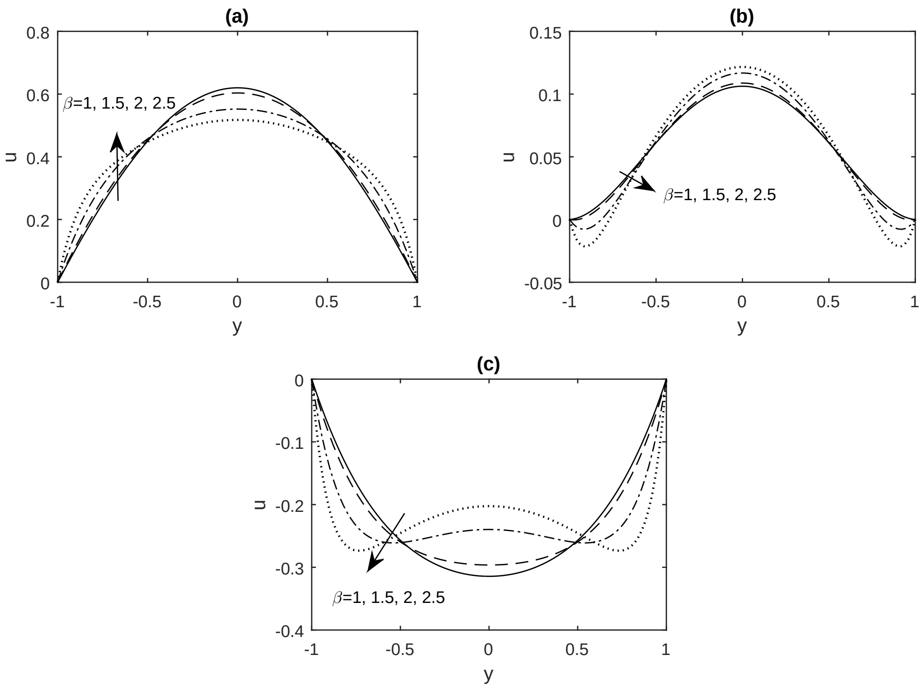

Figure 4.9: Variation in longitudinal velocity as β changes, when $Q_0 = 1$, $\alpha = 0.5$ and $V_0 = 1$ near (a) the entrance of slit, $x = 0.1$, (b) the centre of slit, $x = 0.5$, and (c) the exit of slit, $x = 0.9$

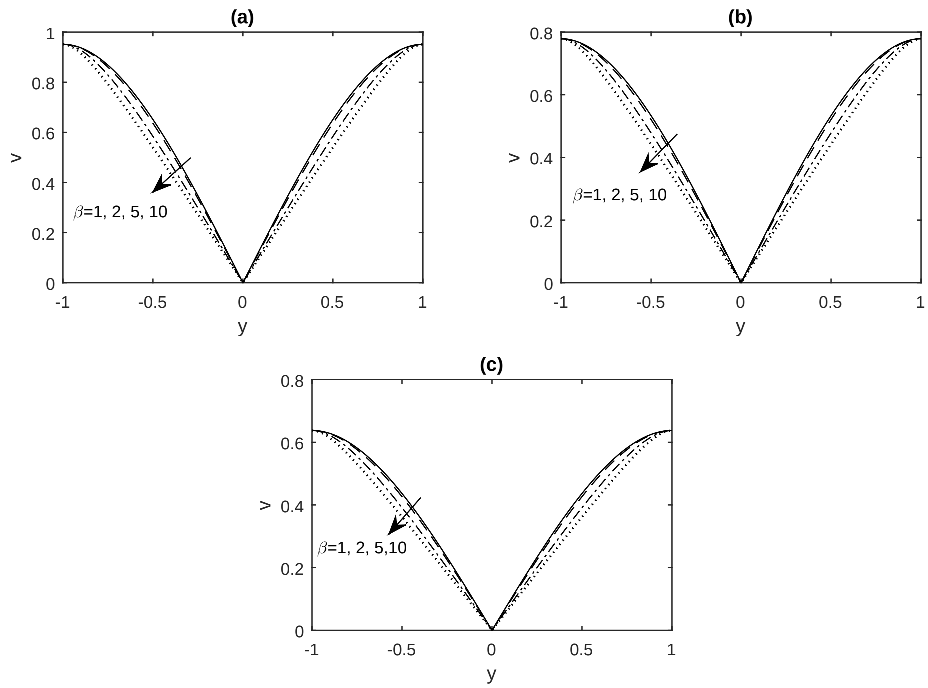

Figure 4.10: Variation in transverse velocity as β changes, when $Q_0 = 1$, $\alpha = 0.5$ and $V_0 = 1$ near (a) the entrance of slit, $x = 0.1$, (b) the centre of slit, $x = 0.5$, and (c) the exit of slit, $x = 0.9$

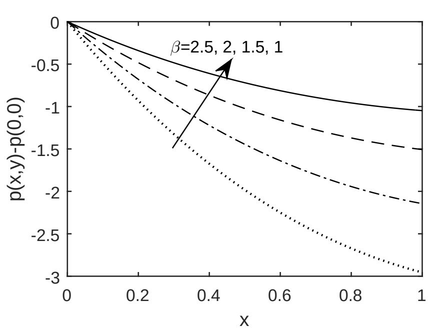

Figure 4.11: Variation in pressure as β changes, when $V_0 = 1$, $\alpha = 0.5$, $Q_0 = 3$ and $y = 0$

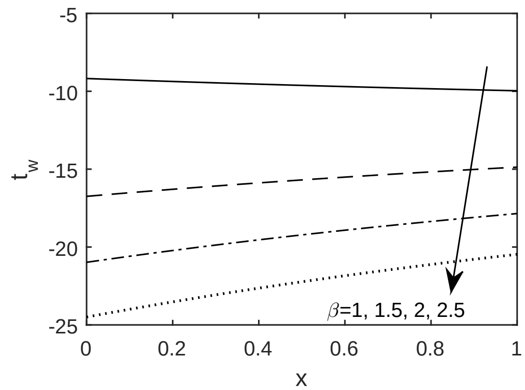

Figure 4.12: Variation in wall shear stress as β changes, when $V_0 = 2$, $\alpha = 0.5$, $Q_0 = 1$ and $y = 1$

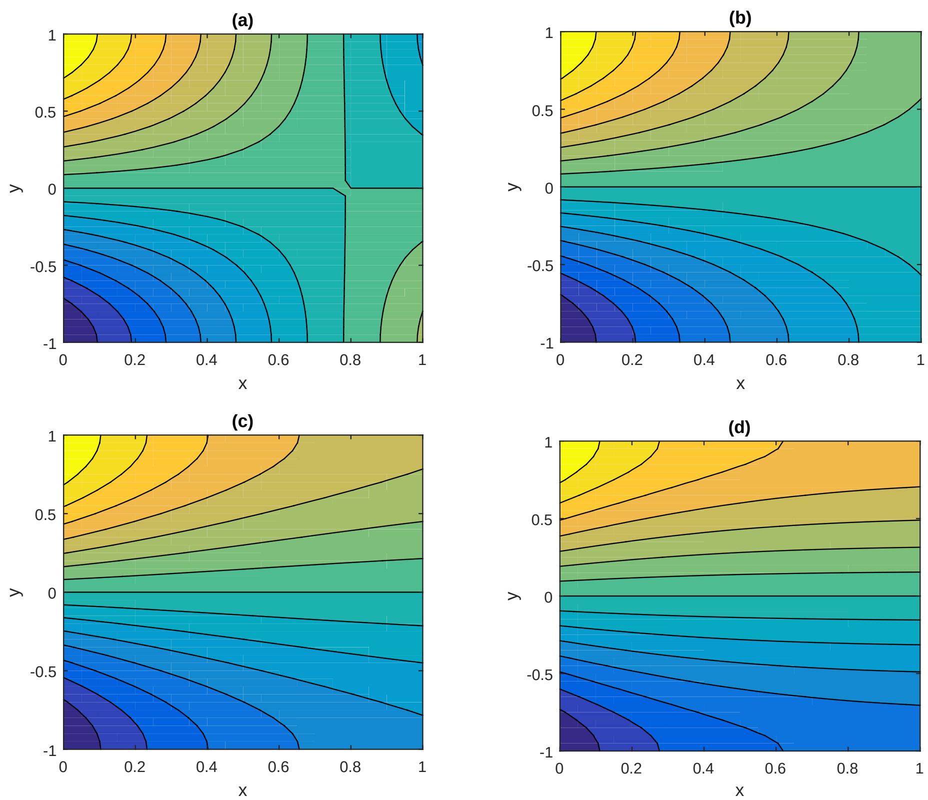

Figure 4.13: Variation in ψ as α changes, when $V_0 = 2$, $\beta = 2$ and $Q_0 = 3$ (a) $\alpha = 0.1$, (b) $\alpha = 1$, (c) $\alpha = 1.9$, (d) $\alpha = 3$

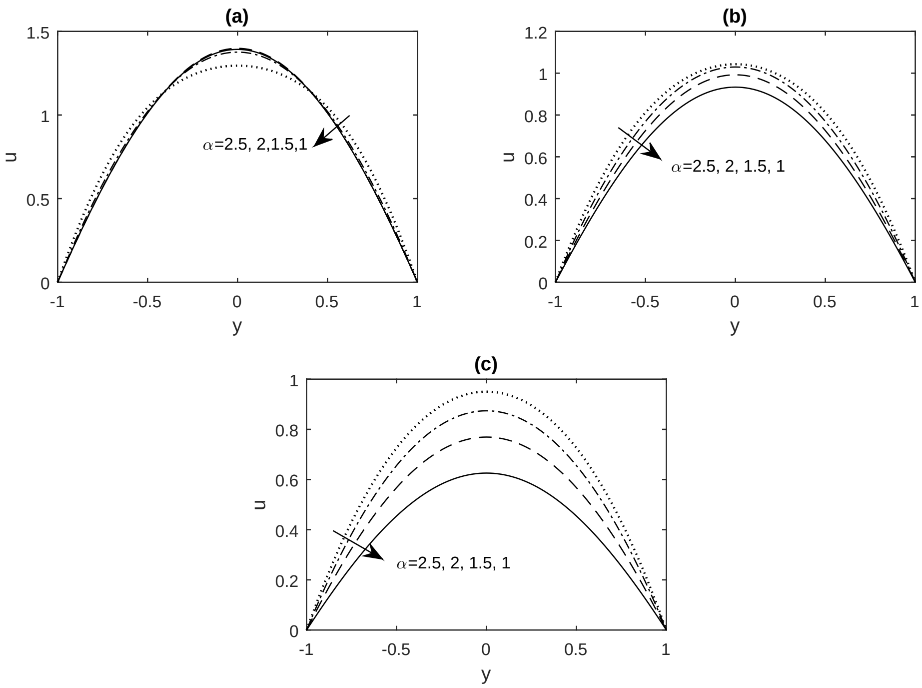

Figure 4.14: Variation in longitudinal velocity as α changes, when $Q_0 = 2$, $\beta = 0.5$ and $V_0 = 1$ near (a) the entrance of slit, $x = 0.1$, (b) the centre of slit, $x = 0.5$, and (c) the exit of slit, $x = 0.9$

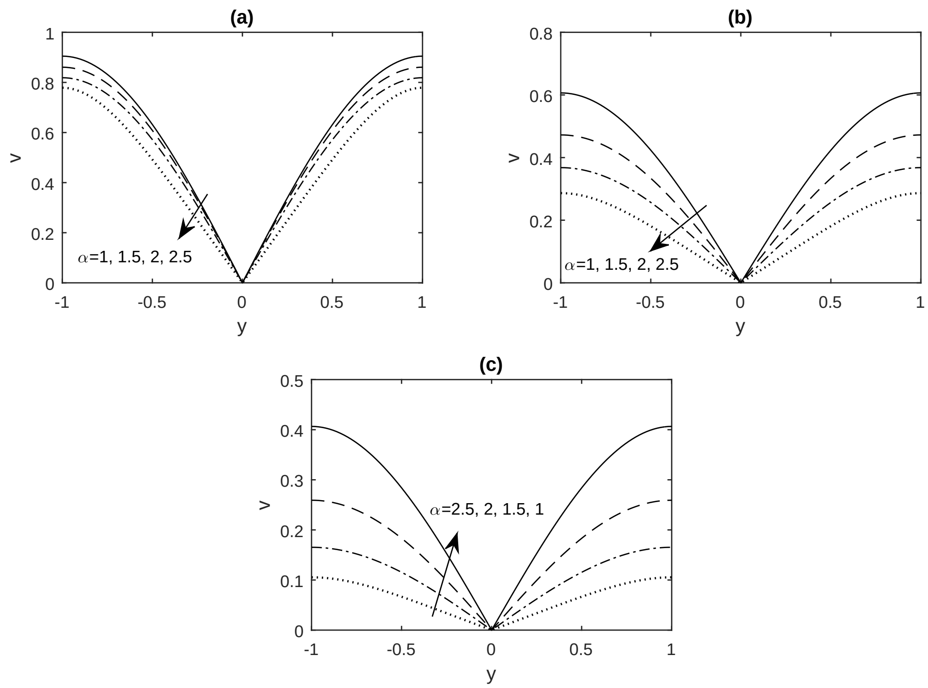

Figure 4.15: Variation in transverse velocity as α changes, when $Q_0 = 2$, $V_0 = 1$ and $\beta = 0.5$ near (a) the entrance of slit, $x = 0.1$, (b) the centre of slit, $x = 0.5$, and (c) the exit of slit, $x = 0.9$

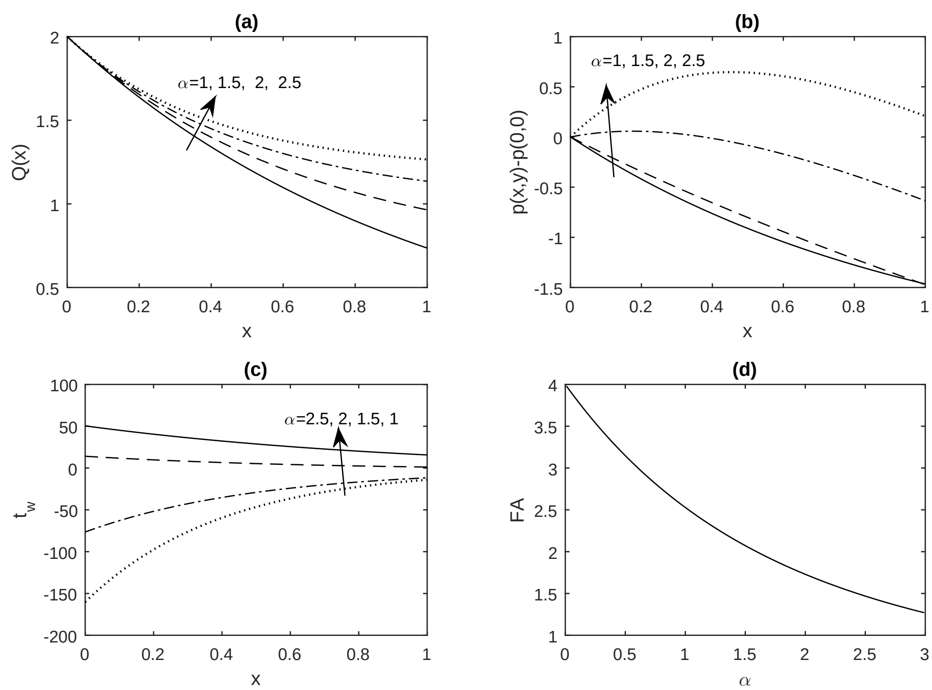

Figure 4.16: Variation in Q(x), pressure, wall shear stress and FA as α changes, when $V_0 = 2$, $\beta = 0.5$, $Q_0 = 1$, $L = 1$ and $y = 0$

Chapter 5

Conclusions

We evaluated the problem of creeping flow of viscous fluid through porous slit with exponential reabsorption. We converted the governing partial differential equations into one partial differential equation with the use of stream function. Exact solution were obtained by using this differential equation. We determined the expression for the components of velocity (both longitudinal and transverse), volume flow rate $Q(x)$, pressure difference $p(x,y)$, wall shear stress τ, fractional reabsorption FA and leakage flux $q(x)$ over the slit for different values of medium porosity. We have concluded the graphical results of our problem. Now we have following conclusions:

- The profile of longitudinal u and transverse velocity v components for exponential reabsorption are dependent on α and the medium porosity β. If we give higher value of α or β, the longitudinal velocity u is decreased and the transverse velocity v increases.

- Volume flow rate $Q(x)$, pressure difference $p(x,y) - p(0,0)$, leakage flux $q(x)$ and fractional reabsorption FA decrease with an increase in absorption velocity.

- When flow rate is increasing then the streamlines ψ becomes flat and simultaneous. Back flow occur when flow rate decreases.

- When V_0 is increasing then the fractional reabsorption shows linearly fluctuation.

• Longitudinal velocity, flow rate and pressure difference are directly proportional to the initial flow rate.

• The volume flow rate $Q(x)$ and pressure $p(x, y)$ increases when the value of α is increases.

• The backward or onward flow fact can not be controlled with the medium porosity parameter (β). When the medium porosity is reduced, the profile of longitudinal velocity becomes parabolic along the axis. While the transverse velocity is affected very minimally by medium porosity.

• When permeability decrease then the pressure difference is increased. Pressure difference is immensely dependent on medium permeability.

• Leakage flux is independent of medium permeability (α), while leakage flux, fractional reabsorption, flow rates are independent of medium permeability (β).

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