# Dynamics of Two Population Synaptic Drive Model with Complex Temporal Kernel

*BY*


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CIIT/FA14-MSMATH-018/LHR


MS Thesis

In

Master of Science in Mathematics


COMSATS Institute Of Information Technology
Lahore, Pakistan


Spring, 2016

COMSATS Institute of Information Technology

# Dynamics of Two Population Synaptic Drive Model with Complex Temporal Kernel

A Thesis Presented to

COMSATS Institute of Information Technology, Lahore

In partial fulfillment

of the requirement for the degree of

## MS Mathematics

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A Post Graduate Thesis submitted to the name of Department of Mathematics as partial fulfillment of the requirement for the award of degree of MS Mathematics


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May, 2016

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# Declaration

I Iqra Khan, Registration ID CIIT/FA14-MSMATH-018/LHR hereby declare that I have produced the work presented in this thesis, during the scheduled period of study. I also declare that I have not taken any material from any source except referred to wherever that amount of plagiarism is within acceptable range. If a violation of HEC rules on research has occurred in this thesis, I shall be liable to punishable action under the plagiarism rules of the HEC

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It is certified that Iqra Khan, Registration ID CIIT/FA14-MSMATH-018/LHR has carried out all the work related to this thesis under my supervision at the Department of Mathematics, COMSATS Institute of Information Technology, Lahore and the work fulfills the requirement for award of MS degree

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# DEDICATION TO

My loving parents, teachers and my family. Who always encouraged me to work hard and guided me towards the right way and destination.

IN THE NAME OF

# ALLAH

THE MOST GRACIOUS COMPASSIONATE

THE MOST MERCIFUL AND BENEFICENT

WHOSE HELP AND GUIDANCE ALWAYS

SOLICIT AT EVERY STEP AND MOMENT.

ACKNOWLEDGEMENTS

**Praise to be ALLAH, the Cherisher and Lord of the World, Most gracious and Most Merciful**

On completion of my MS project, first of all, I would like to thank my ALLAH almighty, The most merciful and beneficent, who gave me the health, courage and energy to accomplish my work in due time and without whose help this study which required untiring effort would not have been possible to complete within the time limit.
Secondly, I would like to thank my supervisor, Dr. M. Yousaf Bhatti, for the patient guidance, encouragement and advice he has provided throughout my time as his student. I have been extremely lucky to have a supervisor who cared so much about my work, and who responded to my questions and queries so promptly. I also would like to thank Dr. Sarfraz Ahmad who gave me opportunity to do this project.
I would also like to express my deepest gratitude to my all friends, my roommates at CIIT for warm working atmosphere, having a nice time, and for the support in my difficult times.
At the end, how can I forget my parents(my world). I would also like to give thanks to my family, my brother, my sister, my cousins and especially my parents who supported me emotionally and financially. I always knew that you believed in me and wanted the best for me. Thank you so much for giving me every thing in my life.

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# ABSTRACT

In this work we have investigated the two population synaptic drive model with a relatively complex temporal kernal for steep firing rate function. We have done this justification by deriving four dimensional autonomous dynamical system from Volterra equations using linear chain trick. Both of the local and global wellposedness of the model has been investigated. The existence of stationary point has also been investigated and the general formula for stationary point is derived. We get two different models for delay and non delay case. Theorem on the existence and stability has also been formulated. The stability issue has also been addressed for both cases for the steepness parameter $q = 0$, and $q > 0$, which leads us to the conclusion that delays in the system with weak kernel does not effect the stability properties of the model.

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# Chapter 1
# Introduction and Background

# 1 Introduction and background

## 1.1 Human nervous system

There are numerous elements in human body that reacts of produced stimuli through nervous system. The human nervous system is the body's conductive system that consist of billion of nerve cells to transfer signals from one part of body to another. These nerve cells starts from brain and branch out all other parts of body to change the internal and external activities. The word stimuli used for this change that may be visual, auditory, electrical, chemical or mechanical [23].
There are two major classes of human nervous system i.e central nervous system(CNS) that consist of brain and spinal cord and peripheral nervous system(PNS) contains nerves that carry input and out put signals to or from the nervous system [52]. These two systems collaborate and collect information from inside and outside of the environment. As a result of which all parts of human body works. Here I will discuss the most important part of central nervous system i.e the Brain

### 1.1.1 Human Brain

The human brain is the most complex part of human nervous system. It is located in head and covered with skull. It is similar to the brain of mammals but cerebral cortex is highly developed in humans [55]. It contains so many different regions which are highly important in their respective functions. The human brain mainly divided into three parts which are the following [51].

- **Fore-brain**
- **Mid-brain**
- **Hind-brain**

**1- The Fore-brain**

The fore-brain contains cerebral cortex which surrounds mid brain and brain stem. Cerebral cortex is the most important and complex structure in our brain. It is grayish in colour because the nerves present in this area have short of insulation due to which most of the other parts appears white. It is densely packed with neurons. It is larger in size and have so many wrinkles [20][49].
Cerebral cortex is the outer surface that covers human brain. It contains two hemisphere, one is left hemisphere which controls our right side of body and other is right hemisphere whose work is associated with left side of body. The outer surface of cortex is about 2 to 4 millimeters thick and it is surrounding round about 1.5 square meter surface area. Its work is associated with thought, perception, awareness, realization and remembrance [20][49][50].
The cerebral cortex is composed of four lobes that are given as (shown in fig 1).

- **Frontal lobe**

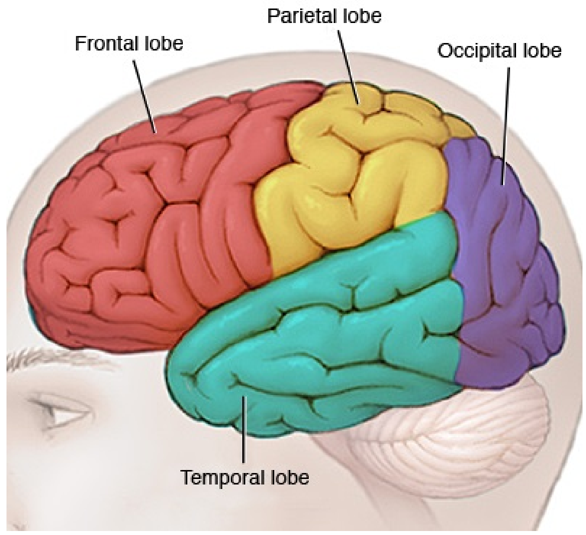


Figure 1: Human Brain with different lobes [54]

- **Parietal lobe**
- **Temporal lobe**
- **Occipital lobe**

Each lobe has its own functions and responsible for it [54].

The Frontal lobe is more developed and larger part in humans that is located in the front of the brain. It controls important human skills like language comprehension, memory, decision making etc. Its work is associated with tasks, short term memory, motivation, planing etc. It is the last lobe to mature due to which children do not effectively take decisions. The frontal lobe is the most common part to happen brain injury due to which a person loses his personality and faces difficulty in taking decisions [60].

The Parietal lobe is major lobe of cerebral cortex present in middle of all lobes. It is a part of cortex due to which it has many responsibilities. It convey message to other parts within seconds. It is responsible for many functions such as makes sense of the world, perceptions, taste, touch or feel etc. When this lobe damaged, a man becomes unable to feel any sensation of touch [57].
The temporal lobe is located behind the left side of other lobes and in bottom middle part of cortex. Its work is purely associated with hearing, listening and understanding sounds having different pitches. A person without temporal lobe cannot understand a meaningful talk. So this is very important lobe of human brain due to which we are able to listen, understand and talk about everything [53].
The last one is occipital lobe that is present in the back of head. It makes a man to being able to correctly understand and seeing things. Similar to the temporal lobe, makes sense

of aural information, this makes sense of visual information [58].

**2- The Mid-brain**

The mid brain is located at the center of brain between fore brain and hind brain. It is composed of brain stem which is base of the brain connected to spinal cord. It plays important role in conveying message from brain to all other parts of body. It controls most important functions such as breathing, temperature, blood pressure, listening and viewing [1][50].

**3- The Hind-brain**

The hind brain is the important part of central nervous system. It consist of medulla oblongata, pons and cerebellum. Medulla works as involuntary function. The pons is the connection between cortex and cerebellum. The transmission of messages in all around body is possible due to pons. Although it is very small in size but without pons, all body movements would not be possible [48].
The cerebellum is the smaller structure (little brain) at the back of the brain. It is small portion of brain and round about ten percent of total weight. It receives sensory information through spinal cord and then regulate these movements. It is responsible for balance and coordination. Any damage to cerebellum cause serious problems such as slower association, lack of stability etc [59].
Infect Our body movements is all due to brain. It controls our thoughts, plans, actions even our body temperature and breathing. Its work is associated with communication of signals, comprehension that and after then decision making. Infect if we say that it is the main body's decision and communication center then it would not be wrong [44].

### 1.1.2 Brain Cells

The human brain is the main communication center and this communication is only possible due to nerve cells that are called Brain cells or neurons. A human brain consist of billions nerve cells that have a work in conveying messages from one part of body to another. Some are the sensory neurons whose work is associated with touch, sound and light. They responds such stimuli and send signals to brain through spinal cord [15].
Neurons are the basic unit of brain in human nervous system. All nerve cells are of two kinds, one of which carry information from the brain to all other parts of body is called efferent neurons(motor nerves) and other carry signals from body to brain called afferent neurons(sensor nerves) [52].
A neuron structure and pre synaptic terminal is shown as in Fig. 2 and Fig. 3 respectively.

It consist of several parts i.e dendrites, soma, axon, synapse and myelin. Dendrites are the input terminals that receives signals from other neurons through its branch like structure and axon is the long tube like output terminal of cell that convey information to other parts of brain. The soma is the main cell body. It is also called processing unit [56].

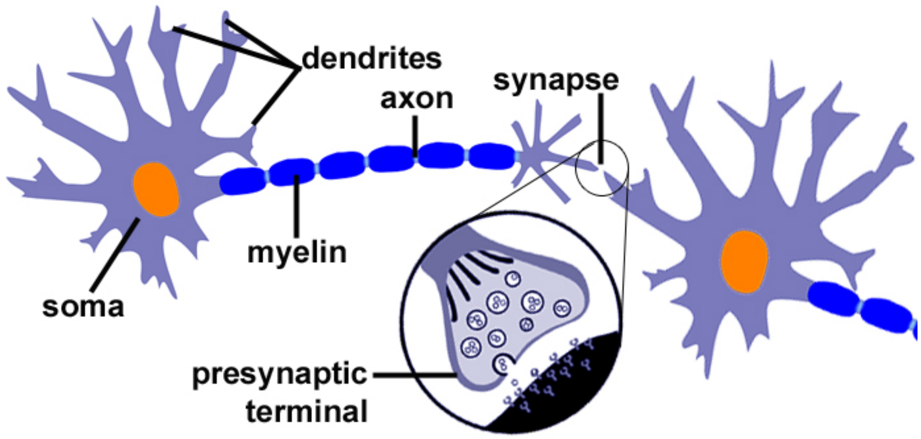


Figure 2: Neurone Structure [54]

Synapse is a small gap contained three parts. The first one is pre-synaptic terminal which contains neurotransmitter, that produces energy in the cell and cell bodies called organelles. The second part is post-synaptic terminal that contains receptors for neurotransmitter. The last one is the space between pre-synaptic and post-synaptic terminal. Thus the pre-synaptic terminal of one neuron is connected to post-synaptic terminal of other neuron through this synaptic gap [46].
Myelin is the necessary coating surrounded axon and helps neuron in conveying signals quickly through long distances [56].
Dendrites catch input waves from other cells and then convey these to soma. Soma interpret this information till the potential approaches to a certain threshold value. Then neurons emits an electrical spark which can be a series of length from a centimeter to about one meter called action potential. This action potential then fastly transmitted to thousand of other neurons through axon. Infect action potential is the way of communication between thousand of neurons [30][44].
Neurons convey message to other part of body through junctions called synapses. Synapses have two forms i.e excitatory and inhibitory. Excitatory synapse brings positive action in whole body while inhibitory decrease the firing action and cause negative action in cells. Excitatory neurons are make up of round about 80% of our settlement and try to bring the affected neurons closed to the action potential while the remaining 20% are inhibitory and they take away neurons from action potential [15].

## 1.2 Different Modeling Approaches

Here I describe different mathematical models to define the transmission of electrical signals that flows through neurons. A model is a mathematical state designed to predict and explain the basic biological process. A mathematical model express the signal transformation of neuron. There are various mathematical models that make a distinction due to their

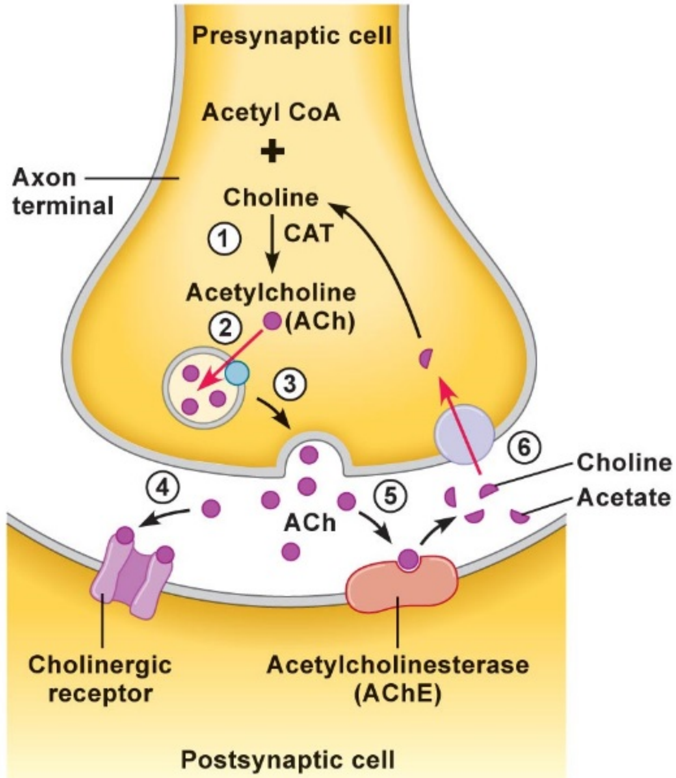


Figure 3: Structure of pre synaptic and post synaptic terminal [45]

extent, ease, amount of biophysical detail included in the depiction and parameters drawn in ([34][43]).
A short report of some familiar mathematical models is given in next sections.

### 1.2.1 compartmental modeling

Compartmental modeling is frequently used to describe the transformation of material in biological system. It is the system which contains many different compartments through which material transformed under certain rules. The following Fig. 4 shows construction of compartmental modeling. This modeling linked with its electrical and geometrical properties then its chemical properties. A compartment model contains a number of sections(compartments) containing fine assorted material. These sections then swap this material under certain rules [43].
So a neuron is divided into many small compartments due to which it comprises a high level of detail. A single section is so small that the electrical potential is considered to be same in each section. Membrane potential is the main variable in this case. Mathematical equation for *n*th compartments is given from Kirchhoff's law of current as

$$\mathbf{g_{n,n+1}}(\mathbf{V_{n+1}}-\mathbf{V_n})-\mathbf{g_{n-1,n}}(\mathbf{V_n}-\mathbf{V_{n-1}})=\mathbf{k_n}\frac{\mathbf{dV_n}}{dt}+\sum_t \mathbf{I}_n^t+\sum_j \mathbf{I}_n^j \tag{1}$$

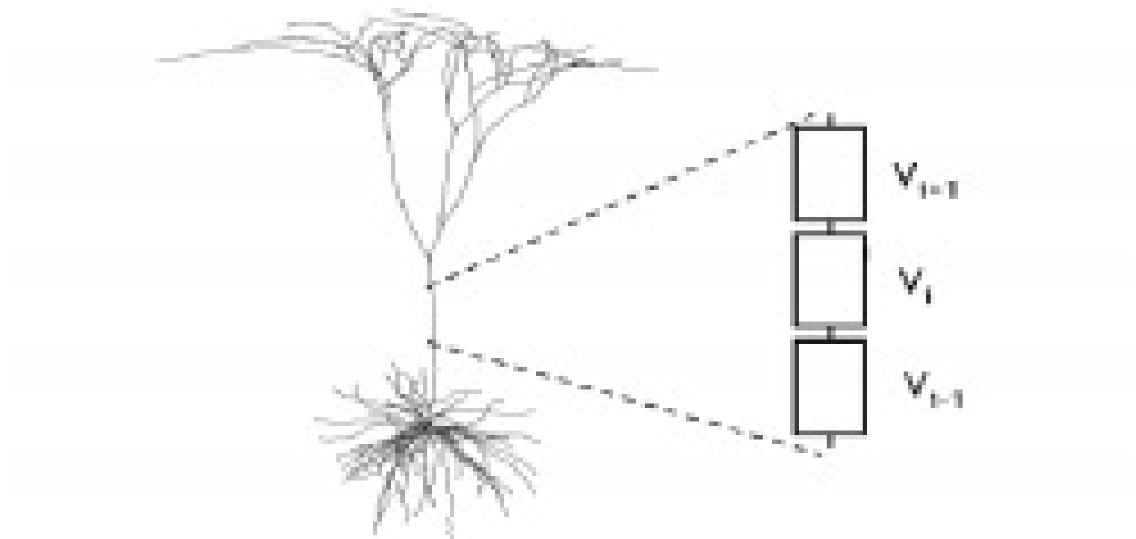


Figure 4: Construction of compartmental model for a neuron [44]

Where the left term of equation is due to the ohmic current between nth and the other two neighboring sections. The first right term gives the membrane capacitance while the other term shows the current injected from the pre-synaptic neurons and the last term is due to the leak ion channels. From end to end these channels current depends on the absorption of certain ions within and outer of the membrane [14][43].

### 1.2.2 Point Neuron Models

Compartmental modeling have many disadvantages due to use of no. of parameters and pricey calculations. This made difficult to compute and run these parameters. Then to overcome this intricacy, the model is reduced to a single compartment by removing a no. of compartments. In this way dendrites and soma malformed into a single point and neuron becomes a one dimensionless component [9][43].
It is a simple form for repetitive firing in neurons. It is just a point and when threshold is approached it fires spike of constant width and amplitude [13].
In moving towards this, the potential and membrane current are taken to be same all through the point neuron. A conversion of compartmental modeling into point neuron is shown in figure 5.

Point neuron model is the simple point model that focus on the sub threshold action and output of neuron. Its well known example is fire models in which spike is recorded when potential of membrane approaches to threshold value. Every single action potential have almost same amplitude of same behavior but there is no information of time period of action potential. Therefore there is no need of calculation of time course every time [44].

The sub-threshold action and output of the neuron are focal point of this model. In a single section model, action potential sub-threshold dynamics of the membrane is given as

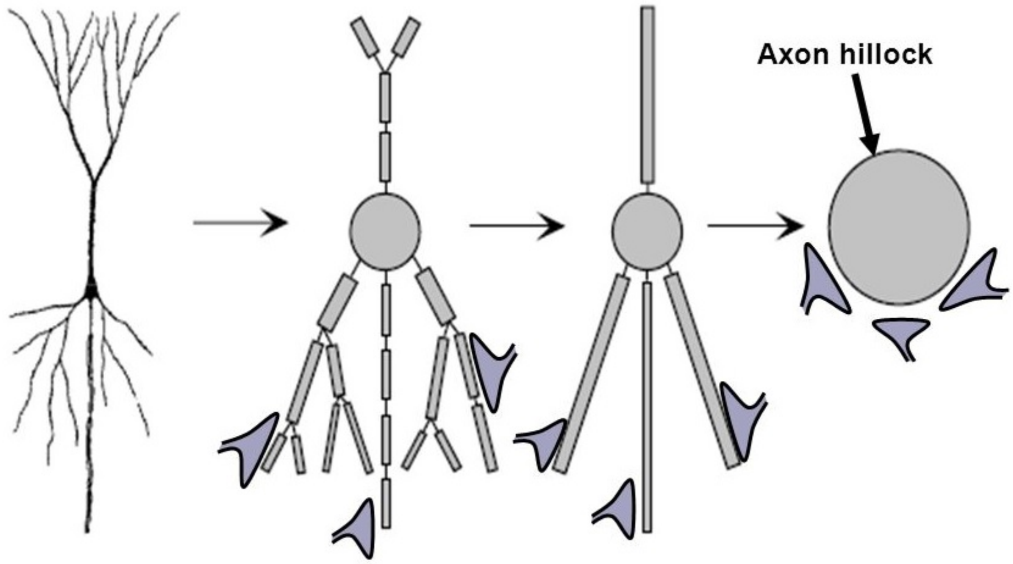


Figure 5: Conversion from compartmental models to point neurons [47]

[9].

$$\mathbf{k}\frac{\mathbf{dV}}{dt} = -\mathbf{g_L}(\mathbf{V} - \mathbf{E_L}) - \sum_t \mathbf{I}^t \tag{2}$$

Here gL and EL represents conductance and resting potential of the neuron, when input current is absent. In this model currents omitted between sections of a single neuron, instead we mull over only the Ohmic leak current. But after converting compartmental modeling into point neuron, it is still not easy to come across critical result for the full dynamics of system for such simple spiking neurons, or even just the settlement rate dynamics [13][43] .

### 1.2.3 Firing Rate Models

To form a large system of neurons, one of the most general method is to apply a generalization called a firing rate model. Instead of spiking of every neuron, one consider the spike rates of groups of neurons inside the circuit. Since these models can represents the whole population of neurons then a single neuron, so they are known as two population models [11].

A neural tissue has a large dynamics which frequently studied by a firing rate models. Firing rate model is actually explains that how group of neurons give firing response to the input signals and what effect produced to other neurons after this response. Two quantities required as input and output of neuron [11].

These models are easy to understand and calculate as they have mathematical formulation of differential and integro-differential developed in applied mathematics. These model-

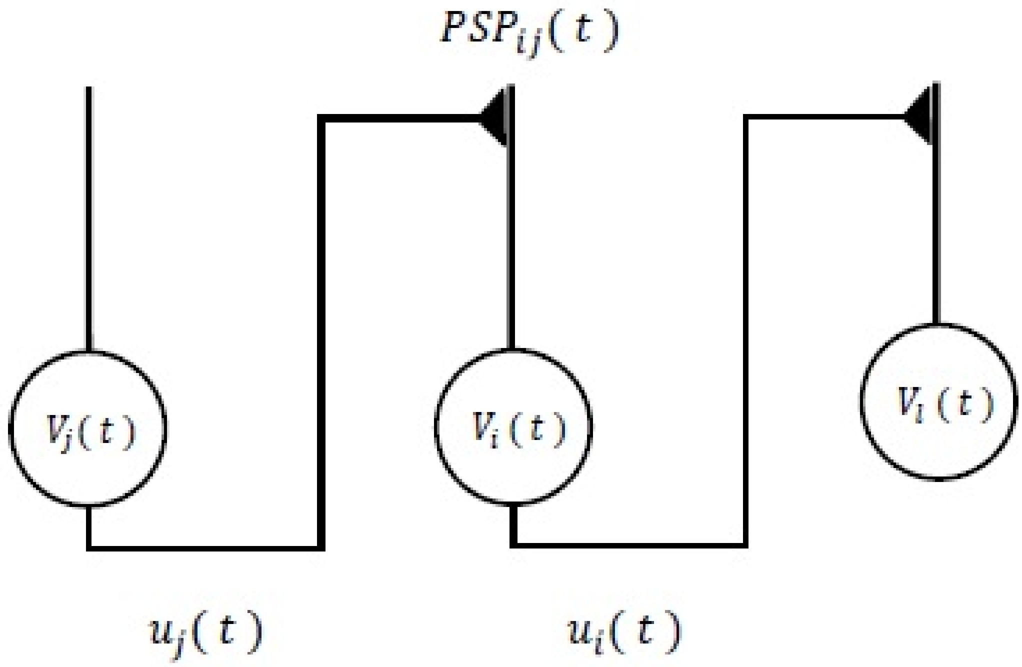


Figure 6: Synaptical interaction between neurons [44]

ing make the analysis simpler and innate understanding. Thus these models, instead of modeling each neuron spikes individually, we focus on modeling the possibility of action potential. The membrane potential is typically transformed to a firing rate by a firing rate function. These models have been usually utilized in examining the network properties of strongly interconnected cortical networks, e.g. the two population synaptic drive model and neural field models [44].

### 1.2.4 Neural network models

In neural field models the cortical tissue is modeled in a continuous lines or sheets of neurons. In other words, it is the network composed of layers [44].
Since all parts of body receives messages and connect each other through nerve tissues called neurons so we develop a space time model for neural network. The easy derivation of neural network is shown in Fig. 6.
The Fig. 6 shows three neurons connected each other in such a way that axon of neuron j is connected to dendrites of neuron i.
The formulation of these models can be carried out by following esay steps [10]. Let $U_i$ represents the firing rate of neuron i at positive time$(t > 0)$ and this is given as

$$\mathbf{u_i}(\mathbf{t}) = \mathbf{P_i}(\mathbf{V_i}(t)) \tag{3}$$

Where $Pi$ is the firing rate function that changes potential $Vi$ into firing rate $Ui$ at time t. This conversion can be delayed due to the distance traveled along axon. This time delay is

denoted by $\delta ij$.
The sum of all action potentials, approaches at the terminal, form the total potential that is defined as

$$\Psi ij = \sum_k \mathbf{PSP_{ij}}(\mathbf{t}_k) \tag{4}$$

- where $t_k$ represents the time of action potential occurred for k=1,2,3...
- Let $S$ be the time representing total potential approached and $t$ represents the time of action potential fired.

The total potential of neuron(i,j) at time t and $t+\triangle t$ is

$$\Psi ij(t) = \int_0^t \mathbf{PSP_{ij}}(\mathbf{t}-\mathbf{s})\mathbf{u_j}(\mathbf{s}-\delta_{\mathbf{ij}})ds \tag{5}$$

We get the following Volterra integral equation as a result of hitting pre-synaptic neurons to the post-synaptic neurons i.

$$\mathbf{V_i}(t) = \sum_j \int_{t_o}^t \mathbf{PSP_{ij}}(t-s)\mathbf{P_j}(\mathbf{V_j}(s-\delta_{ij}))ds \tag{6}$$

Hence the final form of action potential formula of neuron i is

$$\mathbf{U_i}(t) = \mathbf{P_i}(\sum_j \int_{t_o}^t \mathbf{PSP_{ij}}(t-s)\mathbf{P_j}(\mathbf{V_j}(s-\delta_{ij}))ds) \tag{7}$$

### 1.2.5 General Synaptic Drive Model

If the shape of the post synaptic potential (PSP) depends only on the nature of the pre-synaptic neuron, the post synaptic potential at the neuron $i$ is the combined effect of the connection strength and the PSP at $j$-th cell, i.e.,

$$PSP_{ij}(t) = \omega_{ij}P_j(V_j(t)) \tag{8}$$

Further we introduce the time-averaged firing rate of the cell $j$ as

$$U_j(t) = \int_{t_0}^t PSP_j(t-s)\, u_j(s)ds \tag{9}$$

Since the PSP is more oftener assumed to be a function of exponentials and its powers. Integrating (9) yields:

$$L_jU_j(t) = u_j(t) \tag{10}$$

where $L_j$ is linear differential operator. Now by using the equation(3), we end up with

$$L_iU_i(t) = P_i\left(\sum_j \omega_{ij} \int_{t_0}^t PSP_j(t-s)\, U_j(s-\delta_{ij})ds\right) \tag{11}$$

If there is no delay, then the operator $L_i$ is first order linear operator. Hence

$$\tau_i \frac{dU_i}{dt} = -U_i + P_i\left(\sum_j \omega_{ij} U_j\right) \tag{12}$$

where $\tau_i$ is the time constant related to the synaptic decay and $U_i$ is called the synaptic drive [12][28][31].

### 1.2.6 Two Population Synaptic Drive Model (General form)

Communication in the brain is done through the brain cells. This process of conveying messages from neuron to neuron can be described by various mathematical models. One of these models is two population synaptic drive model. This model has been investigated by [22][28][40].
A particular type of integro-differential equation is Volterra equation classified into two categories known as fist and second type. Here I will use first kind Volterra equations with respect to excitatory and inhibitory neurons as

$$u_e = \alpha_e * P_e(\omega_{ee} u_e - \omega_{ei} u_i) \tag{13a}$$

$$\tau u_i = \alpha_i * P_i(\omega_{ie} u_e - \omega_{ii} u_i) \tag{13b}$$

Here $u_e$ and $u_i$ represents average activity levels for excitatory and inhibitory neurons. $\omega_{mn}$ $(m = e,i)$ is the connectivity parameters assumed to be positive and measure the connection strength between neurons.

The Hill function modeled the activation function $P_m$, which is given

$$Z = P_m(x_m) = \frac{x_m^{1/q_m}}{x_m^{1/q_m} + \theta_m^{1/q_m}}; \qquad m = e,i \tag{14}$$

Here $q_m (m = e,i)$ is the steepness parameter. If $q$ is small then function is steep around $\theta$ and $\theta_m$ $(m = e,i)$ is threshold value. Where $H$ is called the Hill function. The parameter $\theta_m$ gives the threshold values, while $q_m$ for $m = e,i$ measures the steepness of the firing rate function and it is assumed to satisfy $0 < q_m \leq 1$. In Fig. 7 the Hill function is plotted for different values of the steepness parameter $q$ for a fixed threshold value $\theta = 0.5$ (for a detailed review see [32][35]. For simplicity, we assume that $q_i = q_e = q$.

The time history of the model is calculated by the convolution integral which is given as

$$(\alpha_m * f_m(x))(t) = \int_{-\infty}^{t} \alpha_m(t-s) f_m(s) ds \tag{15}$$

Here $\alpha_m$ for $m = e,i$ is the temporal kernel. This temporal kernel defines the impact of past neural firing on the present activity levels. So they are typically modeled in terms of exponential decaying functions. In the start their impact is maximum and with the passage of time it decreases exponentially. Simpler temporal functions cover lesser details of the

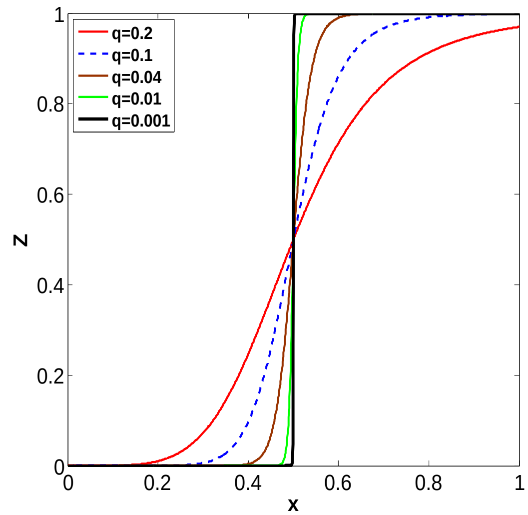


Figure 7: The Hill function for different values of the steepness parameter $q$ at $\theta = 0.5$ [44].

model as compare to the complex ones which covers more details of the model. Some of the examples are given as under:

$$\alpha_e(t) = exp(-t) \tag{16}$$

$$\alpha_i(t) = \frac{1}{\tau} exp(-t/\tau) \tag{17}$$

$$\alpha_m(t) = \frac{t}{\tau_m^2} exp(-t/\tau_m) \tag{18}$$

The system of Volterra equations (13) can be converted to a 2D autonomous dynamical system by means of the linear chain trick [7][26][28].

## 1.3 Some important definitions

Here we revise some basic definitions which have been encountered in the coming sections.

### 1.3.1 Dynamical System

The origin of word dynamical system is from Newtonian mechanics. In mathematics, it is a concept where a function explains the time dependence of a point in a geometrical space.

### 1.3.2 Wellposedness

A problem or a function is said to be wellposed if
1)It contains a solution.
2)It contains a unique solution.
3)The solution incessantly depends on parameters.

### 1.3.3 Stationary Points

A point $x$ is said to be stationary if at that point the derivative of given function $f(x)$ becomes disappear as

$$f'(x) = 0 \tag{19}$$

At this point the function then neither increases nor decreases.

### 1.3.4 Implicit Function Theorem

Let $f$(x,y) be a function of two variables $x$ and $y$ and (a,b) be a point of its domain of definition such that
1)$f(a,b) = 0$
2)The partial derivatives of $f(x,y)$ w.r.t x and y exists and are continuous in a certain neighborhood of $(a,b)$
This is also called existence theorem.

### 1.3.5 Non Generated Jacobian

The Jacobian for a system of equations is said to be non generated if

$$\det(J) \neq 0 \tag{20}$$

This is very important definition to check existence of stationary points by using implicit function theorem.

# Chapter 2
# The Model

# 2 Two Population Synaptic Drive Model

A two population synaptic drive model of excitatory and inhibitory neurons is given by two equations (seen from [28][44]).

$$u'_e = -u_e + P_e(\omega_{ee}u_e - \omega_{ei}u_i) \tag{21a}$$

$$\tau u'_i = -u_i + P_i(\omega_{ie}u_e - \omega_{ii}u_i) \tag{21b}$$

where $u_e$ and $u_i$ represents the excitatory and inhibitory synaptic drives, in that order. The function $\omega_{mn}$ is the correlation power from n to m cells and $Pm$ corresponds the activation (firing rate) function. The kernels for excitatory and inhibitory settlements are considers to be

$$\alpha_e(t) = exp(-t) \tag{22}$$

$$\alpha_i(t) = \frac{1}{\tau}exp(-t/\tau) \tag{23}$$

where $\tau$ is the comparative inhibition time constant and is understood to be strictly positive. Nordb et al [28] have examined the stability properties of the equilibrium states of the model 21 for a verity of temporal coupling functions under the condition of the firing rate function is given in form of piecewise linear function. The stability analysis is carried out by means of the Routh-Hurwitz criterion. The occurrence of Hopfs bifurcation is detected by means of a breakdown of the strictly positivity of one of the Routh-Hurwitz determinants [44].

In the present working we will use the same model but with different temporal kernel.

## 2.1 Autonomous Dynamical System

In this section first, we derived the 4-dimensional autonomous dynamical system from Volterra equations (13) by using linear chain trick for further explanation see [28][36]. The corresponding temporal kernel is given in (18). We end up with the following 4-dimensional autonomous dynamical system

$$u'_e = -u_e + v_e \tag{24a}$$

$$\tau u'_i = -u_i + v_i \tag{24b}$$

$$v'_e = -v_e + f_e(\omega_{ee}u_e - \omega_{ei}u_i) \tag{24c}$$

$$\tau v'_i = -v_i + f_i(\omega_{ie}u_e - \omega_{ii}u_i) \tag{24d}$$

The auxiliary variables $v_e$ and $v_i$ are defined as

$$v_e = \int_{-\infty}^{t} exp(-(t-s))f_e(\omega_{ee}u_e(s) - \omega_{ei}u_i(s))ds \tag{25a}$$

$$v_i = \frac{1}{\tau}\int_{-\infty}^{t} exp(-(t-s)/\tau)f_i(\omega_{ie}u_e(s) - \omega_{ii}u_i(s))ds \tag{25b}$$

## 2.2 wellposedness

Wellposedness of the system (24) depends on its right hand side function which can be expressed in the following matrix form

$$\begin{bmatrix} u_e' \\ u_i' \\ v_e' \\ v_i' \end{bmatrix} = \begin{bmatrix} -1 & 0 & 1 & 0 \\ 0 & -\tau^{-1} & 0 & \tau^{-1} \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -\tau^{-1} \end{bmatrix} \begin{bmatrix} u_e \\ u_i \\ v_e \\ v_i \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ f_e \\ f_i/\tau \end{bmatrix} \tag{26}$$

Which is

$$\mathbf{S}(\mathbf{U}) = \mathbf{D}\,\mathbf{U} + \mathbf{G} \tag{27}$$

where the matrices **U**, **D** and **G** are given as

$$\mathbf{D} = \begin{bmatrix} -1 & 0 & 1 & 0 \\ 0 & -\tau^{-1} & 0 & \tau^{-1} \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -\tau^{-1} \end{bmatrix} \tag{28}$$

$$\mathbf{U} = \begin{bmatrix} u_e \\ u_i \\ v_e \\ v_i \end{bmatrix} \tag{29}$$

$$\mathbf{G} = \begin{bmatrix} 0 \\ 0 \\ f_e \\ f_i/\tau \end{bmatrix} \tag{30}$$

If the function (27) is differentiable for finite $q > 0$, then system (24) is wellposed ([5][38][39]).

### 2.2.1 Local wellposedness

The right hand side of the system (24) consists of two parts, first one is linear whereas the second one is non-linear part (Hill function). The first one is linear and hence differentiable. The non-linear part is always differentiable for finite steepness "q" using the properties of the hill function. Hence the system (24) is locally wellposed ([18][33][37]).

### 2.2.2 Global Wellposedness

It will be shown that the system doesn't grow larger than a linear growth. Now for the global Wellposedness, it will be shown that the right hand side of the system is bounded by some linear function. For the global Wellposedness it will be enough to show that the growth of the function **S** is bounded by some linear function [29]. Since the non-linear part of the function **S** is Hill function which is always bounded for finite $q > 0$, so

$$|\mathbf{G}| \leq \; 1 = C_1 \tag{31}$$

and linear part of $\underline{S}$ is also bounded because all four eigen values of the matrix **D** are negative which results the following boundedness

$$
\begin{aligned}
|\mathbf{D}\,\mathbf{U}| &= |\mathbf{D}| \cdot |\mathbf{U}| && (32)\\
&\leq C_2|\mathbf{U}| && (33)
\end{aligned}
$$

By combining (31) and (32)

$$|\mathbf{S}(\mathbf{U})| \leq C_1 + C_2\,|\mathbf{U}| \tag{34}$$

Hence the dynamical system (24) is globally wellposed.

# Chapter 3
# Existence and Stability

# 3 Stationary points of the model

In this chapter we will investigate the existence and stability of stationary points of the model.

First we have a look at definition of stationary points. As above defined that these are those points where the derivative of a function $f(x)$ disappears (seen from [8]) as.

$$f'(x) = 0 \tag{35}$$

Now by using this definition we will find the stationary points of autonomous dynamical system (24) for $q = 0$ in which the steepness approaches to infinity.
The system (24) is as .

$$u_e' = -u_e + v_e \tag{36a}$$

$$\tau u_i' = -u_i + v_i \tag{36b}$$

$$v_e' = -v_e + f_e(\omega_{ee}u_e - \omega_{ei}u_i) \tag{36c}$$

$$\tau v_i' = -v_i + f_i(\omega_{ie}u_e - \omega_{ii}u_i) \tag{36d}$$

By applying def (35) of stationary points to this system

$$0 = -u_e^0 + v_e^0 \tag{37a}$$

$$0 = -u_i^0 + v_i^0 \tag{37b}$$

$$0 = -v_e^0 + f_e(\omega_{ee}u_e^0 - \omega_{ei}u_i^0) \tag{37c}$$

$$0 = -v_i^0 + f_i(\omega_{ie}u_e^0 - \omega_{ii}u_i^0) \tag{37d}$$

Further, in more general form

$$u_e^0 = v_e^0 \tag{38a}$$

$$u_i^0 = v_i^0 \tag{38b}$$

$$v_e^0 = f_e(\omega_{ee}u_e^0 - \omega_{ei}u_i^0) \tag{38c}$$

$$v_i^0 = f_i(\omega_{ie}u_e^0 - \omega_{ii}u_i^0) \tag{38d}$$

It is shown from these equations that stationary points depends only on two equation that are (38c) and (38d). These equations represents the non delay part of the system.
In order to find the general formula for stationary points, we will transform the non delay part of dynamical system (i.e (24c) and (24d)) to a new set of variables x and y through following transformation.

$$x = \omega_{ee}v_e - \omega_{ei}v_i \tag{39}$$

$$y = \omega_{ie}v_e - \omega_{ii}v_i \tag{40}$$

Its matrix form is

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \omega_{ee} & -\omega_{ei} \\ \omega_{ie} & -\omega_{ii} \end{bmatrix} \begin{bmatrix} v_e \\ v_i \end{bmatrix} \tag{41}$$

which is

$$\mathbf{X} = \Omega\, \mathbf{V} \tag{42}$$

Here $X$, $\Omega$ and $V$ are

$$\Omega = \begin{bmatrix} \omega_{ee} & -\omega_{ei} \\ \omega_{ie} & -\omega_{ii} \end{bmatrix} \tag{43}$$

$$\mathbf{X} = \begin{bmatrix} x \\ y \end{bmatrix} \tag{44}$$

$$\mathbf{V} = \begin{bmatrix} v_e \\ v_i \end{bmatrix} \tag{45}$$

As non delay part of system (24) is

$$v_e' = -v_e + f_e(\omega_{ee}u_e - \omega_{ei}u_i) \tag{46a}$$

$$\tau v_i' = -v_i + f_i(\omega_{ie}u_e - \omega_{ii}u_i) \tag{46b}$$

Converting into matrix form

$$\begin{bmatrix} v_e' \\ v_i' \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -\tau^{-1} \end{bmatrix} \begin{bmatrix} v_e \\ v_i \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & \tau^{-1} \end{bmatrix} \begin{bmatrix} f_e \\ f_i \end{bmatrix} \tag{47}$$

$$\begin{bmatrix} v_e' \\ v_i' \end{bmatrix} = -\begin{bmatrix} 1 & 0 \\ 0 & \tau^{-1} \end{bmatrix} \begin{bmatrix} v_e \\ v_i \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & \tau^{-1} \end{bmatrix} \begin{bmatrix} f_e \\ f_i \end{bmatrix} \tag{48}$$

$$\mathbf{V}' = -\mathbf{C}\,\mathbf{V} + \mathbf{C}\,\mathbf{G}(\mathbf{X}) \tag{49}$$

Where

$$\mathbf{V}' = \begin{bmatrix} v_e' \\ v_i' \end{bmatrix} \tag{50}$$

$$\mathbf{C} = \begin{bmatrix} 1 & 0 \\ 0 & \tau^{-1} \end{bmatrix} \tag{51}$$

$$\mathbf{G(x)} = \begin{bmatrix} f_e \\ f_i \end{bmatrix} \tag{52}$$

$$\mathbf{V} = \begin{bmatrix} v_e \\ v_i \end{bmatrix} \tag{53}$$

Taking derivative of equation (42)

$$\mathbf{X}' = \Omega\, \mathbf{V}' \tag{54}$$

Using value of $\mathbf{V}$ from (49)

$$\mathbf{X}' = \Omega\, (-\mathbf{C}\, \mathbf{V} + \mathbf{C}\, \mathbf{G}(\mathbf{X})) \tag{55}$$

As by assumption $\Omega$ is invertible, we get

$$\mathbf{X}' = \Omega\, (-\mathbf{C}\, \Omega^{-1}\mathbf{X} + \mathbf{C}\, \mathbf{G}(\mathbf{X})) \tag{56}$$

Or

$$\mathbf{X}' = -\Omega\mathbf{C}\Omega^{-1}\mathbf{X} + \Omega\mathbf{C}\mathbf{G}(\mathbf{X}) \tag{57}$$

Or

$$\mathbf{X}' = \mathbf{A} \cdot \mathbf{X} + \mathbf{B}(\mathbf{X}) \tag{58}$$

Where

$$\mathbf{A} = -\Omega\mathbf{C}\, \Omega^{-1} \tag{59}$$

$$\mathbf{B}(\mathbf{X}) = \Omega\mathbf{C}\mathbf{G}(\mathbf{X}) \tag{60}$$

Since

$$\mathbf{A} = -\Omega\mathbf{C}\, \Omega^{-1} \tag{61}$$

And

$$\Omega = \begin{bmatrix} \omega_{ee} & -\omega_{ei} \\ \omega_{ie} & -\omega_{ii} \end{bmatrix} \tag{62}$$

$$\mathbf{C} = \begin{bmatrix} 1 & 0 \\ 0 & \tau{-}1 \end{bmatrix} \tag{63}$$

So

$$\Omega\mathbf{C} = \begin{bmatrix} \omega_{ee} & -\omega_{ei} \\ \omega_{ie} & -\omega_{ii} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & \tau^{-1} \end{bmatrix} \tag{64}$$

$$\Omega\mathbf{C} = \begin{bmatrix} \omega_{ee} & -\tau^{-1}\omega_{ei} \\ \omega_{ie} & -\tau^{-1}\omega_{ii} \end{bmatrix} \tag{65}$$

$$-\Omega\mathbf{C} = \begin{bmatrix} -\omega_{ee} & \tau^{-1}\omega_{ei} \\ -\omega_{ie} & \tau^{-1}\omega_{ii} \end{bmatrix} \tag{66}$$

As

$$\Omega^{-1} = \frac{Adj\Omega}{det\Omega} \tag{67}$$

And

$$Adj(\Omega) = \begin{bmatrix} -\omega_{ii} & \omega_{ei} \\ -\omega_{ie} & \omega_{ee} \end{bmatrix} \tag{68}$$

$$det(\Omega) = -\omega_{ee}\omega_{ii} + \omega_{ei}\omega_{ie} \tag{69}$$

Using these values in Eq. (61)

$$\mathbf{A} = -\Omega\mathbf{C}\,\Omega^{-1} \tag{70}$$

$$\mathbf{A} = \frac{1}{-\omega_{ee}\omega_{ii} + \omega_{ei}\omega_{ie}} \begin{bmatrix} -\omega_{ee} & \tau^{-1}\omega_{ei} \\ -\omega_{ie} & \tau^{-1}\omega_{ii} \end{bmatrix} \begin{bmatrix} -\omega_{ii} & \omega_{ei} \\ -\omega_{ie} & \omega_{ee} \end{bmatrix} \tag{71}$$

Or

$$\mathbf{A} = \frac{1}{d} \begin{bmatrix} \omega_{ee}\omega_{ii} - \tau^{-1}\omega_{ei}\omega_{ie} & -\omega_{ee}\omega_{ei} + \tau^{-1}\omega_{ee}\omega_{ei} \\ \omega_{ii}\omega_{ie} - \tau^{-1}\omega_{ii}\omega_{ie} & -\omega_{ei}\omega_{ie} + \tau^{-1}\omega_{ee}\omega_{ii} \end{bmatrix} \tag{72}$$

Where the parameter $d$ is

$$d = \omega_{ei}\omega_{ie} - \omega_{ee}\omega_{ii} \tag{73}$$

Or

$$\mathbf{A} = \begin{bmatrix} \frac{1}{\tau d}\,(\tau\omega_{ee}\omega_{ii} - \omega_{ei}\omega_{ie}) & \frac{1}{\tau d}\,\omega_{ee}\omega_{ei}(1-\tau) \\ \frac{1}{\tau d}\,\omega_{ii}\omega_{ie}(\tau-1) & \frac{1}{\tau d}\,(\tau\omega_{ee}\omega_{ie} + \omega_{ei}\omega_{ii}) \end{bmatrix} \tag{74}$$

Or

$$\mathbf{A} = \begin{bmatrix} \alpha_1 & \beta_1 \\ \alpha_2 & \beta_2 \end{bmatrix} \tag{75}$$

Where the coefficients $\alpha_1$, $\alpha_2$, $\beta_1$ and $\beta_2$ are given as

$$\alpha_1 = \frac{1}{\tau d}\,(\tau\omega_{ee}\omega_{ii} - \omega_{ei}\omega_{ie}) \tag{76a}$$

$$\beta_1 = \frac{1}{\tau d}\,\omega_{ee}\omega_{ei}(1-\tau) \tag{77a}$$

$$\alpha_2 = \frac{1}{\tau d}\,\omega_{ii}\omega_{ie}(\tau-1) \tag{78a}$$

$$\beta_2 = \frac{1}{\tau d}\,\tau\omega_{ee}\omega_{ie} + \omega_{ei}\omega_{ii} \tag{79a}$$

Now considering Eq. (60). As $\Omega C$ is already calculated given in (65) then

$$\mathbf{B}(\mathbf{x}) = \Omega\mathbf{C}\mathbf{G}(\mathbf{x}) = \begin{bmatrix} \omega_{ee} & -\tau^{-1}\omega_{ei} \\ \omega_{ie} & -\tau^{-1}\omega_{ii} \end{bmatrix} \begin{bmatrix} f_e(x) \\ f_i(y) \end{bmatrix} \tag{80}$$

Or

$$\mathbf{B}(\mathbf{x}) = \begin{bmatrix} \omega_{ee} f_e(x) - \tau^{-1} \omega_{ei} f_i(y) \\ \omega_{ie} f_e(x) - \tau^{-1} \omega_{ii} f_i(y) \end{bmatrix} \tag{81}$$

Using these values of Eq. (81) and Eq. (75) in Eq. (58), we get

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \alpha_1 & \beta_1 \\ \alpha_2 & \beta_2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} \omega_{ee} f_e(x) - \tau^{-1} \omega_{ei} f_i(y) \\ \omega_{ie} f_e(x) - \tau^{-1} \omega_{ii} f_i(y) \end{bmatrix} \tag{82}$$

Or

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \alpha_1 x + \beta_1 y \\ \alpha_2 x + \beta_2 y \end{bmatrix} + \begin{bmatrix} \omega_{ee} f_e(x) - \tau^{-1} \omega_{ei} f_i(y) \\ \omega_{ie} f_e(x) - \tau^{-1} \omega_{ii} f_i(y) \end{bmatrix} \tag{83}$$

From here $x$ and $y$ components are

$$x' = \alpha_1\, x + \beta_1\, y + \omega_{ee}\, f_e(x) - \frac{1}{\tau}\, \omega_{ei}\, f_i(y) \tag{84a}$$

$$y' = \alpha_2\, x + \beta_2\, y + \omega_{ie}\, f_e(x) - \frac{1}{\tau}\, \omega_{ii}\, f_i(y) \tag{84b}$$

For any activation function $f_m$ one can show that the system (84) possess the stationary point $(x_0, y_0)$ given by

$$x_0 = \frac{1}{\alpha_1\beta_2 - \alpha_2\beta_1}\; (\beta_1\omega_{ie} - \beta_2\omega_{ee}) f_e(x) + \frac{1}{\tau}\, (\beta_2\omega_{ei} - \beta_1\omega_{ii}) f_i(y) \tag{85a}$$

$$y_0 = \frac{1}{\alpha_2\beta_1 - \alpha_1\beta_2}\; (\alpha_1\omega_{ie} - \alpha_2\omega_{ee}) f_e(x) + \frac{1}{\tau}\, (\alpha_2\omega_{ei} - \alpha_1\omega_{ii}) f_i(y) \tag{85b}$$

Now consider the following theorem that will guarantee the existence of stationary points for the model for $q > 0$

**Theorem 3.1.** *Assume that the system* (84) *possesses stationary point for $q = 0$. Then the system* (84) *has also stationary point for $q > 0$, if* (84) *has non generated Jacobian for $q = 0$.*

*Proof.* We proof this theorem in two parts, first the existence of the stationary point is proved by using Implicit function theorem and then the condition for non-generated Jacobian.

1. By assumption we have stationary point for $q = 0$, and all these stationary points belong to regular domains regular. The only discontinuity can occur along the threshold lines. Hence by implicit function theorem, the system (84) also have stationary points for small $q > 0$ if we have non-generated Jacobian for $q = 0$, which is given in next step.

2. Here we will find the condition to have non-generated Jacobian for $q = 0$.
The Jacobian for the system of equations (24c) and (24d) is

$$J = \begin{bmatrix} \alpha_1 & \beta_1 \\ \alpha_2 & \beta_2 \end{bmatrix} \tag{86}$$

For non-generated Jacobian the $det(J) \neq 0$. i.e

$$det(J) = \alpha_1\beta_2 - \alpha_2\beta_1 \neq 0 \tag{87}$$

Using the values of the parameters $\alpha_1$, $\alpha_2$, $\beta_1$ and $\beta_2$ given by (76)(77)(78)(79), we end up with the following condition

$$\frac{\omega_{ei}\omega_{ii}(\omega_{ee}^2 - \omega_{ie}^2)}{\omega_{ee}\omega_{ie}(\omega_{ei}^2 - \omega_{ii}^2)} \neq \frac{1}{\tau} \tag{88}$$

□

Hence the system (84) also has stationary point for $q > 0$, if the condition (88) is satisfied.
In the next subsection stability properties of this equilibrium state is discussed.

## 3.1 Stability of the equilibrium state in non-delay case

Here, we first investigated the stability properties of 2-D model (84) by finding the trace and determinant of the Jacobian (86) for $q = 0$, which are given as

$$trace(\mathbf{J}) = T = -(1 + 1/\tau) \tag{89}$$

$$determinant(\mathbf{J}) = D = \frac{\omega_{ei}\omega_{ii}(\omega_{ee}^2 - \omega_{ie}^2)}{\omega_{ie}\omega_{ee}(\omega_{ii}^2 - \omega_{ei}^2)} - \frac{1}{\tau} \tag{90}$$

The stationary point is stable if $T < 0$ and $D > 0$ [4][41]. From inequality (89), we always get trace $T < 0$ because $\tau > 0$ time constant. Hence the stability properties of the dynamical system (84) depends only on the determinant of the Jacobian, which leads us to the following theorem.

**Theorem 3.2.** *Assume that the system* (84) *has stationary point for $q = 0$. If this stationary point is stable for $q = 0$ then it is also stable for small $q > 0$.*

*Proof.* Existence of the stationary point for small $q > 0$ is done by the theorem (3.1). We now look at the stability of the stationary point for $q = 0$. The Jacobian for the non delay case is given by (86). For the stability of the stationary point, the determinant of the Jacobian $\mathbf{J}$ must be positive which leads us to the following condition

$$\frac{\omega_{ei}\omega_{ii}(\omega_{ee}^2 - \omega_{ie}^2)}{\omega_{ie}\omega_{ee}(\omega_{ii}^2 - \omega_{ei}^2)} > \frac{1}{\tau} \tag{91}$$

Since the Jacobian $\mathbf{J}$ depends continuously on the steepness parameter q, Hence by implicit function theorem (91) is also satisfied for small $q > 0$, i.e.

$$det(\mathbf{J}(q)) > 0\ , \quad 0 < q < q_0 \tag{92}$$

where $q_0$ is a small positive number.
In the next subsection a numerical example is given to verify stability results. □

### 3.1.1 Numerical example

We, now find the stationary point of the system (84) and then the stability of this stationary point for specific values of the parameters for $q = 0$ is investigated. i.e.

$$\omega_{ee} = 0.3, \omega_{ei} = 0.1, \omega_{ii} = 0.5, \omega_{ie} = 0.9, \theta_e = 0.5, \theta_i = 0.2, \tau = 0.5 \tag{93}$$

We get two regular stationary points $P_1(0,0)$ and $P_2(1.3,0.4)$ corresponding to above set of parameters using (85) for $q = 0$. Since $P_1(0,0)$ is trivial case, we analyze the stability of the stationary point $P_2$. The invariants of the Jacobian corresponding to (93) are given as

$$T = -3 < 0, \quad D = 2 > 0 \tag{94}$$

From (94), it follows that $P_2$ is a stable stationary point and corresponding Jacobian is also non-generated, which can be verified from Fig. (8).

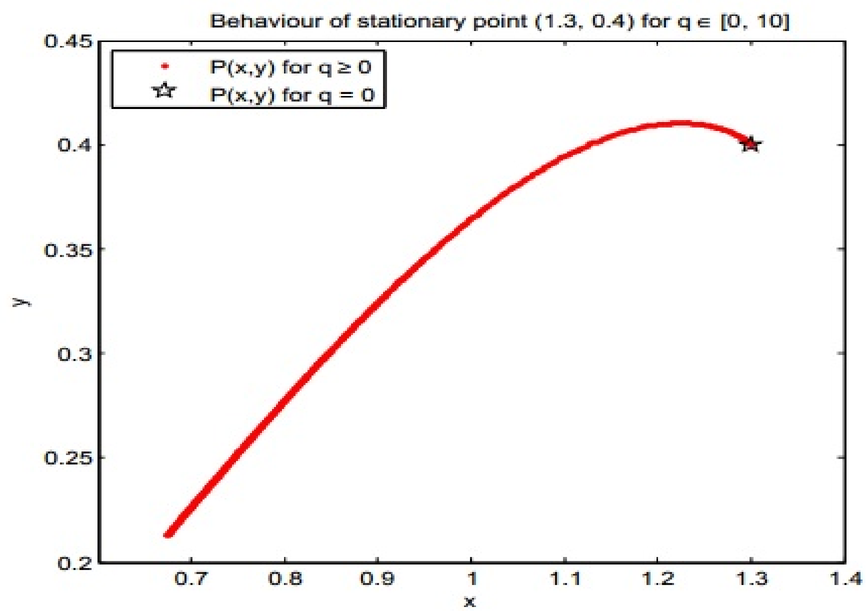


Figure 8: The behavior of the stable stationary point $P(1.3,0.4)$ for larger values of $q \in [0,10]$ by using specific values of parameters given by (93)

## 3.2 Model with delay effects

In this section we checked the stability properties of the model with delay effects. Firstly, We transformed the system in to new set of coordinates like we did in non-delay case. Then we compared the stability results in the delay and non-delay case. In the first step we write down the dynamical system (24) in matrix form, which is given as

$$\begin{bmatrix} u'_e \\ u'_i \\ v'_e \\ v'_i \end{bmatrix} = \begin{bmatrix} -1 & 0 & 1 & 0 \\ 0 & -\tau^{-1} & 0 & \tau^{-1} \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -\tau^{-1} \end{bmatrix} \begin{bmatrix} u_e \\ u_i \\ v_e \\ v_i \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ f_e(x) \\ f_i(y)/\tau \end{bmatrix} \tag{95}$$

$$\mathbf{U}' = \mathbf{D}\,\mathbf{U} + \mathbf{G} \tag{96}$$

Where the matrix $\mathbf{U}$ is given by (26) while $\mathbf{D}$ and $\mathbf{G}$ are as.

$$\mathbf{D} = \begin{bmatrix} -1 & 0 & 1 & 0 \\ 0 & -\tau^{-1} & 0 & \tau^{-1} \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -\tau^{-1} \end{bmatrix} \tag{97}$$

$$\mathbf{G} = \begin{bmatrix} 0 \\ 0 \\ f_e(x) \\ f_i(y)/\tau \end{bmatrix} \tag{98}$$

In the second step we introduce new variables $x_1$, $x_2$, $x_3$ and $x_4$, which are given as

$$x_1 = \omega_{ee}u_e - \omega_{ei}u_i \tag{99}$$

$$x_2 = \omega_{ie}u_e - \omega_{ii}u_i \tag{100}$$

$$x_3 = v_e \tag{101}$$

$$x_4 = v_i \tag{102}$$

Which in matrix form can be written as

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} \omega_{ee} & -\omega_{ei} & 0 & 0 \\ \omega_{ie} & -\omega_{ii} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} u_e \\ u_i \\ v_e \\ v_i \end{bmatrix} \tag{103}$$

$$\mathbf{Y} = \mathbf{E}\,\mathbf{U} \tag{104}$$

Where the matrices $\underline{\mathbf{Y}}$ and $\mathbf{E}$ are given as

$$\mathbf{E} = \begin{bmatrix} \omega_{ee} & -\omega_{ei} & 0 & 0 \\ \omega_{ie} & -\omega_{ii} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{105}$$

$$\mathbf{Y} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \tag{106}$$

Since $det(\mathbf{E}) \neq 0$, there is one to one correspondence between $\mathbf{Y}$ and $\mathbf{U}$. Using derivative of (104) and inserting value of (96) , we get

$$\mathbf{Y}' = \mathbf{E}\mathbf{U}' \tag{107}$$

$$\mathbf{Y}' = \mathbf{E}(\mathbf{D}\mathbf{U} + \mathbf{G}) \tag{108}$$

As

$$\mathbf{U} = \mathbf{E}'\mathbf{Y} \tag{109}$$

$$\mathbf{Y}' = \mathbf{E}\mathbf{D}\mathbf{E}^{-1}\mathbf{Y} + \mathbf{E}\mathbf{G} \tag{110}$$

Or

$$\mathbf{Y}' = \mathbf{A}_1\mathbf{Y} + \mathbf{B}_1 \tag{111}$$

Where

$$\mathbf{A}_1 = \mathbf{E}\mathbf{D}\mathbf{E}^{-1} \tag{112}$$

$$\mathbf{B}_1 = \mathbf{E}\mathbf{G} \tag{113}$$

Now we will find $A_1$ and $B_1$ by Using values of (105) and (97) given as

$$\mathbf{E} = \begin{bmatrix} \omega_{ee} & -\omega_{ei} & 0 & 0 \\ \omega_{ie} & -\omega_{ii} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{114}$$

$$\mathbf{D} = \begin{bmatrix} -1 & 0 & 1 & 0 \\ 0 & -\tau^{-1} & 0 & \tau^{-1} \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -\tau^{-1} \end{bmatrix} \tag{115}$$

$$\mathbf{E}\mathbf{D} = \begin{bmatrix} \omega_{ee} & -\omega_{ei}\tau^{-1} & \omega_{ee} & -\omega_{ei}\tau^{-1} \\ -\omega_{ie} & \omega_{ii}\tau^{-1} & \omega_{ie} & -\omega_{ii}\tau^{-1} \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -\tau^{-1} \end{bmatrix} \tag{116}$$

Since

$$\mathbf{E}^{-1} = \frac{AdjE}{detE} \tag{117}$$

First we will calculate $AdjE$

$$\mathbf{E} = \begin{bmatrix} \omega_{ee} & -\omega_{ei} & 0 & 0 \\ \omega_{ie} & -\omega_{ii} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{118}$$

$$\mathbf{A_{11}} = (-\mathbf{1})^{\mathbf{1+1}} \begin{vmatrix} -\omega_{ii} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = -\omega_{\mathbf{ii}} \tag{119}$$

$$\mathbf{A_{12}} = (-\mathbf{1})^{\mathbf{1+2}} \begin{vmatrix} -\omega_{ie} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = -\omega_{\mathbf{ie}} \tag{120}$$

$$\mathbf{A_{13}} = (-\mathbf{1})^{\mathbf{1+3}} \begin{vmatrix} \omega_{ie} & -\omega_{ii} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{vmatrix} = \mathbf{0} \tag{121}$$

$$\mathbf{A_{14}} = (-\mathbf{1})^{\mathbf{1+4}} \begin{vmatrix} \omega_{ie} & -\omega_{ii} & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{vmatrix} = \mathbf{0} \tag{122}$$

$$\mathbf{A_{21}} = (-\mathbf{1})^{\mathbf{2+1}} \begin{vmatrix} -\omega_{ei} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = \omega_{\mathbf{ei}} \tag{123}$$

$$\mathbf{A_{22}} = (-\mathbf{1})^{\mathbf{2+2}} \begin{vmatrix} -\omega_{ee} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = \omega_{\mathbf{ee}} \tag{124}$$

$$\mathbf{A_{23}} = (-\mathbf{1})^{\mathbf{2+3}} \begin{vmatrix} \omega_{ee} & -\omega_{ei} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{vmatrix} = \mathbf{0} \tag{125}$$

$$\mathbf{A_{24}} = (-\mathbf{1})^{\mathbf{2+4}} \begin{vmatrix} \omega_{ee} & -\omega_{ei} & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{vmatrix} = \mathbf{0} \tag{126}$$

$$\mathbf{A_{31}} = (-\mathbf{1})^{\mathbf{3+1}} \begin{vmatrix} -\omega_{ei} & 0 & 0 \\ -\omega_{ii} & 0 & 0 \\ 0 & 0 & 1 \end{vmatrix} = \mathbf{0} \tag{127}$$

$$\mathbf{A_{32}} = (-\mathbf{1})^{\mathbf{3+2}} \begin{vmatrix} \omega_{ee} & 0 & 0 \\ \omega_{ie} & 0 & 0 \\ 0 & 0 & 1 \end{vmatrix} = \mathbf{0} \tag{128}$$

$$\mathbf{A_{33}} = (-\mathbf{1})^{\mathbf{3+3}} \begin{vmatrix} \omega_{ee} & -\omega_{ei} & 0 \\ \omega_{ie} & -\omega_{ii} & 0 \\ 0 & 0 & 1 \end{vmatrix} = -\omega_{\mathbf{ee}}\omega_{\mathbf{ii}} + \omega_{\mathbf{ei}}\omega_{\mathbf{ie}} \tag{129}$$

$$\mathbf{A_{34}} = (-\mathbf{1})^{\mathbf{3+4}} \begin{vmatrix} \omega_{ee} & -\omega_{ei} & 0 \\ \omega_{ie} & -\omega_{ii} & 0 \\ 0 & 0 & 0 \end{vmatrix} = \mathbf{0} \tag{130}$$

$$\mathbf{A_{41}} = (-\mathbf{1})^{\mathbf{4+1}} \begin{vmatrix} -\omega_{ei} & 0 & 0 \\ -\omega_{ii} & 0 & 0 \\ 0 & 1 & 0 \end{vmatrix} = \mathbf{0} \tag{131}$$

$$\mathbf{A_{42}} = (-\mathbf{1})^{\mathbf{4+2}} \begin{vmatrix} \omega_{ee} & 0 & 0 \\ \omega_{ie} & 0 & 0 \\ 0 & 1 & 0 \end{vmatrix} = \mathbf{0} \tag{132}$$

$$\mathbf{A_{43}} = (-\mathbf{1})^{\mathbf{4+3}} \begin{vmatrix} \omega_{ee} & -\omega_{ei} & 0 \\ \omega_{ie} & -\omega_{ii} & 0 \\ 0 & 0 & 0 \end{vmatrix} = \mathbf{0} \tag{133}$$

$$\mathbf{A_{44}} = (-\mathbf{1})^{\mathbf{4+4}} \begin{vmatrix} \omega_{ee} & -\omega_{ei} & 0 \\ \omega_{ie} & -\omega_{ii} & 0 \\ 0 & 0 & 1 \end{vmatrix} = -\omega_{\mathbf{ee}}\omega_{\mathbf{ii}} + \omega_{\mathbf{ei}}\omega_{\mathbf{ie}} \tag{134}$$

Thus

$$\mathbf{Adj}(\mathbf{E}) = \begin{bmatrix} -\omega_{ii} & -\omega_{ie} & 0 & 0 \\ \omega_{ei} & \omega_{ee} & 0 & 0 \\ 0 & 0 & -\omega_{ee}\omega_{ii} + \omega_{ei}\omega_{ie} & 0 \\ 0 & 0 & 0 & -\omega_{ee}\omega_{ii} + \omega_{ei}\omega_{ie} \end{bmatrix} \tag{135}$$

And

$$\mathbf{det}(\mathbf{E}) = \begin{vmatrix} \omega_{ee} & -\omega_{ei} & 0 & 0 \\ \omega_{ie} & -\omega_{ii} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix} \tag{136}$$

$$\mathbf{det}(\mathbf{E}) \;=\; \omega_{\mathbf{ee}} \begin{vmatrix} -\omega_{ii} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} + \omega_{\mathbf{ei}} \begin{vmatrix} \omega_{ie} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} + \mathbf{0} \begin{vmatrix} \omega_{ie} & \omega_{ii} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{vmatrix} + \mathbf{0} \begin{vmatrix} \omega_{ie} & \omega_{ii} & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{vmatrix}$$

$$\mathbf{det}(\mathbf{E}) = \omega_{\mathbf{ee}}(-\omega_{\mathbf{ii}}) + \omega_{\mathbf{ei}}(\omega_{\mathbf{ie}}) \tag{137}$$

$$\mathbf{det}(\mathbf{E}) = -\omega_{\mathbf{ee}}\omega_{\mathbf{ii}} + \omega_{\mathbf{ei}}\omega_{\mathbf{ie}} \tag{138}$$

By putting these values in (117), we get

$$\mathbf{E}^{-1} = \frac{1}{\mathbf{d}} \begin{bmatrix} -\omega_{ii} & -\omega_{ie} & 0 & 0 \\ \omega_{ei} & \omega_{ee} & 0 & 0 \\ 0 & 0 & -\omega_{ee}\omega_{ii} + \omega_{ei}\omega_{ie} & 0 \\ 0 & 0 & 0 & -\omega_{ee}\omega_{ii} + \omega_{ei}\omega_{ie} \end{bmatrix} \tag{139}$$

Now

$$\mathbf{EDE}^{-1} = \frac{1}{\mathbf{d}} \begin{bmatrix} \omega_{ee} & -\omega_{ei}\tau^{-1} & \omega_{ee} & -\omega_{ei}\tau^{-1} \\ -\omega_{ie} & \omega_{ii}\tau^{-1} & \omega_{ie} & -\omega_{ii}\tau^{-1} \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -\tau^{-1} \end{bmatrix}$$

$$\begin{bmatrix} -\omega_{ii} & -\omega_{ie} & 0 & 0 \\ \omega_{ei} & \omega_{ee} & 0 & 0 \\ 0 & 0 & -\omega_{ee}\omega_{ii} + \omega_{ei}\omega_{ie} & 0 \\ 0 & 0 & 0 & -\omega_{ee}\omega_{ii} + \omega_{ei}\omega_{ie} \end{bmatrix} \tag{140}$$

Or

$$\mathbf{EDE}^{-1} = \frac{1}{\mathbf{d}}$$

$$\begin{bmatrix} -\omega_{ee}\omega_{ii} + \omega_{ei}\omega_{ie}\tau^{-1} & -\omega_{ee}\omega_{ie} + \omega_{ee}\omega_{ie}\tau^{-1} & \omega_{ee}(-\omega_{ee}\omega_{ii} + \omega_{ei}\omega_{ie}) & -\omega_{ei}\tau^{-1}(-\omega_{ee}\omega_{ii} + \omega_{ei}\omega_{ie}) \\ \omega_{ie}\omega_{ii} + \omega_{ii}\omega_{ei}\tau^{-1} & \omega_{ie}^2 + \omega_{ii}\omega_{ee}\tau^{-1} & \omega_{ie}(-\omega_{ee}\omega_{ii} + \omega_{ei}\omega_{ie}) & -\omega_{ii}\tau^{-1}(-\omega_{ee}\omega_{ii} + \omega_{ei}\omega_{ie}) \\ 0 & 0 & -(-\omega_{ee}\omega_{ii} + \omega_{ei}\omega_{ie}) & 0 \\ 0 & 0 & 0 & -\tau^{-1}(-\omega_{ee}\omega_{ii} + \omega_{ei}\omega_{ie}) \end{bmatrix}$$

As

$$\mathbf{A}_1 = \mathbf{EDE}^{-1} \tag{141}$$

Or

$$\mathbf{A}_1 = \begin{bmatrix} \alpha_1 & \beta_1 & \omega_{ee} & -\omega_{ei}/\tau \\ \alpha_2 & \beta_2 & \omega_{ie} & -\omega_{ii}/\tau \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1/\tau \end{bmatrix} \tag{142}$$

Where the coefficients $\alpha_1$, $\alpha_2$, $\beta_1$, $\beta_2$ and $d$ are given by (76)(77)(78)(79)(73).

$$\alpha_1 = \frac{1}{\tau d}\,(\tau\omega_{ee}\omega_{ii} - \omega_{ei}\omega_{ie}) \tag{143a}$$

$$\beta_1 = \frac{1}{\tau d}\,\omega_{ee}\omega_{ei}(1-\tau) \tag{144a}$$

$$\alpha_2 = \frac{1}{\tau d}\,\omega_{ii}\omega_{ie}(\tau - 1) \tag{145a}$$

$$\beta_2 = \frac{1}{\tau d}\,(\tau\omega_{ee}\omega_{ie} + \omega_{ei}\omega_{ii}) \tag{146a}$$

Now we will find the value of $EG$. As from 98 and 105

$$\mathbf{EG} = \begin{bmatrix} \omega_{ee} & -\omega_{ei} & 0 & 0 \\ \omega_{ie} & -\omega_{ii} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ f_e(x) \\ f_i(y)/\tau \end{bmatrix} \tag{147}$$

Or

$$\mathbf{B}_1 = \mathbf{EG} = \begin{bmatrix} 0 \\ 0 \\ f_e(x_1) \\ f_i(x_2) \end{bmatrix} \tag{148}$$

The transformed system (111) to new coordinates in component form can be expressed as

$$\begin{bmatrix} x_1{}' \\ x_2{}' \\ x_3{}' \\ x_4{}' \end{bmatrix} = \begin{bmatrix} \alpha_1 & \beta_1 & \omega_{ee} & -\omega_{ei}/\tau \\ \alpha_2 & \beta_2 & \omega_{ie} & -\omega_{ii}/\tau \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1/\tau \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ f_e(x_1) \\ f_i(x_2) \end{bmatrix} \tag{149}$$

Or

$$\begin{bmatrix} x_1{}' \\ x_2{}' \\ x_3{}' \\ x_4{}' \end{bmatrix} = \begin{bmatrix} \alpha_1 x_1 & \beta_1 x_2 & \omega_{ee} x_3 & -\omega_{ei} x_4/\tau \\ \alpha_2 x_1 & \beta_2 x_2 & \omega_{ie} x_3 & -\omega_{ii} x_4/\tau \\ 0 & 0 & -x_3 & 0 \\ 0 & 0 & 0 & -x_4/\tau \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ f_e(x_1) \\ f_i(x_2) \end{bmatrix} \tag{150}$$

Or

$$x_1' = \alpha_1\, x_1 \ + \beta_1\, x_2 + \omega_{ee} x_3 - \frac{\omega_{ei}}{\tau} x_4 \tag{151a}$$

$$x_2' = \alpha_2\, x_1 \ + \beta_2\, x_2 + \omega_{ie} x_3 - \frac{\omega_{ii}}{\tau} x_4 \tag{151b}$$

$$x_3' = -x_3 + f_e(x_1) \tag{151c}$$

$$x_4' = \frac{1}{\tau}(-x_4 + f_i(x_2)) \tag{151d}$$

### 3.2.1 Stability of the model with delays

In this subsection we investigated the stability of equilibrium states of the system (24) for $q = 0$. The corresponding Jacobian is given as

$$\mathbf{J}_1 = \begin{bmatrix} \alpha_1 & \beta_1 & \omega_{ee} & -\omega_{ei}/\tau \\ \alpha_2 & \beta_2 & \omega_{ie} & -\omega_{ii}/\tau \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1/\tau \end{bmatrix} \tag{152}$$

where the parameters $\alpha_1$, $\beta_1$, $\alpha_2$ and $\beta_2$ are defined by (76)(77)(78)(79). The stability properties of the dynamical system (151) depends on the eigen values of the Jacobian $\mathbf{J}_1$. Two (delay part) of four eigen values of the Jacobian $\mathbf{J}_1$ are always negative for the weak temporal kernel (18).
Thus the eigen values of the Jacobian $\mathbf{J}_1$ depends only on the non-delay part, which leads us to the conclusion that delays in the system with weak kernel (18) does not effect the stability properties of the model.

# Chapter 4
# Results

# 4 Conclusion

The conclusions of the present work can be summarized as follows.

- The model investigated in this work has global Wellposedness.
- The general form of the stationary points were found.
- These stationary points do exist for $q > 0$ and are same for the delay and non-delay cases.
- The stability of the equilibrium state has also been investigated and is found that also same for delay and non-delay case under weak temporal kernel (18).

# Chapter 5
# Discussion

# 5 Discussion

In the present study we studied a two population neuronal synaptic drive model and have investigated its dynamics. In the first step, Wellposedness of the problem has been investigated and found that the model is globally wellposed.

In the second part, we have found the general formula for the stationary points of the system for the steepness parameter $q = 0$ and formulated a theorem which guarantees the existence of stationary point for finite steepness $q > 0$. In the final part we have investigated the stability of the stationary point for both the cases and formulated theorem, which explains that delays in the system with weak kernel does not effect the stability properties of the model.

# Chapter 6
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