COMSATS University Islamabad, Lahore Campus

Stokes Flow Through a Permeable Slit With Periodic Reabsorption and Porous Medium

By

ZERNAB SHAFIQ

CIIT/FA17-RMT-005/LHR

MS Thesis

In

Mathematics

COMSATS University Islamabad,

Lahore Campus

Spring, 2019

STOKES FLOW THROUGH A PERMEABLE SLIT WITH PERIODIC REABSORPTION AND POROUS MEDIUM

A Thesis Presented to

COMSATS University Islamabad,

Lahore Campus

In partial fulfillment

of the requirement for the degree of

MS (Mathematics)

By

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Spring, 2019

STOKES FLOW THROUGH A PERMEABLE SLIT WITH PERIODIC REABSORPTION AND POROUS MEDIUM

A Post Graduate Thesis submitted to the department of Mathematics as partial fulfillment of the requirement for the award of Degree of MS in Mathematics.

Name	Registeration Number
ZERNAB SHAFIQ	CIIT/FA17-RMT-005/LHR

Supervisor

Dr. Qurat-ul-Ain Azim

Assistant Professor Department of Mathematics

COMSATS University Islamabad, Lahore Campus.

Certificate of Approval

This is to certify that the research work presented in this thesis entitled "Slip Effects on Creeping Viscous Flow through Porous Tube with Sinusoidal Reabsorption on the Wall" was conducted by Miss Zernab Shafiq CIIT/FA17-RMT-005/LHR, under the supervision of Dr. Qurat-ul-Ain Azim. No part of this thesis has been submitted anywhere else for any other degree. This thesis is submitted to the Department of Mathematics, COMSATS University Islamabad, Lahore Campus, in partial fulfillment of the requirements for the degree of Master of Science in the field of Mathematics.

Student Name: Zernab Shafiq Signature: ________________

External Committee:

External Examiner 1 Name
(Designation & Office Address)

External Examiner 2 Name
(Designation & Office Address)

Dr. Qurat-ul-Ain Azim
Supervisor,
Department of Mathematics,
CUI, Lahore Campus

Dr. Sarfraz Ahmad
HOD,
Department of Mathematics,
CUI, Lahore Campus

Prof. Dr. Shamsul Qamar
Chairperson,
Department of Mathematics,
CUI, Lahore Campus

Prof. Dr. Arshad Saleem Bhatti
Dean,
Department of Mathematics,
CUI, Lahore Campus

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Zernab Shafiq

CIIT/FA17-RMT-005/LHR

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Zernab Shafiq

CIIT/FA17-RMT-005/LHR

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Date:______________

Supervisor:

____

Dr. Qurat-ul-Ain Azim
Assistant Professor
Department of Mathematics
CUI, Lahore Campus

Head of Department:

____

Dr. Sarfraz Ahmad
Associate Professor
Department of Mathematics
CUI, Lahore Campus

DEDICATION

To

My Loving Parents,

and My Husband

ACKNOWLEDGEMENTS

All praises to ALLAH Almighty who created the universe and knows whatever is in it hidden or evident, who bestowed upon me the intellectual ability and wisdom to search for it's secrets and whose blessings flourished my thoughts and thrived my ambitions and enabled me to contribute sincerely to the sacred wealth of knowledge. Special praise and regards to his last messenger Muhammad (PBUH) who is a tower of guidance and knowledge for humanity.

I feel much pleasure in expressing my heartiest gratitude to my ever affectionate supervisor Dr. Qurat-ul-Ain Azim for her dynamic and friendly supervision. Whose indispensable guidance, deep consideration, constructive comments and active cooperation enabled me to complete my work successfully. All the words I know are insufficient to pay my tribute to her.

I would like to thank and all my teachers for their suggestions, guidance and constant support. I would like to say special thanks to Sadia Karamat for helping me in every situation. Of course, at this stage I cannot forget my whole family especially my respectable Parents Grand parents My Husband and my whole Family and friends for their prayers, continuing support and love. Their prayers have lightened up my spirit to finish this thesis.

COMSATS University Islamabad, Lahore Campus.

Zernab Shafiq
CIIT/FA17-RMT-005/LHR

Abstract

One of the main and most important organs of the body of an organism is Kidney. Nephron is the functional unit of Kidney which has a tubular part called renal tubule. The walls of renal tubules are porous and fluid reabsorption takes place in these tubules. Due to various diseases, the medium inside the tubules becomes porous and the flow permeability is reduced. There are numerous studies in literature that address uniform reabsorption in renal tubules through porous slits. We have done the study of two dimensional reabsorption flow through porous slits in renal tubules that are diseased and are under the effect of biofouling. We model this physical problem into a mathematical form, which result in a high order differential equation as well as associated boundary conditions. The problem then solve exactly, and expressions for various physical quantities are computed. At last, the obtained results are analyzed graphically.

Table of Contents

Chapter 1

Introduction to Fluid Mechanics

1.1 Fluid Mechanics

In Fluid mechanics we study nature of fluids both at rest and in motion.It has applications in many fields like mechanical,civil, synthetic and biomedical engineering, geophysics, astronomy, and biology.

1.2 Types of fluid mechanics

Types of fluid mechanics are :

- Fluid Dynamics

- Fluid Statics

- Fluid Kinematics

1.2.1 Fluid Dynamics

Fluid dynamics is a part of applied science where we study about movement of liquid and gases. In fluid dynamics Aerodynamics and Hydrodynamics is also included.

1.2.2 Fluid Statics

Fluid statics is the part of fluid mechanics that reviews compact fluids on rest. It includes investigation of the conditions on which fluid are at rest in stable balance rather than fluid dynamics, the investigation of fluid in motion.

1.2.3 Fluid Kinematics

Fluid kinematics is a part of physics and mechanics in which we study the flow of fluids and the particular forces that required to flow.Fluids have tendency to flow smoothly, which causes a net movement of molecules starting with one point in space then onto the next point in time.

1.3 Fluids

According to physics, fluid is a material that persistently twists (streams) under a applied shear pressure, or outside force. Fluids are a state of matter and contain liquids, gases and plasma.

1.4 Kind of Fluids

Fluids have following kinds:

- Ideal Fluid

- Real Fluid

- Newtonian Fluid

- Non Newtonian Fluid

1.4.1 Ideal Fluid

- Ideal fluid is incompressible (density is constant).

- Irrotational(the stream is smooth, no disturbance).
- Non viscous(liquid has no internal friction).
- Steady stream (the speed of the fluid at each point is constant in time).

1.4.2 Real Fluid

A fluid, which carry viscosity is called real fluid.

e.g Water, Air.

1.4.3 Newtonian Fluid

Real fluid where the shear stress and the rate of shear strain is directly proportional to each other is called Newtonian fluid.

e.g Water, Benzine.

1.4.4 Non Newtonian Fluid

Real fluid where the shear stress and the rate of shear strain is is not directly proportional to each other is called Non-Newtonian fluid.

e.g Plaster, Pastes.

1.5 Fluid Flow

In Fluid Flow we study about fluid dynamics. Whenever Fluids (Gases and Liquids) are in continious motion it is known as Fluid flow.

1.6 Types of Fluid Flow

Basic kinds of flow are:

• Compressible Flow

- Incompressible Flow
- Steady Flow
- Unsteady Flow
- Laminar Flow
- Turbulent Flow
- Rotational and Irrotational Flow

1.6.1 Compressible Flow

Compressible flow is a type of flow that undergo some variations in density with trending pressure.

1.6.2 Incompressible Flow

In Incompressible flow the fluid density switches from point to point and density is not constant for the fluid.

1.6.3 Steady Flow

A steady flow s a flow whose flow state remain constant with time.

1.6.4 Unsteady Flow

A unsteady flow is a flow whose flow state varies with time is known an unsteady flow.

1.6.5 Laminar Flow

Fluid particles moves in a regular paths is known Laminar flow.

1.6.6 Turbulent Flow

Fluid moves in very irregular paths is known as Turbulent flow.

1.6.7 Rotational and Irrotational Flow

In Rotational flow particles while flowing along also rotate around its axis.

In Irrotational flow particles while flowing along does not rotate around its axis.

1.7 Stream Function

The stream function describes incompressible flows in two dimensional as well as three dimensions with antisymmetry. It is used to plot streamlines of particles in steady flow.

Mathematical form.

$$\psi = \int (udy - vdx) \tag{1.1}$$

1.8 Reynolds Number

The Reynolds number is a ratio between inertial resistance and vicious resistance.

Mathematical form

$$Re = \frac{\rho UL}{\mu} \tag{1.2}$$

1.9 Viscosity

The amount that defines a fluid's resistance to flow is called Viscosity.

1.10 Kinds of Viscosity

Types of viscosity are:

- Kinematic Viscosity
- Dynamic Viscosity

1.10.1 Kinematic Viscosity

The ratio between viscosity of the fluid and the density of fluid is known as kinematic viscosity .

Mathematical form

$$\rho = \frac{mass}{volume} \tag{1.3}$$

$$\rho = \frac{m}{v} \tag{1.4}$$

1.10.2 Dynamic Viscosity

The power required by a fluid to its own interior sub-atomic grating so the fluid will flow is known as Dynamic viscosity .

Mathematical form

$$\eta = \frac{shearingstress}{shearrate} \tag{1.5}$$

$$\eta = \frac{\tau}{\gamma} \tag{1.6}$$

1.10.3 Continuity Equation

The continuity equation expresses that the rate at which mass enters a framework is equivalent to the rate at which mass leaves the framework in addition to the amassing of mass inside the framework.

Mathematical form.

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{V}) = 0 \tag{1.7}$$

There are two cases of equation of continuity given as:

• For steady flow we have, $\frac{\partial \rho}{\partial t} = 0$, the equation of continuity becomes $\nabla \cdot (\rho \mathbf{V}) = 0$

• For incompressible fluid flow, ρ is constant, the equation of continuity becomes $\nabla \cdot \mathbf{V} = 0$

1.10.4 Continuity equation in Cartesian Coordinates

The generalized form of 3-Dimensional continuity equation for each and every point in cartesian coordinates is given as:

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho u_x) + \frac{\partial}{\partial y}(\rho u_y) + \frac{\partial}{\partial z}(\rho u_z) = 0 \tag{1.8}$$

where u_x,u_y and u_z are the velocity components in x, y and z directions.

1.10.5 Continuity equation in Cylindrical Coordinates

The generalized form of 3-Dimensional continuity equation for each and every point in cylindrical coordinates is written as:

$$\frac{1}{r}(\frac{\partial}{\partial r}(rv_r)) + \frac{1}{r}(\frac{\partial v_\theta}{\partial \theta}) + \frac{\partial v_z}{\partial z} = 0 \tag{1.9}$$

1.10.6 Naiver Stokes Equation

In fluid mechanics, a partial differential equation which explains the flow of incompressible fluids is Naiver stokes equation.

$$\frac{\partial u}{\partial t} + \mathbf{u} \cdot \Delta \mathbf{u} = \frac{\Delta boldu}{\rho} + v\Delta^2 \mathbf{u} \tag{1.10}$$

In next chapter, we will discuss about our problem with its complete calculations we will briefly discuss about the previous studies in this matter and then we will explain our problem with problem formulations.

Chapter 2

Problem Description and Formulation

2.1 Introduction

Kidneys are important organs of bodies of complex organisms like Humans. They have two critical functions. First is to separate waste products from the body and second is to adjust the water level within the body. Kidneys have functional units called nephrons.Renal tubules consists of so many pores and where filtered liquid is absorbed through these small pores. Each kidney has about 1 million nephrons. Major parts of nephron are glomerulus tuft and renal tubules. When the body fluid passes through nephrons, all the essential materials are reabsorbed here. Also, 80Daily 200L fluid is filtered via kidney and about 2L of waste fluid is passed out of the human body. There are many small pores of same size present in proximal tubules of a kidney. The reabsorption is through small pores of uniform size in which velocity as well as pressure distributions differ from Poiseuille flow, and fluid has transverse velocity components as well. Sometimes, it so happens that the pores on the tube wall get blocked which means that the reabsorption velocity does not remain uniform. This phenomena is called biofouling and is usually tackled with the help of artificial kidney and dialysis. Moreover, when various proteins, cholesterol and fats are released into the glomerular filtrate, they occasionally get suspended inside the tube cavity thus reducing the medium permeability. This causes the renal tubule to get diseased and the medium inside serves as a porous medium.

Lawrence, Wesson [24] and Burgen [3] hypothetically explained renal tubule structure by using a steady rate of ingestion.There are many more mathematical analysis of flow through porous walls without thinking about renal tubule application done by Yuan [25], Wah [23] , and Terril [22]. Macey [11] investigated the state of steady ingestion, and suggested that the rate fluctuated straightly with distance. Kelman [9] showed exponential fluid flow in renal tubule. Macey [12] then studied increasingly broad range of boundary conditions. He obtained results for the creeping flow in infinitely long cylinder by utilization of exponential saturation. Kozinski [10] extended Macey's work for permeable slits. Marshall and Trowbridge [13] and Palatt et al [15] further worked in similar area with various assumptions on volume flux. Radhakrishnamacharya [16] examined the creeping flow in renal tubule by studying flow through a rounded cylinder with a shifting cross section and saturation in the walls. Muttu [14]examined the stream function in a permeable planar slit with a gradually varying cross section obtained solutions by perturbation technique.There has also been many studies [2],[21] based on analysis of flow through porous walls,but there has been no work on flow in a diseased renal tubule with porous medium. Ahmad and Naseem [1] analyzed the Stokes flow of a thick, laminar, incompressible fluid which flows in a tube with penetrable walls. Krishnaprasad and Chandra [4] considered the flow in unbending containers of gradually changing cross section. Chaturani with Ranganathan [5] worked on fluid flow through meeting tube with variable wall porousness.We will use inverse techniques[17] , [19] and assume stream function to solve our problem. Recently, Siddiqui et al. [6],[7],[18] ,[8], [20] introduced different analysis for creeping flow of Newtonian fluid through permeable tube along different capacities for reabsorption in the wall, with no slip conditions.Usually, these small pores remain to be uniform, but due to some reasons micro organisms grow in these pores and pores become more narrow and with the growth of micro organisms these pores get blocked.

In the past many researchers presented study of fluid flow through renal tubule in plane as well as cylindrical geometry and they have given many changes on the supposition of rate of absorption on the walls of tubules. In this work, we propose study of the laminar, steady creeping flow of a Newtonian fluid along a porous slit, taking into account the effects of befouling as well as porosity. We shall model the equations just for the two-dimensional flow with drag forces in effect due to reduced medium permeability. The equations of motion will then be solved exactly using inverse method and expressions for some relevant physical quantities will be computed. In the end, we shall present our results graphically.

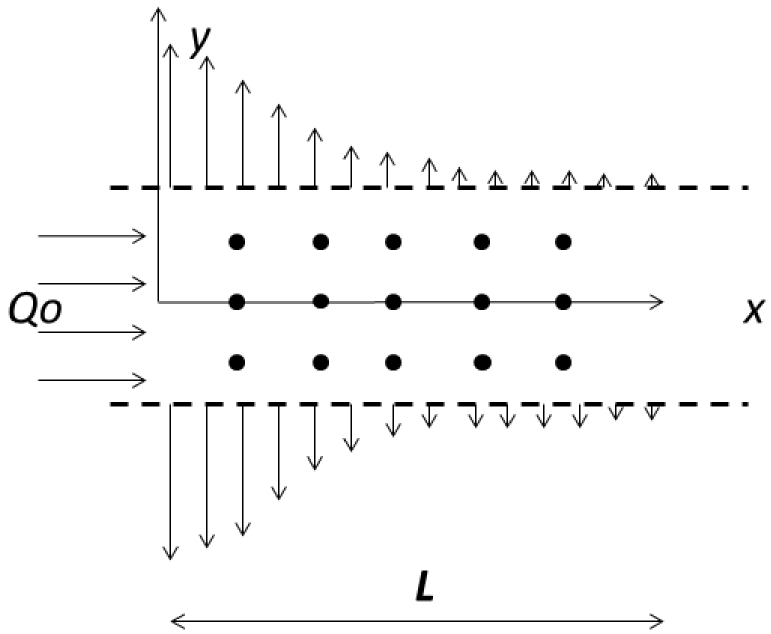

Figure 2.1: Geometry of the Problem

2.2 Problem Formulation

Lets suppose the steady, laminar flow of incompressible fluid passing through a porous slit with L length, having 2W width and W breadth. We will choose a rectangular cartesian coordinate system (x,y,z). We assume middle line of the slit is x-axis and y-axis is perpendicular to it as shown in figure.Fluid is entering in this slit with some volume flow rate Q_0 but due to porosity fluid flow does not remain uniform it becomes periodic. We consider that the fluid is absorbed at the walls of the slit with velocity $V_0 \sin(\alpha x)$ and $V_0 \cos(\alpha x)$. Taking transverse velocity periodic along the wall of slits. The necessary equations for the motion of incompressible Newtonian fluid, ignoring thermal effect and all types of force acting on the body are as follows;

2.2.1 Continuity and Momentum Equation

$$\nabla.\mathbf{V} = 0 \tag{2.1}$$

$$\rho \frac{D\mathbf{V}}{Dt} = -\nabla \rho + \mu \nabla^2 \mathbf{V} \tag{2.2}$$

Here ρ is fluid density constant, μ is coefficient of velocity, v is the velocity of vector, p is for pressure and ∇^2 is for Laplacian notation. The material time derivative is defined as

$$\frac{D(*)}{Dt} = \frac{\partial(*)}{\partial t} + (V \cdot \nabla)(*) \tag{2.3}$$

We follow the velocity profile as;

$$\mathbf{V} = [v(x,y), v(x,y)] \tag{2.4}$$

After applying different suppositions, We obtained Continuity equation as;

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \tag{2.5}$$

Creeping flow and accommodating porosity effects the momentum equation becomes:

$$0 = -\frac{\partial p}{\partial x} + \mu \nabla^2 u - \frac{\mu}{k} u, \tag{2.6}$$

$$0 = -\frac{\partial p}{\partial y} + \mu \nabla^2 v - \frac{\mu}{k} v, \tag{2.7}$$

The boundary conditions are:

2.2.2 Symmetry Condition

$$\frac{\partial u}{\partial y} = 0, v = 0, at y = 0, \tag{2.8}$$

2.2.3 No-slip condition

$$u = 0, \quad v = V_0 \sin(\alpha x), \text{ at } y = H, \tag{2.9}$$

2.2.4 Reabsorption at walls

$$Q_0 = 2W \int_0^H u(0,y)dy, \quad p(0,0) = constant \tag{2.10}$$

V_0 is the velocity of uniform reabsorption.

2.3 Stream function formulation of the problem

Here $\psi(x,y)$ Stream function is given by;

$$u = \frac{\partial \psi}{\partial y}, \quad v = -\frac{\partial \psi}{\partial x}, \tag{2.11}$$

2.3.1 Initial and flow rate condition

Equation (2.6) and (2.7) reduces the following forms:

$$\frac{\partial p}{\partial x} = \mu \nabla^2 \left(\frac{\partial \psi}{\partial y} \right) - \frac{\mu}{k} (\frac{\partial \psi}{\partial y}), \tag{2.12}$$

and

$$\frac{\partial p}{\partial x} = \mu \nabla^2 \left(\frac{\partial \psi}{\partial x} \right) + \frac{\mu}{k} (\frac{\partial \psi}{\partial x}) \tag{2.13}$$

Where,$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ is a Laplacian operator.
Differentiating (2.12) with respect to ”y” and (2.13) w.r.t ”x” we obtain,

$$\frac{\partial^2 p}{\partial x \partial y} = \mu \nabla^2 (\frac{\partial^\psi}{\partial y^2}] - \frac{\mu}{k} \frac{\partial^2 y}{\partial y^2} \tag{2.14}$$

$$\frac{\partial^2 p}{\partial x \partial y} = -\mu \nabla \frac{\partial^2 \psi}{\partial x^2} + \frac{\mu}{k} \frac{\partial^2 \psi}{\partial x^2} \tag{2.15}$$

Comparing (3.45) and (2.15), we get

$$\mu [\nabla^2 \frac{\partial^2 \psi}{\partial y^2} - \frac{1}{k} \frac{\partial^2 \psi}{\partial y^2}] = \mu [-\nabla^2 \frac{\partial^2 \psi}{\partial x^2} + \frac{1}{k} \frac{\partial^2 \psi}{\partial x^2}] \tag{2.16}$$

and

$$\nabla^2 [\frac{\partial^2 \psi}{\partial y^2} + \frac{\partial^2 \psi}{\partial y^2}] - \frac{1}{k} [\frac{\partial^2 \psi}{\partial y^2} + \frac{\partial^2 \psi}{\partial y^2}] = 0 \tag{2.17}$$

$$\nabla^2 (\nabla^2 \psi) - \frac{1}{k} \nabla^2 \psi = 0 \tag{2.18}$$

Lets suppose,

$$\frac{1}{k} = \beta^2 \tag{2.19}$$

Finally, we get

$$\nabla^4 \psi - \beta^2 \left(\nabla^2 \psi \right) = 0, \tag{2.20}$$

where $\nabla^4 = \nabla^2 (\nabla^2)$.

2.4 Boundary Conditions for stream functions

Further, converting the boundary conditions (2.8)-(2.10) in terms of $\psi(x,y)$:

2.4.1 Symmetry Condition

$$\frac{\partial^2 \psi}{\partial y^2} = 0, \quad \frac{\partial \psi}{\partial x} = 0, \quad at \quad y = 0, \tag{2.21}$$

2.4.2 No slip condition

$$\frac{\partial \psi}{\partial y} = 0, \quad -\frac{\partial \psi}{\partial x} = V_0 \sin(\alpha x), \quad at \quad y = H, \tag{2.22}$$

2.4.3 Initial flow rate

$$\frac{Q_0}{2W} = \psi(0,H) - \psi(0,0). \tag{2.23}$$

We assume that,

$$\psi(0,0) = 0, \tag{2.24}$$

and

$$\psi(0,H) = \frac{Q_0}{2W}. \tag{2.25}$$

Now, we are going to use all the above data to solve our problem. We will use all the above boundary conditions. After that we will find out exact solution of our problem in Next chapter.

Chapter 3

Solution of the Problem

Now we are going to use equations (2.6) and (2.7) in this chapter to solve our problem. We will discuss two cases one by one.

3.1 Case 1

In this case we consider sinusoidal reabsorption at the walls $v = V_0 \cos(\alpha x)$. We find out exact solution only for two-dimensional BVP by using inverse solution method. We will get stream function $\psi(x,y)$ of the form;

$$\Psi(x,y) = V_0 \sin(\alpha x) F_1(y) + K_1(y) \tag{3.1}$$

Here unknown functions $F_1(y)$ and $K_1(y)$ are to be find out, By putting the last equation in (2.20),

$$\nabla^4(\sin(\alpha x)F_1(y) + K_1(y)) - \beta^2\nabla^2(\sin(\alpha x)F_1(y) + K_1(y)) \tag{3.2}$$

$$\nabla^2\psi = \alpha^2 \sin(\alpha x)F_1 + sin(\alpha x)\frac{\partial^2 F_1}{\partial y^2} + \frac{\partial^2 K_1}{\partial y^2} \tag{3.3}$$

and

$$\begin{aligned}\nabla^4\psi =&\alpha^4 \sin(\alpha x)F_1 - \alpha^2 \sin(\alpha x)\frac{\partial^2 F_1}{\partial y^2} - \alpha^2 \sin(\alpha x)\frac{\partial^2 F_1}{\partial y^2}\\ &+ \sin(\alpha x)\frac{\partial^4 F_1}{\partial y^4} + \frac{\partial^4 K_1}{\partial y^4}\end{aligned} \tag{3.4}$$

Putting (3.3) and (3.72) in (3.64)

$$\begin{aligned}&\sin(\alpha x)[\frac{\partial^4 F_1}{\partial y^4} - (2\alpha^2 + \beta^2)\frac{\partial^2 F_1}{\partial y^2} + (\alpha^2\beta^2 + \alpha^4)F_1]\\ &+ [\frac{\partial^4 K_1}{\partial y^4} - \beta^2\frac{\partial^2 K_1}{\partial y^2}] = 0\end{aligned} \tag{3.5}$$

$$\sin(\alpha x)[\frac{\partial^4 F_1}{\partial y^4} - (2\alpha^2 + \beta^2)\frac{\partial^2 F_1}{\partial y^2} + (\alpha^2\beta^2 + \alpha^4)F_1] = 0 \tag{3.6}$$

where, $\sin(\alpha x) \neq 0$ than we get,

$$\frac{d^4F_1}{dy^4}+2\alpha^2\frac{d^2F_1}{dy^2}+\alpha^4F_1-\beta^2(\frac{d^2F_1}{dy^2}+\alpha^2F_1)=0, \tag{3.7}$$

and

$$\frac{d^4K_1}{dy^4}-\beta^2\frac{d^2K_1}{dy^2}=0, \tag{3.8}$$

With the help of equation, boundary conditions(2.21), (2.22), (2.24), (2.25) after rearranging become F_1 .

For $y=0$ using (2.21).. Firstly differentiating (3.1) w.r.t y

$$\sin(\alpha x)F_1'(y)+K_1'(y)=0 \tag{3.9}$$

Now we get,

$$\sin(\alpha x)F_1'(y)=0, K_1'(y)=0 \tag{3.10}$$

where $\sin(\alpha x)\neq 0$, so we get

$$F_1'(y)=0 \quad , K_1'(y)=0 \tag{3.11}$$

At $y=H$, above equation becomes

$$F_1'(H)=0 \quad , K_1'(H)=0 \tag{3.12}$$

Now, differentiating (3.1) w.r.t x

$$\alpha\cos(\alpha x)F_1(y)=0 \tag{3.13}$$

using Boundary Condition (2.21) At $y=H$

$$\frac{V_0}{\alpha}=F_1(H) \tag{3.14}$$

Again taking derivative of (3.83).We get

$$\sin(\alpha x)F_1''(y)+K_1''(y)=0 \tag{3.15}$$

We obtained,

$$F_1''(y)=0 \quad ,K_1''(y)=0 \tag{3.16}$$

At $y=0$ in above equation

$$F_1''(0)=0 \quad ,K_1''(0)=0 \tag{3.17}$$

Differentiating (3.1) w.r.t x.

$$\alpha\cos(\alpha x)F_1(y)=0 \tag{3.18}$$

at $y=0$

$$F_1(0)=0 \tag{3.19}$$

For Boundary condition (2.25) putting $x=0$ and $y=0$

$$K_1(H)=\frac{Q_0}{2W} \tag{3.20}$$

and For Boundary Condition (2.24)

$$F_1(0)=0 \tag{3.21}$$

We then obtained the following Boundary Conditions.

$$F_1(0)=0, \quad \frac{d^2F_1(0)}{dy^2}=0 \tag{3.22}$$

$$F_1(H)=-\frac{V_0}{\alpha}, \quad \frac{dF_1(H)}{dy}=0 \tag{3.23}$$

and

$$K_1(0) = 0, \quad \frac{d^2K_1(0)}{dy^2} = 0 \tag{3.24}$$

$$K_1(H) = \frac{Q_0}{2W}, \quad \frac{dK_1(H)}{dy} = 0 \tag{3.25}$$

3.1.1 Solution for ψ involving $F_1(y)$

Now we will find out the value of $F_1(y)$. For this purpose we will solve forth order homogeneous differential equation having constant coefficient or some real roots.

$$\frac{d^4F_1}{dy^4} + 2\alpha^2\frac{d^2 1_2}{dy^2} + \alpha^4 F_1 - \beta^2(\frac{d^2F_1}{dy^2} + \alpha^2 F_1) = 0, \tag{3.26}$$

Above equation takes the form,

$$m^4 - m^2(2\alpha^2 + \beta^2) + \alpha^2\beta^2 + \alpha^4 \tag{3.27}$$

We use Quadratic equation,

$$m^2 = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \tag{3.28}$$

In this we have $a = m^2$, $b = (2\alpha^2 + \beta^2)$ and $c = (\alpha^2\beta^2 + \alpha^4)$ using this data in the above equation. We get,

$$m = \pm\sqrt{\alpha^2 + \beta^2}, m = \pm\alpha \tag{3.29}$$

Lets suppose,

$$\sqrt{\alpha^2 + \beta^2} = \Gamma \tag{3.30}$$

Using this data in (3.1) we obtained following equation.

$$F_1 = C_1 e^{y\Gamma} + C_2 y e^{y\alpha} + C_3 e^{-y\Gamma} + C_4 y e^{-\alpha y} \tag{3.31}$$

Using all the Boundary conditions in above equation. Firstly using $F_1(0) = 0$ which gives

$$F_1 = C_3 e^{y\Gamma} + C_2 y e^{y\alpha} + C_3 e^{-y\Gamma} + C_4 y e^{-\alpha y} \tag{3.32}$$

Now using $\frac{\partial^2 F_1}{\partial y^2} = 0$

$$F_1 = C_3(e^{-y\Gamma} - e^{y\Gamma}) + C_4 y(e^{\alpha y} + e^{-\alpha y}) \tag{3.33}$$

Next using $\frac{\partial F_1}{\partial y} = 0$ We get,

$$C_3 = C_4 \frac{e^{\alpha H} + e^{-\alpha H} + H\alpha e^{\alpha H} - H\alpha e^{-\alpha H}}{\Gamma(e^{-H\Gamma} + e^{H\Gamma})} \tag{3.34}$$

And at last using $F_1 = \frac{-V_0}{\alpha}$

$$C_4 = -\frac{V_0 \Gamma \cosh(H\Gamma)}{2\alpha\Delta_1} \tag{3.35}$$

Finally we obtained our Final result which is

$$F_1 = \frac{\sinh(y\Gamma)\Delta_2 - y\Gamma\cosh(\alpha y)\cosh(H\Gamma)}{\alpha\Delta_1} \tag{3.36}$$

where,

$$\begin{aligned} \Delta_1 =& H\Gamma\cosh(H\Gamma)\cosh(\alpha H) H\alpha\sinh(\alpha H)\sinh(H\Gamma) \\ &- \sinh(H\Gamma)\cosh(\alpha H). \end{aligned} \tag{3.37}$$

and

$$\Delta_2 = \cosh(\alpha H) + h\alpha sinh(\alpha H). \tag{3.38}$$

3.1.2 Solutions of ψ involving $K_1(y)$

To find out $K_1(y)$ we will find out the solution (3.8), we obtained the equation for $\psi(x,y)$ as;

$$\frac{\partial^4 K_1}{\partial y^4} - \beta^2 \frac{\partial^2 K_1}{\partial y^2} = 0 \tag{3.39}$$

Integrating above equation.

$$\frac{\partial^3 K_1}{\partial y^3} - \beta^2 \frac{\partial K_1}{\partial y} = D_1 \tag{3.40}$$

Again integrating

$$\frac{\partial^2 K_1}{\partial y^2} - \beta^2 K_1 = D_1 y + D_2 \tag{3.41}$$

The Auxiliary equation for above equation is given as,

$$m^2 - \beta^2 = 0 \tag{3.42}$$

Which gives

$$m = \pm\beta \tag{3.43}$$

The complementary solution is given by,

$$K_c = A_1 e^{\beta y} + A_2 e^{-\beta y} \tag{3.44}$$

Also assume the particular solution as,

$$K_p = A_3 y + A_4 \tag{3.45}$$

Differentiating (3.45) w.r.t y

$$K_p' = A_3 \tag{3.46}$$

Again Differentiating

$$K_p'' = 0 \tag{3.47}$$

using (3.45) and (3.46) in (3.41).

$$-\beta^2 A_3 y - \beta^2 A_4 = D_1 y + D_2 \tag{3.48}$$

To get the compute solution we have

$$K_1 = K_c + K_p \tag{3.49}$$

Now Putting the values of K_c and K_p in above equation.

$$K_1 = A_1 e^{\beta y} + A_2 e^{-\beta y} - y\frac{D_1}{\beta^2} - \frac{D_2}{\beta^2} \tag{3.50}$$

using the boundary conditions (3.101) and (3.103).We obtained our final value of K_1

$$K_1(y) = \frac{Q_0}{2W\Delta_3}\left[\beta y \cosh(\beta H) - \sinh(\beta y)\right] \tag{3.51}$$

where,

$$\Delta_3 = \beta H \cosh(\beta H) - \sinh(\beta H). \tag{3.52}$$

3.2 Stream Function

Combining (3.36) and (3.51),we obtained the equation for $\psi(x,y)$ as

$$\begin{aligned}\psi(x,y) =& V_0 \sin(\alpha x)\frac{\Delta_2 \sinh(y\Gamma) - y\Gamma\cosh(\alpha y)\cosh(H\Gamma)}{\alpha\Delta_1} \\ &+ \frac{Q_0}{2W\Delta_3}\left[\beta y \cosh(\beta H) - \sinh(\beta y)\right]\end{aligned} \tag{3.53}$$

which is strongly depends upon ”α” and ”Q_0”.

3.3 Velocity Components

We are now going to solve velocity components which are,

3.3.1 Longitudinal Velocity

The velocity components $u(x,y)$ are calculated by using (2.11) after taking derivative of ψ w.r.t y we get,

$$
\begin{aligned}
u(x,y) =& \frac{V_0 \sin(\alpha x)}{\alpha \Delta_1} (\Delta_2 \Gamma \cosh(y\Gamma) - \Gamma \cosh(\alpha y) cosh(H\Gamma) - \alpha y \Gamma \cosh(H\Gamma) \sinh(\alpha y)) \\
& + \frac{Q_0}{2W\Delta_3} (\beta \cosh(\beta H) - \beta \cosh(\beta y))
\end{aligned}
\tag{3.54}
$$

3.3.2 Maximum velocity component of u

Maximum velocity occurs at $y = 0$ From (3.54),we obtained the equation for maximum (longitudinal) velocity at the middle of two walled slit.

$$
\begin{aligned}
u_{max} =& V_0 \sin(\alpha x) \Gamma \frac{\Delta_2 - \cosh(H\Gamma)}{\alpha \Delta_1} \\
& + \frac{Q_0 \beta}{2W\Delta_3} (\cosh(\beta H) - 1)
\end{aligned}
\tag{3.55}
$$

3.3.3 Transverse Velocity

The velocity component $v(x,y)$ obtained by using relation (2.11) taking derivation of ψ w.r.t x.

$$
v(x,y) = \frac{V_0 \cos(\alpha x)}{\Delta_1} (\Gamma y \cosh(\alpha y) \cosh(H\Gamma) - \Delta_2 \sinh(y\Gamma)) \tag{3.56}
$$

3.3.4 Maximum Velocity Component of v

Maximum velocity occurs at $y = H$. Transverse velocity is maximum along the walls that is,

$$
v_{max} = V_0 \cos(\alpha x) \tag{3.57}
$$

3.4 Volume flow rate

The volume flow rate $Q(x)$ is defined as;

$$Q(x) = 2W \int_0^H u(x,y)dy \tag{3.58}$$

Substituting (3.54) in (3.58),we obtained the result for volume flow rate as;

$$\begin{aligned} Q(x) =& \frac{2WV_0 \sin(\alpha x)}{\alpha \Delta_1} \Bigg[\Delta_2 \sinh(y\Gamma) - \frac{\Gamma \sinh(\alpha y) \cosh(H\Gamma)}{\alpha} \\ & - \Gamma y \cosh(H\Gamma) \cosh(\alpha y) + \frac{\sinh(\alpha y)\Gamma \cosh(H\Gamma)}{\alpha} \Bigg] \end{aligned}$$

$$\begin{aligned} Q(x) =& \frac{2WV_0 \sin(\alpha x)}{\alpha \Delta_1} \Bigg[\cosh(\alpha H) \sinh(H\Gamma) + H\alpha \sinh(\alpha H) \sinh(H\Gamma) \\ & - \Gamma H \cosh(H\Gamma) \cosh(\alpha H) \Bigg] \end{aligned}$$

$$Q(x) = Q_0 - \frac{2WV_0 \sin(\alpha x)}{\alpha} + \frac{2WQ_0}{2W\Delta_3}(\beta H \cosh(\beta H) - \frac{\beta \sinh(\beta H)}{\beta}) \tag{3.59}$$

$$Q(x) = Q_0 - \frac{2WV_0 \sin(\alpha x)}{\alpha} \tag{3.60}$$

Axial flow rate at the end of the slit is

$$Q(L) = Q_0 - \frac{2WV_0 \sin(\alpha L)}{\alpha} \tag{3.61}$$

The fluid that is reabsorbed by the walls is given as,

$$Reabsorption = \frac{2WV_0 \sin(\alpha L)}{\alpha} \tag{3.62}$$

And this implies that Reabsorption is depends upon α.

3.5 Pressure Distribution

On making use of (3.1) in (2.12) and (2.13), we get

$$\frac{\partial p}{\partial x} = \mu(-\alpha^2 \sin(\alpha x)\frac{dF_1}{dy} + \sin(\alpha x)\frac{d^3F_1}{dy^3} + \frac{d^3K_1}{dy^3}) - \mu\beta(\sin(\alpha x)\frac{dF_1}{dy} + \frac{dK_1}{dy}) \quad (3.63)$$

or

$$\frac{\partial p}{\partial x} = \mu \sin(\alpha x)(\frac{d^3F_1}{dy^3} - (\alpha^2+\beta^2)\frac{dF_1}{dy}) + \mu(\frac{d^3K_1}{dy^3} - \beta^2\frac{dK_1}{dy}) \quad (3.64)$$

Putting (3.1) (2.13), we obtained

$$\frac{\partial p}{\partial y} = \mu(\alpha^3 \cos(\alpha x)F_1 - \alpha\cos(\alpha x)\frac{d^2F_1}{dy^2}) + \mu\beta^2\alpha\cos(\alpha x)F_1 \quad (3.65)$$

Which gives,

$$\frac{\partial p}{\partial y} = \mu\alpha\cos(\alpha x)[F\Gamma - \frac{d^2F_1}{dy^2} \quad (3.66)$$

Integrating (3.64) w.r.t x, we obtained;

$$p(x,y) = -\mu\frac{\cos(\alpha x)}{\alpha}[\frac{d^3F_1}{dy^3} - (\alpha^2+\beta^2)\frac{dF_1}{dy}] + \mu x[\frac{d^3K}{dy3} - \beta^2\frac{dK_1}{dy}] + M \quad (3.67)$$

Where $M(y)$ is a function of y resulting from integration w.r.t x.

Now, taking derivative of (3.67) w.r.t $'y'$ as $\Gamma^2 = (\alpha^2+\beta^2)$.

$$\frac{\partial p}{\partial y} = -\frac{\mu\cos(\alpha x)}{\alpha}[\frac{d^4F_1}{dy^4} - \Gamma\frac{d^2F_1}{dy^2}] + \mu x[\frac{d^4K_1}{dy^4} - \beta^2\frac{d^2K_1}{dy^2}] + \frac{dM}{dy} \quad (3.68)$$

Now comparing (3.68) and (3.66). We obtained,

$$\begin{aligned} &-\frac{\mu\cos(\alpha x)}{\alpha}[\frac{d^4F_1}{dy^4} - \Gamma\frac{d^2F_1}{dy^2}] + \mu x[\frac{d^4K_1}{dy^4} - \beta^2\frac{d^2K_1}{dy^2}] \\ &+\frac{dM}{dy} = \mu\alpha\cos(\alpha x)[F\Gamma^2 - \frac{d^2F_1}{dy^2} \end{aligned} \quad (3.69)$$

After rearranging we get,

$$\frac{dM}{dy}=\frac{\mu\cos(\alpha x)}{\alpha}[\frac{d^4F_1}{dy^4}-\Gamma\frac{d^2F_1}{dy^2}+\alpha^2F_1\Gamma]+\mu x[\frac{d^4K_1}{dy^4}-\beta^2\frac{d^2K_1}{dy^2}] \tag{3.70}$$

from the above equation after using (3.26) and (3.8), we get

$$\frac{dM}{dy}=0 \tag{3.71}$$

Which implies that $M=C$, where C is a constant. By putting above relation in the (3.67), we get the following equation.

$$p(x,y)=-\mu\frac{\cos(\alpha x)}{\alpha}[\frac{d^3F_1}{dy^3}-\Gamma\frac{dF_1}{dy}]+\mu x[\frac{d^3K}{dy3}-\beta^2\frac{dK_1}{dy}]+C \tag{3.72}$$

Putting the derivatives of F_1 and K_1 in (3.72) which are given below.

$$\frac{\partial F_1}{\partial y}=\frac{V_0}{\alpha\Delta_1}[\Delta_2\Gamma\cosh(y\Gamma)-\Gamma\cosh(\alpha y)\cosh(H\Gamma)-y\Gamma\alpha\sinh(\alpha y)\cosh(H\Gamma)] \tag{3.73}$$

$$\begin{aligned}\frac{\partial^2F_1}{\partial y^2}=&\frac{V_0\Gamma}{\alpha\Delta_1}[\Gamma\Delta_2\sinh(y\Gamma)-\alpha\sinh(\alpha y)\cosh(H\Gamma)-\Gamma\alpha\sinh(\alpha y)\cosh(H\Gamma)\\&-y\alpha^2\cosh(\alpha y)\cosh(H\Gamma)]\end{aligned} \tag{3.74}$$

$$\begin{aligned}\frac{\partial^3y}{\partial y^3}=&\frac{V_0\Gamma}{\alpha\Delta_1}[\Gamma^2\Delta_2\cosh(y\Gamma)-\alpha^2\cosh(\alpha y)\cosh(H\Gamma)-\Gamma\alpha^2\cosh(H\Gamma)\cosh(\alpha y)\\&-\alpha^2\cosh(\alpha y)\cosh(H\Gamma)-y\alpha^3\sinh(\alpha y)\cosh(H\Gamma)]\end{aligned} \tag{3.75}$$

$$\frac{\partial K_1}{\partial y}=\frac{Q_0}{2W\Delta_3}[\beta\cosh(\beta H)-\beta\cosh(\beta y)] \tag{3.76}$$

$$\frac{\partial^2K_1}{\partial y^2}=\frac{Q_0}{2W\Delta_3}[-\beta^2\sinh(\beta y)] \tag{3.77}$$

$$\frac{\partial^3 K_1}{\partial y^3} = \frac{Q_0}{2W\Delta_3}[-\beta^3 \cosh(\beta y)] \tag{3.78}$$

Putting above derivatives in (3.72)

$$\begin{aligned} p =& -\frac{\mu\cos(\alpha x)}{\alpha}[\frac{V_0}{\alpha\Delta_1}(\Delta_2\Gamma^2\cosh(y\Gamma) - 3\alpha^2\cosh(\alpha y)\cosh(H\Gamma) - \alpha^3 y\sinh(\alpha y)\cosh(H\Gamma)) \\ & -\Gamma^2\frac{V_0}{\alpha\Delta_1}(\Delta_2\cosh(y\Gamma) - cosh(\alpha y)\cosh(H\Gamma) - y\alpha\sinh(\alpha y)\cosh(H\Gamma))] \\ & +\mu x[-\frac{Q_0\beta^3}{2W\Delta_3}\cosh(\beta y) - \frac{\beta^3 Q_0}{2W\Delta_3}(cosh(\beta H) - \cosh(\beta y))] \end{aligned} \tag{3.79}$$

$$\begin{aligned} p(x,y) - p(0,0) =& \mu\frac{V_0\cos(\alpha x)}{\alpha H\Delta_1}\left[\beta^2\Gamma\cosh(H\Gamma)\cosh(\alpha H)\right] \\ & -\mu\frac{\alpha Q_0 - 2V_0 W}{2\alpha W\Delta_2}\left[\beta^3\cosh(\beta H)\right]x \end{aligned} \tag{3.80}$$

3.6 Mean Pressure Drop

Mean pressure drop defined as;

$$p(x) = \frac{1}{H}\int_0^H [p(x,y) - p(0,0)]dy \tag{3.81}$$

Putting (4.10) in above expression. we obtained

$$\begin{aligned} p(x) =& \frac{1\mu\cos(\alpha x)V_0\Gamma}{H\alpha^2\Delta_1}[\Gamma\sinh(\alpha H)\frac{(4\alpha^2 - \beta^2)}{2\alpha} \\ & -\frac{\beta^2\Gamma H\cosh(\alpha H)}{2\alpha} - \frac{\alpha Q_0 - 2V_0 W}{2\alpha W}\frac{H\mu x\beta^3\cosh(\beta H)}{\Delta_3}] \end{aligned} \tag{3.82}$$

3.7 Wall Shear Stress

Wall shear stress expression is:

$$\tau_w|_{y=H} = -\mu(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x})_{y=H} \tag{3.83}$$

Using (3.54) and (3.83), we obtained wall shear stress expression,

$$\begin{aligned}\tau_w|_{y=H} = &-\mu[\frac{V_0\sin(\alpha x)}{\Delta_1}(H\Gamma^2\sinh(y\Gamma)\sinh(\alpha H)+\frac{\Gamma^2\sinh(y\Gamma\cosh(\alpha H))}{\alpha}\\ &-\alpha\Gamma y\cosh(H\Gamma)\cosh(\alpha y))-\frac{\alpha V_0\sin(\alpha x)}{\Delta_1}(y\Gamma\cosh(\alpha y)\cosh(\alpha H)\\ &+\frac{\Gamma^2\sinh(H\Gamma)\cosh(\alpha H)}{\alpha}-2\alpha H\cosh(H\Gamma)\cosh(\alpha H))]\end{aligned} \tag{3.84}$$

$$\begin{aligned}\tau_w|_{y=H} = -\mu\Bigg[&[\frac{V_0\sin(\alpha x)}{\alpha\Delta_2}(\Delta_2\Gamma^2\sinh(H\Gamma)-2\Gamma\alpha\sinh(\alpha)\cosh(\Gamma)-\alpha^2\Gamma H\cosh(H\Gamma)\sinh(H\alpha)\\ &-\alpha H\Gamma\cosh(H\alpha)\cosh(H\Gamma)+\delta_2\alpha\sinh(H\Gamma))]+\frac{\mu Q_0\beta^2}{2W\Delta_3}(\cosh(\beta H))\Bigg]\end{aligned}$$

3.8 Fractional Reabsorption

We define Fractional Reabsorption as,

$$FA = \frac{Q(0)-Q(L)}{Q(0)} \tag{3.85}$$

On substituting (3.59) in (3.85),we obtained the fractional reabsorption expression,

$$FA = \frac{2WV_0\sin(\alpha L)}{\alpha Q_0} \tag{3.86}$$

3.9 Leakage flux

We define Leakage flux as,

$$q(x) = -\frac{dQ(x)}{dx} \tag{3.87}$$

The leakage flux is obtained by using (3.59) in (3.87) as

$$q(x) = 2WV_0\cos(\alpha x) \tag{3.88}$$

3.10 Case 2

Now we will discuss our other case which is $v = V_0 \sin(\alpha x)$. we will done all the same calculations as done in above case. Firstly Lets suppose the stream function $\psi(x,y)$ expression:

$$\psi = V_0 \cos(\alpha x) F_2(y) + K_2(y), \tag{3.89}$$

where $F_2(y)$ and $K_2(y)$ are unknown functions that we will find out. Using expression (3.89) in (2.20), we obtain:

$$V_0 \cos(\alpha x)[\frac{\partial^4 F_2}{\partial y^4} - \frac{\partial^2 F_2}{\partial y^2}(2\alpha^2 + \beta^2) + F_2(\alpha^2\beta^2 + \alpha^4)] + \frac{\partial K_2}{\partial y^4} - \beta^2 \frac{\partial^2 K_2}{y^2} \tag{3.90}$$

which can take the form:

$$V_0 \cos(\alpha x)[\frac{\partial^4 F_2}{\partial y^4} - \frac{\partial^2 F_2}{\partial y^2}(2\alpha^2 + \beta^2) + F_2(\alpha^2\beta^2 + \alpha^4)] = 0, \tag{3.91}$$

and

$$\frac{d^4 K_2}{dy^4} - \beta^2 \frac{d^2 K_2}{dy^2} = 0, \tag{3.92}$$

Since $V_0 cos(\alpha x) \neq 0$, therefore,

$$[\frac{d^4 F_2}{dy^4} + 2\alpha^2 \frac{d^2 F_2}{dy^2} + \alpha^4 F_2 - \beta^2(\frac{d^2 F_2}{dy^2} + \alpha^2 F_2)] = 0, \tag{3.93}$$

Using equation (2.22), the boundary conditions (2.12) to (2.21) reduce to:

$$F_2(0) = 0, \quad \frac{d^2 F_2(0)}{dy^2} = 0 \tag{3.94}$$

$$F_2(H) = \frac{V_0}{\alpha}, \quad \frac{dF_2(H)}{dy} = 0 \tag{3.95}$$

and

$$K_2(0) = 0, \quad \frac{d^2K_2(0)}{dy^2} = 0 \tag{3.96}$$

$$K_2(H) = \frac{\alpha Q_0 - 2V_0W}{2\alpha W}, \quad \frac{dK_2(H)}{dy} = 0 \tag{3.97}$$

The solution of equation (3.93) using boundary conditions (3.93) and (3.94) is found to be:

$$F_2(y) = V_0 \frac{[y\Gamma \cosh(H\Gamma)\cos(\alpha y) - \Delta_2(\sinh(y\Gamma))]}{\alpha \Delta_1}, \tag{3.98}$$

where

$$\Delta_1 = H\Gamma \cosh(H\Gamma)\cosh(\alpha H)H\alpha \sinh(\alpha H)\sinh(H\Gamma) - \sinh(H\Gamma)\cosh(\alpha H). \tag{3.99}$$

$$\Delta_2 = \cosh(\alpha H) + h\alpha sinhh(\alpha H). \tag{3.100}$$

and the solution of equation (3.92) using boundary conditions (3.101) and (3.103) is found to be:

$$K_2(y) = \frac{\alpha Q_0 - 2V_0W}{2\alpha W \Delta_2}\left[\beta y \cosh(\beta H) - \sinh(\beta y)\right], \tag{3.101}$$

where,

$$\Delta_3 = \beta H \cosh(\beta H) - \sinh(\beta H). \tag{3.102}$$

To obtaine Stream function $\psi(x,y)$ putting (3.104) and (3.105) in (2.22) and we write as:

$$\begin{aligned}\psi = & \frac{V_0 cos(\alpha x)}{\alpha \Delta_1}\left[\Gamma y \cosh(y\alpha)\cos(H\Gamma) - \Delta_2 \sinh(y\Gamma)\right] \\ & + \frac{\alpha Q_0 - 2V_0W}{2\alpha W \Delta_3}\left[\beta y \cosh(\beta H) - \sinh(\beta y)\right],\end{aligned} \tag{3.103}$$

3.11 Components of velocity

Velocity components calculated by equation (2.9):

$$
\begin{aligned}
u(x,y) =& \frac{V_0\cos(\alpha x)}{\alpha\Delta_1}\Big[\Gamma\cosh(H\Gamma)\alpha y\sin(\alpha y)+\Gamma\cosh(H\Gamma)\cosh(\alpha y) \\
&-\Gamma\cosh(\alpha H)\cosh(y\Gamma)-\Gamma H\alpha\sinh(\alpha H)\cosh(y\Gamma)\Big] \\
&+\frac{\alpha Q_0-2V_0W}{2\alpha W\Delta_3}\beta\left[\cosh(\beta H)-\cosh(\beta y)\right],
\end{aligned} \tag{3.104}
$$

$$
v(x,y)=\frac{V_0 sin(\alpha x)}{\Delta_1}\left[y\Gamma\cosh(H\Gamma)\cos(\alpha y)-\Delta_2\sinh(y\Gamma)\right] \tag{3.105}
$$

The axial volume flow rate $Q(x)$ is find out by the relation:

$$
(\textbf{??})Q(x)=2W\int_0^H u(x,y)dy. \tag{3.106}
$$

Putting (3.104), the above equation becomes:

$$
Q(x)=\frac{2WV_0\cos(alphax)}{\alpha}+\frac{\alpha Q_0-2WV_0}{\alpha} \tag{3.107}
$$

3.12 Pressure Distribution

We will use (2.22) in (2.9) and (2.11) to get:

$$
\frac{\partial p}{\partial x}=\mu[cos(\alpha x)\frac{d^3F_2}{dy^3}-\frac{d^2F_2}{dy^2}-\alpha^2\frac{dF_2}{dy})+(\frac{d^3K_2}{dy^3}-\beta^2\frac{dK_2}{dy})], \tag{3.108}
$$

$$
\frac{\partial p}{\partial y}=-\mu[sin(\alpha x)[\alpha F_2(\alpha^2+\beta^2)-\alpha\frac{d^2F_2}{dy^2})], \tag{3.109}
$$

Integrating (3.108) with x, we get:

$$p(x,y) = \mu[\frac{sin(\alpha x)}{\alpha}\frac{d^3F_2}{dy^3} - \frac{dF_2}{dy}(\alpha^2+\beta^2) + x(\frac{d^3K_2}{dy^3}) - \beta^2\frac{d^2K_2}{dy^2}] + M(y), \quad (3.110)$$

where $M(y)$ is an unknown function that we will determine. By differentiating (2.3) w.r.t y and then comparing this with (3.109) along with the use (2.25) and (3.89), we find that:

$$\frac{dM}{dy} = 0, \Rightarrow M(y) = C, \quad (3.111)$$

where C is a constant of integration.(2.12) can be now written as:
Inserting (3.94) and (3.101), we get:

$$\begin{aligned} p(x,y) - p(0,0) =& \mu\frac{V_0 Sin\alpha x}{\alpha H\Delta_1}\left[\beta^2\Gamma\cosh(H\Gamma)\cos(\alpha H)\right] \\ & - \mu\frac{\alpha Q_0 - 2V_0 W}{2\alpha W\Delta_2}\left[\beta^3\cosh(\beta H)\right]x \end{aligned} \quad (3.112)$$

3.13 Wall shear stress

Wall shear stress is represented as :

$$\tau = -\mu(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}), \quad (3.113)$$

If we substitute (3.104) and (3.105) in the above equation, we obtain:

$$\begin{aligned} \tau =& \frac{-\mu V_0\cos(\alpha x)}{\Delta_1}\Big[2\Gamma\cosh(H\Gamma)\sin(\alpha H) \\ & + 2H\Gamma\alpha\cosh(\alpha H)\cosh(H\Gamma) - \cosh(\alpha H)\sinh(H\gamma)\frac{\beta^2+\alpha^2}{\alpha} - H\sinh(\alpha H)\sinh(H\Gamma)(\beta^2+2\alpha^2)\Big] \\ & + \frac{\alpha Q_0 - 2V_0 W}{2\alpha W\Delta_3}\left[\mu\beta^2\sinh(\beta H)\right] \end{aligned} \quad (3.114)$$

3.14 Fractional reabsorption

Fractional reabsorption ((FA)) is represented by:

$$FA = \frac{Q(0) - Q(L)}{Q(0)}. \tag{3.115}$$

After putting values the above equation becomes:

$$FA = \frac{2WV_0(1 - \cos(\alpha L))}{\alpha Q_0} \tag{3.116}$$

3.15 Leakage flux

Leakage flux $q(x)$ is represented by:

$$q(x) = -\frac{dQ(x)}{dx}. \tag{3.117}$$

After putting values in the above equation,we get:

$$q(x) = 2WV_0 \sin(\alpha x) \tag{3.118}$$

Chapter 4

Graphical results and Description

4.1 Non-Dimensionlization

We will find some more results from expressions that are calculated in previous chapters. We convert all the expression into dimensionless equations. Now we obtained dimensionless expressions using these expressions;

$$x^* = \frac{x}{H}, \quad y^* = \frac{y}{H}, \quad \psi^* = \frac{\psi}{U_0}, \quad x^* = \frac{x}{H} \quad p^* = \frac{pH}{\mu U_0}, \quad p^* = \frac{pH}{\mu U_0} \tag{4.1}$$

$$Q_0^* = \frac{Q_0}{U_0WH}, \quad \beta^* = \beta H, \quad u^* = \frac{u}{U_0}, \quad v^* = \frac{v}{U_0}, \quad q^* = \frac{q}{U_0W} \tag{4.2}$$

$$y^* = \frac{y}{H}, \quad V_0^* = \frac{V_0}{U_0}, \quad \psi^* = \frac{\psi}{U_0H}, \quad Q_0^* = \frac{Q_0}{U_0WH}, \quad \tau_w^* = \frac{\tau H}{\mu U_o} \tag{4.3}$$

Using these dimensionless quantities,we find out the dimensionless expression for Case1 which are stream function, longitudinal velocity, transverse velocity, axial rate, pressure difference, wall shear stres, fractional reabsorption and leakage flux respectively to be:

$$\begin{aligned} \psi^*(x^*, y^*) =& \frac{V_0^* \sin(\alpha^* x^*)}{\alpha^* \Delta_1^*}[\Delta_2^* \sinh(y^*\Gamma^*) - y^*\Gamma^* \cosh(\alpha^* y^*) \cosh(\Gamma^*)] \\ &+ \frac{Q_0^*}{2\Delta_3^*}[\beta^* y^* \cosh(\beta^*) - \sinh(\beta^* y^*)] \end{aligned} \tag{4.4}$$

$$\begin{aligned} u^*(x,y) =& \frac{V_0^* \sin(\alpha^* x^*)}{\alpha^* \Delta_1^*}\Big[\Delta_2^*\Gamma^* \cosh(y^*\Gamma^*) - \Gamma^* \cosh(\alpha^* y^*) \cosh(\Gamma^*) \\ &- \alpha^* y^* \Gamma^* \cosh(\Gamma^*) \sinh(\alpha^* y^*)\Big] \\ &+ \frac{Q_0^*}{2W\Delta_3^*}[cosh(\beta^8) - \cosh(\beta^* y^*)] \end{aligned} \tag{4.5}$$

$$v_0^* = \frac{V_0^* \cos(\alpha^* x^*)}{\Delta_1^*}[\Gamma^* y^* \cosh(\alpha^* y^*) \cosh(\Gamma^*) - \Delta_2^* \sinh(y^*\Gamma^*)] \tag{4.6}$$

$$FA = \frac{2V_0^* \sin(\alpha^*)}{\alpha^* Q_0^*} \tag{4.7}$$

$$q^*(x^*) = 2V_0^* \cos(\alpha^* x^*) \tag{4.8}$$

$$\begin{aligned} \tau_w|_{y=H} = &- \frac{V_0^* \sin(\alpha^* x^*)}{\Delta_1^*} \Bigg[\Gamma^2 * \sinh(\Gamma^*) \sinh(\alpha^*) + \frac{\Gamma^2 * \sinh(\Gamma^*) \cosh(\alpha^*)}{\alpha^*} \\ &- 2\alpha^* \cosh(\Gamma^*) \cosh(\alpha^*) + \alpha^* \sinh(\alpha^*) \sinh(\Gamma^*) - \alpha^* \sinh(\Gamma^*) \cosh(\alpha^*) \Bigg] \\ &+ \frac{Q_0^* \beta^2 *}{2\Delta_3^*} (\sinh(\beta^*)) \end{aligned} \tag{4.9}$$

$$\begin{aligned} p(x,y) - p(0,0) = &\frac{V_0^* \cos \alpha^*}{\alpha^* \Delta_1^*} \left[\beta^* 2\Gamma^* \cosh(\Gamma^*) \cos(\alpha^*) \right] \\ &- \frac{\alpha^* Q_0^* - 2V_0^*}{2\alpha^* \Delta_2^*} \left[\beta^* 3 \cosh(\beta^*) \right], \end{aligned} \tag{4.10}$$

$$Q(x)^* = Q_0^* - \frac{2V_0^* \sin(\alpha^* x^*)}{\alpha^*} \tag{4.11}$$

Similarly, we will get non-dimentionlization equations for Case2.

4.2 Graphs and Discussion

4.2.1 Graphs

Now we will discuss about the graphs of Case1.

Similarly, graphs for case2 also plotted.

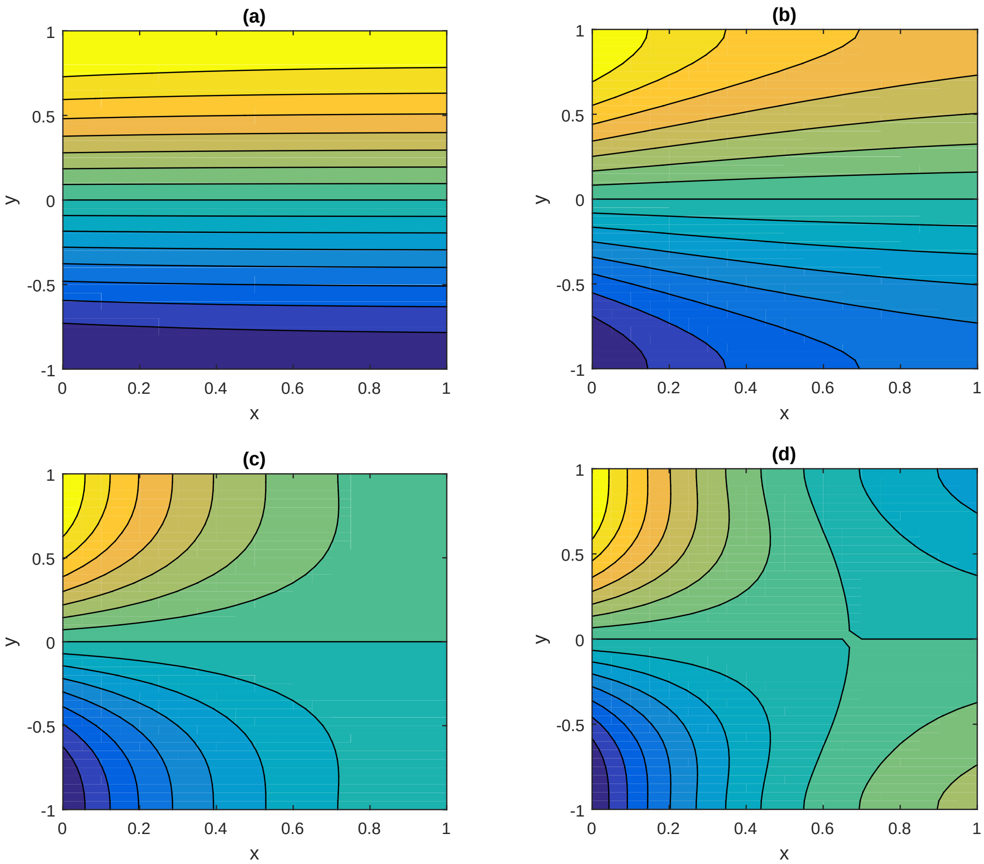

Figure 4.1: Effect of V_0 on ψ when $Q_0 = 3$, $\beta = 2$ and $\alpha = 2$ (a) $V_0 = 1$, (b) $V_0 = 1.5$, (c) $V_0 = 2$, (d) $V_0 = 2.5$

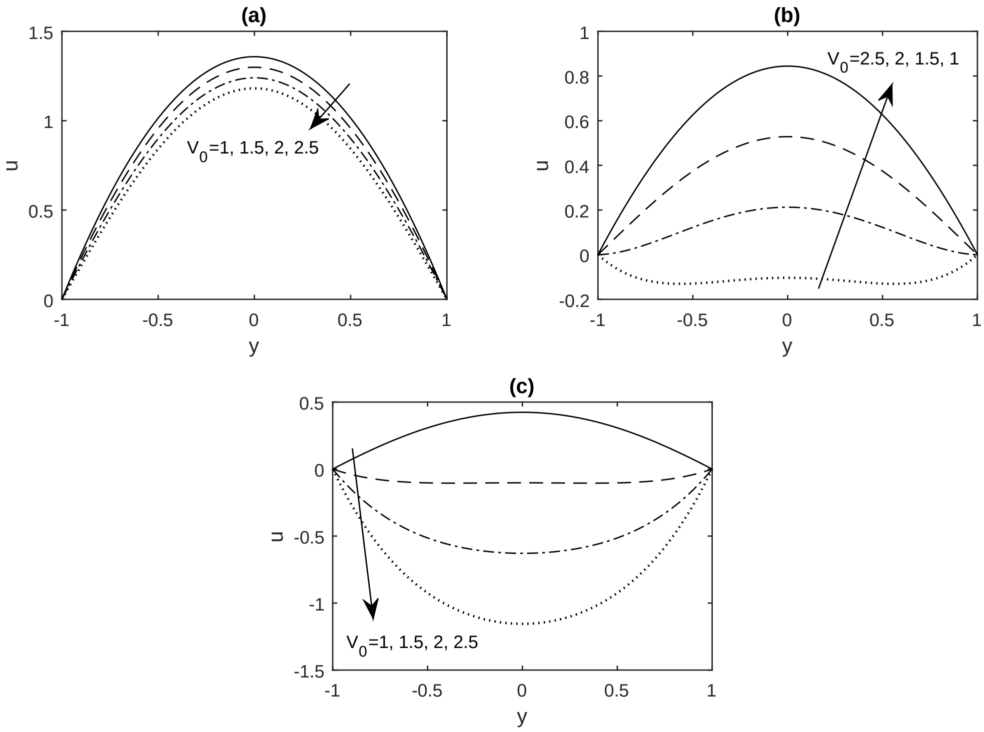

Figure 4.2: Effect of V_0 on longitudinal velocity when $Q_0 = 3$, $\alpha = 0.8$ and $\beta = 1$ near (a) the entrance of slit, $x = 0.1$, (b) the centre of slit, $x = 0.5$, and (c) the end of slit, $x = 0.9$

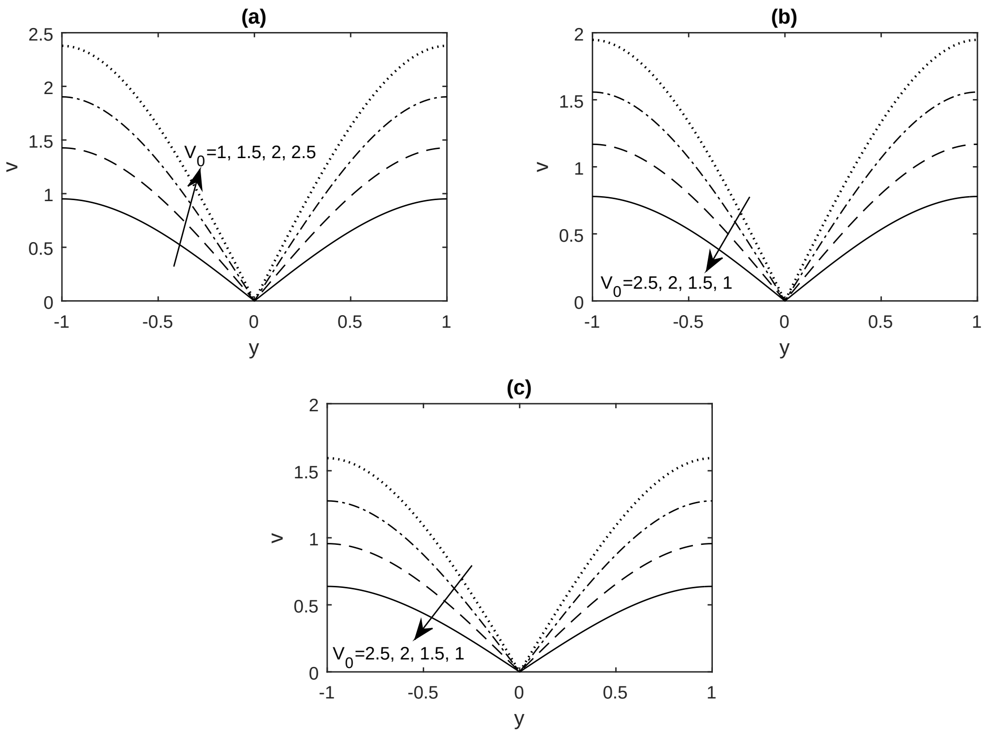

Figure 4.3: Effect of V_0 on transverse velocity when $Q_0 = 3$, $\alpha = 2$ and $\beta = 3$ near (a) the entrance of slit, $x = 0.1$, (b) the centre of slit, $x = 0.5$, and (c) the end of slit, $x = 0.9$

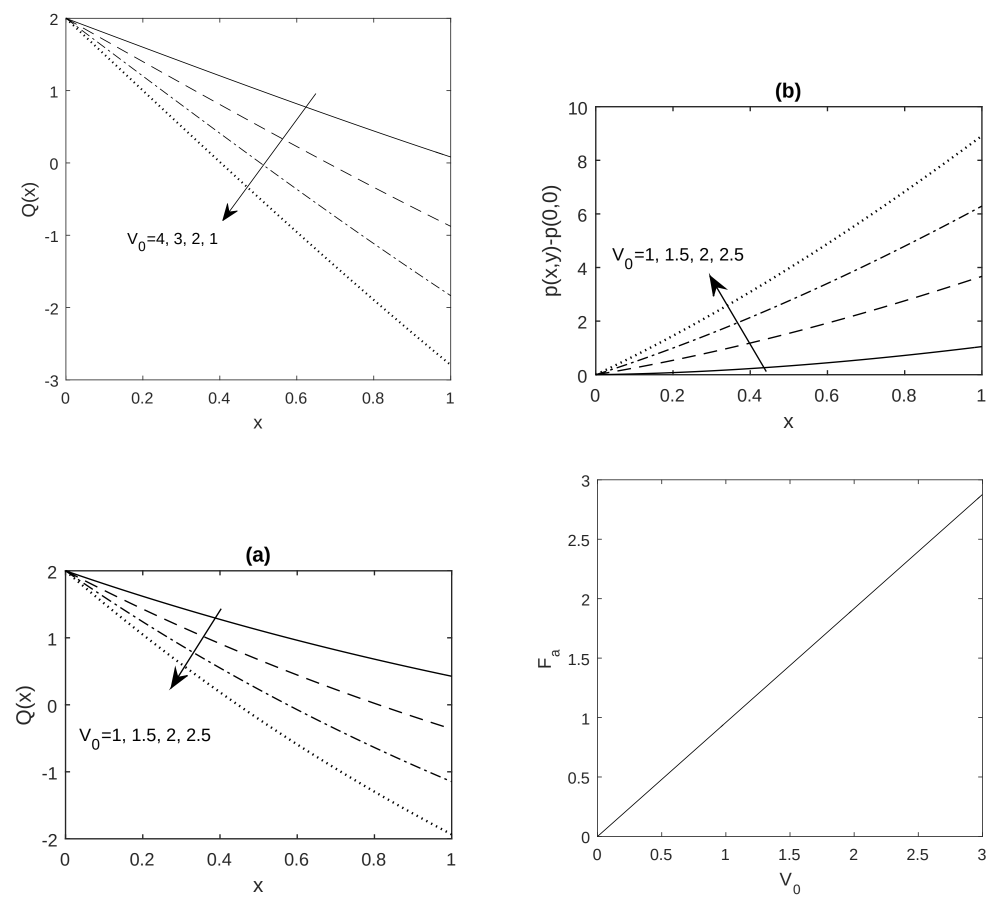

Figure 4.4: Effect of V_0 on $Q(x)$, Pressure difference, $p(x)$, Leakage flux $q(x)$ and FA when $Q_0 = 2$, $\alpha = 0.5$, $\beta = 1$, $L = 1$ and $y = 0$

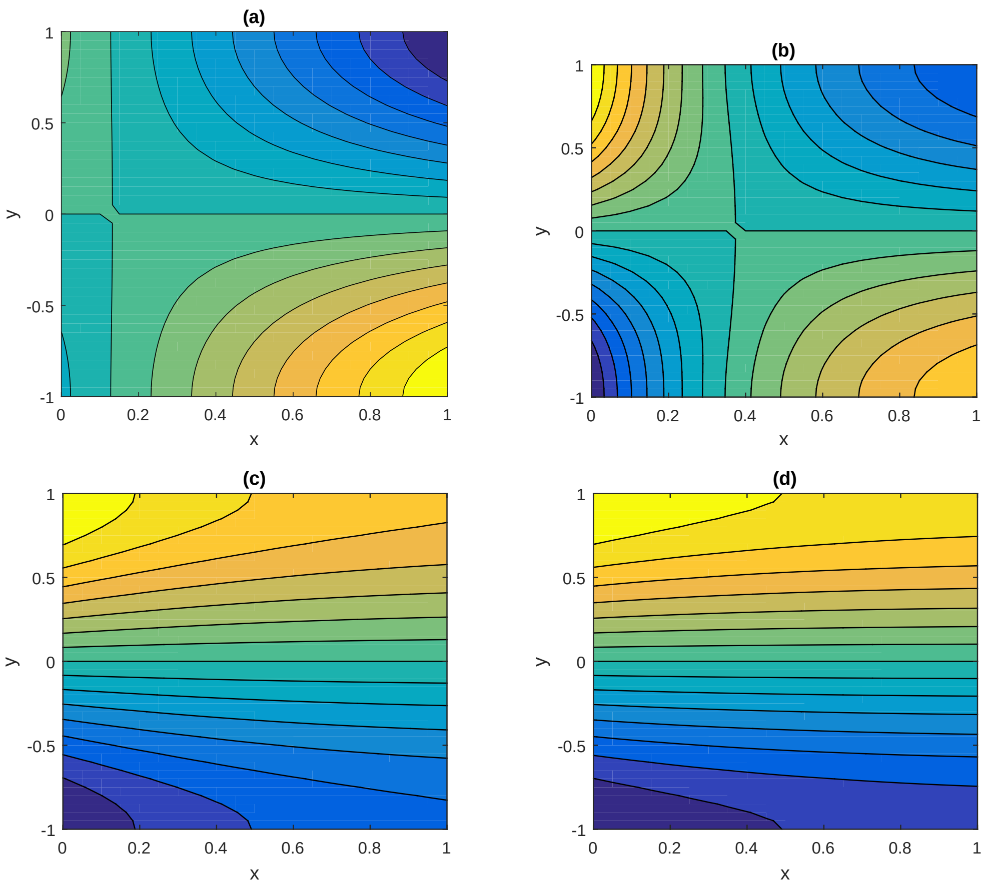

Figure 4.5: Effect of Q_0 on ψ when $V_0 = 1$, $\beta = 2$ and $\alpha = 0.5$ (a) $Q_0 = 1$, (b) $Q_0 = 2$, (c) $Q_0 = 3$, (d) $Q_0 = 4$

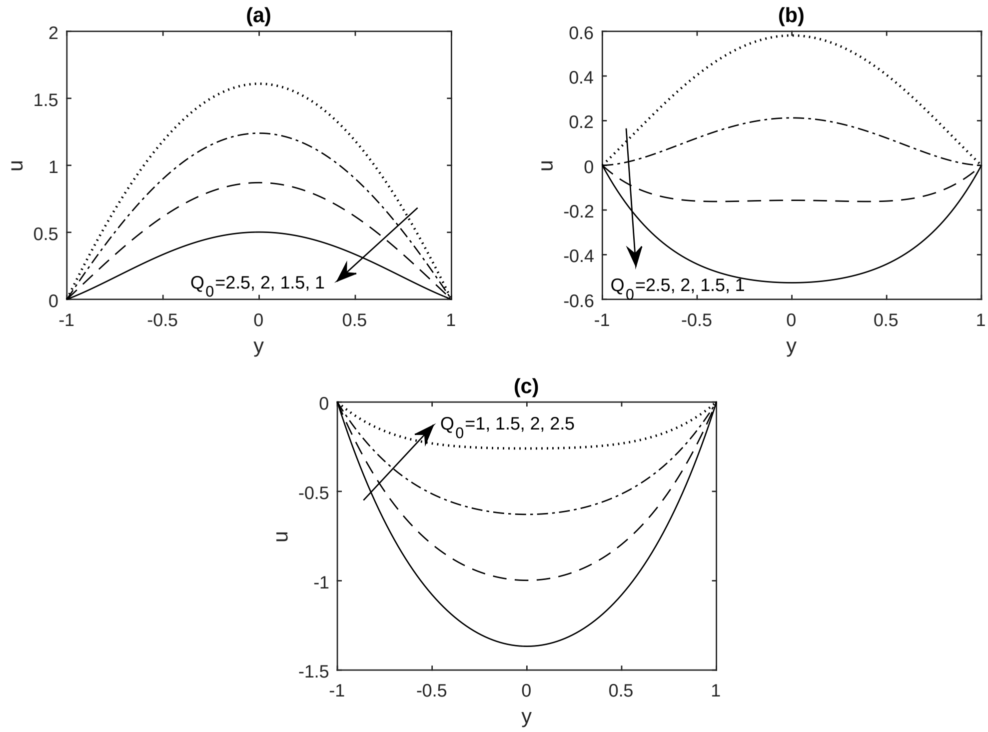

Figure 4.6: Effect of Q_0 on longitudinal velocity when $V_0 = 1$, $\alpha = 0.5$ and $\beta = 2$ near (a) the entrance of slit, $x = 0.1$, (b) the centre of slit, $x = 0.5$, and (c) the end of slit, $x = 0.9$

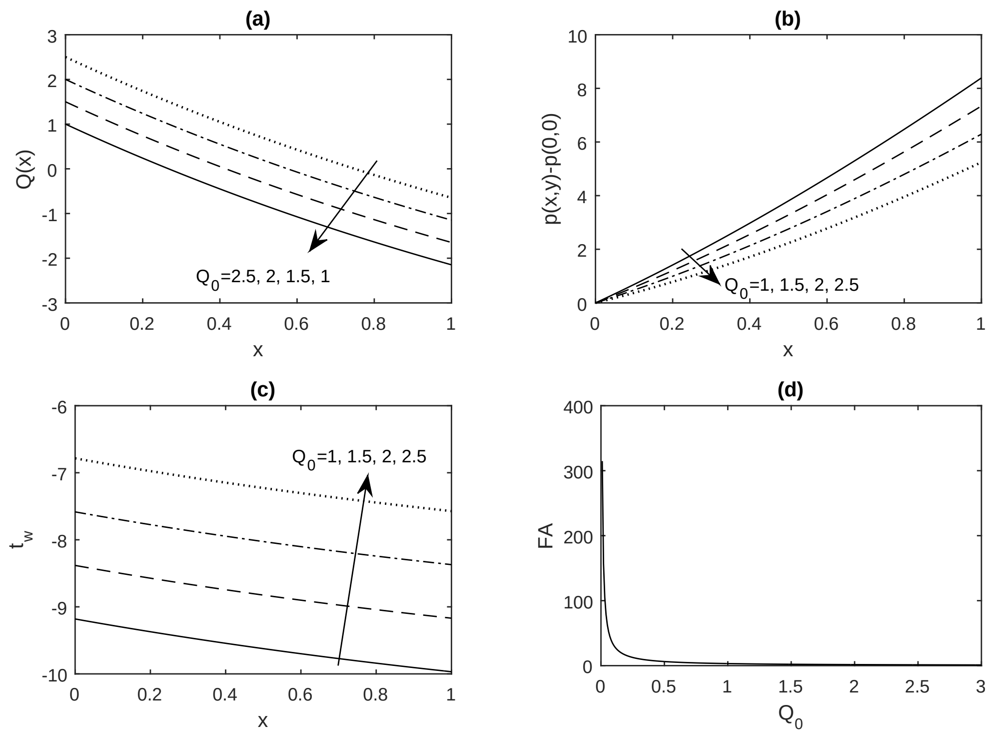

Figure 4.7: Effect of Q_0 on $Q(x)$, pressure, wall shear stress and FA when $V_0 = 0.5$, $\alpha = 2$, $\beta = 1$, $L = 1$ and $y = 0$

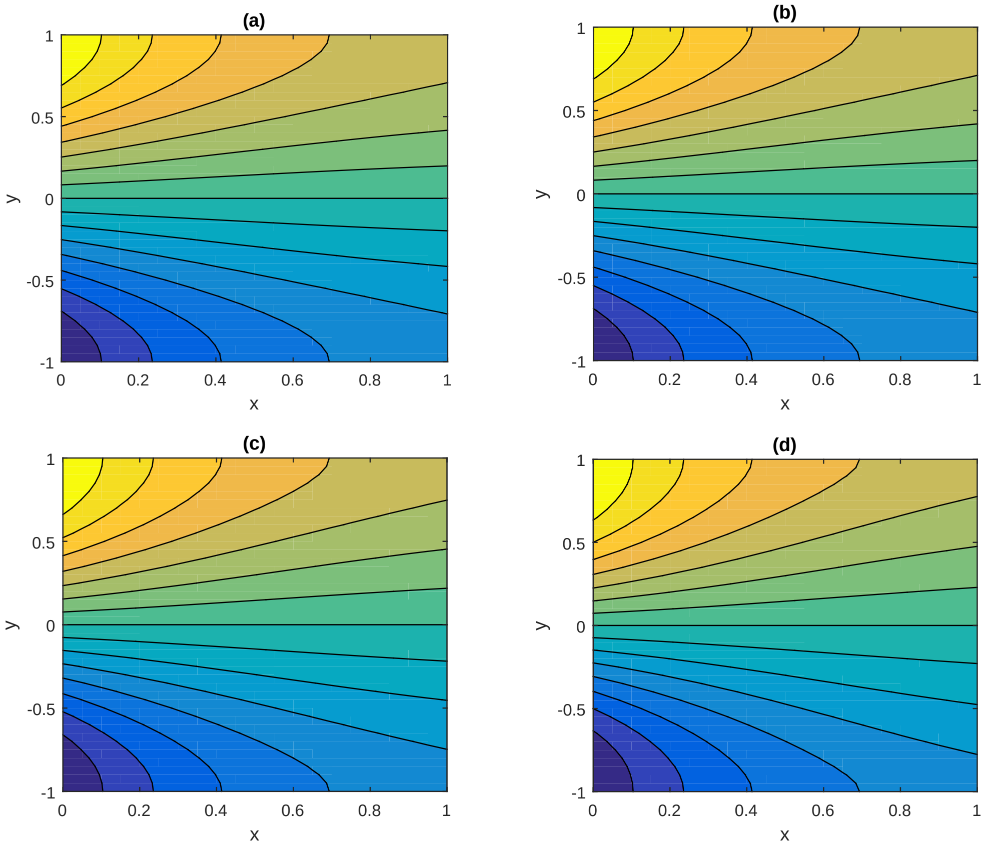

Figure 4.8: Effect of β on ψ when $V_0 = 2$, $Q_0 = 3$ and $\alpha = 2$ (a) $\beta = 0.1$, (b) $\beta = 1$, (c) $\beta = 2$, (d) $\beta = 4$

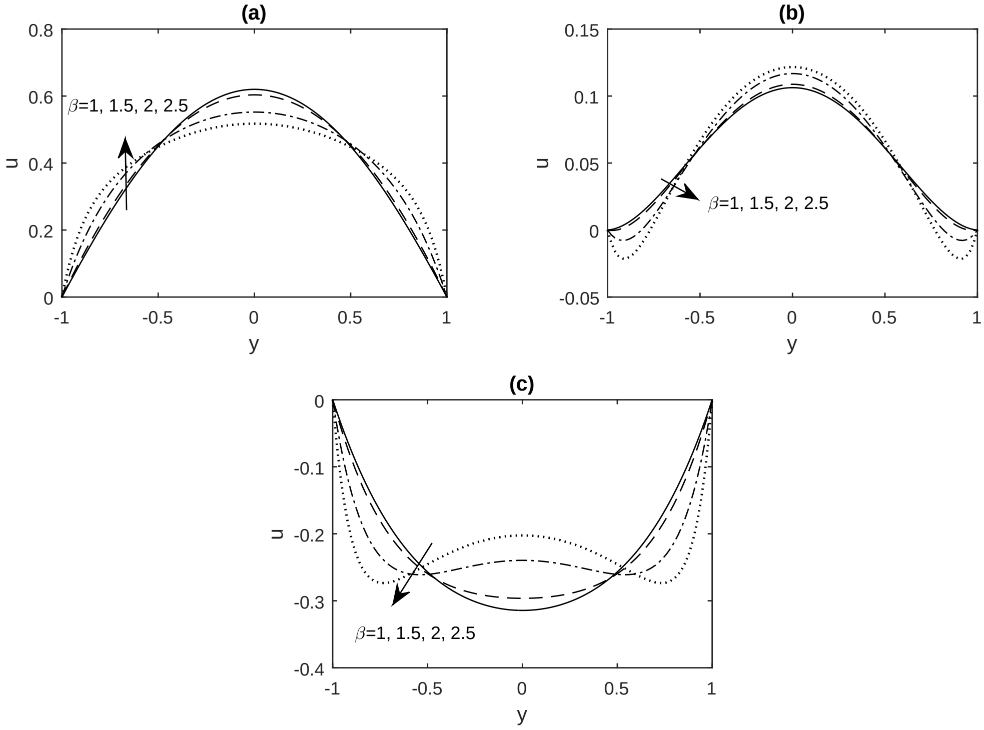

Figure 4.9: Effect of β on longitudinal velocity when $Q_0 = 3$, $\alpha = 0.8$ and $V_0 = 1$ near (a) the entrance of slit, $x = 0.1$, (b) the centre of slit, $x = 0.5$, and (c) the end of slit, $x = 0.9$

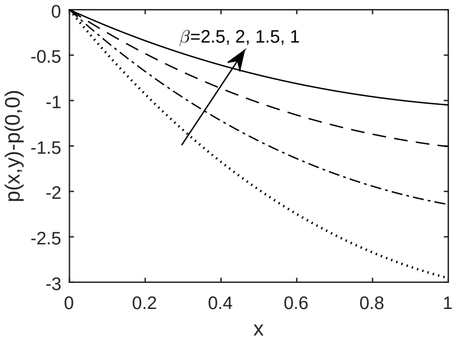

Figure 4.10: Effect of β on pressure when $V_0 = 0.5$, $\alpha = 1$, $Q_0 = 2$ and $y = 0$

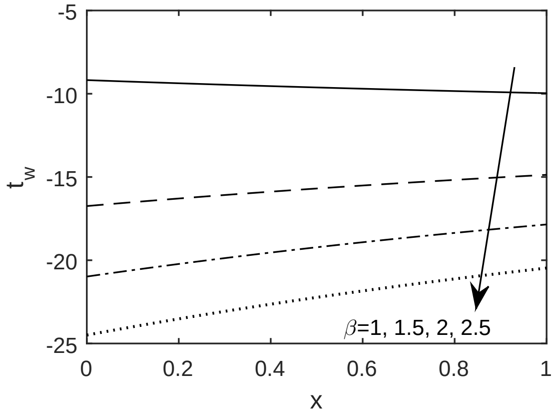

Figure 4.11: Effect of β on wall shear stress when $V_0 = 2$, $\alpha = 1$, $Q_0 = 0.5$ and $y = 0$

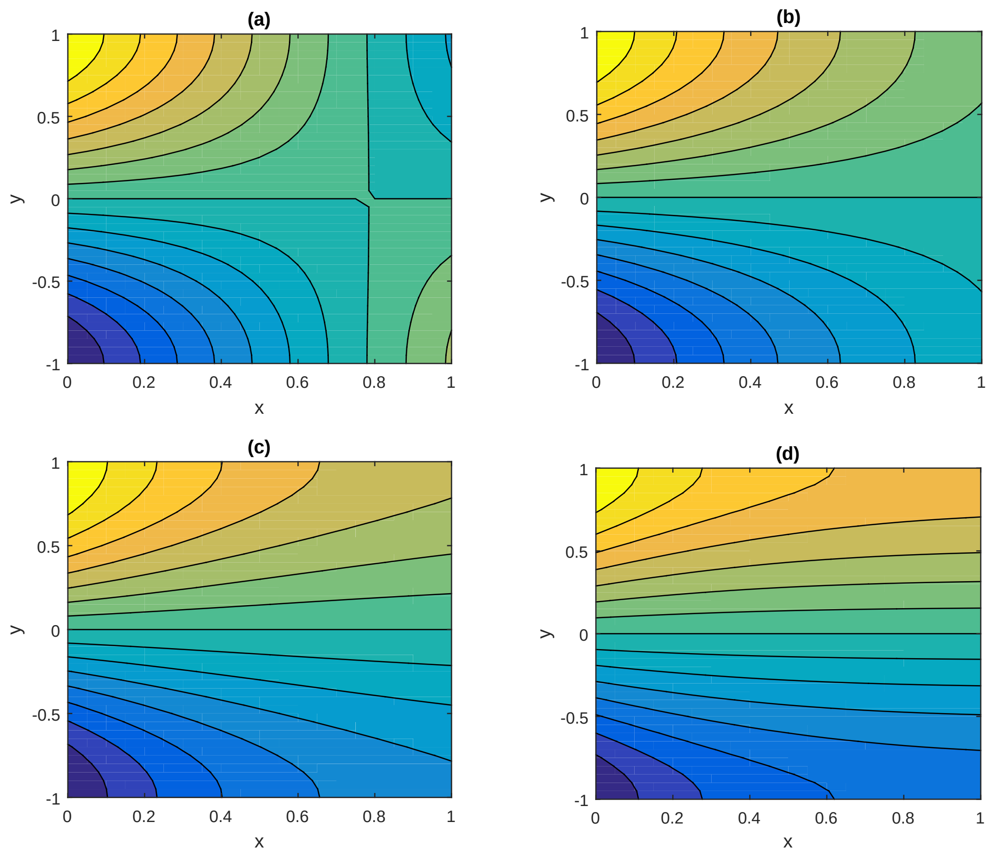

Figure 4.12: Effect of α on ψ when $Q_0 = 2$, $\beta = 2$ and $\alpha = 3$ (a) $\alpha = 0.1$, (b) $\alpha = 1$, (c) $\alpha = 1.5$, (d) $\alpha = 2$

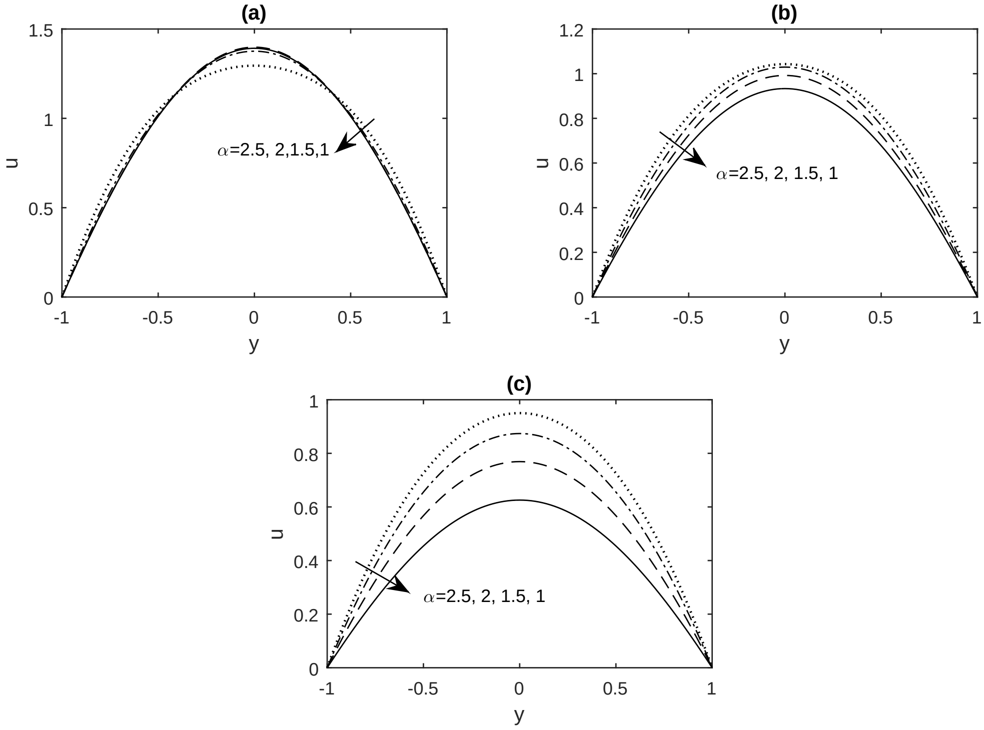

Figure 4.13: Effect of α on longitudinal velocity when $Q_0 = 3$, $V_0 = 2$ and $\beta = 0.5$ near (a) the entrance of slit, $x = 0.1$, (b) the centre of slit, $x = 0.5$, and (c) the end of slit, $x = 0.9$

4.3 Graphical discussion

Figure 4.1 presents the variations of streamlines with different values of V_0, the velocity with which fluid absorbed in the walls of slit. When we use greater values of V_0 reverse flow is detected, as walls absorb fluid in larger quantity. In Figure 4.2 we check the longitudinal velocity $u(x,y)$ as V_0. We will check its effect at three different positions along the slit, at the entrance of slit and at end of the slit and at the middle point of the slit. With the increase in the values of V_0 the longitudinal velocity decreases and at the end of slit with the decrease in velocity giving the possibility of reverse flow. Figure 4.3 shows transverse velocity of V_0 at three different positions and it shows that transverse velocity increases with increase in V_0 values. Figure 4.4 shows shows different effects of V_0 on Pressure difference, Leakage flux, Fractional reabsorption and volume flow rate. We will notice that Pressure difference also increases with increase in V_0 values but at the end of slit its values starts decreasing. We will notice different variations in the flow rate $Q(x)$ with x. Flow rate is effected by V_0 whenever V_0 increases flow rate

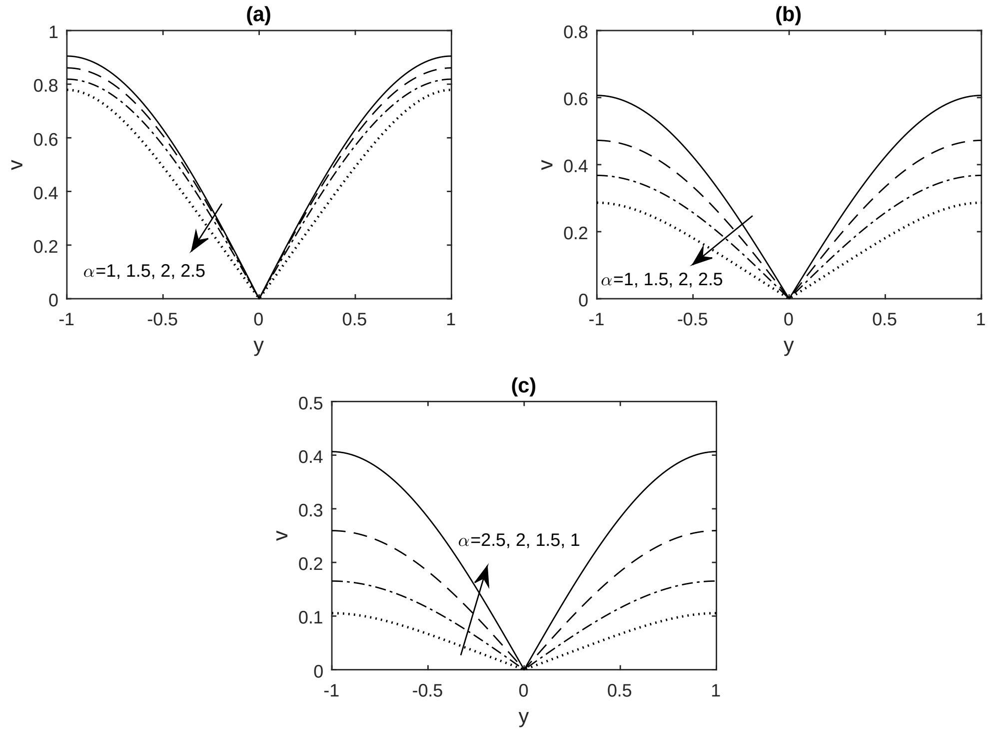

Figure 4.14: Effect of α on transverse velocity when $Q_0 = 2$, $V_0 = 1$ and $\beta = 0.5$ near (a) the entrance of slit, $x = 0.1$, (b) the centre of slit, $x = 0.5$, and (c) the end of slit, $x = 0.9$

decreases.

In figure 4.5 we will check different variations of Q_0 on streamline. There we notice increase in longitudinal flow with increase in the values of Q_0, and similarly with low initial flow fluid start moving in reverse direction. In figure 4.6, we notice longitudinal velocity $u(x,y)$ keep on increasing with the increase in Q_0 values. This takes place at all the points of the slits. next figure 4.7 shows the effect of Q_0 on different quantities. Wall shear stress is effected by Q_0 and increases with increases in the value of Q_0. Similarly pressure is effected by Q_0 in same pattern.

in figure 4.8 The effect of porosity is shown on streamline. We will notice that with the change in the values of β, forward flow does not change to reverse flow and vice versa. Only the flow at along the slit and at the walls of slit is somewhere effected by β when its value increases. Next in figure 4.9, where longitudinal velocity u is plotted against y using different x positions in the slit.we will notice that porosity is increased, while velocity profiles becomes flatter near $x-axis$. this is because of porosity the fluid does not flow freely in the walls of slits. In next Figure 4.10 we will notice changing in

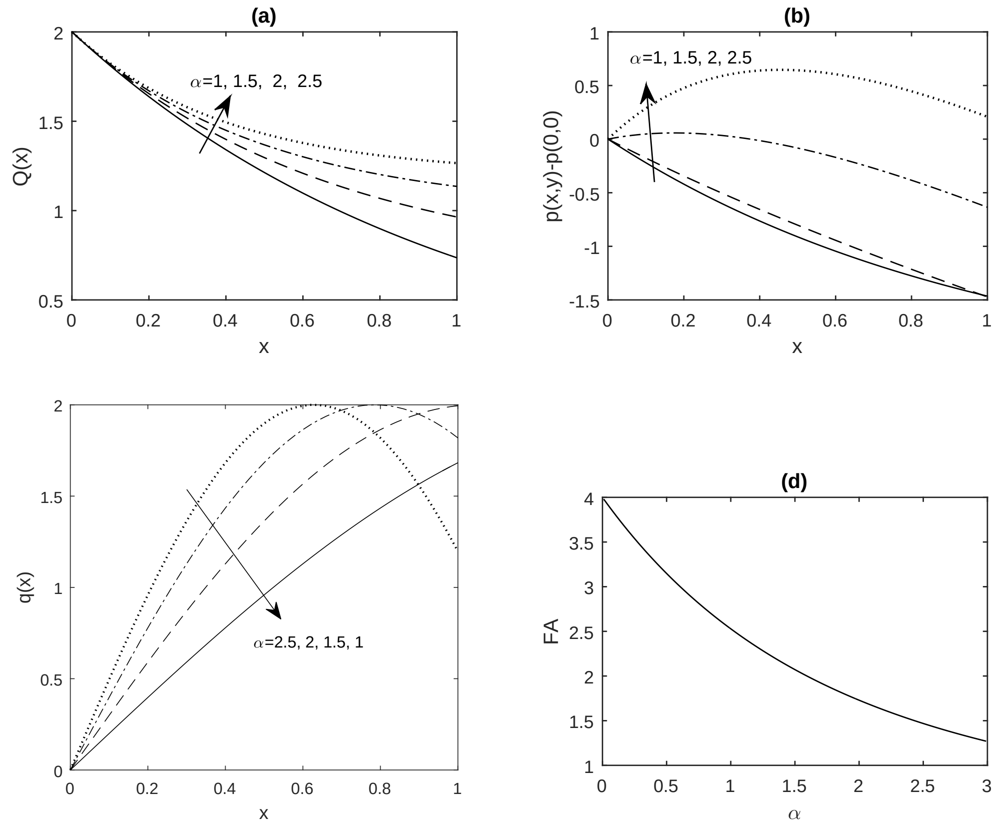

Figure 4.15: Effect of α on Q(x), pressure, q(x) and FA when $Q_0 = 3$, $V_0 = 1$, $\beta = 2$, $L = 1$ and $y = 0$

the pressure and wall shear stress with the change in the values of β. Pressure difference is highly effected by β. When β increases pressure difference also increases but wall shear stress decrease when β vales changes.

In next figure 4.11 we will check out variations of streamline with different vales of α just like β α effect all the quantizes. In figure 4.12 and 4.13 Longitudinal velocity and transverse velocity is plotted on different positions at start middle and end of slit. Figure 4.14 pressure wall shear stress and volume flow rate also effected by α with the increases in the values of α all theses quantities decreases.

All these graphs shows that all these mention properties are dependent on the flow rate.

Chapter 5

Conclusions

In this thesis, we get some exact solutions for Stokes flow in two dimensions along the slits of wall using periodic reabsorption at the walls. We will get Stream function by transferring partial differential equations int single equation. We will find the exact solutions for velocity profile, volume flow rrate, pressure, pressure drop, wall shear stress, fractional reabsorbtion and leakage fluxx for both cases. We will Also get some Graphical analysis of our results. We will observe following observation through this work.

• Longitudinal and Transverse velocities dependent on medium permeability β . Longitudinal velocity also depends upon initial flow rate and absorbtion velocity, but transverse is not depends upon it.

• Pressure difference are highly dependent in medium permeability.

• Volume flow rate, pressure difference and leakage flux decreases .

• Wall shear stress increase along increase in permeability of medium, on the other hand fractional reabsorption as well as leakage flux is not.

• When initial flow rate increase the streamline becomes flatter.it may also results in reverse flow rate only when the value is small.

• Longitudinal velocity decreases if the absorbtion velocity increases, and increases close to the walls of slit.With the decrease in initial flow reverse flow may also takes place.

In the last, we would say our research is about theoretical nature and this work needed more and more work in transport of fluid through renal tubule in diseased state.

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