Ostrwoski Type Inequalities and Related Bounds

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DEDICATION

My Loving Parents and Teachers and my Siblings. Who always encouraged me to Work Hard and Guided me towards the Right Way and Destination.

ACKNOWLEDGEMENTS

Praise to be ALLAH, the Cherisher and Lord of the World, Most gracious and Most Merciful.

Person is not perfect in all the contexts of this life. He has a limited mind and minor thinking approaches. It is the guidance from the almighty ALLAH that shows the man light in the darkness and the person finds his way in the light. Without this helping light, person is nothing but a helpless creature. Same is the case with us, as we experience all these phenomena during the completion of this project and have been successful in fulfilling this duty assigned to us only because of the help of ALLAH. The teaching of the Holy Prophet Muhammad (PBUH) were also the continuous source of guidance for us especially His order of getting knowledge and fulfilling one's duty honestly was the key for motivating us. Credit of my humble efforts goes to my respectable teacher Dr. Saad Ihsaan Butt, under whose supervision I was able to complete this task.

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ABSTRACT

We give simple proofs of Ostrowski's inequality. Then we found some general bounds of a non-negative difference of the Hadamard inequality and an Ostrowski-Gruss type inequalities. We also investigate new inequalities of the Ostrowski-type for the Cebysev functional and give applications for the Taylor's expansion and generalized trapezoid formula.Using these results Fink identity can be investigated.

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Preliminaries and Introduction

1 Preliminaries and Introduction

G. H. Hardy Harald Bohr reported ”all analysts spend half their time hunting through the literature for inequalities which they want to use but cannot prove”. The dynamic field of mathematical inequalities open many doors for analysts. Inequalities play an important role in many areas of mathematics. e.g number theory, geometry, functional analysis, probability, calculus,etc.
In number theory for Diophantine approximations, in engineering and geometry for isoperimetric problems of convex bodies and in probability theory most of the laws put in terms of inequalities. Linear inequalities play a fundamental role in the theory of linear programming. Many inequalities are used in the theory of functions for the derivatives of the polynomials. Linear functional or linear operator in some spaces define or estimate by using classical inequalities. In functional analysis, the definition of norm in a function space requires it to meet the triangle inequality. Some problems in the theory of linear inequalities depends upon the theory of approximations, generated by P. L. Chebyshev. The Cauchy-Schwarz inequality is important because it connects the notion of an inner product with the notion of length. In multivariable calculus, inequalities are used for the gradient vector of a function. Many inequalities are very helpful to solve PDEs and integral equations. Jensen's inequality shows the significant relation between mean and the mean of function values.

1.1 Convex functions

The modern view point on convex functions implies a strong and clear interaction between analysis, engineering and geometry, making the reader to share a sense of excitement. In an article R. J. Gardner [6, 7], described this reality in beautiful words ” convexity like an octopus tentacles and changes its color and shape, cruise area to another and obviously, the research opportunities abound”. In recent years, the concept of convex functions generalized on a large scale. Many fields of modern analysis directly or indirectly includes the applications of convex functions.
Some definitions, characterizations, theorems and remarks about convex functions are given below which are related to our research work. (see [8], Ch:1)

1.1.1 Convex sets

A set $C \subset \Re^n$ is convex if $[\alpha,\beta] \subset C$ for every $\alpha,\beta \in C$ then $\lambda\alpha + (1-\lambda)\beta$ must also belong to C for each $\lambda \in [0,1]$. **Convex functions:** A function $\Psi : I \to \Re$ is called convex (i.e $I \subset \Re$) if for all $\xi_1, \xi_2 \in I$ and $\lambda \in [0,1]$,the inequality

$$\Psi\Big(\lambda\xi_1 + \big(1-\lambda\big)\xi_2\Big) \leq \lambda\Psi(\xi_1) + \big(1-\lambda\big)\Psi(\xi_2),$$

holds. If the strictly inequality holds in above inequality then the function is said to be strictly convex function.

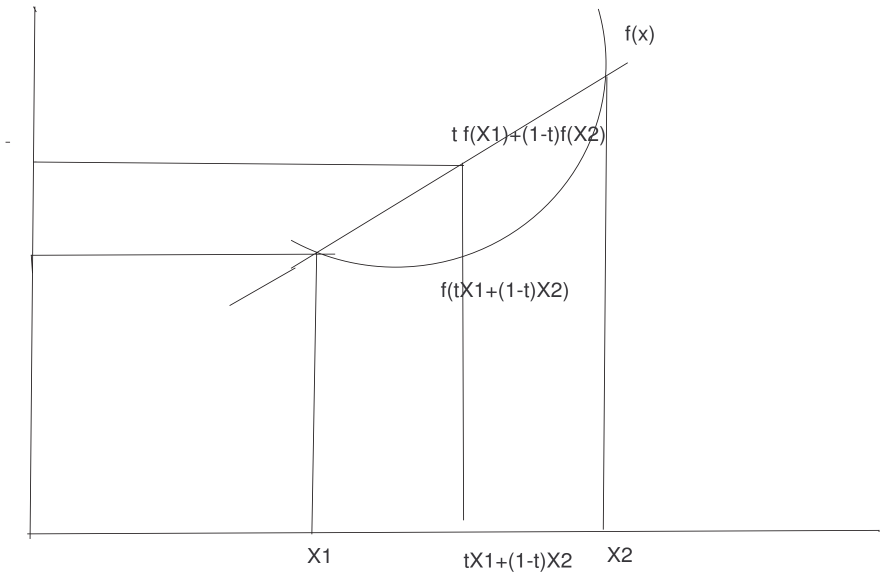
1.1.2 Concave functions

If ”$-\Psi$” is convex then Ψ is called concave function.

1.1.3 Geometric interpretation of convex functions

The geometrical interpretation of convex function is that the graph of Ψ lies below its chords.

$$\Psi\big(\lambda\xi_1+\big(1-\lambda\big)\xi_2\big)\leq\lambda\Psi(\xi_1)+\big(1-\lambda\big)\Psi(\xi_2). \tag{1}$$



The inequality 1 means that the dotted line intersects the graph of a function under the curved line and strictly convex expression can be satisfy into equality in 1 if and only if $\xi_1=\xi_2$.

Examples:

- $e^x, x^p(p\geq 1, x>0), \frac{1}{x}(x\neq 0)$ are convex functions.
- $\log x(x>0), sinx(0\leq x\leq\pi), cosx(\frac{-\pi}{2}\leq x\leq\frac{\pi}{2})$ are concave functions.

Remarks:

1. For $\xi_1,\xi_2\in I, p,q\geq 0, p+q>0, then$

   $$\Psi\big(\lambda\xi_1+\big(1-\lambda\big)\xi_2\big)\leq\lambda\Psi(\xi_1)+\big(1-\lambda\big)\Psi(\xi_2), \tag{2}$$

   is equivalent to

   $$\frac{\Psi(p\xi_1+q\xi_2)}{p+q}\geq\frac{p\Psi(\xi_1)+q\Psi(\xi_2)}{p+q}.$$

2. Ψ is both convex and concave if and only if $\Psi(\xi)=\lambda\xi+\gamma$, for some $\lambda,\gamma\in\Re$.

3. The geometrical interpretation of convexity or convex function is that the graph of Ψ lies below its chords.

4. If $\xi_1, \xi_2.\xi_3$ are three points in I such that $\xi_1 < \xi_2 < \xi_3$, then 2 is equivalent to
$\begin{vmatrix} \xi_1 & \Psi(\xi_1) & 1 \\ \xi_2 & \Psi(\xi_2) & 1 \\ \xi_3 & \Psi(\xi_3) & 1 \end{vmatrix} = (\xi_3 - \xi_2)\Psi(\xi_1) + (\xi_1 - \xi_3)\Psi(\xi_2) + (\xi_2 - \xi_1)\Psi(\xi_3) \geq 0,$
which is equivalent to
$$\Psi(\xi_2) \leq \frac{\xi_2 - \xi_3}{\xi_1 - \xi_3}\Psi(\xi_1) + \frac{\xi_1 - \xi_2}{\xi_1 - \xi_3}\Psi(\xi_3),$$
or more symmetrically and without the condition of monotonicity on ξ_1, ξ_2, ξ_3
$$\frac{\Psi(\xi_1)}{(\xi_1 - \xi_2)(\xi_1 - \xi_3)} + \frac{\Psi(\xi_2)}{(\xi_2 - \xi_3)(\xi_2 - \xi_1)} + \frac{\Psi(\xi_3)}{(\xi_3 - \xi_1)(\xi_3 - \xi_2)} \geq 0.$$

5. If Ψ is a convex function on I and if $\xi_1 \leq \varphi_1, \xi_2 \leq \varphi_2, \xi_1 \neq \xi_2, \varphi_1 \neq \varphi_2$, then the following inequality,
$$\frac{\Psi(\xi_2) - \Psi(\xi_1)}{\xi_2 - \xi_1} \leq \frac{\Psi(\varphi_2) - \Psi(\varphi_1)}{\varphi_2 - \varphi_1},$$
holds.

6. The area of the triangle whose vertices are $(\xi_1, \Psi(\xi_1)), (\xi_2, \Psi(\xi_2))$ and $(\xi_3, \Psi(\xi_3))$ is given by
$$P = \frac{1}{2}\begin{vmatrix} \xi_1 & \Psi(\xi_1) & 1 \\ \xi_2 & \Psi(\xi_2) & 1 \\ \xi_3 & \Psi(\xi_3) & 1 \end{vmatrix},$$
The area of the triangle depends on the rotation if the rotation is anticlockwise then area will be positive and if the rotation is clockwise then area will be negative. $P > 0$ represent a convex function and $P < 0$ express a concave function.

1.2 Important Integral inequalities

Some of the important inequalities cited from [8] are given below:

1.2.1 Means inequality:

Let $\{\xi_i\}_{i=1}^{n} \in \Re^{+}$. Then for these numbers
$$Q.M \geq A.M \geq G.M \geq H.M$$
i.e.
$$\sqrt{\frac{\sum_{i=1}^{n} (\xi_i)^2}{n}} \geq \frac{\sum_{i=1}^{n} (\xi_i)}{n} \geq \sqrt[n]{\prod_{i=1}^{n} (\xi_i)} \geq \frac{n}{\sum_{i=1}^{n} \left(\frac{1}{\xi_i}\right)}.$$

Equality holds if and only if $\xi_1 = \xi_2 = \ldots = \xi_n$.

1.2.2 Triangular inequality:

$\forall \ \{\xi_i\}_{i=1}^{n} \in \Re$. Then

$$|\Sigma_{i=1}^{n} (\xi_i)| \leq \Sigma_{i=1}^{n} |(\xi_i)|.$$

1.2.3 Cauchy Schwarz inequality:

Let $\{\xi_i\}_{i=1}^{n}$ and $\{\varphi_i\}_{i=1}^{n}$ be real numbers. Then

$$|\Sigma_{i=1}^{m} \xi_i \varphi_i|^2 \leq \left(\Sigma_{i=1}^{m} |\xi_i^2|\right)\left(\Sigma_{i=1}^{m} |\varphi_i^2|\right).$$

1.2.4 Minkowski's inequality:

Let $\{\xi_i\}_{i=1}^{n}$ and $\{\varphi_i\}_{i=1}^{n}$ be positive n-tuples and $p > 1$. Then

$$\left(\Sigma_{i=1}^{n} (\xi_i + \varphi_i)^p\right)^{\frac{1}{p}} \leq \left(\Sigma_{i=1}^{n} \xi_i^p\right)^{\frac{1}{p}} + \left(\Sigma_{i=1}^{n} \varphi_i^p\right)^{\frac{1}{p}},$$

if $p < 1 (p \neq 0)$, we have the reverse inequality and equality holds: $\frac{\xi_1}{\varphi_1} = \frac{\xi_2}{\varphi_2} = \ldots \frac{\xi_n}{\varphi_n}$.

1.2.5 Hadamard inequality:

Let $f : I \to R$ be a convex function defined on the interval I of real numbers and $a, b \in I$ with $a < b$. Then the following double inequality

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}. \tag{3}$$

holds. It is known as Hadmard Inequality.History of this inequality begins with the papers of Hermite[4] and Hadamard[4] in the year 1883-1893 [8].

7. textbfHolder's inequality:
Let $\{\xi_i\}_{i=1}^{n}$ and $\{\varphi_i\}_{i=1}^{n}$ be positive n-tuples and p, q be two non-zero numbers such that $\frac{1}{p} + \frac{1}{q} = 1$, then

$$\left(\Sigma_{k=1}^{n} \xi_k \varphi_k\right) \leq \left(\Sigma_{k=1}^{n} \xi_k^p\right)^{\frac{1}{p}} \left(\Sigma_{k=1}^{n} \varphi_k^q\right)^{\frac{1}{q}}.$$

Equality holds if and only if $\frac{\xi_1^p}{\varphi_1^q} = \frac{\xi_2^p}{\varphi_2^q} = \ldots \frac{\xi_n^p}{\varphi_n^q}$.

1.2.6 Cebysev's inequality:

Let $\xi_1 \leq \xi_2 \leq \ldots \leq \xi_n$ and $\varphi_1 \leq \varphi_2 \leq \ldots \leq \varphi_n$ be real numbers. Then we have

$$\left(\Sigma_{i=1}^{n}\xi_i\right)\left(\Sigma_{i=1}^{n}\varphi_i\right) \leq n\left(\Sigma_{i=1}^{n}\xi_i\varphi_i\right).$$

Equality holds if and only if $\xi_1 = \xi_2 = \ldots = \xi_n$ or $\varphi_1 = \varphi_2 = \ldots = \varphi_n$.

1.2.7 Cebysev Functional

For two Lebesgue integrable functions $f, g : [a,b] \to \mathbb{R}$ consider the Cebysev functional

$$T(f,g) = \frac{1}{b-a}\int_a^b f(t)g(t)dt - \frac{1}{b-a}\int_a^b f(t)dt \frac{1}{b-a}\int_a^b g(t)dt. \tag{4}$$

1.3 Gruss inequlity

In 1935 Gruss proved a well known Inequality known as Gruss inequality stated in the following[5]

. Let $f, g : [a,b] \to \mathbb{R}$ be integrable functions such $\phi \leq f(x) \leq \Phi$ and $\gamma \leq g(x) \leq \Gamma$ for all $x \in [a,b]$, where $\phi, \Phi, \gamma, \Gamma$ are constants. Then we have

$$\left|\frac{1}{b-a}\int_a^b f(x)g(x)dx - \frac{1}{b-a}\int_a^b f(x)dx \cdot \frac{1}{b-a}\int_a^b g(x)dx\right| \leq \frac{1}{4}(\Phi - \phi)(\Gamma - \gamma).$$

where the constant 1/4 is sharp.

1.3.1 Other Gruss type inequality:

- In 1934, G.Gruss showed that

  $$T(f,g) = \frac{1}{4}(M-m)(N-n)$$

  provided that m, M, N and n are real numbers with the property

  $$-\infty < m \leq f \leq M < \infty \quad -\infty < n \leq g \leq N < \infty.$$

  The constant $1/4$ is best possible

  $$T(f,g) = \frac{1}{12} \parallel f^{'} \parallel_\infty \parallel g^{'} \parallel_\infty (b-a)^2 \tag{5}$$

  where $\parallel f^{'} \parallel_\infty = Sup_{t\in[a,b]}|f^{'}(t)|$.

1.4 Some Ostrowski's types inequalities:

- In 1970, A.M Ostrowski's [5] proved amongst others the following results that is somehow a mixture of the Cebysev and Gruss results.

$$T(f,g) = \frac{1}{8}(b-a)(M-m) \parallel g^{'} \parallel_{\infty}$$

- Let $f,g : [a,b] \to R$ be integrable functions such $\phi \leq f(x) \leq \Phi$ and $\gamma \leq g(x) \leq \Gamma$ for all $x \in [a,b]$, where ϕ, Φ, γ, Γ are constants. Then we have

$$\left| \frac{1}{b-a} \int_a^b f(x)g(x)dx - \frac{1}{b-a} \int_a^b f(x)dx \cdot \frac{1}{b-a} \int_a^b g(x)dx \right| \leq \frac{1}{4}(\Phi - \phi)(\Gamma - \gamma).$$

- **Ostrowski's Integral inequality for differentiable functions:**
Let I⊂R be an open Innrerval and a,b∈I, a<b. If $f : I \to$R is differntiable function such that $\gamma \leq f^{'}(t) \leq \Gamma$, $\forall t \in [a,b]$, for some constants $\gamma, \Gamma \in \mathbb{R}$, then we have

$$\left| (b-a) \left[\frac{\lambda}{2}(f(a)+f(b)) + (1-\lambda)f(x) - \frac{\Gamma+\gamma}{2}(1-\lambda)\left(x - \frac{a+b}{2a}\right) \right] - \int_a^b f(t)dt \right|$$
$$\leq \frac{\Gamma-\gamma}{2} \left[\frac{(b-a)^2}{4}(\lambda^2 + (1-\lambda^2)) + \left(x - \frac{a+b}{2}\right)^2 \right]$$

where $a + \lambda((b-a)/2) \leq x \leq b - \lambda((b-a)/2)$ and $\lambda \in [0,1]$.

- **Ostrowski's Integral inequality for Absolutely Continuous Mappings:**

$$\left| f(x) - \frac{1}{b-a} \int_a^b f(t)dt \right| \leq \begin{cases} [\frac{1}{4} + \left(\frac{x - \frac{a+b}{2}}{b-a}\right)^2](b-a)\|f'\|_\infty \\ \left\{ \frac{1}{q+1} \left[\left(\frac{x-a}{b-a}\right)^{q+1} + \left(\frac{b-x}{b-a}\right)^{q+1} \right] \right\}^{\frac{1}{q}} \\ \times (b-a)^{\frac{1}{q}} \|f'\|_p, p > 1, \frac{1}{p} + \frac{1}{q} = 1 \\ \left[\frac{1}{2} + \frac{|x - \frac{a+b}{2}|}{b-a} \right] \|f'\|_1 a \end{cases}$$

For all $x \in [a,b]$, where

$$\|f'\|_s := \begin{cases} \left(\int_a^b |f'(t)|^s dt \right)^{\frac{1}{s}}, ifs \in [1,\infty] \\ sup_{t \in (a,b)} |f'(t)| ifs = \infty. \end{cases}$$

- **Ostrowski's Integral inequality for Mappings of Bounded Variation:**
Let $f : [a,b] \to R$ be a mapping of bounded variation on $[a,b]$. Then, we have the inequality

$$\left| f(x) - \frac{1}{b-a} \int_a^b f(t)dt \right| \leq [\frac{1}{2} + \frac{|x - \frac{a+b}{2}|}{b-a}] \bigvee_a^b (f)$$

For all $x\varepsilon[a,b]$, where $\bigvee_a^b(f)$ is the total variation of f on $[a,b]$.

Chapter 2
Straightforward Proofs for Ostrwoski Inequality and Related Results

2 Straightforward Proofs for Ostrwoski Inequality and Related Results

In this chapter we give the detailed proofs of inequalities related Ostrowski's and Gruss used in this paper[4].
We give proofs in explicit way for theorems used in following paper.
Ostrowski's Inequality
In 1938, Ostrowski established the following inequality known as the Ostrowski inequality stated as:

Theorem 2.1. *Let $f : I \to R$, I is an interval in $\mathbb{R}$, be a mapping differentiable in I^0 the interior of I and $a, b \in I^0$, $a < b$. If $|f^{'}(t)| \leq M$, for all $t \in [a,b]$, then we have*

$$\left| f(x) - \frac{1}{b-a}\int_a^b f(t)dt \right| \leq \left[\frac{1}{4} + \frac{(x-(a+b)/2)^2}{(b-a)^2} \right](b-a)M.$$

for all $x \in [a,b]$.

Proof. It is clear that
$(t-a)(f'(t)+M)dt \geq 0 \Rightarrow \int_a^x (t-a)(f'(t)+M)dt \geq 0$
Integrating by parts, we have

$$[((t-a)f(t))|_a^x] - \int_a^x f(t)dt + [(\tfrac{M(t-a)^2}{2})|_a^x] \geq 0$$

$\Rightarrow$

$$(x-a)f(x) - \int_a^x f(t)dt + \frac{M}{2}(x-a)^2 \geq 0. \tag{6}$$

Also,
$\int_x^b (b-t)(M-f'(t))dt \geq 0$

$$\int_x^b (b-t)(M-f'(t))dt \geq 0$$

Integrating by parts, we have

$$[((b-t)f(t))|_x^b] - \int_x^b f(t)dt + [(\tfrac{M(t-a)^2}{2})|_x^b] \geq 0$$

$\Rightarrow$

$$(b-x)f(x) - \int_x^b f(t)dt + \frac{M}{2}(b-x)^2 \geq 0 \tag{7}$$

By adding (6) and (7) one has

$$[(b-x)+(x-a)]f(x) - \left[\int_a^x f(t)dt + \int_x^b f(t)dt\right] + \frac{M}{2}[(b-x)^2+(x-a)^2] \geq 0$$

$$(b-a)f(x)-\int_a^b f(t)dt \geq -\frac{M}{2}[(b-x)^2+(x-a)^2]$$

$$f(x)-\frac{1}{(b-a)}\int_a^b f(t)dt \geq -(b-a)\frac{M}{2}\left[(b-x)^2+(x-a)^2\right] \tag{8}$$

On the other hand using positivity of

$$(b-t)(f'(t)+M)$$

and

$$(t-a)(M-f'(t))$$

, we have

$$\int_a^x (t-a)(M-f'(t))dt \geq 0$$

$$\frac{M(t-a)^2|_a^x}{2}-\left[f(t)(t-a)|_a^x-\int_a^x f(t)dt\right] \geq 0$$

$$\frac{M(x-a)^2}{2}-\left[f(x)(x-a)-\int_a^x f(t)dt\right] \geq 0$$

Multiplying with (-1) we get the inequality

$$f(x)(x-a)-\int_a^x f(t)dt \leq \frac{M(x-a)^2}{2}$$

$$\int_a^x (b-t)(M+f'(t))dt \geq 0$$

$$\frac{-M(b-t)^2|_x^b}{2}+\left[f(t)(b-t)|_x^b+\int_x^b f(t)dt\right] \geq 0$$

$$\frac{-M(b-x)^2}{2}+\left[-f(x)(b-x)+\int_a^x f(t)dt\right] \geq 0$$

Multiplying with (-1) we get the inequality

$$f(x)(b-x)-\int_x^b f(t)dt \leq \frac{M(b-x)^2}{2}$$

$$f(x)-\frac{1}{(b-a)}\int_a^b f(t)dt \leq (b-a)\frac{M}{2}\left[(b-x)^2+(x-a)^2\right] \tag{9}$$

From inequalities in (8) and (9), we have

$$\begin{aligned}&\left|f(x)-\frac{1}{b-a}\int_a^b f(t)dt\right| \\ &= \frac{M}{2(b-a)}\left[(x-a)^2+(x-b)^2\right].\end{aligned} \tag{10}$$

Using the following identity

$$(x-a)^2+(x-b)^2 = [(x^2+a^2-2ax)+(x^2+b^2-2bx)]$$
$$= 2[x^2+(a-b/2)^2+(a+b/2)^2-x(a+b)]$$
$$= 2[(a-b/2)^2+(x-(a+b)/2)^2]$$

Taking common $(a-b)^2$

$$(x-a)^2+(x-b)^2 = 2(a-b)^2\left[\tfrac{1}{4}+\tfrac{(x-(a+b)/2)^2}{(a-b)^2}\right]$$

Using this value in (10)

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(t)dt\right|$$
$$= M(a-b)\left[\frac{1}{4}+\frac{(x-(a+b)/2)^2}{(a-b)^2}\right]$$

□

Theorem 2.2. *Suppose that $f:[a,b]\to\mathbb{R}$ be a twice differentiable function on (a,b) and suppose that $\gamma \le f^{''}(t)\le \Gamma$ for all $t\in(a,b)$[2]. Then we have the double inequality:*

$$\frac{\gamma(b-a)^2}{24} \le \frac{1}{b-a}\int_a^b f(t)dt - f\left(\frac{a+b}{2}\right) \le \frac{\Gamma(b-a)^2}{24}.$$

Proof. It is clear that

$$(t-a)^2(f''(t)-\gamma)\ge 0$$
$$\int_a^{(a+b)/2}(t-a)^2(f''(t)-\gamma)dt \ge 0 \tag{11}$$
$$\left[\int_a^{(a+b)/2}(t-a)^2 f''(t)dt\right]-\left[\int_a^{(a+b)/2}(t-a)^2\gamma dt\right]\ge 0$$

Integrating by parts, we have

$$\left[(t-a)^2 f'(t)|_a^{(a+b)/2} - 2\int_a^{(a+b)/2} f^{'}(t)(t-a)\right]-\left[\frac{\gamma(t-a)^3}{3}|_a^{(a+b)/2}\right]\ge 0$$

$$(\frac{b-a}{2})^2 f'(\frac{a+b}{2}) - 2\left[f(t)(t-a)|_a^{(a+b)/2} - \int_a^{(a+b)/2} f(t)dt\right] - [\frac{\gamma(\frac{b-a}{2})^3 3}{]} \ge 0$$

$$(\frac{b-a}{2})^2 f'(\frac{a+b}{2}) - f(\frac{a+b}{2})(b-a) + 2\int_a^{(a+b)/2} f(t)dt - [\frac{\gamma(b-a)^3}{8*3}]\ge 0$$

$$\left(\frac{b-a}{2}\right)^2 f^{'}\left(\frac{a+b}{2}\right)-(b-a)f\left(\frac{a+b}{2}\right)+2\int_a^{(a+b)/2} f(t)dt - \frac{\gamma(b-a)^3}{24}\ge 0 \tag{12}$$

Now using $(t-b)^2(f^{''}(t)-\gamma)\geq 0$

$$\int_a^{(a+b)/2}(t-b)^2(f''(t)-\gamma)dt\geq 0 \tag{13}$$

integrating on $[(a+b)/2,b]$, we have

$$\left[(t-b)^2 f^{'}(t)|_{(a+b)/2}^{b}-2\int_{(a+b)/2}^{b}f^{'}(t)(t-b)\right]-\left[\frac{\gamma((t-b)/2)^3}{3}|_{(a+b)/2}^{b}\right]\geq 0$$

$$-(\frac{b-a}{2})^2 f^{'}(\frac{a+b}{2})-f(\frac{a+b}{2})(b-a)+2\int_{(a+b)/2}^{b}f(t)dt-\gamma\frac{(b-a)^3}{24}\geq 0 \tag{14}$$

Adding (12) and (14) one can have

$$\left[-(\frac{b-a}{2})^2+(\frac{b-a}{2})^2\right]f^{'}(\frac{a+b}{2})$$
$$-f(\frac{a+b}{2})\left[(b-a)+(b-a)\right]+2\left[\int_{(a+b)/2}^{b}f(t)+\int_a^{(a+b)/2}f(t)\right]dt$$
$$-\gamma\left[\frac{(b-a)^3+(b-a)^3}{24}\right]\geq 0$$

$$-f(\frac{a+b}{2})\left[2(b-a)\right]+2\int_a^b f(t)dt$$
$$-\gamma\left[\frac{2(b-a)(b-a)^2}{24}\right]\geq 0$$

$\Rightarrow$

$$\frac{1}{b-a}\int_a^b f(t)dt-f\left(\frac{a+b}{2}\right)\geq\frac{(b-a^2)\gamma}{24} \tag{15}$$

Similarly using $(t-a)^2(\Gamma-f''(t))\geq 0$ and integrating on $[a,(a+b)/2]$, we have

$$-\left(\frac{b-a}{2}\right)^2 f'\left(\frac{a+b}{2}\right)+(b-a)f\left(\frac{a+b}{2}\right)-2\int_a^{(a+b)/2}f(t)dt+\frac{(b-a)^3\Gamma}{24}\geq 0 \tag{16}$$

Now using $(t-b)^2(\Gamma-f''(t))\geq 0$,integrating on $[(a+b)/2,b]$, we have

$$\left(\frac{b-a}{2}\right)^2 f'\left(\frac{a+b}{2}\right)+(b-a)f\left(\frac{a+b}{2}\right)-2\int_{(a+b)/2}^{b}f(t)dt+\frac{(b-a)^3\Gamma}{24}\geq 0 \tag{17}$$

Adding (15) and (17) we have

$$\frac{1}{b-a}\int_a^b f(t)dt-f\left(\frac{a+b}{2}\right)\leq\frac{(b-a)^2\Gamma}{24} \tag{18}$$

$$\frac{\gamma(b-a)^2}{24} \leq \frac{1}{b-a}\int_a^b f(t)dt - f\left(\frac{a+b}{2}\right) \leq \frac{\Gamma(b-a)^2}{24}. \quad (19)$$

□

Theorem 2.3. *Under the assumptions of above* 2.2 *we have*

$$\frac{\gamma(b-a)^2}{12} \leq \frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(t)dt \leq \frac{\Gamma(b-a)^2}{12}.$$

Theorem 2.4. *Suppose taht $f:[a,b]\to R$ be a twice differentiable function on (a,b) and suppose that $\gamma \leq f^{''}(t) \leq \Gamma$ for all $t \in (a,b)$. Then we have*

$$\frac{3S-2\Gamma}{24}(b-a)^2 \leq \frac{1}{b-a}\int_a^b f(t)dt - f\left(\frac{a+b}{2}\right) \leq \frac{3S-2\gamma}{24}(b-a)^2.$$

where $S = (f'(b)-f'(a))/(b-a)$.

Theorem 2.5. *Under the assumptions of above* 2.4*, we have*

$$\frac{3S-2\Gamma}{24}(b-a)^2 \leq \frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(t)dt \leq \frac{3S-\gamma}{24}(b-a)^2$$

where $S = (f'(b)-f'(a))/(b-a)$.

Proof. It is clear that

$$\left(t-\frac{a+b}{2}\right)^2 (f''(t)-\gamma) \geq 0$$

$$\int_a^b \left(t-\frac{a+b}{2}\right)^2 (f''(t)-\gamma) \geq 0$$

Integrating by parts, we have

$$\left[\int_a^b \left(t-\frac{a+b}{2}\right)^2 f''(t)\right] - \left[\gamma\int_a^b \left(t-\frac{a+b}{2}\right)^2\right] \geq 0$$

$$\left[\left(t-\frac{a+b}{2}\right)^2 f'(t)|_a^b\right]-2\int_a^b f^{'}(t)(t-(\frac{a+B}{2})dt$$

$$-\left[\gamma\frac{\left(t-\frac{a+b}{2}\right)^3}{3}|_a^b\right]\geq 0$$

$$\left[\left(\frac{(b-a)}{2}\right)^2 f'(b)-\left(\frac{(b-a)}{2}\right)^2 f'(a)\right]-2\left[f(b)(\frac{b-a}{2})+f(a)(\frac{b-a}{2})\right]$$

$$+2\int_a^b f(t)dt-\left[\gamma\frac{2\left((\frac{b-a}{2})\right)^3}{3}\right]\geq 0$$

$$\left(\frac{(b-a)}{2}\right)^2\left[f'(b)-f'(a)\right]-(b-a)\left[f(b)+f(a)\right]$$

$$+2\int_a^b f(t)dt-\gamma\frac{(b-a)^3}{12}\geq 0$$

$$\left(\frac{(b-a)^3}{4}\right)\left[\frac{f'(b)-f'(a)}{(b-a)}\right]-2(b-a)\frac{[f(b)+f(a)]}{2}$$

$$+2\int_a^b f(t)dt-\gamma\frac{(b-a)^3}{12}\geq 0$$

$$\left(\frac{1}{4}\right)S-\frac{2}{(b-a)^2}\frac{[f(b)+f(a)]}{2}$$

$$+\frac{2}{(b-a)^3}\int_a^b f(t)dt-\gamma\frac{1}{12}\geq 0$$

$$\left(\frac{3S-\gamma}{12}\right)-\frac{2}{(b-a)^2}\frac{[f(b)+f(a)]}{2}$$

$$+\frac{2}{(b-a)^3}\int_a^b f(t)dt\geq 0$$

dividing and multiplying with $\frac{(b-a)^2}{2}\Rightarrow$

$$\left(\frac{(3S-\gamma)(b-a)^2}{24}\right)-\frac{[f(b)+f(a)]}{2}+\frac{1}{(b-a)}\int_a^b f(t)dt\geq 0$$

multiplying with (-1), we get

$$\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(t)dt\leq\frac{(b-a)^2(3S-\gamma)}{24}\tag{20}$$

Similarly using $(t-(a+b)/2)^2(\Gamma - f''(t)) \geq 0$ and integrating on [a,b], we have

$$\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(t)dt \geq \frac{(b-a)^2(3S-\Gamma)}{24} \tag{21}$$

From (20) and (21) we have our desired result.

$$\frac{3S-2\Gamma}{24}(b-a)^2 \leq \frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(t)dt \leq \frac{3S-\gamma}{24}(b-a)^2.$$

□

Theorem 2.6. *Suppose that $f:[a,b]\to R$ be a twice differentiable function on (a,b) and suppose that $\gamma \leq f''(t) \leq \Gamma$ for all $t \in (a,b)$. Then we have*

$$\frac{(24+\alpha)\gamma - 24S}{24\alpha}(b-a)^2 \leq \frac{1}{b-a}\int_a^b f(t)dt - f\left(\frac{a+b}{2}\right)$$
$$\leq \frac{(24+\alpha)\Gamma - 24S}{24\alpha}(b-a)^2.$$

where $S = (f'(b)-f'(a))/(b-a)$.

Proof. It is clear that

$$\int_a^b (f''(t)-\gamma)dt \geq 0$$
$$\int_a^b f''(t)dt - \gamma\int_a^b dt \geq 0$$
$$f'(t)|_a^b - \gamma t|_a^b \geq 0$$
$$(f'(b)-f'(a)) - \gamma(b-a) \geq 0$$

$\Rightarrow$

$$\frac{f'(b)-f'(a)(b-a)}{b-a} - \gamma(b-a) \geq 0$$
$$S(b-a) - \gamma(b-a) \geq 0$$
$$(b-a)(S-\gamma) \geq 0 \tag{22}$$

Further, we can say for some $\alpha > 0 \Rightarrow \frac{b-a}{\alpha} \geq 0$.
Multiplying (22) with $\frac{b-a}{\alpha} \geq 0$

$$\frac{S-\gamma}{\alpha}(b-a)^2 \geq 0 \tag{23}$$

Adding (15) and (23) one has

$$\frac{1}{b-a}\int_a^b f(t)dt - f\left(\frac{a+b}{2}\right) \geq \frac{(24+\alpha)\gamma - 24S}{24\alpha}(b-a)^2 \tag{24}$$

On the other hand $\int_a^b (\Gamma - f''(t))dt \geq 0$,which gives for $\alpha > 0$

$$\frac{S-\Gamma}{\alpha}(b-a)^2 \leq 0 \tag{25}$$

Adding (17) and (25) one has

$$\frac{1}{b-a}\int_a^b f(t)dt - f\left(\frac{a+b}{2}\right) \leq \frac{(24+\alpha)\Gamma - 24S}{24\alpha}(b-a)^2 \tag{26}$$

From (24) and (26), we have the required inequality.

$$\frac{(24+\alpha)\gamma - 24S}{24\alpha}(b-a)^2 \leq \frac{1}{b-a}\int_a^b f(t)dt - f\left(\frac{a+b}{2}\right) \leq \frac{(24+\alpha)\Gamma - 24S}{24\alpha}(b-a)^2.$$

□

Corollary 2.7. *If one selects, for example,* $\alpha = 24$ *and* $\alpha = 48$ *in* 2.6*, then*

Case1
when $\alpha = 24$

$$\frac{(24+24)\gamma - 24S}{24*24}(b-a)^2 \leq \frac{1}{b-a}\int_a^b f(t)dt - f\left(\frac{a+b}{2}\right) \leq \frac{(24+24)\Gamma - 24S}{24*24}(b-a)^2$$

$$\frac{2\gamma - S}{24}(b-a)^2 \leq \frac{1}{b-a}\int_a^b f(t)dt - f\left(\frac{a+b}{2}\right) \leq \frac{2\Gamma - S}{24}(b-a)^2$$

Case2
when $\alpha = 48$

$$\frac{(48+24)\gamma - 24S}{24*48}(b-a)^2 \leq \frac{1}{b-a}\int_a^b f(t)dt - f\left(\frac{a+b}{2}\right)$$
$$\leq \frac{(24+48)\Gamma - 24S}{24*24}(b-a)^2$$

$$\frac{3\gamma - S}{48}(b-a)^2 \leq \frac{1}{b-a}\int_a^b f(t)dt - f\left(\frac{a+b}{2}\right)$$
$$\leq \frac{3\Gamma - S}{48}(b-a)^2$$

Theorem 2.8. *Let $f:[a,b] \to R$, where I is an interval in $\mathbb{R}$ be a mapping differentiable in I^0, the interior of I and $a,b \in I^0$, $a < b$. If $\gamma \leq f'(t) \leq \Gamma$ for all $t \in (a,b)$, then we have*

$$\left|\frac{1}{2}f(x) - \frac{(x-b)f(b)-(x-a)f(a)}{2(b-a)} - \frac{1}{b-a}\int_a^b f(t)dt\right|$$
$$\leq \frac{(x-a)^2+(b-x)^2}{4(b-a)}(\Gamma - \gamma) \quad (27)$$

for all $x \in [a,b]$.

Proof. It is clear that $(x-t)(f'(t)-\gamma) \geq 0$ and $(t-a)(\Gamma - f'(t)) \geq 0$ for all $t \in [a,x]$. Therefore
$$aa\int_a^x (x-t)(f'(t)-\gamma)dt + \int_a^x (t-a)(\Gamma - f'(t))dt \geq 0$$

After some computation, we have

$$(x-a)f(x)+(x-a)f(a) - 2\int_a^x f(t)dt \leq \frac{1}{2}(x-a)^2(\Gamma-\gamma) \quad (28)$$

Also it is easy to see $(t-x)(\Gamma - f'(t)) \geq 0$ and $(b-t)(f'(t)-\gamma) \geq 0$ for all $t \in [x,b]$. Therefore
$$\int_x^b (t-x)(\Gamma - f'(t))dt + \int_x^b (b-t)(f'(t)-\gamma)dt \geq 0$$

After some computation it can be seen

$$(b-x)f(x) - (x-b)f(b) - 2\int_x^b f(t)dt \leq \frac{1}{2}(b-x)^2(\Gamma-\gamma) \quad (29)$$

By adding (28) and (29) one has

$$\frac{1}{2}f(x)-\frac{(x-a)f(b)-(x-a)f(a)}{2(b-a)}-\frac{1}{b-a}\int_a^b f(t)dt \\ \leq \frac{(x-a)^2+(b-x)^2}{4(b-a)}(\Gamma-\gamma) \quad (30)$$

On the other hand we have

$$\int_a^x (t-a)(f'(t)-\gamma)dt+\int_a^x (x-t)(\Gamma-f'(t))dt \geq 0; \quad t\in[a,x]$$

This gives

$$(x-a)f(x)+(x-a)f(a)-2\int_a^x f(t)dt \geq -\frac{1}{2}(x-a)^2(\Gamma-\gamma) \quad (31)$$

Also we have

$$\int_x^b (b-t)(\Gamma-f'(t))dt+\int_x^b (t-x)(f'(t)-\gamma)dt \geq 0; \quad t\in[x,b],$$

which gives

$$(b-x)f(x)-(x-b)f(b)-2\int_x^b f(t)dt \geq -\frac{1}{2}(b-x)^2(\Gamma-\gamma). \quad (32)$$

Adding (31) and (32), then combining with (30) one can get requred. □

Chapter 3
Some New Ostrwoski-Type Bounds for the Cebysev Functional and Applications

3 Some New Ostrwoski-Type Bounds for the Cebysev Functional and Applications

In this chapter some bounds for cebysev functional are proved in detail[[1]].

Lemma 3.1. *If* $\phi : [a,b] \to \mathbb{R}$ *is an absolutely continuous function with* $(. - a)(b - .)(\phi^{'})^2 \in L[a,b]$ *we have the inequality*

$$T(\phi,\phi) \leq \frac{1}{2(b-a)} \int_a^b (x-a)(b-x)(\phi^{'})^2 dx$$

The constant $\frac{1}{2}$ is best possible.

Proof.
By **Korkine's Identity** ([1],p.242). we have the following expression

$$T(f,g) = \frac{1}{2(b-a)^2} \int_a^b \int_a^b (f(x)-f(y))(g(x)-g(y))dxdy \tag{33}$$

Using (33)

$$T(\phi,\phi) = \frac{1}{2(b-a)^2} \int_a^b \int_a^b (\phi(t)-\phi(s))^2 dtds$$

For $l(x) = x$,and g is differentiable functions

$$T(l,g) = \frac{1}{2(b-a)^2} \int_a^b \int_a^b (x-y)(g(x)-g(y))dxdy$$

Sonin's Identity ([**?**],p.246)

$$T(f,g) = \frac{1}{2(b-a)} \left(\int_a^b f(x) - \frac{1}{2(b-a)} \int_a^b f(y)dy \right) (g(x)-\gamma)dx \tag{34}$$

$$T(f,g) = \left[\frac{1}{(b-a)} \left(\int_a^b f(x) - \frac{1}{(b-a)} \int_a^b f(y)dy \right) (g(x)) \right] - \left[\frac{1}{(b-a)} \left(\int_a^b f(x) - \frac{1}{(b-a)} \int_a^b f(y)dy \right) \gamma dx \right]$$

where $l(x) = x$,and g is differentiable functions
$\frac{1}{2(b-a)} \left(\int_a^b x - \frac{1}{2(b-a)} \int_a^b ydy \right) \gamma dx = 0$
we have the following
$T(l,g) = \frac{1}{2(b-a)} \left(\int_a^b x - \frac{1}{2(b-a)} \int_a^b ydy \right) g(x)dx$

$= \frac{1}{2(b-a)} \left(\int_a^b x - \frac{1}{2(b-a)} \frac{y^2}{2} |_a^b \right) g(x)dx$
$= \frac{1}{(b-a)} \left(\int_a^b x - \frac{b+a}{2} \right) g(x)dx$
$= \left[\frac{1}{2(b-a)} \left(\frac{x^2}{2} - \frac{b+a}{2}x \right) g(x)|_a^b \right]$
$- \left[\frac{1}{2(b-a)} \int_a^b g(x)' \left(\frac{x^2}{2} - \frac{b+a}{2}x|_a^b \right) \right] dx$
$= \left[\frac{-ab(g(b)-g(a))}{2(b-a)} \right] + \left[\frac{1}{2(b-a)} \int_a^b g(x)'(b-x)(x-a)dx \right]$
$+ \left[\frac{ab}{2(b-a)} \int_a^b g(x)'dx \right]$
Applying integrating on left most side of above

$= \left[\frac{-ab(g(b)-g(a))}{2(b-a)} \right]$
$+ \left[\frac{1}{2(b-a)} \int_a^b g(x)'(b-x)(x-a)dx \right] + \left[\frac{ab}{2(b-a)} g(x)|_a bdx \right]$
$= - \left[\frac{ab(g(b)-g(a))}{2(b-a)} \right] + \left[\frac{ab(g(b)-g(a))}{2(b-a)} \right]$
$+ \left[\frac{1}{2(b-a)} \int_a^b g(x)'(b-x)(x-a)dx \right]$
$\Rightarrow$

$$T(l,g) = \frac{1}{2(b-a)} \int_a^b (b-x)(x-a)g(x)'dx \tag{35}$$

since ϕ is an absolutely continuous,$\phi(t) - \phi(s) = \int_s^t \phi'(u)du$
$T(\phi,\phi) = \frac{1}{2(b-a)^2} \int_a^b \int_a^b (t-s)^2 (\frac{(\phi(t)-\phi(s)}{t-s})^2 dtds$
$= \frac{1}{2(b-a)^2} \int_a^b \int_a^b (t-s)^2 (\frac{\int_s^t \phi'(u)du}{t-s})^2 dtds$
by cauchy schaurz inequality

$\leq \frac{1}{2(b-a)^2} \int_a^b \int_a^b (t-s)^2 \left(\int_s^t \phi'(u)du^2 \right) \left(\frac{1}{(t-s)^2} \int_s^t du \right) dtds$a
$\leq \frac{1}{2(b-a)^2} \int_a^b \int_a^b (t-s)^2 \left(\int_s^t \phi'(u)du^2 \right) \left(\frac{1}{(t-s)} \right) dtds$
$\leq \frac{1}{2(b-a)^2} \int_a^b \int_a^b (t-s) \left(\int_s^t \phi'(u)^2 du \right) dtds$
$T(\phi,\phi) \leq \frac{1}{2(b-a)^2} \int_a^b \int_a^b (t-s) \left(\int_s^a \phi'(u)^2 du + \int_a^t \phi'(u)^2 du \right) dtds$
Replacing $t = x, s = y$ in above expression we get

$T(\phi,\phi) \leq \frac{1}{2(b-a)^2} \int_a^b \int_a^b (x-y) \left(- \int_a^y \phi'(u)^2 du + \int_a^x \phi'(u)^2 du \right) dxdy$

As$\int_a^x \phi'(u)^2 du - \int_a^y \phi'(u)^2 du = g(x) - g(y)$
$T(\phi,\phi) \leq \frac{1}{2(b-a)^2} \int_a^b \int_a^b (x-y) (g(x) - g(y)) dxdy$

$T(\phi,\phi) \le T(l,g)$
comparing with (35)

$T(\phi,\phi) \cong \frac{1}{2(b-a)} \int_a^b (b-x)(x-a)g(x)^{'} dx$

$\int_a^b (\phi(u)^{'})^2 = g(x)^{'} \rightarrowtail (\phi(u)^{'})^2 = g(x)$

$$T(\phi,\phi) \le \frac{1}{2(b-a)} \int_a^b (b-x)(x-a)(\phi(u)^{'})^2 dx \tag{36}$$

□

Theorem 3.2. *Let $f,g : [a,b] \to R$ be two continuous functions on $[a,b]$ with $(.-a)(b-.)[f^{'}]^2, (.-a)(b-.)[g^{'}]^2 \in L[a,b]$. Then we have the following inequality*

$$\begin{aligned} T(f,g) &\le \frac{1}{\sqrt{2}}[T(f,f)]^{1/2} * \frac{1}{\sqrt{b-a}} \left(\int_a^b (x-a)(b-x)[g^{'}]^2 \right)^{1/2} \\ &\le \frac{1}{b-a} \left(\int_a^b (x-a)(b-x)[f^{'}]^2 \right)^{1/2} * \left(\int_a^b (x-a)(b-x)[g^{'}]^2 \right)^{1/2} \end{aligned} \tag{37}$$

Proof. Here, we let the following to be hold
$[T(f,g)]^2 \le [T(f,f)][T(g,g)]$
$\Rightarrow$

$$[T(f,g)] \le [T(f,f)]^{1/2}[T(g,g)]^{1/2} \tag{38}$$

Using (36)

$[T(f,f)] \le \frac{1}{2(b-a)} \int_a^b (x-a)(b-x)[f^{'}]^2 dx$
and

$[T(g,g)] \le \frac{1}{2(b-a)} \int_a^b (x-a)(b-x)[g^{'}]^2 dx$

Using above Values in (38)

$[T(f,g)] \le [T(f,f)]^{1/2} \left(\frac{1}{2(b-a)} \int_a^b (x-a)(b-x)[g^{'}]^2 dx \right)^{1/2}$

$[T(f,g)] \le \frac{1}{\sqrt{2(b-a)}} [T(f,f)]^{1/2} \left(\int_a^b (x-a)(b-x)[g^{'}]^2 dx \right)^{1/2}$
$\Rightarrow$

$$[T(f,g)] \leq \frac{1}{2(b-a)} \left(\int_a^b (x-a)(b-x)[f^{'}]^2 dx\right)^{1/2} \left(\int_a^b (x-a)(b-x)[g^{'}]^2 dx\right)^{1/2}$$

□

Theorem 3.3. *Let $g : [a,b] \to R$ is non-decreasing on $[a,b]$ and $g : [a,b] \to R$ is an absolutely continuous with $f^{'} \in L_\infty[a,b]$. Then we have the following inequality*

$$|T(f,g)| \leq \frac{1}{2(b-a)} ||f^{'}|| \int_a^b (x-a)(b-x) dg(x) \tag{39}$$

Proof. Using (33)

$$|T(f,g)| = \frac{1}{2(b-a)^2} \left| \int_a^b \int_a^b (f(x)-f(y))(g(x)-g(y)) dx dy \right|$$

As holds $|\int \hbar(x)| \leq \int |\hbar(x)|$

$$|T(f,g)| \leq \frac{1}{2(b-a)^2} \int_a^b \int_a^b \left| \frac{(f(x)-f(y))}{x-y} \right| |(x-y)(g(x)-g(y)) dx dy|$$

As given in [1]
$||f^{'}|| = sup_{t\in[a,b]} \left| f^{'} \right| = \left| \frac{f(x)-f(y)}{x-y} \right|$

$$|T(f,g)| \leq \frac{1}{2(b-a)^2} ||f^{'}|| \int_a^b \int_a^b |(x-y)(g(x)-g(y)) dx dy|$$

$$|T(f,g)| = ||f^{'}|| \left(\frac{1}{2(b-a)^2} \int_a^b \int_a^b (x-y)(g(x)-g(y)) dx dy \right)$$

Using (33) for the given substitutions $l(x) = x, x \in [a,b]$

$$|T(f,g)| = ||f^{'}||.T(l,g)$$

Using the following identity obtained by Ostrowski's in $([5], p, 366)$ for the monotonic increasing function $y : [a,b] \to R$

$$T(l,y) = \frac{1}{2(b-a)} \int_a^b (x-y) dy(x)$$
$$\Rightarrow$$
$$|T(f,g)| \leq \frac{1}{2(b-a)} ||f^{'}|| \int_a^b (x-y) dg(x)$$

□

Remark 3.4. *In addition to the Hypothesis of* 3.3*, We can assume that the function g is absolutely continuous on* $[a,b]$ *and* $g^{'} \in [a,b]$ *then following holds:*

$$\frac{1}{2(b-a)}\int_a^b (x-a)(b-x)dg(x)dx = \frac{1}{2(b-a)}\int_a^b (x-a)(b-x)g^{'}(x)dx$$

Proof. Using results from 3.2 of L.H.S of above may be written as

$\frac{1}{2(b-a)}\int_a^b (x-a)(b-x)dg(x)dx \leq \frac{1}{2(b-a)}||g^{'}||\int_a^b (x-a)(b-x)df(x)$
As given in[1] $\quad |f^{'}| = 1$
$\Rightarrow$

$\int_a^b (x-a)(b-x)dg(x)dx \leq ||g^{'}||\int_a^b (x-a)(b-x)(1)dx$

$\int_a^b (x-a)(b-x)dg(x)dx \leq ||g^{'}||\left[\frac{b(x)^2}{2} - \frac{x^3}{3} - abx + \frac{a(x)^2}{2}|_a^b\right]$

$\int_a^b (x-a)(b-x)dg(x)dx \leq ||g^{'}||\left[\frac{(b-a)(b^2+a^2-2ab)}{6}\right]$

$\int_a^b (x-a)(b-x)dg(x)dx \leq ||g^{'}||\left[\frac{(b-a)^3}{6}\right]$

$|T(f,g)| \leq \frac{1}{2(b-a)}||f^{'}||.||g^{'}||\frac{(b-a)^3}{6}$
Using (5)
$\Rightarrow$

$|T(f,g)| \leq ||f^{'}||.||g^{'}||\frac{(b-a)^2}{12}$

□

Theorem 3.5. *Assume that* $f,g : [a,b] \to \boldsymbol{R}$ *are continuous on* $[a,b]$ *and differentiable on* $[a,b]$ $g(t) \neq 0$ *for each* $t\varepsilon(a,b)$*.Then we have the Inequality*

$$|T(f,g)| \leq ||\frac{f^{'}}{g^{'}}||.T(g,g) \tag{40}$$

Proof. Applying the Cauchy's Mean Values Theorem for any $t,s \in [a,b]$, with $t \neq s$ there is an η between t and s such that

$$[f(t)-f(s)]g'(\eta) = [g(t)-g(s)]f'(\eta)$$

Thus

$$\left[\frac{f(t)-f(s)}{g(t)-g(s)}\right]=\left[\frac{f^{'}}{g^{'}}\right]$$

By Schawarz Inequality, we can write down above as

$$\left|\frac{f(t)-f(s)}{g(t)-g(s)}\right|\leq\left|\left|\frac{f^{'}}{g^{'}}\right|\right|$$

Using(33)

$$|T(f,g)|\leq\frac{1}{2(b-a)^2}\int_a^b\int_a^b|(f(t)-f(s))(g(t)-g(s))|dtds$$
$$\Rightarrow$$
$$|T(f,g)|\leq\frac{1}{2(b-a)^2}\int_a^b\int_a^b\left|\frac{(f(t)-f(s))}{(g(t)-g(s))}\right|(g(t)-g(s))^2dtds$$

$\Rightarrow$

$$|T(f,g)|\leq\frac{1}{2(b-a)^2}\left|\left|\frac{f^{'}}{g^{'}}\right|\right|.\left(\int_a^b\int_a^b(g(t)-g(s))^2dtds\right)$$

Using (33)

$$|T(f,g)|\leq\left|\left|\frac{f^{'}}{g^{'}}\right|\right|.T(g,g)$$

□

3.1 Applications for the Taylors Expansion

Let $I \subseteq R$ be closed interval, let $a \in I$ and let n be a positive integer.If f:$I \Rightarrow R$ is such that f^n is absolutely continuous, then for each $x \varepsilon I$

$$f(x) = T_n(f;a,x) + R_n(f;a,x) \tag{41}$$

Where $T_n(f;a,x)$ is a Taylors Polynomial

$$T_n(f;a,x) = \sum_{k=0}^{n} \frac{(x-a)^k f^k(a)}{k!} \tag{42}$$

(note that $f^{(0)} = f$ and $0! = 1$), and the reminder is given by

$$R_n(f;a,x) = \frac{1}{n!} \int_a^x (x-t)^n f^{(n+1)}(t)dt \tag{43}$$

We note that, many authors have considered recently different perturbation for Taylors formula and pointed out bounds the reminders that are, some time,better than the ones provided in the classical case.The reader may consult for example $[8],[11]$ and the references there in. This motivates our interest to apply the *Grŭss* type inequalities obtained before in pointing out different bounds for the reminder in the perturbed Taylors formula below.
Using Theorem1 , we may point out the following perturbation of the Taylors expansion.

Theorem 3.6. *Let f:I $\Rightarrow$ R be such that $f^{(n+1)}$ is absolutely continuous and a $\in$I.Then we have the perturbed Taylors formula;*

$$f(x) = T_n(f;a,x) + \frac{(x-a)^{n+1}}{(n+1)!}[f;a,x] + G_n(f;a,x) \tag{44}$$

and the reminder $G_n(f;a,x)$ satisfies the estimation

$$G_n(f;a,x) \le \frac{n}{\sqrt{2}(n+1/2)!}\left|\int_a^x (t-a)(x-t)[f^{(n+2)}(t)]^2\right|^{1/2} \tag{45}$$

for any $x \in I$. where

$$[f^{(n);a,x}] = \frac{f^n(x) - f^n(a)}{x-a}$$

is the divided difference.

Proof. If we apply Using (37) for $f \to (x-a)^n, g \to f^{(n+1)}$ we deduce

$$\left|\frac{1}{x-a}\int_a^x (x-t)^n f^{(n+1)} dt - \frac{1}{x-a}\int_a^x (x-t)^n dt * \frac{1}{x-a}\int_a^x f^{(n+1)} dt\right|$$
$$\le \frac{1}{\sqrt{2}} T_n[(x-a)^n,(x-a)^n]^{1/2} . \frac{1}{\sqrt{(x-a)}}\left|\int_a^x (t-a)(x-t)[f^{(n+1)}(t)]^2\right|^{1/2}$$

since

$T_n[(x-a)^n,(x-a)^n] = \frac{1}{x-a}\int_a^x (x-t)^{2n} dt - [\frac{1}{x-a}\int_a^x (x-t)^n]^2$
integrating on left side we get

$= \left[\frac{1}{x-a}\frac{(x-a)^{2n+1}}{2n+1}\right] - \left[\frac{1}{x-a}\frac{(x-a)^{n+1}}{n+1}\right]^2$

$= \frac{(x-a)^{2n+1}}{x-a}\left[\frac{1}{2n+1} - \frac{1}{(n+1)^2}\right]$

$= \frac{(n)^2(x-a)^{2n}}{(n+1)^2}$

$T_n[(x-a)^n,(x-a)^n] = \frac{(n)^2(x-a)^{2n}}{(n+1)^2}$

Using (37) we get

$$\left|\frac{1}{x-a}\int_a^x (x-t)^n f^{(n+1)} dt - \frac{1}{x-a}\int_a^x (x-t)^n dt * \frac{1}{x-a}\int_a^x f^{(n+1)} dt\right|$$
$$\le \frac{1}{\sqrt{2}}\left(\frac{(n)(x-a)^n}{(n+1)!}\right) . \frac{1}{\sqrt{(x-a)}}\left|\int_a^x (t-a)(x-t)[f^{(n+1)}(t)]^2\right|^{1/2}$$

Multiplying above with $\frac{(x-a)}{n!}$

$$\left|\frac{1}{n!}\int_a^x (x-t)^n f^{(n+1)}dt - \frac{1}{n!(x-a)}\int_a^x (x-t)^n dt * \int_a^x f^{(n+1)(t)}dt\right|$$
$$\leq \frac{1}{\sqrt{2}}\left(\frac{(n)(x-a)^{n+1/2}}{(n+1)!}\right)\cdot\left|\int_a^x (t-a)(x-t)[f^{(n+1)}(t)]^2\right|^{1/2}$$

$$\left|\frac{1}{n!}\int_a^x (x-t)^n f^{(n+1)}dt - \frac{1}{n!(x-a)}\left[(x-t)^n|_a^x * f^{(n)}(t)|_a^x\right]\right|$$
$$\leq \frac{1}{\sqrt{2}}\left(\frac{(n)(x-a)^{n+1/2}}{(n+1)!}\right)\cdot\left|\int_a^x (t-a)(x-t)[f^{(n+1)}(t)]^2\right|^{1/2}$$

$$\left|\frac{1}{n!}\int_a^x (x-t)^n f^{(n+1)}dt - \frac{(x-a)^{n+1}}{(n+1)!}\left[\frac{(f^n(x)-f^n(a))}{x-a}\right]\right|$$
$$\leq \frac{1}{\sqrt{2}}\left(\frac{(n)(x-a)^{n+1/2}}{(n+1)!}\right)\cdot\left|\int_a^x (t-a)(x-t)[f^{(n+1)}(t)]^2\right|^{1/2}$$

$$\left|\frac{1}{n!}\int_a^x (x-t)^n f^{(n+1)}dt\right|$$
$$\leq \frac{1}{\sqrt{2}}\left(\frac{(n)(x-a)^{n+1/2}}{(n+1)!}\right)\cdot\left|\int_a^x (t-a)(x-t)[f^{(n+1)}(t)]^2\right|^{1/2}$$
$$+\frac{(x-a)^{n+1}}{(n+1)!}\left[\frac{(f^n(x)-f^n(a))}{x-a}\right]$$

Where $G_n(f;a,x) \leq \frac{1}{\sqrt{2}}\left(\frac{(n)(x-a)^{n+1/2}}{(n+1)!}\right)\cdot\left|\int_a^x (t-a)(x-t)[f^{(n+1)}(t)]^2\right|^{1/2}$
$\Rightarrow$ Using (43) and (45) we get the expression

$$R_n(f;a,x) \leq G_n(f;a,x) + \frac{(x-a)^{n+1}}{(n+1)!}\left[\frac{(f^n(x)-f^n(a))}{x-a}\right]$$

Adding both sides of above $T_n(f;a,x) = \sum_{k=0}^n \frac{(x-a)^k f^k(a)}{k!}$

$$R_n(f;a,x) + T_n(f;a,x) \leq G_n(f;a,x) + \frac{(x-a)^{n+1}}{(n+1)!}\left[\frac{(f^n(x)-f^n(a))}{x-a}\right] + T_n(f;a,x)$$

Using (41) we get

$$f(x) \leq G_n(f;a,x) + \frac{(x-a)^{n+1}}{(n+1)!}\left[\frac{(f^n(x)-f^n(a))}{x-a}\right] + T_n(f;a,x)$$

$\square$

Theorem 3.7. *Let f:I $\to R$ be such that $f^{(n+1)}$ is absolutely continuous and $f^{(n+1)\geq 0}$ then we have the representation*

(44) with the reminder $G_n(f;a,x)$ that satisfies the bound

$$G_n(f;a,x) \leq \frac{n}{\sqrt{2}(n+1)!}\left|\int_a^x (t-a)(x-t)[f^{(n+2)}(t)]^2\right|^{1/2}. \tag{46}$$

for any x$\in I$.

Proof. :We apply (39) for f$\to$ $(x-a)^n$ and g$\to$ $f^{(n+1)}$, to get

$$\left|\frac{1}{x-a}\int_a^x (x-t)^n f^{(n+1)} dt - \frac{1}{x-a}\int_a^x (x-t)^n dt * \frac{1}{x-a}\int_a^x f^{(n+1)} dt\right|$$
$$\leq \frac{1}{2(b-a)}\left|\left|f^{'}\right|\right|_\infty \int_a^x (t-a)(x-t)[\frac{d}{dt}f^{(n+1)}(t)]dt$$

$$\left|\frac{1}{x-a}\int_a^x (x-t)^n f^{(n+1)} dt - \frac{1}{x-a}\int_a^x (x-t)^n dt * \frac{1}{x-a}\int_a^x f^{(n+1)} dt\right|$$
$$\leq \frac{1}{2(b-a)}[\frac{d}{dx}(x-a)^n]\int_a^x (t-a)(x-t)f^{(n+2)}(t)dt$$

$$\left|\frac{1}{x-a}\int_a^x (x-t)^n f^{(n+1)} dt - \frac{1}{x-a}\int_a^x (x-t)^n dt * \frac{1}{x-a}\int_a^x f^{(n+1)} dt\right|$$
$$\leq \frac{1}{2(x-a)}[n(x-a)^{n-1}]\int_a^x (t-a)(x-t)f^{(n+2)}(t)dt := K \tag{47}$$

Since

$$\int_a^x (t-a)(x-t)f^{(n+2)}(t)dt = (xt-t^2-ax+at)\int_a^x f^{(n+2)}(t)dt$$
$$-\int_a^x\left[\int_a^x f^{(n+2)}(t)dt.\frac{d}{dt}(xt-t^2-ax+at)\right]dt$$

$$\int_a^x (t-a)(x-t)f^{(n+2)}(t)dt = [(x^2-x^2-ax+ax)f^{(n+1)}(x)-(ax-a^2-ax+a^2)f^{(n+1)}(a)]$$
$$+\int_a^x\left[\int_a^x f^{(n+1)}(t).(2t-(x+a))\right]dt$$

$$\int_a^x (t-a)(x-t)f^{(n+2)}(t)dt = \int_a^x \left[\int_a^x f^{(n+1)}(t).(2t-(x+a))\right] dt$$

$$\int_a^x (t-a)(x-t)f^{(n+2)}(t)dt = \left[f^{(n)}(x)(x-a)+f^{(n)}(a)(x-a)\right] - 2\int_a^x f^n(t)dt$$

$$\int_a^x (t-a)(x-t)f^{(n+2)}(t)dt = \left[f^{(n)}(x)(x-a)+f^{(n)}(a)(x-a)\right] - 2[f^{n-1}(x)-f^{n-1}(a)]$$

$$\int_a^x (t-a)(x-t)f^{(n+2)}(t)dt = (x-a)\left[f^{(n)}(x)+f^{(n)}(a)\right] - 2[f^{n-1}(x)-f^{n-1}(a)]$$

$$\int_a^x (t-a)(x-t)f^{(n+2)}(t)dt = 2(x-a)\left[\frac{f^{(n)}(x)+f^{(n)}(a)}{2} - \frac{f^{n-1}(x)-f^{n-1}(a)}{x-a}\right]$$

Using this in (47)

$$\left|\frac{1}{x-a}\int_a^x (x-t)^n f^{(n+1)}dt - \frac{1}{x-a}\int_a^x (x-t)^n dt * \frac{1}{x-a}\int_a^x f^{(n+1)}dt\right|$$
$$\leq [n(x-a)^{n-1}]\left[\frac{f^{(n)}(x)+f^{(n)}(a)}{2} - [f^{(n-1)};a,x]\right]$$

Using (46)For
$G_n(f;a,x) \leq [n(x-a)^{n-1}]\left[\frac{f^{(n)}(x)+f^{(n)}(a)}{2} - [f^{(n-1)};a,x]\right]$

$$f(x) \quad \leq \quad G_n(f;a,x) \quad + \quad \frac{(x-a)^{n+1}}{(n+1)!}\left[\frac{(f^n(xa)-f^n(a))}{x-a}\right] \quad + \quad T_n(f;a,x)$$

□

Theorem 3.8. *Let $f :\to R$ be such that $f^{(n+1)}$ is absolutely continuous .If a,xεI then there exist a constant $M(x)$ such that*

$$\left|f^{(n+2)}(t)\right| \leq M(x)|x-t|^{n-1} \ for \ t\varepsilon[a,x] \tag{48}$$

Then we have the representation (44) and the reminder $G_n(f;a,x)$ that satisfies the estimate

$$\left|G_n(f;a,x)\right| \leq \frac{n}{n+1}\frac{1}{(n+1)!}M(x)|x-a|^{2n+1} \tag{49}$$

Proof. :If We apply (40) for $g \to (x-a)^n$ and $f \to f^{(n+1)}$,to get

$$\left|\frac{1}{x-a}\int_a^x (x-t)^n f^{(n+1)}dt - \frac{1}{x-a}\int_a^x (x-t)^n dt * \frac{1}{x-a}\int_a^x f^{(n+1)}dt\right|$$
$$\leq \left\|\frac{f'}{g'}\right\|_\infty .T(g,g)$$

$$\leq \left| \frac{\frac{d}{dx}(f^{n+1})}{\frac{d}{dx}(x-a)^n} \right| \frac{1}{x-a} \int_a^x (x-t)^{2n} dt - \left[\frac{1}{x-a} \int_a^x (x-t)^n \right]^2$$

$$\leq \left| \frac{(f^{n+2})}{n(x-a)^{n-1}} \right| \left[\frac{1}{x-a} \frac{(x-a)^{2n+1}}{2n+1} \right] - \left[\frac{1}{x-a} \frac{(x-a)^{n+1}}{n+1} \right]^2$$

Using (48) we get the expression

$$\leq \left| \frac{M(x)}{n} \right| \left[\left[\frac{(x-a)^{2n}}{2n+1} \right] - \left[\frac{(x-a)^n}{n+1} \right]^2 \right]$$

$$\leq \left| \frac{M(x)}{n} \right| \left[\frac{n^2 (x-a)^{2n}}{(2n+1)(n+1)^2} \right]$$

$$\left| \frac{1}{x-a} \int_a^x (x-t)^n f^{(n+1)} dt - \frac{1}{x-a} \int_a^x (x-t)^n dt * \frac{1}{x-a} \int_a^x f^{(n+1)} dt \right| \leq \left| \frac{M(x)}{n} \right| \left[\frac{n^2 (x-a)^{2n}}{(2n+1)(n+1)^2} \right]$$

$$\left| \frac{1}{x-a} \int_a^x (x-t)^n f^{(n+1)} dt - \frac{1}{x-a} \int_a^x (x-t)^n dt * \frac{1}{x-a} \int_a^x f^{(n+1)} dt \right| \leq \left| \frac{M(x)}{n} \right| \left[\frac{n^2 (x-a)^{2n}}{(2n+1)(n+1)^2} \right]$$

Multiplying both sides by $\frac{(x-a)}{n!}$

$$\left| \frac{1}{n!} \int_a^x (x-t)^n f^{(n+1)} dt - \frac{1}{n!} \int_a^x (x-t)^n dt * \frac{1}{x-a} \int_a^x f^{(n+1)} dt \right| \leq \left| \frac{M(x)}{n!} \right| \left[\frac{n (x-a)^{2n+1}}{(2n+1)(n+1)^2} \right]$$

$$R_n(f;a,x) \leq \left| \frac{M(x)}{n!} \right| \left[\frac{n (x-a)^{2n+1}}{(2n+1)(n+1)^2} \right] + \frac{(x-a)^{n+1}(f^n;a,x)}{(n+1)!}$$

Adding $T_n(f;a,x) = \sum_{k=0}^{n} \frac{(x-a)^k f^k(a)}{k!}$ on both side we get

$$R_n(f;a,x) + T_n(f;a,x) \leq T_n(f;a,x) + \left| \frac{M(x)}{n!} \right| \left[\frac{n (x-a)^{2n+1}}{(2n+1)(n+1)^2} \right] + \frac{(x-a)^{n+1}(f^n;a,x)}{(n+1)!}$$

$$f(x) \leq T_n(f;a,x) + \left| \frac{M(x)}{n!} \right| \left[\frac{n (x-a)^{2n+1}}{(2n+1)(n+1)^2} \right] + \frac{(x-a)^{n+1}(f^n;a,x)}{(n+1)!}$$

□

3.2 Applications for the Generalized Trapezoid Formula for n-time Differentiable Functions:

let $f:[a,b]\to R$ be a function such that the derivative $f^{(n-1)}$ $(n\geq 1)$ is absolutely continuous on [a,b]. In[[3]], the authors have obtained the following generalisation of the trapezoid formula:

$$\int_a^b f(t)dt=\sum_{k=0}^{n-1}\frac{1}{(k+1)!}\left[(x-a)^{k+1}f^{(k)}(a)+(-1)^k(b-x)^{k+1}f^{(k)}(b)\right] +\frac{1}{n!}\int_a^b (x-t)^n f^{(n)}(t)dt \quad (50)$$

The following perturbed version of (4.1) holds.

Theorem 3.9. *Let $f:[a,b]\to R$ be such that f^n is absolutely continuous on [a,b]. Then we have representation*

$$\int_a^b f(t)dt=\int_a^b f(t)dt=\sum_{k=0}^{n-1}\frac{1}{(k+1)!}\left[(x-a)^{k+1}f^{(k)}(a)+(-1)^k(b-x)^{k+1}f^{(k)}(b)\right] +\frac{(x-a)^{n+1}+(-1)^n(b-x)^{n+1}}{(n+1)!}\left[f^{(n-1)};a,b\right]+S_n(f,x) \quad (51)$$

where the remainder $S_n(f,x)$ satisfies the estimate

$$|S_{n(f,x)}|\leq\frac{1}{\sqrt{2}}\cdot\sqrt{b-a}[B_n(x)]^{\frac{1}{2}}\left(\int_a^b (t-a)(b-t)[f^{(n+1)}(t)]^2dt\right)^{\frac{1}{2}} \quad (52)$$

and $B_n(x)$ is defined by

$$B_n(x)=\frac{(b-x)^{2n+1}+(x-a)^{2n+1}}{(b-a)(2n+1)}-\left[\frac{(b-x)^{n+1}+(-1)^n(x-a)^{n+1}}{(n+1)(b-a)}\right]^2 \quad (53)$$

for any $x\in[a,b]$.

Proof.

If we apply (37) for $f\to(x-\cdot)^n$ and $g\to f^{(n)}$, we obtain

$$\left|\frac{1}{b-a}\int_a^b (x-t)^n f^{(n)}(t)dt-\frac{1}{b-a}\int_a^b (x-t)^n t\cdot\frac{1}{b-a}\int_a^b f^n(t)dt\right| \leq\frac{1}{\sqrt{2}}[T((x-\cdot)^n),((x-\cdot)^n)]^{\frac{1}{2}}\frac{1}{\sqrt{b-a}}\left(\int_a^b (t-a)(b-t)[f^{n+1}(t)]^2dt\right)^{\frac{1}{2}}.$$

Since

$$T((x-\cdot)^n,(x-\cdot)^n)=\frac{1}{b-a}\int_a^b (x-t)^{2n}dt-\left(\frac{1}{b-a}\int_a^b (x-t)^n dt\right)^2=B_n(x)$$

then, by (50) we deduce the representation (51) and the bound (52).
We omit the details.
It is natural to consider the following particular case.

Corollary 3.10. *With the assumptions in* 3.9*, we have*

$$\int_a^b f(t)dt=\sum_{k=0}^{n-1}\frac{1}{(k+1)!}\left(\frac{b-a}{2}\right)^{k+1}[f^{(k)}(a)+(-1)^{(k)}(b)] \\ +\left(\frac{b-a}{2}\right)^n\frac{[1+(-1)^n]}{(n+1)!}[f^{(n-1)};a,b]+S_n(f) \quad (54)$$

and the remainder $S_n(f)$ satisfies the bound:

$$|S_n(f)|\le\frac{1}{n!}\cdot\frac{(b-a)^{n+\frac{1}{2}}}{2^{n+\frac{1}{2}}}\left[\frac{1}{2n+1}-\frac{[1+(-1)^n]^2}{(n+1)^2}\right]^{\frac{1}{2}} \\ \times\left(\int_a^b (t-a)(b-a)[f^{(n+1)}(t)]^2dt\right)^{\frac{1}{2}}. \quad (55)$$

The following result also holds.

Theorem 3.11. *Assume that $f^{(n)}$ $(n\ge 2)$ is absolutely continuous an $f^{(n+1)}\ge 0$ on [a,b]. Then we have the representation* 3.9 *and the remainder $S_n(f,x)$ satisfies the estimate*

$$|S_n(f,x)|\le\frac{1}{(n-1)!}\left[\frac{1}{2}(b-a)+\left|x-\frac{a+b}{2}\right|\right]^{n-1} \\ \times(b-a)\left[\frac{f^{(n-1)}(a)+f^{(n-1)}(b)}{2}-[f^{(n-2)};b,a]\right] \quad (56)$$

for any $x\in[a,b]$.
Proof:
If we use in (39) for $f\to(x-\cdot)^n$ and $g\to f^{(n)}$, we get

$$\left|\frac{1}{b-a}\int_a^b (x-t)^n f^(n)(t)dt-\frac{1}{b-a}\int_a^b (x-t)^n dt\cdot\frac{1}{b-a}\int_a^b f^{(n)}(t)dt\right| \\ \le\frac{1}{2(b-a)}n[max(x-a,b-x)]^{n-1}\int_a^b (t-a)(b-t)f^{(n+1)}(t)dt \quad (57)$$

$$= n[\frac{1}{2}(b-a) + |x - \frac{a+b}{2}|]^{n-1} \left[\frac{f^{(n-1)}(a) + f^{(n-1)}(b)}{2} - [f^{(n-2)}; b, a] \right]$$

Using (50) and (57) we deduce (56).

Corollary 3.12. *With the assumptions in* 3.11*, we have the representation* 3.10*. The remainder $S_n(f)$ satisfies the bound*

$$|S_n(f)| \leq \frac{1}{2^{n-1}(n-1)!}(b-a)^n \left[\frac{f^{(n-1)}(a) + f^{(n-1)}(b)}{2} - [f^{(n-2)}; b, a] \right]$$

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