On Characterization of Spanning Forest Complexes

By

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Declaration

I Muhammad Asif registration number **CIIT/FA14-MSMATH-004/LHR** hereby declare that I have produced the work presented in this thesis, during the scheduled period of study. I also declare that I have not taken any material from any source except referred to wherever due, that amount of plagiarism is within acceptable range. If a violation of HEC rules on research has occurred in this thesis, I shall be liable to punishable action under the plagiarism rules of the HEC.

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It is certified that Muhammad Asif **(CIIT/FA14-MSMATH-004/LHR)** has carried out all the work related to this thesis under my supervision at the Department of Mathematics, COMSATS Institute of Information Technology, Lahore, and the work fulfills the requirement for award of MS degree.

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DEDICATION

To My Late Mother

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ABSTRACT

In this thesis, we discuss the algebraic and combinatorial aspects of spanning forest complex $\Delta_s(G)$ associated to simple disconnected graph G. We mainly emphasis on disjoint union of isomorphic copies graph G and provide special importance to tC_n where tC_n is the disjoint union of t cycles of length n and call this graph t-cycle graph. We give the characterization of $s(tC_n)$, all spanning forests, of tC_n. We intend to provide the algebraic and combinatorial characterization of spanning forest complex $\Delta_s(tC_n)$ associated to tC_n. In particular, we compute the formula for the f-vector of spanning forest complex of t-cycle graph. We also compute the h-vector and Hilbert series of Stanley Riesner ring $k[\Delta_s(tC_n)]$.

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Chapter 1
Introduction

In 1970's, first time R.P Stanley introduced the algebraic study of simplicial complexes. The work of Stanley made the base of the Combinatorial Commutative Algebra. After it Villarreal and his partners initialize the new concept of edge ideal associated to simple graph in commutative algebra. Afterwards, alot of work done by the many people to raise the relation between the combinatorial structure of graph and the algebraic aspects of edge ideal associated to a simple graphs, for this see [10],[11],[29] and [30]. The study of simplicial complexes is one of the central topics in combinatorial commutative algebra because of its connections with algebraic geometry and topology.

Recently in [3], authors discussed the combinatorial structure of spanning trees associated to a simple graph. Motivated from their work Anwar et al. [1] introduced the idea of spanning simplicial complexes associated to spanning trees and discussed its algebraic properties. In [22], they discussed the spanning simplicial complexes of r-cyclic graphs.

Let $G(V, E)$ is a finite simple connected graph, V is the set of vertices and E is the set of edges of G , a spanning tree is a subgraph of simple finite graph G that contains every vertex of G and is also a tree. Edge set of spanning trees represents as

$s(G) = \{E_1, E_2, \ldots, E_r\}$, and facets of spanning simplicial complex $\Delta_s(G)$ are precisely the element of $s(G)$, i.e. $\Delta_s(G) =< E_1, E_2, \ldots, E_r >$. In [1], the algebraic characterization of $\Delta_s(G)$ associated to uni-cyclic graph is discussed.

Later on, Jin Guo in [14] expanded the idea of spanning tree complex and introduced the term spanning forest complex, a complex associated to a disconnected graph. Let $G(V, E)$ is a finite simple disconnected graph having the set of vertices V and set of edges E, a spanning forest of G is a subgraph H of G such that for each connected component K of G, the intersection of H with K is a spanning tree of K i.e. $K \cap H = T_s$, where T_s is a spanning tree of K. The spanning forest complex associated to disconnected graph is generated by all spanning forest of G i.e. $\Delta_s(G) =< E(F_i)|F_i \; be \; the \; spanning \; forest \; of \; G >$.

The algebraic characteristics of spanning forest complex are still unattended. Our aim is to explore some algebraic and combinatorial characteristics of spanning forest complexes associated to the disjoint union of uni-cyclic graphs. This study will extend the concept introduced in [1] and our initial investigations have revealed some very interesting developments in this direction.

In chapter 2, we give the all basic terminologies and notions which are required to understand the recent work. The basic definition and notions are very important to give the advancement. These definitions give a foundation to understand the subject.

In chapter 3, we describe briefly graphs, results and main theorems related to graphs. In this section we give all the basic information about the simplicial complexes associated to simple graphs and algebraic characterization of the simplicial complexes, some theorems and basic definitions related to our work.

In chapter 4, first we will discusses the newly introduced algebraic object spanning simplicial complex associated to a simple connected graph. Then we apply this idea to disconnected graph and investigate the spanning simplicial complexes $\Delta_s(G)$ associated to disconnected graph G. we give the all information on spanning forest associated to simple disconnected graph in order to investigate the algebraic characterization arising from it. Firstly, we discussed about the spanning forests for simple graphs. On the basis of our work, we give some reasonable and important results for spanning forest complexes. We mainly give f-vector, h-vector for the spanning forest complexes.

Chapter 2

Commutative Algebra and its Preliminaries

Commutative algebra is the growing field in mathematical research. Commutative algebra is basically the study of commutative rings. The study of commutative rings having many areas like number theory and algebraic geometry. Commutative algebra deals with the study of monomial ideals, monomial rings and Stanley-Reisner ideals. It is also use to study of combinatorial aspects associated to the graphs. Graphs are the mathematical structures to construct the relationship between objects.
This chapter is designed to give all the basic information and basic notations which are required for the recent work in the field of commutative algebra. The fundamental definitions about the topic are provided in this chapter for the new development. To know clearly and for standard notation, for instant [17], [25], [29], [30],[31].

2.1 Commutative Ring

Definition 2.1.

A set R is a ring with two binary operations (multiplication and addition) such that,

1. R is an abelian group under addition. i.e. R is associative, commutative, has additive identity and additive inverse of each element

2. R associative under multiplication.

3. Multiplication is distributive over addition. i.e. for any three elements $x, y, z \in R$

   (a) $x.(y + z) = xy + xz$

   (b) $(x + y).z = xz + yz$

Definition 2.2.

If R has the multiplicative identity then R is called a ring with identity. i.e. $\exists\ 1 \in R$ s.t.

$$x.1 = 1.x = x \qquad \forall\ \ x \in R$$

Definition 2.3.

A ring R is called commutative ring iff it satisfied the commutative law under multiplication, i.e. $xy = yx$ for all $x, y \in R$.

Example 2.1.1.

- $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{Z}_n$ *and* $\mathbb{C}$ *are the commutative rings with identity.*
- $n\mathbb{Z}$ *is a commutative ring with out identity for* $n > 1$.

Definition 2.4.

A nonzero element a of a ring R is called the zero divisor iff

$$ab = ba = 0$$

for some $b \in R\backslash\{0\}$.

Remark 2.1.1.

R has no zero divisor if cancelation laws hold in R.

Example 2.1.2. 2 *is a zero divisor in* $\mathbb{Z}_{10}$. *Because* $\exists\ 5 \in \mathbb{Z}_{10}$ *such that* $(2).(5) = 0$

Definition 2.5. A zero divisor free ring R is called integral domain.

Example 2.1.3.

- *$\mathbb{Z}$, $\mathbb{R}$ and $\mathbb{C}$ are the integral domains.*

- *$\mathbb{Z}_n$ is an integral domain iff n is a prime.*

Definition 2.6.

Let R be a ring. Two elements $a, b \in R - \{0\}$ called multiplicative inverse of each other iff $a.b = b.a = 1$. a, b called invertible elements or units in R.

Example 2.1.4.

- *In $\mathbb{Z}_7$, 2 and 4 are the inverses of each other and called the invertible elements.*

- *Every non zero element in $\mathbb{R}$ is invertible.*

Definition 2.7.

In a ring R with unity if every non zero element is invertible then R is called a field.

Example 2.1.5.

- *$\mathbb{R}$, $\mathbb{Q}$ and $\mathbb{C}$ are fields.*

- *$\mathbb{Z}$ is not a field.*

2.1.1 Subrings and Ideals

Definition 2.8.

Let R be a ring. A nonempty $S \subseteq R$ is called a subring if S is itself a ring. A non empty $S \subseteq R$ is a subring iff, $\forall \;\; x, y \in S$

1. $x - y \in S$

2. $xy \in S$

Example 2.1.6.

$\mathbb{Z}$ is a subring of $\mathbb{R}$.

Definition 2.9.

A subring I of R is called right ideal in R if $s \in I$ and $a \in R \Rightarrow sa \in I$. similarly, if $as \in I$ then I is called left ideal.

Every ring R has at least two ideals, one is null ideal i.e. $\{0\}$ and second is ring R itself. Both are called improper ideal of R. Other Ideals of R called proper ideal. Any proper ideal is a ring without identity.

Example 2.1.7. $n\mathbb{Z}$ are ideals in the ring of $\mathbb{Z}$ where $n \geq 0$.

Proposition 2.1.

Let R be a ring and $\mathcal{I}$ is the collection of all ideals of R then $\mathcal{I}$ is closed under set intersection.

Definition 2.10.

An ideal m of R is called maximal if m is not contained in any ideal of ring other than R and m itself.

Theorem 2.2. Let R be a commutative ring with unity. Let m be a proper ideal of R, then m is maximal iff R/m is a field.

Example 2.1.8.

(i) In any field, zero ideal is maximal ideal.

(ii) The ideal $n\mathbb{Z}$ is maximal ideal of $\mathbb{Z}$ iff n is prime.

(iii) The ideal (P, x) is maximal in a polynomial ring $\mathbb{Z}[X]$ for each prime P.

Theorem 2.3.

A non-zero commutative ring with unity is a field if it has no proper ideal.

Proof

Let R be a non-zero commutative ring with unity $a(\neq 0) \in R$. then the set Ra is an ideal of R,for $x, y \in Ra \Rightarrow x = r_1a$, $y = r_2a$ for some $r_1, r_2 \in R$ $x - y = r_1a - r_2a = (r_1 - r_2)a \in Ra$ $\because r_1 - r_2 \in R$ and if $r \in R$ then $rx = r\ (r_1a)$ $=(rr_1)a \in Ra \Rightarrow Ra$ is an ideal. Now, if R having no proper ideal then,

$$Ra = R$$

For $1 \in R = Ra$ $\exists$ an element $c \in R$ such that $ca = 1$ i.e R having inverse of each element so R is a field.

Definition 2.11.

Let I be an ideal of R then for $a \in R$, the set

$R/I = \{I + a, a \in R\}$ with the following two operations

(1) $(I + a)) + (I + b) = I + (a + b)$

(2) $(I + a)).(I + b) = I + ab$

is called quotient ring.

Example 2.1.9.

Let $\mathbb{Z}$ be a ring and $n\mathbb{Z}$ be an ideal of $\mathbb{Z}$ then $\mathbb{Z}/n\mathbb{Z}$ is a quotient ring and isomorphic to $\mathbb{Z}_n$.

2.2 Monomial Ideals

We already seen that any ring can serve as a basis for polynomials with n variables. In this section we will investigate the term polynomial, afterward some about ring of polynomials. The study of the ring of polynomials is very useful to understand the concept of commutative algebra. In this section we give the review about monomial ideals and also about monomial ordering.

Definition 2.12.

An algebraic expression of the form

$$P(x) := a_0 + a_1 x + \cdots + a_{n-1}x^{n-1} + a_n x^n = \sum_{j=0}^{n} a_j x^j$$

is called the polynomial of degree n, where all $a_j s \in \mathbb{R}$ and a_n is called the leading coefficient of $P(x)$.

Definition 2.13.

Let R be a commutative ring with unity then the set of all polynomials i.e.,

$$R[x] = \{\sum_{j=0}^{n} a_j x^j \mid a_j \in \mathbb{R} \; and \; n \in \mathbb{Z}^+\}$$

is called the ring of polynomials over R.

Addition of two polynomials:

Let $P(x) = \sum_{j=0}^{n} a_j x^j$ and $Q(x) = \sum_{j=0}^{n} b_j x^j$ from $R[x]$ then we define the addition of two polynomials as,

$$P(x) + Q(x) = \sum_{j=0}^{n} a_j x^j + \sum_{j=0}^{n} b_j x^j = \sum_{j=0}^{n} (a_j + b_j) x^j$$

Multiplication of the two polynomials:

Let $P(x) = \sum_{j=0}^{n} a_j x^j$ and $Q(x) = \sum_{j=0}^{n} b_j x^j$ from $R[x]$ then we define the multiplication of two polynomials as,

$$P(x).Q(x) = \left(\sum_{j=0}^{n} a_j x^j\right) . \left(\sum_{j=0}^{n} b_j x^j\right) = \sum_{j=0}^{n} \left(\sum_{k+l=j} a_k b_l\right) x_j.$$

Definition 2.14.

A monomial in n indeterminates $\{x_1, x_2, ..., x_n\}$ is a power product

$$x_1^{\beta} = x_1^{\beta_1}, ..., x_n^{\beta_n}, \quad \beta = (\beta_1, ..., \beta_n) \in \mathbb{N}^n$$

The set of monomials in n indeterminates represented as

$$Mon(x_1, x_2, ..., x_n) \Rightarrow Mon_n := \{x^{\alpha} : \alpha \in \mathbb{N}^n\}$$

The expression of the polynomials is a linear combination or finite sum of monomials $i.e$,

$$g = \sum_{\alpha} a_{\alpha} x^{\alpha} \ where \ \alpha \ is \ finite$$

we can make the unique order of monomials by choosing a total ordering on the set of monomials. For other applications it is compulsory, however, that the ordering is well matched with the semi group structure on Mon_n.

Definition 2.15.

Let $R = K[x_1, x_2, ..., x_n]$ be a polynomial ring with n indeterminents with the coefficients from the field K. Let $J \subset R$ be an ideal then J is said to be monomial ideal if a set of monomials generates J.

Definition 2.16.

If a set of square free monomials generates an ideal $J \subset R$ then J is called square free ideal i.e.,

$$J = (x^{\alpha} \mid \alpha \in \mathbb{N}^n; \alpha_i = 0 \ or \ \alpha_i = 1)$$

for $\alpha = (\alpha_1, .., \alpha_n) \in \mathbb{N}$

Definition 2.17.

Let $R = K[x_1, x_2, ..., x_n]$ be a polynomial ring then a monomial ideal $I = R$ is said to be *radical ideal* if $I = \sqrt{I}$.

Definition 2.18.

A semigroup ordering (monomial ordering) is a total ordering $>$ on the set of $Mon_n = \{x^\alpha : \alpha \in \mathbb{N}^n\}$

$$x^\alpha > x^\beta \Rightarrow x^\alpha x^\gamma > x^\beta x^\gamma$$

$\forall\, \alpha, \beta, \gamma \in \mathbb{N}^n$.we can also say that $>$ is a monomial ordering on any ring $R[x_1, x_2, x_3, ...,]$, it's means that $>$ is a monomial ordering onMon_n.

Monomial ordering gives us the extra structure of the monomials and also on the polynomial rings. Monomial ordering is also used to give the difficult mathematical proofs of the theorems of commutative algebra. A very important example of monomial ordering is lexicographical ordering on $\mathbb{N}^n$ such that $x^\alpha > x^\beta \Leftrightarrow \alpha - \beta$ is positive. From the monomial ordering , in particular view, we can write a polynomial $g \in K[x]$ in a unique ordered way i.e;

$$g = a_\alpha x^\alpha + a_\beta x^\beta + ... + a_\gamma x^\gamma$$

with $x^\alpha > x^\beta > .... > x^\gamma$ and with non zero coefficients from K.

Definition 2.19.

Let $>$ be a fixed monomial ordering. Write $0 \neq g \in K[x]$, in a unique way

$$g = a_\alpha x^\alpha + a_\beta x^\beta + ... + a_\gamma x^\gamma \ where\ x^\alpha > x^\beta > .... > x^\gamma$$

and all the coefficients from K. we define the terms as:

$LM(g) := x^\alpha$, the *leading monomial* of g,

$LE(g) := \alpha$, the *leading exponent* of g,

$LT(g) := a_\alpha x^\alpha$, the *leading term* of g,

$LC(g) := a_\alpha$, the *leading coefficient* of g.

Further more, there are many types of monomial ordering , some Global Orderings are explained bellow.

Definition 2.20.

Lexicographical Ordering

Let $\alpha, \beta \in \mathbb{N}^n$ where $\alpha = (\alpha_1, \alpha_2, \alpha_3, ..., \alpha_n)$ and $\beta = (\beta_1, \beta_2, ..., \beta_n)$ then,

$x^\alpha <_{lex} x^\beta$ in $Mon(R)$ if $\exists 1 \leq t \leq n$ such that $\alpha_1 = \beta_1, \alpha_2 = \beta_2, ..., \alpha_t - 1 = \beta_t - 1$ and $\alpha_t < \beta_t$ i.e in the difference $\alpha - \beta = (\alpha_1 - \beta_1, \alpha_2 - \beta_2, ..., \alpha_n - \beta_n)$ the most left non-zero component is negative

Example 2.2.1.

Let $n = 3$, $k[x_1, x_2, x_3]$ = Monomials in three variables

x^3y^2z, xy^4z^2 $(x >_{lex} y >_{lex} z)$ so,

$\alpha = (3, 2, 1)$, $\beta = (1, 4, 2) \Rightarrow \alpha - \beta = (2, -2, -1)$

$\Rightarrow x^3y^2z >_{lex} xy^4z^2$

If ordering of the variables is not given, then our assumption will be $x_1 >_{lex} x_2 >_{lex}$ .... $>_{lex} x_n$.

Definition 2.21.

Graded Lexicographical Ordering

Let $\alpha, \beta \in \mathbb{N}^n$ where $\alpha = (\alpha_1, \alpha_2, \alpha_3, ..., \alpha_n)$ and $\beta = (\beta_1, \beta_2, ..., \beta_n)$ then, $x^\alpha <_{grlex} x^\beta$

in $Mon(R)$ if $\mid \alpha \mid<\mid \beta \mid$ where ($\mid \alpha \mid= \sum \alpha_i \ and \ \mid \beta \mid= \sum \beta_i$) also we can say,

$x^\alpha >_{grlex} x^\beta$ if $\mid \alpha \mid>\mid \beta \mid$ OR $x^\alpha <_{lex} x^\beta$ if $\mid \alpha \mid=\mid \beta \mid$

Example 2.2.2.

Let x^2y^2, xy^5z, x^3y^2 be the three monomials then by using definition of Graded Lexicographical ordering we get a result $xy^5z >_{grlex} x^3y^2 >_{grlex} x^2y^2$

Example 2.2.3.

Let $x^2y^2, xy^5z, x^3y^2, xy^2z^3$ be the four different monomials,

Since, $x^3y^2 \ and \ xy^2z^2$ have $\mid \alpha \mid=\mid \beta \mid$, so we will use lex to find ordering between them,

$$\alpha = (3, 2, 0); \beta = (1, 2, 2)$$

$$\alpha - \beta = (2, 0, -2)$$

$\Rightarrow x^\alpha > x^\beta \Rightarrow x^3y^2 >_{lex} xy^2z^2$, so finally we get the ordering

$\Rightarrow x^2y^2 <_{grlex} xy^2z^2 <_{grlex} x^3y^2 <_{grlex} xy^5z$

Definition 2.22.

Graded Reverse Lexicographical Ordering

Let $\alpha, \beta \in \mathbb{N}^n$ where $\alpha = (\alpha_1, \alpha_2, \alpha_3, ..., \alpha_n)$ and $\beta = (\beta_1, \beta_2, ..., \beta_n)$ then,

$x^\alpha <_{Grevlex} x^\beta$ in $Mon(R)$ if $\mid \alpha \mid<\mid \beta \mid$ where ($\mid \alpha \mid= \sum \alpha_i \ and \ \mid \beta \mid= \sum \beta_i$)

OR $\mid \alpha \mid=\mid \beta \mid$ and $\exists \ 1 \leq t \leq n$ such that $\alpha_t > \beta_t, \alpha_{t+1} = \beta_{t+1}, ..., \alpha_n = \beta_n$ i.e. in the difference $\alpha - \beta = \alpha_1 - \beta_1, \alpha_2 - \beta_2, ..., \alpha_n - \beta_n$, the most right non-zero component is negative.

Example 2.2.4.

Let $xy^5z, x^3y^2, xy^2z^2, x^2y^2$ be the four monomials

Since, $x^3y^2 and\ xy^2z^2$ have $\mid \alpha \mid = \mid \beta \mid$ then we can write as,

$\alpha = (3, 2, 0),\ \ \beta = (1, 2, 2)$

$\alpha - \beta = (2, 0, -2)$

$\Rightarrow x^3y^2 > xy^2z^2$,so finally we get the ordering

$\Rightarrow xy^5z >_{Grevlex} x^3y^2 >_{Grevlex} xy^2z^2 >_{Grevlex} x^2y^2$

Chapter 3

Graphs and Simplicial Complexes

3.1 Introduction

One of the oldest subject in this area is the investigation of the relation of graphs and their algebraic aspects. To find the relation of algebraic structure of rings using the graphs associated to them is an interesting case of advance study. Now a days study of algebraic object and its graph is very useful and very interesting. There is a very deep relation between graphs and commutative algebra more particularly the combinatorial part of commutative algebra. In commutative algebra we use the graphs and discuss the algebraic characterization of the graphs which are very useful. In this chapter, firstly we discuss about the graphs and discussed the algebraic object simplicial complex associated to graphs.

3.2 Graphs

A graph G is a set of ordered pairs $G = (V(G), E(G))$,where $V(G)$ and $E(G)$ represents the set of vertices or points and set of edges respectively. The set of vertices or points $V(G)$ can be used to represent real or imaginary entities of all type in the world problems . The set of edges $E(G)$ extend a binary relation between the entities. For example, links of the cities through the joining roads (there is a road between two cities or not), computer's networking (two computers are connected or not). Every edge consists on two end points(vertices).

For ease, V for set of vertices instead of $V(G)$ and E for set of edges instead of $E(G)$ is taken.

Definition 3.1. The degree of a vertex V denoted as $d(v)$, is its number of incident

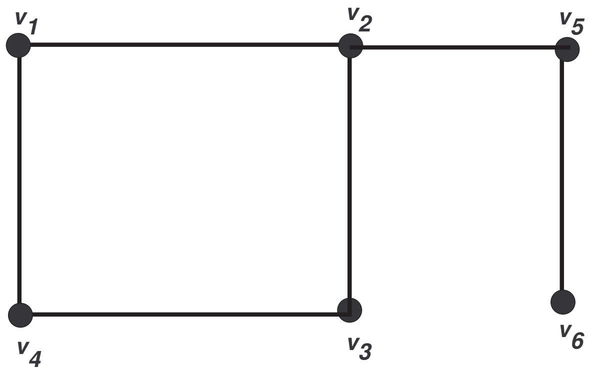

Figure 3.1: Connected Graph

edges.

Proposition 3.1. The sum of the degrees of a graph $G = (V, E)$ is $2 \mid E \mid= 2m$, where m denotes the number of edges.

A geometric representation of the graph G in the fig 3.1 with $V_G = \{v_1, v_2, v_3, v_4, v_5, v_6\}$ and representation of edge set is $E_G = \{v_1v_2, v_1v_4, v_2v_3, v_2v_5, v_3v_4, v_5v_6\}$

Definition 3.2. Subgraphs

A graph H is called subgraph of a graph G if $V_H \subseteq V_G$ and $E_H \subseteq E_G$, denoted as $H \subseteq G$.

Definition 3.3. Span of Graphs

A subgraph H of G $(H \subseteq G)$ spans of G , if $V_G = V_H$ means that if every vertex of G is in H.

Definition 3.4. Induced Graphs

A subgraph H of G is said to be induced graph, if $E_H = E_G \bigcap E(V_H)$, in this , H is induced by its set of vertices.

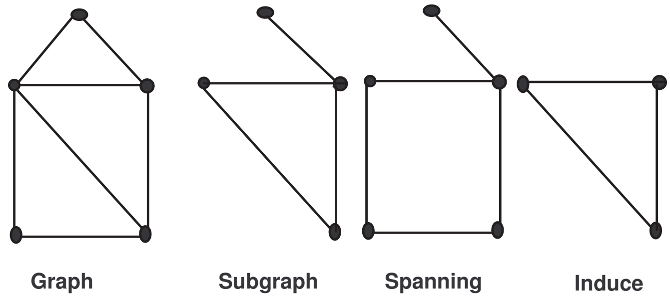

Figure 3.2: Graphs

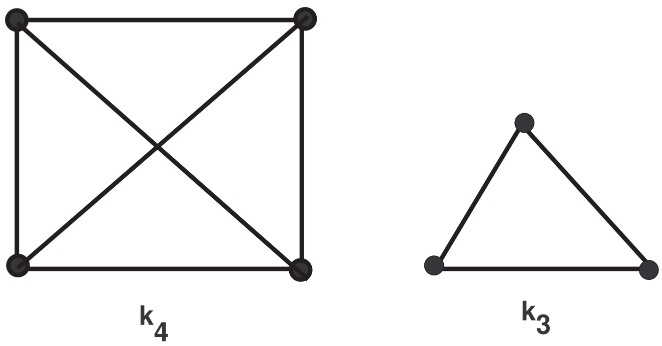

Figure 3.3: Complete Graphs

In the following figure 3.2 we can see that the graphs of above definitions.

Definition 3.5. Complete Graphs

A Graph G is said to be complete graph if every two vertices of G are adjacent.

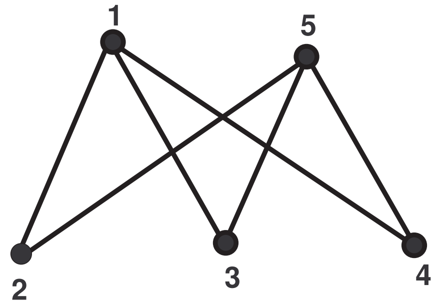

Figure 3.4: Bipartite Graph

Definition 3.6. Bipartite Graphs

Let G be a graph, then G is called bipartite if V_G has a partition in to two sets X and Y such that each $xy \in G$ connect with vertex of X and also with the vertex of Y.In this case , G is $(X, Y) - bipartite$.

Consider the graph in figure 3.4. We can see easily that it is not a complete graph, because each vertex of this graph is not adjacent to every other vertex. However, this is bipartite graph, because the vertices of graph can be divided into two disjoints sets $\{1, 5\}$ and $\{2, 3, 4, \}$, such that every two vertices taken from the same set are not adjacent but every two vertices taken from the both disjoints set are adjacent.

Theorem 3.2. A graph G is called bipartite if and only if G has no odd cycles.

Definition 3.7. (Isomorphism of Graphs)

Two graphs G and H are said to be isomorphic, denoted by $G \cong H$, iff there exist a bijection $f : V_G \to V_H$ such that $xy \in E_G \Leftrightarrow f(x)f(y) \in E_H$.

If the two graphs are isomorphic then they must have:

1) The same number of vertices

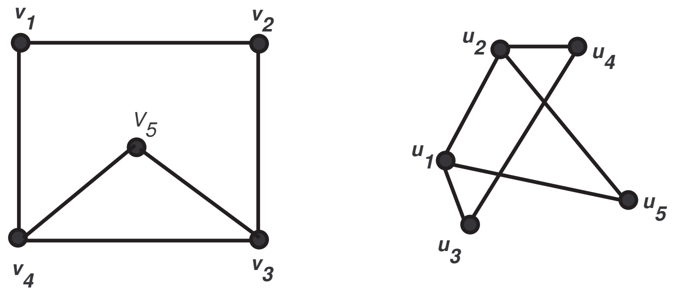

Figure 3.5: Isomorphic Graphs

2) The same number of edges

3) The Same degree for corresponding vertices

4) Number of components must be same

5) Number of loops must be same

6) The same number of parallel edges

7) Number of cycles of same lengths are equal.

In general, it is easy to show that, the two graphs are not isomorphic if one of the property does not fulfill. But if all the above said properties are fulfilled still we can not say that the graph are isomorphic unless we are not able to find a bijection along with the property in the definition.

Example 3.2.1. *The two graphs in figure 3.5 are isomorphic. Indeed , the required isomorphism is given as* $v_1 \mapsto u_1, v_2 \mapsto u_3, v_3 \mapsto u_4, v_4 \mapsto u_2, v_5 \mapsto u_5$.

Definition 3.8. Trees

A graph is acyclic if it has no cycle. A connected acyclic graph is called tree.

The trees on five vertices shown in figure 3.6

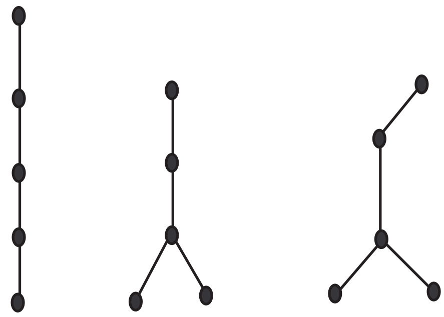

Figure 3.6: Trees

Definition 3.9. Spanning Trees

Let G be a graph, then spanning tree for G is a sub-graph of G which is a tree that contains every vertex of G.

The following is a simple example of a graph with its two spanning trees. we can see easily, the spanning tree of G is the subgraph of G that contains every vertex of G. In the figure 3.7 is a connected graph and in the figure 3.8 are the spanning trees of G.

3.3 Simplicial Complexes

Definition 3.10. Simplicial Complex

Let A be a set such that $A = [n] = \{1, 2, 3, ..., n\}$, a collection "$\Delta$" of subsets of power sets of A is called a simplicial complex if it satisfies the following properties,

1) $\{i\} \in \Delta$;$\forall\ i$ (i.e all the singletons are in Δ)

2) If $B \subseteq C$ and $C \in \Delta \Rightarrow B \in \Delta$

Example 3.3.1.

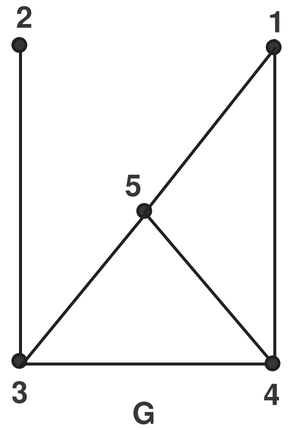

Figure 3.7: Connected Graph

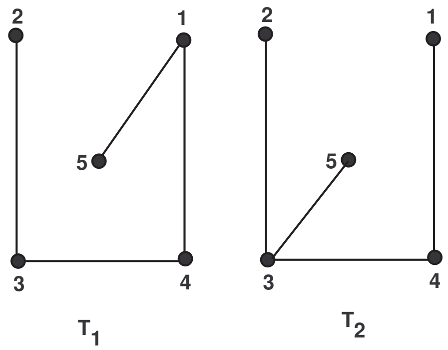

Figure 3.8: Spanning Trees

Consider $n = 4$

Power set is , $\{\phi, \{1\}, \{2\}, \{3\}, \{4\}, \{1, 2\}, \{1, 3\}, \{1, 4\}, \{2, 3\}, \{2, 4\}, \{3, 4\},$ $\{1, 2, 3\}, \{1, 2, 4\}, \{2, 3, 4\}, \{1, 3, 4\}, \{1, 2, 3, 4\}\}$ and now consider,

$\Delta = \{\{1, 2, 3\}, \{1, 2\}, \{2, 3\}, \{1\}, \{2\}, \{3\}, \{4\}\}$ not a simplicial complex as property "1" is not satisfied. Since $\{1, 2, 3\} \in \Delta$ and $\{1, 3\} \subset \{1, 2, 3\}$ but $\{1, 3\}$ not belongs to Δ.

Now, take $\Delta = \{\{1\}, \{2\}, \{3\}, \{4\}, \phi\}$, is a simplicial complex.

Similarly, $\Delta = \{\{1\}, \{2\}, \{3\}, \phi\}$ is a simplicial complex for $n = [3]$

Remark 3.3.1. The elements (subsets) of Δ are called faces.

Definition 3.11. Dimension of a Face

Let $A \in \Delta$ be a face, then dimension of A is denoted by $dim(A)$ and is defined as $dim(A) = |A| - 1$

For example, if $A = \{1, 2, 3\}$

$\Rightarrow \; dim(A) = 3 - 1 = 2$

Number of i-dimensional Faces

Definition: Let Δ be a simplicial complex, then we set $f_i = f_i(\Delta) = number\ of\ i - dimensonal\ faces\ of\ \Delta$

Example:-

Let $\Delta = \langle\{1, 2, 3, 4\}, \{3, 4, 5\}, \{5, 6\}\rangle$ be a simplicial complex as shown in figure 3.13.

$f_0 = 6$ (number of vertices)

$f_1 = 9$ (number of edges)

$f_2 = 5$ (number of triangles)

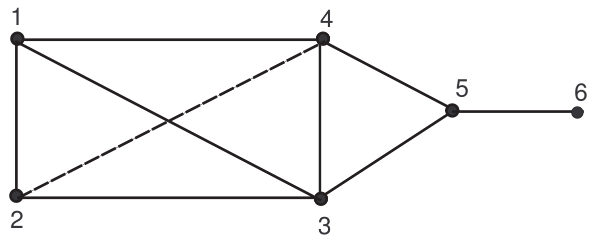

Figure 3.9: Faces of Δ

$f_3 = 1$ (number of tetrahedron)

$f_k = 0$; k ≥ 4

Definition 3.12. Facets

Faces of maximum dimension (inclusionwise) are called facets.

Let $\Delta = \{\{2,3,4\},\{2,3\},\{2,4\},\{3,4\},\{1,2\},\{1\},\{2\},\{3\},\{4\},\phi\}$ be a simplicial complex. Filled triangle in the figure 3.11 shows a face of dimension two.

Definition 3.13. Facet Ideal and Non-face Ideal

Let a simplicial complex Δ over a set of vertices $V = \{v_1, v_2, ...v_n\}$, such that $\{v_i\} \in \Delta$ for all i and R be a polynomial ring over $K[x_1, x_2, ...x_n]$.

(a) We define $\mathcal{F}(\Delta)$ (Facet Ideal) to be ideal of R generated by square-free monomials $\{x_{i_1}....x_{i_s}\}$, where $\{v_{i_1}, ..., v_{i_s}\}$ is a facet of Δ.

(b) We define $\mathcal{N}(\Delta)$ (Non-Face Ideal) to be the ideal of R generated square-free monomials $\{x_{i_1}....x_{i_s}\}$,where $x_{i_1}....x_{i_s}$ is not a face of Δ. We call $\mathcal{N}(\Delta)$ the $non-face\ ideal$ or $Stanley-Reisner$ ideal of Δ.

Example 3.3.2.

Let $\Delta = \{\{x_1, x_2\}, \{x_1, x_3\}, \{x_2, x_3, x_4\}\}$ with the dimension $dim(\Delta) = 3 - 1 = 2$.

Non Faces: $\{\{x_1, x_4\}, \{x_1, x_2, x_4\}, \{x_1, x_3, x_4\}, \{x_1, x_2, x_3\}, \{x_1, x_2, x_3, x_4\}\}$

We write this set of non-faces elements in the form of monomials to get $Stanley - Reisner$ ideal.

$\Rightarrow I_\Delta = \{x_1x_4, x_1x_2x_4, x_1x_3x_4, x_1x_2x_3, x_1x_2x_3x_4\}$

$\Rightarrow I_\Delta = \langle x_1x_4, x_1x_2x_3\rangle$

Example 3.3.3.

Now, we are going to find a simplicial complex against a following ideal;

$$I_\Delta =< x_1x_3, x_1x_4, x_1x_5, x_2x_3x_5, x_3x_4x_5 >$$

$\Rightarrow \Delta = \{\{x_1\}, \{x_2\}, \{x_3\}, \{x_4\}, \{x_1, x_2\}, \{x_2, x_3\}, \{x_2, x_4\}, \{x_2, x_5\}, \{x_3, x_4\}, \{x_4, x_5\}, \{x_1, x_2, x_3\}, \{x_1, x_2, x_4\}, \{x_1, x_2, x_5\}, \{x_2, x_3, x_4\}, \{x_1, x_3, x_5\}, \{x_1, x_4, x_5\}, \{x_2, x_4, x_5\}\}$

$\Rightarrow \Delta = \{\{x_1\}, \{x_2\}, \{x_3\}, \{x_4\}, \{x_1, x_2\}, \{x_2, x_3\}, \{x_2, x_4\}, \{x_2, x_5\}, \{x_3, x_4\}, \{x_3, x_5\}, \{x_4, x_5\}, \{x_2, x_3, x_4\}, \{x_2, x_4, x_5\}\}$

$\Rightarrow \Delta = \langle\{x_1, x_2\}, \{x_3, x_5\}, \{x_2, x_3, x_4\}, \{x_2, x_4, x_5\}\rangle$. For simplicial complex see figure 3.10.

Definition 3.14. Dimension of Simplicial Complex

Supremum (sup) of the dimension of all facets in a simplicial complex is called the dimension of simplicial complex.

A simplicial complex with dimension-3 shown in the figure 3.12, i.e. $\Delta = \langle\{1, 2, 3, 4\}\rangle$

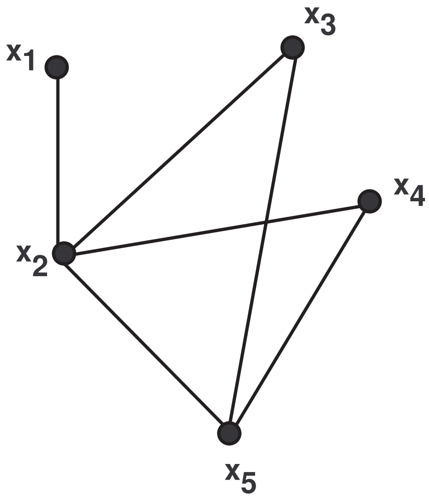

Figure 3.10: Simplicial Complex

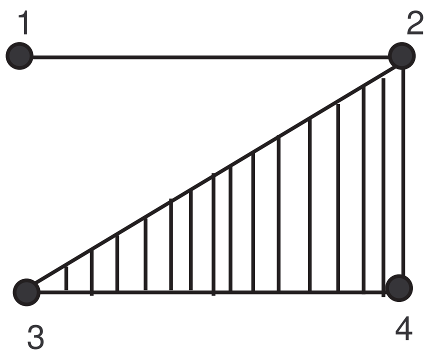

Figure 3.11: Simplicial Complex

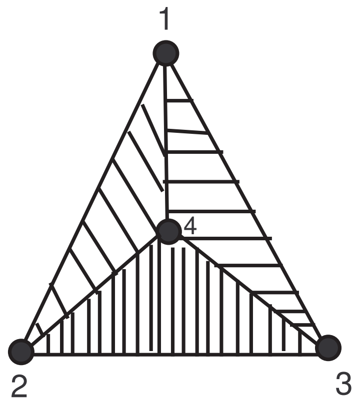

Figure 3.12: Simplicial Complex

Definition 3.15. Simplex

A simplicial complex generated by a single facet is called simplex.

Definition 3.16. Pure Simplicial Complex

A simplicial complex is called pure if it is generated by facets of same dimension.

A simplicial complex shown in figure 3.13 is pure because all the facets having the same dimension.

Definition 3.17. Let Δ be a simplicial complex withe the dimension d, its $f-vector$ is a $d+1$-tuple, defined as:

$$f(\Delta) = (f_0, f_1, ..., f_d)$$

Where f_i denotes the number of $i-dimensional$ faces of Δ.

In [17] and [31], they discussed about graded algebra A, Hilbert function $H(A,t)$ and the Hilbert series $H_t(A)$ of a graded algebra.

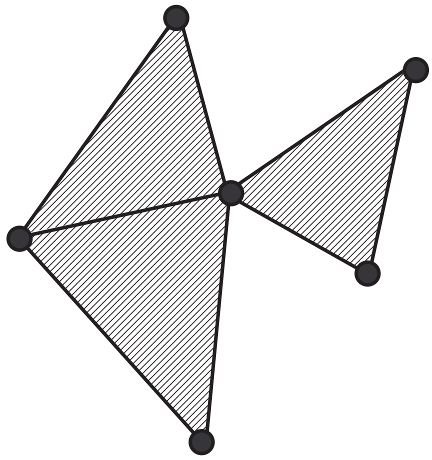

Figure 3.13: Pure Simplicial Complex

Definition 3.18. [1],Let A be a standard graded algebra and the (unique) polynomial with integral coefficients is,

$$h(t) = h_0 + h_1 t + ... + h_r t^r$$

And satisfying

$$H_t(A) = \frac{h(t)}{(1-t)^d}$$

Where $dim(A) = d$. The $h - vector$ of A is defined by $h(A) = (h_0, h_1, h_2, ..., h_r)$.

Chapter 4
Spanning Forest Complex

4.1 Introduction

In this chapter first we will discusses the newly introduced algebraic object spanning simplicial complex associated to a simple connected graph. Then we apply this idea to disconnected graph and investigate the spanning simplicial complexes $\Delta_s(G)$ associated to disconnected graph G. We discuss the union of cycles of same length and give the characterization of all spanning forests of t-cycle graph tC_n. In [15] recall that for a connected graph G, a subgraph H of G is called spanning tree if H is a tree with all vertices of G. In [1], [22] authors use these spanning trees and introduced the term spanning simplicial complex based on the edge sets of all spanning trees of G. Representation of the collection of all the edge-sets of the spanning trees of the graph by $s(G)$, i.e.

$$s(G) = \{E(T_i) \subset E,\ where\ T_i\ is\ a\ spanning\ tree\ of\ the\ graph\}$$

J. Guo and T Wu, [14], extended the idea given in [1] and defined the spanning simplicial complex associated to a disconnected graph. For a disconnected graph G, a spanning forest of G is a subgraph H of G such that for each connected component K of G, the intersection of H with K is a spanning tree of K i.e. $K \cap H = T_s$, where T_s is a spanning tree of K. Representation of the edge-set of the all spanning forests for a simple disconnected graph G is,

$$s(G) = \{E(F) \mid F\ is\ a\ spanning\ forest\ of\ G\}$$

In this chapter we give the entire information on spanning forests associated to simple disconnected graphs in order to describe the algebraic aspects arising from it. Firstly, we discussed about the spanning forests for some simple graphs. For this, we use the results on simplicial complexes of graphs given in [14],[20] and apply the techniques given in [1] to compute characterization of spanning forest complexes associated to the disjoint union of graphs. On the basis of our work, we give some reasonable and applicable computations for spanning forest complexes. We mainly focus the f-vector and h-vector associated to spanning forest complexes.

Spanning forest complex and its algebraic characterization are not known till date. In this chapter, we provide the dimension of spanning forest complex associated to t-cycle graph tC_n, a graph consist of t copies of cycles each of length n. We also formulate the f-vector and h-vector of spanning forest complex. We investigate some other algebraic and combinatorial properties of spanning forest complex in view of defining properties of simplicial complexes for simple graphs discussed in [1],[10],[11],[22],[29],[28].

4.2 Spanning Simplicial Complexes

In [1], Imran Anwar introduced the notion of simplicial complex, a complex based on the spanning trees of connected graph G. In this section, we will discuss the algebra associated to spanning simplicial complex.

Definition 4.1. (Spanning Simplicial Complexes)

Let $s(G) = \{E_1, E_2, ...E_s\}$ be the edge set of all distinct spanning trees of a simple connected graph $G(V, E)$. Then spanning simplicial complex of G is the simplicial complex $\Delta_s(G)$ on E such that the facets of $\Delta_s(G)$ consist of the elements of $s(G)$ i.e.,

$$\Delta_s(G) = \langle E_1, E_2, ..., E_s \rangle.$$

Example 4.2.1.

Let us consider C_6. We have total 6 distinct spanning trees for C_6. One can easily find all these spanning trees by removing one edge of C_6. This method is called *cutting down method*. By using this method one can obtain spanning tree of any graph by removing one edge from each cycle of the graph. So for 4.1 we have the following edge set:

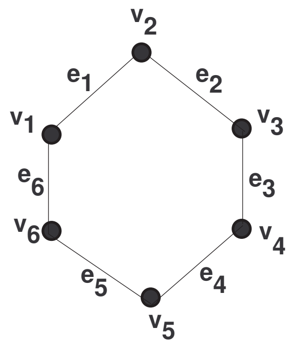

Figure 4.1: C_6

$$s(C_6) = \{\{e_2, e_3, e_4, e_5, e_6\},\ \{e_1, e_3, e_4, e_5, e_6\},\ \{e_1, e_2, e_4, e_5, e_6\},$$
$$\{e_1, e_2, e_3, e_5, e_6\}, \{e_1, e_2, e_3, e_4, e_6\}, \{e_1, e_2, e_3, e_4, e_5\}\}$$

So we have the spanning simplicial complex of C_6 as following

$$\Delta_s(C_6) = \langle\{e_2, e_3, e_4, e_5, e_6\}, \{e_1, e_3, e_4, e_5, e_6\}, \{e_1, e_2, e_4, e_5, e_6\},$$
$$\{e_1, e_2, e_3, e_5, e_6\}, \{e_1, e_2, e_3, e_4, e_6\}, \{e_1, e_2, e_3, e_4, e_5\}\rangle$$

Definition 4.2.

A connected graph $U_{n,m}$ with n vertices is called *uni*-cyclic if it contains exactly one cycle of the length m (with $m \leq n$).

In particular, if $n = m$ then $U_{n,m}$ is simple $n - cyclic$ graph. Throughout in this section for the $uni - cyclic$ graph, edge-labeling $\{e_1, e_2, ..., e_m, e_{m+1}, ..., e_n\}$ is fix of

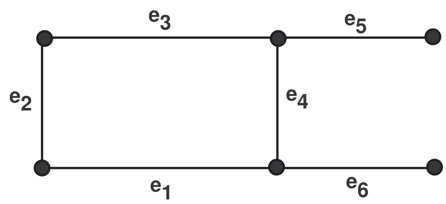

Figure 4.2: $Uni - cyclic\ Graph$

$U_{n,m}$ such that the edge-set of the only cycle in $U_{n,m}$ is $\{e_1, e_2, ..., e_m\}$.

For example, graphs shown in the figure 4.1 and 4.2 are the $uni - cyclic$ graphs.

Lemma 4.1. Characterization of $s(U_{n,m})$*[1]*

Consider a $uni - cyclic$ graph $U_{n,m}$ with $E = \{e_1, e_2, ..., e_n\}$. A subset $E(T_i) \subset E$ will belongs to $s(U_{n,m})$ iff $T_i = E \setminus \{e_i\}$ for some $i \in \{1, 2, 3, ..., m\}$ i.e.

$$s(U_{n,m}) = \{\hat{E}_i \mid \hat{E}_i = E \setminus \{e_i\}\ for\ all\ 1 \leq i \leq m\}$$

Proposition 4.2. *[1]*

Let Δ be a d dimensional simplicial complex over $[n]$ if $f_t = \binom{n}{t+1}$ for some $t \leq d$ then $f_i = \binom{n}{i+1}$ for all $0 \leq i \leq t$

Theorem 4.3. *[1]*

Let $\Delta_s(U_{n,m})$ be the spanning simplicial complex of $uni - cyclic$ graph $U_{n,m}$, then $dim(U_{n,m}) = n - 2$ and f-vector $f(\Delta_s(U_{n,m})) = (f_1, f_1, f_2, ..., f_{n-2})$ with

$$f_i = \begin{cases} \binom{n}{i+1}, & for\ i \leq m-2; \\ \binom{n}{i+1} - \binom{n-m}{i-m+1}, & for\ m-2 < i \leq n-2. \end{cases}$$

In [22], Anwar et al. discussed about $r - cyclic$ graph $G_{n,r}$ for the algebraic properties of *spanning simplicial complex* $\Delta_s(G_{n,r})$. A cyclic graph $G_{n,r}$ is a connected

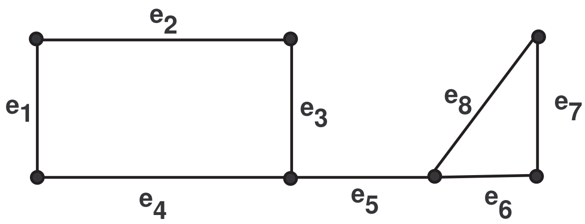

Figure 4.3: $G_{8,2}$

graph having r cycles of different lengths and n edges such that no two cycles having common edge with the edge labeling $\{e_{11},...,e_{1m_1},e_{12},...,e_{2m_2},...,e_{r1},...,e_{rm_r},...,e_1,...,e_t\}$ where $\{e_{i1},...,e_{im_i}\}$ is the edge set of ith cycle for $1\leq i\leq r$.

Lemma 4.4. *[22]*

Let $G_{n,r}$ be the $r-cyclic$ graph with the edge set $E=\{e_{11},...,e_{1m_1},$ $e_{12},...,e_{2m_2},...,e_{r1},...,e_{rm_r},...,e_1,...,e_t\}$. A subset $E(T_{1i_1,...,ri_r})$ of E belongs to $s(G_{n,r})$ iff $E(T_{1i_1,...,ri_r})=E\setminus\{e_{1i_1},...,e_{ri_r}\}$ for some $i_j\in\{1,2,...,m_j\}$ $1\leq j\leq r$.

Example 4.2.2.

Let us consider $G_{8,2}$ *as in figure 4.3. The edge set of* $G_{8,2}$ *is as follows,*

$s(G_{8,2})=\{\{e_2,e_3,e_4,e_5,e_6,e_7\},\{e_2,e_3,e_4,e_5,e_6,e_8\},\{e_2,e_3,e_4,e_5,e_7,e_8\},\{e_1,e_3,e_4,e_5,$ $e_6,e_7\},\{e_1,e_3,e_4,e_5,e_6,e_8\},\{e_1,e_3,e_4,e_5,e_7,e_8\},\{e_1,e_2,e_3,e_5,e_6,e_7\},\{e_1,e_3,e_3,e_5,e_6,e_8\},$ $\{e_1,e_2,e_3,e_5,e_7,e_8\}\}$

So the spanning simplicial complex of $G_{8,2}$ *is*

$\Delta_s(G_{8,2})=\langle\{e_2,e_3,e_4,e_5,e_6,e_7\},\{e_2,e_3,e_4,e_5,e_6,e_8\},\{e_2,e_3,e_4,e_5,e_7,e_8\},\{e_1,e_3,e_4,e_5,e_6,e_7\},$ $\{e_1,e_3,e_4,e_5,e_6,e_8\},\{e_1,e_3,e_4,e_5,e_7,e_8\},\{e_1,e_2,e_3,e_5,e_6,e_7\},\{e_1,e_3,e_3,e_5,e_6,e_8\},$ $\{e_1,e_2,e_3,e_5,e_7,e_8\}\rangle$

Proposition 4.5. *[22]*

Let $\Delta_s(G_{n,r})$ be the spanning simplicial complex of $r-cyclic$ graph with cycles of the length $m_1 \leq m_2 \leq ... \leq m_r$, then the $dim(\Delta_s(G_{n,r})) = n-1-r$ with f-vector,

$$f_i = \binom{n}{t+1} + \sum_{k=1}^{r}(-1)^k \left[\sum_{(j-1,\ldots,j_r)=1, j_1 \neq .. \neq j_r}^{r} \binom{n - \sum_{s=1}^{k} m_{j_s}}{i+1-\sum_{s=1}^{k} m_{j_s}} \right]$$

where $0 \leq i \leq n-r-1$.

4.3 Spanning Forest Complexes

In [14], Gou et al. gave the spanning simplicial complex of the finite graph G. They gave the idea of the edge set and spanning forests complex of the given finite graph G. Recall that a subgraph H of G is called spanning forest of G, if for each connected component M of G, $M \cap H$ is a spanning tree of M. For a simple graph, the edge set of all spanning forests of G is

$$s(G) = \{E(F) \mid F \; is \; a \; spanning \; forest \; of \; G\}$$

Definition 4.3.

Let G be a disconnected graph then the spanning forest complex of G, denoted as $\Delta_s(G)$, is generated from the edge set of all spanning forests of G, i.e.

$$\Delta_s(G) = \langle E(F) \mid F \; is \; a \; spanning \; forest \; of \; G \rangle$$

4.3.1 Spanning Forests Complexes of tC_n

Definition 4.4.

A t-cyclic graph tC_n is a disjoint union of t cycles of length n i.e., $tC_n = \sqcup_{i=1}^{t} C_n$. The number of edges are tn.

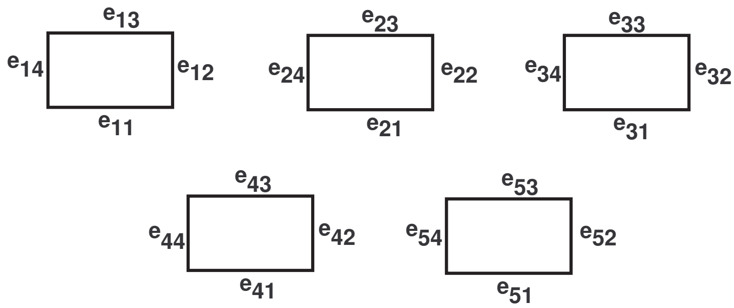

Figure 4.4: $5C_4$

For example, a cyclic graph $5C_4$ is a disconnected graph having 5 copies of the C_4 as shown in figure 4.4.

Theorem 4.6.

Let tC_n be the graph having t disjoints cycles of the length n, then the number of spanning forests are exactly n^t.

Proof. As we know that $tC_n = \sqcup_{i=1}^{t} C_n$. Each cycle of C_n has n number of spanning trees. But here we have t disjoint copies, so the total spanning forests of tC_n are

$$n.n. \ \dots \ (t \ times).n = n^t$$

Example 4.3.1.

The graph shown in the figure 4.4 has five copies of C_4, so by using above theorem the spanning forests of $5C_4$ are $4^5 = 1024$.

Throughout in this section, for tC_n, we fix the edge-labeling $\{e_{11}, e_{12}, ..., e_{1n}, e_{21}, e_{22}, ..., e_{2n}, e_{t1}, e_{t2}, ..., e_{tn}\}$ such that $\{e_{i1}, ..., e_{im_i}\}$ is the edge set of ith-cycle in tC_n for $1 \leq i \leq t$, we give the characterization of $s(tC_n)$.

Lemma 4.7. (Characterization of $s(tC_n)$)

Let tC_n be the t-cycle graph and $E = \{e_{11}, e_{12}, ..., e_{1n}, e_{21}, e_{22}, ..., e_{2n}, e_{t1}, e_{t2}, ..., e_{tn}\}$ of tC_n such that $\{e_{i1}, ..., e_{in}\}$ is the edge set of ith copy of tC_n. A subset $E(F_{(1i_1,2i_2,...,ti_t)})$ of E will belongs $s(tC_n)$ if and only if $E(F_{(1i_1,2i_2,...,ti_t)}) = E \setminus \{e_{1i_1}, ..., e_{ti_t}\}$ for some $i \in \{1, 2, ..., n\}$. In particular,

$$s(tC_n) = \{\hat{E}_{(1i_1,2i_2,...,ti_t)} = E \setminus \{e_{1i_1}, ..., e_{ti_t}\} for\ some\ i \in \{1, 2, ..., n\}\}$$

Proof. As tC_n contains of t cycles with the same length n so for its spanning forests one has to follow the *cutting down method* given in [15] for each copy of C_n i.e. remove exactly one edge from each cycle of tC_n. By doing this we obtain a spanning forest $F_{(1i_1,2i_2,...,ti_t)}$ by removing $\{e_{1i_1}, ..., e_{ti_t}\}$ from edge set of tC_n. So we have

$$s(tC_n) = \{\hat{E}_{(1i_1,2i_2,...,ti_t)} = E \setminus \{e_{1i_1}, ..., e_{ti_t}\} for\ some\ i \in \{1, 2, ..., n\}\}$$

Example 4.3.2.

Let us consider the graph $2C_3$ in figure 4.5, By using the above characterization we have

$$s(2C_3) = \{\{e_{12}, e_{13}, e_{22}, e_{23}\}, \{e_{12}, e_{13}, e_{21}, e_{23}\}, \{e_{12}, e_{13}, e_{21}, e_{22}\}, \{e_{11}, e_{13}, e_{22}, e_{23}\}, \{e_{11}, e_{13}, e_{21}, e_{23}\}, \{e_{11}, e_{13}, e_{21}, e_{22}\}, \{e_{11}, e_{12}, e_{22}, e_{23}\}, \{e_{11}, e_{12}, e_{21}, e_{23}\}, \{e_{11}, e_{12}, e_{21}, e_{22}\}\}$$

Theorem 4.8.

Let $\Delta_s(tC_n)$ be a spanning forest complex of tC_n, then

$$dim(\Delta_s(tC_n)) = t(n-1) - 1$$

Proof.

Let $E = \{e_{ij} \mid 1 \leq i \leq t\ and\ 1 \leq j \leq n\}$ be the edge set of tC_n then,

$$s(tC_n) = \{\hat{E}_{(1i_1,2i_2,...,ti_t)} = E \setminus \{e_{1i_1}, ..., e_{ti_t}\} | for\ some\ 1 \leq i \leq n\}$$

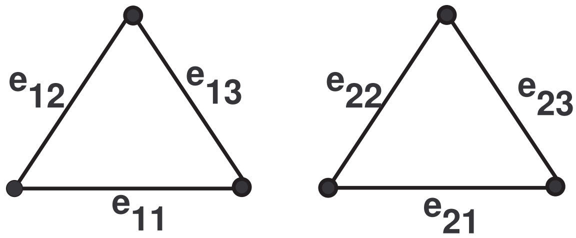

Figure 4.5: $2C_3$

and

$$\Delta_s(tC_n) = \langle \hat{E}_{(1i_1,2i_2,...,ti_t)} = E \setminus \{e_{1i_1}, ..., e_{ti_t}\} \mid for\ some\ 1 \leq i \leq n \rangle$$

be the spanning forest complex. It can be easily seen that each facet $\hat{E}_{(1i_1,...,ti_t)}$ has length $(n-1)t$.

$$\Rightarrow dim\Delta_s(tC_n) = t(n-1) - 1$$

Example 4.3.3.

Let $2C_3$ be a graph shown in figure 4.5, having the two copies of C_3, then $\Delta_s(2C_3)$ is a spanning forest complex of $2C_3$ with the following dimension,

$$dim(\Delta_s(2C_3)) = 2(3-1) - 1 = 3$$

In the following theorems we provide the f-vector for $\Delta_s(2C_n)$ and $\Delta_s(3C_n)$ with out proof. The proof will be provided for the general case 4.12.

Theorem 4.9.

Let $\Delta_s(2C_n)$ be the spanning forest complex of 2-cycle graph $2C_n$ with cycles of same length n,

then the $dim(\Delta_s(2C_n)) = 2n - 3$ with $f - vector$

$$f_i = \binom{2n}{i+1} - 2\binom{n}{i+1-n}$$

where $0 \le i \le 2n-3$

Example 4.3.4.

The spanning forest complex $\Delta_s(2C_3)$ of the graph in the figure 4.5 is:

$\Delta_s(2C_3) = \langle \{e_{12}, e_{13}, e_{22}, e_{23}\}, \{e_{12}, e_{13}, e_{21}, e_{23}\}, \{e_{12}, e_{13}, e_{21}, e_{22}\}, \{e_{11}, e_{13}, e_{22}, e_{23}\},$

$\{e_{11}, e_{13}, e_{21}, e_{23}\}, \{e_{11}, e_{13}, e_{21}, e_{22}\}, \{e_{11}, e_{12}, e_{22}, e_{23}\}, \{e_{11}, e_{12}, e_{21}, e_{23}\}, \{e_{11}, e_{12}, e_{21}, e_{22}\}\rangle$

Now, we are going to calculate the $f-vector$ for the graph given in the figure 4.5,we take a set of all edges of the given graph as:

$$\{e_{11}, e_{12}, e_{13}, e_{21}, e_{22}, e_{23}\}$$

And now take the all possible pairs of two edges,

$\{\{e_{11}, e_{12}\}, \{e_{11}, e_{13}\}, \{e_{11}, e_{21}\}, \{e_{11}, e_{22}\}, \{e_{11}, e_{23}\}, \{e_{12}, e_{13}\}, \{e_{12}, e_{21}\}, \{e_{12}, e_{22}\},$

$\{e_{12}, e_{23}\}, \{e_{13}, e_{21}\}, \{e_{13}, e_{22}\}, \{e_{13}, e_{23}\}, \{e_{21}, e_{22}\}, \{e_{21}, e_{23}\}, \{e_{22}, e_{23}\}\}$

$\Rightarrow f_1 = 15$

Now,take all the possible subsets with the three edges,

$\{\{e_{11}, e_{12}, e_{13}\}, \{e_{11}, e_{12}, e_{21}\}, \{e_{11}, e_{12}, e_{22}\}, \{e_{11}, e_{12}, e_{23}\}, \{e_{11}, e_{13}, e_{21}\}, \{e_{11}, e_{13}, e_{22}\},$

$\{e_{11}, e_{13}, e_{23}\}, \{e_{11}, e_{21}, e_{22}\}, \{e_{11}, e_{21}, e_{23}\}, \{e_{11}, e_{22}, e_{23}\}, \{e_{12}, e_{13}, e_{21}\}, \{e_{12}, e_{13}, e_{22}\},$

$\{e_{12}, e_{13}, e_{23}\}, \{e_{12}, e_{21}, e_{22}\}, \{e_{12}, e_{21}, e_{23}\}, \{e_{12}, e_{22}, e_{23}\}\{e_{13}, e_{21}, e_{22}\}, \{e_{13}, e_{21}, e_{23}\},$

$\{e_{13}, e_{22}, e_{23}\}, \{e_{21}, e_{22}, e_{23}\}\}$

But we can see that $\{e_{11}, e_{12}, e_{13}\}$ and $\{e_{21}, e_{22}, e_{23}\}$ are not the forests, so we will remove these from above.

$\Rightarrow f_2 = 18$

Now, take all the possible subsets with the four edges,

$\{\{e_{11}, e_{12}, e_{13}, e_{21}\}, \{e_{11}, e_{12}, e_{13}, e_{22}\}, \{e_{11}, e_{12}, e_{13}, e_{23}\}, \{e_{11}, e_{12}, e_{21}, e_{22}\}, \{e_{11}, e_{12}, e_{21}, e_{22}\},$

$\{e_{11}, e_{12}, e_{21}, e_{23}\}, \{e_{11}, e_{12}, e_{22}, e_{23}\}, \{e_{11}, e_{13}, e_{22}, e_{23}\}, \{e_{11}, e_{12}, e_{21}, e_{23}\}\{e_{11}, e_{21}, e_{22}, e_{23}\},$

$\{e_{12}, e_{13}, e_{21}, e_{22}\}, \{e_{12}, e_{13}, e_{21}, e_{23}\}, \{e_{12}, e_{13}, e_{22}, e_{23}\}, \{e_{12}, e_{21}, e_{22}, e_{23}\}, \{e_{13}, e_{21}, e_{22}, e_{23}\}\}$

But we ca again see that

$\{e_{11}, e_{12}, e_{13}, e_{21}\}$, $\{e_{11}, e_{12}, e_{13}, e_{22}\}$,$\{e_{11}, e_{12}, e_{13}, e_{23}\}$,$\{e_{11}, e_{21}, e_{22}, e_{23}\}$,

$\{e_{12}, e_{21}, e_{22}, e_{23}\}$ and $\{e_{13}, e_{21}, e_{22}, e_{23}\}$ are not the forests, so we will remove these from above.

$\Rightarrow f_3 = 9$ and then finally we get the following f-vector of the graph in figure 4.5,

$$f = (6, 15, 18, 9)$$

Theorem 4.10.

Let $\Delta_s(3C_n)$ be the spanning forest complex of 3-cycle graph $3C_n$ with cycles of same lengths n, then the $dim(\Delta_s(3C_n)) = 3n - 4$ with f-vector

$$f_i = \binom{3n}{i+1} - 3\binom{2n}{i+1-n} + 3\binom{n}{i+1-2n}$$

where $0 \leq i \leq 3n - 4$

Example 4.3.5.

The spanning forest complex $\Delta_s(3C_4)$ of the graph in the figure 4.6 is:

$\Delta_s(3C_4) = \langle\{\hat{e_{11}}, \hat{e_{21}}, \hat{e_{31}}\}, \{\hat{e_{11}}, \hat{e_{21}}, \hat{e_{32}}\}, \{\hat{e_{11}}, \hat{e_{21}}, \hat{e_{33}}\}, \{\hat{e_{11}}, \hat{e_{21}}, \hat{e_{34}}\}, \{\hat{e_{11}}, \hat{e_{22}}, \hat{e_{31}}\},$

$\{\hat{e_{11}}, \hat{e_{22}}, \hat{e_{32}}\}, \{\hat{e_{11}}, \hat{e_{22}}, \hat{e_{33}}\}, \{\hat{e_{11}}, \hat{e_{22}}, \hat{e_{34}}\}, \{\hat{e_{11}}, \hat{e_{23}}, \hat{e_{31}}\}, \{\hat{e_{11}}, \hat{e_{23}}, \hat{e_{32}}\}, \{\hat{e_{11}}, \hat{e_{23}}, \hat{e_{33}}\},$

$\{\hat{e_{11}}, \hat{e_{23}}, \hat{e_{34}}\}, \{\hat{e_{11}}, \hat{e_{24}}, \hat{e_{31}}\}, \{\hat{e_{11}}, \hat{e_{24}}, \hat{e_{32}}\}, \{\hat{e_{11}}, \hat{e_{24}}, \hat{e_{33}}\}, \{\hat{e_{11}}, \hat{e_{24}}, \hat{e_{34}}\}, \{\hat{e_{12}}, \hat{e_{21}}, \hat{e_{31}}\},$

$\{\hat{e_{12}}, \hat{e_{21}}, \hat{e_{32}}\}, \{\hat{e_{12}}, \hat{e_{21}}, \hat{e_{33}}\}, \{\hat{e_{12}}, \hat{e_{21}}, \hat{e_{34}}\}, \{\hat{e_{12}}, \hat{e_{22}}, \hat{e_{31}}\}, \{\hat{e_{12}}, \hat{e_{22}}, \hat{e_{32}}\}, \{\hat{e_{12}}, \hat{e_{22}}, \hat{e_{33}}\},$

$\{\hat{e_{12}}, \hat{e_{22}}, \hat{e_{34}}\}, \{\hat{e_{12}}, \hat{e_{23}}, \hat{e_{31}}\}, \{\hat{e_{12}}, \hat{e_{23}}, \hat{e_{32}}\}, \{\hat{e_{12}}, \hat{e_{23}}, \hat{e_{33}}\}, \{\hat{e_{12}}, \hat{e_{23}}, \hat{e_{34}}\}, \{\hat{e_{12}}, \hat{e_{24}}, \hat{e_{31}}\},$

$\{\hat{e_{12}}, \hat{e_{24}}, \hat{e_{32}}\}, \{\hat{e_{12}}, \hat{e_{24}}, \hat{e_{33}}\}, \{\hat{e_{12}}, \hat{e_{24}}, \hat{e_{34}}\}, \{\hat{e_{13}}, \hat{e_{21}}, \hat{e_{31}}\}, \{\hat{e_{13}}, \hat{e_{21}}, \hat{e_{32}}\}, \{\hat{e_{13}}, \hat{e_{21}}, \hat{e_{33}}\},$

$\{\hat{e_{13}}, \hat{e_{21}}, \hat{e_{34}}\}, \{\hat{e_{13}}, \hat{e_{22}}, \hat{e_{31}}\}, \{\hat{e_{13}}, \hat{e_{22}}, \hat{e_{32}}\}, \{\hat{e_{13}}, \hat{e_{22}}, \hat{e_{33}}\}, \{\hat{e_{13}}, \hat{e_{22}}, \hat{e_{34}}\}, \{\hat{e_{13}}, \hat{e_{23}}, \hat{e_{31}}\},$

$\{\hat{e_{13}}, \hat{e_{23}}, \hat{e_{32}}\}, \{\hat{e_{13}}, \hat{e_{23}}, \hat{e_{33}}\}, \{\hat{e_{13}}, \hat{e_{23}}, \hat{e_{34}}\}, \{\hat{e_{13}}, \hat{e_{24}}, \hat{e_{31}}\}, \{\hat{e_{13}}, \hat{e_{24}}, \hat{e_{32}}\}, \{\hat{e_{13}}, \hat{e_{24}}, \hat{e_{33}}\},$

$\{\hat{e_{13}}, \hat{e_{24}}, \hat{e_{34}}\}, \{\hat{e_{14}}, \hat{e_{21}}, \hat{e_{31}}\}, \{\hat{e_{14}}, \hat{e_{21}}, \hat{e_{32}}\}, \{\hat{e_{14}}, \hat{e_{21}}, \hat{e_{33}}\}, \{\hat{e_{14}}, \hat{e_{21}}, \hat{e_{34}}\}, \{\hat{e_{14}}, \hat{e_{22}}, \hat{e_{31}}\},$

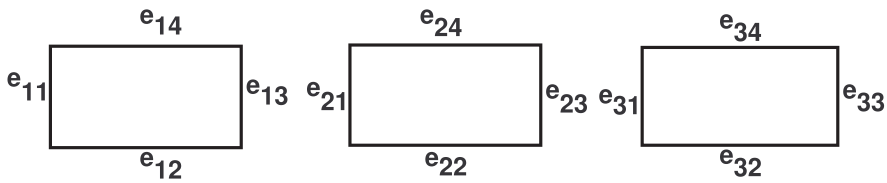

Figure 4.6: $3C_4$

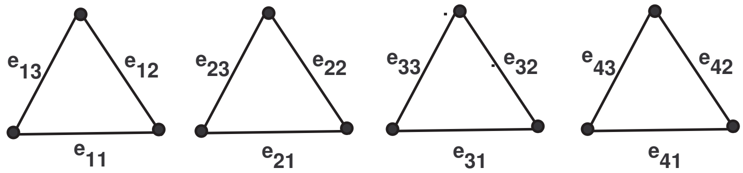

Figure 4.7: $4C_3$

$\{\hat{e_{14}}, \hat{e_{22}}, \hat{e_{32}}\}, \{\hat{e_{14}}, \hat{e_{22}}, \hat{e_{33}}\}, \{\hat{e_{14}}, \hat{e_{22}}, \hat{e_{34}}\}, \{\hat{e_{14}}, \hat{e_{23}}, \hat{e_{31}}\}, \{\hat{e_{14}}, \hat{e_{23}}, \hat{e_{32}}\}, \{\hat{e_{14}}, \hat{e_{23}}, \hat{e_{33}}\},$

$\{\hat{e_{14}}, \hat{e_{23}}, \hat{e_{34}}\}, \{\hat{e_{14}}, \hat{e_{24}}, \hat{e_{31}}\}, \{\hat{e_{14}}, \hat{e_{24}}, \hat{e_{32}}\}, \{\hat{e_{14}}, \hat{e_{24}}, \hat{e_{33}}\}, \{\hat{e_{14}}, \hat{e_{24}}, \hat{e_{34}}\}\rangle$

We calculate the f-vector by using the above theorem,

$$f = (12, 66, 220, 768, 840, 624, 288, 64)$$

Theorem 4.11.

Let $\Delta_s(4C_n)$ be the spanning forest complex of $4-$ *cyclic* graph $4C_n$ with cycles of same length n, then the $dim(\Delta_s(3C_n)) = 4n-5$ with f-vector

$$f_i = \binom{4n}{i+1} - 4\binom{3n}{i+1-n} + 6\binom{2n}{i+1-2n} - 4\binom{n}{i+1-3n}$$

where $0 \leq i \leq 4n-5$

Example 4.3.6.

The spanning forest complex $\Delta_s(4C_3)$ of the graph in the figure 4.6 is:

$\Delta_s(4C_3) = \langle \{\hat{e_{11}}, \hat{e_{21}}, \hat{e_{31}}, \hat{e_{41}}\}, \{\hat{e_{11}}, \hat{e_{21}}, \hat{e_{31}}, \hat{e_{42}}\}, \{\hat{e_{11}}, \hat{e_{21}}, \hat{e_{31}}, \hat{e_{43}}\}, \{\hat{e_{11}}, \hat{e_{21}}, \hat{e_{32}}, \hat{e_{41}}\},$

$\{\hat{e_{11}}, \hat{e_{21}}, \hat{e_{32}}, \hat{e_{42}}\}, \{\hat{e_{11}}, \hat{e_{21}}, \hat{e_{32}}, \hat{e_{43}}\}, \{\hat{e_{11}}, \hat{e_{21}}, \hat{e_{33}}, \hat{e_{41}}\}, \{\hat{e_{11}}, \hat{e_{21}}, \hat{e_{33}}, \hat{e_{42}}\}, \{\hat{e_{11}}, \hat{e_{21}}, \hat{e_{33}}, \hat{e_{43}}\},$

$\{\hat{e_{11}}, \hat{e_{22}}, \hat{e_{31}}, \hat{e_{41}}\}, \{\hat{e_{11}}, \hat{e_{22}}, \hat{e_{31}}, \hat{e_{42}}\}, \{\hat{e_{11}}, \hat{e_{22}}, \hat{e_{31}}, \hat{e_{43}}\}, \{\hat{e_{11}}, \hat{e_{22}}, \hat{e_{32}}, \hat{e_{41}}\}, \{\hat{e_{11}}, \hat{e_{22}}, \hat{e_{32}}, \hat{e_{42}}\},$

$\{\hat{e_{11}}, \hat{e_{22}}, \hat{e_{32}}, \hat{e_{43}}\}, \{\hat{e_{11}}, \hat{e_{22}}, \hat{e_{33}}, \hat{e_{41}}\}, \{\hat{e_{11}}, \hat{e_{22}}, \hat{e_{33}}, \hat{e_{42}}\}, \{\hat{e_{11}}, \hat{e_{22}}, \hat{e_{33}}, \hat{e_{43}}\}, \{\hat{e_{11}}, \hat{e_{23}}, \hat{e_{31}}, \hat{e_{41}}\},$

$\{\hat{e_{11}}, \hat{e_{23}}, \hat{e_{31}}, \hat{e_{42}}\}, \{\hat{e_{11}}, \hat{e_{23}}, \hat{e_{31}}, \hat{e_{43}}\}, \{\hat{e_{11}}, \hat{e_{23}}, \hat{e_{32}}, \hat{e_{41}}\}, \{\hat{e_{11}}, \hat{e_{23}}, \hat{e_{32}}, \hat{e_{42}}\}, \{\hat{e_{11}}, \hat{e_{23}}, \hat{e_{32}}, \hat{e_{43}}\},$

$\{\hat{e_{11}}, \hat{e_{23}}, \hat{e_{33}}, \hat{e_{41}}\}, \{\hat{e_{11}}, \hat{e_{23}}, \hat{e_{33}}, \hat{e_{42}}\}, \{\hat{e_{11}}, \hat{e_{23}}, \hat{e_{33}}, \hat{e_{43}}\}, \{\hat{e_{12}}, \hat{e_{21}}, \hat{e_{31}}, \hat{e_{41}}\}, \{\hat{e_{12}}, \hat{e_{21}}, \hat{e_{31}}, \hat{e_{42}}\},$

$\{\hat{e_{12}}, \hat{e_{21}}, \hat{e_{31}}, \hat{e_{43}}\}, \{\hat{e_{12}}, \hat{e_{21}}, \hat{e_{32}}, \hat{e_{41}}\}, \{\hat{e_{12}}, \hat{e_{21}}, \hat{e_{32}}, \hat{e_{42}}\}, \{\hat{e_{12}}, \hat{e_{21}}, \hat{e_{32}}, \hat{e_{43}}\}, \{\hat{e_{12}}, \hat{e_{21}}, \hat{e_{33}}, \hat{e_{41}}\},$

$\{\hat{e_{12}}, \hat{e_{21}}, \hat{e_{33}}, \hat{e_{42}}\}, \{\hat{e_{11}}, \hat{e_{21}}, \hat{e_{33}}, \hat{e_{43}}\}, \{\hat{e_{11}}, \hat{e_{22}}, \hat{e_{31}}, \hat{e_{41}}\}, \{\hat{e_{12}}, \hat{e_{22}}, \hat{e_{31}}, \hat{e_{42}}\}, \{\hat{e_{12}}, \hat{e_{22}}, \hat{e_{31}}, \hat{e_{43}}\},$

$\{\hat{e_{12}}, \hat{e_{22}}, \hat{e_{32}}, \hat{e_{41}}\}, \{\hat{e_{11}}, \hat{e_{22}}, \hat{e_{32}}, \hat{e_{42}}\}, \{\hat{e_{11}}, \hat{e_{22}}, \hat{e_{32}}, \hat{e_{43}}\}, \{\hat{e_{12}}, \hat{e_{22}}, \hat{e_{33}}, \hat{e_{41}}\}, \{\hat{e_{12}}, \hat{e_{22}}, \hat{e_{33}}, \hat{e_{42}}\},$

$\{\hat{e_{12}}, \hat{e_{22}}, \hat{e_{33}}, \hat{e_{43}}\}, \{\hat{e_{11}}, \hat{e_{23}}, \hat{e_{31}}, \hat{e_{41}}\}, \{\hat{e_{11}}, \hat{e_{23}}, \hat{e_{31}}, \hat{e_{42}}\}, \{\hat{e_{12}}, \hat{e_{23}}, \hat{e_{31}}, \hat{e_{43}}\}, \{\hat{e_{12}}, \hat{e_{23}}, \hat{e_{32}}, \hat{e_{41}}\},$

$\{\hat{e_{12}}, \hat{e_{23}}, \hat{e_{32}}, \hat{e_{42}}\}, \{\hat{e_{11}}, \hat{e_{23}}, \hat{e_{32}}, \hat{e_{43}}\}, \{\hat{e_{11}}, \hat{e_{23}}, \hat{e_{33}}, \hat{e_{41}}\}, \{\hat{e_{12}}, \hat{e_{23}}, \hat{e_{33}}, \hat{e_{42}}\}, \{\hat{e_{12}}, \hat{e_{23}}, \hat{e_{33}}, \hat{e_{43}}\},$

$\{\hat{e_{13}}, \hat{e_{21}}, \hat{e_{31}}, \hat{e_{41}}\}, \{\hat{e_{13}}, \hat{e_{21}}, \hat{e_{31}}, \hat{e_{42}}\}, \{\hat{e_{13}}, \hat{e_{21}}, \hat{e_{31}}, \hat{e_{43}}\}, \{\hat{e_{13}}, \hat{e_{21}}, \hat{e_{32}}, \hat{e_{41}}\}, \{\hat{e_{13}}, \hat{e_{21}}, \hat{e_{32}}, \hat{e_{42}}\},$

$\{\hat{e_{13}}, \hat{e_{21}}, \hat{e_{32}}, \hat{e_{43}}\}, \{\hat{e_{13}}, \hat{e_{21}}, \hat{e_{33}}, \hat{e_{41}}\}, \{\hat{e_{13}}, \hat{e_{21}}, \hat{e_{33}}, \hat{e_{42}}\}, \{\hat{e_{13}}, \hat{e_{21}}, \hat{e_{33}}, \hat{e_{43}}\}, \{\hat{e_{13}}, \hat{e_{22}}, \hat{e_{31}}, \hat{e_{41}}\},$

$\{\hat{e_{13}}, \hat{e_{22}}, \hat{e_{31}}, \hat{e_{42}}\}, \{\hat{e_{13}}, \hat{e_{22}}, \hat{e_{31}}, \hat{e_{43}}\}, \{\hat{e_{13}}, \hat{e_{22}}, \hat{e_{32}}, \hat{e_{41}}\}, \{\hat{e_{13}}, \hat{e_{22}}, \hat{e_{32}}, \hat{e_{42}}\}, \{\hat{e_{13}}, \hat{e_{22}}, \hat{e_{32}}, \hat{e_{43}}\},$

$\{\hat{e_{13}}, \hat{e_{22}}, \hat{e_{33}}, \hat{e_{41}}\}, \{\hat{e_{13}}, \hat{e_{22}}, \hat{e_{33}}, \hat{e_{42}}\}, \{\hat{e_{13}}, \hat{e_{22}}, \hat{e_{33}}, \hat{e_{43}}\}, \{\hat{e_{13}}, \hat{e_{23}}, \hat{e_{31}}, \hat{e_{41}}\}, \{\hat{e_{13}}, \hat{e_{23}}, \hat{e_{31}}, \hat{e_{42}}\},$

$\{\hat{e_{13}}, \hat{e_{23}}, \hat{e_{31}}, \hat{e_{43}}\}, \{\hat{e_{13}}, \hat{e_{23}}, \hat{e_{32}}, \hat{e_{41}}\}, \{\hat{e_{13}}, \hat{e_{23}}, \hat{e_{32}}, \hat{e_{42}}\}, \{\hat{e_{13}}, \hat{e_{23}}, \hat{e_{32}}, \hat{e_{43}}\}, \{\hat{e_{13}}, \hat{e_{23}}, \hat{e_{33}}, \hat{e_{41}}\},$

$\{\hat{e_{13}}, \hat{e_{23}}, \hat{e_{33}}, \hat{e_{42}}\}, \{\hat{e_{13}}, \hat{e_{23}}, \hat{e_{33}}, \hat{e_{43}}\}\rangle$

We calculate the f-vector by using the theorem mentioned above i.e.,

$$f = (12, 66, 216, 455, 648, 696, 324, 81)$$

Now we provide the general expression for the f-vector of any number of copies of C_n.

Theorem 4.12.

Let $\Delta_s(tC_n)$ be the spanning forest complex of $t-$ *cycle* graph tC_n, then the

$dim(\Delta_s(tC_n)) = t(n-1)-1$ with f-vector $f(\Delta_s(tC_n)) = (f_0, f_1, \ldots, f_{t(n-1)-1})$, where,

$$f_i = \binom{tn}{i+1} + \sum_{k=1}^{t-1}(-1)^k\binom{t}{k}\binom{(t-k)n}{i+1-kn}$$

where $0 \le i \le t(n-1)-1$

Proof.

In 4.8 it has already been shown that $dim = t(n-1)-1$.

For f-vector, let E be the edge set of $t(C_n)$ and T_i be the total number of subsets of E of order $i+1$. Let F be any subset of E of order $i+1$ such that F is cycle free i.e. F does not contain any C_n. Now f_i be the total number of such cycle free subset F of E. Let F_{ij} be a subset of E of order $i+1$ such that it contains j number of C_n and $T_{F_{ij}}$ be the total number of such F_{ij}.

Let us recall the Inclusion Exclusion Principle that states for finite sets $S_1, S_2, ..., S_n$,

$$\left|\bigcup_{i=1}^{n} S_i\right| = \sum_{k=1}^{n}(-1)^{k+1}\left(\sum_{1\le i_1\le ....\le i_k\le n}|S_{i_1}\cap ...\cap S_{i_k}|\right)$$

So we have:

$$f_i = T_i - T_{F_{i1}} + T_{F_{i2}} - \cdots + (-1)^{t-1}T_{F_{i(t-1)}}$$

$\Rightarrow$

$$\begin{aligned} f_i &= \binom{tn}{i+1} - \binom{t}{1}\binom{(t-1)n}{i+1-n} + \binom{t}{2}\binom{(t-2)n}{i+1-2n} - \binom{t}{3}\binom{(t-3)n}{i+1-3n} \\ &+ \cdots + (-1)^{t-1}\binom{t}{t-1}\binom{n}{i+1-(t-1)n} \end{aligned}$$

This implies the final result i.e.

$$f_i = \binom{tn}{i+1} + \sum_{k=1}^{t-1}(-1)^k\binom{t}{k}\binom{(t-k)n}{i+1-kn}$$

Theorem 4.13.

Let $\Delta_s(tC_n)$ be a spanning forest complex of tC_n and $h(\Delta_s(tC_n))$ be the h-vector of $k[\Delta_s(tC_n)]$, then

$$h_k = \sum_{i=0}^{k}(-1)^{k-i}\binom{n-1-i}{k-i}\left\{\binom{tn}{i} + \sum_{k=1}^{t-1}(-1)^k\binom{t}{k}\binom{(t-k)n}{i-kn}\right\}$$

for $0 \leq k \leq t(n-1)$. and

$$h_k = 0 \;\; for \;\; k > t(n-1)$$

Proof.

Let $\Delta_s(tC_n)$ be a spanning forest complex of tC_n and $h(\Delta_s(tC_n))$ be the h-vector of $k[\Delta_s(tC_n)]$, then from [31], for a simplicial complex Δ of the dimension d, $h_k = 0$ for $k > d+1$ and

$$h_k = \sum_{i=0}^{k}(-1)^{k-1}\binom{n-1-i}{k-i}f_{i-1} \quad \text{for } 0 \leq k \leq d+1$$

So by using the results from Theorem 4.8 and Theorem 4.12 in the above expression, we have

$$h_k = \sum_{i=0}^{k}(-1)^{k-i}\binom{n-1-i}{k-i}\left\{\binom{tn}{i} + \sum_{k=1}^{t-1}(-1)^k\binom{t}{k}\binom{(t-k)n}{i-kn}\right\}$$

for $0 \leq k \leq t(n-1)$. and

$$h_k = 0 \;\; \text{for} \;\; k > t(n-1)$$

Theorem 4.14.

Let $\Delta_s(tC_n)$ be a spanning forest complex of (tC_n) and $H(k[(\Delta_s(tC_n),t)])$ be the Hilbert series of Stanley-Reisner ring $k[\Delta_s(tC_n)]$ then

$$\begin{aligned} H(k[(\Delta_s(tC_n),t)]) &= 1 + \sum_{i=0}^{t(n-1)-1}\binom{tn}{i+1}\frac{t^{i+1}}{(1-t)^{i+1}} \\ &+ \sum_{i=0}^{t(n-1)-1}\sum_{k=1}^{t-1}(-1)^k\binom{t}{k}\binom{(t-k)n}{i+1-kn}\binom{(t-k)n}{i+1-kn} \end{aligned}$$

Proof.

Let Δ be d dimensional simplicial complex with f-vector

$$f(\Delta) = (f_0, ..., f_d)$$

We know from [31] that the Hilbert series of Stanley-Reisner ring $k[\Delta]$ is

$$H(k[\Delta], t) = 1 + \sum_{i=0}^{d} \frac{f_i t^{i+1}}{(i-d)^{i+1}}$$

which provides the required result by Theorem 4.8 and Theorem 4.12.

Chapter 5

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