Solution of Differential Equations by New Homotopy Perturbation Method

By

Muhammad Azeem

CIIT/FA14-BSMATH-009/LHR

BS

In

Mathematics

COMSATS University Islamabad

Lahore Campus

Spring, 2018

COMSATS University Islamabad

Solution of Differential Equations by New Homotopy Perturbation Method

In partial fulfillment

of the requirement for the degree of

BS (Mathematics)

By

Muhammad Azeem

CIIT/FA14-BSMATH-009/LHR

COMSATS University Islamabad

Lahore Campus

Spring, 2018

Solution of Differential Equations by New Homotopy Perturbation Method

A Final Year Report is submitted to the Department of Mathematics as partial fulfillment of the requirement for the award of Degree of BS Mathematics.

Name	Registration Number
Muhammad Azeem	CIIT/FA14-BSMATH-009/LHR

Supervisor

Dr. Mohsan Hassan
Assistant Professor, Department of Mathematics
COMSATS University Islamabad, Lahore Campus
25 May, 2018

Final Approval

This report titled

Solution of Differential Equations by New Homotopy Perturbation Method

By

Muhammad Azeem

CIIT/FA14-BSMATH-009/LHR

Has been approved

For the COMSATS University Islamabad,Lahore Campus

ExternalExaminer:

Dr.

Supervisor:

Dr. Mohsan Hassan

Department of Mathematics /CUI, Lahore Campus

HoD:

Dr. Sarfaraz Ahmad

Department of Mathematics, Lahore Campus

Declaration

I Muhammad Azeem registration number CIIT/FA14-BSMATH-009/LHR hereby declare that I have produced the work presented in this report, during the scheduled period of study.

Date: 25 May, 2018

Muhammad Azeem
CIIT/FA14-BSMATH-009/LHR

Certificate

It is certified that Muhammad Azeem(CIIT/FA14-BSMATH-009/LHR) has carried out all the work related to this report under my supervision at the Department of Mathematics, COMSATS University Islamabad, Lahore Campus, and the work fulfills the requirement for award of BS degree.

Date:__________________

Supervisor:

____

Dr. Mohsan Hassan
Assistant Professor

Head of Department:

____

Associate Professor
Department ofMathematics

ACKNOWLEDGEMENTS

In the name of Allah Almighty, the most Beneficent, the most Merciful. First and foremost, I want to thank Allah Almighty for giving me the strength to complete my B.S program. Afterwards, I would like to express my gratitude to my respected supervisor Dr. Mohsan Hassan for the continuous support of my B.S study and research. This project would not have been completed without his effort, motivation and generous guidance. It was a pleasure working with him. I would also like to thank our head of mathematics department, Dr. Sarfaraz Ahmad and all faculty members for their guidance during my B.S Course work. I would like to thank my B.S coordinator Dr. Hani Shaker for his coordination in my four year program. I would like to thanks my friends Muhammad Azeem, Adnan Khalil and Shahzeb Maqsood for their help and motivation. Finally yet importantly, I want to give my special thanks to my parents for their concern and moral support encouraged me during my studies.

Muhammad Azeem

CIIT/FA14-BSMATH-009/LHR

Abstract

In this project, a new modification of Homotopy perturbation method (HPM), called New Homotopy Perturbation Method (NHPM), has been applied for solving systems of non-linear partial differential equations. In chapter 2 we solved differential equations by HPM and in chapter 3 we solved same differential equations by NHPM. HPM and NHPM methods have been applied to three examples successfully and achieved approximate solutions. The basic idea described in this project is strong enough to be employed to solve other differential equations. The computations associated with the examples were performed using Mathematica 9.

Contents

Chapter 1

Introduction

1.1Homotopy Perturbation Method

In recent years, the Homotopy perturbation method (HPM), first proposed by dr. Ji Huan He [1], has successfully been applied to solve many types of linear and nonlinear functional equations. This method, which is a combination of Homotopy in topology and classic perturbation techniques, provides us with a convenient way to obtain analytic or approximate solutions for a wide variety of problems arising in different fields. Dr. He used HPM to solve Lighthill equation[2], Duffing equation[3] and Blasius equation[4], and then the idea found its way in sciences and has been used to solve nonlinear wave equations[5], boundary value problems[6], quadratic Riccati differential equations[7], integral equations[8], Klein Gordon[9] and sine Gordon equations[10], initial value problems[11], Schrödinger equation[12], Emden Fowler type equations[13], nonlinear evolution equations[14], differential-difference equations[15], modified KdV equation[16] and many other problems. This wide variety of applications shows the power of HPM in solving functional equation. Studying the method, we understand that the idea is straightforward, but everyone has solved his/her own problem.

1.1.1 Basic Idea of Homotopy Perturbation Method

Homotopy perturbation method (HPM) is a novel and effective method, and can solve various nonlinear equations. To illustrate the basic ideas of the HPM, we consider the following general nonlinear differential equation:

$$A(y) - f(r) = 0 \tag{1.1}$$

with boundary conditions

$$B\left(y, \frac{\partial y}{\partial r}\right) = 0, \quad r \in \lambda \tag{1.2}$$

Where A is a general differential operator, B is a boundary operator, f(r) is a known analytic function, and λ is the boundary of the domain Ω.The operator A can be generally divided into two parts L and N, where L is linear while N is non-linear. Therefore, equation (1.1) can be written as follows:

$$L(y) + N(y) - f(r) = 0 \tag{1.3}$$

We construct homotopy of equation (1.1)

$$y(r, p) : \Omega \times [0,1] \to \mathrm{A} \tag{1.4}$$

$$H(y,p) = (1-p)\left[L(y) + L(y_0)\right] + p[A(y) - f(r)\right] = 0, \quad r \in \Omega \tag{1.5}$$

Where $p \in [0,1]$ is an embedding parameter and y_0 is an initial approximation which satisfies the boundary conditions. From equation (1.4) and (1.5)

$$H(y,0) = L(y) - L(y_0) = 0 \tag{1.6}$$

$$H(y,1) = A(y) - f(r) = 0 \tag{1.7}$$

Thus the changing process from of p 0 to 1 is just that of $y(r,p)$ from $y_0(r)$ to $y(r)$. In topology this is called deformation and $L(y) - L(y_0)$ and $A(y) - f(r)$ are called homotopy. Here the embedding parameter is introduced much more natural, unaffected by artificial factors, further it can be considered as small parameter, for $0 \le p \le 1$. So it is very natural to assume that the solution of equation (1.4) and (1.5) can be expressed as:

$$y(x) = u_0(x) + pu_1(x) + p^2u_2(x) + \ldots \tag{1.8}$$

According to HPM, the approximate solution of (1.5) can be expressed as a series of the power of p[17]

$$y = \lim_{p \to 1} it(y) = u_0 + u_1 + u_2 + \ldots \tag{1.9}$$

1.2New Homotopy Perturbation Method

The new homotopy perturbation method(NHPM) is a modified homotopy perturbation method (HPM).The new homotopy perturbation method (NHPM) was proposed by Biazar an Eslami [18] and has same application as HPM.

1.2.1Basic Idea of New Homotopy Perturbation Method

The general form of a system of PDEs can be considered as the following:

$$\frac{\partial U_j}{\partial t} + N_j(x_1, x_2, \ldots, x_{n-1}, t, u_1, \ldots, u_n) = g_j(x_1, x_2, \ldots, x_n, t), \quad j = 1, \ldots, n \tag{1.10}$$

with the following initial conditions,

$$U_j(x_1, x_2, \ldots, x_{n-1}, t) = f_j(x_1, x_2, \ldots, x_{n-1}), \; j = 1, \ldots, n \tag{1.11}$$

Here $N_1,...,N_n$ are non-linear operators, which usually depend on the functions u_i and their derivatives, and $g_{1,}g_2,...,g_n$ are inhomogeneous terms.

For solving (1.10), by using NHPM, we construct the following homotopy:

$$(1-q)\left(\frac{\partial U_j}{\partial t}-uj,0\right)+q\left(\begin{array}{l}\frac{\partial U_j}{\partial t}\\ +N_j\left(x_1,x_2,...,x_{n-1},t,U_1,U_2,...,U_n\right)-g_j\end{array}\right)=0,\ \ j=1,2,...,n \quad (1.12)$$

or

$$\frac{\partial U_j}{\partial t}=u_{j0}-q\left(u_{j0}+N_j\left(x_1,x_2,...,x_{n-1},U_1,U_2,...,U_n\right)-g_j\right) \quad (1.13)$$

Applying the inverse operator $L^{-1}=\int(.)\,dt$ to both sides of Eq. (1.13), we obtain

$$\begin{aligned}U_j\left(x_1,x_2,...,x_{n-1},t\right)&=U_j\left(x_1,x_2,...,x_{n-1},t_0\right)+\int_{t_0}^{t}u_{j0}dt\\ &-p\int_{t_0}^{t}\left(u_j+N_j\left(x_1,x_2,...,x_{n-1},t,U_j,...,U_n\right)-g_j\right)dt\end{aligned} \quad (1.14)$$

where

$$U_j\left(x_1,x_2,...,x_{n-1},t_0\right)=u_j\left(x_1,x_2,...,x_{n-1},t_0\right) \quad (1.15)$$

$$U_j=U_{j0}+qU_{j1}+q^2U_{j2}..., \quad (1.16)$$

where $U_{ij},\ i=1,2,3,...,n,\ \ j=1,2,3,...,n$ are functions which should be determined. Suppose that the initial approximations of the solutions of Eq. (1.10) are in the following form:

$$U_{j0}\left(x_1,x_2,...,x_{n-1},t\right)=\sum_{j=0}^{\infty}a_{i,j}\left(x_1,x_2,...,x_{n-1}\right)p_j\left(t\right),i=1,2,...,n \quad (1.17)$$

here $a_{i,j}\left(x_1,x_2,...,x_{n-1}\right),\ i=1,2,3,...,n,\ j=1,2,3,...,n$ are unknown coefficients $p_o\left(t\right), p_1\left(t\right)...$ are specific functions. Substituting (1.16) and (6) into (1.14) and equating the coefficients of p with the same powers leads to

$$
\left.\begin{array}{l}
p^0: \;\; U_{i,0}\left(x_1, x_2, ..., x_{n-1}, t\right) = f_1\left(x_1, x_2, ..., x_{n-1}\right) + \sum_{j=0}^{\infty} a_{ij} \int_{t_0}^{t} p_j\left(t\right) dt \\
p^1: \;\; U_{i,1}\left(x_1, x_2, ..., x_{n-1}, t\right) = \\
\quad -\sum_{j=0}^{\infty} a_{ij} \int_{t_0}^{t} p_j\left(t\right) dt - \int_{t_0}^{t} \left(N_i\left(x_1, x_2, ..., x_{n-1}, t, U_{1,0}, U_{2,0}, ..., , U_{n,0}\right) g_j\right) dt \\
p^2: \;\; U_{i,2}\left(x_1, x_2, ..., x_{n-1}, t\right) = \\
\quad -\int_{t_0}^{t} \left(N_i\left(x_1, x_2, ..., x_{n-1}, t, U_{1,0}, U_{2,0}, ..., , U_{n,0}, U_{1,1}, U_{2,1}, ..., , U_{n,1}\right)\right) dt \\
\quad \vdots \\
p^j: \;\; U_{i,j}\left(x_1, x_2, ..., x_{n-1}, t\right) = \\
\quad -\int_{t_0}^{t} \left(N_i\left(x_1, x_2, ..., x_{n-1}, t, U_{1,0}, U_{2,0}, ..., , U_{n,0}, ..., U_{1,j-1}, ..., , U_{n,j-1}\right)\right) dt
\end{array}\right\} \qquad (1.18)
$$

Now if we solve these equations in such a way that $U_{i,1}\left(x_1, x_2, ..., x_{n-1}, t\right) = 0$ then Eq. (1.18) yields

$$
U_{i,2}\left(x_1, x_2, ..., x_{n-1}, t\right) = U_{i,3}\left(x_1, x_2, ..., x_{n-1}, t\right) = ... = 0 \qquad (1.19)
$$

Therefore, the exact solution may be obtained as the following:

$$
\left.\begin{array}{l}
u_i\left(x_1, x_2, ..., x_{n-1}, t\right) = U_{i,0}\left(x_1, x_2, ..., x_{n-1}, t\right) = \\
f_i\left(x_1, x_2, ..., x_{n-1}\right) + \sum_{j=0}^{\infty} a_{ij} \int_{t_0}^{t} p_j\left(t\right) dt, \quad \text{i=1,2,...,n}
\end{array}\right\} \qquad (1.20)
$$

It is worth mentioning that if $g_j\left(x_1, x_2, ..., x_{n-1}, t\right)$ and $u_{i,0}\left(x_1, x_2, ..., x_{n-1}, t\right)$ are analytic around $t = t_0$ then their Taylor series can be defined as:

$$
u_{i,0}\left(x_1, x_2, ..., x_{n-1}, t\right) = \sum_{j=0}^{\infty} a_{ij}\left(x_1, x_2, ..., x_n\right)\left(t - t_0\right) \qquad (1.21)
$$

$$
g_j\left(x_1, x_2, ..., x_{n-1}, t\right) = \sum_{j=0}^{\infty} a^*_{ij}\left(x_1, x_2, ..., x_n\right)\left(t - t_0\right) \qquad (1.22)
$$

Chapter 2

Solution of Differential Equations by Homotopy Perturbation Method

2.1 Example 1

Consider the following system of three-dimensional partial differential equations:

$$\left.\begin{aligned}\frac{\partial u}{\partial t}-v\frac{\partial u}{\partial x}-\frac{\partial v}{\partial t}\frac{\partial u}{\partial y}=1-x+y+t\\ \frac{\partial v}{\partial t}-u\frac{\partial v}{\partial x}-\frac{\partial u}{\partial t}\frac{\partial v}{\partial y}=1-x-y-t\end{aligned}\right\} \tag{2.1}$$

$$u(x,y,0)=x+y-1$$
$$v(x,y,0)=x-y+1$$

Solution:

First we apply the HPM method.

Suppose our linear operator is U_t and V_t.

Homotopy equation is following as:

$$\left.\begin{aligned}(1-q)(U_t-u_0)+q\left(U_t-VU_x-V_tU_y-1+x-y-t\right)=0\\ (1-q)(V_t-v_0)+q\left(V_t-UV_x-U_tV_y-1+x+y+t\right)=0\end{aligned}\right\} \tag{2.2}$$

as we know that,

$$\left.\begin{aligned}U(x,y,t)=u_0+qu_1+q^2u_2+q^3u_3+...\\ V(x,y,t)=v_0+qv_1+q^2v_2+q^3v_3+...\end{aligned}\right\} \tag{2.3}$$

Substituting eq. (2.3) in eq. (2.2)

$$\begin{aligned}&(1-q)\left(u_{0t}+qu_{1t}+q^2u_{2t}+q^3u_{3t}-u_0\right)+\\ &q\left\{\begin{aligned}&\left(u_{0t}+qu_{1t}+q^2u_{2t}+q^3u_{3t}\right)-\\ &\left(v_0+qv_1+q^2v_2+q^3v_3\right)\left(u_{0x}+qu_{1x}+q^2u_{2x}+q^3u_{3x}\right)\\ &-\left(v_{0t}+qv_{1t}+q^2v_{2t}+q^3v_{3t}\right)\left(u_{0y}+qu_{1y}+q^2u_{2y}+q^3u_{3y}\right)\\ &-1+x-y-t\end{aligned}\right\}\end{aligned} \tag{2.4}$$

$$\begin{aligned}&(1-q)\left(v_{0t}+qv_{1t}+q^2v_{2t}+q^3v_{3t}-v_0\right)+\\ &q\left\{\begin{aligned}&\left(v_{0t}+qv_{1t}+q^2v_{2t}+q^3v_{3t}\right)-\\ &\left(u_0+qu_1+q^2u_2+q^3u_3\right)\left(v_{0x}+qv_{1x}+q^2v_{2x}+q^3v_{3x}\right)\\ &-\left(u_{0t}+qu_{1t}+q^2u_{2t}+q^3u_{3t}\right)\left(v_{0y}+qv_{1y}+q^2v_{2y}+q^3v_{3y}\right)\\ &-1+x+y+t\end{aligned}\right\}=0\end{aligned} \tag{2.5}$$

Comparing identical powers of q :

$$q^0 : \begin{cases} u_{0t} - u_0 = 0 \\ v_{0t} - v_0 = 0 \end{cases} \tag{2.6}$$

$$q^1 : \begin{cases} u_{1t} + u_0 - v_0 u_{0x} - v_{0t} u_{0y} - 1 + x - y - t = 0 \\ v_{1t} + v_0 - u_0 v_{0x} - u_{0t} v_{0y} - 1 + x + y + t = 0 \end{cases} \tag{2.7}$$

$$q^2 : \begin{cases} u_{2t} - v_0 u_{1y} - v_1 u_{0x} - v_{0t} u_{1y} - v_{1t} u_{0y} = 0 \\ v_{2t} - u_0 v_{1x} - u_1 v_{0x} - u_{0t} v_{1y} - u_{1t} v_{0y} = 0 \end{cases} \tag{2.8}$$

$$q^3 : \begin{cases} u_{3t} - v_0 u_{2x} - v_1 u_{1x} - v_2 u_{0x} - v_{0t} u_{2y} - v_{1t} u_{1y} - v_{2t} u_{0y} = 0 \\ v_{3t} - u_0 v_{2x} - u_1 v_{1x} - u_2 v_{0y} - u_{0t} v_{2y} - u_{1t} v_{1y} - u_{2t} v_{0y} = 0 \end{cases} \tag{2.9}$$

By solving the above equations, we get:

$$\left. \begin{aligned} U_0 &= x + y - 1 \\ V_0 &= x - y + 1 \end{aligned} \right\} \tag{2.10}$$

$$\left. \begin{aligned} U_1 &= 2t + \frac{1}{2} t^2 \\ V_1 &= -\frac{1}{2} t^2 \end{aligned} \right\} \tag{2.11}$$

$$\left. \begin{aligned} U_2 &= -\frac{1}{6} t^3 - \frac{1}{2} t^2 \\ V_2 &= \frac{1}{2} t^2 + \frac{1}{6} t^3 - 2t \end{aligned} \right\} \tag{2.12}$$

$$\left. \begin{aligned} U_3 &= \frac{1}{3} t^3 + \frac{1}{24} t^4 - \frac{1}{2} t^2 - 2t \\ V_3 &= -\frac{1}{24} t^4 + \frac{1}{2} t^2 \end{aligned} \right\} \tag{2.13}$$

Therefore, the solution will be as follows:

$$\left. \begin{aligned} u &= -1 + x + y + 2t - \frac{1}{5040} t^7 + \frac{1}{40} t^5 - \frac{1}{2} t^3 + \ldots \\ v &= 1 + x - y + 2t + \frac{1}{5040} t^7 - \frac{1}{40} t^5 + \frac{1}{2} t^3 + \ldots \end{aligned} \right\} \tag{2.14}$$

2.2 Example 2

Consider the following system of two non-linear equation:

$$\left.\begin{aligned}\frac{\partial u}{\partial x}-v\frac{\partial u}{\partial t}+u\frac{\partial v}{\partial t}&=-1+e^{x}\sin t\\ \frac{\partial v}{\partial x}-\frac{\partial u}{\partial t}\frac{\partial v}{\partial x}+\frac{\partial v}{\partial t}\frac{\partial u}{\partial x}&=-1-e^{x}\cos t\end{aligned}\right\} \tag{2.15}$$

$$u(0,t)=\sin t$$
$$v(0,t)=\cos t$$

Solution

According to the homotopy perturbation method, we have following homotopies, When linear operator is U_x and V_x:

$$\left.\begin{aligned}(1-q)(U_x-u_0)+q\left(U_x-VU_t+UV_t+1-e^{x}\sin(t)\right)=0\\ (1-q)(V_x-v_0)+q\left(V_x-U_tV_x+V_tU_x+1+e^{-x}\cos(t)\right)=0\end{aligned}\right\} \tag{2.16}$$

as

$$\left.\begin{aligned}U&=u_0+qu_1+q^2u_2+q^3u_3+...\\ V&=v_0+qv_1+q^2v_2+q^3v_3+...\\ U_x&=u_{0x}+qu_{1x}+q^2u_{2x}+q^3u_{3x}+...\\ U_t&=u_{0t}+qu_{1t}+q^2u_{2t}+q^3u_{3t}+...\\ V_x&=v_{0x}+qv_{1x}+q^2v_{2x}+q^3v_{3x}+...\\ V_t&=v_{0t}+qv_{1t}+q^2v_{2t}+q^3v_{3t}+...\end{aligned}\right\} \tag{2.17}$$

By substituting eq. (2.17) in eq. (2.16):

$$(1-q)\left(u_{0x}+qu_{1x}+q^2u_{2x}+q^3u_{3x}-u_0\right)$$
$$+q\begin{pmatrix}u_{0x}+qu_{1x}+q^2u_{2x}+q^3u_{3x}-\\ \left(v_0+qv_1+q^2v_2+q^3v_3+\right)\left(u_{0t}+qu_{1t}+q^2u_{2t}+q^3u_{3t}\right)+\\ \left(u_0+qu_1+q^2u_2+q^3u_3\right)\left(v_{0t}+qv_{1t}+q^2v_{2t}+q^3v_{3t}\right)+1-e^{x}\sin(t)\end{pmatrix}=0 \tag{2.18}$$

$$(1-q)\left(v_{0x}+qv_{1x}+q^2v_{2x}+q^3v_{3x}-v_0\right)+$$
$$+q\begin{pmatrix}v_{0x}+qv_{1x}+q^2v_{2x}+q^3v_{3x}-\\ \left(u_{0t}+qu_{1t}+q^2u_{2t}+q^3u_{3t}\right)\left(v_{0x}+qv_{1x}+q^2v_{2x}+q^3v_{3x}\right)+\\ \left(v_{0t}+qv_{1t}+q^2v_{2t}+q^3v_{3t}\right)\left(u_{0x}+qu_{1x}+q^2u_{2x}+q^3u_{3x}\right)+\\ 1+e^{-x}\cos(t)\end{pmatrix}=0 \tag{2.19}$$

By comparing identical powers of q:

$$q^0:\begin{cases} u_{0x}-u_0=0 \\ v_{0x}-v_0=0 \end{cases} \tag{2.20}$$

$$q^1:\begin{cases} u_{1x}+u_0+v_0u_{0t}+u_0v_{0t}+1-e^x\sin(t)=0 \\ v_{1x}+v_0+u_{0t}v_{0x}+v_{0t}u_{0x}+1+e^{-x}\cos(t)=0 \end{cases} \tag{2.21}$$

$$q^2:\begin{cases} u_{2x}+v_0u_{1t}+v_1u_{0t}+u_0v_{1t}+u_1v_{0t}=0 \\ v_{2x}-u_{0t}v_{1x}+u_{1t}v_{0x}+v_{0t}u_{1x}+v_{1t}u_{0x}=0 \end{cases} \tag{2.22}$$

By solving the above system of differential equations, we will get the following results:

$$\left.\begin{aligned} U_0&=\sin t \\ V_0&=\cos t \end{aligned}\right\} \tag{2.23}$$

$$\left.\begin{aligned} U_1&=e^x\sin t-\sin t \\ V_1&=-x+e^{-x}\cos t-\cos t \end{aligned}\right\} \tag{2.24}$$

$$\left.\begin{aligned} U_2&=e^x-e^{-x}-2x-\frac{1}{2}x^2\cos t \\ V_2&=-e^{-x}\cos^2 t+e^{x^2}\sin^2 t+2\cos{}^2 t+x\cos t-1 \end{aligned}\right\} \tag{2.25}$$

$$\vdots$$

Therefore the solution will be as follows:

$$\left.\begin{aligned} u&=e^x\sin t+e^x-e^{-x}-2x-\frac{1}{2}x^2\cos t+\ldots \\ v&=-e^{-x}\cos^2 t+e^x\sin^2 t+e^{-x}\cos t+2\cos^2 t+x\cos t-x-1\ldots \end{aligned}\right\}. \tag{2.26}$$

2.3 Example 3

Consider the following non-linear system of inhomogeneous partial differential equations:

$$\left.\begin{aligned} &\frac{\partial u}{\partial t}-\frac{\partial w}{\partial x}\frac{\partial v}{\partial t}-\frac{1}{2}\frac{\partial w}{\partial t}\frac{\partial^2 u}{\partial x^2}=-4xt \\ &\frac{\partial v}{\partial t}-\frac{\partial w}{\partial t}\frac{\partial^2 u}{\partial x^2}=6t \\ &\frac{\partial w}{\partial t}-\frac{\partial^2 u}{\partial x}-\frac{\partial v}{\partial x}\frac{\partial w}{\partial t}=4xt-2t-2 \end{aligned}\right\} \tag{2.27}$$

Subject to the initial conditions

$$u(x,0)=x^2+1$$
$$v(x,0)=x^2-1$$
$$w(x,0)=x^2-1$$

Solution:

By using Homotopy Perturbation method.

Suppose our linear operator be U_t, V_t and W_t .

So our Homotopy equation will be as follows:

$$\left.\begin{array}{l}(1-q)(U_t-u_0)+q\left(U_t-W_xV_t-\frac{1}{2}W_tU_{xx}+4xt\right)=0\\(1-q)(V_t-u_0)+q(V_t-W_tU_{xx}-6t)=0\\(1-q)(W_t-u_0)+q(W_t-U_{xx}-V_{xx}W_t-4xt+2t+2)=0\end{array}\right\} \tag{2.28}$$

as,

$$\begin{aligned}U&=u_0+qu_1+q^2u_2+q^3u_3+\ldots\\V&=v_0+qv_1+q^2v_2+q^3v_3+\ldots\\W&=w_0+qw_1+q^2w_2+q^3w_3+\ldots\\U_x&=u_{0x}+qu_{1x}+q^2u_{2x}+q^3u_{3x}+\ldots\\U_{xx}&=u_{0xx}+qu_{1xx}+q^2u_{2xx}+q^3u_{3xx}+\ldots\\U_t&=u_{0t}+qu_{1t}+q^2u_{2t}+q^3u_{3t}+\ldots\\V_x&=v_{0x}+qv_{1x}+q^2v_{2x}+q^3v_{3x}+\ldots\\V_t&=v_{0t}+qv_{1t}+q^2v_{2t}+q^3v_{3t}+\ldots\\W_x&=w_{0x}+qw_{1x}+q^2w_{2x}+q^3w_{3x}+\ldots\\W_t&=w_{0t}+qw_{1t}+q^2w_{2t}+q^3w_{3t}+\ldots\end{aligned} \tag{2.29}$$

So

$$\begin{aligned}&(1-q)\begin{pmatrix}u_{0t}+qu_{1t}\\+q^2u_{2t}+q^3u_{3t}-u_0\end{pmatrix}\\&+q\begin{pmatrix}\left(u_{0t}+qu_{1t}+q^2u_{2t}+q^3u_{3t}\right)-\\\left(w_{0x}+qw_{1x}+q^2w_{2x}+q^3w_{3x}\right)\left(v_{0t}+qv_{1t}+q^2v_{2t}+q^3v_{3t}\right)\\-\frac{1}{2}\begin{pmatrix}w_{0t}+qw_{1t}+\\q^2w_{2t}+q^3w_{3t}\end{pmatrix}\begin{pmatrix}u_{0xx}+qu_{1xx}+\\q^2u_{2xx}+q^3u_{3xx}\end{pmatrix}+4xt\end{pmatrix}=0\end{aligned} \tag{2.30}$$

$$(1-q)\begin{pmatrix} v_{0t}+qv_{1t}+ \\ q^2v_{2t}+q^3v_{3t}-v_0 \end{pmatrix}+ \\ q\begin{pmatrix} \begin{pmatrix} v_{0t}+qv_{1t}+ \\ q^2v_{2t}+q^3v_{3t} \end{pmatrix}-\begin{pmatrix} w_{0t}+qw_{1t}+ \\ q^2w_{2t}+q^3w_{3t} \end{pmatrix} \\ \left(u_{0xx}+qu_{1xx}+q^2u_{2xx}+q^3u_{3xx}\right)-6t \end{pmatrix}=0 \tag{2.31}$$

$$(1-q)\begin{pmatrix} w_{0t}+qw_{1t}+ \\ q^2w_{2t}+q^3w_{3t}-w_0 \end{pmatrix}+ \\ q\begin{pmatrix} \begin{pmatrix} w_{0t}+qw_{1t} \\ +q^2w_{2t}+q^3w_{3t} \end{pmatrix}-\begin{pmatrix} u_{0xx}+qu_{1xx} \\ +q^2u_{2xx}+q^3u_{3xx} \end{pmatrix} \\ -\begin{pmatrix} v_{0xx}+qv_{1xx} \\ +q^2v_{2xx}+q^3v_{3xx} \end{pmatrix}\begin{pmatrix} w_{0t}+qw_{1t}+ \\ q^2w_{2t}+q^3w_{3t} \end{pmatrix} \\ -4xt+2t+2 \end{pmatrix}=0 \tag{2.32}$$

By equating identical powers of q, we will get the following system of equations:

$$q^0:\begin{cases} u_{0t}-u_0=0 \\ v_{0t}-v_0=0 \\ w_{0t}-w_0=0 \end{cases} \tag{2.33}$$

$$q^1:\begin{cases} u_{1t}+u_0-w_{0x}v_{0t}-\frac{1}{2}w_{0t}u_{0xx}+4xt=0 \\ v_{1t}+v_0-w_{0t}u_{0xx}-6t=0 \\ w_{1t}+w_0-u_{0xx}-v_{0xx}w_{0t}-4xt+2t+2=0 \end{cases} \tag{2.34}$$

$$q^2:\begin{cases} u_{2t}-w_{0x}v_{1t}-w_{1x}v_{0t}-\frac{1}{2}w_{0t}u_{1xx}-\frac{1}{2}w_{1t}u_{0xx}=0 \\ v_{2t}-w_{0t}u_{1xx}-w_{1t}u_{0xx}=0 \\ w_{2t}-u_{1xx}-v_{0xx}w_{1t}-v_{1xx}w_{0t}=0 \end{cases} \tag{2.35}$$

By solving the above system of equations we get:

$$\left.\begin{aligned} U_0 &= x^2+1 \\ V_0 &= x^2-1 \\ W_0 &= x^2-1 \end{aligned}\right\} \tag{2.36}$$

$$\left.\begin{aligned} U_1 &= -2xt^2 \\ V_1 &= 3t^2 \\ w_1 &= 2xt^2-t^2 \end{aligned}\right\} \tag{2.37}$$

$$\left.\begin{aligned} U_2 &= \frac{1}{2}(16x-2)t^2 \\ V_2 &= \frac{1}{2}(8x-4)t^2 \\ W_2 &= x(4x-2)t^2 \end{aligned}\right\} \tag{2.38}$$

Then the approximate solutions in series form are:

$$\left.\begin{aligned} u &\approx 1+x^2-t^2 \\ v &\approx -1+x^2+t^2 \\ w &\approx -1+x^2-t^2 \end{aligned}\right\} \tag{2.39}$$

Chapter 3

Solution of Differential Equations by New Homotopy Perturbation Method

3.1 Example 1

Consider the following system of three-dimensional partial differential equations:

$$\left.\begin{aligned} \frac{\partial u}{\partial t}-v\frac{\partial u}{\partial x}-\frac{\partial v}{\partial t}\frac{\partial u}{\partial y}=1-x+y+t \\ \frac{\partial v}{\partial t}-u\frac{\partial v}{\partial x}-\frac{\partial u}{\partial t}\frac{\partial v}{\partial y}=1-x-y-t \end{aligned}\right\} \tag{3.1}$$

$$u(x,y,0)=x+y-1$$
$$v(x,y,0)=x-y+1$$

To solve Eq. (3.1) by using NHPM, we construct the following homotopies:

$$\left.\begin{aligned} \frac{\partial U}{\partial t}(x,y,t)=u_0(x,y,t)-q\left(u_0(x,y,t)-V\frac{\partial U}{\partial x}-\frac{\partial V}{\partial t}\frac{\partial U}{\partial y}-1+x-y-t\right) \\ \frac{\partial V}{\partial t}(x,y,t)=v_0(x,y,t)-q\left(v_0(x,y,t)-U\frac{\partial V}{\partial x}-\frac{\partial U}{\partial t}\frac{\partial V}{\partial y}-1+x+y-t\right) \end{aligned}\right\} \tag{3.2}$$

Applying inverse operator $L^{-1}=\int(.)\,dt$

$$\left.\begin{aligned} U(x,y,t)=U(x,y,0)+\int_0^t u_0(x,y,t)\,dt-q\int_0^t\left(u_0(x,y,t)-V\frac{\partial U}{\partial x}-\frac{\partial V}{\partial t}\frac{\partial U}{\partial y}-1+x-y-t\right)dt \\ V(x,y,t)=V(x,y,0)+\int_0^t v_0(x,y,t)\,dt-q\int_0^t\left(v_0(x,y,t)-U\frac{\partial V}{\partial x}-\frac{\partial U}{\partial t}\frac{\partial V}{\partial y}-1+x+y+t\right)dt \end{aligned}\right\} \tag{3.3}$$

Using the following relations in above equations

$$\begin{aligned} U&=U_0+qU_1+q^2U_2+q^3U_3+\dots \\ \frac{\partial U}{\partial x}&=\frac{\partial U_0}{\partial x}+q\frac{\partial U_1}{\partial x}+q^2\frac{\partial U_2}{\partial x}+q^3\frac{\partial U_3}{\partial x}+\dots \\ \frac{\partial U}{\partial y}&=\frac{\partial U_0}{\partial y}+q\frac{\partial U_1}{\partial y}+q^2\frac{\partial U_2}{\partial y}+q^3\frac{\partial U_3}{\partial y}+\dots \\ \frac{\partial U}{\partial t}&=\frac{\partial U_0}{\partial t}+q\frac{\partial U_1}{\partial t}+q^2\frac{\partial U_2}{\partial t}+q^3\frac{\partial U_3}{\partial t}+\dots \end{aligned} \tag{3.4}$$

$$\begin{aligned} V&=V_0+qV_1+q^2V_2+q^3V_3+\dots \\ \frac{\partial V}{\partial x}&=\frac{\partial V_0}{\partial x}+q\frac{\partial V_1}{\partial x}+q^2\frac{\partial V_2}{\partial x}+q^3\frac{\partial V_3}{\partial x}+\dots \\ \frac{\partial V}{\partial y}&=\frac{\partial V_0}{\partial y}+q\frac{\partial V_1}{\partial y}+q^2\frac{\partial V_2}{\partial y}+q^3\frac{\partial V_3}{\partial y}+\dots \\ \frac{\partial V}{\partial t}&=\frac{\partial V_0}{\partial t}+q\frac{\partial V_1}{\partial t}+q^2\frac{\partial V_2}{\partial t}+q^3\frac{\partial V_3}{\partial t}+\dots \end{aligned} \tag{3.5}$$

and have

$$U_0+qU_1+q^2U_2+q^3U_3=U\left(x,y,0\right)+$$

$$\int_0^t u_0\left(x,y,t\right)dt-q\int_0^t\begin{pmatrix} u_0\left(x,y,t\right)-\left(V_0+qV_1+q^2V_2+q^3V_3\right) \\ \left(\frac{\partial U_0}{\partial x}+q\frac{\partial U_1}{\partial x}+q^2\frac{\partial U_2}{\partial x}+q^3\frac{\partial U_3}{\partial x}\right)- \\ \left(\frac{\partial V_0}{\partial t}+q\frac{\partial V_1}{\partial t}+q^2\frac{\partial V_2}{\partial t}+q^3\frac{\partial V_3}{\partial t}\right) \\ \left(\frac{\partial U_0}{\partial y}+q\frac{\partial U_1}{\partial y}+q^2\frac{\partial U_2}{\partial y}+q^3\frac{\partial U_3}{\partial y}\right)-1+x-y-t \end{pmatrix}dt \tag{3.6}$$

$$V_0+qV_1+q^2V_2+q^3V_3=V\left(x,y,0\right)+$$

$$\int_0^t v_0\left(x,y,t\right)dt-q\int_0^t\begin{pmatrix} v_0\left(x,y,t\right)-\left(U_0+qU_1+q^2U_2+q^3U_3\right) \\ \left(\frac{\partial V_0}{\partial t}+q\frac{\partial V_1}{\partial t}+q^2\frac{\partial V_2}{\partial t}+q^3\frac{\partial V_3}{\partial t}+\right)- \\ \left(\frac{\partial U_0}{\partial t}+q\frac{\partial U_1}{\partial t}+q^2\frac{\partial U_2}{\partial t}+q^3\frac{\partial U_3}{\partial t}\right) \\ \left(\frac{\partial V_0}{\partial y}+q\frac{\partial V_1}{\partial y}+q^2\frac{\partial V_2}{\partial y}+q^3\frac{\partial V_3}{\partial y}\right)-1+x+y+t \end{pmatrix}dt \tag{3.7}$$

By comparing the same power of q:

$$q^0:\begin{cases} U_0\left(x,y,t\right)=U\left(x,y,0\right)+\int_0^t u_0\left(x,y,t\right)dt \\ V_0\left(x,y,t\right)=V\left(x,y,0\right)+\int_0^t v_0\left(x,y,t\right)dt \end{cases} \tag{3.8}$$

$$q^1:\begin{cases} U_1\left(x,y,t\right)=\int_0^t\left(-u_0\left(x,y,t\right)+V_0\frac{\partial U_0}{\partial x}+\frac{\partial V_0}{\partial t}\frac{\partial U_0}{\partial y}+1-x+y+t\right)dt \\ V_1\left(x,y,t\right)=\int_0^t\left(-v_0\left(x,y,t\right)+U_0\frac{\partial V_0}{\partial x}+\frac{\partial U_0}{\partial t}\frac{\partial V_0}{\partial y}+1-x-y-t\right)dt \end{cases} \tag{3.9}$$

$$q^2:\begin{cases} U_2\left(x,y,t\right)=\int_0^t\left(V_0\frac{\partial U_1}{\partial x}+V_1\frac{\partial U_0}{\partial X}+\frac{\partial V_0}{\partial t}\frac{\partial U_1}{\partial y}+\frac{\partial V_1}{\partial t}\frac{\partial U_0}{\partial y}\right)dt \\ V_2\left(x,y,t\right)=\int_0^t\left(U_0\frac{\partial V_1}{\partial x}+U_1\frac{\partial V_0}{\partial X}+\frac{\partial U_0}{\partial t}\frac{\partial V_1}{\partial y}+\frac{\partial U_1}{\partial t}\frac{\partial V_0}{\partial y}\right)dt \end{cases} \tag{3.10}$$

$$\vdots$$

$$q^j:\begin{cases} U_j(x,y,t)=\int_0^t\sum_{k=0}^{j}\left(V_k\,\frac{\partial U_{j-k}}{\partial x}+\frac{\partial V_k}{\partial t}\frac{\partial U_{j-k}}{\partial y}\right)dt \\ V_j(x,y,t)=\int_0^t\sum_{k=0}^{j}\left(U_k\,\frac{\partial V_{j-k}}{\partial x}+\frac{\partial U_k}{\partial t}\frac{\partial V_{j-k}}{\partial y}\right)dt \end{cases} \quad (3.11)$$

Assume

$$\left.\begin{aligned} u_0(x,y,t)&=\sum_{n=0}^{\infty}a_n(x,y)P_n(t),P_n(t)=t^n,U(x,y,0)=u(x,y,o) \\ v_0(x,y,t)&=\sum_{n=0}^{\infty}b_n(x,y)P_n(t),P_n(t)=t^n,V(x,y,0)=v(x,y,o) \end{aligned}\right\} \quad (3.12)$$

Solving the above equations for $U_0(x,y,t)$ and $V_0(x,y,t)$

$$\left.\begin{aligned} U_0(x,y,t)&=u(x,y,0)+\int_0^t\left(a_0(x,y)+a_1(x,y)t+a_2(x,y)t^2+a_3(x,y)t^3\right)dt \\ V_0(x,y,t)&=v(x,y,0)+\int_0^t\left(b_0(x,y)+b_1(x,y)t+b_2(x,y)t^2+b_3(x,y)t^3\right)dt \end{aligned}\right\} \quad (3.13)$$

Now by substituting eq. (3.14) in eq. (3.9)

$$U_1(x,y,t)=\int_0^t\begin{pmatrix} \left(-a_0(x,y)-a_1(x,y)t-a_2(x,y)t^2-a_3(x,y)t^3\right)+ \\ \left(v(x,y,0)+\int_0^t\left(b_0(x,y)+b_1(x,y)t+b_2(x,y)t^2+b_3(x,y)t^3\right)dt\right) \\ \left(u_x(x,y,0)+\int_0^t\left(+a_0(x,y)+a_1(x,y)t+a_2(x,y)t^2+a_3(x,y)t^3\right)dt\right) \end{pmatrix}dt \quad (3.14)$$

$$V_1(x,y,t)=\int_0^t\begin{pmatrix} \left(-b_0(x,y)-b_1(x,y)t-b_2(x,y)t^2-b_3(x,y)t^3\right)+ \\ \left(u(x,y,0)+\int_0^t\left(a_0(x,y)+a_1(x,y)t+a_2(x,y)t^2+a_3(x,y)t^3\right)dt\right) \\ \left(v_x(x,y,0)+\int_0^t\left(+b_0(x,y)+b_1(x,y)t+b_2(x,y)t^2+b_3(x,y)t^3\right)dt\right) \end{pmatrix}dt \quad (3.15)$$

Or

$$U_1(x,y,t)=\int_0^t \begin{pmatrix} \left(-a_0(x,y)-a_1(x,y)t-a_2(x,y)t^2-a_3(x,y)t^3\right)+ \\ \left((x+y-1)+\int_0^t \left(b_0(x,y)+b_1(x,y)t+b_2(x,y)t^2+b_3(x,y)t^3\right)dt\right) \\ \left(1+\int_0^t \left(+a_0(x,y)+a_1(x,y)t+a_2(x,y)t^2+a_3(x,y)t^3\right)dt\right) \\ +1-x+y+t \end{pmatrix} dt \quad (3.16)$$

$$V_1(x,y,t)=\int_0^t \begin{pmatrix} \left(-b_0(x,y)-b_1(x,y)t-b_2(x,y)t^2-b_3(x,y)t^3\right)+ \\ \left(x-y+1+\int_0^t \left(a_0(x,y)+a_1(x,y)t+a_2(x,y)t^2+a_3(x,y)t^3\right)dt\right) \\ \left(1+\int_0^t \left(+b_0(x,y)+b_1(x,y)t+b_2(x,y)t^2+b_3(x,y)t^3\right)dt\right) \end{pmatrix} dt \quad (3.17)$$

0r

$$U_1(x,y,t)=\int_0^t \left(\begin{array}{l} \left(-a_0(x,y)-a_1(x,y)t-a_2(x,y)t^2-a_3(x,y)t^3\right)+ \\ x+y-1+\int_0^t (x+y-1)\begin{pmatrix} a_{0x}(x,y)+a_{1x}(x,y)t+ \\ a_{2x}(x,y)t^2+a_{3x}(x,y)t^3 \end{pmatrix}dt \\ +\int_0^t \left(b_0(x,y)+b_1(x,y)t+b_2(x,y)t^2+b_3(x,y)t^3\right)dt+ \\ \int_0^t\int_0^t \begin{pmatrix} b_0(x,y)a_{0x}(x,y)+b_0(x,y)a_{1x}(x,y)t+b_0(x,y)a_{2x}(x,y)t^2+ \\ b_0(x,y)a_{3x}(x,y)t^3+b_1(x,y)a_{0x}(x,y)t+b_1(x,y)a_{1x}(x,y)t^2+ \\ b_1(x,y)a_{2x}(x,y)t^3+b_1(x,y)a_{3x}(x,y)t^4+b_2(x,y)a_{0x}(x,y)t^2+ \\ b_2(x,y)a_{1x}(x,y)t^3+b_2(x,y)a_{2x}(x,y)t^4+b_2(x,y)a_{3x}(x,y)t^5+ \\ b_3(x,y)a_{0x}(x,y)t^3+b_3(x,y)a_{1x}(x,y)t^4+b_3(x,y)a_{2x}(x,y)t^5+ \\ b_3(x,y)a_{3x}(x,y)t^6+ \end{pmatrix} dtdt+ \\ b_0(x,y)+b_1(x,y)t+b_2(x,y)t^2+b_3(x,y)t^3+ \\ \int_0^t \begin{pmatrix} b_0(x,y)a_{0y}(x,y)+b_0(x,y)a_{1y}(x,y)t+b_0(x,y)a_{2y}(x,y)t^2+ \\ b_0(x,y)a_{3y}(x,y)t^3+b_1(x,y)a_{0y}(x,y)t+b_1(x,y)a_{1y}(x,y)t^2+ \\ b_1(x,y)a_{2y}(x,y)t^3+b_1(x,y)a_{3y}(x,y)t^4+b_2(x,y)a_{0y}(x,y)t^2+ \\ b_2(x,y)a_{1y}(x,y)t^3+b_2(x,y)a_{2y}(x,y)t^4+b_2(x,y)a_{3y}(x,y)t^5+ \\ b_3(x,y)a_{0y}(x,y)t^3+b_3(x,y)a_{1y}(x,y)t^4+b_3(x,y)a_{2y}(x,y)t^5+ \\ b_3(x,y)a_{3y}(x,y)t^6 \end{pmatrix} dt \\ +1-x+y+t \end{array} \right) dt \tag{3.18}$$

$$V_1(x,y,t)=\int_0^t \left(\begin{array}{l} \left(-b_0(x,y)-b_1(x,y)t-b_2(x,y)t^2-b_3(x,y)t^3\right)+ \\ x-y+1+\int_0^t (x-y+1)\begin{pmatrix} b_{0x}(x,y)+b_{1x}(x,y)t+ \\ b_{2x}(x,y)t^2+b_{3x}(x,y)t^3 \end{pmatrix} dt \\ +\int_0^t \left(a_0(x,y)+a_1(x,y)t+a_2(x,y)t^2+a_3(x,y)t^3\right)dt+ \\ \int_0^t\int_0^t \begin{pmatrix} a_0(x,y)b_{0x}(x,y)+a_0(x,y)b_{1x}(x,y)t+a_0(x,y)b_{2x}(x,y)t^2+ \\ a_0(x,y)b_{3x}(x,y)t^3+a_1(x,y)b_{0x}(x,y)t+a_1(x,y)b_{1x}(x,y)t^2+ \\ a_1(x,y)b_{2x}(x,y)t^3+a_1(x,y)b_{3x}(x,y)t^4+a_2(x,y)b_{0x}(x,y)t^2+ \\ a_2(x,y)b_{1x}(x,y)t^3+a_2(x,y)b_{2x}(x,y)t^4+a_2(x,y)b_{3x}(x,y)t^5+ \\ a_3(x,y)b_{0x}(x,y)t^3+a_3(x,y)b_{1x}(x,y)t^4+a_3(x,y)b_{2x}(x,y)t^5+ \\ a_3(x,y)b_{3x}(x,y)t^6+ \end{pmatrix} dtdt+ \\ a_0(x,y)+a_1(x,y)t+a_2(x,y)t^2+a_3(x,y)t^3+ \\ \int_0^t \begin{pmatrix} a_0(x,y)b_{0y}(x,y)+a_0(x,y)b_{1y}(x,y)t+a_0(x,y)b_{2y}(x,y)t^2+ \\ a_0(x,y)b_{3y}(x,y)t^3+a_1(x,y)b_{0y}(x,y)t+a_1(x,y)b_{1y}(x,y)t^2+ \\ a_1(x,y)b_{2y}(x,y)t^3+a_1(x,y)b_{3y}(x,y)t^4+a_2(x,y)b_{0y}(x,y)t^2+ \\ a_2(x,y)b_{1y}(x,y)t^3+a_2(x,y)b_{2y}(x,y)t^4+a_2(x,y)b_{3y}(x,y)t^5+ \\ a_3(x,y)b_{0y}(x,y)t^3+a_3(x,y)b_{1y}(x,y)t^4+a_3(x,y)b_{2y}(x,y)t^5+ \\ a_3(x,y)b_{3y}(x,y)t^6 \end{pmatrix} dt \\ +1-x-y-t \end{array} \right) dt \qquad (3.19)$$

By vanishing of $U_1(x,y,t)$, $V_1(x,y,t)$, the coefficients $a_n(x,y)$, $b_n(x,y)$, $n=1,2,3,...$ as determined,

$$\left. \begin{array}{l} a_0(x,y)=1, a_1(x,y)=a_2(x,y)=a_3(x,y)=a_4(x,y)=...=0 \\ b_0(x,y)=-1, b_1(x,y)=b_2(x,y)=b_3(x,y)=b_4(x,y)=...=0 \end{array} \right\} \qquad (3.20)$$

Therefore, we obtain the solution,

$$\left.\begin{array}{l} u(x,y,t)=U_0(x,y,t)=x+y-1+a_0(x,y)t+\frac{1}{2}a_1(x,y)t^2+ \\ \qquad \frac{1}{3}a_2(x,y)t^3+\frac{1}{4}a_3(x,y)t^4+\ldots=x+y+t-1 \\ v(x,y,t)=V_0(x,y,t)=x-y+1+b_0(x,y)t+\frac{1}{2}b_1(x,y)t^2+ \\ \qquad \frac{1}{3}b_2(x,y)t^3+\frac{1}{4}b_3(x,y)t^4+\ldots=x-y-t+1 \end{array}\right\} \tag{3.21}$$

3.2 Example 2

Now we have to solve by NHPM, first we construct homotopy

$$\left.\begin{array}{l} \frac{\partial U}{\partial x}(x,t)=u_0(x,t)-p\left(u_0(x,t)-V\frac{\partial U}{\partial t}+U\frac{\partial V}{\partial t}+1-e^x\sin(t)\right) \\ \frac{\partial V}{\partial x}(x,t)=v_0(x,t)-p\left(v_0(x,t)+\frac{\partial U}{\partial t}\frac{\partial V}{\partial x}+\frac{\partial V}{\partial t}\frac{\partial U}{\partial x}+1+e^x\cos(t)\right) \end{array}\right\} \tag{3.22}$$

Applying inverse operator $L^{-1}=\int_0^x(.)\,dx$ to both sides of the above equation, we obtain,

$$\left.\begin{array}{l} U(x,t)=U(0,t)+\int_0^x u_0(x,t)dx-p\int_0^x\left(u_0(x,t)-V\frac{\partial U}{\partial t}+U\frac{\partial V}{\partial t}+1-e^x\sin(t)\right)dx \\ V(x,t)=V(0,t)+\int_0^x v_0(x,t)dx-p\int_0^x\left(v_0(x,t)+\frac{\partial U}{\partial t}\frac{\partial V}{\partial x}+\frac{\partial V}{\partial t}\frac{\partial U}{\partial x}+1+e^x\cos(t)\right)dx \end{array}\right\} \tag{3.23}$$

As we know that

$$\begin{array}{l} U=U_0+qU_1+q^2U_2+q^3U_3+\ldots \\ V=V_0+qV_1+q^2V_2+q^3V_3+\ldots \\ \frac{\partial U}{\partial x}=\frac{\partial U_0}{\partial x}+q\frac{\partial U_1}{\partial x}+q^2\frac{\partial U_2}{\partial x}+q^3\frac{\partial U_3}{\partial x}+\ldots \\ \frac{\partial U}{\partial t}=\frac{\partial U_0}{\partial t}+q\frac{\partial U_1}{\partial t}+q^2\frac{\partial U_2}{\partial t}+q^3\frac{\partial U_3}{\partial t}+\ldots \\ \frac{\partial V}{\partial x}=\frac{\partial V_0}{\partial x}+q\frac{\partial V_1}{\partial x}+q^2\frac{\partial V_2}{\partial x}+q^3\frac{\partial V_3}{\partial x}+\ldots \\ \frac{\partial V}{\partial t}=\frac{\partial V_0}{\partial t}+q\frac{\partial V_1}{\partial t}+q^2\frac{\partial V_2}{\partial t}+q^3\frac{\partial V_3}{\partial t}+\ldots \end{array} \tag{3.24}$$

Substituting eq. (3.25) in eq. (3.24)

$$U_0+qU_1+q^2U_2+q^3U_3=U\left(0,t\right)+\int_0^x u_0\left(x,t\right)dx-$$
$$p\int_0^x\begin{pmatrix} u_0\left(x,t\right)-\left(V_0+qV_1+q^2V_2+q^3V_3+...\right) \\ \left(\frac{\partial U}{\partial t}=\frac{\partial U_0}{\partial t}+q\frac{\partial U_1}{\partial t}+q^2\frac{\partial U_2}{\partial t}+q^3\frac{\partial U_3}{\partial t}+...\right)+ \\ \left(U_0+qU_1+q^2U_2+q^3U_3\right)\left(\frac{\partial V}{\partial t}=\frac{\partial V_0}{\partial t}+q\frac{\partial V_1}{\partial t}+q^2\frac{\partial V_2}{\partial t}+q^3\frac{\partial V_3}{\partial t}\right) \\ +1-e^x\sin\left(t\right)\end{pmatrix}dx \tag{3.25}$$

$$V_0+qV_1+q^2V_2+q^3V_3=V\left(0,t\right)+\int_0^x v_0\left(x,t\right)dx-$$
$$p\int_0^x\begin{pmatrix} v_0\left(x,t\right)+\left(\frac{\partial U}{\partial t}=\frac{\partial U_0}{\partial t}+q\frac{\partial U_1}{\partial t}+q^2\frac{\partial U_2}{\partial t}+q^3\frac{\partial U_3}{\partial t}\right) \\ \left(\frac{\partial V}{\partial x}=\frac{\partial V_0}{\partial x}+q\frac{\partial V_1}{\partial x}+q^2\frac{\partial V_2}{\partial x}+q^3\frac{\partial V_3}{\partial x}\right)+ \\ \left(\frac{\partial V}{\partial t}=\frac{\partial V_0}{\partial t}+q\frac{\partial V_1}{\partial t}+q^2\frac{\partial V_2}{\partial t}+q^3\frac{\partial V_3}{\partial t}\right) \\ \left(\frac{\partial U}{\partial x}=\frac{\partial U_0}{\partial x}+q\frac{\partial U_1}{\partial x}+q^2\frac{\partial U_2}{\partial x}+q^3\frac{\partial U_3}{\partial x}\right)+1+e^x\cos\left(t\right)\end{pmatrix}dx \tag{3.26}$$

Now equating the identical powers of q, we will obtain the following results,

$$q^0:\begin{cases}U_0\left(x,t\right)=U\left(0,t\right)+\int_0^x u_0\left(x,t\right)dx \\ V_0\left(x,t\right)=V\left(0,t\right)+\int_0^x v_0\left(x,t\right)dx\end{cases} \tag{3.27}$$

$$q^1:\begin{cases}U_1\left(x,t\right)=\int_0^x\left(-u_0\left(x,t\right)+V_0\frac{\partial U_0}{\partial t}-1+e^x\sin\left(t\right)\right)dx \\ \mathrm{V}_1\left(x,t\right)=\int_0^x\left(-v_0\left(x,t\right)-\frac{\partial U_0}{\partial t}\frac{\partial V_0}{\partial x}-\frac{\partial V_0}{\partial t}\frac{\partial U_0}{\partial x}-1-e^x\cos\left(t\right)\right)dx\end{cases} \tag{3.28}$$

$$q^2:\begin{cases}U_2\left(x,t\right)=\int_0^x\left(V_0\frac{\partial U_1}{\partial t}+V_1\frac{\partial U_0}{\partial t}-U_0\frac{\partial V_1}{\partial t}-U_1\frac{\partial V_1}{\partial t}\right)dx \\ \mathrm{V}_2\left(x,t\right)=\int_0^x\left(-\frac{\partial U_0}{\partial t}\frac{\partial V_1}{\partial x}-\frac{\partial U_1}{\partial t}\frac{\partial V_0}{\partial x}-\frac{\partial V_0}{\partial t}\frac{\partial U_1}{\partial x}-\frac{\partial V_1}{\partial t}\frac{\partial U_0}{\partial x}\right)dx\end{cases} \tag{3.29}$$

$$\vdots$$

$$q^j:\begin{cases} U_j(x,t)=\int_0^x \sum_{k=0}^{j} V_k\left(\frac{\partial U_{j-k}}{\partial t}-\frac{U_k\partial V_{j-k}}{\partial t}\right)dx \\ \mathrm{V}_j(x,t)=\int_0^x \sum_{k=0}^{j}\left(\frac{-\partial U_k}{\partial t}\frac{-\partial V_{j-k}}{\partial x}-\frac{-\partial V_k}{\partial t}\frac{\partial V_{j-k}}{\partial x}\right)dx \end{cases} \tag{3.30}$$

Assume that

$$\left.\begin{aligned} u_0(x,y,t)&=\sum_{n=0}^{\infty}a_n(t)P_n(x),P_n(x)=x^n,U(x,y,0)=u(x,y,o)\\ v_0(x,y,t)&=\sum_{n=0}^{\infty}b_n(t)P_n(x),P_n(x)=x^n,V(x,y,0)=v(x,y,o)\end{aligned}\right\} \tag{3.31}$$

Using eq. (2.54) in eq. (2.53) to find $U_1(x,t)$ and $\mathrm{V}_1(x,t)$.

$$\left.\begin{aligned} U_0(x,t)&=U(0,t)+\int_0^x\left(a_0(t)+a_1(t)x+a_2(t)x^2+a_3(t)x^3\right)dx\\ V_0(x,t)&=V(0,t)+\int_0^x\left(b_0(t)+b_1(t)x+b_2(t)x^2+b_3(t)x^3\right)dx\end{aligned}\right\} \tag{3.32}$$

Solving the above equations,

$$\begin{aligned} U_1(x,t)=\left(-a_0(t)+\sin(t)\right)x+&\left(\begin{aligned}&-\frac{1}{2}a_1(t)+\frac{a_0'(t)\cos(t)}{2}+\frac{b_0(t)\cos(t)}{2}-\\ &\frac{b_0'(t)\sin(t)}{2}+\frac{a_0(t)\sin(t)}{2}+\frac{\sin(t)}{2}\end{aligned}\right)x^2+\\ &\left(\begin{aligned}&-\frac{1}{3}a_2(t)+\frac{a_1'(t)\cos(t)}{6}+\frac{b_1(t)\cos(t)}{6}+\\ &\frac{b_0(t)a_0'(t)}{3}-\frac{b_1'(t)\sin(t)}{6}+\frac{a_1(t)\sin(t)}{6}-\\ &-\frac{b_0'(t)a_0(t)}{3}+\frac{\sin(t)}{6}\end{aligned}\right)x^3+\end{aligned} \tag{3.33}$$

…

$$V_1(x,t)=\begin{pmatrix}-b_0(t)-b_0(t)\cos(t)+\\ a_0(t)\sin(t)-\cos(t)-1\end{pmatrix}x+\begin{pmatrix}-\frac{1}{2}b_1(t)-\frac{b_1(t)\cos(t)}{2}-\frac{b_0(t)a_0'(t)}{2}+\\ \frac{a_1(t)\sin(t)}{2}-\frac{b_0'(t)a_0(t)}{2}+\frac{\cos(t)}{2}\end{pmatrix}x^2+$$
$$\begin{pmatrix}-\frac{1}{3}b_2(t)-\frac{b_2(t)\cos(t)}{3}-\frac{b_0(t)a_1'(t)}{6}-\\ \frac{b_1(t)a_0'(t)}{3}+\frac{a_2(t)\sin(t)}{3}-\frac{b_1'(t)a_0(t)}{6}-\\ -\frac{b_0'(t)a_1(t)}{3}-\frac{\cos(t)}{6}\end{pmatrix}x^3+ \qquad (3.34)$$

$\ldots$

By vanishing of $U_1(x,t)$ and $V_1(x,t)$ the coefficients $a_n(t),b_n(t)$ n=1,2,3,... are obtained as follows

$$a_0(t)=\sin(t),a_1(t)=\sin(t),a_2(t)=\frac{1}{2}\sin(t),a_3(t)=\frac{1}{6}\sin(t),$$
$$a_4(t)=\frac{1}{24}\sin(t),...$$
$$b_0(t)=-\cos(t),b_1(t)=\cos(t),b_2(t)=-\frac{1}{2}\cos(t),a_3(t)=\frac{1}{6}\cos(t),$$
$$a_4(t)=-\frac{1}{24}\cos(t),...$$

This implies that

$$u(x,t)=U_0(x,t)=\sin(t)+a_0(t)x+\frac{1}{2}a_1(t)x^2+\frac{1}{3}a_2(t)x^3+$$
$$\frac{1}{4}a_3(t)x^4+...=e^x\sin(t)$$
$$v(x,t)=V_0(x,t)=\cos(t)+b_0(t)x+\frac{1}{2}b_1(t)x^2+\frac{1}{3}b_2(t)x^3+ \qquad (3.35)$$
$$\frac{1}{4}b_3(t)x^4+...=e^{-x}\cos(t)$$

3.3 Example 3

Consider the following non-linear system of inhomogeneous partial differential equations:

$$\left.\begin{aligned}
&\frac{\partial u}{\partial t}-\frac{\partial w}{\partial x}\frac{\partial v}{\partial t}-\frac{1}{2}\frac{\partial w}{\partial t}\frac{\partial^2 u}{\partial x^2}=-4xt\\
&\frac{\partial v}{\partial t}-\frac{\partial w}{\partial t}\frac{\partial^2 u}{\partial x^2}=6t\\
&\frac{\partial w}{\partial t}-\frac{\partial^2 u}{\partial x}-\frac{\partial v}{\partial x}\frac{\partial w}{\partial t}=4xt-2t-2
\end{aligned}\right\} \tag{3.36}$$

Subject to the initial conditions,

$$\begin{aligned}
u(x,o)&=x^2+1\\
v(x,o)&=x^2-1\\
w(x,o)&=x^2-1
\end{aligned}$$

Solution:

For solving this system by using NHPM, we consider the following homotopy:

$$\left.\begin{aligned}
&\frac{\partial U}{\partial t}(x,t)=u_0(x,t)-q\left(u_0(x,t)-\frac{\partial W}{\partial x}\frac{\partial V}{\partial t}-\frac{1}{2}\frac{\partial W}{\partial t}\frac{\partial^2 U}{\partial x^2}+4xt\right)\\
&\frac{\partial V}{\partial t}(x,t)=v_0(x,t)-q\left(v_0(x,t)-\frac{\partial w}{\partial t}\frac{\partial^2 U}{\partial x^2}-6t\right)\\
&\frac{\partial W}{\partial t}(x,t)=w_0(x,t)-q\left(w_0(x,t)-\frac{\partial^2 U}{\partial x^2}-\frac{\partial V}{\partial x}\frac{\partial W}{\partial t}-4xt+2t+2\right)
\end{aligned}\right\} \tag{3.37}$$

Applying inverse operator $L^{-1}=\int_0^t(.)dt$ to both sides of (2.71) leads to the results,

$$\left.\begin{aligned}
&U(x,t)=U(x,0)+\int_0^t u_0(x,t)\,dt-\\
&\qquad q\int_0^t\left(u_0(x,t)-\frac{\partial W}{\partial x}\frac{\partial V}{\partial t}-\frac{1}{2}\frac{\partial W}{\partial t}\frac{\partial^2 U}{\partial x^2}+4xt\right)dt\\
&V(x,t)=V(x,0)+\int_0^t v_0(x,t)\,dt-\\
&\qquad q\int_0^t\left(v_0(x,t)-\frac{\partial w}{\partial t}\frac{\partial^2 U}{\partial x^2}-6t\right)dt\\
&W(x,t)=W(x,0)+\int_0^t w_0(x,t)\,dt-\\
&\qquad q\int_0^t\left(w_0(x,t)-\frac{\partial^2 U}{\partial x^2}-\frac{\partial V}{\partial x}\frac{\partial W}{\partial t}-4xt+2t+2\right)dt
\end{aligned}\right\} \tag{3.38}$$

Again using (2.7) in (2.72) and then equating the same powers of q, we have

$$q^0:\begin{cases} U_0(x,t) = U_0(x,0) + \int_0^t u_0(x,t)dt \\ V_0(x,t) = V_0(x,0) + \int_0^t v_0(x,t)dt \\ W_0(x,t) = w_0(x,0) + \int_0^t w_0(x,t)d \end{cases} \tag{3.39}$$

$$q^1:\begin{cases} U_1(x,t) = \int_0^t \left(-u_0(x,t) + \frac{\partial W_0}{\partial x}\frac{\partial V_0}{\partial t} + \frac{1}{2}\frac{\partial W_0}{\partial t}\frac{\partial^2 U_0}{\partial x^2} - 4xt \right)dt \\ V_1(x,t) = \int_0^t \left(-v_0(x,t) + \frac{\partial W_0}{\partial t}\frac{\partial^2 U_0}{\partial x^2} + 6t \right)dt \\ W_1(x,t) = \int_0^t \left(-w_0(x,t) + \frac{\partial^2 U_0}{\partial x^2} + \frac{\partial V_0}{\partial x}\frac{\partial W_0}{\partial t} + 4xt - 2t - 2 \right)dt \end{cases} \tag{3.40}$$

$$q^2:\begin{cases} U_2(x,t) = \int_0^t \left(\frac{\partial W_1}{\partial x}\frac{\partial V_0}{\partial t} + \frac{\partial W_0}{\partial x}\frac{\partial V_1}{\partial t} + \frac{1}{2}\frac{\partial w_0}{\partial t}\frac{\partial^2 U_1}{\partial x^2} + \frac{1}{2}\frac{\partial w_1}{\partial t}\frac{\partial^2 U_0}{\partial x^2} \right)dt \\ V_2(x,t) = \int_0^t \left(\frac{\partial W_0}{\partial x}\frac{\partial^2 U_1}{\partial x^2} + \frac{\partial W_1}{\partial x}\frac{\partial^2 U_0}{\partial x^2} \right)dt \\ W_2(x,t) = \int_0^t \left(\frac{\partial^2 U_1}{\partial x^2} + \frac{\partial V_0}{\partial x}\frac{\partial W_1}{\partial t} + \frac{\partial V_1}{\partial x}\frac{\partial w_0}{\partial t} \right)dt \end{cases} \tag{3.41}$$

$$\vdots$$

$$q^j:\begin{cases} U_j(x,t) = \int_0^t \left(\sum_{k=0}^{j} \left(\frac{\partial W_k}{\partial x}\frac{\partial V_{j-k}}{\partial t} + \frac{1}{2}\frac{\partial W_k}{\partial t}\frac{\partial U_{j-k}}{\partial x^2} \right) \right) \\ V_j(x,t) = \int_0^t \left(\sum_{k=0}^{j} \left(\frac{\partial W_k}{\partial x}\frac{\partial U_{j-k}}{\partial x^2} \right) \right)dt \\ W_j(x,t) = \int_0^t \left(\frac{\partial U_j}{\partial x^2} + \sum_{k=0}^{j} \frac{\partial V_k}{\partial x}\frac{\partial W_{j-k}}{\partial t} \right)dt \end{cases} \tag{3.42}$$

As we know that,

$$\left.\begin{aligned} u_0(x,t) &= \sum_{n=0}^{\infty} a_n(t)t^k, U(x,0) = u(x,o) \\ v_0(x,t) &= \sum_{n=0}^{\infty} b_n(t)t^k, V(x,0) = v(x,o) \\ w_0(x,t) &= \sum_{n=0}^{\infty} c_n(t)t^k, V(x,0) = v(x,o) \end{aligned}\right\} \tag{3.43}$$

If we set the Taylor series of $U_1(x,t)$, $V_1(x,t)$ and $W_1(x,t)$ at $t=0$,then

$$\begin{aligned} &U_1(x,y) = \left(-a_0(x) + 2xb_0(x) + c_0(x)\right)t + \\ &\begin{pmatrix} -\frac{1}{2}a_1(x) + xb_1(x) + \frac{1}{2}b_0(x)c_0'(x) + \frac{1}{2}c_1(x) + \\ \frac{1}{4}a_0''(x)c_0(x) - 2x \end{pmatrix} t^2 + \\ &\begin{pmatrix} -\frac{1}{3}a_2(x) + \frac{1}{2}xb_3(x) + \frac{1}{3}b_1(x)c_0'(x) + \frac{1}{6}b_0(x)c_1'(x) + \\ \frac{1}{3}c_2(x) + \frac{1}{12}a_1''(x)c_0(x) + \frac{1}{6}a_0''(x) + c_1(x) \end{pmatrix} t^3 \\ &+\ldots \end{aligned} \tag{3.44}$$

$$\begin{aligned} &V_1(x,y) = \left(-b_0(x) + 2c_0(x)\right)t + \\ &\left(-\frac{1}{2}b_1(x) + c_1(x) + \frac{1}{2}c_0(x)a_0''(x) + 3\right)t^2 + \\ &\left(-\frac{1}{3}b_2(x) + \frac{2}{3}c_2(x) + \frac{1}{6}c_0(x)a_1''(x) + \frac{1}{3}c_1(x)a_0''(x)\right)t^3 \\ &+\ldots. \end{aligned} \tag{3.45}$$

$$\begin{aligned} &W_1(x,y) = \left(-c_0(x) + 2xc_0(x)\right)t + \\ &\begin{pmatrix} -\frac{1}{2}c_1(x) + \frac{1}{2}a_0''(x) + xc_1(x) + b_0'(x)c_0(x) + \\ 2x - 1 \end{pmatrix} t^2 + \\ &\begin{pmatrix} -\frac{1}{3}c_2(x) + \frac{1}{6}a_1''(x) + \frac{2}{3}xc_2(x) + \frac{1}{3}b_0(x)c_1(x) \\ +\frac{1}{6}b_1'(x)c_0(x) \end{pmatrix} t^3 \\ &+\ldots.. \end{aligned} \tag{3.46}$$

It follows easily

$$\left.\begin{array}{l} a_0(x)=0, a_1(x)=2, a_2(x)=a_3(x)=a_4(x)=....=0 \\ b_0(x)=0, b_1(x)=2, b_2(x)=b_3(x)=b_4(x)=....=0 \\ c_0(x)=0, c_1(x)=-2, c_2(x)=c_3(x)=c_4(x)=....=0 \end{array}\right\} \tag{3.47}$$

$$\begin{aligned} u(x,t)&=U_0(x,t)=x^2+1+a_0(x)t+\frac{1}{2}a_1(x)t^2+\frac{1}{3}a_2(x)t^3+ \\ &\quad \frac{1}{4}a_3(x)t^4+...=x^2-t^2+1 \\ v(x,t)&=V_0(x,t)=x^2-1+b_0(x)t+\frac{1}{2}b_1(x)t^2+\frac{1}{3}b_2(x)t^3+ \\ &\quad \frac{1}{4}b_3(x)t^4+...=x^2-t^2-1 \\ w(x,t)&=W_0(x,t)=x^2+1+c_0(x)t+\frac{1}{2}c_1(x)t^2+\frac{1}{3}c_2(x)t^3+ \\ &\quad \frac{1}{4}c_3(x)t^4+...=x^2-t^2-1 \end{aligned} \tag{3.48}$$

References

[1] J.H. He, Homotopy perturbation technique, Comp. Meth. Appl. Mech. Eng. 178 (1999) 257_262.

[2] J.H. He, A coupling method of homotopy technique and a perturbation technique for non linear problems, Int. J. Non linear Mech. 35 (2000) 37_43.

[3] J.H. He, Homotopy perturbation method: A new nonlinear analytical technique, Appl. Math. Comput. 135 (2003) 73_79.

[4] J.H. He, A simple perturbation approach to Blasius equation, Appl. Math. Comput. 140 (2003) 217_222.

[5] J.H. He, Application of homotopy perturbation method to nonlinear wave equations, Chaos Solitons Fractals 26 (2005) 295_700.

[6] J.H. He, Homotopy perturbation method for solving boundary value problems, Phys. Lett. A 350 (2006) 87_88.

[7] A. Yildirim, Solution of BVPs for fourth-order integro-differential equations by using homotopy perturbation method, Comput. Math. Appl. 56 (12)
(2008) 3175_3180.

[8] S. Abbasbandy, Homotopy perturbation method for quadratic Riccati differential equation and comparison with Adomian's decomposition method,
Appl. Math. Comput. 172 (2006) 485_490.

[9] A. Golbabai, B. Keramati, Modified homotopy perturbation method for solving Fredholm integral equations, Chaos Solitons Fractals 37 (5) (2008)
1528_1537.

[10] M. Ghasemi, M. Tavassoli Kajani, E. Babolian, Numerical solutions of nonlinear Volterra_Fredholm integral equations by using homotopy perturbation
method, Appl. Math. Comput. 188 (2007) 446_449.

[11] J. Biazar, H. Ghazvini, He's homotopy perturbation method for solving systems of Volterra integral equations, Chaos Solitons Fractals (2007).

[12] Z. Odibat, S. Momani, A reliable treatment of homotopy perturbation method for Klein_Gordon equations, Phys. Lett. A 365 (2007) 351_357.

[13] M.S.H. Chowdhury, I. Hashim, Application of homotopy perturbation method to Klein_Gordon and sine_Gordon equations, Chaos Solitons Fractals(2007).

[14] M.S.H. Chowdhury, I. Hashim, Solutions of class of singular second-order IVPs by homotopy perturbation method, Phys. Lett. A 365 (2007) 439_447.

[15] O. Abdul Aziz, I. Hashim, S. Momani, Application of homotopy perturbation method to fractional IVPs, J. Comput. Appl. Math. 216 (2) (2008) 574_584.

[16] J. Biazar, H. Ghazvini, Exact solution of nonlinear Schrödinger equations by He's homotopy perturbation method, Phys. Lett. A 366 (2007) 79_84.

[17] M.S.H. Chowdhury, I. Hashim, Solutions of time-dependent Emden-Fowler type equations by homotopy perturbation method, Phys. Lett. A 368 (2007)
305_313.

[18] D.D. Ganji, H. Tari, M. Bakhshi Jooybari, Variational iteration method and homotopy perturbation method for nonlinear evolution equations, Int. J.
Comput. Math. Appl. 54 (2007) 1018_1027.

[19] Y. Cherruault, G. Adomian, Decomposition methods: a new proof of convergence, Mathematical and Computer Modeling 18 (1993) 103–106.

[20] A.M. Wazwaz, The decomposition method applied to systems of partial differential equations and to the reaction–diffusion Brusselator model, Applied
Mathematics and Computation 110 (2000) 251–264.

[21] J.H. He, Variational iteration method a kind of non-linear analytical technique: some examples, International Journal of Non-Linear Mechanics 34
(1999) 699–708.