Face Anti-magic Labelings and Deficiency of Graphs

By

Ali Tabraiz

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Ali Tabraiz

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[redacted]	[redacted]
Ali Tabraiz	CIIT/FA12-PMATH-001/LHR

Supervisor

Dr. Muhammad Hussain

Associate Professor Department of Mathematics

COMSATS Institute of Information Technology (CIIT), Lahore.

Certificate of Approval

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Student Name: Ali Tabraiz

Signature: ____________

External Committee:

External Examiner 1:
Prof. Dr. Muhammad Aslam
GC University, Lahore

External Examiner 2:
Prof. Dr. Muhammad Ozair
The University of Lahore

Dr. Muhammad Hussain
Supervisor
Department of Mathematics,
CIIT, Lahore

Dr. Kashif Ali
HoD, Department of Mathematics,
CIIT, Lahore

17/10/2017

Prof. Dr. Moiz ud Din Khan
Chairperson, Department of Mathematics,
CIIT

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Date: 2-10-17.

Supervisor:

Dr. Muhammad Hussain
Associate Professor
Department of Mathematics

Head of Department:

Dr. Kashif Ali
Associate Professor
Department of Mathematics.

DEDICATION

To my Parents

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COMSATS Institute of Information Technology, Lahore Campus.

Ali Tabraiz

CIIT/FA12-PMATH-001/LHR

ABSTRACT

Face Anti-magic Labelings and Deficiency of Graphs

Labeling of graph is an attractive and gorgeous procedure of Mathematics having wide applications in various field of sciences. A labeling of a graph is a correspondence from elements of graphs to integers. Face anti-magic labeling is a correspondence from elements of the given graph to integers in such a way that the face weight forms an arithmetic progression having b is the least weight and c is their common difference. In graph Labeling, while assigning symbols to elements of a graph sometime a situation arises, when labeling become neither anti-magic nor magic but it slightly obeys the condition of the anti-magic labeling and that laid the idea of deficiency of the graph.

This dissertation studies the construction of face anti-magic labeling of snake graphs, magic and anti-magic labeling of grid graph and edge magic deficiency of subdivided banana tree and subdivided w tree.

Key Words:

Face anti-magic labeling, face magic labeling, snake graph, subdivided triangular ladder graph, deficiency, subdivided banana tree, subdivided w tree.

TABLE OF CONTENT

1. Introduction 1

2. Basic Concepts 7

2.1 Basic Definitions 8

2.1.1 Preliminaries 8

2.1.2 Operations on Graph 8

2.1.3 Types of Graphs 9

3. Graph Labeling and Edge Magic Deficiency 11

3.1 Labelings 12

3.2 Face Magic and Face Anti-magic Labelings 12

3.2.1 Some Know Results on Face Magic and Face Anti-magic Labeling 12

3.3 Edge Magic Deficiency 13

4. Super d-Anti magic Labeling of Subdivided qC_5 15

4.1 Introduction 16

4.2 Super Anti-magic Labeling of qC_5 - Snake Graph 16

4.3 Super Anti-magic Labeling of Subdivided qC_5 - Snake Graph 20

5. Face Anti-magic Labeling of Generalized qC_n-Snake Graph.............32

5.1 Introduction 33

5.2 Super Anti-magic Labeling of qC_n - Snake Graph 33

5.3 Super Anti-magic Labeling of subdivided qC_n - Snake Graph 36

6. Magic and Anti-magic Labeling on Subdivision of $G \cong P_n \times P_m$ 39

6.1 Introduction 40

6.2 Face Magic Labeling of Grid Graph 40

6.3 Face Anti-magic Labeling of Type (1,1,1) on Subdivided Triangular Ladder 47

7. Super Anti-magic Labeling of Type(1,1,1) on Subdivided qC_4 55

7.1 Introduction 56

7.2 Super Face Anti-magic Labeling of qC_4 – Snake Graph 56

7.3 Super Face Anti-magic Labeling of Subdivided kC_4 – Snake Graph 60

7.4 Super Face Anti-magic Labeling of Triangular qC_4 – Snake Graph 68

8. On the Deficiencies of Some Trees 73

8.1 Introduction 74

8.2 Super Edge-Magic Deficiency of Subdivided Banana Tree 74

$BT(n_1, n_2, \cdots, n_q)$ 74

8.3 Super Edge-Magic Deficiency of Subdivided w Tree $w(n, q)$ 77

9. Conclusions and Open Problems 84

9.1 Conclusions 85

9.2 Open Problems 85

10. References 87

LIST OF FIGURES

Figure 1: Operations of graphs 8
Figure 2: A tree 9
Figure 3: $3C_5$ snake graph having string (1,1,1) 16
Figure 4: Face anti-magic labeling of $3C_5$ snake graph having string (1,1,1) 19
Figure 5: Face anti-magic labeling of $3C_5$ snake graph with 1 subdivision 22
Figure 6: Subdivided grid $H \cong P_4 \times P_3$ with 1 subdivision 41
Figure 7: Subdivided grid graph $H \cong P_3 \times P_3$ with 1 subdivision 42
Figure 8: Subdivided triangular ladder $H \cong P_3 \times P_2$ with 4 subdivisions 47
Figure 9: $3C_4$ snake graph having string (1,1,1) 56
Figure 10: Face anti-magic labeling of $3C_4$ snake graph having string(1,1,1) 58
Figure 11: $3C_4$ snake graph with 1 Subdivision, having string (1,1,1) 60
Figure 12: Triangular $4C_4$ snake graph having string (1,1,1,1) 68
Figure 13: Triangular $3C_4$ snake graph having string (1,1,1) 69
Figure 14: A banana tree $BT(n_1, n_2, \cdots, n_k)$ 75
Figure 15: A w tree w (3) 78

LIST OF ABBREVIATIONS

$V(G)$	Vertex set of the graph G
$E(G)$	Edge set of the graph G
$F(G)$	Face set of the graph G
qC_n	q copies of C_n graph
ε	Epsilon
ξ	Zeta
η	Eta
λ	Lambda
μ	Mu
∞	Infinity

Chapter 1
Introduction

Konigsberg, after words named as Kaliningrad, located on Pregel River is one of the biggest commercial and industrial center of Russia. River Pregel divides the town into four portions. Seven bridges built to adjoin these four parts of the town. People used to travel across the village through these bridges that took long hours. People wanted to travel all the seven bridges by startup from a point and reaching back at the same point without any bridge repetition. Leonhard Euler was the first, who resolved this issue and invented a new branch of mathematics, known as Graph Theory. Euler gave a solution by considering the sequence in which bridges were crossed, that idea actually reformulated the above problem. Euler resolved this problem in 1735 not too long ago from today but as far as the applications and significance is concerned Graph Theory is one of most popular branch of mathematics now days.

Graph Theory is becoming popular and significant as we use it in multiple areas of Science, Mathematics, Technology, Biochemistry, Computer science, Electrical engineering, Operation research and many more fields.

Graph Theory plays a vital role in many fields for example, to model chemical bounds in Chemistry, to model some sequences of cell samples in Biochemistry, to model numerous algorithms which are used to solve different problems in Computer applications, to model a network through which airlines and trains can connect countless cities in an efficient way and a large number of passengers can move with the fewest possible trips in Flight and Rail networks, to model a migration network of species in Biology, to model a best timetable which provide equal number of lectures to each teacher with no overlapping in time table of educational institute.

Graph theory can be applied to road networks, a way to reduce rush of traffic. The intelligent transportation systems can help us by assembling location of persons through smart phones and guide them where and how fast to move in order to avoid rush and accidents. If this idea is successful it can save billions yearly which are lost due to the time wasted on the road. It can also avoid road accidents in the first place and make the life safer and allowing emergency services to move quickly. Flight networks are already utilizing Graph Theory. Airlines want, maximum passengers moving throughout the globe with least possible trips and connect cities in an efficient way. Moreover, air traffic controllers use Graph Theory and computers to make sure thousands of planes are at the right time and right destination safely which is a massive task that would nearly be impossible without these techniques.

The internet is another important example of Graph Theory. Every page on internet can be considered as a vertex of a graph and the link between each page can be considered as an edge among the adjacent vertices. The subsequent graph is indeed a massive one. Initially it was a difficult task for web search engines while searching for a particular keyword, they may get thousands of pages but one cannot determine whether that page is authentic or not, or useful or not. For example if you searched for Lahore there are many websites of various shops in Lahore or various offices in Lahore but no official information is obtained. The solution of this problem was given by google in such a way that any page that has many other pages linked to it is considered as a good page or a page that is visited more often is considered as a good page. That provided a method to rank the website and a technique to display the best results in descending order showing the most wanted at the top. Facebook can also be considered as an example of Graph Theory. All Facebook user scan be considered as vertices and an edge between two users' means that they are Facebook friends. Efficiency of social networking sites and web developers is improved using techniques of Graph Theory.

Graphs are used in geometry and other branches of Mathematics like Statistics. In Operation research the successful planning and scheduling of large complicated projects is possible by suitable networks. In Algebra, graphs are useful to understand coset diagram.

Graph labeling is extensively used in various areas of life, for example, Networks representation, Computer science, Circuit design, Data base management, Communication network and many more. In Organic Chemistry the elements of any organic compound can be considered as the vertices of the graphs and the bonds between them are the edges either single edge or multiple edges depending upon the number of bonds between these elements. Same is the case of covalent bonds where the numbers of edges represents the number of shared bonds between the two elements. Theory of graph is also extensively used in industry as well as other areas of life. As far as the industrial applications are concerned Graph Theory is used to study demand and supply, market needs and the level of customer satisfaction, profit and loss of a company and comparison of one company with its Competitors. In educational institutions time table management is a big task as no student or teacher can have more than one lectures at the same time and the thing happens in the examination that no student can have more than one papers at the same time and no teacher can have more than one invigilation's at the same time it is a very painful job to manage all these things but graph theory made it easy due to its versatility. Theory of graphs is also applicable in computer networking, all computers are connected with a single computer is an example of bipartite graph and if each computer is connected with one another then it is a complete graph. Speed and range of computers can be controlled through networking which is also an application of graph theory. Same logic is applied by cellular companies and they offer various internet packages for number of days at assorted speed of internet which a consumer is using according to their need. With the help of graph theory complicated and complex proves of mathematical problems can easily resolved by powerful combinatorial methods. For example, Hasse diagram is used to represent a finite partially ordered set in order theory, for a partially ordered set $(S, \leq)$ each element of S is represented as a vertex in the plane and whenever y covers x there is an edge from x to y.

Moreover, Graph Theory is used to resolve the problems in many fields to find out optimal solutions in Statistics, Economics and various fields of sciences. In short, Graph Theory has its unique impact in various fields and is growing with every passing day. Many problems in Graph Theory can be formulated in terms of graph labeling.

Labeling is a powerful tool of graph theory. Labeling of a graph is a technique in which we assign symbols to the graph elements according to our requirement, sometime it is not necessary to assign labels to each and every member of the graph. In this case we label only the required elements of the graph according where needed. Labeling of a graph has vast applications in numerous fields of life and science. Some of these applications are discussed as under. Labeling can be applied to model a best network which ensures the smooth flow of traffic without hindrance in road signals network, to model a suitable internet networking system for an office, institute or commercial use to provide controlled speed for the user, to resolve industrial problems to find out optimal solutions of the problem, to model an appropriate circuit in network analysis or circuit analysis, to control air traffic in civil aviation, to find shortest path between desired locations using vertex coloring or different methods, to model an effective flow of communication signals in mobile networks for educational as well as commercial purpose, to model flow charts to give best optimal solution of an engineering problem, to model useful algorithms to solve various problems in various field of life, simply one can understand the versatility of this field. There are different types of labelings. This dissertation consist of some new results of face magic labeling, face anti-magic labeling and edge magic deficiency of different graphs.

A correspondence from elements of the graphs to numbers (non-negative or positive integers) is termed graph labeling. A labeling having type $(\varepsilon, \zeta, \eta)$ (where $\varepsilon, \zeta, \eta \in \{0,1\}$)from the elements of the graph to the numbers(sum of cardinality of elements of graph (vertices, edge and face)) in a specific way that every element of the graph take a unique image and none of the images are repeated is termed as face labeling.

$$\lambda : V(H) \cup E(H) \cup F(H) \rightarrow \{1, 2, \ldots, |V(H)| + |E(H)| + |F(H)|\}$$

The face weight in a graph is the sum of numbers assigned to that face and all the edges and vertices covering that face. A face labeling of the graph G is termed d-anti-magic if the face weight form an arithmetic sequence starting having a is the least weight and their common difference is d, also $a > 0$ and $d \geq 0$ both are integers. In particular for $d =, 0$ Lih [44] named such type of labeling is face-magic. If smallest number is assigned to the vertices then labeling is called super d-anti-magic .

Sometime when each and every member of the graph (vertices, edges and/or faces) are assign symbols a situation comes that correspondence become neither anti-magic nor magic but it slightly obey the condition of the antimagic labeling and that laid the idea of deficiency of the graph. Kotzig and Rosa [40] proved that, for some non-negative integer n, every graph G admits edge magic labeling having property $G \cong G \cup nK_1$, which is also called deficiency. Edge magic deficiency of a graph G represented by $\mu(G)$.

This dissertation is organized as follows:

In the 2nd Chapter, some basic definition, terminologies and notations are discussed.

In the 3rd Chapter, a survey concerning some known results for face magic labeling, face anti-magic labeling and deficiency of graph are presented.

In 4th Chapter, various results of face anti-magic labeling of qC_5- snake graph are presented.

In 5th Chapter, face anti-magic labeling of generalized qC_n snake graph are discussed.

In 6th Chapter, face magic and face anti-magic total labeling on subdivision of $G \cong P_o \times P_r$ and subdivided triangular ladder are presented.

In 7th Chapter, face anti-magic labeling of qC_4 are presented.

In 8th Chapter, super edge-magic deficiency of subdivided banana tree $BT(n_1, n_2, \ldots, n_k)$ and subdivided w tree $w(n,k)$ are presented.

In 9th Chapter, we gave conclusion and some open problems.

Chapter 2

Basic Concepts

Some basic concepts, terminologies and notations regarding Graph Theory are presented in this chapter which are necessary for the present study. Basic concepts regarding Graph Theory is explicitly given in [49, 50].

2.1 Basic Definitions

2.1.1 Preliminaries

Graph consists of various components named as *vertices*, *edges* and *faces* having representation $G(V,E,F)$ respectively. If p is any vertex of a graph G then it has illustration $p \in V(G)$. *Order* means the number of elements present in the vertex set and *size* means the number of elements present in the edge set of that graph G. *Degree* of a vertex means the sum of edges with that vertex as terminal point.

2.1.2 Operations on Graphs

Two graphs G_1 and G_2 having set of edges and vertices E_1,E_2 and V_1,V_2 respectively. The *union* of two graphs $G = G_1 \cup G_2$ has edge set $E_1 \cup E_2$ and vertex set $V_1 \cup V_2$. The join $G_1 + G_2$ consists of all edges joining V_1,V_2 and $G_1 \cup G_2$. $G_1 \times G_2$ represents the cartesian product of two graphs G_1 and G_2.

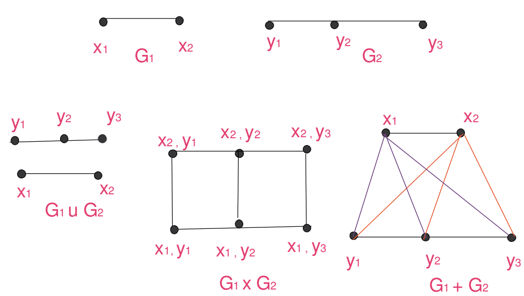

Figure 2.1: Operations of graphs

2.1.3 Types of Graphs

An approach from one vertex to another vertex is called *walk. Length* means how many edges traveled in the walk. A walk whose terminal vertex is also its initial vertex is a *closed walk*. A walk having no repeated edge is a *trail*. A trail with same initial and terminal vertex is called *circuit*. A trail with each vertex appears once is called *path*. A path in which initial and terminal vertices are same is called *cycle*. A graph having no cycle is *acyclic.*

If each pair of vertices are connected by a path (sequence of adjacent edges) then the graph G is called *connected graph.* A graph having more than one pieces is called *diconnected graph.* A disconnected graph having no cycle is a *forest* and *tree* is a components of the forest. *Caterpillar* is formed if removal of endpoints of tree leaves a path. A graph is called *regular graph* if each vertex in the graph have same degree.

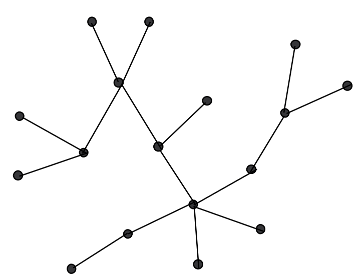

Figure 2.2: A tree

Suppose s is the degree of each vertex of the graph, then graph is called s-regular. A regular graph having n vertices is denoted by R_n. If graph is regular having degree 0, then it is termed as Null graph. A graph which is regular of degree 2 is called a cycle graph. Complete graph denoted by K_n, is regular of degree $n-1$, is formed if all vertices present in the graph are adjacent.

A *Path graph* is obtained by removing one edge of a cycle graph. A path having n vertices is written as P_n. A *wheel* graph can be defined as $W_n = C_n + K_1$. A graph G that can be redrawn without crossing, is called *planar graph*. A graph G without any multiple edge or loop is termed as *simple graph.* A *complete graph* is formed if all distinct vertices are adjacent in a simple graph. A connected graph G whose each vertex has degree equal to 2 is known as *cycle graph*. *Bipartite graph* is formed if it is possible to split the set of vertices of any graph into two separate sets in such a way that an edge is formed having both vertices from different sets.

Chapter 3

Graph Labeling and Edge Magic Deficiency

3.1 Labelings

A correspondence from graph elements to numbers is called *Labeling* (or *valuation*). The correspondence is called *total* labelings if edge set and vertex set is its domain.

$$\lambda : V(H) \cup E(H) \to \{1, 2, \ldots, |V(H)| + |E(H)|\}$$

If either of face set or edge set or vertex set are assigned symbols then such a labeling is *face-labeling*, *edge-labeling* and *vertex-labeling* respectively. Harmonious labelings, cordial labelings, graceful labelings and anti-magic labelings are examples of other types of labeling.

In this chapter, some results about *face-magic labelings*, *face anti-magic labelings* and *deficiency* of the graph G are presented, which are helpful in the formulation of some new results.

3.2 Face Magic and Face Anti-magic Labelings

A labeling of type $(\varepsilon, \zeta, \eta)$ (where $\varepsilon, \zeta, \eta \in \{0, 1\}$) is a process in which symbols are assigned to the graph elements in a specific way that each element of the graph receive a unique image and neither of the image has been repeated. The face weight in a graph is sum of the numbers assigned to that face and all the edges and vertices covering that face. If all m-sided faces in a graph has equal weight then labeling is called face-magic while if weight of m-sided faces form an arithmetic progression having c is their common difference and b is their starting number such a labeling is called (b, c) face antimagic furthermore if the label of vertices is the least possible number then labeling is called super magic and anti-magic accordingly.

3.2.1 Some Known Results on Face Magic and Face Anti-magic Labeling

Lih [44] gave the concept of magic labelings of plane graphs. Magic labeling of antiprism, M$\ddot{o}$bius ladder, convex polytops and plane graph has been discussed by Ba$\check{c}$a in [8]- [15]. For wheel like families, magic labelings of (a,b,c) type has been discussed in [3]. Face-magic labeling for honeycomb, grid graphs, convex polytopes and for special classes of plane graphs has been discussed in [4] -[7]. For 3D graphs, face magic labeling of $(1,0,0)$ type has been discussed in [43]. Magic labeling of, one external infinite face and families of planar graphs having $3,5$ and 6-sided faces has been discussed by Katheresan and Gokulakrishnan [42].

In 1999, Ba$\check{c}$a defined the face anti-magic labeling and discussed the face anti-magic labeling of convex polytopes in [25]. Ba$\check{c}$a et al. [26] investigated the (a,d) face anti-magic labeling of convex polytopes $P_{m+1} \times C_n$ in 2002. The class Q_n^m of convex polytopes was defined by Ba$\check{c}$a and Miller [27] in 2002. They proved that for odd m and even n, Q_n^m the above class is face anti-magic. In 2003, Ba$\check{c}$a [16] discussed the anti-magic labeling of planer graphs. Face anti-magic labeling for disjoint union of antiprisms, prisms and plane graphs were determined in [17]- [19]. For a special Hamilton path for disconnected plane graphs, super $d-$anti-magic labeling has been studied in [21]. Face anti-magic labeling of C_b^a planer graphs has been discussed in [24].

Super $d-$anti-magic labelings of toroidal polyhex H_m^n $(1,1,1)$ type has been studied by Ba$\check{c}$a et al. [20]. For friendship graphs F_n, $n \geq 2$, and $d \in 2$, $\{1,3,5,7,9,11,13\}$ super d-anti-magic labeling of $(1,1,0)$ type is determined in [28].

Face anti-magic labeling of disjoint union of prisms has been presented by Ali et al. in [2]. Face anti-magic labeling of uniform subdivision of wheel for certain differences d has been determined by Imran et al. [38]. Face anti-magic labeling of planer graph G in generalized form has been studied by Ahmad et al. [1]. Face anti-magic labeling of Jahangir graph was determined by Siddiqui et al. [47] for certain difference d. A general survey of graph labelings has been discussed in [33].

3.3 Edge Magic Deficiency

In graph Labeling, while assigning symbols to elements of a graph sometime a situation arises, when labeling become neither anti-magic nor magic but it slightly obeys the condition of the anti-magic labeling and that laid the idea of deficiency of the graph. Kotzig and Rosa, gave the idea of deficiency of graph in [40] for some non-negative integer n, every graph G admits edge magic labeling having property $G \cong G \cup nK_1$, which is also called deficiency. Edge magic deficiency of a graph G represented by $\mu(G)$.

Upper bound of edge-magic deficiency in terms of Fibonaci numbers is given in [40]. Giaro et al. [34] discussed the deficiency of some bipartite graphs. Edge magic deficiency of different graphs has been discussed in [32]. Ngurah et al. [45] also determined edge magic deficiency of some families of graphs.

Chapter 4

Super d-Anti-magic Labeling of Subdivided qC_5

4.1 Introduction

A qC_4-snake graph was first introduced by Barrientos [30] , Rosa [46] gave the generalized concept of triangular snake. A qC_n-snake can be defined as a connected graph with q blocks, each of the blocks is isomorphic to the cycle C_n, such that the block-cut-vertex graph is a path.

Super face anti-magic labeling of qC_5-snake graph, subdivision of qC_5-snake graph and isomorphic copies of qC_5-snake graph, as well as subdivision of isomorphic copies of qC_5-snake graph having string $(1,1,\ldots,1)$ and $(2,2,\ldots,2)$, are discussed here. In graph theory, a string graph is an intersection graph of curves in the plane, each curve is called a string.

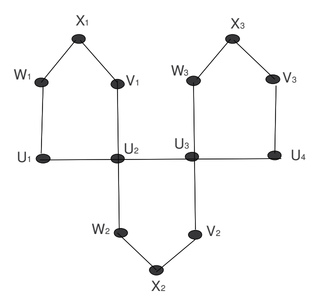

Figure 4.1: $3C_5$ snake graph having string $(1,1,1)$

$3C_5$ shown in figure 4.1 having 13 vertices, 15 edges and 3 faces while qC_5 has $4q+1$ vertices, $5q$ edges and q faces.

4.2 Super Anti-magic Labeling of qC_5- Snake Graph

Super face anti-magic labeling of qC_5- snake graph as well as isomorphic copies of qC_5 -snake graph are formulated in this section. Throughout this section $1 \leq s \leq q$.

Theorem 4.2.1. *Snake graph* $H \cong qC_5, q \geq 2$ *having string* $(2,2,\ldots,2)$ *admits super* 1*-anti-magic labeling. [36]*

Proof.

Let $m = |V(H)|, o = |E(H)|$ and $p = |F(H)|$. Then $m = 4q+1, o = 5q$ and $p = q$.
$\lambda : V(H) \cup E(H) \cup F(H) \to \{1, 2, \ldots, m+o+p\}$

$$\lambda(u_s) = s, \quad 1 \leq s \leq q+1 \tag{4.2.1}$$

$$\lambda(v_s) = (q+1)+s, \tag{4.2.2}$$

$$\lambda(w_s) = 3q+2-s, \tag{4.2.3}$$

$$\lambda(x_s) = 4q+2-s, \tag{4.2.4}$$

For edges

$$\lambda(w_s x_s) = m+s, \tag{4.2.5}$$

$$\lambda(x_s u_{s+1}) = m+2q+1-s, \tag{4.2.6}$$

$$\lambda(u_{s+1} v_s) = m+3q+1-s, \tag{4.2.7}$$

$$\lambda(u_s v_s) = m+3q+s, \tag{4.2.8}$$

$$\lambda(u_s w_s) = m+4q+s, \tag{4.2.9}$$

For faces

$$\lambda(f_s) = m+o+q+1-s, \tag{4.2.10}$$

Theorem 4.2.2. *Snake graph $H \cong qC_5, q \geq 3$ having string $(1, 1, \ldots, 1)$ admits super 1-anti-magic labeling. [36]*

Proof.

Let $m = |V(H)|, o = |E(H)|$ and $p = |F(H)|$. Then $m = 4q+1, o = 5q$ and $p = q$.

$\lambda : V(H) \cup E(H) \cup F(H) \to \{1, 2, \ldots, m+o+p\}$

$$\lambda(u_s) = s, \quad 1 \leq s \leq q+1 \tag{4.2.11}$$

$$\lambda(v_s) = (q+1)+s, \tag{4.2.12}$$

$$\lambda(w_s) = 3q+2-s, \tag{4.2.13}$$

$$\lambda(x_s) = 4q+2-s, \tag{4.2.14}$$

We symbolize the edges of H as follows:

$$\lambda(u_s u_{s+1}) = m+q+1-s, \tag{4.2.15}$$

$$\lambda(u_s w_s) = m+q+s, \tag{4.2.16}$$

$$\lambda(w_s x_s) = m+2q+s, \tag{4.2.17}$$

$$\lambda(x_s v_s) = m+4q+1-s, \tag{4.2.18}$$

$$\lambda(v_s u_{s+1}) = m+5q+1-s, \tag{4.2.19}$$

We symbolize the faces of H as follows:

$$\lambda(f_s) = m+o+s, \tag{4.2.20}$$

Theorem 4.2.3. *Snake graph $H \cong rqC_5, q \geq 2$ having string $(2, 2, \ldots, 2)$ admits super 1-anti-magic labeling. [36]*

Proof.

Let $m = |V(H)|$, $o = |E(H)|$ and $p = |F(H)|$.Then we have $m = n(4q+1)$, $o = 5nq$ and $p = nq$, $1 \leq r \leq n$. $\lambda : V(H) \cup E(H) \cup F(H) \to \{1, 2, \ldots, m+o+p\}$

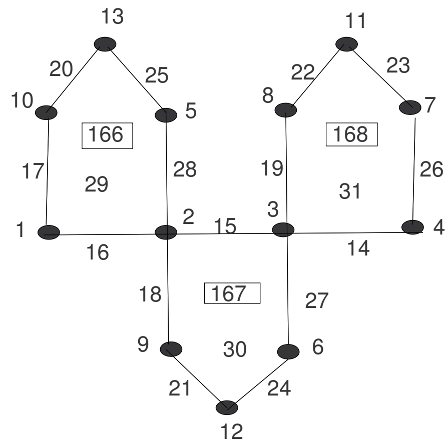

Figure 4.2: Face anti-magic labeling of $3C_5$ snake graph having string $(1,1,1)$

For vertices

$$\lambda(u_s^r) = s + (r-1)(q+1) \quad 1 \leq s \leq q+1 \tag{4.2.21}$$

$$\lambda(v_s^r) = 2nq + n + 1 - s - (r-1)q \tag{4.2.22}$$

$$\lambda(x_s^r) = 3nq + n + 1 - s - (r-1)q \tag{4.2.23}$$

$$\lambda(w_s^r) = 4nq + n + 1 - s - (r-1)q \tag{4.2.24}$$

For edges

$$\lambda(u_s^r w_s^r) = m + (r-1)q + s \tag{4.2.25}$$

$$\lambda(v_s^r u_{s+1}^r) = m + 2nq + 1 - s - (r-1)q \tag{4.2.26}$$

$$\lambda(u_s^r v_s^r) = m + 3nq + 1 - (r-1)q - s \tag{4.2.27}$$

$$\lambda(w_s^r x_s^r) = m + 3nq + (r-1)q + s \tag{4.2.28}$$

$$\lambda(x_s^r u_{s+1}^r) = m + 4nq + (r-1)q + s \tag{4.2.29}$$

For faces

$$\lambda(f_s^r) = m + o + n - (r-1) + (s-1)n \tag{4.2.30}$$

Theorem 4.2.4. *Snake graph $H \cong rqC_5, q \geq 3$ having string $(1,1,\ldots,1)$ admits super 1-anti-magic labeling. [36]*

Proof.

Let $m = |V(H)|$, $o = |E(H)|$ and $p = |F(H)|$.Then we have $m = n(4q+1)$, $o = 5nq$ and $p = nq$, $1 \leq r \leq n$. $\lambda : V(H) \cup E(H) \cup F(H) \rightarrow \{1,2,\ldots,m+o+p\}$

For vertices

$$\lambda(u_s^r) = s + (r-1)(q+1) \quad 1 \leq s \leq q+1 \tag{4.2.31}$$

$$\lambda(v_s^r) = 2nq + n + 1 - (r-1)q - s \tag{4.2.32}$$

$$\lambda(x_s^r) = 3nq + n + 1 - s - (r-1)q \tag{4.2.33}$$

$$\lambda(w_s^r) = 4nq + n + 1 - s - (r-1)q \tag{4.2.34}$$

For edges

$$\lambda(u_s^r w_s^r) = m + (r-1)q + s \tag{4.2.35}$$

$$\lambda(u_{s+1}^r v_s^r) = m + 2nq + 1 - s - (r-1)q \tag{4.2.36}$$

$$\lambda(u_s^r u_{s+1}^r) = m + 3nq + 1 - (r-1)q - s \tag{4.2.37}$$

$$\lambda(w_s^r x_s^r) = m + 3nq + (r-1)q + s \tag{4.2.38}$$

$$\lambda(x_s^r v_s^r) = m + 4nq + (r-1)q + s \tag{4.2.39}$$

For faces

$$\lambda(f_s^r) = m + o + n - (r-1) + (s-1)n \tag{4.2.40}$$

4.3 Super Anti-magic Labeling of Subdivided qC_5- Snake Graph

Super face anti-magic labeling of subdivided qC_5- snake graph as well as isomorphic copies of subdivided qC_5 -snake graph are formulated in this section. Throughout this section $1 \leq s \leq q$.

Theorem 4.3.1. *Snake graph $H \cong qC_5, q \geq 3$ having string $(1,1,\ldots,1)$ with 1 subdivision admits super 1-anti-magic labeling. [36]*

Proof.

Let $m = |V(H)|$, $o = |E(H)|$ and $p = |F(H)|$. Then $m = 9q+1$, $o = 10q$ and $p = q$.
$\lambda : V(H) \cup E(H) \cup F(H) \to \{1,2,...,m+o+p\}$

$$\lambda(u_s) = s, \quad 1 \leq s \leq q+1 \tag{4.3.41}$$

$$\lambda(v_s) = (q+1)+s, \tag{4.3.42}$$

$$\lambda(w_s) = 3q+2-s, \tag{4.3.43}$$

$$\lambda(x_s) = 4q+2-s, \tag{4.3.44}$$

For partitions

$$\lambda(a_s) = 5q+2-s, \tag{4.3.45}$$

$$\lambda(e_s) = 6q+2-s, \tag{4.3.46}$$

$$\lambda(b_s) = 7q+2-s, \tag{4.3.47}$$

$$\lambda(c_s) = 7q+1+s, \tag{4.3.48}$$

$$\lambda(d_s) = 9q+2-s, \tag{4.3.49}$$

For edges

$$\lambda(u_s a_s) = m+s, \tag{4.3.50}$$

$$\lambda(a_s u_{s+1}) = m+2q+1-s, \tag{4.3.51}$$

$$\lambda(v_s e_s) = m+3q+1-s, \tag{4.3.52}$$

$$\lambda(e_s u_{s+1}) = m+3q+s, \tag{4.3.53}$$

$$\lambda(x_s d_s) = m + 5q + 1 - s, \tag{4.3.54}$$

$$\lambda(d_s v_s) = m + 5q + s, \tag{4.3.55}$$

$$\lambda(w_s c_s) = m + 7q + 1 - s, \tag{4.3.56}$$

$$\lambda(c_s x_s) = m + 7q + s, \tag{4.3.57}$$

$$\lambda(w_s b_s) = m + 9q + 1 - s, \tag{4.3.58}$$

$$\lambda(b_s u_s) = m + 9q + s, \tag{4.3.59}$$

For faces

$$\lambda(f_s) = m + o + s, \tag{4.3.60}$$

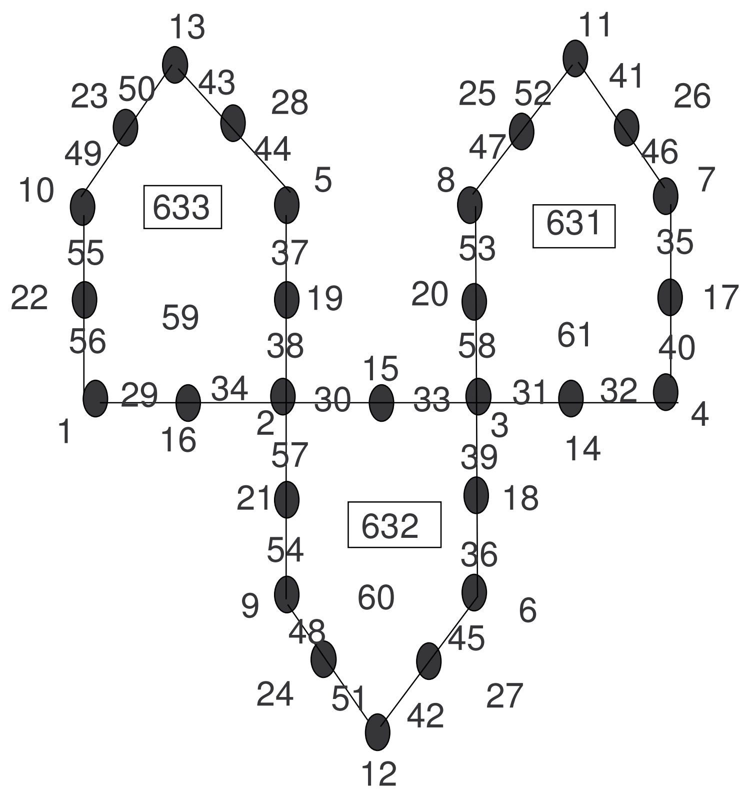

Figure 4.3: Face anti-magic labeling of $3C_5$ snake graph with 1 subdivision

Theorem 4.3.2. *Snake graph $H \cong qC_5, q \geq 2$ having string $(2,2,\ldots,2)$ with 1 subdivision, admits super 1-anti-magic labeling. [36]*

Proof.

Let $m = |V(H)|, o = |E(H)|$ and $p = |F(H)|$. Then $m = 9q+1$, $o = 10q$ and $p = q$.
$\lambda : V(H) \cup E(H) \cup F(H) \rightarrow \{1,2,\ldots,m+o+p\}$

$$\lambda(u_s) = s, \quad 1 \leq s \leq q+1 \tag{4.3.61}$$

$$\lambda(v_s) = (q+1)+s, \tag{4.3.62}$$

$$\lambda(w_s) = 3q+2-s, \tag{4.3.63}$$

$$\lambda(x_s) = 4q+2-s, \tag{4.3.64}$$

For partitions

$$\lambda(a_s) = 5q+2-s, \tag{4.3.65}$$

$$\lambda(e_s) = 6q+2-s, \tag{4.3.66}$$

$$\lambda(b_s) = 7q+2-s, \tag{4.3.67}$$

$$\lambda(c_s) = 7q+1+s, \tag{4.3.68}$$

$$\lambda(d_s) = 9q+2-s, \tag{4.3.69}$$

For edges

$$\lambda(u_s a_s) = m+s, \tag{4.3.70}$$

$$\lambda(a_s v_s) = m+2q+1-s, \tag{4.3.71}$$

$$\lambda(u_{s+1} e_s) = m+3q+1-s, \tag{4.3.72}$$

$$\lambda(e_s v_s) = m+3q+s, \tag{4.3.73}$$

$$\lambda(x_s d_s) = m + 5q + 1 - s, \tag{4.3.74}$$

$$\lambda(d_s u_{s+1}) = m + 5q + s, \tag{4.3.75}$$

$$\lambda(w_s c_s) = m + 7q + 1 - s, \tag{4.3.76}$$

$$\lambda(c_s x_s) = m + 7q + s, \tag{4.3.77}$$

$$\lambda(w_s b_s) = m + 9q + 1 - s, \tag{4.3.78}$$

$$\lambda(b_s u_s) = m + 9q + s, \tag{4.3.79}$$

For faces

$$\lambda(f_s) = m + o + s, \tag{4.3.80}$$

Theorem 4.3.3. *Snake graph $H \cong qC_5, q \geq 3$ having string $(1,1,\ldots,1)$ with 2 subdivisions, admits super 1-anti-magic labeling. [36]*

Proof.

Let $m = |V(H)|, o = |E(H)|$ and $p = |F(H)|$. Then $m = 14q+1, o = 15q$ and $p = q$.
$\lambda : V(H) \cup E(H) \cup F(H) \rightarrow \{1,2,\ldots,m+o+p\}$

$$\lambda(u_s) = s, \quad 1 \leq s \leq q+1 \tag{4.3.81}$$

$$\lambda(v_s) = (q+1) + s, \tag{4.3.82}$$

$$\lambda(w_s) = 3q + 2 - s, \tag{4.3.83}$$

$$\lambda(x_s) = 4q + 2 - s, \tag{4.3.84}$$

For partitions

$$\lambda(a_{s,1}) = 5q + 2 - s, \tag{4.3.85}$$

$$\lambda(a_{s,2}) = 5q + 1 + s, \tag{4.3.86}$$

$$\lambda(b_{s,1}) = 7q+2-s, \tag{4.3.87}$$

$$\lambda(b_{s,2}) = 7q+1+s, \tag{4.3.88}$$

$$\lambda(c_{s,1}) = 9q+2-s, \tag{4.3.89}$$

$$\lambda(c_{s,2}) = 9q+1+s, \tag{4.3.90}$$

$$\lambda(d_{s,1}) = 11q+2-s, \tag{4.3.91}$$

$$\lambda(d_{s,2}) = 11q+1+s, \tag{4.3.92}$$

$$\lambda(e_{s,1}) = 13q+2-s, \tag{4.3.93}$$

$$\lambda(e_{s,2}) = 13q+1+s, \tag{4.3.94}$$

For edges

$$\lambda(u_s a_{s,1}) = m+s, \tag{4.3.95}$$

$$\lambda(a_{s,1}a_{s,2}) = m+q+s, \tag{4.3.96}$$

$$\lambda(a_{s,2}u_{s+1}) = m+2q+s, \tag{4.3.97}$$

$$\lambda(v_s e_{s,1}) = m+4q+1-s, \tag{4.3.98}$$

$$\lambda(e_{s,1}e_{s,2}) = m+5q+1-s, \tag{4.3.99}$$

$$\lambda(e_{s,2}u_{s+1}) = m+6q+1-s, \tag{4.3.100}$$

$$\lambda(x_s d_{s,1}) = m+6q+s, \tag{4.3.101}$$

$$\lambda(d_{s,1}d_{s,2}) = m+7q+s, \tag{4.3.102}$$

$$\lambda(d_{s,2}v_s) = m+8q+s, \tag{4.3.103}$$

$$\lambda(x_s c_{s,1}) = m+10q+1-s, \tag{4.3.104}$$

$$\lambda(c_{s,1}c_{s,2}) = m + 11q + 1 - s, \tag{4.3.105}$$

$$\lambda(c_{s,2}w_s) = m + 12q + 1 - s, \tag{4.3.106}$$

$$\lambda(w_s b_{s,1}) = m + 13q + 1 - s, \tag{4.3.107}$$

$$\lambda(b_{s,1}b_{s,2}) = m + 14q + 1 - s, \tag{4.3.108}$$

$$\lambda(b_{s,2}u_s) = m + 15q + 1 - s, \tag{4.3.109}$$

For faces

$$\lambda(f_s) = m + o + s, \tag{4.3.110}$$

Theorem 4.3.4. *Snake graph $H \cong qC_5, q \geq 2$ having string $(2,2,\ldots,2)$ with 2 subdivisions, admits super 1-anti-magic labeling. [36]*

Proof.
Let $m = |V(H)|, o = |E(H)|$ and $p = |F(H)|$. Then $m = 14q + 1$, $o = 15q$ and $p = q$.
$\lambda : V(H) \cup E(H) \cup F(H) \to \{1, 2, \ldots, m + o + p\}$

$$\lambda(u_s) = s, \quad 1 \leq s \leq q + 1 \tag{4.3.111}$$

$$\lambda(v_s) = (q + 1) + s, \tag{4.3.112}$$

$$\lambda(w_s) = 3q + 2 - s, \tag{4.3.113}$$

$$\lambda(x_s) = 4q + 2 - s, \tag{4.3.114}$$

For partitions

$$\lambda(a_{s,1}) = 5q + 2 - s, \tag{4.3.115}$$

$$\lambda(a_{s,2}) = 5q + 1 + s, \tag{4.3.116}$$

$$\lambda(b_{s,1}) = 7q + 2 - s, \tag{4.3.117}$$

$$\lambda(b_{s,2}) = 7q+1+s, \tag{4.3.118}$$

$$\lambda(c_{s,1}) = 9q+2-s, \tag{4.3.119}$$

$$\lambda(c_{s,2}) = 9q+1+s, \tag{4.3.120}$$

$$\lambda(d_{s,1}) = 11q+2-s, \tag{4.3.121}$$

$$\lambda(d_{s,2}) = 11q+1+s, \tag{4.3.122}$$

$$\lambda(e_{s,1}) = 13q+2-s, \tag{4.3.123}$$

$$\lambda(e_{s,2}) = 13q+1+s, \tag{4.3.124}$$

For edges

$$\lambda(u_s a_{s,1}) = m+s, \tag{4.3.125}$$

$$\lambda(a_{s,1} a_{s,2}) = m+q+s, \tag{4.3.126}$$

$$\lambda(a_{s,2} v_s) = m+2q+s, \tag{4.3.127}$$

$$\lambda(u_{s+1} e_{s,1}) = m+4q+1-s, \tag{4.3.128}$$

$$\lambda(e_{s,1} e_{s,2}) = m+5q+1-s, \tag{4.3.129}$$

$$\lambda(e_{s,2} v_s) = m+6q+1-s, \tag{4.3.130}$$

$$\lambda(x_s d_{s,1}) = m+6q+s, \tag{4.3.131}$$

$$\lambda(d_{s,1} d_{s,2}) = m+7q+s, \tag{4.3.132}$$

$$\lambda(d_{s,2} u_{s+1}) = m+8q+s, \tag{4.3.133}$$

$$\lambda(x_s c_{s,1}) = m+10q+1-s, \tag{4.3.134}$$

$$\lambda(c_{s,1} c_{s,2}) = m+11q+1-s, \tag{4.3.135}$$

$$\lambda(c_{s,2}w_s) = m + 12q + 1 - s, \tag{4.3.136}$$

$$\lambda(w_sb_{s,1}) = m + 13q + 1 - s, \tag{4.3.137}$$

$$\lambda(b_{s,1}b_{s,2}) = m + 14q + 1 - s, \tag{4.3.138}$$

$$\lambda(b_{s,2}u_s) = m + 15q + 1 - s, \tag{4.3.139}$$

For faces

$$\lambda(f_s) = m + o + s, \tag{4.3.140}$$

Theorem 4.3.5. *Snake graph $H \cong rqC_5, q \geq 3$ having string $(1,1,\dots,1)$ with 1 subdivision, admits super 1-anti-magic labeling. [36]*

Proof.
Let $m = |V(H)|$, $o = |E(H)|$ and $p = |F(H)|$.Then we have $m = n(9q+1)$, $o = 10nq$ and $f = nq$, $1 \leq r \leq n$. $\lambda : V(H) \cup E(H) \cup F(H) \to \{1, 2, \dots, m+o+p\}$
For vertices

$$\lambda(u_s^r) = s + (r-1)(q+1) \quad 1 \leq s \leq q+1 \tag{4.3.141}$$

$$\lambda(v_s^r) = 2nq + n + 1 - s - (r-1)q \tag{4.3.142}$$

$$\lambda(x_s^r) = 3nq + n + 1 - s - (r-1)q \tag{4.3.143}$$

$$\lambda(w_s^r) = 4nq + n + 1 - s - (r-1)q \tag{4.3.144}$$

For partitions

$$\lambda(a_s^r) = 4nq + n + s + (r-1)q, \tag{4.3.145}$$

$$\lambda(d_s^r) = 6nq + n + 1 - s - (r-1)q, \tag{4.3.146}$$

$$\lambda(e_s^r) = 7nq + n + 1 - s - (r-1)q, \tag{4.3.147}$$

$$\lambda(b_s^r) = 7nq + n + s + (r-1)q, \tag{4.3.148}$$

$$\lambda(c_s^r) = 8nq + n + s + (r-1)q, \tag{4.3.149}$$

For edges

$$\lambda(u_s^r e_s^r) = m + nq + 1 - s - (r-1)q, \tag{4.3.150}$$

$$\lambda(e_s^r u_{s+1}^r) = m + nq + s + (r-1)q, \tag{4.3.151}$$

$$\lambda(v_s^r d_s^r) = m + 3nq + 1 - s - (r-1)q, \tag{4.3.152}$$

$$\lambda(d_s^r u_{s+1}^r) = m + 4nq + 1 - s - (r-1)q, \tag{4.3.153}$$

$$\lambda(w_s^r a_s^r) = m + 4nq + s + (r-1)q, \tag{4.3.154}$$

$$\lambda(a_s^r u_s^r) = m + 6nq + 1 - s - (r-1)q, \tag{4.3.155}$$

$$\lambda(w_s^r b_s^r) = m + 7nq + 1 - s - (r-1)q, \tag{4.3.156}$$

$$\lambda(b_s^r x_s^r) = m + 7nq + s + (r-1)q, \tag{4.3.157}$$

$$\lambda(x_s^r c_s^r) = m + 8nq + s + (r-1)q, \tag{4.3.158}$$

$$\lambda(c_s^r v_s^r) = m + 9nq + s + (r-1)q, \tag{4.3.159}$$

For faces

$$\lambda(f_s^r) = m + o + nq - n(s-1) - (r-1), \tag{4.3.160}$$

Theorem 4.3.6. *Snake graph $H \cong rqC_5, q \geq 2$ having string $(2,2,\ldots,2)$ with 1 subdivision, admits super 1-anti-magic labeling. [36]*

Proof.

Let $m = |V(H)|, o = |E(H)|$ and $p = |F(H)|$.Then we have $m = n(9q+1), o = 10nq$ and $p = nq$, $1 \leq r \leq n$. $\lambda : V(H) \cup E(H) \cup F(H) \to \{1,2,\ldots,m+o+p\}$

For vertices

$$\lambda(u_s^r) = s + (r-1)(q+1), \quad 1 \leq s \leq q+1 \tag{4.3.161}$$

$$\lambda(v_s^r) = 2nq + n + 1 - s - (r-1)q, \tag{4.3.162}$$

$$\lambda(x_s^r) = 3nq + n + 1 - s - (r-1)q, \tag{4.3.163}$$

$$\lambda(w_s^r) = 4nq + n + 1 - s - (r-1)q, \tag{4.3.164}$$

For partitions

$$\lambda(a_s^r) = 4nq + n + s + (r-1)q, \tag{4.3.165}$$

$$\lambda(d_s^r) = 6nq + n + 1 - s - (r-1)q, \tag{4.3.166}$$

$$\lambda(e_s^r) = 7nq + n + 1 - s - (r-1)q, \tag{4.3.167}$$

$$\lambda(b_s^r) = 7nq + n + s + (r-1)q, \tag{4.3.168}$$

$$\lambda(c_s^r) = 8nq + n + s + (r-1)q, \tag{4.3.169}$$

For edges

$$\lambda(u_s^r e_s^r) = m + nq + 1 - s - (r-1)q, \tag{4.3.170}$$

$$\lambda(e_s^r v_s^r) = m + nq + s + (r-1)q, \tag{4.3.171}$$

$$\lambda(u_{s+1}^r d_s^r) = m + 3nq + 1 - s - (r-1)q, \tag{4.3.172}$$

$$\lambda(d_s^r v_s^r) = m + 4nq + 1 - s - (r-1)q, \tag{4.3.173}$$

$$\lambda(w_s^r a_s^r) = m + 4nq + s + (r-1)q, \tag{4.3.174}$$

$$\lambda(a_s^r u_s^r) = m + 6nq + 1 - s - (r-1)q, \tag{4.3.175}$$

$$\lambda(w_s^r b_s^r) = m + 7nq + 1 - s - (r-1)q, \tag{4.3.176}$$

$$\lambda(b_s^r x_s^r) = m + 7nq + s + (r-1)q, \tag{4.3.177}$$

$$\lambda(x_s^r c_s^r) = m + 8nq + s + (r-1)q, \tag{4.3.178}$$

$$\lambda(c_s^r u_{s+1}^r) = m + 9nq + s + (r-1)q, \tag{4.3.179}$$

For faces

$$\lambda(f_s^r) = m + o + nq - n(s-1) - (r-1), \tag{4.3.180}$$

Chapter 5

Face Anti-magic Labeling of Generalized qC_n-Snake Graph

5.1 Introduction

A qC_n-snake is a connected graph with q blocks, each one isomorphic to cycle C_n, such that the block-cutpoint graph is a path. We call these graphs cyclic snakes. The definition of qC_4-snake graphs introduced Barrientos in [30] as a natural extension of triangular snake graphs defined by Rosa [46]. The order and size of qC_n-snake graph are defined as $(n-1)q+1$ and nq. In this chapter, we discuss super face anti-magic labeling of generalized qC_n snake graph as well as partition of generalized qC_n snake graph.

5.2 Super Face Anti-magic Labeling of qC_n- Snake Graph

Super face anti-magic labeling of qC_n snake graphs as well as subdivision of qC_n snake graphs with string $(1,1,\ldots,1)$ and string $(2,2,\ldots,2)$ are investigated in this section. Throughout this section $1 \leq s \leq q$.

Theorem 5.2.1. *Snake graph $H \cong qC_n, q \geq 2$ and even n, having string $(2,2,\ldots,2)$ and $(1,1,\ldots,1)$, admits super 1-anti-magic labeling. [37]*

Proof.
Let $m = |V(H)|, o = |E(H)|$ and $|F(H)| = p$. Then $m = (n-1)q+1, o = nq$ and $p = q$.
$\lambda : V(H) \cup E(H) \cup F(H) \rightarrow \{1,2,\ldots,m+o+p\}$

$$\lambda(u_{s,1}) = s, \quad 1 \leq s \leq q+1 \tag{5.2.1}$$

$$\lambda(u_{s,2}) = 2q+2-s, \tag{5.2.2}$$

$$\lambda(u_{s,3}) = 4q+2-s, \tag{5.2.3}$$

$$\vdots \qquad \vdots$$

$$\lambda(u_{s,\frac{n}{2}}) = (n-2)q+2-s, \tag{5.2.4}$$

$$\lambda(u_{s,\frac{n+4}{2}}) = 2q+1+s, \tag{5.2.5}$$

$$\lambda(u_{s,\frac{n+6}{2}}) = 4q+1+s, \tag{5.2.6}$$

$$\vdots \qquad \vdots \tag{5.2.7}$$

$$\lambda(u_{s,n-1}) = (n-2)q+1+s, \tag{5.2.8}$$

For edges

$$\lambda(u_{s,1}u_{s,2}) = m+s, \tag{5.2.9}$$

$$\lambda(u_{s,2}u_{s,3}) = m+2q+s, \tag{5.2.10}$$

$$\vdots \qquad \vdots$$

$$\lambda(u_{s,\frac{n}{2}}u_{s+1,1}) = m+(n-1)q+s, \tag{5.2.11}$$

$$\lambda(u_{s,1}u_{s,\frac{n+4}{2}}) = m+2q+1-s, \tag{5.2.12}$$

$$\lambda(u_{s,\frac{n+4}{2}}v_{s,\frac{n+6}{2}}) = m+4q+1-s, \tag{5.2.13}$$

$$\vdots \qquad \vdots$$

$$\lambda(u_{s,n-1}v_{s+1,2}) = m+nq+1-s, \tag{5.2.14}$$

For faces

$$\lambda(f_s) = m+o+p+1-s, \tag{5.2.15}$$

Theorem 5.2.2. *Snake graph $H \cong qC_n, q \geq 2$ and odd n, having string $(2,2,\ldots,2)$ and $(1,1,\ldots,1)$, admits super 1-anti-magic labeling. [37]*

Proof.

Let $m = |V(H)|, o = |E(H)|$ and $p = |F(H)|$. Then $m = (n-1)q+1, o = nq$ and $f = q$. $\lambda : V(H) \cup E(H) \cup F(H) \to \{1,2,\ldots,m+o+p\}$

$$\lambda(u_{s,1}) = s, \quad 1 \leq s \leq q+1 \tag{5.2.16}$$

$$\lambda(u_{s,2}) = 2q+1+s, \tag{5.2.17}$$

$$\lambda(u_{s,3}) = 4q+1+s, \tag{5.2.18}$$

$$\vdots \qquad \vdots$$

$$\lambda(u_{s,\frac{n-1}{2}}) = (n-2)q+1+s, \tag{5.2.19}$$

$$\lambda(u_{s,\frac{n+3}{2}}) = 2q+2-s, \tag{5.2.20}$$

$$\lambda(u_{s,\frac{n+5}{2}}) = 4q+2-s, \tag{5.2.21}$$

$$\vdots \qquad \vdots$$

$$\lambda(u_{s,n}) = (n-2)q+2-s, \tag{5.2.22}$$

For edges

$$\lambda(u_{s,1}u_{s,2}) = m+s, \tag{5.2.23}$$

$$\lambda(u_{s,2}u_{s,3}) = m+2q+1-s, \tag{5.2.24}$$

$$\lambda(u_{s,3}u_{s,4}) = m+2q+s, \tag{5.2.25}$$

$$\lambda(u_{s,4}u_{s,5}) = m+4q+1-s, \tag{5.2.26}$$

$$\vdots \qquad \vdots$$

$$\lambda(u_{s,n}v_{s,1}) = m+(n-1)q+s, \tag{5.2.27}$$

For faces

$$\lambda(f_s) = m + o + p + 1 - s, \tag{5.2.28}$$

5.3 Super Anti-magic Labeling of Subdivided qC_n- Snake Graph

Super anti-magic labeling of qC_n- snake graph of string $(1,1,\ldots,1)$ and string $(2,2,\ldots,2)$ with 1 subdivision has been formulated in this section. Throughout this section $1 \leq s \leq q$.

Theorem 5.3.1. *Snake graph $H \cong qC_n, q \geq 2$ and even n, having string $(2,2,\ldots,2)$ and $(1,1,\ldots,1)$ with 1 subdivision, admits super 1-anti-magic labeling. [37]*

Proof.
Let $m = |V(H)|, o = |E(H)|$ and $p = |F(H)|$. Then $m = 2nq - q + 1, o = 2nq$ and $p = q$.
$\lambda : V(H) \cup E(H) \cup F(H) \to \{1, 2, \ldots, m+o+p\}$

$$\lambda(u_{s,1}) = s, \quad 1 \leq s \leq q+1 \tag{5.3.29}$$

$$\lambda(u_{s,2}) = 4q + 2 - s, \tag{5.3.30}$$

$$\lambda(u_{s,3}) = 6q + 2 - s, \tag{5.3.31}$$

$$\vdots \qquad \vdots$$

For partitions

$$\lambda(a_{s,1}) = 2q + 2 - s, \tag{5.3.32}$$

$$\lambda(a_{s,2}) = 2q + 1 + s, \tag{5.3.33}$$

$$\lambda(a_{s,3}) = 4q + 1 + s, \tag{5.3.34}$$

$$\vdots \qquad \vdots$$

For edges

$$\lambda(u_{s,1}a_{s,1}) = m+s, \tag{5.3.35}$$

$$\lambda(a_{s,1}u_{s,2}) = m+2q+1-s, \tag{5.3.36}$$

$$\lambda(u_{s,2}a_{s,2}) = m+2q+s, \tag{5.3.37}$$

$$\lambda(a_{s,2}u_{s,3}) = m+4q+1-s, \tag{5.3.38}$$

$$\vdots \qquad \vdots$$

For faces

$$\lambda(f_s) = m+o+p+1-s, \tag{5.3.39}$$

Theorem 5.3.2. *Snake graph $H \cong qC_n, q \geq 2$ and odd n, having string $(2,2,\ldots,2)$ and $(1,1,\ldots,1)$ with 1 subdivision, admits super 1-anti-magic labeling. [37]*

Proof.
Let $m = |V(H)|, o = |E(H)|$ and $p = |F(H)|$. Then $m = 2nq-q+1, o = 2nq$ and $p = q$.
$\lambda : V(H) \cup E(H) \cup F(H) \to \{1,2,\ldots,m+o+p\}$

$$\lambda(u_{s,1}) = s, \quad 1 \leq s \leq q+1 \tag{5.3.40}$$

$$\lambda(u_{s,2}) = 2q+2-s, \tag{5.3.41}$$

$$\lambda(u_{s,3}) = 2q+1+s, \tag{5.3.42}$$

$$\lambda(u_{s,4}) = 4q+2-s, \tag{5.3.43}$$

$$\lambda(u_{s,5}) = 4q+1+s, \tag{5.3.44}$$

$$\vdots \qquad \vdots$$

For partitions, here $y=(n-1)q+1$,

$$\lambda(a_{s,1}) = y+q+1-s, \tag{5.3.45}$$

$$\lambda(a_{s,2}) = y+q+s, \tag{5.3.46}$$

$$\lambda(a_{s,3}) = y+3q+1-s, \tag{5.3.47}$$

$$\vdots \qquad \vdots$$

For edges

$$\lambda(u_{s,1}a_{s,1}) = m+q+1-s, \tag{5.3.48}$$

$$\lambda(a_{s,1}u_{s,2}) = m+q+s, \tag{5.3.49}$$

$$\lambda(u_{s,2}a_{s,2}) = m+3q+1-s, \tag{5.3.50}$$

$$\lambda(a_{s,2}u_{s,3}) = m+3q+s, \tag{5.3.51}$$

$$\vdots \qquad \vdots$$

For faces

$$\lambda(f_s) = m+o+p+1-s, \tag{5.3.52}$$

Chapter 6

Magic and Anti-magic Labeling on Subdivision of $G \cong P_o \times P_r$

6.1 Introduction

Face magic labeling of grid, super face magic labeling of subdivided grid graph as well as subdivided triangular ladder are formulated in this chapter.

6.2 Face Magic Labeling of Grid Graph

Theorem 6.2.1. *Grid graph $H \cong P_4 \times P_o, o \geq 2$ admits super magic labeling. [48]*

Proof.

Let $m = |V(H)|$. Then $m = 4o$.

For vertice:

$V(H) = \{v_{cr} \mid 1 \leq c \leq 4; 1 \leq r \leq o\}$. $\lambda : V \to \{1, 2, \ldots, m\}$

For $1 \leq r \leq o$

$$\lambda(v_{1,r}) = \begin{cases} 2r-1 & for \quad r \ odd \\ 2r & for \quad r \ even \end{cases} \tag{6.2.1}$$

$$\lambda(v_{2,r}) = \begin{cases} m-2r+2 & for \quad r \ odd \\ m-2r+1 & for \quad r \ even \end{cases} \tag{6.2.2}$$

$$\lambda(v_{3,r}) = \begin{cases} 2r & for \quad r \ odd \\ 2r-1 & for \quad r \ even \end{cases} \tag{6.2.3}$$

$$\lambda(v_{4,r}) = \begin{cases} m-2r+1 & for \quad r \ odd \\ m-2r+2 & for \quad r \ even \end{cases} \tag{6.2.4}$$

Subdivided grid

Subdivided grid graph can be obtained if we insert more vertices on the vertices, edges and faces of subdivided grid graph as shown in Figure 6.1.

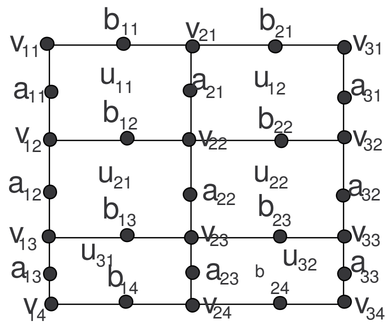

Figure 6.1: Subdivided grid $H \cong P_4 \times P_3$ with 1 subdivision.

Theorem 6.2.2. *Subdivided grid graph $H \cong P_3 \times P_o, o \geq 2$, with 1 subdivision, admits super magic labeling of type $(1,0,0)$. [48]*

Proof.

Let $|V(H)| = m$ and $p = |F(H)|$. Then $m = 4o + 2p + 1$ and $p = 2o - 2$.

$\lambda : V \to \{1, 2, \ldots, m\}$.

For $1 \leq t \leq o$

$$\lambda(v_{1,t}) = \begin{cases} 2t - 1 & for \quad t \;\; odd \\ 2t & for \quad t \;\; even \end{cases} \tag{6.2.5}$$

$$\lambda(v_{2,t}) = 3o - (t-1), \quad 1 \leq t \leq o \tag{6.2.6}$$

$$\lambda(v_{3,t}) = \begin{cases} 2t & for \quad t \;\; odd \\ 2t - 1 & for \quad t \;\; even \end{cases} \tag{6.2.7}$$

$1 \leq t \leq o - 1$

$$\lambda(a_{1,t}) = m - 2o - (3t - 3), \tag{6.2.8}$$

$$\lambda(a_{2,t}) = m - (2o + 1) - (3t - 3), \tag{6.2.9}$$

$$\lambda(a_{3,t}) = m - (2o + 2) - (3t - 3), \tag{6.2.10}$$

$1 \leq t \leq o$

$$\lambda(b_{1,t}) = m - 2o + (2t - 1), \tag{6.2.11}$$

$$\lambda(b_{2,t}) = m - 2o + 2t, \tag{6.2.12}$$

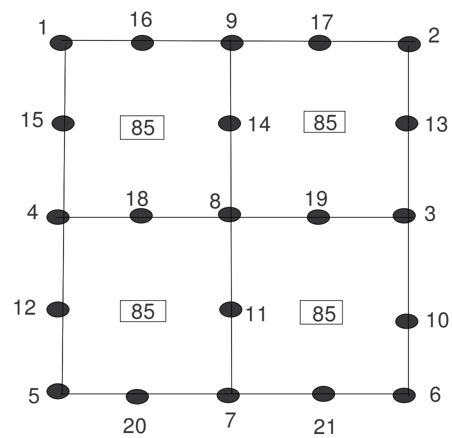

Figure 6.2: Subdivided grid graph $H \cong P_3 \times P_3$ with 1 subdivision.

Theorem 6.2.3. *Subdivided grid graph $H \cong P_3 \times P_o, o \geq 2$, with 1 subdivision, admits super magic labeling of type $(1,1,0)$. [48]*

Proof.

Let $m = |V(H)|$, $d = |E(H)|$ and $p = |F(H)|$. Then $m = 4o + 2p + 1$, $d = 4o + 3p$ and $p = 2o - 2$. $\lambda : V \cup E \to \{1, 2, \ldots, m + d\}$

For $1 \leq t \leq o$

$$\lambda(v_{1,t}) = \begin{cases} 2t - 1 & for \quad t \;\; odd \\ 2t & for \quad t \;\; even \end{cases} \tag{6.2.13}$$

$$\lambda(v_{2,t}) = 3o - (t - 1), \quad 1 \leq t \leq o \tag{6.2.14}$$

$$\lambda(v_{3,t}) = \begin{cases} 2t & for \quad t \;\; odd \\ 2t - 1 & for \quad t \;\; even \end{cases} \tag{6.2.15}$$

For $1 \leq t \leq o - 1$

$$\lambda(a_{1,t}) = 5o - p + 3t - 2 \tag{6.2.16}$$

$$\lambda(a_{2,t}) = 5o - p + 3t - 3 \tag{6.2.17}$$

$$\lambda(a_{3,t}) = 5o - p + 3t - 4 \tag{6.2.18}$$

$1 \leq t \leq o$

$$\lambda(b_{1,t}) = 2o + 2p + 2t, \tag{6.2.19}$$

$$\lambda(b_{2,t}) = 2o + 2p + 2t + 1, \tag{6.2.20}$$

For $s = 1$. $\quad 1 \leq t \leq o - 1$

$$\lambda(v_{s,t}a_{s,t}) = m + 4o + 2 + (4t - 4), \tag{6.2.21}$$

$$\lambda(a_{s,t}v_{s,t+1}) = m + 4o + 1 + (4t - 4), \tag{6.2.22}$$

$$\lambda(v_{s+1,t}a_{s+1,t}) = m + d - (2t - 2), \tag{6.2.23}$$

$$\lambda(a_{s+1,t+1}v_{s+1,t+1}) = m + d - 1 - (2t - 2), \tag{6.2.24}$$

$$\lambda(v_{s+2,t}a_{s+2,t}) = m + 4o + 3 + (4t - 4), \tag{6.2.25}$$

$$\lambda(a_{s+2,t}v_{s+2,t+1}) = m + 4o + 4 + (4t - 4), \tag{6.2.26}$$

Case 1: when o is odd. $\quad 1 \leq t \leq o$.

For t odd.

$$\lambda(v_{s,t}b_{s,t}) = 8o + 2p - 4t + 4 \tag{6.2.27}$$

$$\lambda(b_{s,t}v_{s+1,t}) = 8o + 2p - 4t + 5 \tag{6.2.28}$$

$$\lambda(v_{s+1,t}b_{s+1,t}) = 8o + 2p - 4t + 2 \tag{6.2.29}$$

$$\lambda(b_{s+1,t}v_{s+2,t}) = 8o + 2p - 4t + 3 \tag{6.2.30}$$

For t even.

$$\lambda(v_{s,t}b_{s,t}) = 8o + 2p - 4t + 6 \tag{6.2.31}$$

$$\lambda(v_{s,t}b_{s+1,t}) = 8o + 2p - 4t + 5 \tag{6.2.32}$$

$$\lambda(v_{s+1,t}b_{s+1,t}) = 8o + 2p - 4t + 4 \tag{6.2.33}$$

$$\lambda(b_{s+1,t}v_{s+2,t}) = 8o + 2p - 4t + 3 \tag{6.2.34}$$

Case 2: when o is even. $1 \leq t \leq o$.

For t odd.

$$\lambda(v_{s,t}b_{s,t}) = 8o + 2p - 4t + 5 \tag{6.2.35}$$

$$\lambda(b_{s,t}v_{s+1,t}) = 8o + 2p - 4t + 4 \tag{6.2.36}$$

$$\lambda(v_{s+1,t}b_{s+1,t}) = 8o + 2p - 4t + 2 \tag{6.2.37}$$

$$\lambda(b_{s+1,t}v_{s+2,t}) = 8o + 2p - 4t + 3 \tag{6.2.38}$$

For t even.

$$\lambda(v_{s,t}b_{s,t}) = 8o + 2p - 4t + 3 \tag{6.2.39}$$

$$\lambda(v_{s,t}b_{s+1,t}) = 8o + 2p - 4t + 4 \tag{6.2.40}$$

$$\lambda(v_{s+1,t}b_{s+1,t}) = 8o + 2p - 4t + 5 \tag{6.2.41}$$

$$\lambda(b_{s+1,t}v_{s+2,t}) = 8o + 2p - 4t + 6 \tag{6.2.42}$$

Theorem 6.2.4. *Subdivided grid graph $H \cong P_3 \times P_o, o \geq 2$, with 1 subdivision, admits super magic labeling of type $(1,1,1)$. [48]*

Proof.

Let $m = |V(H)|, d = |E(H)|$ and $p = |F(H)|$. Then $m = 4o + 2p + 1, d = 4o + 3p$ and $p = 2o - 2$. $\lambda : V \cup E \cup F \rightarrow \{1, 2, \ldots, m + d + p\}$.

For $1 \leq t \leq o$

$$\lambda(v_{1,t}) = \begin{cases} 2t-1 & for \quad t \ odd \\ 2t & for \quad t \ even \end{cases} \tag{6.2.43}$$

$$\lambda(v_{2,t}) = 3o-(t-1), \quad 1 \leq t \leq o \tag{6.2.44}$$

$$\lambda(v_{3,t}) = \begin{cases} 2t & for \quad t \ odd \\ 2t-1 & for \quad t \ even \end{cases} \tag{6.2.45}$$

For $1 \leq t \leq o-1$

$$\lambda(a_{1,t}) = \begin{cases} m-3p+o+(3t-3) & for \quad t \ odd \\ m-3p+o+3+(3t-6) & for \quad t \ even \end{cases} \tag{6.2.46}$$

$$\lambda(a_{2,t}) = \begin{cases} m-3p+o-1+(3t-3) & for \quad t \ odd \\ m-3p+o+2+(3t-6) & for \quad t \ even \end{cases} \tag{6.2.47}$$

$$\lambda(a_{3,t}) = \begin{cases} m-3p+o-2+(3t-3) & for \quad t \ odd \\ m-3p+o+1+(3t-6) & for \quad t \ even \end{cases} \tag{6.2.48}$$

$1 \leq t \leq o$

$$\lambda(b_{1,t}) = m-2o+1+(2t-2) \tag{6.2.49}$$

$$\lambda(b_{2,t}) = m-2o+2+(2t-2) \tag{6.2.50}$$

For $s = 1$ and $1 \leq t \leq o-1$.

$$\lambda(v_{s,t}a_{s,t}) = m+4o+2+(4t-4), \tag{6.2.51}$$

$$\lambda(a_{s,t}v_{s,t+1}) = m+4o+3+(4t-4), \tag{6.2.52}$$

$$\lambda(v_{s+1,t}a_{s+1,t}) = m+d-(2t-2), \tag{6.2.53}$$

$$\lambda(a_{s+1,t+1}v_{s+1,t+1}) = m+d-1-(2t-2), \tag{6.2.54}$$

$$\lambda(v_{s+2,t}a_{s+2,t}) = m+4o+1+(4t-4), \tag{6.2.55}$$

$$\lambda(a_{s+2,t}v_{s+2,t+1}) = m+4o+4+(4t-4), \tag{6.2.56}$$

Case 1: when o is odd.

For $1 \le t \le o$

$$\lambda(v_{s,t}b_{s,t}) = \begin{cases} m+4o-3-(4t-4) & for \quad t \ odd \\ m+4o-4-(4t-8) & for \quad t \ even \end{cases} \tag{6.2.57}$$

$$\lambda(b_{s,t}v_{s+1,t}) = \begin{cases} m+4o-2-(4t-4) & for \quad t \ odd \\ m+4o-5-(4t-8) & for \quad t \ even \end{cases} \tag{6.2.58}$$

$$\lambda(v_{s+1,t}b_{s+1,t}) = \begin{cases} m+4o-1-(4t-4) & for \quad t \ odd \\ m+4o-6-(4t-8) & for \quad t \ even \end{cases} \tag{6.2.59}$$

$$\lambda(b_{s+1,t}v_{s+2,t}) = \begin{cases} m+4o-(4t-4) & for \quad t \ odd \\ m+4o-7-(4t-8) & for \quad t \ even \end{cases} \tag{6.2.60}$$

Case 2: when o is even.

For $1 \le t \le o$

$$\lambda(v_{s,t}b_{s,t}) = \begin{cases} m+4o-(4t-4) & for \quad t \ odd \\ m+4o-7-(4t-8) & for \quad t \ even \end{cases} \tag{6.2.61}$$

$$\lambda(b_{s,t}v_{s+1,t}) = \begin{cases} m+4o-1-(4t-4) & for \quad t \ odd \\ m+4o-6-(4t-8) & for \quad t \ even \end{cases} \tag{6.2.62}$$

$$\lambda(v_{s+1,t}b_{s+1,t}) = \begin{cases} m+4o-2-(4t-4) & for \quad t \ odd \\ m+4o-5-(4t-8) & for \quad t \ even \end{cases} \tag{6.2.63}$$

$$\lambda(b_{s+1,t}v_{s+2,t}) = \begin{cases} m+4o-3-(4t-4) & for \quad t \ odd \\ m+4o-4-(4t-8) & for \quad t \ even \end{cases} \tag{6.2.64}$$

For s,t both even or both odd.

$1 \leq s \leq o-1$ and $1 \leq t \leq 2$

$$\lambda(u_{s,t}) = 8o + 6p - 2s + 2 \tag{6.2.65}$$

Either one of s and t are odd and other is even.
$1 \leq s \leq o-1$ and $1 \leq t \leq 2$

$$\lambda(u_{s,t}) = 8o + 6p - 2s + 3 \tag{6.2.66}$$

6.3 Face Anti-magic Labeling of Type $(1,1,1)$ on Subdivided Triangular Ladder

The elements of the graph (vertices, edges and faces) of subdivided triangular ladder graph can be symbolized as

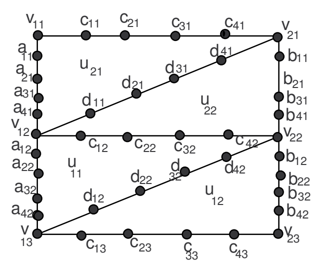

Figure 6.3: Subdivided triangular ladder $H \cong P_3 \times P_2$ with 4 subdivisions.

Theorem 6.3.1. *Subdivided triangular ladder graph $H \cong P_2 \times P_o, o \geq 2$, with 1 subdivision, admits super 1-anti-magic labeling. [48]*

Proof.
Let $m = |V(H)|, g = |E(H)|$ and $p = |F(H)|$. Then $m = 4o + p - 1$, $d = 4o + 2p - 2$ and $p = 2o - 2$. $\lambda : V \cup E \cup F \to \{1, 2, \ldots, m + g + p\}$
For $1 \leq t \leq o$

$$\lambda(v_{1,t}) = 2t - 1, \tag{6.3.67}$$

$$\lambda(v_{2,t}) = 2t, \tag{6.3.68}$$

$$\lambda(c_{1,t}) = 3o - (t-1), \tag{6.3.69}$$

For $1 \leq t \leq o-1$

$$\lambda(a_{1,t}) = 4o + p - 2t + 1, \tag{6.3.70}$$

$$\lambda(b_{1,t}) = 4o + p - 2t, \tag{6.3.71}$$

$$\lambda(d_{1,t}) = 4o - t, \tag{6.3.72}$$

For $s = 1, \quad 1 \leq t \leq o-1.$

$$\lambda(v_{s,t}a_{s,t}) = 6o + p + 1 + (2t-2), \tag{6.3.73}$$

$$\lambda(a_{s,t}v_{s,t+1}) = 6o + 2p + 1 + (2t-2), \tag{6.3.74}$$

$$\lambda(v_{s+1,t}b_{s,t}) = 6o + p + (2t-2), \tag{6.3.75}$$

$$\lambda(b_{s,t}v_{s+1,t+1}) = 6o + 2p + (2t-2), \tag{6.3.76}$$

$$\lambda(v_{s,t}c_{s,t}) = 5o + p - 1 - (t-1), \tag{6.3.77}$$

$$\lambda(c_{s,t}v_{s+1,t}) = 5o + p + (t-1), \tag{6.3.78}$$

$$\lambda(v_{s,t+1}d_{s,t}) = m + d - (2t-2), \tag{6.3.79}$$

$$\lambda(d_{s,t}v_{s+1,t}) = m + d - 1 - (2t-2), \tag{6.3.80}$$

Theorem 6.3.2. *Subdivided triangular ladder graph $H \cong P_2 \times P_o, o \geq 2$, with 2 subdivision, admits super 1-anti-magic labeling. [48]*

Proof.

Let $m = |V(H)|, g = |E(H)|$ and $p = |F(H)|$. Then $m = 4o + 3p, g = 6o + 2p + 3$ and $p = 2o - 2$. $\lambda : V \cup E \cup F \rightarrow \{1, 2, \ldots, m + g + p\}$

For $1 \leq t \leq o$

$$\lambda(v_{1,t}) = 2t - 1, \tag{6.3.81}$$

$$\lambda(v_{2,t}) = 2t, \tag{6.3.82}$$

$$\lambda(c_{1,t}) = p + o - t + 3, \tag{6.3.83}$$

$$\lambda(c_{2,t}) = p + o + t + 2, \tag{6.3.84}$$

For $1 \leq t \leq o - 1$

$$\lambda(a_{1,t}) = 4o + 3p - 2t + 1, \tag{6.3.85}$$

$$\lambda(a_{2,t}) = 4o + 2p - 2t + 1, \tag{6.3.86}$$

$$\lambda(b_{1,t}) = 4o + 3p - 2t + 2, \tag{6.3.87}$$

$$\lambda(b_{2,t}) = 4o + 2p - 2t + 2, \tag{6.3.88}$$

$$\lambda(d_{1,t}) = 2p + o - t + 4, \tag{6.3.89}$$

$$\lambda(d_{2,t}) = 2p + o - t + 3, \tag{6.3.90}$$

For $s = 1, \quad 1 \leq t \leq o - 1.$

$$\lambda(v_{s,t}a_{s,t}) = m + 3o + 2 + (2t - 2), \tag{6.3.91}$$

$$\lambda(a_{s,t}a_{s+1,t}) = m + 5o + (2t - 2), \tag{6.3.92}$$

$$\lambda(a_{s+1,t}v_{s,t+1}) = m + 7o - 2 + (2t - 2), \tag{6.3.93}$$

$$\lambda(v_{s+1,t}b_{s,t}) = m + 3o + 1 + (2t - 2), \tag{6.3.94}$$

$$\lambda(b_{s,t}b_{s+1,t}) = m + 5o - 1 + (2t - 2), \tag{6.3.95}$$

$$\lambda(b_{s+1,t}v_{s+1,t+1}) = m + 7o - 3 + (2t - 2), \tag{6.3.96}$$

$$\lambda(v_{s,t}c_{s,t}) = m + o - (t - 1), \tag{6.3.97}$$

$$\lambda(c_{s,t}c_{s+1,t}) = m + o + 1 + (t - 1), \tag{6.3.98}$$

$$\lambda(c_{s+1,t}v_{s+1,t}) = m + 3o - (t - 1), \tag{6.3.99}$$

$$\lambda(v_{s,t+1}d_{s,t}) = m+g-2-(3t-3), \tag{6.3.100}$$

$$\lambda(d_{s,t}d_{s+1,t}) = m+g-1-(3t-3), \tag{6.3.101}$$

$$\lambda(d_{s+1,t}v_{s+1,t}) = m+g-(3t-3), \tag{6.3.102}$$

Theorem 6.3.3. *Subdivided triangular ladder graph $H \cong P_2 \times P_o, o \geq 2$, with 3 subdivision, admits super 1-anti-magic labeling. [48]*

Proof.

Let $m = |V(H)|, g = |E(H)|$ and $p = |F(H)|$. Then $m = 4o+5p+1, g = 4o+6p$ and $p = 2o-2$. $\lambda : V \cup E \cup F \to \{1,2,\dots,m+g+p\}$

For $1 \leq t \leq o$

$$\lambda(v_{1,t}) = 2t-1, \tag{6.3.103}$$

$$\lambda(v_{2,t}) = 2t, \tag{6.3.104}$$

$$\lambda(c_{1,t}) = 3o-(t-1), \tag{6.3.105}$$

$$\lambda(c_{2,t}) = 3o+1+(t-1), \tag{6.3.106}$$

$$\lambda(c_{3,t}) = 5o-(t-1), \tag{6.3.107}$$

For $1 \leq t \leq o-1$

$$\lambda(a_{1,t}) = m-3p+1+(2t-2), \tag{6.3.108}$$

$$\lambda(b_{1,t}) = m-3p+2+(2t-2), \tag{6.3.109}$$

$$\lambda(d_{1,t}) = m-4p-(t-1), \tag{6.3.110}$$

$$\lambda(a_{2,t}) = m-2p+1+(2t-2), \tag{6.3.111}$$

$$\lambda(b_{2,t}) = m-2p+2+(2t-2), \tag{6.3.112}$$

$$\lambda(d_{2,t}) = m-4p+1+(t-1), \tag{6.3.113}$$

$$\lambda(a_{3,t}) = m-p+1+(2t-2), \tag{6.3.114}$$

$$\lambda(b_{3,t}) = m - p + 2 + (2t-2), \tag{6.3.115}$$

$$\lambda(d_{3,t}) = m - 3p - (t-1), \tag{6.3.116}$$

For $s = 1, \quad 1 \le t \le o-1$.

$$\lambda(v_{s,t}a_{s,t}) = m + 4o + 2 + (2t-2), \tag{6.3.117}$$

$$\lambda(a_{s,t}a_{s+1,t}) = m + 6o + (2t-2), \tag{6.3.118}$$

$$\lambda(a_{s+1,t}a_{s+2,t}) = m + 8o - 2 + (2t-2), \tag{6.3.119}$$

$$\lambda(a_{s+2,t}v_{s,t+1}) = m + 10o - 4 + (2t-2), \tag{6.3.120}$$

$$\lambda(v_{s+1,t}b_{s,t}) = m + 4o + 1 + (2t-2), \tag{6.3.121}$$

$$\lambda(b_{s,t}b_{s+1,t}) = m + 6o - 1 + (2t-2), \tag{6.3.122}$$

$$\lambda(b_{s+1,t}b_{s+2,t}) = m + 8o - 3 + (2t-2), \tag{6.3.123}$$

$$\lambda(b_{s+2,t}v_{s+1,t+1}) = m + 10o - 5 + (2t-2), \tag{6.3.124}$$

$$\lambda(v_{s,t}c_{s,t}) = m + o - (t-1), \tag{6.3.125}$$

$$\lambda(c_{s,t}c_{s+1,t}) = m + o + 1 + (t-1), \tag{6.3.126}$$

$$\lambda(c_{s+1,t}c_{s+2,t}) = m + 3o - (t-1), \tag{6.3.127}$$

$$\lambda(c_{s+2,t}v_{s+1,t}) = m + 3o + 1 + (t-1), \tag{6.3.128}$$

$$\lambda(v_{s,t+1}d_{s,t}) = m + g - 3 - (4t-4), \tag{6.3.129}$$

$$\lambda(d_{s,t}d_{s+1,t}) = m + g - 2 - (4t-4), \tag{6.3.130}$$

$$\lambda(d_{s+1,t}d_{s+2,t}) = m + g - 1 - (4t-4), \tag{6.3.131}$$

$$\lambda(d_{s+2,t}v_{s+1,t}) = m + g - (4t-4), \tag{6.3.132}$$

Theorem 6.3.4. *Subdivided triangular ladder graph $H \cong P_2 \times P_o, o \geq 2$, with 4 subdivision, admits super 1-anti-magic labeling. [48]*

Proof.

Let $m = |V(H)|, g = |E(H)|$ and $p = |F(H)|$. Then $m = 6o + 6p, g = 6o + 7p - 1$ and $p = 2o - 2$. $\lambda : V \cup E \cup F \rightarrow \{1, 2, \ldots, m + g + p\}$

For $1 \leq t \leq o$

$$\lambda(v_{1,t}) = 2t - 1, \tag{6.3.133}$$

$$\lambda(v_{2,t}) = 2t, \tag{6.3.134}$$

$$\lambda(c_{1,t}) = 3o - (t - 1), \tag{6.3.135}$$

$$\lambda(c_{2,t}) = 3o + 1 + (t - 1), \tag{6.3.136}$$

$$\lambda(c_{3,t}) = 5o - (t - 1), \tag{6.3.137}$$

$$\lambda(c_{4,t}) = 5o + 1 + (t - 1), \tag{6.3.138}$$

For $1 \leq t \leq o - 1$

$$\lambda(a_{1,t}) = m - 4p + 1 + (2t - 2), \tag{6.3.139}$$

$$\lambda(b_{1,t}) = m - 4p + 2 + (2t - 2), \tag{6.3.140}$$

$$\lambda(d_{1,t}) = 5o + p + 1 - (t - 1), \tag{6.3.141}$$

$$\lambda(a_{2,t}) = m - 3p + 1 + (2t - 2), \tag{6.3.142}$$

$$\lambda(b_{2,t}) = m - 3p + 2 + (2t - 2), \tag{6.3.143}$$

$$\lambda(d_{2,t}) = 5o + p + 2 + (t - 1), \tag{6.3.144}$$

$$\lambda(a_{3,t}) = m - 2p + 1 + (2t - 2), \tag{6.3.145}$$

$$\lambda(b_{3,t}) = m - 2p + 2 + (2t - 2), \tag{6.3.146}$$

$$\lambda(d_{3,t}) = m - 4p - o + 1 - (t - 1), \tag{6.3.147}$$

$$\lambda(a_{4,t}) = m - p + 1 + (2t - 2), \tag{6.3.148}$$

$$\lambda(b_{4,t}) = m - p + 2 + (2t - 2), \tag{6.3.149}$$

$$\lambda(d_{4,t}) = m - 4p - o + 2 - (t - 1), \tag{6.3.150}$$

For $s = 1, \quad 1 \leq t \leq o - 1$.

$$\lambda(v_{s,t}a_{s,t}) = m + 5o + 2 + (2t - 2), \tag{6.3.151}$$

$$\lambda(a_{s,t}a_{s+1,t}) = m + 7o + (2t - 2), \tag{6.3.152}$$

$$\lambda(a_{s+1,t}a_{s+2,t}) = m + 9o - 2 + (2t - 2), \tag{6.3.153}$$

$$\lambda(a_{s+2,t}a_{s+3,t}) = m + 11o - 4 + (2t - 2), \tag{6.3.154}$$

$$\lambda(a_{s+3,t}v_{s,t+1}) = m + 13o - 6 + (2t - 2), \tag{6.3.155}$$

$$\lambda(v_{s+1,t}b_{s,t}) = m + 5o + 1 + (2t - 2), \tag{6.3.156}$$

$$\lambda(b_{s,t}b_{s+1,t}) = m + 7o - 1 + (2t - 2), \tag{6.3.157}$$

$$\lambda(b_{s+1,t}b_{s+2,t}) = m + 9o - 3 + (2t - 2), \tag{6.3.158}$$

$$\lambda(b_{s+2,t}b_{s+3,t}) = m + 11o - 5 + (2t - 2), \tag{6.3.159}$$

$$\lambda(b_{s+3,t}v_{s+1,t+1}) = m + 13o - 7 + (2t - 2), \tag{6.3.160}$$

$$\lambda(v_{s,t}c_{s,t}) = m + o - (t - 1), \tag{6.3.161}$$

$$\lambda(c_{s,t}c_{s+1,t}) = m + o + 1 + (t - 1), \tag{6.3.162}$$

$$\lambda(c_{s+1,t}c_{s+2,t}) = m + 3o - (t - 1), \tag{6.3.163}$$

$$\lambda(c_{s+2,t}c_{s+3,t}) = m + 3o + 1 + (t - 1), \tag{6.3.164}$$

$$\lambda(c_{s+3,t}v_{s+1,t}) = m + 5o - (t - 1), \tag{6.3.165}$$

$$\lambda(v_{s,t+1}d_{s,t}) = m+g-4-(5t-5), \tag{6.3.166}$$

$$\lambda(d_{s,t}d_{s+1,t}) = m+g-3-(5t-5), \tag{6.3.167}$$

$$\lambda(d_{s+1,t}d_{s+2,t}) = m+g-2-(5t-5), \tag{6.3.168}$$

$$\lambda(d_{s+2,t}d_{s+3,t}) = m+g-1-(5t-5), \tag{6.3.169}$$

$$\lambda(d_{s+3,t}v_{s+1,t}) = m+g-(5t-5), \tag{6.3.170}$$

Chapter 7

Super Anti-magic Labeling of Type $(1,1,1)$ on Subdivision of qC_4

7.1 Introduction

The basic defination of qC_4-snake graph has been discussed in the introduction of privious chapters. The formulation of face anti-magic labeling of qC_4 - snake graph are discussed in this chapter.

7.2 Super Face Anti-magic Labeling of qC_4- Snake Graph

Super face anti-magic labeling of qC_4- snake graph as well as isomorphic copies of qC_4 -snake graph has been discussed in this section. Throughout this section $1 \leq s \leq q$.

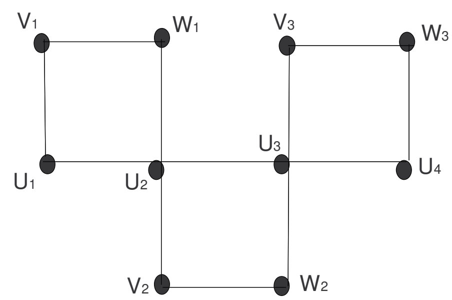

Figure 7.1: $3C_4$ snake graph having string $(1,1,1)$

Theorem 7.2.1. *Snake graph $H \cong qC_4, q \geq 2$ having string $(2,2,\ldots,2)$, admits super 1-anti-magic labeling.*

Proof.
Let $m = |V(H)|, o = |E(H)|$ and $p = |F(H)|$. Then $m = 3q+1, o = 4q$ and $p = q$.
$\lambda : V(H) \cup E(H) \cup F(H) \to \{1,2,\ldots,m+o+p\}$

$$\lambda(u_s) = s, \quad 1 \leq s \leq q+1 \tag{7.2.1}$$

$$\lambda(v_s) = 3(q+1) - 2s, \tag{7.2.2}$$

$$\lambda(w_s) = 3q + 2 - 2s, \tag{7.2.3}$$

For edges

$$\lambda(u_s v_s) = m + s, \tag{7.2.4}$$

$$\lambda(u_s w_s) = m + q + s, \tag{7.2.5}$$

$$\lambda(u_{s+1} w_s) = m + 3q + s, \tag{7.2.6}$$

$$\lambda(u_{s+1} v_s) = m + 2q + s, \tag{7.2.7}$$

For faces

$$\lambda(f_s) = m + o + q + 1 - s, \tag{7.2.8}$$

Theorem 7.2.2. *Snake graph* $H \cong qC_4, q \geq 3$ *having string* $(1,1,\ldots,1)$, *admits super* 1-*anti-magic labeling.*

Proof.

Let $m = |V(H)|, o = |E(H)|$ and $p = |F(H)|$. Then $m = 3q+1, o = 4q$ and $p = q$.
$\lambda : V(H) \cup E(H) \cup F(H) \to \{1,2,\ldots,m+o+p\}$

$$\lambda(u_s) = s, \quad 1 \leq s \leq q+1 \tag{7.2.9}$$

$$\lambda(v_s) = 3(q+1) - 2s, \tag{7.2.10}$$

$$\lambda(w_s) = 3q + 2 - 2s, \tag{7.2.11}$$

For edges

$$\lambda(u_s v_s) = m + s, \tag{7.2.12}$$

$$\lambda(u_s u_{s+1}) = m + q + s, \tag{7.2.13}$$

$$\lambda(u_{s+1} w_s) = m + 3q + s, \tag{7.2.14}$$

$$\lambda(v_s w_s) = m + 2q + s, \tag{7.2.15}$$

For faces

$$\lambda(f_s) = m + o + q + 1 - s, \tag{7.2.16}$$

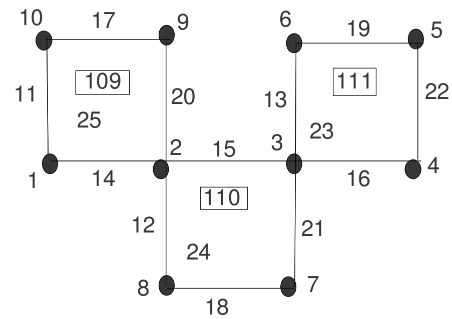

Figure 7.2: Face anti-magic labeling of $3C_4$ snake graph having string $(1,1,1)$

Theorem 7.2.3. *Snake graph $H \cong rqC_4, q \geq 2$ having string $(2,2,\ldots,2)$, admits super 1-anti-magic labeling.*

Proof.

Let $m = |V(H)|, o = |E(H)|$ and $p = |F(H)|$.Then we have $m = n(3q+1)$, $o = 4nq$ and $p = nq$, $1 \leq r \leq n$.

$\lambda : V(H) \cup E(H) \cup F(H) \rightarrow \{1,2,\ldots,m+o+p\}$

For vertices

$$\lambda(u_s^r) = s + (r-1)(q+1) \quad 1 \leq s \leq q+1 \tag{7.2.17}$$

$$\lambda(v_s^r) = n(q+1) + (r-1)q + s \tag{7.2.18}$$

$$\lambda(w_s^r) = m - (r-1)q + 1 - s \tag{7.2.19}$$

For edges

$$\lambda(v_s^r u_{s+1}^r) = m + nq - (r-1)q + 1 - s \tag{7.2.20}$$

$$\lambda(u_s^r w_s^r) = m + 2nq - (r-1)q + 1 - s \tag{7.2.21}$$

$$\lambda(u_s^r v_s^r) = m + 3nq - (r-1)q + 1 - s \tag{7.2.22}$$

$$\lambda(w_s^r u_{s+1}^r) = m + 3nq + (r-1)q + s \tag{7.2.23}$$

For faces

$$\lambda(f_s^r) = m + o + p - (r-1) - (s-1)n \tag{7.2.24}$$

Theorem 7.2.4. *Snake graph $H \cong rqC_4, q \geq 3$ having string $(1,1,\ldots,1)$, admits super 1-anti-magic labeling.*

Proof.

Let $m = |V(H)|, o = |E(H)|$ and $p = |F(H)|$.Then we have $m = n(3q+1), o = 4nq$ and $p = nq$, $1 \leq r \leq n$.

$\lambda : V(H) \cup E(H) \cup F(H) \to \{1, 2, \ldots, m+o+p\}$

For vertices

$$\lambda(u_s^r) = s + (r-1)(q+1) \quad 1 \leq s \leq q+1 \tag{7.2.25}$$

$$\lambda(v_s^r) = n(q+1) + (r-1)q + s \tag{7.2.26}$$

$$\lambda(w_s^r) = m - (r-1)q + 1 - s \tag{7.2.27}$$

For edges

$$\lambda(v_s^r u_s^r) = m + nq - (r-1)q + 1 - s \tag{7.2.28}$$

$$\lambda(u_{s+1}^r w_s^r) = m + 2nq - (r-1)q + 1 - s \tag{7.2.29}$$

$$\lambda(v_s^r w_s^r) = m + 3nq - (r-1)q + 1 - s \tag{7.2.30}$$

$$\lambda(u_s^r u_{s+1}^r) = m + 3nq + (r-1)q + s \tag{7.2.31}$$

For faces

$$\lambda(f_s^r) = m + o + p - (r-1) - (s-1)n \tag{7.2.32}$$

7.3 Super Face Anti-magic Labeling of Subdivided qC_4- Snake Graph

Super face anti-magic labeling for subdivision of qC_4- snake graph as well as isomorphic copies for subdivision of qC_4 -snake graph are discussed in this section. Throughout this section $1 \leq s \leq q$.

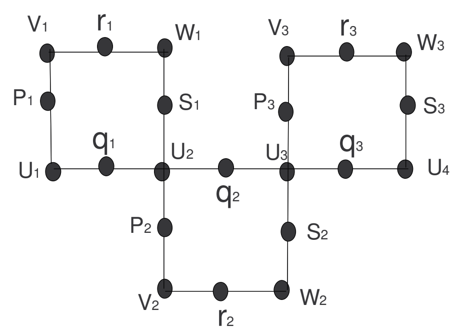

Figure 7.3: $3C_4$ snake graph with 1 subdivision, having string $(1,1,1)$

Theorem 7.3.1. *Snake graph $H \cong qC_4, q \geq 3$ having string $(1,1,\ldots,1)$ with 1 subdivision, admits super 1-anti-magic labeling.*

Proof.
Let $m = |V(H)|, o = |E(H)|$ and $p = |F(H)|$. Then $m = 7q+1$, $o = 8q$ and $p = q$.
$\lambda : V(H) \cup E(H) \cup F(H) \to \{1,2,\ldots,m+o+p\}$

$$\lambda(u_s) = s, \quad 1 \leq s \leq q+1 \tag{7.3.33}$$

$$\lambda(v_s) = 3(q+1) - 2s, \tag{7.3.34}$$

$$\lambda(w_s) = 3q+2-2s, \tag{7.3.35}$$

For partitions

$$\lambda(p_s) = 3q+1+s, \tag{7.3.36}$$

$$\lambda(S_s) = 5q+2-s, \tag{7.3.37}$$

$$\lambda(r_s) = 5q+1+s, \tag{7.3.38}$$

$$\lambda(q_s) = 6q+1+s, \tag{7.3.39}$$

For edges

$$\lambda(v_s p_s) = 7q+1+s, \tag{7.3.40}$$

$$\lambda(p_s u_s) = 9q+2-s, \tag{7.3.41}$$

$$\lambda(v_s r_s) = 10q+2-s, \tag{7.3.42}$$

$$\lambda(r_s w_s) = 11q+2-s, \tag{7.3.43}$$

$$\lambda(u_s q_s) = 11q+1+s, \tag{7.3.44}$$

$$\lambda(q_s u_{s+1}) = 12q+1+s, \tag{7.3.45}$$

$$\lambda(S_s u_{s+1}) = 14q+2-s, \tag{7.3.46}$$

$$\lambda(w_s S_s) = 15q+2-s, \tag{7.3.47}$$

For faces

$$\lambda(f_s) = m+o+s, \tag{7.3.48}$$

Theorem 7.3.2. *Snake graph $H \cong qC_4, q \geq 2$ having string $(2,2,\ldots,2)$ with 1 subdivision, admits super 1-anti-magic labeling.*

Proof.

Let $m = |V(H)|, o = |E(H)|$ and $p = |F(H)|$. Then $m = 7q+1, o = 8q$ and $p = q$.
$\lambda : V(H) \cup E(H) \cup F(H) \rightarrow \{1,2,\ldots,m+o+p\}$

$$\lambda(u_s) = s, \quad 1 \leq s \leq q+1 \tag{7.3.49}$$

$$\lambda(v_s) = 3(q+1) - 2s, \tag{7.3.50}$$

$$\lambda(w_s) = 3q + 2 - 2s, \tag{7.3.51}$$

For partitions

$$\lambda(p_s) = 3q + 1 + s, \tag{7.3.52}$$

$$\lambda(S_s) = 5q + 2 - s, \tag{7.3.53}$$

$$\lambda(r_s) = 6q + 2 - s, \tag{7.3.54}$$

$$\lambda(q_s) = 7q + 2 - s, \tag{7.3.55}$$

For edges

$$\lambda(w_s p_s) = 8q + 2 - s, \tag{7.3.56}$$

$$\lambda(p_s u_s) = 8q + 1 + s, \tag{7.3.57}$$

$$\lambda(u_{s+1} S_s) = 10q + 2 - s, \tag{7.3.58}$$

$$\lambda(S_s v_s) = 11q + 2 - s, \tag{7.3.59}$$

$$\lambda(u_s r_s) = 11q + 1 + s, \tag{7.3.60}$$

$$\lambda(r_s v_s) = 12q + 1 + s, \tag{7.3.61}$$

$$\lambda(w_s q_s) = 13q + 1 + s, \tag{7.3.62}$$

$$\lambda(q_s u_{s+1}) = 14q + 1 + s, \tag{7.3.63}$$

For faces

$$\lambda(f_s) = m + o + s, \tag{7.3.64}$$

Theorem 7.3.3. *Snake graph $H \cong qC_4, q \geq 3$ having string $(1,1,\ldots,1)$ with 2 subdivision, admits super 1-anti-magic labeling.*

Proof.

Let $m = |V(H)|, o = |E(H)|$ and $p = |F(H)|$. Then $m = 11q+1, o = 12q$ and $p = q$.

$\lambda : V(H) \cup E(H) \cup F(H) \to \{1, 2, \ldots, m+o+p\}$

$$\lambda(u_s) = s, \quad 1 \leq s \leq q+1 \tag{7.3.65}$$

$$\lambda(v_s) = 3(q+1) - 2s, \tag{7.3.66}$$

$$\lambda(w_s) = 3q + 2 - 2s, \tag{7.3.67}$$

For partitions

$$\lambda(p_{s1}) = 3q + 1 + s, \tag{7.3.68}$$

$$\lambda(p_{s2}) = 4q + 1 + s, \tag{7.3.69}$$

$$\lambda(S_{s1}) = 5q + 1 + s, \tag{7.3.70}$$

$$\lambda(S_{s2}) = 6q + 1 + s, \tag{7.3.71}$$

$$\lambda(r_{s1}) = 8q + 2 - s, \tag{7.3.72}$$

$$\lambda(r_{s2}) = 9q + 2 - s, \tag{7.3.73}$$

$$\lambda(q_{s1}) = 10q + 2 - s, \tag{7.3.74}$$

$$\lambda(q_{s2}) = 11q + 2 - s, \tag{7.3.75}$$

For edges

$$\lambda(u_s p_{s1}) = 11q + 1 + s, \tag{7.3.76}$$

$$\lambda(p_{s1} p_{s2}) = 12q + 1 + s, \tag{7.3.77}$$

$$\lambda(p_{s2} v_s) = 13q + 1 + s, \tag{7.3.78}$$

$$\lambda(u_{s+1} S_{s1}) = 14q + 1 + s, \tag{7.3.79}$$

$$\lambda(S_{s1} S_{s2}) = 15q + 1 + s, \tag{7.3.80}$$

$$\lambda(S_{s2} w_s) = 16q + 1 + s, \tag{7.3.81}$$

$$\lambda(v_s r_{s1}) = 18q + 2 - s, \tag{7.3.82}$$

$$\lambda(r_{s1} r_{s2}) = 19q + 2 - s, \tag{7.3.83}$$

$$\lambda(r_{s2} w_s) = 20q + 2 - s, \tag{7.3.84}$$

$$\lambda(u_s q_{s1}) = 21q + 2 - s, \tag{7.3.85}$$

$$\lambda(q_{s1} q_{s2}) = 22q + 2 - s, \tag{7.3.86}$$

$$\lambda(q_{s2} u_{s+1}) = 23q + 2 - s, \tag{7.3.87}$$

For faces

$$\lambda(f_s) = m + o + s, \tag{7.3.88}$$

Theorem 7.3.4. *Snake graph $H \cong qC_4, q \geq 2$ having string $(2,2,\ldots,2)$ with 2 subdivision, admits super 1-anti-magic labeling.*

Proof.

Let $m = |V(H)|, o = |E(H)|$ and $p = |F(H)|$. Then $m = 11q + 1, o = 12q$ and $p = q$.
$\lambda : V(H) \cup E(H) \cup F(H) \rightarrow \{1, 2, \ldots, m + o + p\}$

$$\lambda(u_s) = s, \quad 1 \leq s \leq q + 1 \tag{7.3.89}$$

$$\lambda(v_s) = 3(q + 1) - 2s, \tag{7.3.90}$$

$$\lambda(w_s) = 3q + 2 - 2s, \tag{7.3.91}$$

For partitions

$$\lambda(p_{s1}) = 3q + 1 + s, \tag{7.3.92}$$

$$\lambda(p_{s2}) = 4q + 1 + s, \tag{7.3.93}$$

$$\lambda(S_{s1}) = 6q + 2 - s, \tag{7.3.94}$$

$$\lambda(S_{s2}) = 7q + 2 - s, \tag{7.3.95}$$

$$\lambda(r_{s1}) = 8q + 2 - s, \tag{7.3.96}$$

$$\lambda(r_{s2}) = 9q + 2 - s, \tag{7.3.97}$$

$$\lambda(q_{s1}) = 9q + 1 + s, \tag{7.3.98}$$

$$\lambda(q_{s2}) = 10q + 1 + s, \tag{7.3.99}$$

For edges

$$\lambda(w_s p_{s1}) = 12q + 2 - s, \tag{7.3.100}$$

$$\lambda(p_{s1} p_{s2}) = 13q + 2 - s, \tag{7.3.101}$$

$$\lambda(p_{s2} u_s) = 14q + 2 - s, \tag{7.3.102}$$

$$\lambda(u_{s+1} s_{s1}) = 15q + 2 - s, \tag{7.3.103}$$

$$\lambda(s_{s1} s_{s2}) = 16q + 2 - s, \tag{7.3.104}$$

$$\lambda(s_{s2} w_s) = 17q + 2 - s, \tag{7.3.105}$$

$$\lambda(u_s r_{s1}) = 17q + 1 + s, \tag{7.3.106}$$

$$\lambda(r_{s1} r_{s2}) = 18q + 1 + s, \tag{7.3.107}$$

$$\lambda(r_{s2} v_s) = 19q + 1 + s, \tag{7.3.108}$$

$$\lambda(u_{s+1} q_{s1}) = 20q + 1 + s, \tag{7.3.109}$$

$$\lambda(q_{s1} q_{s2}) = 21q + 1 + s, \tag{7.3.110}$$

$$\lambda(r_{s2} v_s) = 22q + 1 + s, \tag{7.3.111}$$

For faces

$$\lambda(f_s) = m + o + s, \tag{7.3.112}$$

Theorem 7.3.5. *Snake graph $H \cong rqC_4, q \geq 3, r \geq 1$ having string $(1, 1, \ldots, 1)$ with 1 subdivision, admits super 1-anti-magic labeling.*

Proof.

Let $m = |V(H)|, o = |E(H)|$ and $p = |F(H)|$. Then $m = n(7q+1), o = 8nq$, and $p = nq$, $1 \leq r \leq n$

$\lambda : V(H) \cup E(H) \cup F(H) \to \{1, 2, \ldots, m+o+p\}$

$$\lambda(u_s^r) = s + (r-1)(q+1), \quad 1 \leq s \leq q+1 \tag{7.3.113}$$

$$\lambda(w_s^r) = n(3q+1) - (r-1)q + 1 - s, \tag{7.3.114}$$

$$\lambda(v_s^r) = n(q+1) + (r-1)q + s, \tag{7.3.115}$$

For partitions

$$\lambda(p_s^r) = 4nq + n + 1 - s - (r-1)q, \tag{7.3.116}$$

$$\lambda(s_s^r) = 5nq + n + 1 - s - (r-1)q, \tag{7.3.117}$$

$$\lambda(r_s^r) = 5nq + n + s + (r-1)q, \tag{7.3.118}$$

$$\lambda(q_s^r) = 7nq + n + 1 - s - (r-1)q, \tag{7.3.119}$$

We symbolize the edges of H as follows:

$$\lambda(v_s^r p_s^r) = n(7q+1) + s + (r-1)q, \tag{7.3.120}$$

$$\lambda(p_s^r u_s^r) = n(8q+1) + s + (r-1)q, \tag{7.3.121}$$

$$\lambda(u_{s+1}^r S_s^r) = n(10q+1) + 1 - s - (r-1)q, \tag{7.3.122}$$

$$\lambda(s_s^r w_s^r) = n(11q+1) + 1 - s - (r-1)q, \tag{7.3.123}$$

$$\lambda(v_s^r r_s^r) = n(12q+1) + 1 - s - (r-1)q, \tag{7.3.124}$$

$$\lambda(r_s^r w_s^r) = n(13q+1) + 1 - s + (r-1)q, \tag{7.3.125}$$

$$\lambda(u_s^r q_s^r) = n(13q+1) + s + (r-1)q, \tag{7.3.126}$$

$$\lambda(q_s^r u_{s+1}^r) = n(14q+1)+s+(r-1)q, \tag{7.3.127}$$

For faces

$$\lambda(f_s^r) = m+o+p-(r-1)-(s-1)n, \tag{7.3.128}$$

Theorem 7.3.6. *Snake graph $H \cong rqC_4, q \geq 2, r \geq 1$ having string $(2,2,\dots,2)$ with 1 subdivision, admits super 1-anti-magic labeling.*

Proof.

Let $m = |V(H)|, o = |E(H)|$ and $p = |F(H)|$. Then $m = n(7q+1), o = 8nq$ and $p = nq, 1 \leq r \leq n$

$\lambda : V(H) \cup E(H) \cup F(H) \to \{1,2,\dots,m+o+p\}$

$$\lambda(u_s^r) = s+(r-1)(q+1), \quad 1 \leq s \leq q+1 \tag{7.3.129}$$

$$\lambda(w_s^r) = n(3q+1)-(r-1)q+1-s, \tag{7.3.130}$$

$$\lambda(v_s^r) = n(q+1)+(r-1)q+s, \tag{7.3.131}$$

For partitions

$$\lambda(p_s^r) = 4nq+n+1-s-(r-1)q, \tag{7.3.132}$$

$$\lambda(S_s^r) = 5nq+n+1-s-(r-1)q, \tag{7.3.133}$$

$$\lambda(r_s^r) = 5nq+n+s+(r-1)q, \tag{7.3.134}$$

$$\lambda(q_s^r) = 7nq+n+1-s-(r-1)q, \tag{7.3.135}$$

For edges

$$\lambda(w_s^r p_s^r) = n(7q+1)+s+(r-1)q, \tag{7.3.136}$$

$$\lambda(p_s^r u_s^r) = n(8q+1)+s+(r-1)q, \tag{7.3.137}$$

$$\lambda(u_{s+1}^r S_s^r) = n(10q+1)+1-s-(r-1)q, \tag{7.3.138}$$

$$\lambda(S_s^r v_s^r) = n(11q+1)+1-s-(r-1)q, \tag{7.3.139}$$

$$\lambda(u_s^r r_s^r) = n(12q+1)+1-s-(r-1)q, \tag{7.3.140}$$

$$\lambda(r_s^r v_s^r) = n(13q+1)+1-s+(r-1)q, \tag{7.3.141}$$

$$\lambda(u_{s+1}^r q_s^r) = n(13q+1)+s+(r-1)q, \tag{7.3.142}$$

$$\lambda(q_s^r w_s^r) = n(14q+1)+s+(r-1)q, \tag{7.3.143}$$

For faces

$$\lambda(f_s^r) = m+o+p-(r-1)-(s-1)n, \tag{7.3.144}$$

7.4 Super Face Anti-magic Labeling of Triangular qC_4- Snake Graph

Super face anti-magic labeling of triangular qC_4- snake graph is discussed in this section. Triangular qC_4- snake graph can be made by joining vertex u_i with vertex w_i with an edge for string $(1,1,\ldots,1)$ and joining vertex v_i with vertex w_i with an edge for string $(2,2,\ldots,2)$. Throughout this section $1 \leq s \leq q$ otherwise it is mentioned.

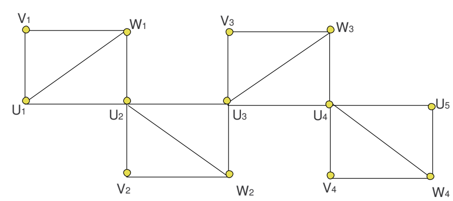

Figure 7.4: Triangular $4C_4$ having string $(1,1,1,1)$

Theorem 7.4.1. *The triangular snake graph* $H \cong qC_4, q \geq 3,$ *having string* $(1,1,\ldots,1)$, *admits super* 1*-anti-magic labeling.*

Proof.

Let $m = |V(H)|, o = |E(H)|$ and $p = |F(H)|$. Then $m = 3q+1, o = 5q$ and $p = q$.

$\lambda : V(H) \cup E(H) \cup F(H) \to \{1,2,\ldots,m+o+p\}$

$$\lambda(u_s) = 2q + s, \quad 1 \leq s \leq q+1 \tag{7.4.145}$$

$$\lambda(v_s) = s, \tag{7.4.146}$$

$$\lambda(w_s) = 2q + 1 - s, \tag{7.4.147}$$

For edges

$$\lambda(u_{s+1}w_s) = m + q + 1 - s, \tag{7.4.148}$$

$$\lambda(v_s u_s) = m + 2q + 1 - s, \tag{7.4.149}$$

$$\lambda(u_s u_{s+1}) = m + 2q + s, \tag{7.4.150}$$

$$\lambda(v_s w_s) = m + 3q + s, \tag{7.4.151}$$

$$\lambda(u_s w_s) = m + 4q + s, \tag{7.4.152}$$

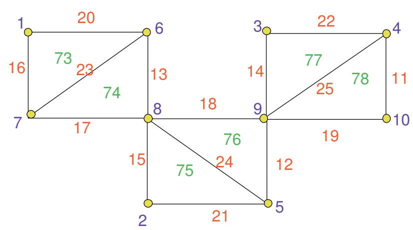

Figure 7.5: Triangular $3C_4$ having string $(1,1,1)$

Theorem 7.4.2. *The triangular snake graph $H \cong qC_4, q \geq 2$, having string $(2,2,\ldots,2)$, admits super 1-anti-magic labeling.*

Proof.

Let $m = |V(H)|, o = |E(H)|$ and $p = |F(H)|$. Then $m = 3q+1, o = 5q$ and $p = q$.
$\lambda : V(H) \cup E(H) \cup F(H) \to \{1, 2, \ldots, m+o+p\}$

$$\lambda(u_s) = s, \quad 1 \leq s \leq q+1 \tag{7.4.153}$$

$$\lambda(w_s) = 2q + 2 - s, \tag{7.4.154}$$

$$\lambda(v_s) = 2q + 1 + s, \tag{7.4.155}$$

For edges

$$\lambda(u_{s+1}w_s) = m + q + 1 - s, \tag{7.4.156}$$

$$\lambda(v_s u_s) = m + q + s, \tag{7.4.157}$$

$$\lambda(u_s w_s) = m + 3q + 1 - s, \tag{7.4.158}$$

$$\lambda(v_s u_{s+1}) = m + 3q + s, \tag{7.4.159}$$

$$\lambda(v_s w_s) = m + 4q + s, \tag{7.4.160}$$

Theorem 7.4.3. *The triangular snake graph* $H \cong qC_4, q \geq 3$, *having string* $(1,1,\ldots,1)$ *with* 1 *subdivision admits super* 1*-anti-magic labeling.*

Proof.

Let $m = |V(H)|, o = |E(H)|$ and $p = |F(H)|$. Then $m = 8q+1, o = 10q$ and $p = q$.
$\lambda : V(H) \cup E(H) \cup F(H) \to \{1, 2, \ldots, m+o+p\}$

$$\lambda(u_s) = 2q + s, \quad 1 \leq s \leq q+1 \tag{7.4.161}$$

$$\lambda(v_s) = s, \tag{7.4.162}$$

$$\lambda(w_s) = 2q + 1 - s, \tag{7.4.163}$$

For partitions

$$\lambda(b_s) = 4q + 2 - s, \tag{7.4.164}$$

$$\lambda(d_s) = 4q + 1 + s, \tag{7.4.165}$$

$$\lambda(a_s) = 6q + 2 - s, \tag{7.4.166}$$

$$\lambda(c_s) = 6q + 1 + s, \tag{7.4.167}$$

$$\lambda(g_s) = 8q + 2 - s, \tag{7.4.168}$$

For edges

$$\lambda(u_s g_s) = m + s, \tag{7.4.169}$$

$$\lambda(g_s w_s) = m + 2q + 1 - s, \tag{7.4.170}$$

$$\lambda(u_s a_s) = m + 2q + s, \tag{7.4.171}$$

$$\lambda(a_s u_{s+1}) = m + 6q + 1 - s, \tag{7.4.172}$$

$$\lambda(v_s c_s) = m + 4q + 1 - s, \tag{7.4.173}$$

$$\lambda(c_s w_s) = m + 4q + s, \tag{7.4.174}$$

$$\lambda(v_s d_s) = m + 6q + s, \tag{7.4.175}$$

$$\lambda(d_s u_s) = m + 10q + 1 - s, \tag{7.4.176}$$

$$\lambda(w_s b_s) = m + 8q + 1 - s, \tag{7.4.177}$$

$$\lambda(b_s u_{s+1}) = m + 8q + s, \tag{7.4.178}$$

Theorem 7.4.4. *The triangular snake graph $H \cong qC_4, q \geq 2$, having string $(2,2,\ldots,2)$ with 1 subdivision admits super 1-anti-magic labeling.*

Proof.

Let $m = |V(H)|, o = |E(H)|$ and $p = |F(H)|$. Then $m = 8q+1, o = 10q$ and $p = q$.
$\lambda : V(H) \cup E(H) \cup F(H) \to \{1, 2, \ldots, m+o+p\}$

$$\lambda(u_s) = s, \quad 1 \leq s \leq q+1 \tag{7.4.179}$$

$$\lambda(w_s) = 2q + 2 - s, \tag{7.4.180}$$

$$\lambda(v_s) = 2q + 1 + s, \tag{7.4.181}$$

For partitions

$$\lambda(b_s) = 4q + 2 - s, \tag{7.4.182}$$

$$\lambda(d_s) = 4q + 1 + s, \tag{7.4.183}$$

$$\lambda(a_s) = 6q + 2 - s, \tag{7.4.184}$$

$$\lambda(c_s) = 6q + 1 + s, \tag{7.4.185}$$

$$\lambda(g_s) = 8q + 2 - s, \tag{7.4.186}$$

For edges

$$\lambda(v_s g_s) = m + s, \tag{7.4.187}$$

$$\lambda(g_s w_s) = m + 2q + 1 - s, \tag{7.4.188}$$

$$\lambda(v_s d_s) = m + 2q + s, \tag{7.4.189}$$

$$\lambda(d_s u_s) = m + 6q + 1 - s, \tag{7.4.190}$$

$$\lambda(u_{s+1} b_s) = m + 4q + 1 - s, \tag{7.4.191}$$

$$\lambda(b_s w_s) = m + 4q + s, \tag{7.4.192}$$

$$\lambda(v_s c_s) = m + 8q + 1 + s, \tag{7.4.193}$$

$$\lambda(c_s u_{s+1}) = m + 8q + s, \tag{7.4.194}$$

$$\lambda(u_s a_s) = m + 6q + s, \tag{7.4.195}$$

$$\lambda(a_s w_s) = m + 9q + s, \tag{7.4.196}$$

Chapter 8

On the Deficiencies of Some Trees

8.1 Introduction

A graph G is called SEM if there exists a Correspondence from elements of graph to integers equal to the sum of the cardinality of vertices and edges of the graph such that for all edges $uv \in E(G)$, have a fix weight $f(u) + f(uv) + f(v) = c(f)$.

Since not all graphs are SEM, we define, the SEM deficiency of a graph G as either least integer n such that $G \cup nK_1$ is a SEM graph or $+\infty$ if there is no such n EM deficiency of the graph Grepresented by $\mu(G)$.

The edge magic deficiency of subdivided banana trees and subdivided w trees is in this chapter.

Lemma 8.1.1. *A graph H having t edges and s vertices is SEM if the function g : $V(G) \to \{1, 2, \dots, s\}$ is onto function as well as $1-1$ function having property that the set $L = g(m) + g(q)|mq \in E(G)$ contains t successive integers. If such case occurs the function g lead towards SEM labeling of H having magic constant $c = s + t + min(L)$.*

8.2 Super Edge-Magic Deficiency of Subdivided Banana Tree $BT(n_1, n_2, \dots, n_q)$

Super edge-magic deficiency of subdivided banana tree $BT(n_1, n_2, \cdots, n_q)$ with 1 subdivision is discussed in this section and subdivision of the banana tree is denoted by b_{st}.

Hussain et al. [35] discuss the SEM labeling of banana tree. Figure below shows a banana tree $BT(n_1, n_2, ..., n_k)$.

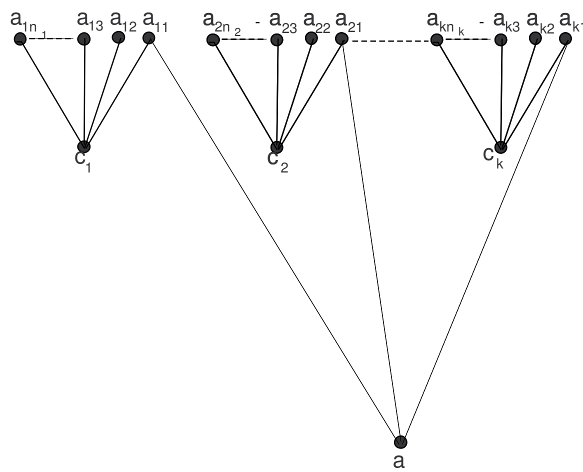

Figure 8.1: A banana tree $BT(n_1, n_2, ..., n_k)$.

Theorem 8.2.1. *For every even* $n_s, 1 \leq s \leq q$, *and* $n_1 = n_2 = \cdots = n_q = u$ *with* 1 *subdivision*

$$\mu_s(BT(n_1, n_2, \cdots, n_q)) \leq uq.$$

Proof.

We consider the banana tree $BT(n_1, n_2, \ldots, n_q)$ and all n_i are even with 1 subdivision. We take the case $n_1 = n_2 = \ldots = n_q = u$ and subdivision is denoted by (b_{st}).
Let $s = |V(G)|$, and $e = |E(G)|$. $\quad \lambda : V(G) \to \{1, 2, \ldots, o\}$

$$\lambda(a) = q + 1 \tag{8.2.1}$$

$1 \leq s, t \leq q$

$$\lambda(c_s) = s, \tag{8.2.2}$$

$$\lambda(b_{st}) = (q+1) + t + u(s-1), \tag{8.2.3}$$

$$\lambda(a_s) = (q+1) + uq + t + u(s-1), \tag{8.2.4}$$

It is easy verify that the mapping λ form a consecutive sequence of numbers and by using the lemma 1 we get our desired result.

Theorem 8.2.2. *For every even $n_s, 1 \leq s \leq 2$, and $n_1 \neq n_2$ with 1 subdivision*

$$\mu_s(BT(n_1,n_2)) \leq n_1+n_2+3.$$

Proof.

We consider the banana tree $BT(n_1,n_2)$ and both n_s are even with 1 subdivision. We take the case where $n_1 \neq n_2$ and subdivision is denoted by (b_{st}). Let $s = |V(G)|$, and $e = |E(G)|$. $\lambda : V(G) \rightarrow \{1,2,\ldots,o\}$

$$\lambda(a) = 3 \tag{8.2.5}$$

$$\lambda(c_s) = s, \quad 1 \leq s \leq 2 \tag{8.2.6}$$

$$\lambda(b_{st}) = \begin{cases} 4+n_1-t, & s=1, 1 \leq t \leq n_1 \\ 4+n_1+n_2-t, & s=2, 1 \leq t \leq n_2. \end{cases} \tag{8.2.7}$$

$$\lambda(a_{st}) = \begin{cases} 4+2n_1+n_2-t, & s=1,\ 2 \leq t \leq n_1 \\ 4+2(n_1+n_2)-t, & s=2,\ 2 \leq t \leq n_2 \\ 4+2(n_1+n_2), & s=1, t=1 \\ 6+2(n_1+n_2), & s=2, t=1. \end{cases} \tag{8.2.8}$$

In the above mapping λ we take only $BT(n_1,n_2)$ and take $n_1 \neq n_2$. We believe a better deficiency of this mapping could be find and its a open problem for reader.
In the following theorem we formulate the upper bound of super edge-magic deficiency of subdivided Banana tree $BT(n_1,n_2,\ldots,n_q)$ with 2 subdivision and subdivision is denoted by b_{st}.

Theorem 8.2.3. *For every even $n_s, 1 \leq s \leq 2$, and $n_1 = n_2 = u$ with 2 subdivision*

$$\mu_s(BT(n_1,n_2)) \leq 4u+3.$$

Proof.

We consider the banana tree $BT(n_1,n_2)$ and both n_s are even with 2 subdivision. We take the case $n_1 = n_2 = u$ and subdivision is denoted by (b_{st}^q). Let $s = |V(G)|$, and $e = |E(G)|$. Now, we define labeling $\lambda : V(G) \to \{1,2,\ldots,o\}$ as follows:

$$\lambda(a) = 3 \tag{8.2.9}$$

$$\lambda(c_s) = s, \quad 1 \leq s \leq 2 \tag{8.2.10}$$

$$\lambda(b_{st}^1) = \begin{cases} 4+u-t, & s=1, 1 \leq t \leq u \\ 4+2u-t, & s=2, 1 \leq t \leq u. \end{cases} \tag{8.2.11}$$

$$\lambda(b_{st}^2) = \begin{cases} 4+3n-t, & s=1, 1 \leq t \leq u \\ 4+4n-t, & s=2, 1 \leq t \leq u. \end{cases} \tag{8.2.12}$$

$$\lambda(a_{st}) = \begin{cases} 4+5n-t, & s=1, 2 \leq t \leq u \\ 4+6n-t, & s=2, 2 \leq t \leq u \\ 4+6u, & s=1, t=1 \\ 6+6u, & s=1, t=2 \end{cases} \tag{8.2.13}$$

8.3 Super Edge-Magic Deficiency of Subdivided w Tree $w(u,q)$

In this section we formulate the upper bound of super edge magic deficiency of subdivided w tree $w(u,q)$ with 1 subdivision and subdivision is denoted by b_{st}. Javaid et al. [39] defined w tree as follows and discuss the super edge magic total labeling of $w-$tree and it is denoted by W_n.

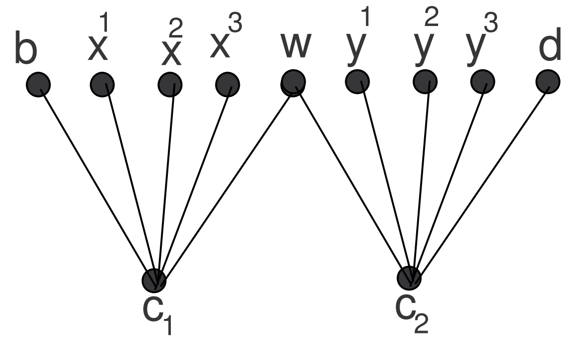

Figure 8.2: A w-tree $W(3)$.

Theorem 8.3.1. *For every u with 1 subdivision*

$$\mu_s(w(u,2)) \leq 6u+1.$$

Proof.

We consider the w tree $w(u,2)$ with 1 subdivision. We take the case for all u. We subdivided the leafs only and subdivision is denoted by (b_{st}). Let $s = |V(w(u,2))|$, and $e = |E(w(u,2))|$. $\lambda : V(w(u,2)) \to \{1,2,\ldots,o\}$

$$\lambda(a) = 5 \tag{8.3.14}$$

$$\lambda(c_{st}) = \begin{cases} t, & s = 1, 1 \leq t \leq 2 \\ 2+t, & s = 2, 1 \leq t \leq 2. \end{cases} \tag{8.3.15}$$

$1 \leq s \leq u, 1 \leq t \leq 2$

$$\lambda(p_{st}) = 5+s+2(t-1)u, \tag{8.3.16}$$

$$\lambda(q_{st}) = 5+u+s+2(t-1)u, \tag{8.3.17}$$

$$\lambda(X_t^s) = 5+4u+s+2u(t-1), \tag{8.3.18}$$

$$\lambda(Y_t^s) = 5 + 5u + s + 2n(t-1), \tag{8.3.19}$$

$$\lambda(w_s) = 11 + 14u + s \tag{8.3.20}$$

$1 \leq t \leq u$

$$\lambda(b_t) = 5 + 8u + 2t - 1, \tag{8.3.21}$$

$$\lambda(d_t) = 10 + 12u + t - 4(t-1), \tag{8.3.22}$$

Here the mapping λ form a consecutive sequence of numbers and by using the lemma 1 we get our desired result.

Theorem 8.3.2. *For every q with 1 subdivision*

$$\mu_s(w(2,q)) \leq 8q - 3.$$

Proof.

We consider the w tree $w(2,q)$ with 1 subdivision and for all q. We subdivided the leafs only and subdivision is denoted by (b_{st}). Let $s = |V(w(2,q))|$, and $e = |E(w(2,q))|$. $\lambda : V(w(2,k)) \to \{1,2,\ldots,o\}$

$$\lambda(a) = 2q + 1 \tag{8.3.23}$$

$$\lambda(c_{st}) = \begin{cases} 2(s-1)+1, & t = 1, 1 \leq s \leq q \\ 2s, & t = 2, 1 \leq s \leq q. \end{cases} \tag{8.3.24}$$

$1 \leq s \leq 2, 1 \leq t \leq q$

$$\lambda(p_t^s) = 2q + 1 + s + 4(t-1), \tag{8.3.25}$$

$$\lambda(q_t^s) = 2q + 3 + s + 4(t-1), \tag{8.3.26}$$

$$\lambda(X_t^s) = 6q + 1 + s + 4(t-1), \tag{8.3.27}$$

$$\lambda(Y_t^s) = 6q + 3 + s + 4(t-1), \tag{8.3.28}$$

$$\lambda(b_s) = 10q + 2s, \tag{8.3.29}$$

$$\lambda(w_s) = 16q + 1 + s, \tag{8.3.30}$$

$$\lambda(d_s) = 19q + 2(s-1) \tag{8.3.31}$$

We consider the w tree $w(u,q)$ and take $u = 2$ with 1 subdivision and use lemma 1 for our result. A better upper bound for $w(2,q)$ could be find and its an open problem.

Theorem 8.3.3. *For every u with 1 subdivision*

$$\mu_s(w(u,3)) \leq 6(u+1) + 3.$$

Proof.

We consider the w tree $w(u,3)$ with 1 subdivision. We take the case for all u. We subdivided the leafs only and subdivision is denoted by (b_{st}). Let $s = |V(w(u,3))|$, and $e = |E(w(u,3))|$. $\lambda : V(w(u,3)) \to \{1, 2, \ldots, o\}$

$$\lambda(a) = 7 \tag{8.3.32}$$

$$\lambda(c_{st}) = \begin{cases} t, & s = 1, 1 \leq t \leq 2 \\ 2 + t, & s = 2, 1 \leq t \leq 2 \\ 4 + t, & s = 3, 1 \leq t \leq 2. \end{cases} \tag{8.3.33}$$

$1 \leq s \leq u, 1 \leq t \leq 2$

$$\lambda(p_t^s) = 7 + s + 2(t-1)u, \tag{8.3.34}$$

$$\lambda(q_t^s) = 7 + n + s + 2(t-1)u, \tag{8.3.35}$$

$$\lambda(X_t^s) = 7 + 6n + s + 2u(t-1), \tag{8.3.36}$$

$$\lambda(Y_t^s) = 7 + 7n + s + 2u(t-1), \tag{8.3.37}$$

$$\lambda(b_s) = 7 + 12u + 2s - 1, \tag{8.3.38}$$

$$\lambda(w_s) = 14 + 18u + (s-1), \tag{8.3.39}$$

$$\lambda(d_s) = 21 + 18u + 2(s-1), \tag{8.3.40}$$

Now we consider extended w tree $w(u,q,m)$ with 1 subdivision and subdivision is denoted by b_{st}.

Theorem 8.3.4. *For every r with 1 subdivision*

$$\mu_s(w(2,2,m)) \leq 7m-1.$$

Proof.

We consider the extended w tree $w(2,2,m)$ with 1 subdivision. We subdivided leafs only and all leafs are represented by X_t^s and all subdivision is represented by (p_t^s we take the case for all m). Let $s = |V(w(2,2,m))|$, and $e = |E(w(2,2,m))|$. λ : $V(w(2,2,m)) \to \{1,2,\ldots,o\}$

$$\lambda(a) = 2m+1 \tag{8.3.41}$$

$$\lambda(c_{st}) = \begin{cases} t, & s=1, 1 \leq t \leq m \\ m+t, & s=2, 1 \leq t \leq m. \end{cases} \tag{8.3.42}$$

$1 \leq s \leq 2, 1 \leq t \leq m$

$$\lambda(p_t^s) = 2m+1+s+2m(t-1)+2(t-1), \tag{8.3.43}$$

$$\lambda(X_t^s) = 6m+1+s+2m(t-1)+2(t-1), \tag{8.3.44}$$

$$\lambda(b_s) = 10m+1+s+m(s-1), \tag{8.3.45}$$

$$\lambda(w_t^s) = 4+16m+m(s-1)+(t-1) \tag{8.3.46}$$

$$\lambda(d_s) = 2+19m-(5m-1)(s-1), \tag{8.3.47}$$

It is easy verify that the mapping λ form a consecutive sequence of numbers by using the lemma 1 we get our desired result.

In the following theorem we formulate the upper bound of super edge-magic deficiency of subdivided w tree $w(u,q)$ with 2 subdivision and subdivision is denoted by b_{st}.

Theorem 8.3.5. *For every u with 2 subdivision*

$$\mu_s(w(u,2)) \leq 10u+2.$$

Proof.

We consider the w tree $w(u,2)$ with 2 subdivision. we take the case for all u. We subdivided the leafs only and subdivision is denoted by (b_{st}). Let $s = |V(w(u,2))|$, and $e = |E(w(u,2))|$. $\lambda : V(w(u,2)) \to \{1,2,\ldots,o\}$

$$\lambda(a) = 5 \tag{8.3.48}$$

$$\lambda(c_{st}) = \begin{cases} t, & s = 1, 1 \leq t \leq 2 \\ 2+t, & s = 2, 1 \leq t \leq 2. \end{cases} \tag{8.3.49}$$

$$\lambda(p_{st}^q) = \begin{cases} 5+s+2(t-1)n, & q = 1, 1 \leq s \leq u, 1 \leq t \leq 2 \\ 5+4u+s+2(t-1)u, & q = 2, 1 \leq s \leq u, 1 \leq t \leq 2. \end{cases} \tag{8.3.50}$$

$$\lambda(Q_{st}^q) = \begin{cases} 5+u+s+2(t-1)n, & q = 1, 1 \leq s \leq u, 1 \leq t \leq 2 \\ 5+5u+s+2(t-1)n, & q = 2, 1 \leq s \leq u, 1 \leq t \leq 2. \end{cases} \tag{8.3.51}$$

$1 \leq s \leq u, 1 \leq t \leq 2$

$$\lambda(X_t^s) = 5+8u+s+2u(t-1), \tag{8.3.52}$$

$$\lambda(Y_t^s) = 5+9u+s+2u(t-1), \tag{8.3.53}$$

$$\lambda(b_t) = 4+12u+2t, \tag{8.3.54}$$

$$\lambda(d_t) = 8+20u+2t, \tag{8.3.55}$$

$$\lambda(w_t) = 11+22u+2t \tag{8.3.56}$$

It is easy verify that the mapping λ form a consecutive sequence of numbers by using the lemma 1 we get our desired result.

Theorem 8.3.6. *For every q with 2 subdivision*

$$\mu_s(w(2,q)) \leq 12q-2.$$

Proof.

We consider the w tree $w(2,q)$ with 2 subdivision. We take the case for all q. We subdivided the leafs only and subdivision is denoted by (b_{st}). Let $s = |V(w(2,q))|$, and $e = |E(w(2,q))|$. $\lambda : V(w(2,q)) \to \{1,2,\ldots,o\}$

$$\lambda(c_{st}) = \begin{cases} 2(s-1)+1, & t=1, 1 \leq s \leq q \\ 2s, & t=2, 1 \leq s \leq q. \end{cases} \tag{8.3.57}$$

$$\lambda(a) = 2q+1 \tag{8.3.58}$$

$$\lambda(p_{st}^{q}) = \begin{cases} 2q+1+s+4(t-1), & q=1, 1 \leq s \leq 2, 1 \leq t \leq q \\ 6q+1+s+4(t-1), & q=2, 1 \leq s \leq 2, 1 \leq t \leq q. \end{cases} \tag{8.3.59}$$

$$\lambda(q_{st}^{q}) = \begin{cases} 2q+3+s+4(t-1), & q=1, 1 \leq s \leq 2, 1 \leq t \leq q \\ 6q+3+s+4(t-1), & q=2, 1 \leq s \leq 2, 1 \leq t \leq q. \end{cases} \tag{8.3.60}$$

$1 \leq s \leq 2, 1 \leq t \leq q$

$$\lambda(X_t^s) = 10q+1+s+4(t-1), \tag{8.3.61}$$

$$\lambda(Y_t^s) = 10q+3+s+4(t-1), \tag{8.3.62}$$

$1 \leq s \leq q$

$$\lambda(b_s) = 14q+2s, \tag{8.3.63}$$

$$\lambda(d_s) = 24q+2s, \tag{8.3.64}$$

$$\lambda(w_s) = 28q+2s, \tag{8.3.65}$$

λ form a consecutive sequence of numbers using the lemma 1 we get our desired result.

Chapter 9

Conclusions and Open Problems

9.1 Conclusions

This dissertation studied the construction of super face anti-magic labeling of $(1,1,1)$ type for qC_5 - snake graph, subdivision of qC_5 - snake graph and isomorphic copies of qC_5 - snake graph. Face anti-magic labeling of generalized qC_n Snake graph and subdivision of generalized qC_n Snake graph. Super face anti-magic labeling of $(1,1,1)$ type on subdivision of qC_4, triangular qC_4 as well as subdivision of triangular qC_4. Magic and anti-magic total labeling on subdivision of $G \cong P_o \times P_4$, super anti-magic labeling of $(1,1,1)$ type on subdivided triangular ladder.
Super edge-magic deficiency of banana tree $BT(n_1, n_2, \dots, n_q)$, subdivided banana tree $BT(n_1, n_2, \dots, n_q)$ and subdivided w tree $w(u,q)$ has also been discussed in this thesis.

For advance studies in this field some open problems are identified.

9.2 Open Problems

Open Problem 1.
Snake graph $H \cong rqC_o, o \geq 4, q \geq 2$, having p subdivisions and $(2,2,\dots,2)$ string , admits face anti-magic labeling or not.
Open Problem 2. Triangular Snake graph $H \cong rqC_4, q \geq 2$ of different strings with p subdivisions, admits super face anti-magic labeling or not.
Open Problem 3. Subdivided Grid graph, $H \cong P_o \times P_r, o, r \geq 3$ admits face anti-magic labeling or not.
Open Problem 4. Subdivided triangular grid graph $H \cong P_o \times P_r, o, r \geq 3$ admits face anti-magic labeling or not.
Open Problem 5. For every $n_s, 1 \leq s \leq 2, n_1 = n_2 = n$ and p subdivisions find an upper bound of $\mu_s(BT(n_1, n_2))$
Open Problem 6. For every $n_s, 1 \leq s \leq 2$,and $n_1 \neq n_2$ and p subdivisions find a better upper bound of $\mu_s(BT(n_1, n_2))$.

Open Problem 7. For every o, q with p subdivision find a upper bound of $\mu_s(w(o,q))$.

Open Problem 8. For every o, q, r with p subdivision find an upper bound of $\mu_s(w(o,q,r))$.

Chapter 10

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