On Algebraic Aspects of Spanning Complexes

By

ASIF ALLAH-DITTA

CIIT/FA17-RMT-027/LHR

MS Thesis

In

Mathematics

COMSATS University Islamabad

Lahore Campus

Spring, 2019

COMSATS University Islamabad

On Algebraic Aspects of Spanning Complexes

A Thesis Presented to

COMSATS University Islamabad, Lahore Campus

In Partial fulfillment

of the requirement for the degree of

MS MATHEMATICS

By

ASIF ALLAH-DITTA

CIIT/FA17-RMT-027/LHR

Spring, 2019

On Algebraic Aspects of Spanning Complexes

A Post Graduate Thesis submitted to the Department of Mathematics as partial fulfillment of the requirement for the award of Degree of M.S in Mathematics.

Name	Registration Number
ASIF ALLAH-DITTA	CIIT/FA17-RMT-027/LHR

Supervisor

Dr. Hani Shaker

Associate Professor

Department of Mathematics,

Lahore Campus.

COMSATS University Islamabad, Lahore Campus

June, 2019

Declaration

I ASIF ALLAH-DITTA (CIIT/FA17-RMT-027/LHR) hereby declare that I have produced the work presented in this thesis, during the scheduled period of study. I also declare that I have not taken any material from any source except referred to wherever due that amount of plagiarism is within acceptable range. If a violation of HEC rules on research has occurred in this thesis. I shall be liable to punishable action under the plagiarism rules of the HEC.

Date:_____________________

June, 2019

Signature of the student:

____

ASIF ALLAH-DITTA

(CIIT/FA17-RMT-027/LHR)

Certificate

It is certified that ASIF ALLAH-DITTA (CIIT/FA17-RMT-027/LHR) has carried out all the work related to this thesis under my supervision at the Department of Mathematics, COMSATS University Islamabad, Lahore Campus and the work fulfills the requirement for award of MS degree.

Date:______________________

Supervisor:

____

Dr. Hani Shaker
Associate Professor

Head of Department:

____

Dr. Sarfraz Ahmad
Associate Professor
Department of Mathematics

DEDICATION

To

My beloved parents and respected teachers

.

ACKNOWLEDGEMENTS

First of all, I pay my gratitude to Allah Almighty, the most Gracious and the most Merciful, Who helped me and gave me the courage and strength to complete my thesis. My very humble praise is to the beloved Holy Prophet Muhammad S.A.W whose love is light to me.

First of all, I would like to express my sincere gratitude to my supervisor Dr. Hani Shaker for his continuous guidance and support throughout my MS study and related research. His patience, motivation, immense knowledge, precious time and vision ensured the accuracy, effectiveness and success of my research. His guidance helped me in all the time of research work and writing of Project report. Secondly, we am really grateful to Dr. Sarfraz Ahmad, HOD (Department of Mathematics, CUI, Lahore) for his moral support.

Besides my advisor, I would like to say thanks to all my teachers of Mathematics Department of CUI, Lahore, for their insightful comments and encouragement.

I would also like to express deepest gratitude to my father and my family for bringing me up to this level where I could finish this demanding errand. It is only due to their struggle and sacrifice that I am at this stage. My father has been a continuous source of motivation all along.

I am very thankful to all those who helped me directly or indirectly and for their cooperation.

ASIF ALLAH-DITTA (CIIT/FA17-RMT-002/LHR)

ABSTRACT

On Algebraic Aspects of Spanning Complexes

In this thesis the main motive is to characterize some of the algebraic and combinatorial properties of spanning forest complexes $\Delta_s(G)$ of some simple finite disconnected graphs G. Spanning forests complexes play a key role in algebraic, topological and geometric combinatorics but, as compared to simplicial complexes, little is known about their geometric and combinatorial structure. We characterize some algebraic properties of forests complexes related to different book graphs on fixed ground set. We provide the characterization of f-vector for the spanning forest complex of k disjoint copies of a book graph.

Table of Contents

List of Figures

Chapter 1

Introduction

Algebra is now a vast discipline in mathematics. There is a deep interaction of algebraic combinatorics with commutative algebra since $1790's$. The practical work in this discipline has developed during the $19th$ and $20th$ century with the advancement of applied graph theory. The Combinatorial techniques in pure mathematics and graph theory have been employed to many problems. Combinatorial Commutative Algebra is raised as an emerging subject of pure and applied mathematics.

The discussion on Simplicial complexes has become the main topic in combinatorial commutative algebra because of its interactions with modern algebra, algebraic geometry and graph theory. Richard P. Stanley first explained the combinatorial study of simplicial complexes in $1970's$. His work have a prominent role in making interaction of combinatorial commutative algebra. Villarreal et. al [30] gave the new concept by relating edge ideals with some simple finite graphs in commutative algebra. Later on, many researchers worked on commutative algebra to develop the relations between combinatorial graph structures and algebraic properties of edge ideals associated to simple finite graphs. [12, 13, 31, 32].

Imran Anwar et al. [1] introduced spanning simplicial complexes of some finite cycle graph G. They give some combinatorial characteristics of spanning complexes related to some *uni*-cycle or r-cycle connected graphs [24]. Jin Guo in [16] used the term spanning forest complex associated to a finite disconnected graph which is the extension of spanning tree complex. The algebraic characterization of spanning forest complexes associated to the disjoint union of finite uni-cyclic graphs is explained by M. Asif [3] and the algebraic aspects of spanning forest complexes associated to disjoint union of multi-cyclic graphs is explained by Rabia Nazir. They discussed f-vector, h-vector and some other algebraic properties of spanning forests complexes of disjoint union of uni-cyclic and multi-cyclic graphs.

We will discuss the algebraic aspects of spanning forests complexes associated with disjoint union of multi-cyclic graphs with some common vertices and edges. We want to analyze some algebraic properties of spanning forest complexes in view of the algebraic discussion of simplicial complexes for simple finite graphs discussed in [1, 2, 3, 4, 24]. Our motive is to extend the work of Rabia Nazir and Guangjun Zhu et. al [4] in algebraic direction and algebraic characterization of spanning forest complexes associated to the disjoint union of book graphs and triangular book graphs.

In Chapter 2, We give some basic preliminaries and terminologies of commutative algebra that will enable the reader to understand the basic concepts related to commutative algebra. These terms and relevant study give rise to have understanding about the topics which are going to discuss in next chapters of this thesis.

Chapter 3, have some basic introduction of graphs. We define basic types of graphs and discuss some essential results and theorems about graphs, a Brief description connected graphs, disconnected graphs, spanning subgraphs, spanning trees and their other basic characteristics .

In Chapter 4,We give a brief overview of simplicial complexes Δ. We explain some basic characteristics about simplicial complexes. We discuss results and theorems related to this topic which helps the reader to go through the next sections of this thesis. We discuss an important algebraic object spanning complex of cyclic graphs. To understand this topic, the reader should know about the concepts discussed in previous chapters. This section also consist of basic results and theorems about spanning complexes related to finite graphs.

Chapter 5, contains the discussion on spanning forest complexes of finite disconnected graphs. Firstly, we will describe the results of spanning forest complexes related to multiple copies of uni-cyclic graphs and algebraic properties arising from it. After this, we will explain our work about the spanning forest complexes related to multiple copies of same type of book graphs. Finally, we will discuss spanning forests complexes associated to triangular book graphs with some fixed size and their algebraic properties like formulas for finding number of spanning forests, their vertex set, edge set, dimension of complex and mainly we will characterize f-vector.

Chapter 2

Algebraic Combinatorics and Commutative Algebra

2.1 Combinatorics

Combinatorics is generally referred as a field about counting and combining objects and surely counting is a major part of combinatorics. Some questions that mostly arises have counting problems like, In how many ways can the specific elements be combined? Or some questions like, although a specific combination exists, or somehow the specific combination is the best.

2.1.1 Permutations and Combinations

First lets have a look on *counting*. The word "counting" sounds to be simple to study. When we think of counting, we actually mean of determining the size of a set, or might be the sizes of some sets, with some common things, but different sizes may have one or more parameters.

Definition 2.1. Permutations

A permutation is a mathematical tool that determines the number of possible arrangements in a set when order of objects is important.

Permutations are often confused with another mathematical technique that is combinations. A permutation of some objects is a particular linear ordering of the objects.

Formula for computation of permutations

$$^{n}P_{r} = n!/(n-r)! \tag{2.1.1}$$

Example 2.1.1.

$$^{8}P_{3} = 8!/(8-3)! = 8 \times 7 \times 6 = 336 \tag{2.1.2}$$

Definition 2.2. Combinations

In algebraic combinatorics, combination is a mathematical technique that determines the number of possible arrangements in a set when r objects are combined from n objects. The order of objects in Combinations doesn't matter.

Formula for computation of Combinations

$$^{n}C_r = n!/(n-r)!r! \tag{2.1.3}$$

Example 2.1.2.

$$^{6}C_2 = 6!/(6-2)! * 2! = (6 \times 5) = 15 \tag{2.1.4}$$

Definition 2.3. Algebraic Combinatorics is a branch of mathematics that employs methods of modern abstract algebra precisely graph theory and some other discrete mathematical structures in various combinatorial extents and conversely, applies combinatorial techniques to problems in algebra.

2.2 Commutative Algebra

Commutative algebra is now an emerging field of mathematics related to geometries. Since last few years a number of exciting developments are seen with the combination of algebraic combinatorics and commutative algebra. Algebraic geometry and algebraic number theory, both have deep interaction with the broad field of commutative algebra. Some of the fundamental topics in commutative algebra are monomial ideals, monomial rings and Stanley-Reisner rings. We provide a brief description to some basic concepts about combinatorial methods, terminologies and notations about commutative algebra to understand polynomial rings, subrings, semi-group rings and ideals. For More understanding and standard notation, we refer [19], [27], [31], [32],[33].

2.3 Commutative Ring

Definition 2.4.

Consider R be a non-empty set acquiring the binary operations (sum'+' and product'.') would be a ring only if R fulfils these relations:

1. R w.r.t sum be the abelian group.
2. R is monoid w.r.t product i.e. R satisfies associative property w.r.t multiplication.
3. Distributive laws are satisfied in R
   (a) $l(m+n) = lm + ln$
   (b) $(l+m)n = ln + mn$

$\forall l, m, n \in R$

Definition 2.5.

If product operation is commutative in R then the ring R will be a commutative ring. that is, $m.n = n.m, \quad \forall \quad m, n \in R$.

Definition 2.6.

Two elements a, b of a ring R will be the zero divisors iff

$$ab = ba = 0$$

for some $a, b \neq 0$.

In a commutative ring left and right zero divisors are same.

Example 2.3.1. *Consider $\mathbb{Z}_6$. As both $2, 3 \in \mathbb{Z}_6$ and $(2).(3) = 0$, so 2 and 3 are zero divisors*

Definition 2.7. If $\mathbb{R}$ does not have any zero divisor implies $\mathbb{R}$ will be integral domain.

Example 2.3.2.

$\mathbb{C}$, $\mathbb{R}$ and $\mathbb{Z}$ are the integral domains.

$\mathbb{Z}_n$ is called an integral domain iff n is a prime.

Definition 2.8. If the non-zero elements are invertible in $\mathbb{R}$ then $\mathbb{R}$ will be a division ring.

Example 2.3.3. *$\mathbb{R}$ is the division ring then the elements in $\mathbb{R}$ that are non-zero will be invertible .*

Definition 2.9.

A division ring having commutative property w.r.t product will be a field.

Definition 2.10.

Let R and Q be two rings. Then the rings R and Q will be homomorphic rings if, we have a map $f : R \to S$ such that $f(x+y) = f(x)+f(y)$ and $f(xy) = f(x)f(y)$ for all $x, y \in R$. If the mapping f is bijective then the rings R and Q will be **isomorphic rings**.

2.3.1 Subrings and Ideals

Definition 2.11.

A subset Q of a ring R is a subring of R only if the set Q also satisfy the properties of a ring w.r.t sum and product defined in the ring R.

Example 2.3.4.

$\mathbb{Z}$ integers set will be the subring of $\mathbb{R}$.

Definition 2.12.

Let I as a subring of ring R, then it will be the left ideal in R, only if $\forall\ x \in R$ and $a \in I$, we have $xa \in I$. In the same way the subring I be the right ideal in R only if $ax \in I$. Any subring will be the **ideal** of a ring R if is the both left and right ideal.

Every ring contains two basic ideals, a null ideal i.e. $\{0\}$ and ring ideal i.e. $\{R\}$. These both ideals are improper ideals of R. Any ideal I of R except $\{0\}$ and {R} will be a proper ideal.

Example 2.3.5. $n\mathbb{Z}$ be the ideals in $\mathbb{Z}$ where $n \geq 0$.

Definition 2.13.

A proper ideal I will be the **maximal ideal** of R it is not contained in the proper ideals of R.

Theorem 2.1. *Let R be the commutative ring with unity, let s be a proper ideal of R, then s will be maximal iff R/s be a field.*

Definition 2.14.

Let J is an ideal in a ring R then J will be a **principal ideal** if there exists an $p \in J$ such that $I = \{\ pq \mid q \in R\ \}$. Then it can be said the ideal J is generated by p and written as $j =< p >$.

Example 2.3.6.

(i) Zero is maximal ideal in any field.

(ii)For n is prime the ideal $n\mathbb{Z}$ will be the maximal ideal of $\mathbb{Z}$.

(iii) The ideal (P, y) be the maximal in a polynomial ring $\mathbb{Z}[Y]$ for all prime numbers P.

Theorem 2.2.

A commutative ring that is non-zero and have product identity will be field if it does not have any proper ideal.

Proof:

Consider R a non-zero commutative ring with identity $a \neq 0 \in R$. then the set Ra is an ideal of R,for $p, q \in Ra \Rightarrow p = x_1 a$, $q = x_2 a$ for few $x_1, x_2 \in R$ $p - q = x_1 a - x_2 a = (x_1 - x_2)a \in Ra$ $\because x_1 - x_2 \in R$ and if $x \in R$ then $xp = x$ $(x_1 a)$ =$(xx_1)a \in Ra \Rightarrow Ra$ is an ideal. Now, if R having no proper ideal then,

$$Ra = R$$

For $1 \in R = Ra$ $\exists$ an element $c \in R$ such that $ca = 1$ i.e R having inverse of each element so R will be field.

Example 2.3.7.

Consider $\mathbb{Z}$ is a ring and $m\mathbb{Z}$ be the ideal in $\mathbb{Z}$ implies $\mathbb{Z}/m\mathbb{Z}$ will be a quotient ring and isomorphic to $\mathbb{Z}_m$.

2.4 Monomial Ideals

The study of ideals of a polynomial ring $K[x_1, ..., x_n]$ such that these ideals are created by square-free monomials. These ideals are called Stanley-Reisner ideals, that are quotient by these ideals will be Stanley-Reisner rings. The combinatoric behaviour of these algebraic objects related to their intimate connections with algebraic topology.

In this section we will study about polynomial rings. The study of polynomial ring is very useful to learn the idea of commutative algebra.

Definition 2.15.

A general mathematical expression of polynomials in the form of algebraic terms is given by

$$P(x) := a_0 + a_1 s + \cdots + a_{n-1}s^{n-1} + a_n s^n = \sum_{j=0}^{n} a_j s^j$$

where n is the degree of polynomials , where all $a_j s \in \mathbb{R}$ and a_n is a leading coefficient of $P(s)$.

Definition 2.16.

Consider k a field and $S = K[s] = \{\sum_{i=0}^{n} a_i s^i \mid a_i \in K \ and \ n \in \mathbb{Z}^+\}$. is a polynomial ring over this field k in n indeterminates $\mathbf{s} = s_1, ..., s_n$.

Addition of two polynomials:

Let $P(s) = \sum_{j=0}^{n} a_j s^j$ and $Q(s) = \sum_{j=0}^{n} b_j s^j$ from $R[s]$ then addition of two polynomials describe as,

$$P(s) + Q(s) = \sum_{j=0}^{n} a_j s^j + \sum_{j=0}^{n} b_j s^j = \sum_{j=0}^{n} (a_j + b_j) s^j$$

Multiplication of the two polynomials:

Let $P(s) = \sum_{j=0}^{n} a_j s^j$ and $Q(s) = \sum_{j=0}^{n} b_j s^j$ from $R[s]$ then the multiplication of two polynomials describe as,

$$P(s).Q(s) = \left(\sum_{j=0}^{n} a_j s^j \right) . \left(\sum_{j=0}^{n} b_j s^j \right) = \sum_{j=0}^{n} \left(\sum_{k+l=j} a_k b_l \right) s_j.$$

Definition 2.17.

An algebraic expression having only one term is called monomial. It is a power product of variables. A monomial with n indeterminants $\{s_1, s_2, ..., s_n\}$ express as

$$s_1^{\beta_1}, ..., s_n^{\beta_n}, \quad \beta = (\beta_1, ..., \beta_n) \in \mathbb{N}^n$$

The monomial set with n indeterminants express as

$$Mon(s_1, s_2, ..., s_n) \Rightarrow Mon_n := \{s^\alpha : \alpha \in \mathbb{N}^n\}$$

The expression of the polynomials is a linear combination or finite sum of monomials *i.e*,

$$g = \sum_\alpha a_\alpha s^\alpha \ where \ \alpha \ is \ finite$$

we can make the unique order of monomials by selecting the total ordering for a set of monomials. For other applications it is compulsory, however, that the ordering is well matched with the semi group structure on Mon_n.

Definition 2.18. [5]

"Let I be an ideal of a polynomial ring $R = K[x_1, x_2, ..., x_n]$ with n indeterminates over field K. An ideal $I \subseteq K[x]$ is called a monomial ideal if it is generated by monomials."

Definition 2.19. [5]

"An ideal generated by square free monomials is called square free monomial ideal. i.e.,

$$J = (x^\alpha \mid \alpha \in \mathbb{N}^n; \alpha_i = 0 \ or \ \alpha_i = 1)$$

for $\alpha = (\alpha_1, .., \alpha_n) \in \mathbb{N}$."

Definition 2.20. [5]

"An ideal I of a polynomial ring $R = K[x_1, x_2, ..., x_n]$ such that $I = R$ is termed as *radical ideal* if $I = \sqrt{I}$."

Chapter 3

Preliminaries of Graphs

3.1 Introduction

The advanced study of graphs deals with the close relationship between graphs and their algebraic properties. Graph theory have vast applications in all major fields of science like biology, chemistry and Computer science. In algebraic combinatorics we characterize the algebraic properties of graphs and other discrete geometric objects. Here we will give an brief overview of graphs and discuss its main algebraic and combinatorial properties of simple finite graphs.

3.2 Graph Theory

Graphs are mathematical structures which show the pair wise relationship between different objects. In structural view, objects are called vertices,nodes or points and relations or connections between them are named as edges or lines. The basic terminology vertex is used for a point or an object and edge for a line. It becomes more convenient to understand things and structures in the form of graphs.

Definition 3.1. Graph

A graph can be defined as combination of two sets (V, E), here V be the vertex set and E is for the edge set generated by the pair of vertices. We can denote a graph as $G = G(V, E)$.

Definition 3.2. A graph is represented as diagram through points and lines, these points are called vertices. The Vertices may be isolated in a graph or any two vertices may be connected by some edges. A graph that carries a single vertex will be a trivial graph.

Definition 3.3. The lines or arcs that connect vertices, objects or represent the connection of any circuit are called edges. Each edge has two vertices on its both ends. An edge is said to be a loop if its both ends have the same vertex. The edge set of G is denoted as E.

For example: In Chemical graphs, the vertices represents the atoms and the edges represents the chemical bonds between them.

The geometric representation of a graph is shown in figure 3.1.

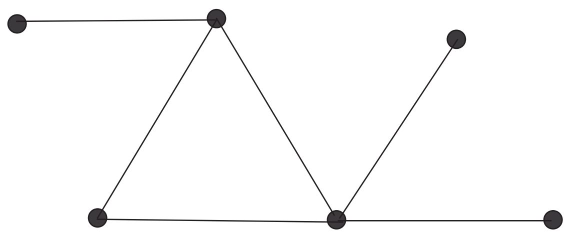

Figure 3.1: Connected Graph

We generally label vertices with letters (for example: $a, b, c, ...or v_1, v_2, ...$) or numbers $1, 2, ...$ and V for the vertex and E edge set, while edges are denoted by pair of vertices. E is a multi-set of edges, in other words, its edges can exist more than one time. We use letters $a, b, c, ...or e_1, e_2, ...$) or numbers $1, 2, ..$ to label the edges.

Definition 3.4. Any graph will be the **simple graph** if graph is loopless and no multiple edge exists in it. The figure 3.2 is a simple graph.

Definition 3.5. Count of edges connected to a specific vertex is termed as degree of that vertex and is represented by $d(v)$. in figure 3.1 $d(v_1) = 1$, $d(v_3) = 2$, and $d(v_4) = 5$.

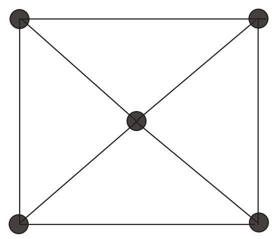

Figure 3.2: Simple Graph

Definition 3.6. The total vertices contained by a graph will be the graph order. Mathematically, it is written as $\mid V \mid$. The graph in figure 3.1 $\mid V \mid= 6$.

Definition 3.7. The cardinality of the set of edges of a graph named as the graph size. It is denoted as $\mid E \mid$. In figure 3.1 $\mid E \mid= 6$.

For simplicity, we denote the cardinality of vertex set to be $\mid V \mid$=n and that of edge set $\mid E \mid= m$

Remark: The two edges (v_1, v_2) and (v_2, v_1) are the same. In other words, the pair of vertices is not ordered.

Corollary 3.1. ***Hand Shaking Lemma*** *[1] "The graph $G = (V, E)$, where $V = v_1, ..., v_n$ and $E = e_1, ..., e_m$ satisfies;* $\sum_{i=1}^{n} d(v_i) = 2m$*"*

Definition 3.8. Subgraphs

A graph F will be subgraph of another graph G, only if both the set of vertices and the edges of F be subsets in G i,e. $V_F \subseteq V_G$ and $E_F \subseteq E_G$.

Definition 3.9. Spanning Subgraphs

A subgraph S of a graph G will be the spanning subgraph of G if $V_G = V_S$ and $E_S \subseteq E_G$.

Definition 3.10. Induced Graph

Any subgraph M of a connected graph will be induced subgraph, such that whenever we remove any vertex from G the corresponding edges will also be removed from the resulting subgraph M.

The spanning subgraph and induced subgraph are shown in fugure 3.3.

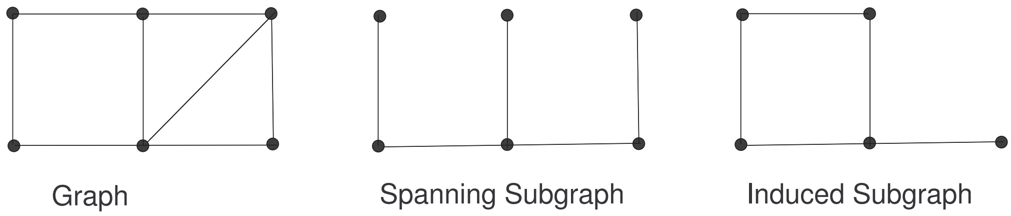

Figure 3.3: Subgraphs

Definition 3.11. Connected Graph

A graph G is called a connected if we can find a path between each pair of vertices in the graph G. The graphs in figure 3.3 are the connected finite graphs. The graph which is not connected or there is infinite distance between any of the pair of vertices then the graph is disconnected.

Definition 3.12. Regular Graphs

A graph whose every vertex carries the equal degree called the regular graph. Any graph G is m-regular graph all the vertices of the graph carries the degree m. For example only cycle graphs are 2 regular, null graphs are 0 regular graphs, complete graphs k_n are $n-1$ regular graphs.

Definition 3.13. Complete Graphs

A finite simple and connected graph that carries all possible edges between the vertices and every pair of unique vertices is adjacent will be the complete graph. We generally represent complete graph with m vertices by K_m. A complete graph K_m has the size $m(m-1)/2$. In the figure 3.4 some complete graphs are given.

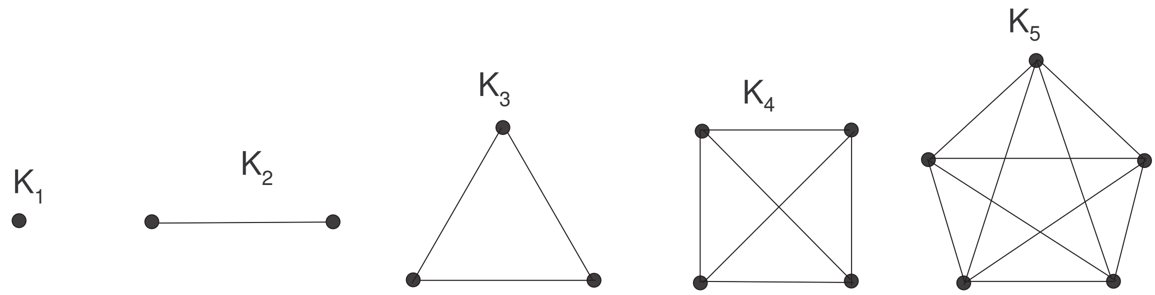

Figure 3.4: Complete Graphs

Definition 3.14. Walk in a graph

A **walk** is the alternate series of the vertices and the edges of in graph G from an starting vertex to a last vertex i,e,.

$$v_{i_0}, e_{j_1}, v_{i_1}, e_{j_2}, ..., e_{j_n}, v_{i_n} \tag{3.2.1}$$

v_{i_0} is the starting vertex and v_{i_n} is the ending vertex. n will the length of that walk. Literally we can speak of a walk from v_{i_0} to v_{i_n}. A walk will be just a vertex if it have the zero length. The vertices and the edges can be repeated in walk. A walk is open if $v_{i_0} \neq v_{i_n}$ and is closed if $v_{i_0} = v_{i_n}$.

Definition 3.15. In a graph a **trail** will be the walk with the property that all the edges are covered at most one time. A trail of a graph is said to be a **path** if all the vertices are covered at most once other than possibly a starting and an end vertex, while a closed path is a **circuit** in specific graph.

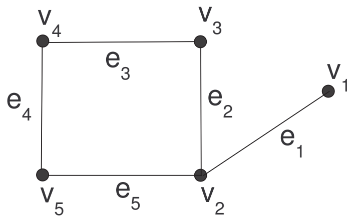

Figure 3.5: Trail in a graph

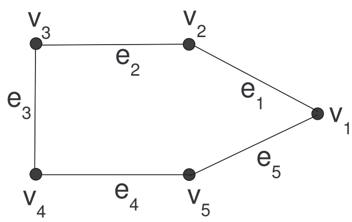

Figure 3.6: Closed path in graph

The walk $v_1, e_1, v_2, e_2, v_3, e_3, v4, e_4, v_5, e_5, v_2$ in figure 3.5 is a trail and the trail $v_1, e_1, v_2, e_2, v_3, e_3, v_4$ in figure 3.6 is a path and also a circuit.

Definition 3.16. Null graph

A null graph that which has the edge set as empty set. We generally represent any null graph on m order as N_m. A null graph on five vertices N_5 is shown in Fig 3.7. We can see that a null graph carries all the isolated vertices.

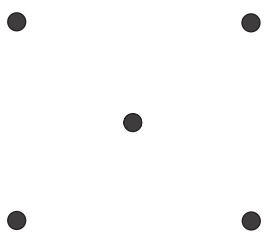

Figure 3.7: Null Graph

Definition 3.17. Cyclic Graph

A graph that consists of some cycles is called a cyclic graph, in other words, some of its vertices are connected in a closed path. The cycle graphs are regular with the degree 2. If the graph contains only one cycle then the graph will be a uni-cyclic graph. It is generally represented as C_n, where n denotes the order of graph G. A uni-cyclic graph has equal order and size. A graph that do not have cycle is called acyclic graph.

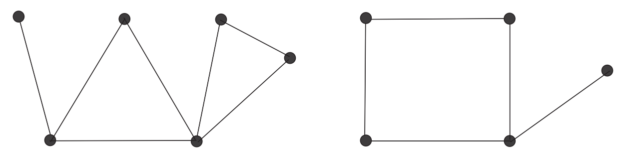

Figure 3.8: Cyclic Graphs

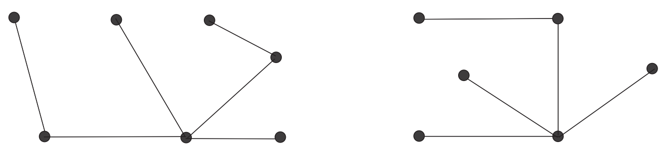

Figure 3.9: Acyclic Graphs

Definition 3.18. Bipartite Graphs

A graph G will be bipartite if we can divide the set of vertices of G into two distinct sets S and T such thateach edge in G connects the vertex in S with the vertex in T. There is at least one vertex of S is joined to set T by an edge.

Definition 3.19. A **complete bipartite graph** in which all vertices of set S is adjacent with all vertices in set T and is connected by just a single edge. We represent it by $K_{m,n}$. For example, the complete bipartite graphs $K_{1,4}, K_{2,3}$ and $K_{3,3}$ are given in figure 3.10. Note that $K_{m,n}$ has $m+n$ number of vertices and mn edges.

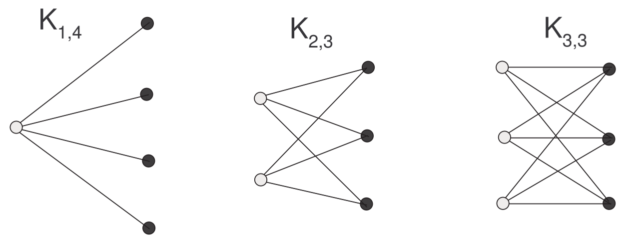

Figure 3.10: Complete Bipartite Graphs

Definition 3.20. (Eulerian Graphs)

A graph G is said to be Eulerian graph if we have a closed trail covering all edges of G once only,a closed trail of this type is called the Eulerian trail and simple graph G will be semi-Eulerian only if it consists of an open trail traversing every edge.

Theorem 3.2. *(Euler 1736). "A connected graph G is Eulerian if and only if the degree of each vertex of G is even."*

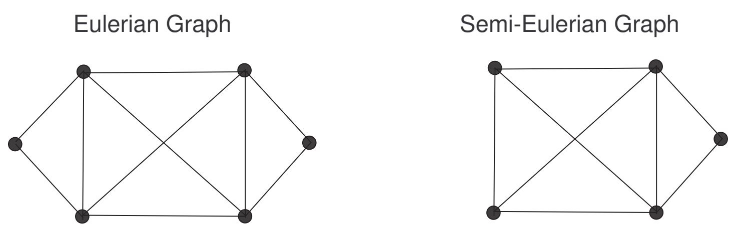

Figure 3.11: Eulerian and Semi-Eulerian graphs

Corollary 3.3. *"A connected graph is semi-Eulerian if and only if it has exactly two vertices of odd degree."*

Definition 3.21. Trees

A tree that is simple non-trivial finite cycle-free graph with the property that all pair of its vertices are connected through a unique path. A tree on n vertices is generally represented as T_n.

Trees on five and six vertices are shown in Fig 3.12

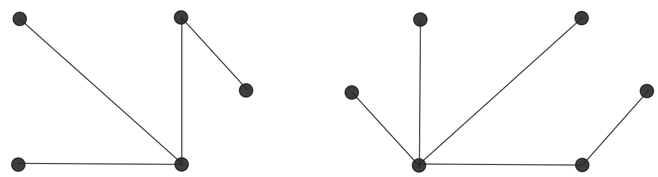

Figure 3.12: Tree Graphs

Definition 3.22. Spanning Trees

A subgraph L of a connected graph G will be a spanning tree if L is a connected tree, having the same vertices as of G.

Example 3.2.1. *All the spanning trees of graph 3.13 are shown in figure 3.14*

Spanning trees of the graph are obtain by removing an edge or edges, until there is no more cycle in the graph. This method is called **cutting down method.**

Definition 3.23. (Isomorphism of Graphs)

Two graphs W and X will be the isomorphic if both the graphs have

1. Same vertex and edge sets

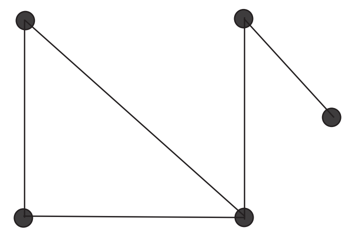

Figure 3.13: Simple Connected Tree

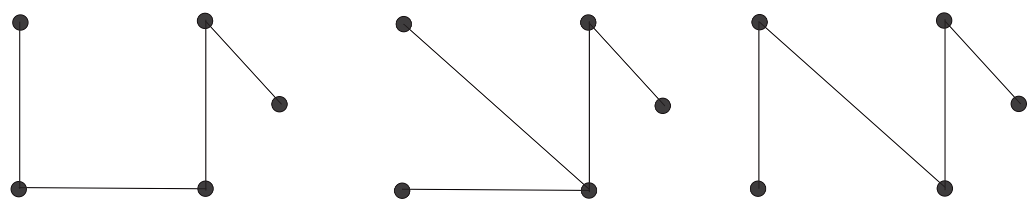

Figure 3.14: Spanning Trees

2. Same degree sequence

3. Same edge connectivity

We represents the isomorphic graphs as $w \equiv X$

The isomorphic graphs preserve the bijection to each other.

Chapter 4

Simplicial Complexes of graphs

4.1 Introduction

This chapter will provide the interesting concepts about simplicial complexes and spanning simplicial complexes. The idea of spanning simplicial complex is introduced by Anwar et al. [1]. They also explained the algebratc properties of unicyclic graphs in [1] and r-cyclic graphs in [24].They formulated the formulae for f-vector, h-vector for some simple connected graphs having disjoint cycles.

For complete understanding, we should be familiar about spanning trees and simplicial complex. The combinatorial Study of Spanning complexes be an algebraic property related to some simple connected graphs.

4.2 Simplicial Complexes

Consider $V = \{v_1, v_2, ..., v_n\}$ be the point set, we can define a simplicial complex Δ on ground set V be the union of those subsets from the set V, that satisfy these properties

1. $\{v_i\} \in \Delta, \quad \forall \quad i = 1, 2, ..., m$.
2. Any face $\sigma \in \Delta$ & $\tau \subseteq \sigma$ then $\tau \subseteq \Delta$.

Example 4.2.1. *Consider a complex on five vertices i,e,* $n = 5$ *shown in the Figure 4.1,*

Take

$\Delta = \{\{3,4,5\},\{3,4\},\{3,5\},\{4,5\},\{1,2\},\{2,3\},\{1,3\},\{1\},\{2\},\{3\},\{4\},\{5\}\}$ is a simplicial complex this set satisfies all axioms of simplicial complex.

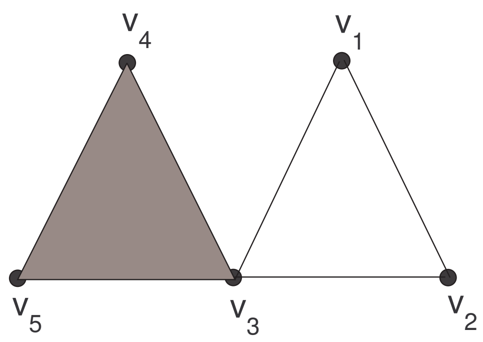

Figure 4.1: Simplicial Complex 1

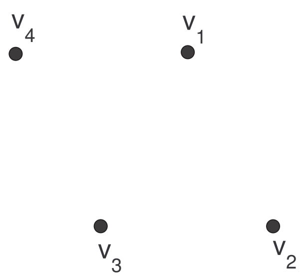

Figure 4.2: Trivial Simplicial Complex on a graph having all vertices isolated

Similarly, $\Delta = \{\{1\}, \{2\}, \{3\}, \{4\}, \phi\}$ in Fig 4.2 is also a simplicial complex

The elements in a simplicial complex are simplices or faces of Δ. The maximal faces of simplicial complex Δ under the inclusion principle are the facets of Δ.

Definition 4.1. Simplex

A simplicial complex composed of a single facet will be a simplex.

Definition 4.2. Pure Simplicial Complex

A simplicial complex Δ will be pure only if its every facets contain same dimension.

Definition 4.3. Dimension of a Face The dimension of any face F of Δ is one less than the cardinality of that face F. Mathematically, it is given by $dim(F) = \mid F \mid -1$.

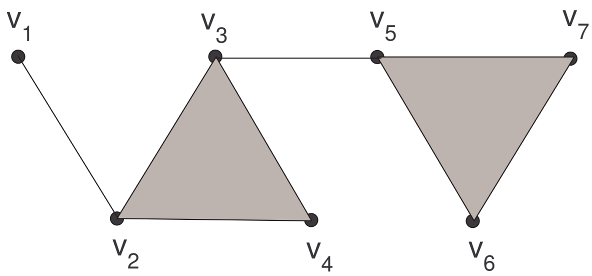

Figure 4.3: simplicial complex 2

For example, the dimension of a face $F = \{3, 4, 5\}$ in Fig 4.1 is

$\Rightarrow \ dim(F) = 3 - 1 = 2$

and dimension of vertices is zero, dimension of an edges is 1, dimension of a filled triangle is 2.

Definition 4.4. Computation of f-Vector

Consider a complex Δ with d be its dimension, then its face vector or simply $f-vector$ be the $d+1$-tuple, given by

$$f(\Delta) = (f_{-1}, f_0, f_1, ..., f_{d-1})$$

Here f_i records the count faces of Δ with dimension i and $0 \leq i \leq d$. Also $f_i = f(\Delta)$

Example 4.2.2. *Consider a simplicial complex Δ shown in Fig 4.3 with seven vertices with dimension $dim(\Delta) = 2$, its i-dimensional faces of Δ*

$f_0 = 7$ *(number of vertices)*

$f_1 = 8$

$f_2 = 2$ *The f-vector of the simplicial complex Δ shown in Fig 4.3 is $f = (7, 8, 2)$*

Definition 4.5. Computation of h-Vector

Suppose $dim(\Delta) = d$ and $f(\Delta) = (f_{-1}, f_0, f_1, ..., f_{d-1})$. The $h-vector$ $h(\Delta) = (h_0, h_1, ..., h_d)$ is a $d+1$-tuple, where its components are given by

$$h_k(\Delta) = \sum_{i=1}^{k} (-1)^{k-i} \tbinom{d-i}{k-i} f_{i-1}$$

For k=0,1,...,d.

Example 4.2.3. *Consider a simplicial complex Δ in Fig 4.3 with seven vertices with dimension $dim(\Delta) = 2$, its h-vector can be computed as;*

$$h_k(\Delta) = \sum_{i=1}^{k} (-1)^{k-i} \tbinom{d-i}{k-i} f_{i-1}$$

, where $h_0 = 1$

$$h_1(\Delta) = \sum_{i=1}^{1} (-1)^{0} \tbinom{2-1}{1-1} f_0$$

$$h_1(\Delta) = 1 \times 1 \times f_0$$

$$h_1(\Delta) = 7$$

and

$$h_1(\Delta) = (-1)^{0} \tbinom{2-1}{1-1} f_0 + (-1)^{1} \tbinom{2-1}{2-2} f_1$$

$$h_1(\Delta) = 7 - 8$$

$$h_1(\Delta) = -1$$

Hence, $h(\Delta) = (1, 7, -1)$

The f-vector and the $h$vector may determine each other uniquely by a linear equation,

$$\sum_{i=1}^{k} (x-1)^{d-i} f_{i-1} = \sum_{k=0}^{d} h_k \; x^{d-k}$$

Let $R = k[\Delta]$ be a Stanley-Reisner ring of a spanning complex Δ. Then its Hilbert-Poincare series can be expressed as,

$$P_R(x) = \sum_{i=0}^{d} f_{i-1}x^i/(1-x)^i = h_0 + h_1x + ... + h_dx^d/(1-x)^d$$

Definition 4.6. [12] **Facet Ideal and Non-face Ideal**

"Let Δ be a simplicial complex over a set $\{v_1, v_2, ...v_n\}$, and R a polynomial ring over $K[x_1, x_2, ...x_n]$.

(a) An ideal of R, defined as $\mathcal{F}(\Delta)$ (Facet Ideal), if it is generated by square free monomials $\{x_{i_1}....x_{i_s}\}$, so that$\{v_{i_1}, ..., v_{i_s}\}$ is a facet of Δ.

(b) An ideal of R, defined as We $\mathcal{N}(\Delta)$ (Non-Face Ideal), if generated square-free monomials $\{x_{i_1}....x_{i_s}\}$ is called not a facet ideal, where $x_{i_1}....x_{i_s}$ not belongs to Δ. We call $\mathcal{N}(\Delta)$ the $non-face\ ideal$ or $Stanley-Reisner$ ideal of Δ".

4.3 Spanning Simplicial Complexes

A brief description of spanning subgraphs and spanning treed is given in graphs section. In [1], Imran Anwar explained the idea of spanning simplicial complex, a complex associated with the spanning trees of connected graph G. In this section, we are going to have a brief overview on the algebra of spanning simplicial complexes.

Definition 4.7. Spanning Simplicial Complexes

Consider $s(G) = \{E_1, E_2, ..., E_m\}$ as the edge set the distinct spanning trees for a simple graph G and E_i be the edge set of T_i, the i-th spanning tree of G. Then spanning simplicial complex related to this spanning set $s(G)$ is the complex $\Delta_s(G)$ defined as $\Delta_s(G) = \langle E_1, E_2, ..., E_s\rangle$ where the facets of $\Delta_s(G)$ are actually the elements of spanning set $s(G)$.

Example 4.3.1.

Let a unicycle graph C_6 in Fig 4.4. There are total 6 spanning trees of C_6 by using

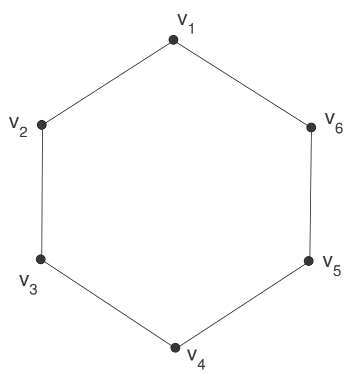

Figure 4.4: Cyclic Graph C_6

cutting down method we have.

We denote the edges sets of all the distinct spanning trees of C_6 *as* $s(C_6)$*. Hence we can write*

$$s(C_6) = \{\{v_2v_3, v_3v_4, v_4v_5, v_5v_6, v_6v_1\}, \{v_1v_2, v_3v_4, v_4v_5, v_5v_6, v_6v_1\}, \{v_1v_2, v_2v_3, , v_4v_5, v_5v_6, v_6v_1\},$$

$$\{v_1v_2, v_2v_3, v_3v_4, v_5v_6, v_6v_1\}, \{v_1v_2, v_2v_3, v_3v_4, v_4v_5, , v_6v_1\}, \{v_1v_2, v_2v_3, v_3v_4, v_4v_5, v_5v_6, \}\}$$

and the spanning simplicial complex for $\Delta_s(C_6)$ *is written as :-*

$$\Delta_s(C_6) = \langle\{v_2v_3, v_3v_4, v_4v_5, v_5v_6, v_6v_1\}, \{v_1v_2, v_3v_4, v_4v_5, v_5v_6, v_6v_1\}, \{v_1v_2, v_2v_3, , v_4v_5, v_5v_6, v_6v_1\},$$

$$\{v_1v_2, v_2v_3, v_3v_4, v_5v_6, v_6v_1\}, \{v_1v_2, v_2v_3, v_3v_4, v_4v_5, , v_6v_1\}, \{v_1v_2, v_2v_3, v_3v_4, v_4v_5, v_5v_6, \}\rangle$$

Definition 4.8.

A finite connected graph G having a single cycle is called uni-cycle graph. Consider $U_{n,m}$ is a uni-cycle graph ,where m represents the cycle length in graph and n is the total number of vertices in $U_{n,m}$. $(m \leq n)$

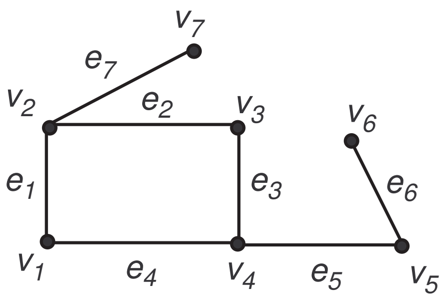

Figure 4.5: Uni-Cyclic Graph

For this section, edge labelling of *unicyclic* graphs is fix as $\{e_1, e_2, ..., e_m, e_{m+1}, ..., e_n\}$ so that the edge labelling of the only cycle in the $U_{n,m}$ is $\{e_1, e_2, ..., e_m\}$.

For example, graph shown in the figure 4.5

Lemma 4.1. Algebraic properties of $s(U_{n,m})$ *[1]*

Consider $E = \{e_1, e_2, ..., e_m\}$ is the edge set of uni-cycle graph denoted as $U_{n,m}$ and $E(T_i) \subseteq E$ is the set edges of the i-th spanning tree of $U_{n,m}$, such that $T_i = E \setminus \{e_i\}$ for some $i \in \{1, 2, 3, ..., m\}$. In other words

$$s(U_{n,m}) = \{\hat{E}_i \mid \hat{E}_i = E \setminus \{e_i\} \; for \; all \; 1 \leq i \leq m\}.$$

$(\hat{E}_i = E(T_i))$

Theorem 4.2. *[1]*

Consider $\Delta_s(U_{n,m})$ be a spanning complex related to a *uni*-cycle graph $U_{n,m}$, having Dimension $dim(\Delta_s(U_{n,m})) = n - 2$, then the face vector of $\Delta_s(U_{n,m})$ can be given by the following formula $f(\Delta_s(U_{n,m})) = (f_0, f_1, ..., f_{n-2})$, with

$$f_i = \begin{cases} \binom{n}{i+1}, & for \; i \leq m-2; \\ \binom{n}{i+1} - \binom{n-m}{i-m+1}, & for \; m-2 < i \leq n-2. \end{cases}$$

In [24], Anwar et al. explained the algebraic aspects of spanning simplicial complexes related to simple connected $r - cyclic$ finite graphs noted as $G_{n,r}$.

Consider a connected graph $G_{n,r}$, where n be the order of graph and r be the number of cycles respectively. The length of cycles can be different and there is no common edge.

The edge labeling for $G_{n,r}$ will be as $\{e_{11}, ..., e_{1m_1}, e_{12}, ..., e_{2m_2}, ..., e_{r1}, ..., e_{rm_r}, ..., e_1, ..., e_t\}$ where the ith cycle's edge set is $\{e_{i1}, ..., e_{im_i}\}$ for $1 \leq i \leq r$.

Lemma 4.3. *[24]*

Consider a graph $G_{n,r}$ having r cycles with no common edge between them and n is the number of edges with edge labelling as $E = \{e_{11}, ..., e_{1m_1},$

$e_{12}, ..., e_{2m_2}, ..., e_{r1}, ..., e_{rm_r}, ..., e_1, ..., e_t\}$ and $E(T_{1i_1,...,ri_r}) \subset E$ then $E(T_{1i_1,...,ri_r}) \in s(G_{n,r})$ iff $E(T_{1i_1,...,ri_r}) = E \setminus \{e_{1i_1}, ..., e_{ri_r}\}$ for few $i_j \in \{1, 2, ..., m_j\}$ $1 \leq j \leq r$.

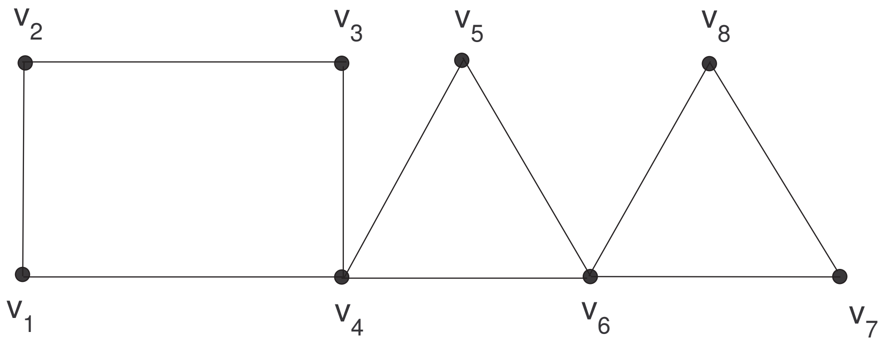

Figure 4.6: Multi-Cyclic Graph

Proposition 4.4. *[24]*

Let $G_{n,r}$ be an $r-cyclic$ connected graph having cycles of length $m_1 \leq m_2 \leq ... \leq m_r$, and $\Delta_s(G_{n,r})$ be a spanning simplicial complex of multi-cycle graph $G_{n,r}$, then its dimension is given by $dim(\Delta_s(G_{n,r})) = n - r - 1$ with face vector,

$$f_i = \binom{n}{t+1} + \sum_{k=1}^{r} (-1)^k \left[\sum_{(j-1,...,j_r)=1, j_1 \neq .. \neq j_r}^{r} \binom{n - \sum_{s=1}^{k} m_{j_s}}{i + 1 - \sum_{s=1}^{k} m_{j_s}} \right]$$

where $0 \leq i \leq n - r - 1$.

Chapter 5

Spanning Forest Complexes of Book Graphs

5.1 Introduction

In previous chapter 4, we discussed the spanning simplicial complexes of a simple connected graphs.Here we explain the extended idea of spanning simplicial complexes to Spanning Forest Complexes, and replacing spanning trees of simple finite connected graphs with spanning forests of finite disconnected graphs.

In [16], Gou et al. present the idea of spanning forest complexes associated to disconnected graphs. Firstly we explain the spanning forest complexes associated to disjoint union of cyclic graphs and discuss about some algebraic properties like the number spanning forests, dimension of spanning forest complex, its face vector and h-vector. We also describe some algebraic properties and results related to spanning forest complexes associated to book graphs and triangular book graphs with fixed number of pages and k-copies. Moreover, we give some characterization of f-vector taking k copies of triangular books of fixed size.

5.2 Spanning Forest Complexes

Definition 5.1. Let G be a finite disconnected graph, a subgraph F is said to be the spanning forest of G if it gives a spanning tree as a result of $F \cap K$, where K is one of the connected components of G. The edge set of spanning forests of G can be written as,

$$s(G) = \{E(F) \mid F \textit{ is a spanning forest of } G\}$$

Example 5.2.1. *Consider a graph shown in Fig 5.1 The set edges of all spanning forests of G is*

$s(G) = \{\{e_1, e_2, e_4, e_5\}, \{e_1, e_2, e_4, e_6\}, \{e_1, e_2, e_5, e_6\}, \{e_1, e_3, e_4, e_5\}, \{e_1, e_3, e_4, e_6\},$

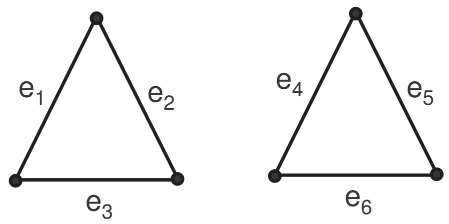

Figure 5.1: Disconnected Graph $G = 2C_3$

$\{e_1, e_3, e_5, e_6\}, \{e_2, e_3, e_4, e_5\}, \{e_2, e_3, e_4, e_6\}, \{e_2, e_3, e_5, e_6\}\}$.

Definition 5.2. The spanning forest complex of disconnected graph G is noted as $\Delta_s(G)$, is generated by the set of edges of the spanning forests of G i.e.

$$\Delta_s(G) = \langle F_i \mid F_i \in s(G)\rangle$$

Example 5.2.2. *The spanning forest complex of above example is*
$\Delta_s(G) = \langle\{e_1, e_2, e_4, e_5\}, \{e_1, e_2, e_4, e_6\}, \{e_1, e_2, e_5, e_6\}, \{e_1, e_3, e_4, e_5\}, \{e_1, e_3, e_4, e_6\},$
$\{e_1, e_3, e_5, e_6\}, \{e_2, e_3, e_4, e_5\}, \{e_2, e_3, e_4, e_6\}, \{e_2, e_3, e_5, e_6\}\rangle$

5.2.1 Spanning Forests Complexes of kC_n

Definition 5.3.

Let kC_n be the graph having k disjoint copies of cycle graph C_n each of length n, defined as $kC_n = \sqcup_{i=1}^{k} C_n$. Total edges in the graph kC_n are kn.

For example, in figure 5.3 there are two disconnected cyclic graphs $2C_3$ and $3C_5$. In the graph $2C_3$ there are two copies of C_3 and $3C_5$ have three copies of C_5.

Theorem 5.1.

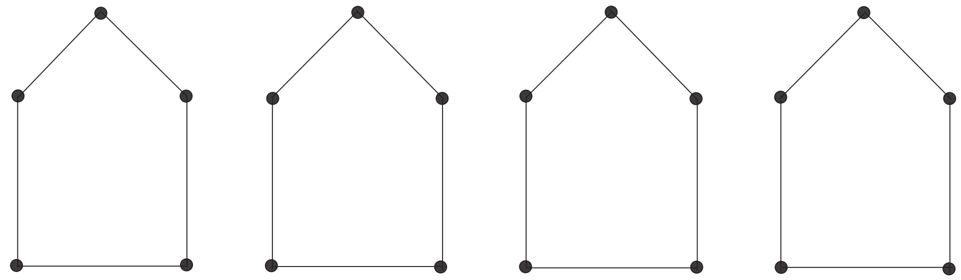

Figure 5.2: Disconnected cycles C_5

Let kC_n be the graph having k disjoint cycles of length n, then the number of spanning forests of kC_n are exactly n^k.

Proof. As we know that $kC_n = \sqcup_{i=1}^{k} C_n$. Each cycle of C_n has n number of spanning trees. But here we have k disjoint copies, so total spanning forests of kC_n are

$$n.n. \ \ldots \ (k \ times).n = n^k$$

Example 5.2.3. In the figure 5.2 the graph has four copies of C_5, using the above result the number of spanning forests of $4C_5$ are $5^4 = 625$.

Lemma 5.2. (combinatorics of $s(kC_n)$)

Let kC_n be the k-cycle graph and $E = \{e_{11}, e_{12}, ..., e_{1n}, e_{21}, e_{22}, ..., e_{2n}, e_{k1}, e_{k2}, ..., e_{kn}\}$ of kC_n in such a way that $\{e_{i1}, ..., e_{in}\}$ is the set of edge i-th copy of kC_n. A subset $E(F_{(1i_1, 2i_2, ..., ki_k)})$ of E will belongs $s(kC_n)$ if and only if $E(F_{(1i_1, 2i_2, ..., ki_k)}) = E \setminus \{e_{1i_1}, ..., e_{ki_k}\}$ for some $i \in \{1, 2, ..., n\}$. Specially,

$$s(kC_n) = \{\hat{E}_{(1i_1, 2i_2, ..., ki_k)} = E \setminus \{e_{1i_1}, ..., e_{ki_k}\} for \ some \ i \in \{1, 2, ..., n\}\}$$

Example 5.2.4. Consider the graph $2C_3$ in fig5.3, and by using above result the edge set of the spanning forests, is given as

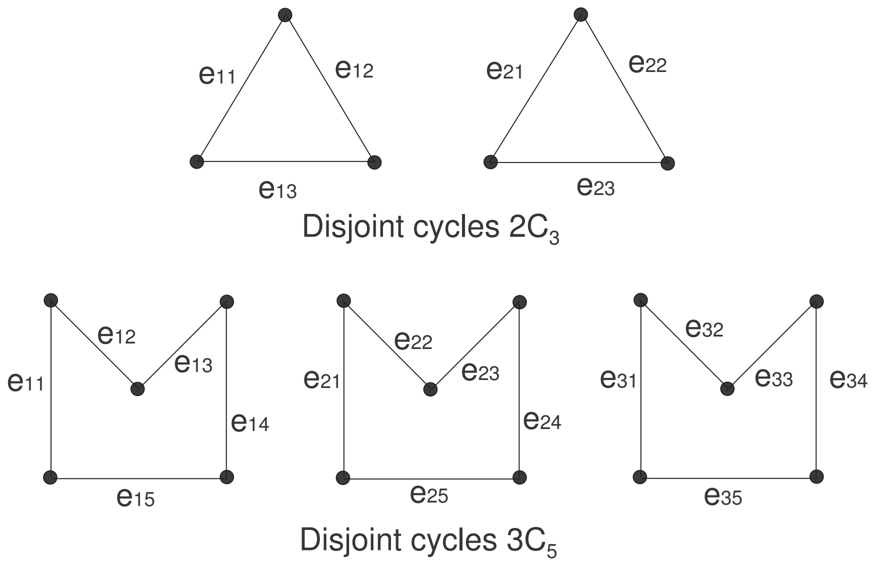

Figure 5.3: Edge labeling of disconnected Uni-Cyclic graphs

$s(2C_3) = \{\{e_{12}, e_{22}, e_{13}, e_{23}\}, \{e_{12}, e_{21}, e_{13}, e_{23}\}, \{e_{12}, e_{21}, e_{13}, e_{22}\}, \{e_{11}, e_{22}, e_{13}, e_{23}\},$
$\{e_{11}, e_{21}, e_{13}, e_{23}\}, \{e_{11}, e_{21}, e_{13}, e_{22}\}, \{e_{22}, e_{11}, e_{23}, e_{12}\}, \{e_{11}, e_{21}, e_{12}, e_{23}\},$
$\{e_{11}, e_{21}, e_{12}, e_{22}\}\}$

Theorem 5.3.

Let $\Delta_s(tC_n)$ be a spanning forest complex of tC_n, then

$$dim(\Delta_s(tC_n)) = t(n-1) - 1$$

Example 5.2.5. Consider a graph $2C_3$ shown in figure 5.3, the dimension of $\Delta_s(2C_3)$ is,

$$dim(\Delta_s(2C_3)) = 2(3-1) - 1 = 3$$

Theorem 5.4. Consier $\Delta_s(tC_n)$ is the forest complex of t copies of cycle C_n, so its $dim(\Delta_s(tC_n)) = t(n-1)-1$ and its f-vector given by $f(\Delta_s(tC_n)) = (f_0, f_1, \ldots, f_{t(n-1)-1})$, where,

$$f_i = \binom{tn}{i+1} \ + \ \sum_{k=1}^{t-1}(-1)^k\binom{t}{k}\binom{(t-k)n}{i+1-kn}$$

where $0 \leq i \leq t(n-1)-1$

5.2.2 Spanning Forests Complexes of $kC_{n,t}$

Definition 5.4.

Let $kC_{n,t}$ be a disconnected graph having t disjoint copies of a cyclic graph $C_{n,t}$ with t number of cycles each of length n ,where $n \geq 3$ and $t \geq 2$ We can defined it as $kC_{n,t} = \sqcup_{i=1}^{k} C_{n,t}^{i}$

Theorem 5.5.

Let $kC_{n,t}$ be a disconnected graph having k disjoint copies of a cyclic graph $C_{n,t}$ with t number of cycles each of length n , then the number of spanning forests are exactly n^p.

Proof. As we know that each C_n has n number or spanning trees and $C_{n,t}$ has t copies of cycles C_n, so the number of spanning forests of each copy of $C_{n,t}$ are

$$n.n. \ \ldots \ (t \ times).n = n^t.$$

Now by the definition of $kC_{n,t}$ we have

$$kC_{n,t} = \sqcup_{i=1}^{k} C_{n,t}^{i}$$

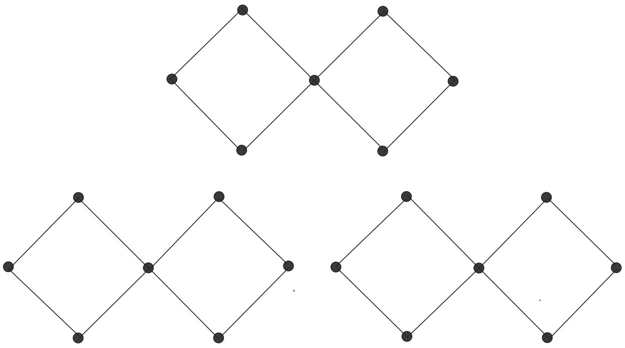

Figure 5.4: Cycle Graphs connected by a vertex

and since each copy of $C_{n,t}$ has total n^t spanning trees , so the total number of spanning forests of $kC_{n,t}$ are

$$n^t.n^t. \ \ldots \ (k \ times).n = n^{tk}$$

since $nt = p$ so

$$n^t.n^t. \ \ldots \ (k \ times).n^t = n^p$$

Example 5.2.6. *The graph shown in figure 5.4 have three copies of $C_{4,2}$ connected with a vertex and total number of cycles in $3C_{4,2}$ are six , so the number of forests in $3C_{4,2}$ will be $4^{3\times 2} = 4^6 = 4096$.*

5.2.3 Spanning Forests Complexes of Book Graphs

Definition 5.5. ***Book Graph*** $B_{t,n}$

We can define an n-Book (book graph with n pages) as the cartesian product of graphs $S_{n+1} \times P_2$, here S_{n+1} be the star graph on $n+1$ vertices and P_2 be the path graph. One can define a book graph as a graph consisting of n cycles sharing a common edge. here t be the length of each cycle and the n be the count of pages of the book.

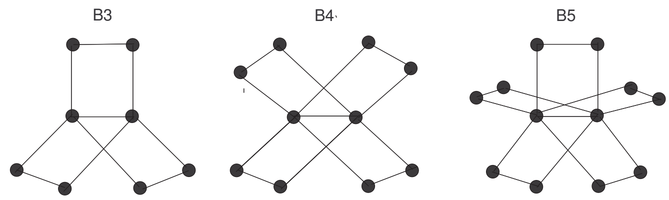

Figure 5.5: Book Graphs

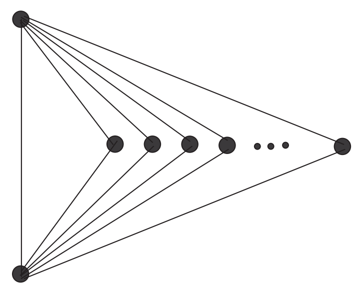

Figure 5.6: Triangular Book on n-pages

Example 5.2.7.

The graphs shown in figure 5.5 are rectangular books having three pages, four pages and five pages.

5.2.4 Spanning Forests Complexes of Triangular Book Graphs

Definition 5.6. ***Triangular Book Graph*** $B_{3,n}$

A triangular book graph is a specific type of book graph that is complete tripartite graph $K_{1,1,n}$. In other words, a triangular book with n-pages is formed by n copies of C_3 a cycle graph having a common edge.

Example 5.2.8.

A triangular book with n-pages is shown in figure5.6.

Here we have some important notations related to our graphs.

- t=length of each cycle
- n=number of cycles in one copy of book graph
- k=number of copies of graph
- $B_{t,n}$=Book graph
- $B_{3,n}$=Triangular Book Graph with n pages
- $kB_{3,n}$= k-copies of Triangular Book Graph
- $|E| = |E(B_{3,n})|$= Total number of edges in one copy of triangular book
- $p = k \times |E|$= Total number of edges in k books

Proposition 5.6. *k Disjoint books, each with n pages is an $n-$cyclic disconnected graph having the order $k\{nt - 2(n-1)\}$ and the size $k\{nt - (n-1)\}$, we get by connecting n cycle graphs by a common edge.*

Example 5.2.9.

A triangular book graph with three pages shown in figure 5.7

Here we have $t = 3$, $n = 3$, $k = 1$, the number of vertices are

$$|V| = k\{nt - 2(n-1)\} = 1 \times \{3 \times 3 - 2(3-1)\} = 5$$

and number of edges are

$$|E| = k\{nt - (n-1)\} = 1 \times \{3 \times 3 - (3-1)\} = 7$$

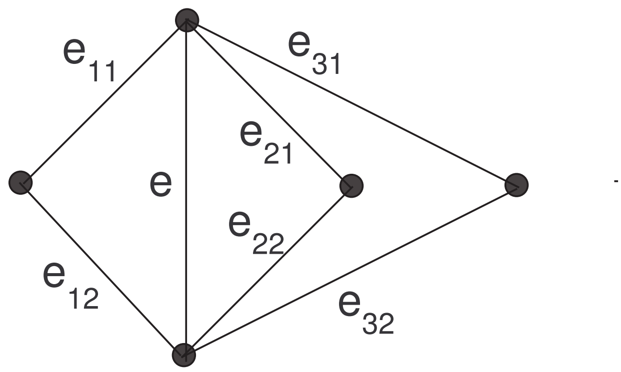

Figure 5.7: Triangular Book with three pages

Example 5.2.10.

A triangular book graph with three pages shown in figure 5.9

Here we have $t = 3$*,* $n = 3$*,* $k = 4$*, the number of vertices are*

$$|V| = k\{nt - 2(n-1)\} = 4 \times \{3 \times 3 - 2(3-1)\} = 20$$

and number of edges are

$$|E| = k\{nt - (n-1)\} = 4 \times \{3 \times 3 - (3-1)\} = 28$$

Definition 5.7. A spanning forests complex Δ will be a pure forest complex, if Δ posses all the facets of same dimension.

Proposition 5.7. *[24]*

A spanning forests complex $\Delta_s(kB_{t,n})$ of k-copies of book graphs with n-pages in each copy, is a pure spanning forests complex with dimension

$$dim(\Delta_s(kB_{t,n})) = k\{tn - (2n-1)\} - 1$$

Proof. Consider one copy of book graph having n pages with t-cycle length then the total edges will be tn, obtained connecting n cycles by a common edge, n-edges will be contracted to 1-edge i,e $n-1$ edges are subtracted from t-cycles. Transforming book graph into spanning forests, we need to remove one edge from each cycle, so n more edges will be removed from n-cycles, summarizing this information we conclude that each facet of spanning complex of one copy of $B_{t,n}$ will have $tn-(n-1)-n = tn-(2n-1)$ edges. As we are considering k-copies of $B_{t,n}$ so each facet of spanning forests complex will have $k\{tn-(2n-1)\}$ edges. Thus the result follows

$$dim(\Delta_s(B_{t,n})) = \mid F \mid -1$$

implies that,

$$dim(\Delta_s(B_{t,n})) = k\{tn-(2n-1)\}-1$$

Moreover each facet have the same dimension this implies that the spanning forests complex of book graph $B_{t,n}$ is a pure complex.

Example 5.2.11.

Consider a spanning forest complex $\Delta_s(2B_{3,3})$ shown in figure 5.8 the dimension of $\Delta_s(kB_{3,3})$ is calculated as,

$$dim(\Delta_s(kB_{3,3})) = k\{tn-(2n-1)\}-1$$

$$dim(\Delta_s(2B_{3,3})) = 2\{3\times 3-(2\times 3-1)\}-1$$

$$dim(\Delta_s(2B_{3,3})) = 7$$

Example 5.2.12.

Consider a spanning forest complex $\Delta_s(4B_{3,3})$ shown in figure 5.9 the dimension of $\Delta_s(4B_{3,3})$ is calculated as,

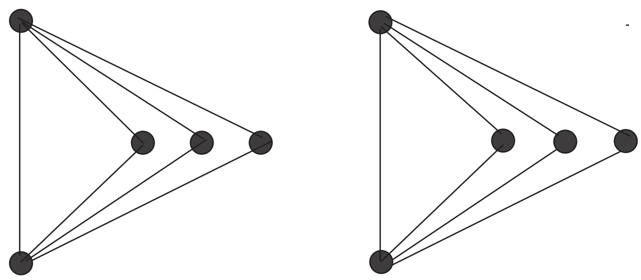

Figure 5.8: Two copies of $B_{3,3}$

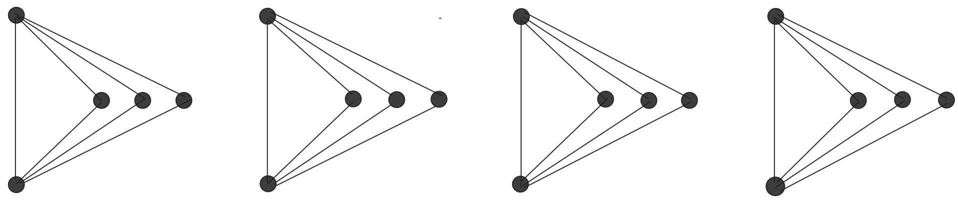

Figure 5.9: Multiple copies of $B_{3,3}$

$$dim(\Delta_s(4B_{3,3})) = 4\{3 \times 3 - (2 \times 3 - 1)\} - 1$$

$$dim(\Delta_s(4B_{3,3})) = 15$$

5.2.5 Computation of f-vector of $\Delta_s(kB_{3,3})$

In previous chapter we computed f-vector for the simplicial complexes and spanning simplicial complexes related to some simple finite and connected graphs. Now we give the formulas to compute the number of $i-th$ dimensional faces of spanning forests complexes related to some triangular book graphs with fixed size.

Theorem 5.8.

Let $\Delta_s(kB_{3,3})$ is the spanning forests complex on k copies of triangular book graphs $B_{3,3}$ then its component of face vector f_0 is given by $f_0(\Delta_s(kB_{3,3})) = k \times |E| = p$.

Proof. Since one copy of triangular book graph with three pages $B_{3,3}$ has $|E|$ edges and there we have k-copies. As f_0 is the count of zero-dimensional faces (edges) of spanning forests complex $\Delta_s(kB_{3,3})$ and there are total $k \times |E|$ edges in k-copies, implies that $f_0(\Delta_s(kB_{3,3}) = k \times |E| = p$.

Example 5.2.13.

Consider a spanning forest complex $\Delta_s(4B_{3,3})$ shown in figure 5.9 then its f_0 can be calculated as,

$$f_0 = k \times |E| = 4 \times 7 = 28.$$

Theorem 5.9.

Let $\Delta_s(kB_{3,3})$ be the spanning forests of k copies of $B_{3,3}$ then number of its one dimensional faces will be $f_1(\Delta_s(kB_{3,3})) = \sum_{i=1}^{p-1} i$ where $p = k \times |E|$.

Proof. As f_1 is the number of all the subsets of $\Delta_s(kB_{3,3})$ with two elements. Since $kB_{3,3}$ have total p edges. The first element have $p-1$ combinatorial ways to form pairs of edges, the second element have $p-2$ possibilities, the third one have$p-3$ possibilities and so on. Proceeding recursively, we obtain an arithmetic series as, $(p-1)+(p-2)+(p-3)+...+3+2+1$. Writing in compact form we have, $f_1 = \sum_{i=1}^{p-1} i$. As we know that n-arithmetic series of sum can be written as $n(n+1)/2$. Using this result we have,

$f_1(\Delta_s(kB_{3,3})) = \sum_{i=1}^{p-1} i = p(p-1)/2,$, where $p = k \times |E|$.

Example 5.2.14.

Consider a spanning forest complex $\Delta_s(4B_{3,3})$ shown in figure 5.9 then using the above theorem its f_1 can be computed as,

$$f_1(\Delta_s(kB_{3,3}) = \sum_{i=1}^{p-1} i = p(p-1)/2$$

$$f_1 = \sum_{i=1}^{(4\times|E|-1)} i$$

$$f_1 = (4 \times |E|)(4 \times |E| - 1)/2$$

$$f_1 = (4 \times 7)(4 \times 7 - 1)/2$$

$$f_1 = (28)(27)/2 = 378$$

Theorem 5.10.

Let $\Delta_s(kB_{3,3})$ be the spanning forests complex of k copies of $B_{3,3}$ then its f_2 is given by the following formula $f_2(\Delta_s(kB_{3,3})) = k\{(k-1)|E| \times 21 + 32\}$

Proof. Since f_2 be the number of all possible cycle free subsets from the edge set of triangular book graph with cardinality three. For one copy of triangular book graph i,e, $k = 1$ we get that there are 32 such subsets. Taking two copies of triangular book (for $k = 2$) computed that there are $2(|E| \times 21 + 32) = 358$ cycle free subsets with three elements. For $k = 3$ we will have $3(2|E| \times 21 + 32) = 978$ such cycle free subsets with cardinality three in total. Hence, using the combinatorial formula for $k = 1, 2, 3, ..., n$, we have

$$f_2(\Delta_s(kB_{3,3})) = k\{(k-1)|E| \times 21 + 32\}$$

Example 5.2.15.

Consider a spanning forest complex $\Delta_s(4B_{3,3})$ shown in figure 5.9 then using the above result we can compute f_2 as,

$$f_2 = k\{(k-1)|E| \times 21 + 32\}$$

$$f_2 = 4 \times \{(4-1) \times 7 \times 21 + 32\}$$

$$f_2 = 1892$$

Theorem 5.11.

Let $\Delta_s(kB_{3,3})$ be the spanning forests complex of $kB_{3,3}$ then its f_3 is given by the following formula,

$f_3(\Delta_s(kB_{3,3})) = k\{(k-1)|E| \times 32 + 20\} + C_2^k \times (\sum_{i=1}^{|E|-1} i)^2 + \{k(k-2) \times \sum_{i=1}^{|E|-1} i \times |E|^2\} + \{(k-3) \times |E|^4\}$

Proof. Since we are computing components of f-vector taking k copies of triangular book graph with three pages.For f_3 we will have to calculate the number of all the cycle free subsets of $E(kB_{3,3})$ with four elements. For this we have to exclude the subsets containing C_3 and C_4. If we consider one copy of $B_{3,3}$, we calculated there are 20 such cycle free subsets with cardinality four. Taking two copies of $B_{3,3}$ and taking three elements from one copy and one element from other copy we get that there are $2 \times (|E| \times 32 + 20)$ such cycle free subsets, while if we take two elements from each copy of $B_{3,3}$ we get that there $(\sum_{i=1}^{|E|-1} i)^2$ cycle free subsets with four elements. Hence, for $k = 2$ we have $f_3 = 2 \times (|E| \times 32 + 20) + (\sum_{i=1}^{|E|-1} i)^2$. Taking three copies of $B_{3,3}$ and taking three elements from one copy and one element from other two copies we get that there are $3 \times (2|E| \times 32 + 20)$ such subsets. Taking two elements from one copy and two elements from any of other two copies, we get that there are $3(\sum_{i=1}^{|E|-1} i)^2$ such subsets in total. If we take two elements from one copy and one element from each remaining copy we will have $3\sum_{i=1}^{|E|-1} i \times |E|^2$. Hence, for $k = 3$ we have $f_3 = 3 \times (2|E| \times 32 + 20) + 3(\sum_{i=1}^{|E|-1} i)^2 + 3\sum_{i=1}^{|E|-1} i \times |E|^2$. Considering four copies of $B_{3,3}$ and taking three elements from one copy and one element from other three copies we get that there are $4 \times (3|E| \times 32 + 20)$ total cycle free subsets with cardinality four, taking two elements from one copy and two elements from any of other three copies, we get that there total are $6(\sum_{i=1}^{|E|-1} i)^2$ such subsets. If we take two elements from one copy and one element from each remaining copy we will have $(4 \times 2) \sum_{i=1}^{|E|-1} i \times |E|^2$ total cycle free subsets. Taking one element from each of the copy we get that there are total $|E|^4$ such subsets. Hence, for $k = 4$ get that $f_3 = 4 \times (3|E| \times 32 + 20) + 6(\sum_{i=1}^{|E|-1} i)^2 + (4 \times 2) \sum_{i=1}^{|E|-1} i \times |E|^2 + |E|^4$ Continuing in this algebraic and combinatorial manner we conclude that,

$f_3(\Delta_s(kB_{3,3})) = k\{(k-1)|E| \times 32 + 20\} + C_2^k \times (\sum_{i=1}^{|E|-1} i)^2 + \{k(k-2) \times \sum_{i=1}^{|E|-1} i \times |E|^2\} + \{(k-3) \times |E|^4\}$

Example 5.2.16.

Consider a spanning forest complex $\Delta_s(4B_{3,3})$ in figure 5.9 using the above theorem we can compute its f_3 as,

$$f_3(\Delta_s(kB_{3,3}) = k\{(k-1)|E| \times 32 + 20\} + C_2^k \times (\sum_{i=1}^{|E|-1} i)^2 + \{k(k-2) \times \sum_{i=1}^{|E|-1} i \times |E|^2\} + \{(k-3) \times |E|^4\}$$

$$f_3 = 4\{(4-1)7 \times 32 + 20\} + C_2^4 \times (\sum_{i=1}^{7-1} i)^2 + \{4(4-2) \times \sum_{i=1}^{7-1} i \times 7^2\} + \{(4-3) \times 7^4\}$$

$$f_3 = \{4 \times 692\} + \{6 \times 441\} + \{8 \times 21 \times 49\} + \{(4-3) \times 2401\}$$

$$f_3 = 16047$$

Chapter 6

References

[1] Anwar, I., Raza, Z., Kashif, A.(2015) Spanning Simplicial Complexes of Uni -Cyclic graphs, *Algebra Colloquium,* 22, 707-710.

[2] Nazir. R, (2018) Spanning Forest Complex of union of multicycle graphs, *(MS Thesis).*

[3] Asif. M (2016), On The Characterization Of Spanning Forest Complexes, *(MS Thesis).*

[4] Zhu, G., Shi, F., Geng, Y.(2014) Spanning Simplicial Complexes of n-Cyclic graphs with a common edge *International Electronic Journal of Algebra*,Volume 15, 132-144.

[5] Atiyah, M. F., Macdonald, I. G.(1969) Introduction to Commutative Algebra, *Addison- Wesley publishing company.*

[6] Ballesteros, J. J. M., Campo E. R., 05 (2013) On the Hetrochromatic Number of Hypergraphs Associated to Geometric Graphs and to Matroids, Graphs and Combinatrics, 29, 1517-1522.

[7] Caboara, M., Faridi, S., Selinger, P.,(2007) Simplicial cycles and the computation of simplicial trees. *J. Symbolic Computation* 42, 74-88.

[8] Campo, E. R.,(1998) Note on the Existence of Plane Spanning Trees of Geometric Graphs, Discrete and computational geometry *Lecture Notes in Comput. Sci. 1763* (2000), 274-277.

[9] David, M. and Burton,(1970) A first course in rings and ideals, *Addison-Wesley, reading,Massachusctts.*

[10] Dummit, D. S. and Foote, R. M.,(1999) Abstract Algebra, *John Wiley and Sons,Inc.*

[11] Eisenbud, D.,(2004) Commutative Algebra with a View Toward Algebraic Geometry. *Springer Science,Business Media, Inc.*

[12] Faridi, S.,(2002) The facet ideal of a simplicial complex, *Manuscripta Mathematica,* 109, 159-174.

[13] Faridi, S.,(2004) Simplicial Tree are sequentially Cohen-Macaulay, *Journal of Pure and Applied Algebra* 190, 121-136.

[14] Froberg,(1990) On Stanley-Reisner ring, Topics in Algebra(S. Balcerzyk et. al., Eds), part 2, *Polish Scientific Publishers*, 57- 70.

[15] Grigorescu, E.,(2003) Hilbert Series and Free Resolutions, *A Senior Project submitted to The Division of Natural Sciences and Mathematics of Bard College.*

[16] J. Gou, Tongsuo Wu,(2015) On The Spanning Complex of a Finite Graph, available online *(www.researchgate.net/publication/26207008.)*

[17] Harary, F.,(1994) Graph Theory. Reading, MA: *Addison-Wesley.*

[18] Herzog J. and Hibi. T.,December(2007) Monomials. *Essen, Osaka.*

[19] Herzog, J., Hibi, T.,(2009) Monomial Algebra, *Springer-Verlag New York Inc.*

[20] Hochster, M.(1977) C-M rings, Combinatorics and Simplicial Complexes, *Lecture notes in pure and applied math. 26, Macel and Dekker, New York,* 171-223.

[21] Hungerford, T. W.,(1987) Algebra, Springer-Verlag, *New York-Heidelberg, Berlin,* .

[22] Jonsson, J.,(2005) Simplicial Complexes of Graphs, *Doctoral Thesis, Department of Mathematics, KTH,*

[23] Kaplansky, I.,(1974) Commutative Rings, *University of chicago press.*

[24] Kashif, A., I. Anwar, Z. Raza,(2014) On the Algebraic Study of Spanning Simplicial Complexes of r-cyclic Graph, *ARS Combinatoria* 115, 89-99.

[25] Matsumura, H.,(1980) Commutative Algebra, 2nd edition, *The Banjamin Publishing Company London.*

[26] Matsumura, H.,1986 Commutative Ring Theory, *Cambridge.*

[27] Miller, E., Sturmfels, B.,(2005) Combinatorial Commutative Algebra, *Springer-Verlag New York Inc.*

[28] Peeva, I.,(2011) Graded Syzygies, Algebra and Applications , *Springer-Verlag London Limited,* 14.

[29] Seward, B.,(2011) Burnside's Problem, Spanning Tress and Tilings. *arXIV: 1104.1231v1.* (available online on www.arxiv.org)

[30] Simis, A., Vasconcelos, W., Villarreal, R.,(1994) On Ideal Theory of Graphs , *J. Algebra* 167, no 2, 389-416.

[31] Stanley, R. P.,1996 Combinatorics and commutative algebra, Second edition. *Progress in Mathematics, 41. Birkhuser Boston, MA, x+164 pp*. ISBN: 0-8176-3836-9.

[32] Stanley, R. P.,(1975) Cohen-Macaulay Rings and constructible polytopes, *Bull. Amer. Math*. Soc.81,133-142

[33] Villareal, R. H.,2001 Monomial Algebras, *Dekker, New York*.

[34] Zariski, O., Samueal, P.,1960 Commutative Algebra, *Vol. II, Van Nostrand, Princeton*.