Thermodynamics of FRW Universe in Various Theories of Gravity Inspired by Entropy Corrections

By

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It is certified that I Salman Rafique & CIIT/SP16-RMT-004/LHR has carried out all the work related to this thesis under my supervision at the Department of Mathematics, COMSATS Institute of Information Technology, Lahore campus and the work fulfills the requirement for award of MS degree.

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DEDICATED TO

My Parents

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Abstract

In this thesis, we discuss the thermodynamical analysis for gravitationally induced particle creation scenario in the framework of DGP braneworld and Chern-Simons gravity. For this purpose, we consider apparent horizon as the boundary of the universe. We take three types of entropy such as Bakenstein, logarithmic corrected and power law corrected entropies with ordinary creation rate Γ. We analyze the first law and generalized second law of thermodynamics for these entropies which hold under some constraints. The behavior of total entropy in each case is also discussed which implies the validity of generalized second law of thermodynamics. Also, we check the thermodynamical equilibrium condition for two phases of creation rate Γ, that is constant and variable and found its vitality in all cases of entropy.

Chapter 1

Introduction

Cosmology is the branch of astronomy which deals with the origin and advancement in the universe, including procedure to get a more profound comprehension of the laws of physics expected to hold all through the universe. The word cosmology derived from two Greek words cosmos means world and logia means study off. In cosmology, we study about the universe, its nature, center structure, physical aspects and the reaction. Newton proposed a physical mechanism for Kepler's laws of planetary motion. His law of universal gravitation resolved the anomalies caused by gravitational interaction between the planets.

The idea of static universe is presented by Einstein by adding cosmological constant in the field equation of general relativity. After Einstein, famous scientist de Sitter, Schwarzschild and Eddington promoted cosmology. In 1922, the theory of relativity is rejected by Friedmann and he introduced the idea of expanding universe. After that, researchers started debates on expanding universe. The difference in these ideas came to an end when Hubble detected novae in the Andromeda galaxy and then showed its distance to be way beyond the Milky Way's boundaries. The high-Z Supernova Search Team and the Supernova Cosmology Project are two independent teams were sent into the space to collect data about astronomical objects. The result was unexpected, because the obtained results showed accelerated expansion of the universe.

Recently, different observations such as Supernova Ia [1, 2], cosmic microwave background [3] and Sloan Digital Sky Survey [4] show the accelerating phase of the universe. Through these observations, the researchers suggested that this cosmic acceleration is due to unknown energy dubbed dark energy (DE) [5]-[8]. After this, a lot of research is being done to find the source and properties of DE but unfortunately, scientists failed to introduces the valid argument till now. The only known property of DE is the repulsive force. The simplest candidate of DE is the cosmological constant but it is plagued with two basic problems such as fine-tuning and cosmic coincidence. Alternative approaches such as modified theories of gravity and dynamical DE models have been adopted to elaborate the accelerated expansion scenario. The dynamical DE models include quintom [9]-[11], quintessence [12]-[14], family of Chaplygin gass [15]-[17], K-essence [18]-[20], new agegraphic DE [21, 22], pilgrim DE (PDE) [23]-[25], holographic DE (HDE) and its modified models [26, 27], etc. The proposed modified theories of gravity include Gauss-Bonnet gravity with $\mathcal{G}$ invariant [28], $f(R)$ takes Ricci scalar R [29], $f(T)$ inherits torsion scalar T [30], $f(R,\mathcal{T})$ with $\mathcal{T}$ as the trace of stress-energy tensor [31], $f(R,\mathcal{G})$ carries both R and $\mathcal{G}$ [32], Brans-Dicke [33], Kaluza-Klein theory [34], dynamical Chern-Simons modified gravity [35, 36], braneworld model etc.

Investigations on the profound association between gravity and thermodynamics have been reached out to braneworld gravity. Dvali et al. [37] purposed that the 3-brane is embedded in a space-time with infinite size extra dimensions, with the hope that this picture could shed new light on the standing problem of the cosmological constant as well as on supersymmetry breaking [37, 38]. The DGP display expects the presence of a $4+1-$dimensional Minkowski space, inside which customary $3+1-$dimensional Minkowski space is inserted. The model containing the two terms Einstein-Hilbert action, which involves only the $4-D$ space-time dimensions and other is the equivalent of the Einstein-Hilbert action, as extended to all five dimensions.

Chern-Simons modified gravity is an extension of general relativity that captures leading order, gravitational parity violation due to the parity violating correction term given by the Pontryagin density [39, 47]. Pontryagin density is a topological term in $4D$ where the coupling constant promoted to a scaler field which did not behave like a constant. We obtain the Chern-Simons action by adding Einstein-Hilbert term with product of dynamical scalar and Pontryagin density plus the action of scaler field. We add a Chern-Simons term which is coupled to gravity using scaler field and adding usual Einstein-Hilbert action, we get modified form. Non-dynamical and dynamical formulation are two theories of Chern-Simons which exist independently. Also this modification displays violation of parity symmetry in the Einstein-Hilbert action and consists of pontryagin density in addition.

The concept of thermodynamics in cosmological system originates through black hole (BH) physics. It was suggested [48] that the temperature of Hawking radiations emitting from black holes is proportional to their corresponding surface gravity on the event horizon. Jacobson [49] found a relation between thermodynamics and the Einstein field equations. He derived this relation on the basis of entropy-horizon area proportionality relation along with first law of thermodynamics (also called Clausius relation) $dQ = TdS$, where dQ, T and dS indicate the exchange in energy, temperature and entropy change for a given system respectively. It was shown that the field equations for any spherically symmetric spacetime can be expressed as $TdS = dE + PdV$ (E, P and V represent the internal energy, pressure and volume of the spherical system) for any horizon [50].

The generalized second law of thermodynamics (GSLT) has been studied extensively in the scenario of expanding behavior of the universe. The

GSLT states that *the entropy of matter inside the horizon plus entropy of the horizon remains positive and increases with the passage of time* [51]. In order to discuss GSLT, horizon entropy of the universe can be taken as one quarter of its horizon area [52] or power law corrected [53] or logarithmic corrected [54] forms. Many people have explored the validity of GSLT of different systems including interaction of two fluid components like DE and dark matter [55], as well as interaction of three components of fluid [56] in the FRW universe by using simple horizon entropy of the universe.

The gravitationally induced particle creation is another well-known mechanism which was firstly introduced by Schrodinger [57] on microscopic level. This mechanism was further extended by Parker et al. towards quantum field theory in curved spacetimes [58]-[62]. The macroscopic description of particle creation mechanism induced by gravitational field was presented by Prigogine et al. [63]. Later on, the covariant description of this mechanism was developed [64, 65] as well as the physical difference between particle creation and bulk viscosity was also given [66]. The particle creation process can be described with the inclusion of backreaction term in the Einstein field equations whose negative pressure may help in explaining the cosmic acceleration. In this way, various phenomenological models of particle creation have been presented [67]-[72]. It is also shown that phenomenological particle production [73]-[76] help in explaining the cosmic acceleration and paved the alternative way to the concordance ΛCDM model.

The purpose of this thesis is to investigate the validity of GSLT in the framework of different laws corresponding Bekenstein, logarithmic and power law corrected entropies. Further, we investigate the stability of thermodynamical equilibrium for constant as well as variable Γ. The thesis is organized as follows:

- In chapter **2**, some basic material is provided which is helpful to understand our work.
- In chapter **3** and **4**, we consider Bekenstein entropy, logarithmic corrected entropy and power law corrected entropy with particle creation rate Γ in the frame work of DGP braneworld and dynamical Chern-Simons modified gravity respectively. We also check the validity of GSLT and stability of thermodynamical equilibrium.
- In the last section, we summarize our results.

Chapter 2

Preliminaries

This chapter has the basic concepts for understanding the thesis work. The basic definitions are very useful for the calculations.

2.1 Cosmology

Cosmology is the study of the dynamical structure of the universe considered as a whole. Contemporary cosmological models depend on the possibility that the universe is, all things considered, the same overall. This principle is called the cosmological principle, which is a generalization of the Copernican principle, the cosmological principle states that "at each epoch, the universe presents the same aspect from every point, except for local irregularities" [77]. The universe and matter in the universe should be isotropic and homogeneous when averaged over sufficiently large volumes. These characteristics are defined as

- Isotropy states that space looks the same, no matter what direction one looks at (direction independent).

- Homogeneity is the statement that the location is the same throughout the space (position independent)

Astronomical observations reveal that the universe is homogeneous and isotropic on the largest scales. Traditionally this homogeneity has been assumed up to small fluctuations that are large enough to include clusters of galaxies. The big-bang theory reveals the expansion of the universe from a denser point with temperature at extreme limit around 13.7 billion years ago. After this, the inflationary paradigm came into being which is an extremely rapid exponential expansion. This paradigm was supposed to express the accelerating expansion of the universe through negative pressure. Observations indicate that after inflation, the universe passed through three dominant phases during its evolution. These are given as radiation, matter and DE dominated phases.

2.2 Cosmic Acceleration

In 1998 two independent team high-Z supernova search project and the Supernova cosmology were sent into the space. Their mission was to collect data about astronomical objects. This discovery was unexpected, because obtained results were showing accelerating expansion of universe. This was the biggest achievement of that time, therefor three members of these groups awarded by noble prize due to this achievement. After that some observational data like cosmic microwave background, large scale structure, Sloan Digital Sky Survey, the integrated Sachs Wolfe effect, gravitational lensing and baryonic acoustic oscillations, also showed the expansion of universe is accelerating. These observational data suggest that the main cause of this expansion is an unknown force, called DE. Unfortunately properties of DE are unknown. As scientists have made different observations and conclude that universe is accelerating. different theories have been presented about this cosmic acceleration.

2.3 Dark Energy and Dark Matter

Dark energy is a mysterious force which acts to push rather than pull things, which caused the accelerated expansion of universe. This energy is the biggest puzzle of current time. Astronomer are trying to find the nature and properties of DE but we just known that DE have negative pressure (repulsive force). According to latest observation two-third of our universe consist of DE and remaining part include baryonic and non-barynoic dark matter (DM).

Cosmologists have made a lot of efforts in making consensus about the types of DM constituents in the universe and their quantities. Dark matter exists in baryonic and non-baryonic forms. Astronomers can easily detect some types of DM through the emission of photons in all directions, because stars emit light in infrared, visible and ultraviolet range of electromagnetic spectrum. This type of matter lies in the category of baryonic DM but it contributes 4 percent (predicted contribution of baryonic DM in the universe) in the overall density of the universe. However, the second major constituent of the universe is non-baryonic DM and its contribution is approximately 23%. This matter cannot be detected directly because it does not absorb, emit or scatter light of any wavelength.

2.4 Scale Factor

By looking at distant galaxies, they seem to be receding from our galaxy. It appears that the universe is not static, but changing with time. The spatial size of universe at given time is called the scale factor and it is denoted by $a(t)$. According to latest cosmological observation, it can be defined as "universe is expanding" so, $a(t) > 0$.

2.5 Hubble Parameter

The "rate of expansion" is characterized by the Hubble parameter

$$H = \frac{\dot{a}}{a}, \tag{2.5.1}$$

The Hubble parameter relates with speed for which universe is expanding. The value of the Hubble parameter at the present epoch is the Hubble constant H_0. Recently, there is a lot of discussion have done about its actual value, which is given by

$$H_0 = 100\ h \ \ km/s/Mpc \quad \text{and} \quad 1Mpc \cong 3 \times 10^{24} cm, \quad h = 0.71^{+0.04}_{-0.03}$$

2.6 FRW Geometry

The Friedmann-Robertson-Walker (FRW) metric is based on the assumption of homogeneity and isotropy of the universe which is approximately true on large scales. This homogeneous and isotropic cosmological model is called the FRW geometry and is given by

$$dS^2 = -dt^2 + a(t)^2 \left(\frac{dr^2}{1 - kr^2} + r^2 \big(d\theta^2 + \sin^2\theta d\phi^2 \big) \right) \tag{2.6.1}$$

where a is scale factor and k represents curvature which involves three cases as follow

- $k = -1$ represents the negative curvature (hyperboloid)
- $k = 0$ corresponds to no curvature (flat surface)
- $k = 1$ corresponds to positive curvature (for the 3-sphere)

2.7 Black Hole Thermodynamics

Black hole thermodynamics provides the relationship among laws of thermodynamics and BH mechanics. In general relativity, BHs comply with specific laws which have numerical similarity with common laws of thermodynamics. In classical theory, they are excellent absorber and don't discharge anything yielding absolute zero temperature and in quantum theory, they emit Hawking radiation with a perfect thermal spectrum. In BH mechanics, one can relate entropy of BH with quarter of its area by taking temperature of Hawking radiation. In perspective of BH mechanics, the laws of BH thermodynamics are given as follows [78]:

2.7.1 Zeroth law of black hole thermodynamics

In BH mechanics Zeroth law is defined as the surface gravity of a stationary BH horizon is constant. The zeroth law in classical thermodynamics defines that the temperature in a equilibrium system remains constant. Moreover, In the presence of Hawking radiation, the surface gravity is directly proportional to the temperature of a BH, which concludes that the surface gravity is the measure of temperature of a BH. Hence, constant surface gravity of stationary BH leads to constant temperature. This is called zeroth law of BH.

2.7.2 First law of black hole thermodynamics

First law of BH mechanics is defined as

$$dM = \frac{\kappa}{8\pi}dA + \Omega dJ + \phi dq, \tag{2.7.1}$$

where M, A, Ω, J, ϕ and q are the mass, horizon area, angular velocity, angular momentum, electrostatic potential and electric charge of a BH respectively. The last two terms dJ and dq have the interpretation of work required to turn up BH by an amount dJ keeping in mind the end goal to expand its charge by dq.

2.7.3 Generalized second law Of black hole thermodynamics

The second law of BH thermodynamics states that, in any process, the area of BH and its entropy S do not decrease, i.e.,

$$\delta S \geq 0. \tag{2.7.2}$$

2.7.4 Third law of black hole thermodynamics

It can be defined as "*the temperature of a BH cannot be reduced to absolute zero by a finite number of operations*".

2.8 Entropy

Entropy is the thermodynamical quantity that represents the flow of energy in a thermodynamic process. It explains the nature of heat why it flows from a cold object to a warm one. A physical system will always evolve towards a state of thermal equilibrium. This empirical fact is formulated in the important second law of thermodynamics which reads "entropy tends to increase". Mathematically,

$$\Delta S \geq 0. \tag{2.8.1}$$

2.8.1 Bekenstein entropy

Bekenstein entropy of the apparent horizon is giving as

$$S_{\mathrm{A}} = \left(\frac{c^3}{G\hbar}\right)\frac{A}{4}, \tag{2.8.2}$$

where $A = 4\pi R_A^2$ is the area and $G,\ \hbar,\ R_A$ are the Newton's gravity constant, the Planck-Dirac constant and radius of apparent horizon respectively.

2.8.2 Logarithmic corrected entropy

Logarithmic corrections arises from the loop quantum gravity due to thermal equilibrium fluctuations and quantum fluctuations

$$S_{\mathrm{A}} = \frac{A}{4L_{\mathrm{p}}^2} + \alpha \ln \frac{A}{4L_{\mathrm{p}}^2} + \beta\frac{4L_{\mathrm{p}}^2}{A},$$

where α and β are dimensionless constants and L_p stands for Planck length.

2.8.3 Power law corrected entropy

The horizon entropy of the universe is the basic entity for discussing the GSLT of the expanding universe. In this respect, the correction is found in the power law form as

$$S_{\mathrm{A}} = \frac{A}{4L_{\mathrm{p}}^2}\left(1 - K_\alpha A^{1-\frac{\alpha}{2}}\right),$$

where α is dimensionless constant and

$$K_\alpha = \frac{\alpha\left(4\pi\right)^{\frac{\alpha}{2}-1}}{\left(4-\alpha\right)r_c^{4-\alpha}},$$

with r_c is a crossover scale.

2.9 Cosmological Thermodynamics

Thermodynamics is the branch of science in which we study energy, heat, work and their relationship. In any physical system, thermodynamics is discussed at two different levels such as microscopic and macroscopic. To study the microscopic level, statistical thermodynamics is used which provides the behavioral knowledge of individual particles, while the classical thermodynamics help us to study the physical system at macroscopic level such as pressure, volume, temperature, entropy and internal energy are discussed. In the following, we define the laws of thermodynamics which are helpful to investigate the thermodynamical states of a system [78].

2.9.1 Zeroth law of thermodynamics

According to this law, all the components of a system have the same temperature until the system is in equilibrium.

2.9.2 First law of thermodynamics

First law is also known as the law of conservation of energy. It stats that the total amount of energy in a system remains constant but can change from one form to another, i.e,

$$dE = TdS + dW.$$

It merits seeing that Eq.(2.7.1) has a striking likeness to the first law of thermodynamics, which concludes that first law of black hole mechanics is also the first law of thermodynamics by keeping $\kappa \propto T$ and $A \propto S$.

2.9.3 Generalized second law of thermodynamics

Thermodynamics of BH plays an important role to discuss the behavior of thermodynamical interpretation of the universe. Black hole thermodynamics demonstrates that there is association between attractive energy and thermodynamics. First of all Jacobson [79] drived Einstein field equation with the help of Clausius relation $TdS = dQ$ together with entropy proportional to the horizon area ($S = \frac{A}{4G}$). For the flat quasi de-Sitter geometry, Frolov and Kofman [80] showed Friedmann equations can result from $dE = TdS$ for slowly rolling scalar field. Further Padmanabhan [81] explored such connection in case of general spherically symmetric spacetime and showed that field equations can be stated in the form

$$TdS = dE + PdV, \tag{2.9.1}$$

where P is the pressure and V is volume, also,

$$V = \frac{4\pi R_h^3}{3}, \qquad T = \frac{1}{2\pi R}, \qquad E = \frac{4\pi R_h^3}{3}\left(\rho_\Lambda + \rho_m\right). \tag{2.9.2}$$

Hence, the total entropy turns out to be

$$\dot{S_{tot}} = \dot{S} + \dot{S}_{BH}, \tag{2.9.3}$$

where $\dot{S}$ denotes background entropy and $\dot{S}_{BH}$ shows BH entropy. That is the sum of BH entropy and the background entropy must be an increasing quantity with time.

2.10 DGP Brabeworld Gravity

A most particular version was proposed by Dvali et al. [37], in which the four dimensional FRW universe is enclosed in a five dimensional Minkoski bulk with infinite size. The gravitational laws were obtained by adding an Einstein-Hilbert term to the action of brane computed with the brane's intrinsic curvature. The presence of such a term in the action is generically induced by quantum corrections coming from the bulk gravity and its coupling with matter living on the brane and must be included for a large class of theories for self-consistency [82, 83]. Here, we consider 3-brane embedded in a $5D$ space-time with an intrinsic curvature term included in the brane whose action can be written as

$$S_{(5)} = -\frac{1}{2\kappa^2}\int d^5X\sqrt{-\tilde{g}}\tilde{R} + \int d^5X\sqrt{-\tilde{g}}L_m, \tag{2.10.1}$$

where L_m is the brane curvature term, given by

$$L_m = -\frac{1}{2\mu^2}\int d^4x\sqrt{-g}R, \tag{2.10.2}$$

and $\kappa^2 = 8\pi G_{(5)} = M_{(5)}^{-3}$, $\mu^2 = 8\pi G_{(4)} = M_{(4)}^{-2}$. The Eq.(2.10.1) represents the Einstein-Hilbert action in five dimensions for a five-dimensional metric $\tilde{g}_{\rm AB}$ (bulk metric) of Ricci scalar $\tilde{R}$. Similarly, Eq.(2.10.2) indicates the Einstein-Hilbert action for the induced metric $\tilde{g}_{\rm cd}$ on the brane with R appeared as its scalar curvature. From Eq.(2.10.1), we can get modified Friedmann equation as [84]

$$H^2 + \frac{k}{a^2} = \left(\sqrt{\frac{\rho}{3M_p^2} + \frac{1}{4r_c^2}} + \frac{\epsilon}{2r_c}\right)^2, \tag{2.10.3}$$

where $\rho = \rho_m + \rho_D$, the subscripts m and D represent the energy densities corresponding to DM and DE respectively, r_c is the crossover length which represents the scale that has length away from which gravity starts opening into the bulk [84]. Moreover it is the distance scale follow the comparison among $4D$ and $5D$ effects of gravity and can be written as [84]

$$r_{\rm c} = \frac{M_p^2}{2M_5^3}, \tag{2.10.4}$$

where M_5^2 stands for the $5D$ Planck mass and M_{p}^2 is the $4D$ Planck mass.

2.11 Chern-Simons Modified Gravity

Dynamical Chern-Simons modified theory of gravity is the extension of general relativity. This theory can be obtained explicitly from string theory [85]. We add Chern-Simons term with couple to gravity using scaler field than adding usual Einstein-Hilbert action, we get modified form. Non-dynamical and dynamical formulation are two theories of Chern-Simons which exist independently. The action of Chern-Simons theory is given by [86]-[88]

$$\begin{aligned} S &= \frac{1}{16\pi G}\int d^4x\bigg(\sqrt{-g}R+\frac{\ell}{4}\theta^{\star}R^{\rho\sigma\mu\nu}R_{\rho\sigma\mu\nu}-\frac{1}{2}g^{\mu\nu}\nabla_{\mu}\theta\nabla\nu\theta+V(\theta)\bigg) \\ &+ S_{mat}, \end{aligned} \tag{2.11.1}$$

where ${}^{\star}R^{\rho\sigma\mu\nu}R_{\rho\sigma\mu\nu}$, ℓ, θ, S_{mat}, and $V(\theta)$ are topological invariant called the Pontryagin term, the coupling constant, the dynamical variable, the action of matter and the potential, respectively.

Fields equations describe by the action of dynamical Chern-Simons modified gravity are,

$$G_{\mu\nu}+lC_{\mu\nu} = 8\pi GT_{\mu\nu}, \tag{2.11.2}$$

$$g^{\mu\nu}\nabla_{\mu}\nabla_{\nu}\theta = -\frac{l}{64\pi}{}^{\star}R^{\rho\sigma\mu\nu}R_{\rho\sigma\mu\nu}. \tag{2.11.3}$$

Here $G_{\mu\nu}$ is the Einstein tensor while $C_{\mu\nu}$ is the cotton tensor which can be defined as follows

$$C^{\mu\nu}=-\frac{1}{2\sqrt{-g}}\Big((\nabla_{p}\theta)\varepsilon^{\rho\beta\tau(\mu}\nabla_{\tau}R^{\nu)}_{\beta}\Big)+(\nabla_{\sigma}\nabla_{\rho}\theta)^{\star}R^{\rho(\mu\nu)\sigma}. \tag{2.11.4}$$

Also $\hat{T}_{\mu\nu}$ (energy-momentum tensor corresponding to the scaler field) and $T_{\mu\nu}$ (energy-momentum tensor corresponding to the matter field) are

$$\begin{aligned} \hat{T}^{\theta}_{\mu\nu} &= \nabla_\mu \nabla_\nu \theta - \frac{1}{2} g^{\mu\nu} \nabla^\rho \theta \nabla_\rho \theta, \\ T_{\mu\nu} &= (\rho + p) u_\mu u_\nu + p g_{\mu\nu}. \end{aligned} \tag{2.11.5}$$

Chapter 3

Thermodynamics in DGP Braneworld Scenario

In this chapter, we discuses first law of thermodynamics, GSLT and thermodynamical equilibrium scenario by assuming different entropy corrections such as Bekenstein entropy, logarithmic corrected entropy and power law corrected entropy for constant and variable Γ. The results of this chapter have been accepted for publication in the form of a research paper [89].

3.1 Field Equations

For the spatially flat DGP braneworld ($k = 0$), Eq.(2.10.3) reduces to

$$H^2 - \frac{\epsilon}{r_c} H = \frac{\rho}{3M_p^2}. \tag{3.1.1}$$

There exist two different branches for the DGP model depending on the sign of ϵ. These are as follows:

- For $\epsilon = +1$, there is a de Sitter solution for Eq.(3.1.1) with constant Hubble parameter, i.e., $H = \frac{1}{r_c} \Rightarrow a(t) = a_0 e^{\frac{t}{r_c}}$ in the absence of any kind of energy or matter field on the brane (i.e., $\rho = 0$). However, this branch faces some problems like ghost instabilities [90].

- For $\epsilon = -1$, the accelerated expansion of the universe can only be explained through the inclusion of DE component in the DGP scenario.

We consider the latter case in the present work. The equation of continuity for this model will become

$$\dot{\rho} + \Theta\big(\rho + P + \Pi\big) = 0, \tag{3.1.2}$$

where Π is a particle creation pressure which represents the gravitationally induced process of particle creation and $\Theta = 3H$ is the fluid expansion.

Differentiating Eq.(3.1.1) and replacing the value of $\dot{\rho}$ using Eq.(3.1.2), we obtain

$$\dot{H} = \frac{-H\big(\rho + p + \Pi\big)}{M_p^2(2H - \frac{\epsilon}{r_c})}. \tag{3.1.3}$$

The respective EoS for this model is giving by $p = \big(\gamma - 1\big)\rho$ with $\frac{2}{3} \leq \gamma \leq 2$. The non-conservation of the total rate of change of number of particles, $N = na^3$ with comoving volume a^3 and n is the number density of particle production in an open thermodynamical system yields

$$\dot{n} + \Theta n = n\Gamma, \tag{3.1.4}$$

where Γ is a particle creation rate has negative and positive phases. Negative Γ represents the particle destruction and positive Γ describes the elimination of particles. Furthermore, a non-zero Γ produces effective bulk viscous pressure [91]-[97].

Now using the Eqs.(3.1.2), (3.1.4) and Gibbs relation, we get

$$Tds = d\bigg(\frac{\rho}{n}\bigg) + pd\bigg(\frac{1}{n}\bigg). \tag{3.1.5}$$

An equation related to the creation pressure Π and the creation rate Γ has the form

$$\Pi = -\frac{\Gamma}{\Theta}\big(\rho + p\big). \tag{3.1.6}$$

Under traditional assumption that the specific entropy of each particle is constant, i.e., the process is adiabatic or isentropic. This implies a dissipative fluid is similar to a perfect fluid with a non-conserved particle number. To discuss cosmological parameters, we insert Eqs.(3.1.6) and $p = (\gamma - 1)\rho$ in Eq.(3.1.3), to obtain

$$\frac{\dot{H}}{H^2} = -\frac{3\gamma\big(H - \frac{\epsilon}{r_c}\big)\big(1 - \frac{\Gamma}{3H}\big)}{\big(2H - \frac{\epsilon}{r_c}\big)}. \tag{3.1.7}$$

The deceleration parameter q can be written as

$$q = -\frac{\dot{H}}{H^2} - 1 = \frac{3\gamma(H - \frac{\epsilon}{r_c})\left(1 - \frac{\Gamma}{3H}\right)}{(2H - \frac{\epsilon}{r_c})} - 1. \tag{3.1.8}$$

The effective EoS parameter for this model turns out to be

$$\omega_{\text{eff}} = \frac{p + \Pi}{\rho} = \gamma\left(1 - \frac{\Gamma}{3H}\right) - 1. \tag{3.1.9}$$

This parameter has ability to explain the different phases of the universe on the basis of Γ, i.e., if $\Gamma < 3H$, then we have quintessence era of the universe ($\omega_{\text{eff}} < -1$), if $\Gamma > 3H$ then effective EoS parameter represents the phantom era of the universe ($\omega_{\text{eff}} > -1$) while for $\Gamma = 3H$, effective EoS parameter exhbits the cosmological constant ($\omega_{\text{eff}} = -1$).

In the following, we will discuss first and second thermodynamical laws in the presence of particle creation rate Γ on the apparent horizon.

3.2 Thermodynamical Analysis with Usual Entropy

For flat FRW universe, Hubble parameter coincides with the apparent horizon as $R_{\text{A}} = \frac{1}{H}$. Differentiating the apparent horizon with respect to time, we get

$$\dot{R_A} = -\frac{\dot{H}}{H^2} = \frac{3\gamma(H - \frac{\epsilon}{r_c})\left(1 - \frac{\Gamma}{3H}\right)}{(2H - \frac{\epsilon}{r_c})}. \tag{3.2.1}$$

The Bekenstein entropy and Hawking temperature of the apparent horizon are given by $(8\pi = G = 1)$

$$S_{\text{A}} = \frac{A}{4} = \frac{R_A^2}{8} \quad \text{and} \quad T_{\text{A}} = \frac{1}{2\pi R_A} = \frac{4}{R_A}, \tag{3.2.2}$$

where $A = 4\pi R_A^2$. The first law of thermodynamics at the horizon can be obtained through the Clausius relation as

$$-dE_{\mathrm{A}} = T_{\mathrm{A}} dS_{\mathrm{A}}. \tag{3.2.3}$$

For the sake of convenance, we consider $\Xi = T_{\mathrm{A}} dS_{\mathrm{A}} + dE_{\mathrm{A}}$. The differential dE_{A} is the amount of energy crossing the apparent horizon can be evaluated as [98]

$$-dE_{\mathrm{A}} = \frac{1}{2} R_{\mathrm{A}}^3 (\rho + p) H dt = \frac{3\gamma\big(H - \frac{\epsilon}{r_c}\big)}{2H} dt. \tag{3.2.4}$$

From Eq.(3.2.2), the differential of surface entropy at apparent horizon yields

$$dS_A = \frac{3\gamma}{4H} \frac{(H - \frac{\epsilon}{r_c})(1 - \frac{\Gamma}{3H})}{(2H - \frac{\epsilon}{r_c})}, \tag{3.2.5}$$

which leads to

$$T_{\mathrm{A}} dS_{\mathrm{A}} = \frac{3\gamma\big(H - \frac{\epsilon}{r_c}\big)\big(1 - \frac{\Gamma}{3H}\big)}{\big(2H - \frac{\epsilon}{r_c}\big)} dt. \tag{3.2.6}$$

Thus, Ξ turns out to be

$$\Xi = 3\gamma\big(H - \frac{\epsilon}{r}\big)\left(\frac{1 - \frac{\Gamma}{3H}}{2H - \frac{\epsilon}{r_c}} - \frac{1}{2H}\right). \tag{3.2.7}$$

From this relation, it can be seen that first law of thermodynamics holds (i.e., $\Xi \to 0$) at the apparent horizon for $\Gamma = 3H\big(1 - \frac{2H - \frac{\epsilon}{r_c}}{2H}\big)$.

Next, we will discuss the GSLT and thermodynamical equilibrium of a system containing perfect fluid distribution bounded by apparent horizon in DGG brane-world scenario. For GSLT, the total entropy of the system can not be decrease, i.e., $d(S_{\mathrm{A}} + S_{\mathrm{f}}) \geq 0$. In this relation, S_{A} and S_{f} appear as the entropy at apparent horizon and the entropy of cosmic fluid enclosed within the horizon, respectively. The Gibbs equation is given by

$$T_{\mathrm{f}} dS_{\mathrm{f}} = dE_{\mathrm{f}} + pdV, \tag{3.2.8}$$

where $T_{\rm f}$ is the temperature of the cosmic fluid and $E_{\rm f}$ is the energy of the fluid ($E_{\rm f} = \rho V$). The evolution equation for fluid temperature having constant entropy can be described as [99]

$$\frac{\dot{T}_{\rm f}}{T_{\rm f}} = (\Gamma - \Theta)\frac{\partial p}{\partial \rho}. \tag{3.2.9}$$

Eq.(3.1.7) leads to $\Gamma - \Theta = \frac{\dot{H}(2H - \frac{\epsilon}{r_{\rm c}})}{\gamma H(H - \frac{\epsilon}{r_{\rm c}})}$ which inserting in Eq.(3.2.9) gives the following equation

$$\ln\left(\frac{T_{\rm f}}{T_0}\right) = \frac{(\gamma - 1)}{\gamma}\int \frac{\dot{H}(2H - \frac{\epsilon}{r_{\rm c}})}{\gamma H(H - \frac{\epsilon}{r_{\rm c}})} \quad \Rightarrow \quad T_{\rm f} = T_0\left(H^2 - \frac{\epsilon}{r_{\rm c}}H\right)^{\frac{(\gamma-1)}{\gamma}} \tag{3.2.10}$$

where T_0 is the constant of integration. The differential of the fluid entropy can be obtained by using the Eq.(3.2.8) as follows

$$\begin{aligned} dS_{\rm f} &= -\frac{3\gamma T_0^{-1}}{(2H - \frac{\epsilon}{r_{\rm c}})}\big(H - \frac{\epsilon}{r_{\rm c}}\big)\big(1 - \frac{\Gamma}{3H}\big)\big(H^2 - \frac{\epsilon}{r_{\rm c}}H\big)^{\frac{1-\gamma}{\gamma}} \\ &\times \left(1 - \frac{1}{2Hr_{\rm c}} - \frac{3\gamma}{2H}\big(H - \frac{\epsilon}{r_{\rm c}}\big)\right)dt. \end{aligned} \tag{3.2.11}$$

Using Eqs.(3.2.5) and (3.2.11), we get the rate of change of total entropy as

$$\begin{aligned} \dot{S}_T &= \frac{3\gamma\big(H - \frac{\epsilon}{r_{\rm c}}\big)\big(1 - \frac{\Gamma}{3H}\big)}{4H\big(2H - \frac{\epsilon}{r_{\rm c}}\big)}\Bigg[1 + 4T_0^{-1}H\big(H^2 - \frac{\epsilon}{r_{\rm c}}H\big)^{\frac{1}{\gamma}-1} \\ &\times \left(\frac{1}{2H}\frac{\epsilon}{r_{\rm c}} - \frac{3\gamma}{2H}\big(H - \frac{\epsilon}{r_{\rm c}}\big) - 1\right)\Bigg], \end{aligned} \tag{3.2.12}$$

where $S_T = S_{\rm A} + S_{\rm f}$. We discuss the validity of GSLT on the basis of Γ such that

- $\underline{\Gamma < 3H}$: The GSLT holds if the following constraint

  $$1 > 4T_0^{-1}\big(H^2 - \frac{\epsilon}{r_{\rm c}}H\big)^{\frac{1}{\gamma}-1}\left(H - \frac{\epsilon}{2r_{\rm c}} + \frac{3\gamma}{2}\big(H - \frac{\epsilon}{r_{\rm c}}\big)\right)$$

  satisfies. This shows that the GSLT satisfies in the qunitessence era of the evolving universe.

- $\underline{\Gamma > 3H}$: For this case, we have the constraint

$$1 < 4T_0^{-1}\big(H^2 - \frac{\epsilon}{r_c}H\big)^{\frac{1}{\gamma}-1}\left(H - \frac{\epsilon}{2r_c} + \frac{3\gamma}{2}\big(H - \frac{\epsilon}{r_c}\big)\right),$$

which implies the GSLT holds in phantom era of the universe.

- $\underline{\Gamma = 3H}$: This case implies $\dot{S}_T = 0$ in the cosmological constant era.

Replacing dt to $\frac{dH}{\dot{H}}$ and integrating Eq.(3.2.12), we get

$$\begin{aligned}
S_{\mathrm{T}} &= S_{\mathrm{A}} + S_{\mathrm{f}} \\
&= -\frac{1}{8H^2\epsilon^2 r_c T_0(1-\gamma)}\big(1 - \frac{\epsilon}{H^2 r_c}\big)^{\frac{-1}{\gamma}}\big(1 - \frac{H^2 r_c}{\epsilon}\big)^{\frac{-1}{\gamma}}\Big[6\gamma^2\epsilon^2\ {}_2F1(\frac{-1+\gamma}{\gamma} \\
&, -\frac{1}{\gamma}, 2 - \frac{1}{\gamma}, \frac{\epsilon}{H^2 r_c})(H^2 - \frac{\epsilon}{r_c})^{\frac{1}{\gamma}}(1 - \frac{H^2 r_c}{\epsilon})^{\frac{1}{\gamma}} + 2(H^2 - \frac{\epsilon}{r_c})^{\frac{1}{\gamma}} r_c^2\Big\{ - 4H\epsilon \\
&\times\ (-1+\gamma)\ {}_2F1(-\frac{1}{2}, -\frac{1}{\gamma}, \frac{1}{2}, \frac{H^2 r_c}{\epsilon},)(1 - \frac{\epsilon}{H^2 r_c})^{\frac{1}{\gamma}} + \gamma\Big(6H^3(-1+\gamma) \\
&\times\ {}_2F1(\frac{1}{2}, \frac{-1+\gamma}{\gamma}\frac{3}{2}, \frac{H^2 r_c}{\epsilon})(1 - \frac{\epsilon}{H^2 r_c})^{\frac{1}{\gamma}} + \epsilon^2\ {}_2F1(\frac{-1+\gamma}{\gamma}, -\frac{1}{\gamma}, 2 - \frac{1}{\gamma} \\
&, \frac{\epsilon}{H^2 r_c})(1 - \frac{H^2 r_c}{\epsilon})^{\frac{1}{\gamma}}\Big)\Big\} + 2H^2 r_c^3(H^2 - \frac{\epsilon}{r_c})^{\frac{1}{\gamma}}(-1+\gamma)\Big(4H\ {}_2F1(\frac{1}{2} \\
&, \frac{-1+\gamma}{\gamma}, \frac{3}{2}, \frac{H^2 r_c}{\epsilon})(1 - \frac{\epsilon}{H^2 r_c})^{\frac{1}{\gamma}} + \gamma\epsilon\Big(-\ {}_2F1(-\frac{1}{\gamma}, -\frac{1}{\gamma}, \frac{-1+\gamma}{\gamma}, \frac{\epsilon}{H^2 r_c}) \\
&\times\ (1 - \frac{H^2 r_c}{\epsilon})^{\frac{1}{\gamma}} + (1 - \frac{\epsilon}{H^2 r_c})^{\frac{1}{\gamma}}(-1 + (1 - \frac{H^2 r_c}{\epsilon})^{\frac{1}{\gamma}})\Big)\Big) + (-1+\gamma)\epsilon \\
&\times\ r_c\Big\{6\gamma H(H^2 - \frac{\epsilon}{r_c})^{\frac{1}{\gamma}}\Big(- 2(1 - \frac{\epsilon}{H^2 r_c})^{\frac{1}{\gamma}}\ {}_2F1(-\frac{1}{2}, \frac{-1}{\gamma}, \frac{1}{2}, \frac{H^2 r_c}{\epsilon}) - H\gamma \\
&\times\ (1 - \frac{H^2 r_c}{\epsilon})^{\frac{1}{\gamma}}\ {}_2F1(-\frac{1}{\gamma}, -\frac{1}{\gamma}, \frac{-1+\gamma}{\gamma}, \frac{\epsilon}{H^2 r_c}) + (1 - \frac{\epsilon}{H^2 r_c})^{\frac{1}{\gamma}} \\
&\times\ (-1 + (1 - \frac{H^2 r_c}{\epsilon})^{\frac{1}{\gamma}}\Big) - \epsilon(1 - \frac{\epsilon}{H^2 r_c})^{\frac{1}{\gamma}}(1 - \frac{H^2 r_c}{\epsilon})^{\frac{1}{\gamma}}\Big\}\Big].
\end{aligned} \tag{3.2.13}$$

The plot between total entropy and parameter γ is shown in Figure **3.1** for three values of T by setting constant values as $H = 67$, $r_c = \frac{1}{67}$, $\epsilon = -1$. We observe that $S_T \geq 0$ for all the values of T which leads to the validity of GSLT.

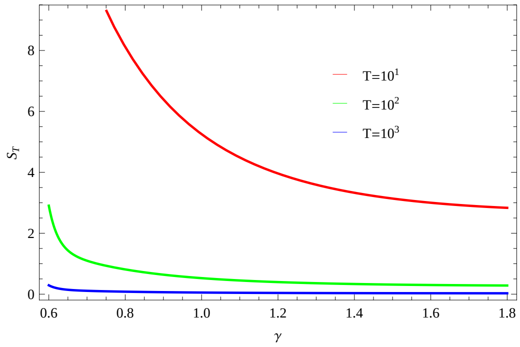

Figure 3.1: Plot of total entropy versus γ for usual entropy.

3.2.1 Thermal Equilibrium Scenario

Further, we will discuss the thermal equilibrium scenario in the present case. For thermodynamical equilibrium, the entropy function attains a maximum value and satisfies the condition $d^2S_T = d^2(S_\mathrm{A} + S_\mathrm{f}) < 0$. For this purpose, we consider two cases of particle creation rate (Γ).

Case 1: $\underline{\Gamma = \text{constant}}$

Firstly, we consider particle creation rate Γ as a constant. Under this scenario, differentiating Eq.(3.2.12) w.r.t time, it results the following second order differential equation

$$\begin{aligned}
\ddot{S_T} \;=\; & -\frac{3\gamma\big(1-\frac{\Gamma}{3H}\big)\big(-\frac{\epsilon}{r_c}+H\big)\lambda\dot{H}}{2H\big(-\frac{\epsilon}{r_c}+2H\big)^2} + \frac{3\gamma\big(1-\frac{\Gamma}{3H}\big)\lambda\dot{H}}{4H\big(-\frac{\epsilon}{r_c}+2H\big)} + \gamma\Gamma\big(1-\frac{\Gamma}{3H}\big) \\
\times\; & \frac{\big(-\frac{\epsilon}{r_c}+H\big)\lambda\dot{H}}{4H^3\big(-\frac{\epsilon}{r_c}+2H\big)} - \frac{3\gamma\big(1-\frac{\Gamma}{3H}\big)\big(-\frac{\epsilon}{r_c}+H\big)\lambda\dot{H}}{4H^2\big(-\frac{\epsilon}{r_c}+2H\big)} + \frac{3\gamma}{4H\big(-\frac{\epsilon}{r_c}+2H\big)} \\
\times\; & \big(1-\frac{\Gamma}{3H}\big)\big(-\frac{\epsilon}{r_c}+H\big)\Bigg(\frac{4\big(-\frac{\epsilon}{r_c}+H^2\big)^{\frac{1-\gamma}{\gamma}}}{T_0}\Bigg(-1+\frac{\epsilon}{2r_cH}-\frac{3\gamma}{2H} \\
\times\; & \big(-\frac{\epsilon}{r_c}+H\big)\Bigg)\dot{H} + \frac{4(1-\gamma)H}{\gamma T_0}\big(-\frac{\epsilon H}{r_c}+H^2\big)^{-1+\frac{1-\gamma}{\gamma}}\Bigg(-1+\frac{\epsilon}{2Hr_c} \\
-\; & \frac{3\gamma\big(-\frac{\epsilon}{r_c}+H\big)}{2H}\Bigg)\big(-\frac{\epsilon}{r_c}+2H\dot{H}\big) + 4H\big(-\frac{\epsilon H}{r_c}+H^2\big)^{\frac{1-\gamma}{\gamma}}
\end{aligned}$$

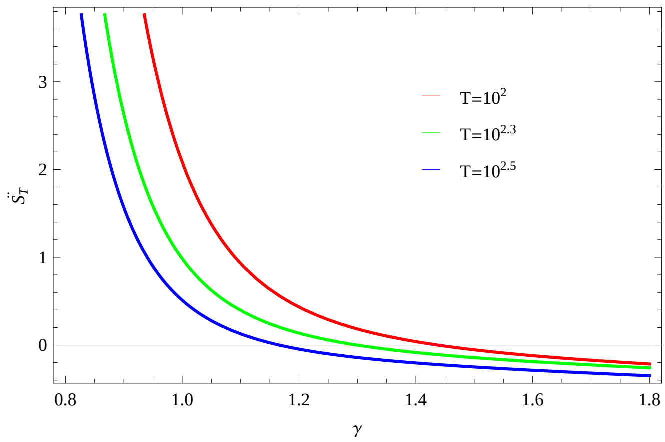

Figure 3.2: Plot of $\ddot{S_T}$ versus γ for usual entropy when $\Gamma = constant$.

$$\times \frac{\Big(-\frac{\epsilon\dot{H}}{2r_cH^2} - \frac{3\gamma\dot{H}}{2H} + \frac{3\gamma\left(-\frac{\epsilon}{r_c}+H\right)}{2H^2}\Big)}{T_0}\Bigg). \tag{3.2.14}$$

where $\lambda = 1 + \frac{4H}{T_0}\big(-\frac{\epsilon H}{r_c} + H^2\big)^{\frac{1-\gamma}{\gamma}}\big(-1 + \frac{\epsilon}{2r_cH} - \frac{3\gamma(-\frac{\epsilon}{r_c}+H)}{2H}\big)$. The plot between $\ddot{S_T}$ versus γ for three values of T with constant values of $H = 67$, $r_c = \frac{1}{67}$. $\epsilon = -1$, $q = -0.53$ as shown in Figure **3.2**. One can observe that the thermodynamical equilibrium condition holds for all values of T with specific ranges of γ. For example, for $T = 10^2$, thermodynamical equilibrium holds for the range $1.4 < \gamma \leq 1.8$ and does not obey for $0.6 \leq \gamma \leq 1.4$. For $T = 10^{2.3}$, thermal equilibrium holds for the range $1.3 < \gamma \leq 1.8$ and does not showing the validity for $0.6 \leq \gamma \leq 1.3$. However, for $T = 10^{2.5}$, thermodynamic equilibrium condition holds for the range $1.2 < \gamma \leq 1.8$ and disobey for the range $0.6 \leq \gamma < 1.2$.

Case 2: $\underline{\Gamma = \Gamma(t)}$

Here we take Γ as variable parameter, i.e., $\Gamma = \Gamma(t)$, for which Eq.(3.2.12) becomes

$$\begin{aligned}
\ddot{S_T} &= -\frac{3\gamma\lambda(-\frac{\epsilon}{r_c}+H)(1-\frac{\Gamma}{3H})\dot{H}}{2H(-\frac{\epsilon}{r_c}+2H)^2} + \frac{3\gamma\lambda(1-\frac{\Gamma}{3H})\dot{H}}{4H(-\frac{\epsilon}{r_c}+2H)} - 3\gamma\lambda(1-\frac{\Gamma}{3H})\dot{H} \\
&\times \frac{(-\frac{\epsilon}{r_c}+H)}{4H^2(-\frac{\epsilon}{r_c}+2H)} + \frac{3\gamma(-\frac{\epsilon}{r_c}+H)(1-\frac{\Gamma}{3H})}{4H(-\frac{\epsilon}{r_c}+2H)}\Bigg(4(-\frac{\epsilon H}{r_c}+H^2)^{\frac{1-\gamma}{\gamma}} \\
&\times \frac{\big(-1+\frac{\epsilon}{2r_cH}-\frac{3\gamma(-\frac{\epsilon}{r_c}+H)}{2H}\big)\dot{H}}{T_0} + \frac{4\big(1-\gamma\big)H}{\gamma T_0}(-\frac{\epsilon H}{r_c}+H^2)^{-1+\frac{1-\gamma}{\gamma}} \\
&\times 4H\big(1-\gamma\big)\big(-1+\frac{\epsilon}{2r_cH}-\frac{3\gamma(-\frac{\epsilon}{r_c}+H)}{2H}\big)\big(-\frac{\epsilon\dot{H}}{r_c}+2H\dot{H}\big) \\
&+ \frac{4H(-\frac{\epsilon H}{r_c}+H^2)^{\frac{1-\gamma}{\gamma}}\big(-\frac{\epsilon\dot{H}}{2r_cH^2}-\frac{3\gamma\dot{H}}{2H}+\frac{3\gamma(-\frac{\epsilon}{r_c}+H)\dot{H}}{2H^2}\big)}{T_0}\Bigg) \\
&+ \frac{3\gamma\lambda(-\frac{\epsilon}{r_c}+H)\big(\frac{\Gamma\dot{H}}{3H^2}-\frac{\dot{\Gamma}}{3H}\big)}{4H(-\frac{\epsilon}{r_c}+2H)}, \qquad (3.2.15)
\end{aligned}$$

where

$$\dot{\Gamma} = -\frac{3\gamma H\Gamma\big(\frac{\Gamma}{3H}\big)\big(H-\frac{\epsilon}{r_c}\big)\big(1-\frac{\Gamma}{3H}\big)}{2H-\frac{\epsilon}{r_c}} + 3H\gamma\Gamma\big(1-\frac{\Gamma}{3H}\big).$$

The plot of $\ddot{S_T}$ versus γ for three values of T as shown in Figure **3.3** by keeping the same constant values as in previous case. We observe that the thermodynamic equilibrium holds for all values of T with different ranges of γ. For example, for $T = 10^2$, thermodynamic equilibrium holds for the range $1.4 < \gamma \leq 1.8$ and does not satisfying $0.6 \leq \gamma \leq 1.4$. For $T = 10^{2.3}$, it leads to the validity of thermodynamic equilibrium for the range $1.3 < \gamma \leq 1.8$ and does not valid for $0.6 \leq \gamma \leq 1.3$. However, for $T = 10^{2.5}$, thermodynamic equilibrium holds for the range $1.2 \leq \gamma \leq 1.8$ and does not satisfying within $0.6 \leq \gamma < 1.2$.

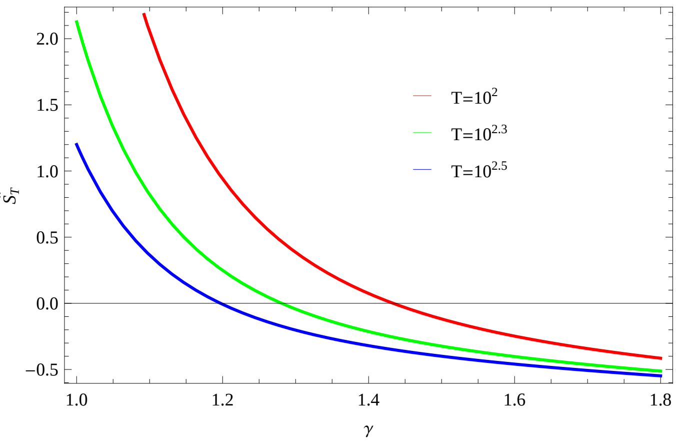

Figure 3.3: Plot of $\ddot{S_T}$ versus γ for usual entropy when $\Gamma = \Gamma(t)$.

3.3 Logarithmic Corrected Entropy

Quantum gravity allows the logarithmic corrections in the presence of thermal equilibrium fluctuations and quantum fluctuations [100]-[106]. The logarithmic entropy corrections can be defined as

$$S_{\rm A} = \frac{A}{4L_{\rm p}^2} + \alpha \ln \frac{A}{4L_{\rm p}^2} + \beta \frac{4L_{\rm p}^2}{A}, \tag{3.3.1}$$

where α and β are constants whose values are still under consideration. The differential form of above equation leads to

$$dS_{\rm A} = -\frac{3\gamma\big(H - \frac{\epsilon}{r_{\rm c}}\big)\big(1 - \frac{\Gamma}{3H}\big)}{\big(2H - \frac{\epsilon}{r_{\rm c}}\big)}\left(\frac{1}{4HL_{\rm p}^2} + 2\alpha H - 16\beta H^3 L_{\rm p}^2\right)dt, \tag{3.3.2}$$

which gives

$$T_{\rm A} dS_{\rm A} = \frac{3\gamma\big(H - \frac{\epsilon}{r_{\rm c}}\big)\big(1 - \frac{\Gamma}{3H}\big)}{\big(2H - \frac{\epsilon}{r_{\rm c}}\big)}\left(\frac{1}{L_{\rm p}^2} + 8H^2\alpha - 64\beta H^4 L_{\rm p}^2\right)dt. \tag{3.3.3}$$

In view of this entropy, the quantity Ξ takes the form

$$\Xi = 3\gamma(H - \frac{\epsilon}{r_c})\left(-\frac{1}{2H} + \frac{(1 - \frac{\Gamma}{3H})}{(2H - \frac{\epsilon}{r_c})}\big(\frac{1}{L_{\rm p}^2} + 8H^2\alpha - 64\beta H^4 L_{\rm p}^2\big)\right). \tag{3.3.4}$$

It can be observed from Eq.(3.3.4) that the first law of thermodynamics holds when $\Gamma = 3H\big(1 - \frac{(2H - \frac{\epsilon}{r_c})}{2H}(\frac{1}{L_{\rm p}^2} + 8H^2\alpha - 64\beta H^4 L_{\rm p}^2)\big)$. To discuss the

GSLT for logarithmic corrected entropy of horizon, we obtain the total entropy by using Eqs.(3.2.11) and (3.3.2) as follows

$$\begin{aligned}\dot{S_T} &= \frac{3\gamma\big(H-\frac{\epsilon}{r_\mathrm{c}}\big)\big(1-\frac{\Gamma}{3H}\big)}{\big(2H-\frac{\epsilon}{r_\mathrm{c}}\big)}\Bigg(4HL_\mathrm{p}^2+2\alpha H-16\beta H^3L_\mathrm{p}^2-T_0^{-1}\\ &\times \big(H^2-\frac{\epsilon}{r_\mathrm{c}}H\big)^{\frac{1-\gamma}{\gamma}}\Big(1-\frac{1}{2H}\frac{\epsilon}{r_\mathrm{c}}-\frac{3\gamma}{2H}\big(H-\frac{\epsilon}{r_\mathrm{c}}\big)\Big)\Bigg). \qquad (3.3.5)\end{aligned}$$

The GSLT will hold under these constraints.

- For the case $\underline{\Gamma < 3H}$, the GSLT satisfy in the quintessence region of the universe if the following constraint holds

$$\begin{aligned}4L_\mathrm{p}^2+2\alpha &> 16\beta H^2L_\mathrm{p}^2+(HT_0)^{-1}\Big(1-\frac{1}{2H}\frac{\epsilon}{r_\mathrm{c}}-\frac{3\gamma}{2H}\big(H-\frac{\epsilon}{r_\mathrm{c}}\big)\Big)\\ &\times \big(H^2-\frac{\epsilon}{r_\mathrm{c}}H\big)^{\frac{1-\gamma}{\gamma}}.\end{aligned}$$

- $\underline{\Gamma > 3H}$ For this case, we obtain the following constraint

$$\begin{aligned}4L_\mathrm{p}^2+2\alpha &< 16\beta H^2L_\mathrm{p}^2+(HT_0)^{-1}\Big(1-\frac{1}{2H}\frac{\epsilon}{r_\mathrm{c}}-\frac{3\gamma}{2H}\big(H-\frac{\epsilon}{r_\mathrm{c}}\big)\Big)\\ &\times \big(H^2-\frac{\epsilon}{r_\mathrm{c}}H\big)^{\frac{1-\gamma}{\gamma}}.\end{aligned}$$

  which implies the GSLT holds in phantom era of the universe.

- The case $\underline{\Gamma = 3H}$ means $\dot{S_T} = 0$ in the cosmological constant era.

The expression of total entropy in the form of Hubble parameter is given by

$$\begin{aligned}S_\mathrm{T} &= S_\mathrm{A}+S_\mathrm{f}\\ &= 8H^2\beta L_\mathrm{p}^2-2\ln(H)(\alpha+2L_\mathrm{p}^2)+\frac{\gamma\big(H(H-\frac{\epsilon}{r_c})\big)^{\frac{1}{\gamma}}}{2H^3T_0}. \qquad (3.3.6)\end{aligned}$$

The plot of total entropy S_T versus γ with respect to three values of T is shown in Figure **3.4** with constant values as $\alpha = -2$, $\beta = -0.00001$, $L_p = 1$. It is observed that the total entropy is positive, i.e, $S_T > 0$ which leads to the validity of GSLT for all values of T .

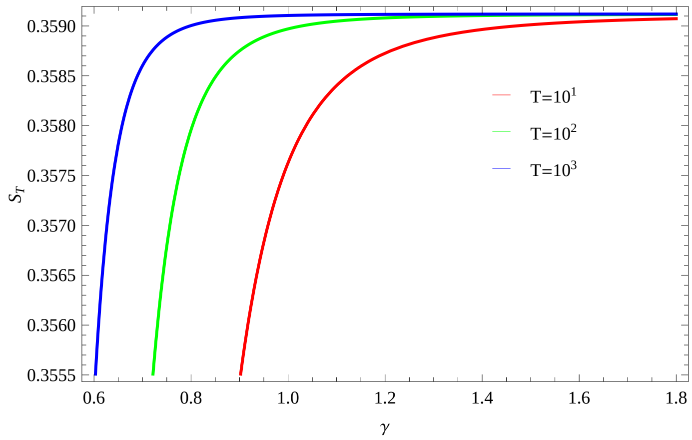

Figure 3.4: Plots of total entropy versus γ for logarithmic corrected entropy.

3.3.1 Thermal Equilibrium Scenario

Now we will discuss the thermodynamic equilibrium by assuming two cases for particle creation rate Γ as follows:

Case 1: <u>Γ = constant</u>

In this way, the second order differential equation can be obtained from Eq.(3.3.5) for Γ as a constant

$$\begin{aligned}
\ddot{S_T} &= -\frac{6\gamma(1-\frac{\Gamma}{3H})\big(H-\frac{\epsilon}{r_c}\big)}{\big(2H-\frac{\epsilon}{r_c}\big)}\big(4HL_{\mathrm{p}}^2-\lambda_2+2\alpha H-16L_{\mathrm{p}}^2\beta H^3\big)\dot{H}+3\gamma \\
&\times \frac{(1-\frac{\Gamma}{3H})\big(4HL_{\mathrm{p}}^2-\lambda_2+2\alpha H-16L_{\mathrm{p}}^2\beta H^3\big)\dot{H}}{\big(2H-\frac{\epsilon}{r_c}\big)}+\gamma\Gamma\big(H-\frac{\epsilon}{r_c}\big) \\
&\times \frac{\big(4HL_{\mathrm{p}}^2-\lambda_2+2\alpha H-16L_{\mathrm{p}}^2\beta H^3\big)}{H^2\big(2H-\frac{\epsilon}{r_c}\big)}+\frac{3\gamma(1-\frac{\Gamma}{3H})\big(H-\frac{\epsilon}{r_c}\big)}{\big(2H-\frac{\epsilon}{r_c}\big)} \\
&\times \Bigg(4L_{\mathrm{p}}^2\dot{H}-\frac{\big(1-\gamma\big)\big(1-\frac{3\gamma\left(H-\frac{\epsilon}{r_c}\right)}{2H}-\frac{\epsilon}{2Hr_c}\big)\big(H^2-\frac{\epsilon H}{r_c}\big)^{-1+\frac{1-\gamma}{\gamma}}}{\gamma T_0} \\
&\times \big(2H\dot{H}-\frac{\epsilon\dot{H}}{r_c}\big)-\frac{\big(H^2-\frac{\epsilon}{r_c}\big)^{\frac{1-\gamma}{\gamma}}\big(-\frac{3\gamma\dot{H}}{2H}+\frac{3\gamma\left(H-\frac{\epsilon}{r_c}\right)}{2H^2}+\frac{\epsilon\dot{H}}{2H^2r_c}\big)}{T_0} \\
&+ 2\alpha\dot{H}-48L_{\mathrm{p}}^2\beta H^2\beta\dot{H}\Bigg), \qquad (3.3.7)
\end{aligned}$$

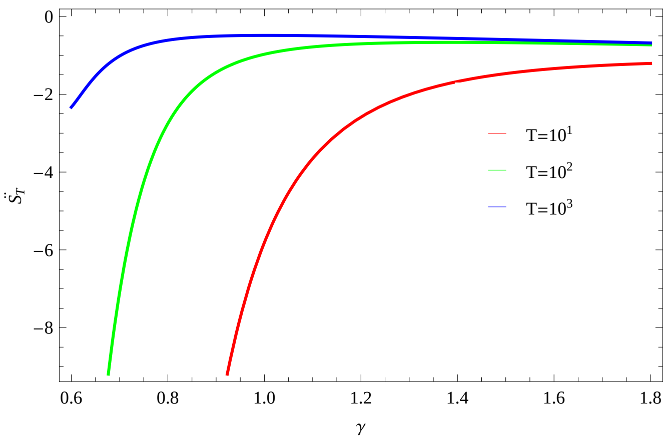

Figure 3.5: Plot of $\ddot{S_T}$ versus γ for logarithmic corrected entropy when $\Gamma = constant$.

where

$$\lambda_2 = \frac{\left(1 - \frac{3\gamma\left(H - \frac{\epsilon}{r_c}\right)}{2H} - \frac{\epsilon}{2Hr_c}\right)\left(H^2 - \frac{\epsilon H}{r_c}\right)^{\frac{1-\gamma}{\gamma}}}{T_0}.$$

The plot between $\ddot{S_T}$ and γ for three values of T by fixing the constant values $q = -0.53$, $\alpha = -2$, $\beta = -0.000000001$ and $L_p = 1$ as shown in Figure **3.5**. It can be seen that thermodynamical equilibrium is obeying the condition $\ddot{S_T} < 0$ for all values of T which leads to the thermal equilibrium.

Case 2: $\underline{\Gamma = \Gamma(t)}$

Taking Γ as a function of t, Eq.(3.3.5) yields

$$\begin{aligned}
\ddot{S_T} &= -\frac{6\gamma(1 - \frac{\Gamma}{3H})\left(H - \frac{\epsilon}{r_c}\right)}{\left(2H - \frac{\epsilon}{r_c}\right)}\left(4HL_{\rm p}^2 - \lambda_2 + 2\alpha H - 16L_{\rm p}^2\beta H^3\right)\dot{H} + 3\gamma \\
&\times \frac{(1 - \frac{\Gamma}{3H})\left(4HL_{\rm p}^2 - \lambda_2 + 2\alpha H - 16L_{\rm p}^2\beta H^3\right)\dot{H}}{\left(2H - \frac{\epsilon}{r_c}\right)} + \frac{3\gamma(1 - \frac{\Gamma}{3H})}{\left(2H - \frac{\epsilon}{r_c}\right)} \\
&\times \left(H - \frac{\epsilon}{r_c}\right)\Bigg(4L_{\rm p}^2\dot{H} - \frac{\left(1 - \frac{3\gamma\left(H - \frac{\epsilon}{r_c}\right)}{2H} - \frac{\epsilon}{2Hr_c}\right)\left(H^2 - \frac{\epsilon H}{r_c}\right)^{-1+\frac{1-\gamma}{\gamma}}}{\gamma T_0} \\
&\times (1-\gamma)\left(2H\dot{H} - \frac{\epsilon\dot{H}}{r_c}\right) - \frac{\left(-\frac{3\gamma\dot{H}}{2H} + \frac{3\gamma\left(H - \frac{\epsilon}{r_c}\right)}{2H^2} + \frac{\epsilon\dot{H}}{2H^2r_c}\right)}{T_0}
\end{aligned}$$

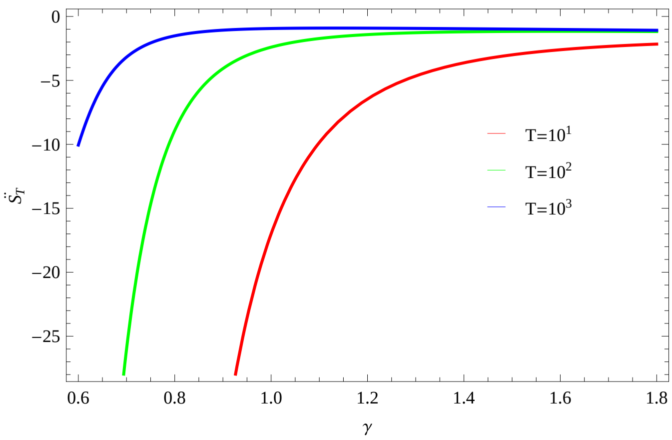

Figure 3.6: Plot of $\ddot{S_T}$ versus γ for logarithmic corrected entropy when $\Gamma = \Gamma(t)$.

$$\times \quad \left(H^2 - \frac{\epsilon}{r_c}\right)^{\frac{1-\gamma}{\gamma}} + 2\alpha\dot{H} - 48L_{\rm p}^2\beta H^2\beta\dot{H}\Bigg) + 3\gamma\left(H - \frac{\epsilon}{r_c}\right)$$
$$\times \quad \frac{1}{\left(2H - \frac{\epsilon}{r_c}\right)}\left(4HL_{\rm p}^2 - \lambda_2 + 2\alpha H - 16L_{\rm p}^2\beta H^3\right)\left(\frac{\Gamma\dot{H}}{3H^2} - \frac{\dot{\Gamma}}{3H}\right). \quad (3.3.8)$$

Figure **6** reperesents the plot between $\ddot{S_T}$ and γ for three values of T for variable Γ for same constant values. Figure **3.6** indicates that the trajectories of $\ddot{S_T}$ corresponding to all the values of T ensure the validity of thermodynamical equilibrium.

3.4 Power Law Corrected Entropy

The power-law correction to the entropy-area law comes from association of the wave-function of the scalar field between the ground state and the exited state [107]-[109]. The correction term is also more significant for higher excitations. It is important to note that the correction term decreases faster with A and hence in the semi-classical limit (large area) the entropy-area law is recovered. The power entropy can b given as

$$S_{\mathrm{A}}=\frac{A}{4L_{\mathrm{p}}^{2}}\big(1-K_{\delta}A^{1-\frac{\delta}{2}}\big),\quad K_{\delta}=\frac{\delta\big(4\pi\big)^{\frac{\delta}{2}-1}}{\big(4-\delta\big)r_{c}^{4-\delta}}\tag{3.4.1}$$

where δ is dimensionless constant and r_c is the crossover scale. From Eq.(3.4.1) the differential of surface entropy at horizon can be expressed as

$$dS_{\mathrm{A}}=\frac{3\gamma\big(H-\frac{\epsilon}{r_{c}}\big)\big(1-\frac{\Gamma}{3H}\big)}{\big(2H-\frac{\epsilon}{r_{c}}\big)}\bigg(\frac{1}{4HL_{\mathrm{p}}^{2}}-\frac{K_{\delta}}{4L_{\mathrm{p}}^{2}}\big(2-\frac{\delta}{2}\big)\big(\frac{1}{H}\big)^{3-\delta}\bigg)dt,\tag{3.4.2}$$

which gives

$$T_{\mathrm{A}}dS_{\mathrm{A}}=\frac{3\gamma\big(H-\frac{\epsilon}{r_{c}}\big)\big(1-\frac{\Gamma}{3H}\big)}{\big(2H-\frac{\epsilon}{r_{c}}\big)}\bigg(\frac{1}{L_{\mathrm{p}}^{2}}-\big(2-\frac{\delta}{2}\big)K_{\delta}R^{2-\delta}\bigg)dt.\tag{3.4.3}$$

In this entropy correction, we have

$$\Xi=3\gamma(H-\frac{\epsilon}{r_{c}})\bigg(-\frac{1}{2H}+\frac{(1-\frac{\Gamma}{3H})}{(2H-\frac{\epsilon}{r_{c}}}\big(\frac{1}{L_{\mathrm{p}}^{2}}-(2-\frac{\delta}{2})K_{\delta}R^{2-\delta}\big)\bigg),\tag{3.4.4}$$

This equation shows that the first law of thermodynamics (i.e., $\Xi=0$) holds for $\Gamma=3H\big(1-\frac{(2H-\frac{\epsilon}{r_{c}})}{2H\big(\frac{1}{L_{\mathrm{p}}^{2}}-(2-\frac{\delta}{2})K_{\delta}R^{2-\delta}\big)}\big)$.

In the presence of power law corrected entropy, the rate of change of total entropy takes the form

$$\begin{aligned}\dot{S_{T}}=&\frac{3\gamma\big(H-\frac{\epsilon}{r_{c}}\big)\big(1-\frac{\Gamma}{3H}\big)}{\big(2H-\frac{\epsilon}{r_{c}}\big)}\bigg(\frac{1}{4HL_{\mathrm{p}}^{2}}-\frac{K_{\delta}}{4L_{\mathrm{p}}^{2}}\big(2-\frac{\delta}{2}\big)\big(\frac{1}{H}\big)^{3-\delta}\\&-T_{0}^{-1}\big(H^{2}-\frac{\epsilon}{r_{c}}H\big)^{\frac{1-\gamma}{r_{c}}}\big(1-\frac{1}{2H}\frac{\epsilon}{r_{c}}-\frac{3\gamma}{2H}(H-\frac{\epsilon}{r})\big)\bigg).\end{aligned}\tag{3.4.5}$$

To discuss the validity of GSLT, the following constraints must satisfy. These are as

- The condition $\underline{\Gamma<3H}$ implies that

$$\begin{aligned}\frac{1}{4L_{\mathrm{p}}^{2}}\ >&\ \frac{HK_{\delta}}{4L_{\mathrm{p}}^{2}}\big(2-\frac{\delta}{2}\big)\big(\frac{1}{H}\big)^{3-\delta}+HT_{0}^{-1}\big[1-\frac{1}{2H}\frac{\epsilon}{r_{c}}-\frac{3\gamma}{2H}(H-\frac{\epsilon}{r})\big]\\\times&\ \big(H^{2}-\frac{\epsilon}{r_{c}}H\big)^{\frac{1-\gamma}{r_{c}}}\end{aligned}$$

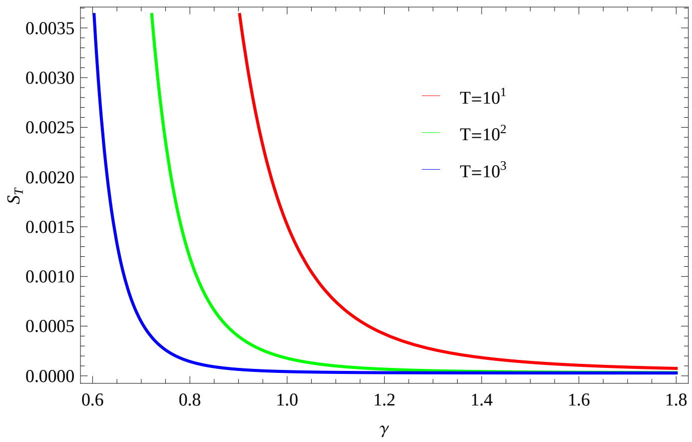

Figure 3.7: Plot of total entropy versus γ for power law corrected entropy.

- $\underline{\Gamma > 3H}$ For this case, we obtain the following constraint which indicates the validity of GSLT in phantom phase. It is given as

$$\begin{aligned}\frac{1}{4L_{\mathrm{p}}^2} &< \frac{HK_\delta}{4L_{\mathrm{p}}^2}\big(2-\frac{\delta}{2}\big)\big(\frac{1}{H}\big)^{3-\delta} + HT_0^{-1}\big[1-\frac{1}{2H}\frac{\epsilon}{r_c} - \frac{3\gamma}{2H}(H-\frac{\epsilon}{r})\big] \\ &\times \ \big(H^2 - \frac{\epsilon}{r_c}H\big)^{\frac{1-\gamma}{r_c}}\end{aligned}$$

- The case $\underline{\Gamma = 3H}$ gives $\dot{S}_T = 0$ in the cosmological constant era.

The equation (3.4.5) in terms of Hubble parameter takes the following form

$$\begin{aligned} S_T &= S_{\mathrm{A}} + S_{\mathrm{f}} \\ &= \frac{1}{4}\left(\frac{1}{2H^2L_{\mathrm{p}}^2} - \frac{(\frac{1}{H})^{4-\delta}K_\delta}{2L_{\mathrm{p}}^2} + \frac{2\gamma(H(H-\frac{\epsilon}{r_c}))^{\frac{1}{\gamma}}}{H^3T_0}\right). \end{aligned} \tag{3.4.6}$$

The plot of total entropy (S_T) is shown in Figure **3.7** versus γ for three values of T by assigning the constant values as $L = 1$, $H = 67$, $\frac{1}{67}$, $\alpha = -2$, and $\epsilon = -1$. We observe that total entropy is positive, i.e, $(S_T) > 0$ which leads to the validity of GSLT for all values of T.

3.4.1 Thermal Equilibrium Scenario

For thermodynamic equilibrium, we assume two cases of Γ, i.e., Γ is constant and Γ is variable.

Case 1: $\underline{\Gamma = \text{constant}}$

For constant Γ, the second order differential equation of total entropy leads to

$$\begin{aligned}
\ddot{S}_T &= -\frac{6\gamma(1-\frac{\Gamma}{3H})(H-\frac{\epsilon}{r_c})\lambda_3}{(2H-\frac{\epsilon}{r_c})^2} + \frac{3\gamma(1-\frac{\Gamma}{3H})\lambda_3}{(2H-\frac{\epsilon}{r_c})} + \frac{\gamma\Gamma(H-\frac{\epsilon}{r_c})\lambda_3}{H^2(2H-\frac{\epsilon}{r_c})} \\
&+ \frac{3\gamma(1-\frac{\Gamma}{3H})(H-\frac{\epsilon}{r_c})}{(2H-\frac{\epsilon}{r_c})}\Bigg(-\frac{\dot{H}}{4H^2L_{\mathrm{p}}^2} + \frac{\big(3-\delta\big)\big(2-\frac{\delta}{2}\big)\big(\frac{1}{4H}\big)^{4-\delta}K_\delta\dot{H}}{4L_{\mathrm{p}}^2} \\
&- \frac{\big(1-\gamma\big)\big(1-\frac{3\gamma\left(H-\frac{\epsilon}{r_c}\right)}{2H}-\frac{\epsilon}{2Hr_c}\big)\big(H^2-\frac{\epsilon H}{r_c}\big)^{-1+\frac{1-\gamma}{\gamma}}\big(2H\dot{H}-\frac{\epsilon\dot{H}}{r_c}\big)}{\gamma T_0} \\
&- \frac{\big(H^2-\frac{\epsilon H}{r_c}\big)^{\frac{1-\gamma}{\gamma}}\big(-\frac{3\gamma\dot{H}}{2H}+\frac{3\gamma\left(H-\frac{\epsilon}{r_c}\right)\dot{H}}{2H^2}+\frac{\epsilon\dot{H}}{2H^2r_c}\big)}{T_0}\Bigg),
\end{aligned} \tag{3.4.7}$$

where

$$\begin{aligned}
\lambda_3 &= \Bigg(\frac{1}{4HL_{\mathrm{p}}^2} - \frac{\big(2-\frac{\delta}{2}\big)\big(\frac{1}{H}\big)^{3-\delta}K_\delta}{4L_{\mathrm{p}}^2} - \big(1-\frac{3\gamma\big(H-\frac{\epsilon}{r_c}\big)}{2H}-\frac{\epsilon}{2Hr_c}\big) \\
&\times \frac{\big(H^2-\frac{\epsilon H}{r_c}\big)^{\frac{1-\gamma}{\gamma}}}{T_0}\Bigg)\dot{H}.
\end{aligned}$$

The plot between $\ddot{S}_T$ and γ for three values of T by setting the constant values as $H = 67$, $r_c = \frac{1}{67}$, $\epsilon = -1$, $q = -0.53$, $\delta = -2$, $L = 1$ as shown in Figure **3.8**. It is found that the trajectories in this plot satisfy the condition $\ddot{S}_T < 0$ for all values of T, which leads to the thermodynamical equilibrium of the system.

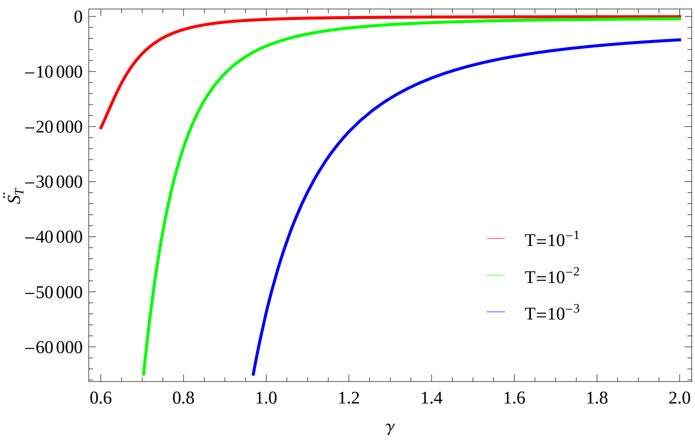

Figure 3.8: Plot of $\ddot{S_T}$ versus γ for power law corrected entropy when Γ=constant.

Case 2: $\underline{\Gamma = \Gamma(t)}$

For variable Γ, second order differential equation of total entropy turns out to be

$$\begin{aligned}
\ddot{S_T} &= -\frac{6\gamma(1-\frac{\Gamma}{3H})(H-\frac{\epsilon}{r_c})\lambda_3}{(2H-\frac{\epsilon}{r_c})^2} + \frac{3\gamma(1-\frac{\Gamma}{3H})\lambda_3}{(2H-\frac{\epsilon}{r_c})} + \frac{3\gamma(1-\frac{\Gamma}{3H})(H-\frac{\epsilon}{r_c})}{(2H-\frac{\epsilon}{r_c})} \\
&\times \Bigg(-\frac{\dot{H}}{4H^2L_{\rm p}^2} + \frac{(3-\delta)\left(2-\frac{\delta}{2}\right)\left(\frac{1}{4H}\right)^{4-\delta}K_\delta \dot{H}}{4L_{\rm p}^2} - \left(2H\dot{H} - \frac{\epsilon\dot{H}}{r_c}\right) \\
&\times \frac{(1-\gamma)\left(1-\frac{3\gamma\left(H-\frac{\epsilon}{r_c}\right)}{2H} - \frac{\epsilon}{2Hr_c}\right)\left(H^2-\frac{\epsilon H}{r_c}\right)^{-1+\frac{1-\gamma}{\gamma}}}{\gamma T_0} - \left(H^2 - \frac{\epsilon H}{r_c}\right)^{\frac{1-\gamma}{\gamma}} \\
&\times \frac{\left(-\frac{3\gamma\dot{H}}{2H} + \frac{3\gamma\left(H-\frac{\epsilon}{r_c}\right)\dot{H}}{2H^2} + \frac{\epsilon\dot{H}}{2H^2r_c}\right)}{T_0}\Bigg) + \frac{3\gamma\left(H-\frac{\epsilon}{r_c}\right)\left(\frac{\Gamma\dot{H}}{3H^2} - \frac{\dot{\Gamma}}{3H}\right)\lambda_3}{2H-\frac{\epsilon}{r_c}} \qquad (3.4.8)
\end{aligned}$$

The plot between $\ddot{S_T}$ and γ for three values of T by setting the constant values as $H = 67$, $r_c = \frac{1}{67}$, $\epsilon = -1$, $q = -0.53$, $\delta = -2$, $L = 1$ as shown in Figure **3.9**. We observe that thermodynamical equilibrium satisfying the condition $\ddot{S_T} < 0$ for all values of T. For example, thermodynamical equilibrium is satisfying the condition $\frac{d^2S_T}{dt^2} < 0$ for $1 \leq \gamma \leq 2$ and does showing stability for $0 \leq \gamma < 1$ with $T = 10^1$ and $T = 10^2$ respectively. On other hand, For $T = 10^3$ thermal equilibrium is stable for $1.1 < \gamma \leq 2$ and

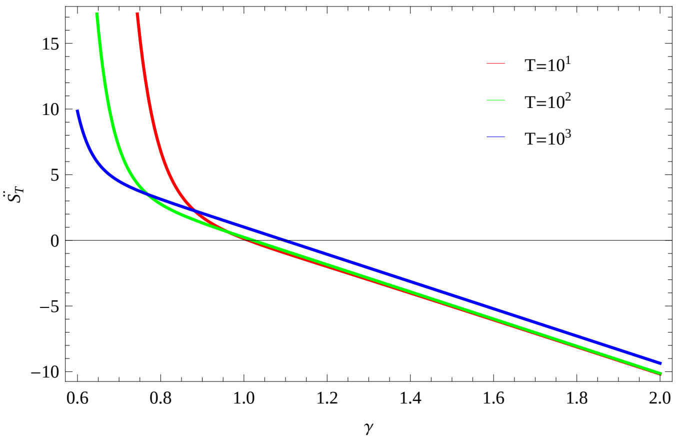

Figure 3.9: Plot of $\ddot{S}_T$ versus γ for power law corrected entropy when $\Gamma = \Gamma(t)$.

unstable for $0 \leq \gamma \leq 1.1$.

Chapter 4

Thermodynamics in Dynamical Chern-Simons Modified Gravity

This chapter contains the discussion on thermodynamics including first law of thermodynamics, GSLT and thermal equilibrium for constant as well as variable Γ in the presence of Bekenstein entropy, logarithmic corrected entropy and power law corrected entropy. The results of this chapter are compiled and submitted in the form of a paper.

4.1 Basic equations

The Eq.(2.11.1) will reduce to Freidmann equations for flat universe as [110]

$$H^2 = \frac{1}{3}\rho + \frac{c^2}{6a^6}, \tag{4.1.1}$$

here c is constant, $H = \frac{\dot{a}(t)}{a(t)}$ is a Hubble parameter and $a(t)$ is the scale factor. Differentiation of Eq.(4.1.1) will give

$$\dot{H} = -\frac{1}{2}\bigg((\rho + p + \Pi) - H\frac{c^2}{a^6}\bigg). \tag{4.1.2}$$

Inserting Eq.(3.1.6) and $p = (\gamma - 1)\rho$ in above equation, we get

$$\frac{\dot{H}}{H^2} = -\frac{1}{2H^2}\bigg(\gamma\big(3H^2 - \frac{c^2}{2a^6}\big)\big(1 - \frac{\Gamma}{3H}\big) + \frac{c^2}{a^6}\bigg). \tag{4.1.3}$$

The deceleration parameter q is of the form

$$q = -\frac{\dot{H}}{H^2} - 1 = \frac{1}{2H^2}\bigg(\gamma\big(3H^2 - \frac{c^2}{2a^6}\big)\big(1 - \frac{\Gamma}{3H}\big) + \frac{c^2}{a^6}\bigg) - 1. \tag{4.1.4}$$

For this model the EoS parameter can be stated as

$$\omega_{\text{eff}} = \frac{p + \Pi}{\rho} = \gamma\big(1 - \frac{\Gamma}{3H}\big) - 1. \tag{4.1.5}$$

In following discussion, we analyze the validity of first law of thermodynamics with Gibbs relation, GSLT and thermodynamical equilibrium by assuming the different entropy corrections such as Bekenstein entropy, logarithmic corrected entropy, power law correction.

4.2 Thermodynamical Analysis with Usual Entropy

For flat FRW universe, Hubble parameter relates with the apparent horizon as $R_{\mathrm{A}} = \frac{1}{H}$. Differentiating the apparent horizon with respect to time, we have

$$\dot{R}_A = -\frac{\dot{H}}{H^2} = \frac{1}{2H^2}\Big(\gamma\big(3H^2 - \frac{c^2}{2a^6}\big)\big(1 - \frac{\Gamma}{3H}\big) + \frac{c^2}{a^6}\Big). \qquad (4.2.1)$$

The differential dE_{A} is the amount of energy crossing the apparent horizon can be evaluated by using Eq.(3.2.3)

$$-dE_{\mathrm{A}} = \frac{\gamma}{2H^2}\big(3H^2 - \frac{c^2}{2a^6}\big). \qquad (4.2.2)$$

From Eq.(3.2.2), the differential of surface entropy at apparent horizon is given by

$$dS_A = \frac{1}{8H^3}\Big(\gamma\big(3H^2 - \frac{c^2}{2a^6}\big)\big(1 - \frac{\Gamma}{3H}\big) + \frac{c^2}{a^6}\Big), \qquad (4.2.3)$$

the above equation with horizon temperature will leads to

$$T_A dS_A = \frac{1}{2H^2}\Big(\gamma\big(3H^2 - \frac{c^2}{2a^6}\big)\big(1 - \frac{\Gamma}{3H}\big) + \frac{c^2}{a^6}\Big). \qquad (4.2.4)$$

Hence Ξ will becomes

$$\Xi = -\frac{\gamma}{2H^2}\big(3H^2 - \frac{c^2}{2a^6}\big) + \frac{1}{2H^2}\Big(\gamma\big(3H^2 - \frac{c^2}{2a^6}\big)\big(1 - \frac{\Gamma}{3H}\big) + \frac{c^2}{a^6}\Big). \qquad (4.2.5)$$

With the help of Eq.(4.2.5) we can observe that first law of thermodynamics is hold when $\Gamma = 3H\Bigg(\frac{c^2}{\gamma a^6}\big(3H^2 - \frac{c^2}{2a^6}\big)^{-1}\Bigg)$.

Further, we discuss the GSLT and thermodynamical equilibrium of an isolated macroscopic physical system having maximum entropy state. Hence, the total entropy S_T must satisfies the following conditions $d(S_{\mathrm{A}} +$

$S_{\rm f}) \geq 0$ i.e., entropy function cannot be decrease and $d(S_{\rm A}+S_{\rm f}) < 0$ i.e., entropy function attains a maximum thermodynamical equilibrium. In this relation, $S_{\rm A}$ and $S_{\rm f}$ appear as the entropy at apparent horizon and the entropy of cosmic fluid enclosed within the horizon, respectively. Using Eq.(4.1.3) we can get $(\Gamma-\Theta)=\frac{6H\dot{H}+\frac{3Hc^2}{a^6}}{\gamma\left(3H^2-\frac{c^2}{2a^6}\right)}$, hence Eq.(3.2.9) leads to the integral

$$\ln\left(\frac{T_{\rm f}}{T_0}\right)=\frac{\gamma-1}{\gamma}\int\frac{\left(2H\dot{H}+\frac{Hc^2}{a^6}\right)}{\left(H^2-\frac{c^2}{6a^6}\right)}dH, \tag{4.2.6}$$

integrating above equation, we obtain

$$T_{\rm f}=T_0\left(H^2-\frac{c^2}{6a^6}\right)^{\frac{\gamma-1}{\gamma}}, \tag{4.2.7}$$

where T_0 is the constant of integration. From Eq.(3.2.8) we can get the differential of fluid entropy as

$$dS_f=\frac{T_0^{-1}}{H^2}\left(H^2-\frac{c^2}{6a^6}\right)^{\frac{1-\gamma}{\gamma}}\left(\left(2\dot{H}+\frac{c^2}{a^6}\right)-\frac{\gamma\dot{H}}{H^2}\left(3H^2-\frac{c^2}{2a^6}\right)\right). \tag{4.2.8}$$

By using Eqs.(4.2.3) and (4.2.8) we have the differential of total entropy as

$$\begin{aligned}\dot{S_T} &= \frac{1}{8H^3}\left(\gamma\left(3H^2-\frac{c^2}{2a^6}\right)\left(1-\frac{\Gamma}{3H}\right)+\frac{c^2}{a^6}\right)+\frac{T_0^{-1}}{H^2}\left(H^2-\frac{c^2}{6a^6}\right)^{\frac{1-\gamma}{\gamma}}\\ &\times \left(\left(2\dot{H}+\frac{c^2}{a^6}\right)-\frac{\gamma\dot{H}}{H^2}\left(3H^2-\frac{c^2}{2a^6}\right)\right),\end{aligned} \tag{4.2.9}$$

where $S_T=S_{\rm A}+S_{\rm f}$. The plot between $\dot{S_T}$ versus γ as shown in Figure **4.1** for three values of T by setting the constant values $H=67$, $c=-1$, $a=-1$ and $q=-0.53$. We observe that $\dot{S_T}\geq 0$ for all values of T leads to the validity of GSLT.

Now, we discuss two scenarios for thermodynamical equilibrium by taking $\Gamma=constant$ and $\Gamma=\Gamma(t)$. Firstly, we will consider Γ as constant and observe the behavior of thermodynamical equilibrium. For constant Γ we

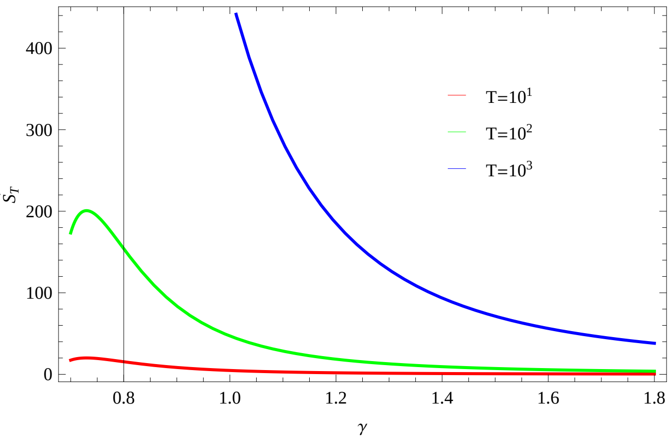

Figure 4.1: Plot of $\dot{S}_T$ versus γ for usual entropy

can get second order differential equation by using Eq.(4.2.9)

$$
\begin{aligned}
\ddot{S}_T \;=\;& -\frac{3\dot{H}\Big(\frac{c^2}{a^6}+\gamma\big(1-\frac{\Gamma}{3H}\big)\big(-\frac{c^2}{2a^6}+3H^2\big)\Big)}{8H^4} - 2\dot{H}\Big(-\frac{c^2}{6a^6}+H^2\Big)^{\frac{1-\gamma}{\gamma}}\\
\times\;& \frac{\Big(-\gamma\big(1-\frac{\Gamma}{3H}\big)\big(3H^2-\frac{c^2}{2a^6}\big)+\frac{\gamma\big(3H^2-\frac{c^2}{2a^6}\big)\big(\frac{c^2}{a^6}+\gamma\big(1-\frac{\Gamma}{3H}\big)\big(3H^2-\frac{c^2}{2a^6}\big)\big)}{2H^2}\Big)}{H^3T_0}\\
+\;& \frac{\Big(-\gamma\big(1-\frac{\Gamma}{3H}\big)\big(3H^2-\frac{c^2}{2a^6}\big)+\frac{\gamma\big(3H^2-\frac{c^2}{2a^6}\big)\big(\frac{c^2}{a^6}+\gamma\big(1-\frac{\Gamma}{3H}\big)\big(3H^2-\frac{c^2}{2a^6}\big)\big)}{2H^2}\Big)}{\gamma H^2T_0}\\
\times\;& \Big(\frac{c^2\dot{a}}{a^7}+2H\dot{H}\Big)(1-\gamma)\Big(H^2-\frac{c^2}{6a^6}\Big)^{-1+\frac{1-\gamma}{\gamma}}+\gamma\Gamma\dot{H}\Big(3H^2-\frac{c^2}{2a^6}\Big)\frac{1}{24H^5}\\
-\;& \frac{6c^2\dot{a}}{8H^3a^7}+\frac{\gamma\big(1-\frac{\Gamma}{3H}\big)\big(\frac{3c^2\dot{a}}{a^7}+6H\dot{H}\big)}{8H^3}+\frac{\big(H^2-\frac{c^2}{6a^6}\big)^{\frac{1-\gamma}{\gamma}}}{H^2T_0}\Big(-\frac{\gamma\Gamma\dot{H}}{3H^2}\\
\times\;& \Big(3H^2-\frac{c^2}{2a^6}\Big)-\frac{\gamma\dot{H}\big(3H^2-\frac{c^2}{2a^6}\big)\big(\frac{c^2}{a^6}+\gamma\big(1-\frac{\Gamma}{3H}\big)\big(3H^2-\frac{c^2}{2a^6}\big)\big)}{H^3}\\
-\;& \gamma\Big(1-\frac{\Gamma}{3H}\Big)\Big(\frac{3c^2\dot{a}}{a^7}+6H\dot{H}\Big)+\frac{\gamma\big(\frac{c^2}{a^6}+\gamma\big(1-\frac{\Gamma}{3H}\big)\big(3H^2-\frac{c^2}{2a^6}\big)\big)}{2H^2}\\
\times\;& \Big(\frac{3c^2\dot{a}}{a^7}+6H\dot{H}\Big)+\Bigg(\frac{-\frac{6c^2\dot{a}}{a^7}+\frac{\gamma\Gamma\dot{H}\left(3H^2-\frac{c^2}{2a^6}\right)}{3H^2}}{2H^2}+\frac{\big(\frac{3c^2\dot{a}}{a^7}+6H\dot{H}\big)}{2H^2}\\
\times\;& \gamma\Big(1-\frac{\Gamma}{3H}\Big)\Bigg)\Big(3H^2-\frac{c^2}{2a^6}\Big)\gamma\Bigg).
\end{aligned}
\tag{4.2.10}
$$

The graphical behavior between $\ddot{S}_T$ versus γ as shown in Figure **4.2** by keeping constant values $H = 67$, $c = -1$, $a = -1$ and $q = -0.53$ for

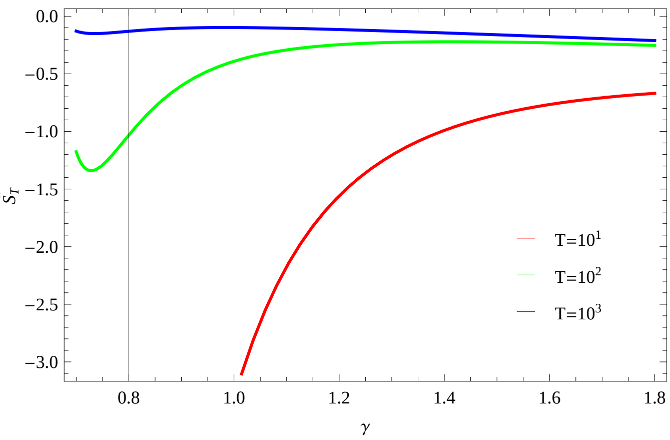

Figure 4.2: Plot of $\ddot{S_T}$ versus γ for usual entropy when Γ=constant.

different values of parameter T. One can see that $\ddot{S_T} < 0$ for all values of T which leads to the validity of thermodynamical equilibrium.

Secondly, we will consider Γ as a variable, i.e., $\Gamma = \Gamma(t)$ for which Eq.(4.2.9) reduces to second order differential equation as

$$
\begin{aligned}
\ddot{S_T} &= -\frac{3\dot{H}\big(\frac{c^2}{a^6}+\gamma(1-\frac{\Gamma}{3H})(-\frac{c^2}{2a^6}+3H^2)\big)}{8H^4} - 2\dot{H}\Big(-\frac{c^2}{6a^6}+H^2\Big)^{\frac{1-\gamma}{\gamma}}\\
&\times \left(\frac{-\gamma\big(3H^2-\frac{c^2}{2a^6}\big)\big(1-\frac{\Gamma}{3H}\big)+\frac{\gamma\left(3H^2-\frac{c^2}{2a^6}\right)\left(\frac{c^2}{a^6}+\gamma\left(3H^2-\frac{c^2}{2a^6}\right)\left(1-\frac{\Gamma}{3H}\right)\right)}{2H^2}}{H^3T_0}\right)\\
&+ \left(\frac{-\gamma\big(3H^2-\frac{c^2}{2a^6}\big)\big(1-\frac{\Gamma}{3H}\big)+\frac{\gamma\left(3H^2-\frac{c^2}{2a^6}\right)\left(\frac{c^2}{a^6}+\gamma\left(3H^2-\frac{c^2}{2a^6}\right)\left(1-\frac{\Gamma}{3H}\right)\right)}{2H^2}}{\gamma H^2T_0}\right)\\
&\times (1-\gamma)\Big(\frac{c^2\dot{a}}{a^7}+2H\dot{H}\Big)\Big(H^2-\frac{c^2}{6a^6}\Big)^{-1+\frac{1-\gamma}{\gamma}} - \frac{6c^2\dot{a}}{8H^3a^7} + \frac{\big(\frac{3c^2\dot{a}}{a^7}+6H\dot{H}\big)}{8H^3}\\
&\times \gamma\Big(1-\frac{\Gamma}{3H}\Big) + \frac{\gamma\big(3H^2-\frac{c^2}{2a^6}\big)\big(\frac{\Gamma\dot{H}}{3H^2}-\frac{\dot{\Gamma}}{3H}\big)}{8H^3} + \frac{\big(H^2-\frac{c^2}{6a^6}\big)^{\frac{1-\gamma}{\gamma}}}{H^2T_0}\\
&\times \left(-\frac{\gamma\dot{H}\big(3H^2-\frac{c^2}{2a^6}\big)\big(\frac{c^2}{a^6}+\big(3H^2-\frac{c^2}{2a^6}\big)\big(1-\frac{\Gamma}{3H}\big)\big)}{H^3} - \gamma\Big(1-\frac{\Gamma}{3H}\Big)\right.\\
&\times \Big(\frac{3c^2\dot{a}}{a^7}+6H\dot{H}\Big) + \frac{\gamma\big(\frac{c^2}{a^6}+\gamma\big(3H^2-\frac{c^2}{2a^6}\big)\big(1-\frac{\Gamma}{3H}\big)\big)\big(\frac{3c^2\dot{a}}{a^7}+6H\dot{H}\big)}{2H^2}\\
&- \gamma\Big(3H^2-\frac{c^2}{2a^6}\Big)\Big(\frac{\Gamma\dot{H}}{3H^2}-\frac{\dot{\Gamma}}{3H}\Big) + \left(\frac{-\frac{6c^2\dot{a}}{a^7}+\gamma\big(1-\frac{\Gamma}{3H}\big)\big(\frac{3c^2\dot{a}}{a^7}+6H\dot{H}\big)}{2H^2}\right.\\
&\left.\left. + \frac{\gamma\big(3H^2-\frac{c^2}{2a^6}\big)\big(\frac{\Gamma\dot{H}}{3H^2}-\frac{\dot{\Gamma}}{3H}\big)}{2H^2}\right)\Big(3H^2-\frac{c^2}{2a^6}\Big)\gamma\right). \qquad (4.2.11)
\end{aligned}
$$

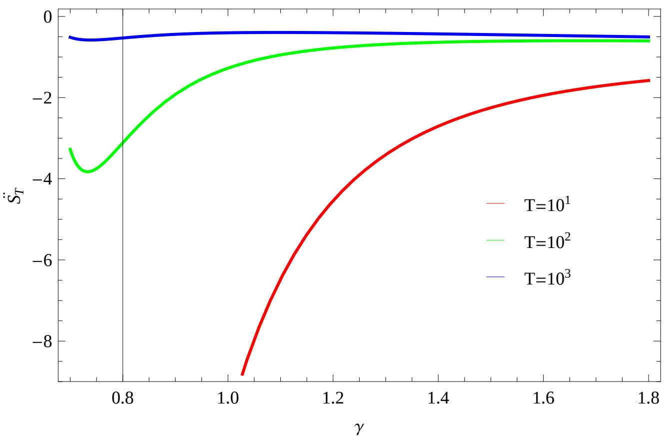

Figure 4.3: Plot of $\ddot{S_T}$ versus γ for usual entropy when $\Gamma = \Gamma(t)$.

The plot between $\ddot{S_T}$ versus γ for three values of T as shown in Figure **4.3** by keeping the constant values same as for constant Γ. It is observed that the thermodynamical is obeying the condition $\ddot{S_T} < 0$ which leads to the validity of thermodynamical equilibrium.

4.3 Entropy Corrections

In this section we will analyze the first law, GSLT and thermodynamical equilibrium for different entropy corrections such as logarithmic entropy, power law correction, Higher order entropy and reyni entropy.

4.3.1 Logarithmic Corrected Entropy

To study the expansion of entropy of the universe, we discuss the addition of entropy related to the horizon. Quantum gravity allows the logarithmic corrections in the presence of thermal equilibrium fluctuations and quantum fluctuations.

The differential form of Logarithmic Corrected Entropy is given by

Eqs.(3.2.2)

$$dS_A = \frac{\Big(\gamma\big(3H^2-\frac{c^2}{2a^6}\big)\big(1-\frac{\Gamma}{3H}\big)+\frac{c^2}{a^6}\Big)\Big(\frac{1}{4HL_p^2}+2\alpha H-16\beta H^3L_p^2\Big)}{2H^2}dt, \qquad (4.3.1)$$

which leads to

$$T_A dS_A = \frac{1}{2H^2}\Big(\gamma\big(3H^2-\frac{c^2}{2a^6}\big)\big(1-\frac{\Gamma}{3H}\big)+\frac{c^2}{a^6}\Big)\Big(\frac{1}{L_\mathrm{p}^2}+8H^2\alpha-64\beta H^4L_\mathrm{p}^2\Big) \qquad (4.3.2)$$

Combining Eqs.(4.2.2) and (4.3.2) we get,

$$\begin{aligned}\Xi &= \frac{1}{2H^2}\Big(\gamma\big(3H^2-\frac{c^2}{2a^6}\big)\big(1-\frac{\Gamma}{3H}\big)+\frac{c^2}{a^6}\Big)\Big(\frac{1}{L_\mathrm{p}^2}+8H^2\alpha-64\beta H^4L_\mathrm{p}^2\Big)\\ &- \frac{\gamma}{2H^2}\big(3H^2-\frac{c^2}{2a^6}\big). \qquad (4.3.3)\end{aligned}$$

From above equation we can observe the validity of first law of thermodynamics for $\Gamma = 3H\Bigg(1-\frac{1}{\gamma}\big(3H^2-\frac{c^2}{a^6}\big)^{-1}\Big(\frac{\gamma(3H^2-\frac{c^2}{2a^6})}{\frac{1}{L_\mathrm{p}^2}+8H^2\alpha-64\beta H^4L_\mathrm{p}^2}-\frac{c^2}{a^6}\Big)\Bigg)$.

Now we will observe the validity GSLT and thermodynamical equilibrium in the presence of Logarithmic Corrected Entropy satisfying the conditions $dS_T \geq 0$ and $d^2S_T < 0$ respectively. Using Eqs.(4.2.8) and (4.3.1) we can get differential of total entropy as

$$\begin{aligned}\dot{S_T} &= \frac{1}{2H^2}\Big(\gamma\big(3H^2-\frac{c^2}{2a^6}\big)\big(1-\frac{\Gamma}{3H}\big)+\frac{c^2}{a^6}\Big)\Big(\frac{1}{4HL_p^2}+2\alpha H-16\beta H^3L_p^2\Big)dt\\ &+ \frac{T_0^{-1}}{H^2}\Big(H^2-\frac{c^2}{6a^6}\Big)^{\frac{1-\gamma}{\gamma}}\Big(\big(2\dot{H}+\frac{c^2}{a^6}\big)-\frac{\gamma\dot{H}}{H^2}\big(3H^2-\frac{c^2}{2a^6}\big)\Big)dt. \qquad (4.3.4)\end{aligned}$$

The plot of $\dot{S_T}$ versus γ with respect to the three values of T by fixing the constant values $H = 67$, $c = -1$, $a = -1$, $q = -0.53$, $\alpha = 1$, $\beta = -0.00001$ and $L_P = 1$ as shown in Figure **4.4**. It can be seen that GSLT is obeying the condition $\dot{S_T} \geq 0$ which leads to the validity of GSLT.

Here, we will discuss the thermodynamical equilibrium by assuming two scenario for particle creation rate Γ. First of all we will observe the validity

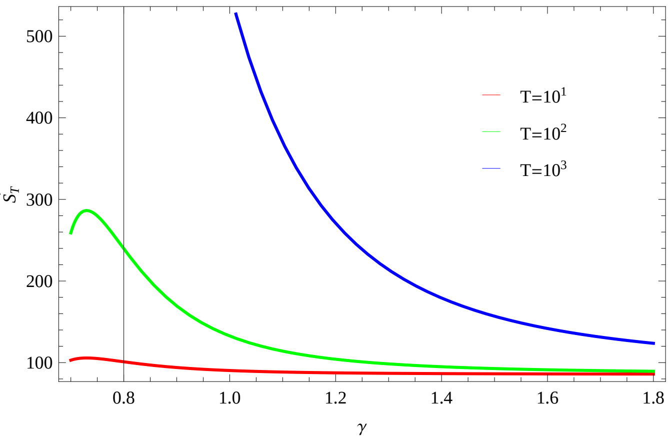

Figure 4.4: Plot of $\dot{S_T}$ versus γ for logarithmic corrected entropy.

of thermodynamical equilibrium by keeping the particle creation rate Γ as constant, in this way second order differential equation can be obtained from Eq.(4.3.4) as

$$\begin{aligned}
\ddot{S_T} &= -\frac{\dot{H}\big(\frac{1}{4HL_p^2}+2\alpha H-16L_p^2\beta H^3\big)\big(\frac{c^2}{a^6}+\gamma\big(1-\frac{\Gamma}{3H}\big)\big(3H^2-\frac{c^2}{2a^6}\big)\big)}{H^3} \\
&- \Bigg(\frac{-\gamma\big(1-\frac{\Gamma}{3H}\big)\big(3H^2-\frac{c^2}{2a^6}\big)+\frac{\gamma\left(3H^2-\frac{c^2}{2a^6}\right)\left(\frac{c^2}{a^6}+\gamma\left(1-\frac{\Gamma}{3H}\right)\left(3H^2-\frac{c^2}{2a^6}\right)\right)}{2H^2}}{H^3T_0}\Bigg) \\
&\times 2\dot{H}\big(H^2-\frac{c^2}{6a^6}\big)^{\frac{1-\gamma}{\gamma}}+(1-\gamma)\big(\frac{c^2\dot{a}}{a^7}+2H\dot{H}\big)\big(H^2-\frac{c^2}{6a^6}\big)^{-1+\frac{1-\gamma}{\gamma}} \\
&\times \Bigg(\frac{-\gamma\big(1-\frac{\Gamma}{3H}\big)\big(3H^2-\frac{c^2}{2a^6}\big)+\frac{\gamma\left(3H^2-\frac{c^2}{2a^6}\right)\left(\frac{c^2}{a^6}+\gamma\left(1-\frac{\Gamma}{3H}\right)\left(3H^2-\frac{c^2}{2a^6}\right)\right)}{2H^2}}{\gamma H^2T_0}\Bigg) \\
&+ \frac{\big(\frac{c^2}{a^6}+\gamma\big(1-\frac{\Gamma}{3H}\big)\big(3H^2-\frac{c^2}{2a^6}\big)\big)\big(2\alpha\dot{H}-\frac{\dot{H}}{4L_p^2H^2}-48L_p^2\beta H^2\dot{H}\big)}{2H^2} \\
&+ \big(\frac{1}{4HL_p^2}+2\alpha H-16L_p^2\beta H^3\big)\Bigg(\frac{-\frac{6c^2\dot{a}}{a^7}+\gamma\big(1-\frac{\Gamma}{3H}\big)\big(\frac{3c^2\dot{a}}{a^7}+6H\dot{H}\big)}{2H^2} \\
&+ \frac{\gamma\Gamma\dot{H}\big(3H^2-\frac{c^2}{2a^6}\big)}{6H^4}\Bigg)+\frac{1}{H^2T_0}\big(H^2-\frac{c^2}{6a^6}\big)^{\frac{1-\gamma}{\gamma}}\Bigg(-\frac{\gamma\Gamma\dot{H}\big(3H^2-\frac{c^2}{2a^6}\big)}{3H^2} \\
&- \frac{\gamma\dot{H}\big(3H^2-\frac{c^2}{2a^6}\big)\big(\frac{c^2}{a^6}+\gamma\big(1-\frac{\Gamma}{3H}\big)\big(3H^2-\frac{c^2}{2a^6}\big)\big)}{H^3}-\gamma\big(1-\frac{\Gamma}{3H}\big) \\
&\times \big(\frac{3c^2\dot{a}}{a^7}+6H\dot{H}\big)+\frac{\gamma\big(\frac{3c^2\dot{a}}{a^7}+6H\dot{H}\big)\big(\frac{c^2}{a^6}+\gamma\big(1-\frac{\Gamma}{3H}\big)\big(3H^2-\frac{c^2}{2a^6}\big)\big)}{2H^2}
\end{aligned}$$

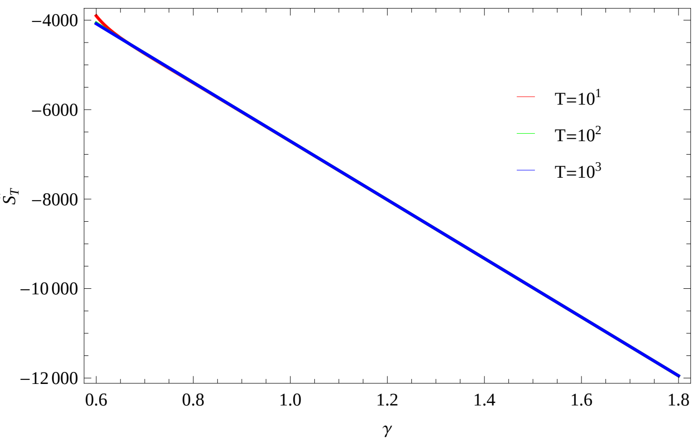

Figure 4.5: Plot of $\ddot{S_T}$ versus γ for logarithmic corrected entropy when Γ=constant.

$$\begin{aligned} &+ \quad \gamma\big(3H^2-\frac{c^2}{2a^6}\big)\Bigg(\frac{\frac{\gamma\Gamma\dot{H}\left(3H^2-\frac{c^2}{2a^6}\right)}{3H^2}+\gamma\big(1-\frac{\Gamma}{3H}\big)\big(\frac{3c^2\dot{a}}{a^7}+6H\dot{H}\big)}{2H^2} \\ &- \quad \frac{6c^2\dot{a}}{2H^2a^7}\Bigg)\Bigg). \end{aligned} \tag{4.3.5}$$

The graphical behavior between $\ddot{S_T}$ and γ as shown in Figure **4.5** for three values of T keeping the constant values as $H = 67$, $c = -1$, $a = -1$, $q = -0.53$, $\alpha = 1$, $\beta = -0.00001$ and $L_P = 1$. One can observe that $\ddot{S_T}$ remains negative at early and present stage, which exhibits the validity of thermodynamical equilibrium. Further, we discuss the stability analysis of thermal equilibrium in the presence of Logarithmic Corrected Entropy by taking Γ as variable for which Eq.(4.3.4) will reduce to second order differential equation as

$$\begin{aligned} \ddot{S_T} \;=\; &-\frac{\dot{H}\big(\frac{1}{4HL_p^2}+2\alpha H-16L_p^2\beta H^3\big)\big(\frac{c^2}{a^6}+\gamma\big(1-\frac{\Gamma}{3H}\big)\big(3H^2-\frac{c^2}{2a^6}\big)\big)}{H^3} \\ &- \left(\frac{-\gamma\big(1-\frac{\Gamma}{3H}\big)\big(3H^2-\frac{c^2}{2a^6}\big)+\frac{\gamma\left(3H^2-\frac{c^2}{2a^6}\right)\left(\frac{c^2}{a^6}+\gamma\left(1-\frac{\Gamma}{3H}\right)\left(3H^2-\frac{c^2}{2a^6}\right)\right)}{2H^2}}{H^3T_0}\right) \\ &\times \; 2\dot{H}\big(H^2-\frac{c^2}{6a^6}\big)^{\frac{1-\gamma}{\gamma}}+(1-\gamma)\big(\frac{c^2\dot{a}}{a^7}+2H\dot{H}\big)\big(H^2-\frac{c^2}{6a^6}\big)^{-1+\frac{1-\gamma}{\gamma}} \end{aligned}$$

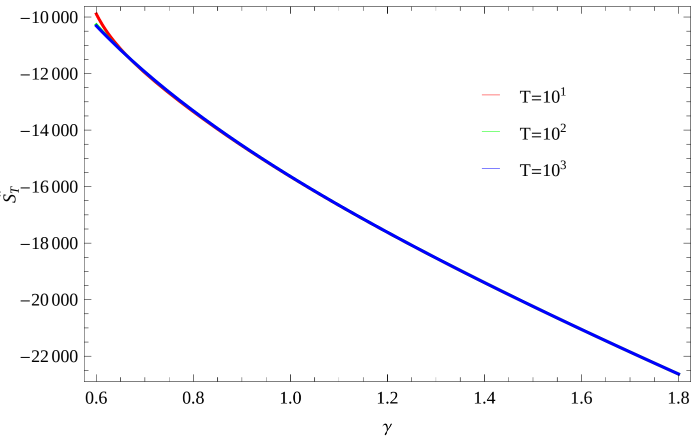

Figure 4.6: Plot of $\ddot{S_T}$ versus γ for logarithmic corrected entropy when $\Gamma = \Gamma(t)$.

$$
\begin{aligned}
\times\;& \Bigg(\frac{-\gamma\big(1-\frac{\Gamma}{3H}\big)\big(3H^2-\frac{c^2}{2a^6}\big)+\frac{\gamma\left(3H^2-\frac{c^2}{2a^6}\right)\left(\frac{c^2}{a^6}+\gamma\left(1-\frac{\Gamma}{3H}\right)\left(3H^2-\frac{c^2}{2a^6}\right)\right)}{2H^2}}{\gamma H^2 T_0}\Bigg)\\
+\;& \frac{\big(\frac{c^2}{a^6}+\gamma\big(1-\frac{\Gamma}{3H}\big)\big(3H^2-\frac{c^2}{2a^6}\big)\big)\big(2\alpha\dot{H}-\frac{\dot{H}}{4L_p^2H^2}-48L_p^2\beta H^2\dot{H}\big)}{2H^2}\\
+\;& \Big(\frac{1}{4HL_p^2}+2\alpha H-16L_p^2\beta H^3\Big)\Bigg(\frac{-\frac{6c^2\dot{a}}{a^7}+\gamma\big(1-\frac{\Gamma}{3H}\big)\big(\frac{3c^2\dot{a}}{a^7}+6H\dot{H}\big)}{2H^2}\\
+\;& \frac{\gamma\big(3H^2-\frac{c^2}{2a^6}\big)\big(\frac{\Gamma\dot{H}}{3H^2}-\frac{\dot{\Gamma}}{3H}\big)}{2H^2}\Bigg)+\frac{1}{H^2T_0}\Big(H^2-\frac{c^2}{6a^6}\Big)^{\frac{1-\gamma}{\gamma}}\Bigg(-\Big(3H^2-\frac{c^2}{2a^6}\Big)\\
\times\;& \frac{\gamma\dot{H}\big(\frac{c^2}{a^6}+\gamma\big(1-\frac{\Gamma}{3H}\big)\big(3H^2-\frac{c^2}{2a^6}\big)\big)}{H^3}-\gamma\Big(1-\frac{\Gamma}{3H}\Big)\Big(\frac{3c^2\dot{a}}{a^7}+6H\dot{H}\Big)\\
+\;& \frac{\gamma\big(\frac{c^2}{a^6}+\gamma\big(1-\frac{\Gamma}{3H}\big)\big(3H^2-\frac{c^2}{2a^6}\big)\big)\big(\frac{3c^2\dot{a}}{a^7}+6H\dot{H}\big)}{2H^2}-\gamma\Big(3H^2-\frac{c^2}{2a^6}\Big)\\
\times\;& \Big(\frac{\Gamma\dot{H}}{3H^2}-\frac{\dot{\Gamma}}{3H}\Big)+\frac{\gamma\big(3H^2-\frac{c^2}{2a^6}\big)\big(-\frac{6c^2\dot{a}}{a^7}+\gamma\big(1-\frac{\Gamma}{3H}\big)\big(\frac{3c^2\dot{a}}{a^7}+6H\dot{H}\big)\big)}{2H^2}\\
+\;& \frac{\gamma^2\big(3H^2-\frac{c^2}{2a^6}\big)^2\big(\frac{\Gamma\dot{H}}{3H^2}-\frac{\dot{\Gamma}}{3H}\big)}{2H^2}\Bigg). \qquad (4.3.6)
\end{aligned}
$$

The plot of $\ddot{S_T}$ versus γ as shown in Figure **4.6** for three values of T by keeping the same values as in above case, we can observe that thermal equilibrium is obeying the condition $\ddot{S_T} < 0$ which leads to the validity of thermodynamical equilibrium.

4.3.2 Power Law Corrected Entropy

Thermodynamics of apparent horizon in the standard FRW cosmology, the geometric entropy is assumed to be proportional to its horizon area ($S_A = \frac{A}{4}$). The quantum corrections provided to the entropy-area relationship lead to the curvature correction in the Einstein-Hilbert action and vice versa.

The differential of Eq.(3.2.2) in the presence of power law corrected entropy is given by

$$dS_A = \frac{1}{2H^2}\Bigg(\gamma\big(3H^2-\frac{c^2}{2a^6}\big)\big(1-\frac{\Gamma}{3H}\big)+\frac{c^2}{a^6}\Bigg)\Bigg(\frac{1}{4HL_{\mathrm{p}}^2}-\frac{K_\delta}{4L_{\mathrm{p}}^2}(2-\frac{\delta}{2})\big(\frac{1}{H}\big)^{3-\delta}\Bigg)dt \quad (4.3.7)$$

which leads to

$$T_A dS_A = \frac{1}{2H^2}\Bigg(\gamma\big(3H^2-\frac{c^2}{2a^6}\big)\big(1-\frac{\Gamma}{3H}\big)+\frac{c^2}{a^6}\Bigg)\Bigg(\frac{1}{L_{\mathrm{p}}^2}-\big(2-\frac{\delta}{2}\big)\frac{K_\delta}{L_p^2}\big(\frac{1}{H}\big)^{2-\delta}\Bigg) \quad (4.3.8)$$

We can get Ξ is of the form

$$\begin{aligned}\Xi &= \frac{1}{2H^2}\Bigg(\gamma\big(3H^2-\frac{c^2}{2a^6}\big)\big(1-\frac{\Gamma}{3H}\big)+\frac{c^2}{a^6}\Bigg)\Bigg(\frac{1}{L_{\mathrm{p}}^2}-\big(2-\frac{\delta}{2}\big)\frac{K_\delta}{L_p^2}\big(\frac{1}{H}\big)^{2-\delta}\Bigg)\\ &- \frac{\gamma}{2H^2}\big(3H^2-\frac{c^2}{2a^6}\big). \end{aligned} \quad (4.3.9)$$

From Eq.(4.3.9) we can analyze that first law of thermodynamics is valid for

$$\Gamma = 3H\Bigg(1-\frac{1}{\gamma}\big(3H^2-\frac{c^2}{2a^6}\big)^{-1}\big(\gamma(3H^2-\frac{c^2}{2a^6})\big)\Bigg(\frac{1}{L_{\mathrm{p}}^2}-(2-\frac{\delta}{2})\frac{K_\delta}{L_p^2}(\frac{1}{H})^{2-\delta}\Bigg)-\frac{c^2}{a^6}\Bigg).$$

We discuss the validity of GSLT and thermodynamical equilibrium with following conditions $dS_T \geq 0$ and $dS_T^2 < 0$ respectively. From Eqs.(4.2.8) and (4.3.8) we can can get the differential of total entropy as

$$\begin{aligned}\dot{S_T} &= \frac{1}{2H^2}\Bigg(\gamma\big(3H^2-\frac{c^2}{2a^6}\big)\big(1-\frac{\Gamma}{3H}\big)+\frac{c^2}{a^6}\Bigg)\Bigg(\frac{1}{4HL_{\mathrm{p}}^2}-\frac{K_\delta}{4L_{\mathrm{p}}^2}(2-\frac{\delta}{2})\big(\frac{1}{H}\big)^{3-\delta}\Bigg)dt\\ &+ \frac{T_0^{-1}}{H^2}\Bigg(H^2-\frac{c^2}{6a^6}\Bigg)^{\frac{1-\gamma}{\gamma}}\Bigg(\big(2\dot{H}+\frac{c^2}{a^6}\big)-\frac{\gamma\dot{H}}{H^2}\big(3H^2-\frac{c^2}{2a^6}\big)\Bigg)dt. \end{aligned} \quad (4.3.10)$$

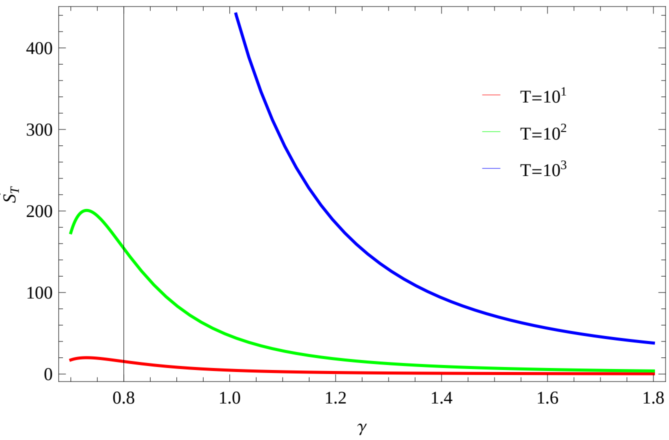

Figure 4.7: Plot of $\dot{S_T}$ versus γ for power law corrected entropy.

The plot of $\dot{S_T}$ versus γ for three values of T by setting the constant values as $H = 67$, $c = -1$, $a = -1$, $q = -0.53$, $\delta = -1$, $r_c = \frac{1}{67}$, and $L_P = 1$ as shown in Figure **4.7**. One can see $\dot{S_T}$ remains positive for all values of T which leads to the validity of GSLT. To find the validity of thermodynamical equilibrium we take $\Gamma = constant$ for which second order differential equation can be express with the help of Eq.(4.3.10)

$$\begin{aligned}
\ddot{S_T} &= -\frac{\Big(\frac{c^2}{a^6}+\gamma\big(1-\frac{\Gamma}{3H}\big)\big(3H^2-\frac{c^2}{2a^6}\big)\Big)\Big(\frac{1}{4HL_p^2}-\frac{\big(2-\frac{\delta}{2}\big)\big(\frac{1}{H}\big)^{3-\delta}K_\delta}{4L_p^2}\Big)\dot{H}}{H^3} \\
&- \left(\frac{-\gamma\big(1-\frac{\Gamma}{3H}\big)\big(3H^2-\frac{c^2}{2a^6}\big)+\frac{\gamma\big(3H^2-\frac{c^2}{2a^6}\big)\big(\frac{c^2}{a^6}+\gamma\big(1-\frac{\Gamma}{3H}\big)\big(3H^2-\frac{c^2}{2a^6}\big)\big)}{2H^2}}{H^3T_0}\right) \\
&\times\ \dot{H}\Big(H^2-\frac{c^2}{6a^6}\Big)^{\frac{1-\gamma}{\gamma}}+(1-\gamma)\Big(\frac{c^2\dot{a}}{a^7}+2H\dot{H}\Big)\Big(H^2-\frac{c^2}{6a^6}\Big)^{-1+\frac{1-\gamma}{\gamma}} \\
&\times \left(\frac{-\gamma\big(1-\frac{\Gamma}{3H}\big)\big(3H^2-\frac{c^2}{2a^6}\big)+\frac{\gamma\big(3H^2-\frac{c^2}{2a^6}\big)\big(\frac{c^2}{a^6}+\gamma\big(1-\frac{\Gamma}{3H}\big)\big(3H^2-\frac{c^2}{2a^6}\big)\big)}{2H^2}}{\gamma H^2T_0}\right) \\
&+ \frac{\Big(\frac{c^2}{a^6}+\gamma\big(1-\frac{\Gamma}{3H}\big)\big(3H^2-\frac{c^2}{2a^6}\big)\Big)\Big(-\frac{\dot{H}}{4HL_p^2}+\frac{(3-\delta)\dot{H}\big(2-\frac{\delta}{2}\big)\big(\frac{1}{H}\big)^{4-\delta}K_\delta}{4L_p^2}\Big)}{2H^2} \\
&+ \left(\frac{1}{4HL_p^2}-\frac{\big(2-\frac{\delta}{2}\big)\big(\frac{1}{H}\big)^{3-\delta}K_\delta}{4HL_p^2}\right)\left(\frac{\big(-\frac{6c^2\dot{a}}{a^7}\big)+\gamma\big(1-\frac{\Gamma}{3H}\big)\big(\frac{3c^2\dot{a}}{a^7}+6H\dot{H}\big)}{2H^2}\right.
\end{aligned}$$

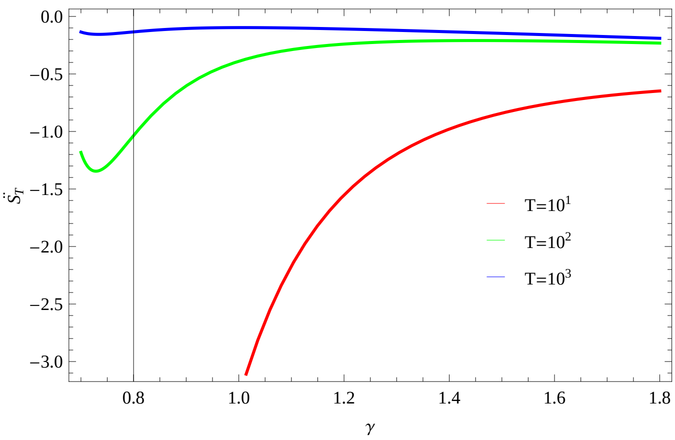

Figure 4.8: Plot of $\ddot{S_T}$ versus γ for power law corrected entropy when Γ=constant.

$$
\begin{aligned}
&+ \frac{\gamma\Gamma\dot{H}\big(3H^2-\frac{c^2}{2a^6}\big)}{6H^4}\Big) + \frac{\big(H^2-\frac{c^2}{6a^6}\big)^{-\frac{1-\gamma}{\gamma}}}{H^2T_0}\Big(\frac{\gamma\Gamma\dot{H}\big(3H^2-\frac{c^2}{2a^6}\big)}{3H^2} - \gamma\big(\frac{3c^2\dot{a}}{a^7}\\
&+ 6H\dot{H}\big)\big(1-\frac{\Gamma}{3H}\big) - \frac{\gamma\dot{H}\big(3H^2-\frac{c^2}{2a^6}\big)\big(\frac{c^2}{a^6}+\gamma\big(1-\frac{\Gamma}{3H}\big)\big(3H^2-\frac{c^2}{2a^6}\big)\big)}{H^3}\\
&+ \frac{\gamma\big(\frac{3c^2\dot{a}}{a^7}+6H\dot{H}\big)\big(\frac{c^2}{a^6}+\gamma\big(1-\frac{\Gamma}{3H}\big)\big(3H^2-\frac{c^2}{2a^6}\big)\big)}{2H^2} + \gamma\big(3H^2-\frac{c^2}{2a^6}\big)\\
&\times \Bigg(\frac{-\frac{6c^2\dot{a}}{a^7}+\frac{\gamma\Gamma\dot{H}\big(3H^2-\frac{c^2}{2a^6}\big)}{3H^2}+\gamma\big(1-\frac{\Gamma}{3H}\big)\big(\frac{3c^2\dot{a}}{a^7}+6H\dot{H}\big)}{2H^2}\Bigg)\Bigg). \qquad (4.3.11)
\end{aligned}
$$

The plot of $\ddot{S_T}$ versus γ as shown in Figure **4.8** for three values of T we observe that thermodynamical equilibrium is obeying the condition $\ddot{S_T} < 0$ for all values of T by assigning the constant values $H = 67$, $c = -1$, $a = -1$, $q = -0.53$, $\delta = -1$, $r_c = \frac{1}{67}$, and $L_P = 1$ which leads to validity of thermodynamical equilibrium. For variable Γ is constant second order differential equation can be expressed by using Eq.(4.3.10)

$$
\begin{aligned}
\ddot{S_T} &= -\frac{\Big(\frac{1}{4HL_p^2}-\frac{\big(2-\frac{\delta}{2}\big)\big(\frac{1}{H}\big)^{3-\delta}K_\delta}{4L_p^2}\Big)\big(\frac{c^2}{a^6}+\gamma\big(1-\frac{\Gamma}{3H}\big)\big(3H^2-\frac{c^2}{2a^6}\big)\big)\dot{H}}{H^3}\\
&- \Bigg(\frac{-\gamma\big(1-\frac{\Gamma}{3H}\big)\big(3H^2-\frac{c^2}{2a^6}\big)+\frac{\gamma\big(3H^2-\frac{c^2}{2a^6}\big)\big(\frac{c^2}{a^6}+\gamma\big(1-\frac{\Gamma}{3H}\big)\big(3H^2-\frac{c^2}{2a^6}\big)\big)}{2H^2}}{H^3T_0}\Bigg)\\
&\times\ 2\dot{H}\big(H^2-\frac{c^2}{6a^6}\big)^{\frac{1-\gamma}{\gamma}} + (1-\gamma)\big(\frac{c^2\dot{a}}{a^7}+2H\dot{H}\big)\big(H^2-\frac{c^2}{6a^6}\big)^{-1+\frac{1-\gamma}{\gamma}}
\end{aligned}
$$

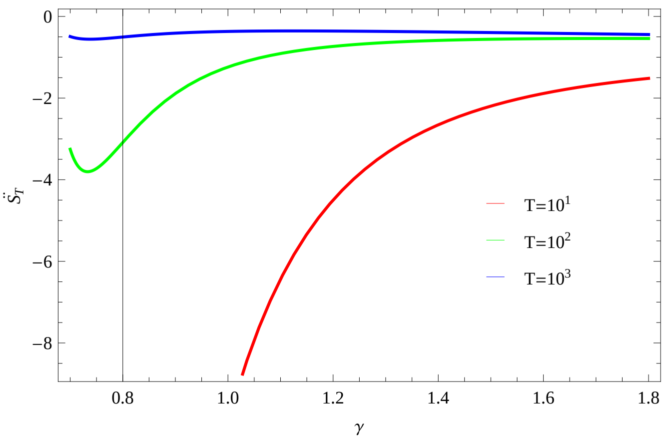

Figure 4.9: Plot of $\ddot{S_T}$ versus γ for power law corrected entropy when $\Gamma = \Gamma(t)$

$$
\begin{aligned}
&\times \left(\frac{-\gamma\big(3H^2-\frac{c^2}{2a^6}\big)\big(1-\frac{\Gamma}{3H}\big)+\frac{\gamma\left(3H^2-\frac{c^2}{2a^6}\right)\left(\frac{c^2}{a^6}+\gamma\left(3H^2-\frac{c^2}{2a^6}\right)\left(1-\frac{\Gamma}{3H}\right)\right)}{2H^2}}{\gamma H^2 T_0}\right)\\
&+ \frac{\Big(-\frac{\dot{H}}{4HL_p^2}+\frac{\dot{H}(3-\delta)\left(2-\frac{\delta}{2}\right)\left(\frac{1}{H}\right)^{4-\delta}K_\delta}{4L_p^2}\Big)\left(\frac{c^2}{a^6}+\gamma\big(3H^2-\frac{c^2}{2a^6}\big)\big(1-\frac{\Gamma}{3H}\big)\right)}{2H^2}\\
&+ \Big(\frac{1}{4HL_p^2}-\frac{\left(2-\frac{\delta}{2}\right)\left(\frac{1}{H}\right)^{3-\delta}K_\delta}{4L_p^2}\Big)\left(\frac{-\frac{6c^2\dot{a}}{a^7}+\gamma\big(1-\frac{\Gamma}{3H}\big)\big(\frac{3c^2\dot{a}}{a^7}+6H\dot{H}\big)}{2H^2}\right.\\
&+ \left.\frac{\gamma\big(3H^2-\frac{c^2}{2a^6}\big)\big(\frac{\Gamma\dot{H}}{3H^2}-\frac{\dot{\Gamma}}{3H}\big)}{2H^2}\right)+\frac{\big(H^2-\frac{c^2}{6a^6}\big)^{\frac{1-\gamma}{\gamma}}}{H^2T_0}\Big(-\big(\frac{3c^2\dot{a}}{a^7}+6H\dot{H}\big)\\
&\times\ \gamma\big(1-\frac{\Gamma}{3H}\big)-\frac{\gamma\dot{H}\big(3H^2-\frac{c^2}{2a^6}\big)\big(\frac{c^2}{a^6}+\gamma\big(3H^2-\frac{c^2}{2a^6}\big)\big(1-\frac{\Gamma}{3H}\big)\big)}{H^3}\\
&+ \frac{\gamma\big(\frac{3c^2\dot{a}}{a^7}+6H\dot{H}\big)\big(\frac{c^2}{a^6}+\gamma\big(3H^2-\frac{c^2}{2a^6}\big)\big(1-\frac{\Gamma}{3H}\big)\big)}{2H^2}-\gamma\big(3H^2-\frac{c^2}{2a^6}\big)\\
&\times \big(\frac{\Gamma\dot{H}}{3H^2}-\frac{\dot{\Gamma}}{3H}\big)+\frac{\gamma\big(3H^2-\frac{c^2}{2a^6}\big)}{2H^2}\Big(+\gamma\big(1-\frac{\Gamma}{3H}\big)\big(\frac{3c^2\dot{a}}{a^7}+6H\dot{H}\big)\\
&+ \gamma\big(3H^2-\frac{c^2}{2a^6}\big)\big(\frac{\Gamma\dot{H}}{3H^2}-\frac{\dot{\Gamma}}{3H}\big)-\frac{6c^2\dot{a}}{a^7}\Big)\Big). \qquad (4.3.12)
\end{aligned}
$$

The plot of $\ddot{S_T}$ versus γ for three values of T by keeping the same values as in above, one can observe easily the validity of thermodynamical equilibrium for all values of T with $\ddot{S_T} < 0$ (Figure **4.9**).

Concluding Remarks

We have studied the thermodynamics on the apparent horizon for gravitationally induced particle creation scenario by assuming entropy corrections (Bakenstein entropy, logarithmic corrected entropy and power law corrected entropy) in DGP braneworld and dynamical Chern-Simons modified gravity. Considering the perfect fluid EoS $p = (\gamma - 1)\rho$, we have analyzed the first law of thermodynamics, GSLT and thermodynamic equilibrium. The results of first law of thermodynamics are given as follows:

For Bakenstein entropy, the first law of thermodynamics holds at apparent horizon when $\Gamma = 3H\big(1 - \frac{(2H-\frac{\epsilon}{r_c})}{2H}\big)$, $\Gamma = 3H\Bigg(\frac{c^2}{\gamma a^6}\big(3H^2 - \frac{c^2}{2a^6}\big)^{-1}\Bigg)$ in DGP braneworld and Chern-Simons gravity respectively.

For logarithmic corrected entropy, it has observed that first law of thermodynamics holds at apparent horizon in DGP braneworld for $\Gamma = 3H\big(1 - \frac{(2H-\frac{\epsilon}{r_c})}{2H}(\frac{1}{L_{\mathrm{p}}^2} + 8H^2\alpha - 64\beta H^4 L_{\mathrm{p}}^2)\big)$. In the presence of Chern-Simons modified gravity first law of thermodynamics is showing the validity when $\Gamma = 3H\Bigg(1 - \frac{1}{\gamma}\big(3H^2 - \frac{c^2}{a^6}\big)^{-1}\Bigg(\frac{\gamma\Big(3H^2-\frac{c^2}{2a^6}\Big)}{\Big(\frac{1}{L_{\mathrm{p}}^2}+8H^2\alpha-64\beta H^4L_{\mathrm{p}}^2\Big)} - \frac{c^2}{a^6}\Bigg)\Bigg)$.

For power law entropy corrected, the first law of thermodynamics satisfies in DGP braneworld when $\Gamma = 3H\big(1 - \frac{(2H-\frac{\epsilon}{r_c})}{2H\Big(\frac{1}{L_{\mathrm{p}}^2}-(2-\frac{\delta}{2})K_\delta R^{2-\delta}\Big)}\big)$ and Chern-Simons gravity for $\Gamma = 3H\Bigg(1 - \frac{1}{\gamma}\big(3H^2 - \frac{c^2}{2a^6}\big)^{-1}\big(\gamma\big(3H^2 - \frac{c^2}{2a^6}\big)\big)\big(\frac{1}{L_{\mathrm{p}}^2} - \big(2 - \frac{\delta}{2}\big)\frac{K_\delta}{L_p^2}\big(\frac{1}{H}\big)^{2-\delta}\big) - \frac{c^2}{a^6}\Bigg)$.

The remaining results have been summarized in the form of tables as follows:

Table 4.1: Validity of GSLT in DGP braneworld gravity

Models	Entropies	Constraints	Validity
DGP braneworld	Bekenstein	for $0.6 \leq \gamma \leq 1.8$	Hold
DGP braneworld	Logarithamic	for $0.6 \leq \gamma \leq 1.8$	Hold
DGP braneworld	Power law	for $0.6 \leq \gamma \leq 1.8$	Hold

Table 4.2: Thermodynamical equilibrium in DGP braneworld gravity when Γ=constant

Models	Entropies	Constraints	Stability
DGP braneworld	Bekenstein	for $0.8 \leq \gamma < 1.2$	Unstable
DGP braneworld	Bekenstein	for $1.2 \leq \gamma \leq 1.8$	Stable
DGP braneworld	Logarithmic	for $0.6 \leq \gamma \leq 1.8$	Stable
DGP braneworld	Power law	for $0.6 \leq \gamma \leq 1.8$	Stable

Table 4.3: Thermodynamical equilibrium in DGP braneworld gravity when $\Gamma = \Gamma(t)$.

Models	Entropies	Constraints	Stability
DGP braneworld	Bekenstein	for $1.0 \leq \gamma < 1.2$	Unstable
DGP braneworld	Bekenstein	for $1.2 \leq \gamma \leq 1.8$	Stable
DGP braneworld	Logarithamic	for $0.6 \leq \gamma \leq 1.8$	Stable
DGP braneworld	Power law	for $0.6 \leq \gamma < 1.0$	Unstable
DGP braneworld	Power law	for $1.0 \leq \gamma \leq 2$	Stable

Table 4.4: Validity of GSLT in Chern-Simons modified gravity

Models	Entropies	Constraints	Validity
Chern-Simons theory	Bekenstein	for $0.7 \leq \gamma \leq 1.8$	Hold
Chern-Simons theory	Logarithamic	for $0.7 \leq \gamma \leq 1.8$	Hold
Chern-Simons theory	Power law	for $0.7 \leq \gamma \leq 1.8$	Hold

Table 4.5: Thermodynamical equilibrium in Chern-Simons modified gravity when Γ=constant.

Models	Entropies	Constraints	Stability
Chern-Simons theory	Bekenstein	for $0.7 \leq \gamma \leq 1.8$	Stable
Chern-Simons theory	Logarithmic	for $0.6 \leq \gamma \leq 1.8$	Stable
Chern-Simons theory	Power law	for $0.7 \leq \gamma \leq 1.8$	Stable

Table 4.6: Thermodynamical equilibrium in Chern-Simons modified gravity when $\Gamma = \Gamma(t)$.

Models	Entropies	Constraints	Stability
Chern-Simons theory	Bekenstein	for $0.7 \leq \gamma \leq 1.8$	Stable
Chern-Simons theory	Logarithamic	for $0.6 \leq \gamma \leq 1.8$	Stable
Chern-Simons theory	Power law	for $0.7 \leq \gamma \leq 1.8$	Stable

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