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Chapter 1

Introduction

1.1 Introduction

Mechanics is very important branch of science which deals with the physical objects at rest or motion.And also fluid mechanics is of great importance in order to understand fluid flow problems. Mechanics can be divided into clases: rigid body mechanics and Mechanics of non-rigid bodies. Moreover, the mechanics of non-rigid bodies can be classified into Fluid Mechanics and Solid Mechanics. The branch of physics which deals with the study of various fluids and forces on them is known as fluid mechanics. Fluid mechanics is consist of two main branches one is Fluid Dynamics and the other is Fluid Statics. As the names suggest fluid dynamics deals with the fluids exist in motion while fluid statics deals with the fluid present at rest.

Fluid mechanics is concerned with understanding,predicting, and controlling the behavior of fluid. Oftenly, it is said that there is both science and art to practise with Fluid Mechanics. Fluid Mechanics ia the part of everyday life it has supported us in many fields such as Oceanography, fluid control system, plasma fluid, applied mathematics, bio-engineering, climatology and food industry. Fluid Mechanics is helpful in understanding the nature of the fluid like velocity, pressure, shear stress, temperature etc. Like other subjects fluid Mechanics also needs

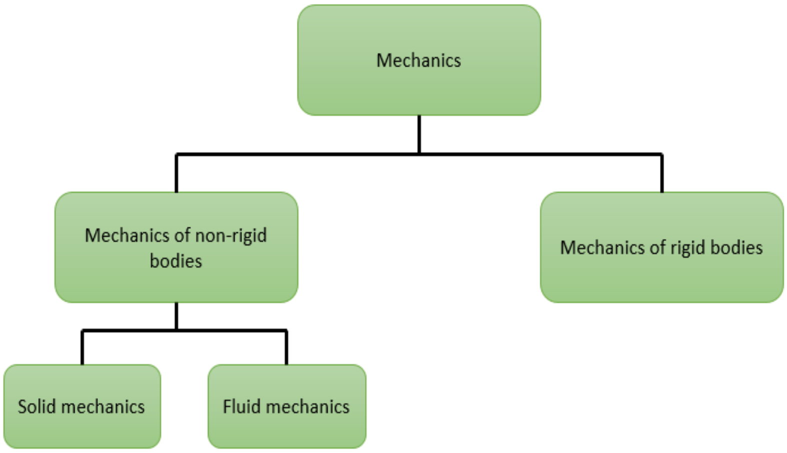

Figure 1.1: Flow chart of Mechanics

mathematical calculations, analysis, physical and graphical description.

Over the last few decades, porous channel flow played very important role in our daily and technological life. A porous channel is made up of porous walls and filled with the fluid. The study of porous channel has fascinated us in many areas such as: irrigation, transpiration and cooling system, gaseous diffusion, biomedical engineering and water purification system.

The human interest in understanding the behavior of fluids is very old. People of that time had enough knowledge to solve their problem using the Fluid Mechanics and they used the Fluid Mechanics to design the arrows, spears, boats and for the protection of flood, irrigation and water supply. The most illustrious name in Engineering is Archimedes, examined the internal structure of the fluid and stated that fluids don't have space inside. He also gave the idea of pressure of the fluid, its continuity and proposed the Laws of Buoyancy. Furthermore, these laws were applied to floating and submerged bodies .

Fluid flow is the movement of gases and liquids, the particles of fluid are freely moving, they are not rigidly combined with each other rather they are loosely packed. Fluids play significant roles in our daily life; they are penetrating in numerous zones of our life on account oftheir promising properties e.g., in energy generation (hydroelectric dams), Aerodynamics of automobile, centrifuges, hydraulics (pump and motor), meteorology (flow of atmosphere), hydrology (flow of water), particle dynamics (blood through arteries), quantum physics etc. H. L. Weissberg [1] explored the development of reverse flow at the wall of the pipe for the values whose solution does not exist and they expressed the motion of fluid with uniform wall suction as dimensionless parameter which describes the distance from the inlet of the pipe. G. S. Beavers [2] presented a theory with effect slip velocity proportional to the exterior velocity gradient and compared these results with experimental results. R. M. Terrill and P.W. Thomas [3] used graphs to present skin-friction and the velocity profiles for laminar flow in a porous pipe. Similarly, Masliyah [4] used Stokes and Berman equation to pass the creeping flow o a solid sphere. They also compared their experimental results with analytical solutions for the composite sphere and found correspondence between these results. N. M. Bujurke [5] analyzed the numerical and theoretical solutions to laminar flow in a circular pipe with constant suction at the wall and with the help of similarity transformation reduces the leading equations to non-linear ordinary differential equation. S. Sisavath et al [6] studied different fluids with different radius of tubes and derived the closed form to estimate the pressure drop of tubes and compared the results with sinusoidally constricted tubes. M. E. Erdogan and C. E.Imrak [7] examined laminar flow of an incompressible fluid and graphically presented the velocity, volume flux and vorticity. O. D. Makinde [8] examined axisymetric wall driven steady flow of a viscous fluid, they used perturbation series to solve the Navier Stokes equations and energy equation. They also find the relation among acceleration Reynolds numbers, wall shear stress and the rate of heat transfer across the wall. M. E. Erdogan and C. E. Imrak [9] considers an incompressible viscous fluid in a uniformly porous pipe with suction and injection and solved the Navier Stoke's equation using Bessel function of first kind to get parameters like velocity, volume flux, vorticity and stress. Moreover, A. Putz and I. A. Frigaard

[10] discussed the sedimentation problems of creeping flow of Bingham fluid i.e., the resistance problem and the mobility problem and find the relationship between these two problems and derive the symmetry and reversibility properties and use augmented Lagrangian algorithm to solve them. C. Pozrikidis [11] used integral form of Stokes flow and boundary-element method to formulate the problem of fluid passing through the walls and find accurate results. A. Maali and B. Bhushan [12] presented the theoretical models and techniques for measurement of the large slip length on superhydrophobic surfaces. The scientists studied the two dimensional flow of incompressible fluid by employing perturbation method under specific boundary conditions of first order slip flow. J. D. Schieber et al [13] found that Sir George Stokes (1856) was the first to derive the time-dependent creeping flow around a sphere. They provide its simple derivation and showed the correlation between creeping flow and linear viscoelastic flow solutions. T. Haroon et al [14] proposed a slit flow model of creeping flow with a uniform form of wall reabsorption and they used equations of motion with boundary conditions and find the solutions for Velocity field, fractional reabsorption, leakage flux etc. A.M. Siddiqui [15] considered the Stokes flow through uniform porous tubes and finds accurate solutions to the Stokes flow and derives the expressions for volume flux, pressure distribution, leakage flux, wall shear stress etc. and draw conclusions for both low and high permeability.

Soundalgekar and Divekar[16] examined the fluid flow problem of two-dimensional , incompressible viscous fluid through circular porous pipe with the help of perturbation method by using the first order slip flow boundary conditions. Masliyah [4] studied creeping flow which is passing through over a solid sphere with the aid of stream function. Bujurke. et al, [17] experimented the fluid flow problem which passed through uniformly porous pipe and calculated the governing equations to non-linear ordinary differential equation by using of similarity transformation. Makinde. et al, [18] studied the perturbation series for low Reynolds number of flow which is passing through uniform porous tube of circular cross section to solve the governing equations. Erdogan and Imrak [19] experimented the problem of uniformly porous pipe with reabsorption and found the exact solution of Navier Stokes equation to get velocity profile by using Bessel function of first kind. Moreover, Putz and Frigaard [20] observed the problems of creeping flow of Bingham fluid related to mobility and resistance, during the sedimentation as well as with the static stability limit.

However, another research was conducted to elaborate the Burger's creeping flow by employing DBEM (stands for direct boundary element method). In DBEM the sphere surface was distributed into various quadrilateral elements[21]. Alongside, distribution of velocity was also estimated. DBEM was used to calculate flow field regarding two and three dimensional. Creeping flow problem is known as stoke;s law as it was firstly elaborated by stokes. A comparison of calculated results was carried out with previous findings in which fluid passing via a peristaltic tube was studied by employing Homotopy method and discovered approximate analytical solution in terms of volumetric flow rate, axial velocity, stream function, mechanical efficiency and pressure gradient. It was done under the approximation of long wavelength. Earlier, mutual effects of slip, compressibility and inertia was studied by Kountouriotis [22]. All these three factors were dependent on the extrudate shape and exit correction factor which is the extra pressure losses in the system. In the exploration, by employing the time-dependent calculation, it was noticed that moderate and higher compressibility in terms of Reynolds numbers, results in steady state and stable solutions with wavy extrudate surface. Alongside, oscillatory extrudate contraction and shapes was noticed with contrasting effect of inertia at moderate slip and Reynolds numbers. Whilst, at different slip numbers and relatively higher Reynolds numbers swell ratios were found. Likewise, simplified Phan-Then-Tanner model was employed to describe the flow of viscoelastic fluid under slip boundary conditions at the wall of the channel [23]. In another scientific study, slip coefficients of porous media was calculated and it was concluded that those were higher in porous systems as compared to non-porous systems [24].

One of their peers, studied the slip flow regime via Navier-Stokes equations with particle based methods and slip velocity boundary [25]. Furthermore, Siddiqui [26] adopted Berman strategy to calculate the creeping flow via porous tubes along with observing the solution and its ultimate effects on the radial component of flow [27]. Under low Reynolds number the slip length reduced the drag. Met-analyses were carried out focusing on the measurement of slip length by direct visualization via inferred [28].

Chapter 2

Preliminaries

In this chapter, some primary definitions will be discussed which are related to given problem.

2.1 Introduction to fluid mechanics

Fluid mechanics is defined as "the branch of science which describes the effect and behavior of various forces on fluids like gas and liquid either existing at resting phase or in motion. Fluid mechanics is divided into three main branches:

1. Static Fluids
2. Dynamics Fluids
3. Kinematics of Fluids

2.2 Fluid

Fluid is defined as"any substance that go under continuous deformation under the forces applied". It's examples are any liquid and gasses form like blood, smoke, honey etc.

2.3 Flow

Flow is described as fluid's quantity which passes through a point per unit time.

2.4 Types of flow

Depending upon the nature of flow, it is classified in many types

2.4.1 Steady flow

A flow is said to be steady if any particular property of fluid at each point remains the same with respect to time and the flow pattern remains same. Mathematically it can be expressed as:

$$\frac{\partial P}{\partial t} = 0$$

Here, P be any property of the fluid e.g velocity, pressure and density etc.

2.4.2 Unsteady flow

A flow is said to be unsteady if any particular property of the fluid does not remain constant with respect to time i.e

$$\frac{\partial P}{\partial t} \neq 0$$

2.4.3 Laminar flow

It is a simple flow in which at the same speed, all the molecules of the fluid flow in the same direction having smooth sliding of adjacent layers. Laminar flow occurs only when fluid flows in parallel layers and there is no disruption in layers.

2.4.4 Turbulent flow

Turbulent flow is a flow in which particles of the fluid cross the path of other particle and do not follow the same path.

2.4.5 Rotational flow and irrotational flow

When fluid particle rotate around its own axes then it is called as rotational flows otherwise it is said to be irrotational.

2.5 Characteristics of fluid

2.5.1 Stress

Stress can be defined as the internal force of particles which they exert continuously on each other. Mathematically it can b defined as

$$stress = \frac{F}{A}$$

Where F is the magnitude of force and A is cross sectional area.

2.5.2 Shear stress

Shear stress or tangential stress is the component of force which is coplanar with the material cross section. Mathematically it can be represented as:

$$\tau = \frac{F}{A}$$

Where F is the magnitude of force and A is area of the material cross section.

2.5.3 Strain

Strain is defined as the fraction of change in length by total length.

$$strain = \frac{\triangle L}{L}$$

$\triangle L$ denotes change in length and L is the actual length of the object.

2.5.4 Pressure

Force applied per unit area is known as pressure. Its mathematical representation is:

$$P = \frac{F}{A}$$

2.5.5 Density

Density can be defined as ratio of mass by volume. Its symbol is ρ it is expressed as:

$$\rho = \frac{m}{v}$$

Here, m is the total mass and v be volume of the substance.

2.5.6 Temperature

Temperature is known as the heat intensity of any substance.

2.6 Viscosity

Thickness measurement of fluid is termed ad viscosity. Viscosity is basically the resistance faced by moving fluid. For example Ketchup is more thick than water so ketchup is more viscous than water.

2.7 Types of viscosity

There are two types of viscosity i.e Dynamic viscosity ad Kinematic viscosity

2.7.1 Dynamic viscosity

The relation between strain and shear stress is called as dynamic viscosity. It is represented by the symbol μ.

$$\mu = \frac{\tau}{\chi}$$

Here τ is the shear stress and χ is the strain.

2.7.2 Kinematic viscosity

Kinematic viscosity is known as the fraction of dynamic viscosity to density. It is represented by the symbol ν and it can be quoted as:

$$\nu = \frac{\mu}{\rho}$$

where μ is dynamic viscosity and ρ is density of the fluid.

2.8 Types of fluid

Generally, fluids are categorized into the following types;

2.8.1 Compressible fluid

When temperature and pressure applied on the fluid then their density does not remain constant. These fluids are called as compressible fluid. Usually gases are considered as compressible fluid.
For incompressible fluids:

$$\frac{\partial \rho}{\partial t} \neq 0 \tag{2.1}$$

where, ρ is a density of the fluid and t is a time.

2.8.2 Incompressible fluid

On temperature and pressure application, the fluids which do not change density are called incompressible fluids e.g liquids.

For incompressible fluids:

$$\frac{\partial \rho}{\partial t} = 0 \tag{2.2}$$

2.8.3 Newtonian fluid

Newtonian fluid is the fluid in which the velocity gradient is linearly proportional to the shear stress i.e.

$$\tau_{yx} = \mu \frac{du}{dy}$$

Here τ represents the value of shear stress μ is the fluid viscosity and $\frac{du}{dy}$ is the rate of change of velocity component along with the stress direction. Air as well as water is the example of Newtonian fluid.

2.8.4 Non-newtonian fluid

When shear stress is not linearly proportional to velocity gradient then the fluid is known as Non-Newtonian fluids e.g. oils, blood, ketchup, custard , toothpaste and paints etc. The flow p roperties of the Non-Newtonian fluids differ from Newtonian fluid.

2.8.5 Ideal fluid

Fluids having negligible thermal conductivity and zero viscosity is known as ideal fluid. Practically no ideal flow exist.

2.9 Slip and no slip conditions

In no slip condition the boundary points of the fluid and the solid boundary have the same velocity. It means that the fluid velocity becomes zero at solid boundary. But in no slip condition the velocity of the boundary points of the fluid and the solid boundary is different and the fluid velocity is not zero at boundary. In general for Newtonian fluids, the slip velocity at surface is proportional to the shear rate at boundary. This velocity is actually connected with the thin layer of stream wise moving fluid in the boundary region just below the surface. slip velocity for solid surface is given as

$$u - u_{wall} = \beta \frac{\partial u}{\partial y} \tag{2.3}$$

Here β is the slip parameter and u is the velocity of fluid.

2.10 Porous channel

Walls of the porous channel contain pores. The walls are made up of matrix solid in nature while the fluid moves across the walls.

2.11 Reynold's number

The ratio of internal forces of fluid to the viscous forces is called as Reynolds number, it is a dimensionless quantity. In fluid mechanics, Reynold's number is used to predict whether the fluid flow is laminar or turbulent. Reynold's number is not a precise physical quantity. Mathematically it can be written as;

$$R_e = \frac{v \rho L}{\mu}$$

Here, R_e is wall Reynolds number, v is velocity and ν Kinematic viscosity and L is the traveled length of fluid. If $R_e < 2000$ then flow is laminar otherwise turbulent and for transient flow $2300 R_e < 4000$

2.12 Equation of continuity

The differential form of the equation is

$$\frac{\partial \rho}{\partial t} + \nabla.(\rho \mathbf{v}) = 0$$

where $\nabla = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)$ If density field is constant then we can write above equation as:

$$\nabla.\mathbf{v} = 0$$

It is compulsory condition for incompressible fluids.

2.13 Navier stokes equation

Newton's second law regarding motion is the base of Navier-Stokes equations. In vector form the equation of fluid differential momentum can be narrated as

$$\rho\frac{D\mathbf{v}}{Dt} = \nabla.\mathbf{T} \tag{2.4}$$

Where $\frac{D\mathbf{v}}{Dt}$ represents material derivative w.r.t time, density is written as ρ and 3 dimensional unsteady velocity of fluid is denoted as $\mathbf{v}$ and $\mathbf{T}$ is the Cauchy stress tensor

$$\mathbf{T} = \begin{bmatrix} \sigma_{xx} & \tau_{yx} & \tau_{zx} \\ \tau_{xy} & \sigma_{yy} & \tau_{zy} \\ \tau_{xz} & \tau_{yz} & \sigma_{zz} \end{bmatrix}$$

Where, σ_{ij} is normal stress component & τ_{ij} is shear stress.

Equation (2.13.1) in component form can be written as

$$\rho(u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} + u\frac{\partial u}{\partial t}) = \frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} \tag{2.5}$$

$$\rho(u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} + \frac{\partial v}{\partial t}) = \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \tau_{zy}}{\partial z} \tag{2.6}$$

$$\rho(u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} + \frac{\partial w}{\partial t}) = \frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \sigma_{zz}}{\partial z} \tag{2.7}$$

where

$$\begin{aligned} \tau_{xy} &= \tau_{yx} = \mu\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right) \\ \sigma_{xx} &= -p - \frac{2}{3}\mu\nabla.v + 2\mu\frac{\partial u}{\partial x} \end{aligned}$$

For incompressible flow with constant viscosity, equations (2.5)-(2.7) become:

$$\rho(u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}+u\frac{\partial u}{\partial t}) = -\frac{\partial p}{\partial x}+\mu(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}) \tag{2.8}$$

$$\rho(u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+w\frac{\partial v}{\partial z}+\frac{\partial v}{\partial t}) = -\frac{\partial p}{\partial y}+\mu\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}+\frac{\partial^2 v}{\partial z^2}\right) \tag{2.9}$$

$$\rho(u\frac{\partial w}{\partial x}+v\frac{\partial w}{\partial y}+w\frac{\partial w}{\partial z}+\frac{\partial w}{\partial t}) = -\frac{\partial p}{\partial z}+\mu\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}+\frac{\partial^2 w}{\partial z^2}\right) \tag{2.10}$$

These equations are known as Navier Stokes equation in Cartesian coordinates.

Chapter 3

Flow Through Circular Porous Pipe With Linear Re-Absorption

3.1 Introduction

This chapter is the review of the paper "Two Dimentional Flow in Renal Tubules with Linear Model", L.N. Achala and K.R. Shreenivas [29]. This chapter includes the study of viscous, incompressible and two dimensional flow within the tube between two parallel flat ultrafiltration membrane having reabsorption at walls. The solution is formed by using stream function(Beramn strategy) to obtain the velocity profiles and pressure gradient.we have been discussed Graphically the effect of velocity components and pressure distribution.

3.2 Problem formulation

Let us consider the fluid is flowing through a porous tube as shown in figure 2.1.The flow is supposed to be two-dimensional, steady, fully developed laminar with negligible body forces. Now governing equations of motion for steady axisymmetric flow without body forces.In this figure x-aixes is the aixes of the tube and y-aixes is the perpendicular of the tube and walls of the tube are porous.

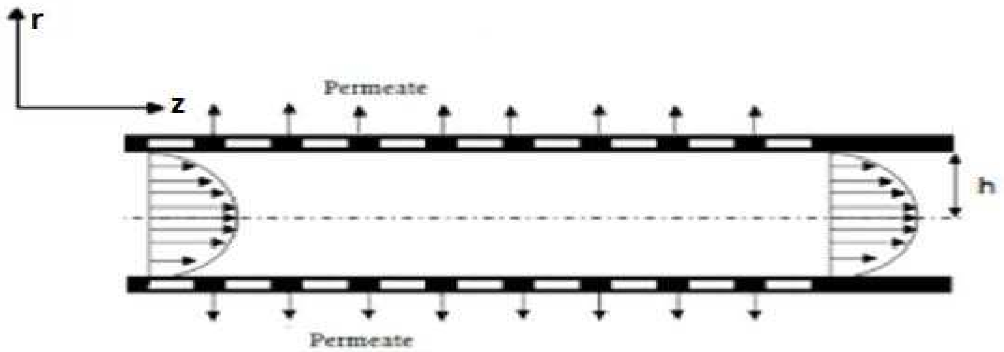

Figure 3.1: Geometry of the porous tube

Assumptions:

$$U = [u_r(r,z), u_z(r,z)]$$

$$p = p\,(r, z)$$

$$\frac{1}{r}\frac{\partial(ru_r)}{\partial r} + \frac{\partial u_z}{\partial z} = 0 \tag{3.1}$$

$$\frac{\partial p}{\partial r} = \mu\left[\frac{\partial^2 u_r}{\partial r^2} + \frac{1}{r}\frac{\partial u_r}{\partial r} - \frac{u_r}{r^2} + \frac{\partial^2 u_r}{\partial z^2}\right] \tag{3.2}$$

$$\frac{\partial p}{\partial z} = \mu[\frac{\partial^2 u_z}{\partial r^2} + \frac{1}{r}\frac{\partial u_r}{\partial r} - \frac{u_r}{r^2} + \frac{\partial^2 u_z}{\partial z^2}] \tag{3.3}$$

Equivalently (3.2) and (3.3) may be expressed as

$$\frac{\partial p}{\partial r} = \mu[\frac{\partial}{\partial r}\{\frac{1}{r}\frac{\partial}{\partial r}(ru_r)\} + \frac{\partial^2 u_r}{\partial z^2}] \tag{3.4}$$

$$\frac{\partial p}{\partial z} = \mu[\frac{1}{r}\{\frac{\partial}{\partial r}(r\frac{\partial u_z}{\partial r})\} + \frac{\partial^2 u_z}{\partial z^2}]$$

From (3.1)

$$\frac{1}{r}\frac{\partial}{\partial r}(ru_r) = -\frac{\partial u_z}{\partial z} \tag{3.5}$$

using this relation in (3.4)

$$\frac{\partial p}{\partial r} = \mu[\frac{\partial}{\partial r}(-\frac{\partial u_z}{\partial z}) + \frac{\partial^2 u_r}{\partial z^2}]$$

or

$$\frac{\partial p}{\partial r} = \mu[\frac{\partial}{\partial z}\{\frac{\partial u_r}{\partial z} - \frac{\partial u_z}{\partial z}\}]$$

On defining vorticity function $\Omega(r, z)$such that we write $\frac{\partial p}{\partial r} = -\mu\frac{\Omega}{\partial z}$

similarly

$$\frac{1}{r}\frac{\partial}{\partial r}(ru_r) + \frac{\partial u_z}{\partial z} = 0$$

we write,

$$\frac{\partial u_r}{\partial r} + \frac{u_r}{r} + \frac{\partial u_z}{\partial z} = 0$$

or

$$\frac{\partial}{\partial z}[\frac{\partial u_r}{\partial r}+\frac{u_r}{r}+\frac{\partial u_z}{\partial z}]=0$$

or

$$\frac{\partial^2 u_r}{\partial r\partial z}+\frac{1}{r}\frac{\partial u_r}{\partial z}+\frac{\partial^2 u_z}{\partial z^2}=0 \tag{3.6}$$

we now express equation (3.3) in alternate form

$$\frac{\partial p}{\partial z}=\mu[\frac{1}{r}\frac{\partial}{\partial r}(r\frac{\partial u_z}{\partial r})+\frac{\partial^2 u_z}{\partial z^2}] \tag{3.7}$$

$$=\mu[\frac{1}{r}\{r\frac{\partial^2 u_z}{\partial r^2}+\frac{\partial u_z}{\partial r}\}+\frac{\partial^2 u_z}{\partial z^2}] \tag{3.8}$$

$$=\mu[\frac{\partial^2 u_z}{\partial r^2}+\frac{1}{r}\frac{\partial u_z}{\partial r}+\frac{\partial^2 u_z}{\partial z^2}] \tag{3.9}$$

using (3.6) by expressing it as

$$\frac{\partial^2 u_z}{\partial z^2}=-\frac{\partial^2 u_r}{\partial r\partial z}-\frac{1}{r}\frac{\partial u_r}{\partial z}$$

in the above,

$$\frac{\partial p}{\partial z}=\mu\{\frac{\partial^2 u_z}{\partial r^2}+\frac{1}{r}\frac{\partial u_z}{\partial r}-\frac{\partial^2 u_r}{\partial r\partial z}-\frac{1}{r}\frac{\partial u_r}{\partial z}\}$$

$$=\mu\{(\frac{\partial^2 u_z}{\partial r^2}-\frac{\partial^2 u_r}{\partial r\partial z})+\frac{1}{r}(\frac{\partial u_z}{\partial r}-\frac{\partial u_r}{\partial z})\}$$

$$=\mu\{\frac{\partial}{\partial r}(\frac{\partial u_z}{\partial r}-\frac{\partial u_r}{\partial r})+\frac{1}{r}(\frac{\partial u_z}{\partial r}-\frac{\partial u_r}{\partial z})\}$$

$$\frac{\partial p}{\partial z}=\mu\{\frac{\partial \Omega}{\partial r}+\frac{1}{r}\Omega\}$$

where,

$$\Omega=\frac{\partial u_z}{\partial r}-\frac{\partial u_r}{\partial z}$$

$$\frac{\partial p}{\partial r}=-\mu\frac{\partial \Omega}{\partial z}$$

$$\frac{\partial p}{\partial z}=\mu[\frac{\partial \Omega}{\partial r}+\frac{\Omega}{r}]$$

where,$\Omega=\frac{\partial u_z}{\partial r}-\frac{\partial u_r}{\partial z}$

Defining stream function$\psi(r,z)$ such that,

$$u_r=\frac{1}{r}\frac{\partial \psi}{\partial z}$$

$$u_z = -\frac{1}{r}\frac{\partial\psi}{\partial r}$$

satisfied identically and the vorticity function.

$$\Omega(r,z) = \frac{\partial u_z}{\partial r} - \frac{\partial u_r}{\partial z}$$

becomes,

$$\Omega(r,z) = \frac{\partial}{\partial r}\{-\frac{1}{r}\frac{\partial\psi}{\partial r}\} - \frac{\partial}{\partial z}(\frac{1}{r}\frac{\partial\psi}{\partial z})$$
$$= \frac{1}{r^2}\frac{\partial\psi}{\partial r} - \frac{1}{r}\frac{\partial^2\psi}{\partial r^2} - \frac{1}{r}\frac{\partial^2\psi}{\partial z^2}$$
$$\Omega(r,z) = -\frac{1}{r}[\frac{\partial^2}{\partial r^2} - \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2}]\psi$$

or

$$\Omega(r,z) = -\frac{F^2\psi}{r}$$

where,

$$F^2 = \frac{\partial^2}{\partial r^2} - \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2}$$

3.3 Equation of motion in terms of stream function

$$\frac{\partial p}{\partial r} = -\mu\frac{\partial}{\partial z}\{-\frac{F^2\psi}{r}\}$$

and

$$\frac{\partial p}{\partial z} = \mu\{(\frac{\partial}{\partial r} + \frac{1}{r})(-\frac{F^2\psi}{r})\}$$

or

$$\frac{\partial p}{\partial r} = \frac{\mu}{r}\frac{\partial}{\partial z}[F^2\psi]$$
$$\frac{\partial p}{\partial z} = -\mu\{(\frac{\partial}{\partial r} + \frac{1}{r})(\frac{F^2\psi}{r})\}$$

since

$$(\frac{\partial}{\partial r} + \frac{1}{r})(\frac{F^2\psi}{r}) = \frac{\partial}{\partial r}(\frac{F^2\psi}{r}) + \frac{1}{r^2}F^2\psi$$
$$= -\frac{1}{r^2}F^2\psi + \frac{1}{r}\frac{\partial}{\partial r}(F^2\psi) + \frac{1}{r^2}F^2\psi$$

$$= \frac{1}{r}\frac{\partial}{\partial r}(F^2\psi)$$

Differentiating first equation w.r.t z and 2nd equation w.r.t r ,we have

$$\frac{\partial^2 p}{\partial r \partial z} = \frac{\mu}{r}\frac{\partial^2}{\partial z^2}(F^2\psi)$$

$$\frac{\partial^2 p}{\partial z \partial r} = -\mu\frac{\partial}{\partial r}[\frac{1}{r}\frac{\partial}{\partial r}(F^2\psi)]$$

subtraction yield

$$0 = \mu[\frac{\partial}{\partial r}\{\frac{1}{r}\frac{\partial}{\partial r}(F^2\psi)\} + \frac{1}{r}\frac{\partial^2}{\partial z^2}(F^2\psi)]$$

or

$$\mu[\frac{1}{r}\frac{\partial^2}{\partial r^2}(F^2\psi) - \frac{1}{r^2}\frac{\partial}{\partial r}(F^2\psi) + \frac{1}{r}\frac{\partial^2}{\partial z^2}(F^2\psi)] = 0 \tag{3.10}$$

$\mu \neq 0$

$$\frac{1}{r}[(\frac{\partial^2}{\partial r^2} - \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2})F^2\psi] = 0$$

$$\frac{1}{r}[F^2(F^2\psi)] = 0$$

$$F^4\psi = 0 \tag{3.11}$$

which is knwon as compatibility equation.

3.4 Basic equation and boundary conditions

The following equations of continuity and motion.

$$\frac{1}{r}\frac{\partial}{\partial r}(ru_r) + \frac{\partial u_z}{\partial z} = 0 \tag{3.12}$$

$$\frac{1}{\mu}\frac{\partial p}{\partial r} = \frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(ru_r)\right) + \frac{\partial^2 u_r}{\partial z^2} \tag{3.13}$$

$$\frac{1}{\mu}\frac{\partial p}{\partial z} = \frac{1}{r}\frac{\partial}{\partial r}(r\frac{\partial u_z}{\partial r}) + \frac{\partial^2 u_z}{\partial z^2} \tag{3.14}$$

where $u_r(r, z)$ and $u_z(r, z)$ are respectively the radial and axial components of velocities , μ is the coefficient of viscosity and $p(r, z)$ is the pressure distribution. These are total of three equations with three unknowns.

The boundary conditions are:

(a) On velocity,

$$at \quad r = 0, \qquad u_r = 0 \tag{3.15}$$

$$at \quad r = 0, \qquad \frac{\partial u_z}{\partial r} = 0 \tag{3.16}$$

$$at \quad r = R, \quad u_r = \varphi_z \tag{3.17}$$

$$at \quad r = R \quad u_z = 0 \tag{3.18}$$

$$at \quad z = 0 \quad 2\pi \int_o^R r u_z(0.r) dr \tag{3.19}$$

(b) On pressure

$$at \quad z = 0 \quad p = p_0 \tag{3.20}$$

and

$$at \quad z = 0 \quad p = p_L \tag{3.21}$$

Now we differentiate equation(4.12)with respect to z and equation (4.13) with respect to r

$$\frac{1}{\mu}\frac{\partial^2 p}{\partial r \partial z} = \frac{\partial}{\partial z}[\frac{\partial}{\partial r}(\frac{1}{r}\frac{\partial}{\partial r}(r u_r))] + \frac{\partial^3 u_r}{\partial z^3}$$

$$\frac{1}{\mu}\frac{\partial^2 p}{\partial r \partial z} = \frac{\partial}{\partial r}[\frac{1}{r}\frac{\partial}{\partial r}(r\frac{\partial u_z}{\partial r}))] + \frac{\partial^3 u_z}{\partial r \partial z^2}$$

eliminating pressure field with subtraction

$$\frac{\partial^2 p}{\partial r \partial z}(\frac{1}{r}\frac{\partial}{\partial r}(r u_r))] + \frac{\partial^3 u_z}{\partial z^3} = \frac{\partial}{\partial r}[\frac{1}{r}\frac{\partial}{\partial r}(r\frac{\partial u_z}{\partial r}))] + \frac{\partial^3 u_z}{\partial r \partial z^2} \tag{3.22}$$

Taking partial derivative of(4.23) w.r.t z

$$\frac{\partial^2 p}{\partial z^2}[\frac{\partial}{\partial r}(\frac{1}{r}\frac{\partial}{\partial r}(r u_r)] + \frac{\partial^4 u_r}{\partial z^4} = \frac{\partial}{\partial z}[\frac{1}{r}\frac{\partial}{\partial r}(r\frac{\partial u_z}{\partial r}))] + \frac{\partial^4 u_z}{\partial r \partial z^3} \tag{3.23}$$

$$[\frac{\partial}{\partial r}(\frac{1}{r}\frac{\partial}{\partial r}(\frac{\partial^2}{\partial z^2}))](r u_r)\frac{1}{r} + \frac{\partial^4}{\partial z^4}(r u_r) = \frac{1}{r}\frac{\partial}{\partial r}(r\frac{\partial^2 u_z}{\partial r \partial z}) + \frac{\partial^4 u_z}{\partial r \partial z^3} \tag{3.24}$$

recall equation (4.11)

$$\frac{1}{r}\frac{\partial}{\partial r}(r u_r) = -\frac{\partial u_z}{\partial z}$$

and

$$\frac{\partial}{\partial r}\left\{\frac{1}{r}\frac{\partial}{\partial r}(ru_r) = -\frac{\partial^2 u_z}{\partial z \partial r}\right\} \tag{3.25}$$

$$\left[\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(\frac{\partial^2}{\partial z^2})\right)\right](ru_r) + \frac{1}{r}\frac{\partial^4}{\partial z^4}(ru_r) = -\frac{1}{r}\frac{\partial}{\partial r}\left[r\left\{\frac{\partial}{\partial r}\left[\frac{1}{r}\frac{\partial}{\partial r}\right](ru_r)\right\}\right] + \frac{\partial^4 u_z}{\partial r \partial z^3} \tag{3.26}$$

Again recall,(4.11)

$$\frac{1}{r}\frac{\partial}{\partial r}(ru_r) = -\frac{\partial u_z}{\partial z}$$

$$\frac{\partial^2}{\partial z^2}\left\{\frac{1}{r}\frac{\partial}{\partial r}(ru_r)\right\} = -\frac{\partial^3 u_z}{\partial z^3} \tag{3.27}$$

or

$$\frac{1}{r}\frac{\partial}{\partial r}\left(\frac{\partial^2}{\partial z^2}\right)(ru_r) = -\frac{\partial^3 u_z}{\partial z^3}$$

and

$$\frac{\partial}{\partial r}\left\{\frac{1}{r}\frac{\partial}{\partial r}\left(\frac{\partial^2}{\partial z^2}\right)\right\}(ru_r) = -\frac{\partial^4 u_z}{\partial r \partial z^3} \tag{3.28}$$

Therefore,

$$\left[\frac{\partial}{\partial r}\left\{\frac{1}{r}\frac{\partial}{\partial r}\frac{\partial^2}{\partial z^2}\right\}\right](ru_r)+\frac{1}{r}\frac{\partial^4}{\partial z^4}(ru_r) = -\frac{1}{r}\frac{\partial}{\partial r}\left[r\left\{\frac{\partial}{\partial r}\left[\frac{1}{r}\frac{\partial}{\partial r}\right]\right\}\right](ru_r)-\frac{\partial}{\partial r}\left\{\frac{1}{r}\frac{\partial}{\partial r}\left(\frac{\partial^2}{\partial z^2}\right)\right\}(ru_r \tag{3.29}$$

hence,

$$\frac{1}{r}\frac{\partial}{\partial r}\left[r\left\{\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}\right)\right\}\right](ru_r) + 2\left[\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}\left(\frac{\partial^2}{\partial z^2}\right)\right)\right](ru_r) + \frac{1}{r}\frac{\partial^4}{\partial z^4}(ru_r) = 0 \tag{3.30}$$

We note that if $u_r = e(r)h(r)$,then the form of (4.24) suggests that an analytic solution may be possible if

$$h(z) = B_0 + B_1 z + B_2 z^2 \tag{3.31}$$

3.5 Solution when radial velocity at wall decrease linearly with z

We solve the compatibility equation

$$F^4\psi = 0 \tag{3.32}$$

Where,

$$F^2 = \left(\frac{\partial^2}{\partial r^2} - \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2}\right)$$

and

$$\frac{1}{r}\frac{\partial \psi}{\partial z} = u_r \quad , \quad u_z = -\frac{1}{r}\frac{\partial \psi}{\partial r}$$

by trying the solution

$$\psi(r,z) = \left(b_0 z + \frac{1}{2} b_1 z^2\right) E(r) + H(r) \tag{3.33}$$

Accordingly,

$$\frac{\partial \psi}{\partial z} = (b_0 + b_1 z)\, E(r)$$

and

$$\frac{\partial^2 \psi}{\partial z^2} = b_1 E(r)$$

$$\frac{\partial \psi}{\partial z} = \left(b_0 z + \frac{1}{2} b_1 z^2\right) E'(r) + H'(r)$$

$$\frac{\partial^2 \psi}{\partial z^2} = \left(b_0 z + \frac{1}{2} b_1 z^2\right) E''(r) + H''(r)$$

and

$$u_r = \frac{1}{r}\frac{\partial \psi}{\partial z} = \frac{1}{r} E(r)\,(b_0 + b_1 z) \tag{3.34}$$

$$u_z = -\frac{1}{r}\frac{\partial \psi}{\partial r} = -\left[\frac{1}{r} E'(r)\left(b_0 z + \frac{1}{2} b_1 z^2\right) + \frac{G'(r)}{r}\right] \tag{3.35}$$

so,

$$F^2\psi = \left[\left(b_0 z + \frac{1}{2} b_1 z^2\right) E''(r) + H''(r)\right] - \frac{1}{r}\left[\left(b_0 z + \frac{1}{2} b_1 z^2\right) E'(r) + H'(r)\right] + b_1 E(r)$$

$$F^2\psi = \left(b_0 z + \frac{1}{2} b_1 z^2\right)\left[\frac{d^2}{dr^2} - \frac{1}{r}\frac{d}{dr}\right] E(r) + \left[\frac{d^2}{dr^2} - \frac{1}{r}\frac{d}{dr}\right] H(r) + b_1 E(r) \tag{3.36}$$

3.6 Stream function form of equation of motion

$$F^4\psi = F^2(F^2\psi) = \left(\frac{\partial^2}{\partial r^2} - \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2}\right) F^2\psi$$

$$\frac{\partial^2}{\partial r^2}\left[\left(b_0 z + \frac{1}{2} b_1 z^2\right)\left[\frac{d^2}{dr^2} - \frac{1}{r}\frac{d}{dr}\right] E(r) + \left[\frac{d^2}{dr^2} - \frac{1}{r}\frac{d}{dr}\right] H(r) + b_1 E(r)\right]$$

$$-\frac{1}{r}\frac{\partial}{\partial r}\left[\left(b_0 z+\frac{1}{2}b_1 z^2\right)\left[\frac{d^2}{dr^2}-\frac{1}{r}\frac{d}{dr}\right]E(r)+\left[\frac{d^2}{dr^2}-\frac{1}{r}\frac{d}{dr}\right]H(r)+b_1E(r)\right]+$$
$$\frac{\partial^2}{\partial z^2}\left[\left(b_0 z+\frac{1}{2}b_1 z^2\right)\left[\frac{d^2}{dr^2}-\frac{1}{r}\frac{d}{dr}\right]E(r)+\left[\frac{d^2}{dr^2}-\frac{1}{r}\frac{d}{dr}\right]H(r)+b_1E(r)\right]$$

$F^4\psi = 0$ leads to

$$\left(b_0 z+\frac{1}{2}b_1 z^2\right)\left[\left[\frac{d^2}{dr^2}-\frac{1}{r}\frac{d}{dr}\right]\left[\frac{d^2}{dr^2}-\frac{1}{r}\frac{d}{dr}\right]E(r)\right]+2a_1[\left[\frac{d^2}{dr^2}-\frac{1}{r}\frac{d}{dr}\right]E(r)+$$
$$\left[\left[\frac{d^2}{dr^2}-\frac{1}{r}\frac{d}{dr}\right][\frac{d^2}{dr^2}-\frac{1}{r}\frac{d}{dr}\right]H(r)=0 \tag{3.37}$$

From (4.27) we get two equations:

$$\left[\frac{d^2}{dr^2}-\frac{1}{r}\frac{d}{dr}\right]\left[\frac{d^2}{dr^2}-\frac{1}{r}\frac{d}{dr}\right]E(r)=0 \tag{3.38}$$

$$\left[\left[\frac{d^2}{dr^2}-\frac{1}{r}\frac{d}{dr}\right]\left[\frac{d^2}{dr^2}-\frac{1}{r}\frac{d}{dr}\right]\right]H(r)$$
$$+2b_1[\left[\frac{d^2}{dr^2}-\frac{1}{r}\frac{d}{dr}\right]E(r)=0 \tag{3.39}$$

We shall solve these differential equations subjects to associated boundary conditions (4.14) to (4.17)

3.7 Boundary conditions using stream functions

$$\psi(r,z)=\left(b_0 z+\frac{b_1}{2}z^2\right)E(r)+H(r)$$

with

$$u_r=\frac{1}{r}\frac{\partial\psi}{\partial z} \quad ; u_z= \quad -\frac{1}{r}\frac{\partial\psi}{\partial r}$$

or

$$u_r=\frac{1}{r}E(r)\left(b_0+b_1 z\right)$$
$$u_z=-\left[\left(b_0 z+\frac{1}{2}b_1 z^2\right)\frac{E'(r)}{r}+\frac{H'(r)}{r}\right]$$

B.C.1 at $r=0,$ $u_r=0$

or at $r=0$, $\frac{E(r)}{r}\left(b_0+b_1 z\right)=0$ or

$$at \qquad r=0 \quad ; \quad \frac{E(r)}{r}=0$$

B.C.2 at $r = 0, \quad -\frac{\partial u_z}{\partial r} = 0$

at $r = 0 \quad -\frac{\partial}{\partial z}\left[\left(b_0 z + \frac{1}{2}b_1 z^2\right)\frac{E'(r)}{r} + \frac{H'(r)}{r}\right] = 0$

or at $r = 0$, $\frac{d}{dr}\left[\frac{1}{r}\frac{d}{dr}\right]E(r) = 0$, $\frac{d}{dr}\left[\frac{1}{r}\frac{d}{dr}\right]H(r) = 0$

B.C.3 at $r = R, \quad u_r = b_0 + b_1 z$

or at $r = R$, $\frac{E(r)}{r}(b_0 + b_1 z) = b_0 + b_1 z$

or at $r = R$, $\frac{E(r)}{r} = 1$

or at $r = R$, $E(r) = R$

B.C.4 at $r = R$, $u_z = 0$

at $r = R, \quad -\left[\left(b_0 z + \frac{1}{2}b_1 z^2\right)\frac{E'(r)}{r} + \frac{H'(r)}{r}\right] = 0$

or at $r = R$, $\frac{1}{r}\frac{d}{dr}E(r) = 0$, $\frac{1}{r}\frac{d}{dr}H(r) = 0$

or at $r = R$, $E'(R) = 0 \qquad H'(R) = 0$

we sumarize the boundary conditions as:

3.8 Boundary condition interms of E(r)

1 at $r = 0 \quad \frac{E(r)}{r} = 0$ or $E(r) = 0$

2 at $r = 0 \quad \frac{d}{dr}\left[\frac{1}{r}\frac{d}{dr}\right]E(r) = 0$

3 at $r = R \quad E(r) = R$

4 at $r = R \quad E'(r) = R$

3.9 Boundary conditions interms of H(r)

1 at $r = 0$;

$$\frac{d}{dr}\left[\frac{1}{r}\frac{d}{dr}\right]H(r) = 0 \tag{3.40}$$

2 at $r = R$; $H'(R) = 0$

3.10 Additional boundary condition for H(r)

at $\quad z=0 \quad 2\pi \int\limits_0^R r u_z(0,r) dr = Q_0$

at $\quad z=0 \quad , \quad \int\limits_0^R \left[\left(b_0 z + \frac{1}{2} b_1 z^2\right) \frac{E'(r)}{r} + \frac{H'(r)}{r} \right] = \frac{Q_0}{2\pi}$

at $\quad z=0 \quad , \quad \int\limits_0^R H'(r) dr = -\frac{Q_0}{2\pi}$

at $\quad z=0 \quad , \quad [H(r)]_0^R = -\frac{Q_0}{2\pi}$

3 $\quad H(R) - H(0) = -\frac{Q_0}{2\pi}$

4 $\quad H(0) = 0$

3.11 Solution of the equation involving E(r)

$$\left(\frac{d^2}{dr^2} - \frac{1}{r}\frac{d}{dr}\right)\left(\frac{d^2}{dr^2} - \frac{1}{r}\frac{d}{dr}\right) E(r) = 0$$

We note that

$$\left(\frac{d^2}{dr^2} - \frac{1}{r}\frac{d}{dr}\right) = r\frac{d}{dr}\left(\frac{1}{r}\frac{d}{dr}\right)$$

and wrtie this equation in alternate form

$$\left[r\frac{d}{dr}\left(\frac{1}{r}\frac{d}{dr}\right)\right]\left[r\frac{d}{dr}\left(\frac{1}{r}\frac{d}{dr}\right)\right] E(r) = 0 \tag{3.41}$$

multiplying by $\frac{1}{r}$ then we get,

$$\frac{d}{dr}\left(\frac{1}{r}\frac{d}{dr}\right)\left[r\frac{d}{dr}\left(\frac{1}{r}\frac{d}{dr}\right)\right] E(r) = 0$$

Integrating once w.r.t r

$$\left(\frac{1}{r}\frac{d}{dr}\right)\left[r\frac{d}{dr}\left(\frac{1}{r}\frac{d}{dr}\right)\right] E(r) = B_1$$

or

$$\left(\frac{d}{dr}\right)\left[r\frac{d}{dr}\left(\frac{1}{r}\frac{d}{dr}\right)\right] E(r) = B_1 r$$

Integrating again w.r.t "r"

$$\left[r\frac{d}{dr}\left(\frac{1}{r}\frac{d}{dr}\right)\right] E(r) = \frac{B_1 r^2}{2} + B_2$$

dividing both sides by r

$$\left[\frac{d}{dr}\left(\frac{1}{r}\frac{d}{dr}\right)\right]E(r) = \frac{B_1 r}{2} + \frac{B_2}{r} \tag{3.42}$$

before,

Integrating one more time w.r.t r ,we use boundary conditon B.C.2

or at $r = 0$, $\frac{d}{dr}\left[\frac{1}{r}\frac{d}{dr}\right]E(r) = 0$

We get $B_2 = 0$ (must)

thus,

$$\left[\frac{d}{dr}\left(\frac{1}{r}\frac{d}{dr}\right)\right]E(r) = \frac{B_1 r}{2} \tag{3.43}$$

Now we integrate again

$$\left(\frac{1}{r}\frac{d}{dr}\right)E(r) = \frac{B_1 r^2}{4} + B_3$$

or

$$\frac{d}{dr}E(r) = \frac{B_1 r^3}{4} + B_3 r \tag{3.44}$$

Integrating fourth time w.r.t "r"

$$E(r) = \frac{B_1 r^4}{16} + \frac{B_3 r^2}{2} + B_4 \tag{3.45}$$

Using the boundary condition B.C.1 at $r = 0$,$E(r) = 0$

We find that $B_4 = 0$

Thus ,

$$E(r) = \frac{B_1 r^4}{16} + \frac{B_3 r^2}{2} \tag{3.46}$$

Applying the B.C. 3 and 4

at $r = R$ $E(R) = R$ we get,

$$R = \frac{B_1 R^4}{16} + \frac{B_3 R^2}{2}$$

$$1 = \frac{B_1 R^3}{16} + \frac{B_3 R}{2} \tag{3.47}$$

at $r = R$, $E'(R) = 0$ leads to,

$$0 = \frac{B_1 R^3}{16} + \frac{B_3 R}{2} \tag{3.48}$$

We solve (4.34) and (4.33) for B_1 and B_3

$$\frac{B_1R^3}{16} + \frac{B_3R}{2} = 1 \tag{3.49}$$

$$\frac{B_1R^3}{16} + \frac{B_3R}{2} = 0 \tag{3.50}$$

multiplying (4.35) by 4 and subtracting from (3.50)

then,

$$B_3R = 4$$

$$B_3 = \frac{4}{R} \tag{3.51}$$

Now,mulityplying (3.50) by $\frac{1}{2}$ and subtracting from (4.35)

then,

$$B_1R^3\left[\frac{1}{16} - \frac{1}{8}\right] = 1$$

$$B_1\left[-\frac{1}{16}\right] = \frac{1}{R^3}$$

$$B_1 = -\frac{16}{R^3} \tag{3.52}$$

using (3.51) and(3.52) in (4.32) ,we get the final form of solution (particular)

$$E(r) = \frac{B_1r^4}{16} + \frac{B_3r^2}{2}$$

$$= -\frac{16r^4}{16R^3} + \frac{1}{2}\left[\frac{4}{R}r^2\right] \tag{3.53}$$

$$E(r) = R\left[2\left(\frac{r}{R}\right)^2 - \left(\frac{r}{R}\right)^4\right] \tag{3.54}$$

3.12 Solution of equation (4.6) for$H(r)$

$$\left[\left[\frac{d^2}{dr^2} - \frac{1}{r}\frac{d}{dr}\right]\left[\frac{d^2}{dr^2} - \frac{1}{r}\frac{d}{dr}\right]\right]H(r)$$
$$+2b_1\left[\frac{d^2}{dr^2} - \frac{1}{r}\frac{d}{dr}\right]E(r) = 0$$

We know that,

$$E(r) = -\frac{r^4}{R^3} + \frac{2r^2}{R}$$

$$\begin{aligned}
\frac{dE(r)}{dr} &= -\frac{4r^3}{R^3} + \frac{4r}{R} \\
\frac{1}{r}\frac{dE(r)}{dr} &= -\frac{4r^2}{R^3} + \frac{4}{R} \\
\frac{d^2E(r)}{dr} &= -\frac{12r^2}{R^3} + \frac{4}{R}
\end{aligned}$$

thus,

$$\begin{aligned}
\frac{d^2E(r)}{dr^2} - \frac{1}{r}\frac{dE(r)}{dr} &= \left(\frac{-12r^2}{R^3} + \frac{4}{R}\right) - \left(-\frac{4r^2}{R^3} + \frac{4}{R}\right) \\
\left(\frac{d^2}{dr^2} - \frac{1}{r}\frac{d}{dr}\right) E(r) &= -\frac{8r^2}{R^3} \qquad (3.55)
\end{aligned}$$

Substituting (4.40) back in (4.6) we obtain

$$\left[\left[\frac{d^2}{dr^2} - \frac{1}{r}\frac{d}{dr}\right]\left[\frac{d^2}{dr^2} - \frac{1}{r}\frac{d}{dr}\right]\right] H(r) + 2b_1\left[-\frac{8r^2}{R^3}\right] = 0$$

or

$$\left[\left[\frac{d^2}{dr^2} - \frac{1}{r}\frac{d}{dr}\right]\left[\frac{d^2}{dr^2} - \frac{1}{r}\frac{d}{dr}\right]\right] H(r) = 16b_1\frac{r^2}{R^3} \qquad (3.56)$$

Introducing an operator

$$\left(\frac{d^2}{dr^2} - \frac{1}{r}\frac{d}{dr}\right) = r\frac{d}{dr}\left(\frac{1}{r}\frac{d}{dr}\right)$$

We express(4.40) in operator form

$$\left[r\frac{d}{dr}\left(\frac{1}{r}\frac{d}{dr}\right)\right]\left[r\frac{d}{dr}\left(\frac{1}{r}\frac{d}{dr}\right)\right] H(r) = 16b_1\frac{r^2}{R^3} \qquad (3.57)$$

Dividing both sides by r,

$$\left[\frac{d}{dr}\left(\frac{1}{r}\frac{d}{dr}\right)\right]\left[r\frac{d}{dr}\left(\frac{1}{r}\frac{d}{dr}\right)\right] H(r) = 16b_1\frac{r}{R^3}$$

Integrating with respect to r leads to,

$$\left(\frac{1}{r}\frac{d}{dr}\right)\left[r\frac{d}{dr}\left(\frac{1}{r}\frac{d}{dr}\right)\right] H(r) = 8b_1\frac{r^2}{R^3} + A_1 \qquad (3.58)$$

Where $A_1$is a constant of integration

Mulityplying r on bothsides ,

$$\left(\frac{d}{dr}\right)\left[r\frac{d}{dr}\left(\frac{1}{r}\frac{d}{dr}\right)\right] H(r) = 8b_1\frac{r^3}{R^3} + A_1 r \qquad (3.59)$$

integration (4.43) with respect to r yields

$$\left[r\frac{d}{dr}\left(\frac{1}{r}\frac{d}{dr}\right)\right]H(r) = 8b_1\frac{r^3}{R^3} + \frac{A_1r^2}{2} + C_1 \tag{3.60}$$

Where C_1 is a constant of integration .

Dividing (4.51) by r on bothsides,

$$\left[r\frac{d}{dr}\left(\frac{1}{r}\frac{d}{dr}\right)\right]H(r) = \frac{8b_1}{4}\frac{r^4}{R^3} + \frac{A_1r^2}{2} + C_1 \tag{3.61}$$

where C_1 is a constant of integration.

Dividing (4.44) by r on bothsides

$$\left[\frac{d}{dr}\left(\frac{1}{r}\frac{d}{dr}\right)\right]H(r) = 2b_1\frac{r^3}{R^3} + \frac{A_1r}{2} + \frac{C_1}{r} \tag{3.62}$$

When the boundary condition (4.9) is applied on (4.9)

at $r = 0$

$$\frac{d}{dr}\left[\frac{1}{r}\frac{d}{dr}\right]H(r) = 0$$

We notice that we must have

$$C_1 = 0 \tag{3.63}$$

Therefore (4.45) reduces to,

$$\left[\frac{d}{dr}\left(\frac{1}{r}\frac{d}{dr}\right)\right]H(r) = 2b_1\frac{r^3}{R^3} + \frac{A_1r}{2} \tag{3.64}$$

Integrating of (4.9) with respect to r, gives us

$$\left(\frac{1}{r}\frac{d}{dr}\right)H(r) = 2b_1\frac{r^4}{4R^3} + \frac{A_1r^2}{4} + D_1 \tag{3.65}$$

where D_1 is an arbitrary constant

On multiplying bothsides of(4.47) by r, we get

$$\frac{dH(r)}{dr} = b_1\frac{r^5}{2R^3} + \frac{A_1r^3}{4} + D_1r \tag{3.66}$$

Now we use the boundary condition (2) at $r = R$; $H'(R) = 0$

Then (4.48) becomes,

$$b_1\frac{R^2}{2} + \frac{A_1R^3}{4} + D_1R = 0 \tag{3.67}$$

or

$$\frac{A_1R^3}{4}+D_1R=-b_1\frac{R^2}{2} \tag{3.68}$$

Where R can be taken common

Finally we get ,after integration of (4.48)

$$H(r)=b_1\frac{r^6}{12R^3}+\frac{A_1r^4}{16}+\frac{D_1r^2}{2}+G \tag{3.69}$$

Where G is a constant added due to integration

Applying the boundary condition at $z=0$, $[H(r)]_0^R=-\frac{Q_0}{2\pi}$

On (4.53) ,it turns out that

$$-\frac{Q_0}{2\pi}=\left[b_1\frac{R^6}{12R^3}+\frac{A_1R^4}{16}+\frac{D_1R^2}{2}+G\right]-\left[b_1\frac{(0)^6}{12R^3}+\frac{A_1(0)^4}{16}+\frac{D_1(0)^2}{2}+G\right]$$

$$-\frac{Q_0}{2\pi}=b_1\frac{R^3}{12}+\frac{A_1R^4}{16}+\frac{D_1R^2}{2}$$

$$\frac{A_1R^2}{16}+\frac{D_1}{2}=-\frac{Q_0}{2\pi R^2}-b_1\frac{R}{12} \tag{3.70}$$

In order to evaluate the constant in (4.49) and (3.70).we solve,

$$\frac{A_1R^2}{4}+D_1=-b_1\frac{R}{2} \tag{3.71}$$

$$+\frac{D_1}{2}=-\frac{Q_0}{2\pi R^2}-b_1\frac{R}{12} \tag{3.72}$$

for A_1 and D_1multiplying first equation with $\frac{1}{2}$ on both sides and subtract it from 2nd equation

$$\frac{A_1R^2}{16}=\frac{Q_0}{2\pi R^2}-2b_1\frac{R}{12}$$

$$A_1=\left[\frac{8Q_0}{\pi R^4}-\frac{8b_1}{3R}\right]$$

$$A_1=8\left[\frac{Q_0}{\pi R^4}-\frac{b_1}{3R}\right]$$

$$A_1=\frac{8}{R}\left[\frac{Q_0}{\pi R^4}-\frac{b_1}{3R}\right] \tag{3.73}$$

putt this value in (3.70)

$$\left[\frac{Q_0}{\pi R^2}-\frac{b_1R}{3}\right]+\frac{D_1}{2}=-\frac{Q_0}{2\pi R^2}-b_1\frac{R}{12} \tag{3.74}$$

$$D_1 = \frac{b_1 R}{6} - \frac{2Q_0}{\pi R^2} \tag{3.75}$$

$$H(0) = 0 \tag{3.76}$$

Therefore,

$$G = 0$$

putt all values of constants in this equation,

$$H(r) = b_1 \frac{r^6}{12R^3} + \frac{A_1 r^4}{16} + \frac{D_1 r^2}{2} + G$$

$$H(r) = b_1 \frac{r^6}{12R^3} + \frac{r^4}{16}\left[\frac{8}{R^2}\left(\frac{Q_0}{\pi R^2} - \frac{b_1 R}{3}\right)\right] + \frac{r^2}{2}\left[\frac{b_1 R}{6} + \frac{2Q_0}{\pi R^2}\right] \tag{3.77}$$

$$H(r) = \frac{b_1 r^6}{12R^3} + \frac{r^4}{R^2}\left[\frac{Q_0}{2\pi R^2} - \frac{b_1 R}{6}\right] + r^2\left[\frac{b_1 R}{12} + \frac{Q_0}{\pi R^2}\right] \tag{3.78}$$

Our assumption was:

$$\psi(r, z) = \left(b_0 z + \frac{b_1}{2} z^2\right) E(r) + H(r)$$

Making use of (3.54) and (3.78) in the solution

$$\psi(r, z) = \left(b_0 z + \frac{b_1 z^2}{2}\right)\left[\frac{2r^2}{R} - \frac{r^4}{R^3}\right] + r^2\left[\frac{b_1 R}{12} - \frac{Q_0}{\pi R^2}\right] + \frac{r^4}{R^2}\left[\frac{Q_0}{2\pi R^2} - \frac{b_1 R}{6}\right] + \frac{b_1 r^6}{R^3} \tag{3.79}$$

3.13 Our radial component of velocity is defined through :

$$u_r = \frac{1}{r}\frac{\partial \psi}{\partial z}$$

$$u_r = [b_0 + b_1 z]\left[\frac{2r}{R} - \left(\frac{r}{R}\right)^3\right] \tag{3.80}$$

3.14 Our axial component of velocity is defined through :

$$u_z = -\frac{1}{r}\frac{\partial \psi}{\partial r}$$

$$u_z = -4\left[\left(\frac{1}{R} - \frac{r^2}{R^3}\right)\left(b_0 z + \frac{b_1 z^2}{2}\right)\right] - 2\left[\frac{b_1 R}{12} - \frac{Q_0}{\pi R^2}\right] - \frac{4r^2}{R^2}\left[\frac{Q_0}{2\pi R^2} - \frac{b_1 R}{6}\right] - \frac{b_1 r^4}{2R^3} \tag{3.81}$$

3.15 Volume flow rate formula

$$Q(z) = \int_0^R 2\pi r u_z(r,z)dr \tag{3.82}$$

We recall that (4.5)

$$u_z = -\frac{1}{r}\frac{\partial \psi}{\partial r} = -\left[\frac{1}{r}E'(r)\left(b_0 z + \frac{1}{2}b_1 z^2\right) + \frac{H'(r)}{r}\right]$$

Differentiating (4.56)

$$\frac{dQ(z)}{dz} = -2\pi \int_0^R r\frac{d}{dz}\left\{\frac{E'(r)}{r}\left(b_0 z + \frac{1}{2}b_1 z^2\right) + \frac{1}{r}H'(r)\right\} dr$$

or

$$\frac{dQ(z)}{dz} = -2\pi \int_0^R E'(r)\left(b_0 + b_1 z\right) dr$$

or

$$\frac{dQ(z)}{dz} = -2\pi\left(b_0 + b_1 z\right) \int_0^R E'(r)dr$$

or

$$\frac{dQ(z)}{dz} = -2\pi\left(b_0 + b_1 z\right)\left[E(r)\right]_0^R$$

$$\frac{dQ(z)}{dz} = -2\pi\left(b_0 + b_1 z\right)\left[E(R) - E(0)\right]$$

since

$$E(r) = \frac{2r^2}{R} - \frac{r^4}{R^3}$$

$$E(r) = 2R - R = R$$

Also

$$E(0) = 0$$

thus,

$$\frac{dQ(z)}{dz} = 2\pi R\,(b_0 + b_1 z) \tag{3.83}$$

$$Q(z) = Q_0 - \pi R\left(2b_0 z + b_1 z^2\right) \tag{3.84}$$

Proof:

$$\int_0^z \frac{dQ(z)}{dz}dz = -2\pi R\int_0^z (b_0 + b_1 z)\,dz$$

$$[Q(z)]_0^z = -2\pi R\left[b_0 z + \frac{b_1 z^2}{2}\right]$$

$$Q(z) - Q(0) = -\pi R\left[2b_0 z + b_1 z^2\right]$$

$$Q(z) = Q(0) - \pi R\left[2b_0 z + b_1 z^2\right]$$

$$\tag{3.85}$$

Chapter 4

Slip Effects on Creeping Flow Through a Tube With Linear Re-Absorption on the Wall

4.1 Introduction

In this chapter we are studying the general solution for two dimensional, laminar and fully developed flow through a tube with linear re-absorption.We will discuss the effects of slip velocity at tubular membrane surface and solve the governing equations to get velocity profile and pressure gradient along with shear stress and slip effects. This parametric equation is simplified by using stream function. From this general solution expression for axial components of velocity , radial components of velocity and slip velocity are developed. We also observed the effect of slip velocity at a porous surface. Their graphical solutions are also observed for different values of slip coefficient $''\lambda''$ and different values of slip velocity.

4.2 Problem formulation

The problem which is analyzed in this chapter involves the steady, laminar, 2-D, in-compressible and axisymmetric fluid flow through the structure of porous tubular membrane. The fluid is supposed to be re-absorped through the surface of the tube. The geometry of flow under consideration is given in Figure 4.1 Assumptions:

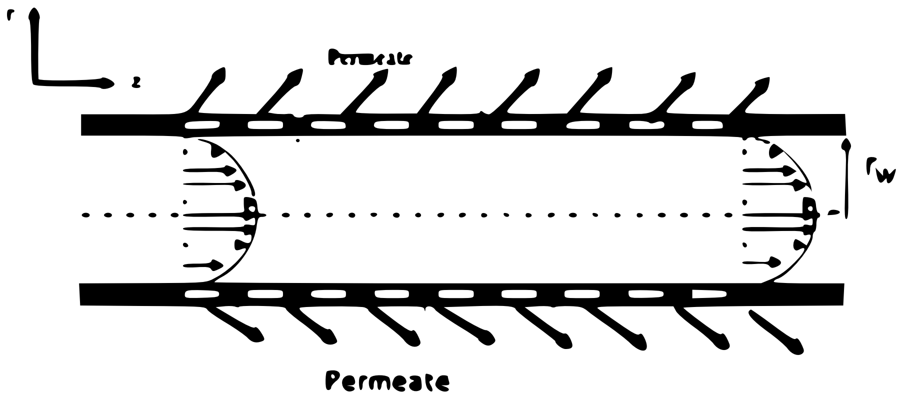

Figure 4.1: Geometry of tubular membrane

We are taking the following assumptions

$$U = [u_r(r,z), u_z(r,z)]$$

$$p = p\,(r,z)$$

$$\frac{1}{r}\frac{\partial(ru_r)}{\partial r} + \frac{\partial u_z}{\partial z} = 0 \tag{4.1}$$

$$\frac{\partial p}{\partial r} = \mu\left[\frac{\partial^2 u_r}{\partial r^2} + \frac{1}{r}\frac{\partial u_r}{\partial r} - \frac{u_r}{r^2} + \frac{\partial^2 u_r}{\partial z^2}\right] \tag{4.2}$$

$$\frac{\partial p}{\partial z} = \mu[\frac{\partial^2 u_z}{\partial r^2} + \frac{1}{r}\frac{\partial u_z}{\partial r} + \frac{\partial^2 u_z}{\partial z^2}] \tag{4.3}$$

Equivalently (4.2) and (4.3) may be expressed as

$$\frac{\partial p}{\partial r} = \mu[\frac{\partial}{\partial r}\{\frac{1}{r}\frac{\partial}{\partial r}(ru_r)\} + \frac{\partial^2 u_r}{\partial z^2}] \tag{4.4}$$

$$\frac{\partial p}{\partial z} = \mu[\frac{1}{r}\{\frac{\partial}{\partial r}(r\frac{\partial u_z}{\partial r})\} + \frac{\partial^2 u_z}{\partial z^2}]$$

From (4.1)

$$\frac{1}{r}\frac{\partial}{\partial r}(ru_r) = -\frac{\partial u_z}{\partial z} \tag{4.5}$$

using this relation in (4.4)

$$\frac{\partial p}{\partial r} = \mu[\frac{\partial}{\partial r}(-\frac{\partial u_z}{\partial z}) + \frac{\partial^2 u_r}{\partial z^2}]$$

or

$$\frac{\partial p}{\partial r} = \mu[\frac{\partial}{\partial z}\{\frac{\partial u_r}{\partial z} - \frac{\partial u_z}{\partial z}\}] \tag{4.6}$$

On defining vorticity function $\Omega(r,z)$such that

we write $\frac{\partial p}{\partial r} = -\mu\frac{\Omega}{\partial z}$

similarly

$$\frac{1}{r}\frac{\partial}{\partial r}(ru_r) + \frac{\partial u_z}{\partial z} = 0$$

we write,

$$\frac{\partial u_r}{\partial r} + \frac{u_r}{r} + \frac{\partial u_z}{\partial z} = 0$$

or

$$\frac{\partial}{\partial z}[\frac{\partial u_r}{\partial r}+\frac{u_r}{r}+\frac{\partial u_z}{\partial z}]=0$$

or

$$\frac{\partial^2 u_r}{\partial r\partial z}+\frac{1}{r}\frac{\partial u_r}{\partial z}+\frac{\partial^2 u_z}{\partial z^2}=0 \tag{4.7}$$

we now express equation (4.3) in alternate form

$$\frac{\partial p}{\partial z}=\mu[\frac{1}{r}\frac{\partial}{\partial r}(r\frac{\partial u_z}{\partial r})+\frac{\partial^2 u_z}{\partial z^2}]$$

$$=\mu[\frac{1}{r}\{r\frac{\partial^2 u_z}{\partial r^2}+\frac{\partial u_z}{\partial r}\}+\frac{\partial^2 u_z}{\partial z^2}]$$

$$=\mu[\frac{\partial^2 u_z}{\partial r^2}+\frac{1}{r}\frac{\partial u_z}{\partial r}+\frac{\partial^2 u_z}{\partial z^2}]$$

using (4.7) by expressing it as

$$\frac{\partial^2 u_z}{\partial z^2}=-\frac{\partial^2 u_r}{\partial r\partial z}-\frac{1}{r}\frac{\partial u_r}{\partial z}$$

in the above,

$$\frac{\partial p}{\partial z}=\mu\{\frac{\partial^2 u_z}{\partial r^2}+\frac{1}{r}\frac{\partial u_z}{\partial r}-\frac{\partial^2 u_r}{\partial r\partial z}-\frac{1}{r}\frac{\partial u_r}{\partial z}\}$$

$$=\mu\{(\frac{\partial^2 u_z}{\partial r^2}-\frac{\partial^2 u_r}{\partial r\partial z})+\frac{1}{r}(\frac{\partial u_z}{\partial r}-\frac{\partial u_r}{\partial z})\}$$

$$=\mu\{\frac{\partial}{\partial r}(\frac{\partial u_z}{\partial r}-\frac{\partial u_r}{\partial r})+\frac{1}{r}(\frac{\partial u_z}{\partial r}-\frac{\partial u_r}{\partial z})\}$$

$$\frac{\partial p}{\partial z}=\mu\{\frac{\partial \Omega}{\partial r}+\frac{1}{r}\Omega\}$$

where,

$$\Omega=\frac{\partial u_z}{\partial r}-\frac{\partial u_r}{\partial z}$$

$$\frac{\partial p}{\partial r}=-\mu\frac{\partial \Omega}{\partial z}$$

$$\frac{\partial p}{\partial z}=\mu[\frac{\partial \Omega}{\partial r}+\frac{\Omega}{r}]$$

where,$\Omega=\frac{\partial u_z}{\partial r}-\frac{\partial u_r}{\partial z}$

Defining stream function$\psi(r,z)$ such that,

$$u_r=\frac{1}{r}\frac{\partial \psi}{\partial z}$$

$$u_z = -\frac{1}{r}\frac{\partial\psi}{\partial r}$$

satisfied identically and the vorticity function.

$$\Omega(r,z) = \frac{\partial u_z}{\partial r} - \frac{\partial u_r}{\partial z}$$

becomes,

$$\Omega(r,z) = \frac{\partial}{\partial r}\{-\frac{1}{r}\frac{\partial\psi}{\partial r}\} - \frac{\partial}{\partial z}(\frac{1}{r}\frac{\partial\psi}{\partial z})$$
$$= \frac{1}{r^2}\frac{\partial\psi}{\partial r} - \frac{1}{r}\frac{\partial^2\psi}{\partial r^2} - \frac{1}{r}\frac{\partial^2\psi}{\partial z^2}$$
$$\Omega(r,z) = -\frac{1}{r}[\frac{\partial^2}{\partial r^2} - \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2}]\psi$$

or

$$\Omega(r,z) = -\frac{F^2\psi}{r} \tag{4.8}$$

where,

$$F^2 = \frac{\partial^2}{\partial r^2} - \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2} \tag{4.9}$$

4.3 Equation of motion in terms of stream function

$$\frac{\partial p}{\partial r} = -\mu\frac{\partial}{\partial z}\{-\frac{F^2\psi}{r}\}$$

and

$$\frac{\partial p}{\partial z} = \mu\{(\frac{\partial}{\partial r} + \frac{1}{r})(-\frac{F^2\psi}{r})\}$$

or

$$\frac{\partial p}{\partial r} = \frac{\mu}{r}\frac{\partial}{\partial z}[F^2\psi]$$
$$\frac{\partial p}{\partial z} = -\mu\{(\frac{\partial}{\partial r} + \frac{1}{r})(\frac{F^2\psi}{r})\}$$

since

$$(\frac{\partial}{\partial r} + \frac{1}{r})(\frac{F^2\psi}{r}) = \frac{\partial}{\partial r}(\frac{F^2\psi}{r}) + \frac{1}{r^2}F^2\psi$$
$$= -\frac{1}{r^2}F^2\psi + \frac{1}{r}\frac{\partial}{\partial r}(F^2\psi) + \frac{1}{r^2}F^2\psi$$

$$= \frac{1}{r}\frac{\partial}{\partial r}(F^2\psi)$$

Differentiating first equation w.r.t z and 2nd equation w.r.t r ,we have

$$\frac{\partial^2 p}{\partial r \partial z} = \frac{\mu}{r}\frac{\partial^2}{\partial z^2}(F^2\psi)$$

$$\frac{\partial^2 p}{\partial z \partial r} = -\mu\frac{\partial}{\partial r}[\frac{1}{r}\frac{\partial}{\partial r}(F^2\psi)]$$

subtraction yield

$$0 = \mu[\frac{\partial}{\partial r}\{\frac{1}{r}\frac{\partial}{\partial r}(F^2\psi)\} + \frac{1}{r}\frac{\partial^2}{\partial z^2}(F^2\psi)]$$

or

$$\mu[\frac{1}{r}\frac{\partial^2}{\partial r^2}(F^2\psi) - \frac{1}{r^2}\frac{\partial}{\partial r}(F^2\psi) + \frac{1}{r}\frac{\partial^2}{\partial z^2}(F^2\psi)] = 0$$

$\mu \neq 0$

$$\frac{1}{r}[(\frac{\partial^2}{\partial r^2} - \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2})F^2\psi] = 0$$

$$\frac{1}{r}[F^2(F^2\psi)] = 0$$

$$F^4\psi = 0 \tag{4.10}$$

which is knwon as compatibility equation.

4.4 Basic equation and boundary conditions

The following equations of continuity and motion.

$$\frac{1}{r}\frac{\partial}{\partial r}(ru_r) + \frac{\partial u_z}{\partial z} = 0 \tag{4.11}$$

$$\frac{1}{\mu}\frac{\partial p}{\partial r} = \frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(ru_r)\right) + \frac{\partial^2 u_r}{\partial z^2} \tag{4.12}$$

$$\frac{1}{\mu}\frac{\partial p}{\partial z} = \frac{1}{r}\frac{\partial}{\partial r}(r\frac{\partial u_z}{\partial r}) + \frac{\partial^2 u_z}{\partial z^2} \tag{4.13}$$

where $u_r(rs, z)$ and $u_z(r, z)$ are respectively the radial and axial components of velocities ,μ is the coefficient of viscosity and $p(r, z)$ is the pressure distribution. These are total of three equations with three unknowns.

The boundary conditions are:

(a) On velocity,

$$at \quad r = 0, \qquad u_r = 0 \tag{4.14}$$

$$at \quad r = 0, \qquad \frac{\partial u_z}{\partial r} = 0 \tag{4.15}$$

$$at \quad r = R, \quad u_r = \varphi_z \tag{4.16}$$

$$at \quad r = R \quad u_z = -\lambda \frac{\partial u_z}{\partial r} \tag{4.17}$$

$$at \quad z = 0 \quad 2\pi \int_o^R r u_z(0.r) dr \tag{4.18}$$

(b) On pressure

$$at \quad z = 0 \quad p = p_0 \tag{4.19}$$

and

$$at \quad z = 0 \quad p = p_L \tag{4.20}$$

Now we differentiate equation (4.12)with respect to z and equation (4.13) with respect to r

$$\frac{1}{\mu}\frac{\partial^2 p}{\partial r \partial z} = \frac{\partial}{\partial z}[\frac{\partial}{\partial r}(\frac{1}{r}\frac{\partial}{\partial r}(r u_r))] + \frac{\partial^3 u_r}{\partial z^3} \tag{4.21}$$

aa

$$\frac{1}{\mu}\frac{\partial^2 p}{\partial r \partial z} = \frac{\partial}{\partial r}[\frac{1}{r}\frac{\partial}{\partial r}(r\frac{\partial u_z}{\partial r}))] + \frac{\partial^3 u_z}{\partial r \partial z^2} \tag{4.22}$$

eliminating pressure field with subtraction

$$\frac{\partial^2 p}{\partial r \partial z}(\frac{1}{r}\frac{\partial}{\partial r}(r u_r))] + \frac{\partial^3 u_z}{\partial z^3} = \frac{\partial}{\partial r}[\frac{1}{r}\frac{\partial}{\partial r}(r\frac{\partial u_z}{\partial r}))] + \frac{\partial^3 u_z}{\partial r \partial z^2} \tag{4.23}$$

Taking partial derivative of(4.23) w.r.t z

$$\frac{\partial^2 p}{\partial z^2}[\frac{\partial}{\partial r}(\frac{1}{r}\frac{\partial}{\partial r}(r u_r)] + \frac{\partial^4 u_r}{\partial z^4} = \frac{\partial}{\partial z}[\frac{1}{r}\frac{\partial}{\partial r}(r\frac{\partial u_z}{\partial r}))] + \frac{\partial^4 u_z}{\partial r \partial z^3}$$

$$[\frac{\partial}{\partial r}(\frac{1}{r}\frac{\partial}{\partial r}(\frac{\partial^2}{\partial z^2})](r u_r)\frac{1}{r} + \frac{\partial^4}{\partial z^4}(r u_r) = \frac{1}{r}\frac{\partial}{\partial r}(r\frac{\partial^2 u_z}{\partial r \partial z}) + \frac{\partial^4 u_z}{\partial r \partial z^3}$$

recall equation (4.11)

$$\frac{1}{r}\frac{\partial}{\partial r}(r u_r) = -\frac{\partial u_z}{\partial z}$$

and

$$\frac{\partial}{\partial r}\left\{\frac{1}{r}\frac{\partial}{\partial r}(ru_r) = -\frac{\partial^2 u_z}{\partial z \partial r}\right\}$$

$$\left[\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(\frac{\partial^2}{\partial z^2})\right)\right](ru_r) + \frac{1}{r}\frac{\partial^4}{\partial z^4}(ru_r) = -\frac{1}{r}\frac{\partial}{\partial r}\left[r\left\{\frac{\partial}{\partial r}\left[\frac{1}{r}\frac{\partial}{\partial r}\right](ru_r)\right\}\right] + \frac{\partial^4 u_z}{\partial r \partial z^3}$$

Again recall,(4.11)

$$\frac{1}{r}\frac{\partial}{\partial r}(ru_r) = -\frac{\partial u_z}{\partial z}$$

$$\frac{\partial^2}{\partial z^2}\left\{\frac{1}{r}\frac{\partial}{\partial r}(ru_r)\right\} = -\frac{\partial^3 u_z}{\partial z^3}$$

or

$$\frac{1}{r}\frac{\partial}{\partial r}\left(\frac{\partial^2}{\partial z^2}\right)(ru_r) = -\frac{\partial^3 u_z}{\partial z^3}$$

and

$$\frac{\partial}{\partial r}\left\{\frac{1}{r}\frac{\partial}{\partial r}\left(\frac{\partial^2}{\partial z^2}\right)\right\}(ru_r) = -\frac{\partial^4 u_z}{\partial r \partial z^3}$$

Therefore,

$$\left[\frac{\partial}{\partial r}\left\{\frac{1}{r}\frac{\partial}{\partial r}\frac{\partial^2}{\partial z^2}\right\}\right](ru_r)\frac{1}{r}\frac{\partial^4}{\partial z^4}(ru_r) = -\frac{1}{r}\frac{\partial}{\partial r}\left[r\left\{\frac{\partial}{\partial r}\left[\frac{1}{r}\frac{\partial}{\partial r}\right]\right\}\right](ru_r) - \frac{\partial}{\partial r}\left\{\frac{1}{r}\frac{\partial}{\partial r}\left(\frac{\partial^2}{\partial z^2}\right)\right\}(ru_r)$$

hence,

$$\frac{1}{r}\frac{\partial}{\partial r}\left[r\left\{\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}\right)\right\}\right](ru_r) + 2\left[\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}\left(\frac{\partial^2}{\partial z^2}\right)\right)\right](ru_r) + \frac{1}{r}\frac{\partial^4}{\partial z^4}(ru_r) = 0 \tag{4.24}$$

We note that if $u_r = e(r)h(r)$,then the form of (4.24) suggests that an analytic solution may be possible if

$$h(z) = B_0 + B_1 z + B_2 z^2$$

4.5 Solution when radial velocity at wall decrease linearly with z

We solve the compatibility equation

$$F^4\psi = 0 \tag{4.25}$$

Where,

$$F^2 = \left(\frac{\partial^2}{\partial r^2} - \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2}\right)$$

and

$$\frac{1}{r}\frac{\partial \psi}{\partial z} = u_r \quad , \quad u_z = -\frac{1}{r}\frac{\partial \psi}{\partial r}$$

by trying the solution

$$\psi(r,z) = \left(b_0 z + \frac{1}{2}b_1 z^2\right) E(r) + H(r)$$

Accordingly,

$$\frac{\partial \psi}{\partial z} = (b_0 + b_1 z)\, E(r)$$

and

$$\frac{\partial^2 \psi}{\partial z^2} = b_1 E(r)$$

$$\frac{\partial \psi}{\partial z} = \left(b_0 z + \frac{1}{2}b_1 z^2\right) E'(r) + H'(r)$$

$$\frac{\partial^2 \psi}{\partial z^2} = \left(b_0 z + \frac{1}{2}b_1 z^2\right) E''(r) + H''(r)$$

and

$$u_r = \frac{1}{r}\frac{\partial \psi}{\partial z} = \frac{1}{r}E(r)\,(b_0 + b_1 z)$$

$$u_z = -\frac{1}{r}\frac{\partial \psi}{\partial r} = -\left[\frac{1}{r}E'(r)\left(b_0 z + \frac{1}{2}b_1 z^2\right) + \frac{G'(r)}{r}\right]$$

so,

$$F^2\psi = \left[\left(b_0 z + \frac{1}{2}b_1 z^2\right) E''(r) + H''(r)\right] - \frac{1}{r}\left[\left(b_0 z + \frac{1}{2}b_1 z^2\right) E'(r) + H'(r)\right] + b_1 E(r)$$

$$F^2\psi = \left(b_0 z + \frac{1}{2}b_1 z^2\right)\left[\frac{d^2}{dr^2} - \frac{1}{r}\frac{d}{dr}\right] E(r) + \left[\frac{d^2}{dr^2} - \frac{1}{r}\frac{d}{dr}\right] H(r) + b_1 E(r) \quad (4.26)$$

4.6 Stream function form of equation of motion

$$F^4\psi = F^2(F^2\psi) = \left(\frac{\partial^2}{\partial r^2} - \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2}\right) F^2\psi$$

$$\frac{\partial^2}{\partial r^2}\left[\left(b_0 z + \frac{1}{2}b_1 z^2\right)\left[\frac{d^2}{dr^2} - \frac{1}{r}\frac{d}{dr}\right] E(r) + \left[\frac{d^2}{dr^2} - \frac{1}{r}\frac{d}{dr}\right] H(r) + b_1 E(r)\right]$$

$$-\frac{1}{r}\frac{\partial}{\partial r}\left[\left(b_0 z+\frac{1}{2}b_1 z^2\right)\left[\frac{d^2}{dr^2}-\frac{1}{r}\frac{d}{dr}\right]E(r)+\left[\frac{d^2}{dr^2}-\frac{1}{r}\frac{d}{dr}\right]H(r)+b_1E(r)\right]+$$
$$\frac{\partial^2}{\partial z^2}\left[\left(b_0 z+\frac{1}{2}b_1 z^2\right)\left[\frac{d^2}{dr^2}-\frac{1}{r}\frac{d}{dr}\right]E(r)+\left[\frac{d^2}{dr^2}-\frac{1}{r}\frac{d}{dr}\right]H(r)+b_1E(r)\right]$$

$F^4\psi=0$ leads to

$$\left(b_0 z+\frac{1}{2}b_1 z^2\right)\left[\left[\frac{d^2}{dr^2}-\frac{1}{r}\frac{d}{dr}\right]\left[\frac{d^2}{dr^2}-\frac{1}{r}\frac{d}{dr}\right]E(r)\right]+2a_1[\left[\frac{d^2}{dr^2}-\frac{1}{r}\frac{d}{dr}\right]E(r)+$$
$$\left[\left[\frac{d^2}{dr^2}-\frac{1}{r}\frac{d}{dr}\right][\frac{d^2}{dr^2}-\frac{1}{r}\frac{d}{dr}\right]H(r)=0 \tag{4.27}$$

From (4.27) we get two equations:

$$\left[\frac{d^2}{dr^2}-\frac{1}{r}\frac{d}{dr}\right]\left[\frac{d^2}{dr^2}-\frac{1}{r}\frac{d}{dr}\right]E(r)=0 \tag{4.28}$$

$$\left[\left[\frac{d^2}{dr^2}-\frac{1}{r}\frac{d}{dr}\right]\left[\frac{d^2}{dr^2}-\frac{1}{r}\frac{d}{dr}\right]\right]H(r)$$
$$+2b_1\left[\frac{d^2}{dr^2}-\frac{1}{r}\frac{d}{dr}\right]E(r)=0 \tag{4.29}$$

We shall solve these differential equations subjects to associated boundary conditions (4.14) to (4.17)

4.7 Boundary conditions using stream functions

$$\psi(r,z)=\left(b_0 z+\frac{b_1}{2}z^2\right)E(r)+H(r)$$

with

$$u_r=\frac{1}{r}\frac{\partial\psi}{\partial z}\quad ; u_z=\quad -\frac{1}{r}\frac{\partial\psi}{\partial r}$$

or

$$u_r=\frac{1}{r}E(r)\left(b_0+b_1 z\right)$$
$$u_z=-\left[\left(b_0 z+\frac{1}{2}b_1 z^2\right)\frac{E'(r)}{r}+\frac{H'(r)}{r}\right]$$

B.C.1 at $r=0,\quad u_r=0$

or at $r=0$, $\frac{E(r)}{r}\left(b_0+b_1 z\right)=0$ or

$$at\qquad r=0\quad ;\quad \frac{E(r)}{r}=0$$

B.C.2 at $r=0$, $-\frac{\partial u_z}{\partial r}=0$

at $r=0$ $-\frac{\partial}{\partial z}\left[\left(b_0 z+\frac{1}{2}b_1 z^2\right)\frac{E'(r)}{r}+\frac{H'(r)}{r}\right]=0$

or at $r=0$, $\frac{d}{dr}\left[\frac{1}{r}\frac{d}{dr}\right]E(r)=0$, $\frac{d}{dr}\left[\frac{1}{r}\frac{d}{dr}\right]H(r)=0$

B.C.3 at $r=R$, $u_r=b_0+b_1 z$

or at $r=R$, $\frac{E(r)}{r}(b_0+b_1 z)=b_0+b_1 z$

or at $r=R$, $\frac{E(r)}{r}=1$

or at $r=R$, $E(r)=R$

B.C.4 at $r=R$, $u_z=-\lambda\frac{\partial u_z}{\partial r}$

at $r=R$,

$$u_z=-\lambda\frac{\partial}{\partial r}\left[\frac{1}{r}\frac{\partial\psi}{\partial r}\right]$$

$$-\left[\left(b_0 z+\frac{1}{2}b_1 z^2\right)\frac{E'(r)}{r}+\frac{H'(r)}{r}\right]=\lambda\frac{\partial}{\partial r}\left[\left(b_0 z+\frac{1}{2}b_1 z^2\right)\frac{E'(r)}{r}+\frac{H'(r)}{r}\right]$$

$$-\left[\left(b_0 z+\frac{1}{2}b_1 z^2\right)\frac{E'(r)}{r}+\frac{H'(r)}{r}\right]=\left[\left(b_0 z+\frac{1}{2}b_1 z^2\right)\frac{rE''(r)-E'(r)}{r^2}+\frac{rh''(r)-h'(r)}{r^2}\right]$$

$$b_0 z\frac{E'(r)}{r}+\frac{1}{2}b_1 z^2\frac{E'(r)}{r}+\frac{h'(r)}{r}=-\lambda\left[b_0 z\frac{E''(r)}{r}-b_0 z\frac{E'(r)}{r^2}+\frac{1}{2}b_1 z^2\frac{E''(r)}{r}-\frac{1}{2}b_1 z^2\frac{E'(r)}{r^2}\right]$$
$$-\lambda\left[\frac{h''(r)}{r}-\frac{h'(r)}{r^2}\right] \tag{4.30}$$

$$\left[b_0\frac{E'(r)}{r}-\lambda b_0\frac{E'(r)}{r^2}+\lambda b_0\frac{E''(r)}{r}\right]z+\frac{z^2}{2}\left[b_1\frac{E'(r)}{r}-\lambda b_1\frac{E'(r)}{r^2}+\lambda b_1\frac{E''(r)}{r}\right]$$
$$+\frac{h'(r)}{r}+\lambda\frac{h''(r)}{r}-\lambda\frac{h'(r)}{r^2}=0$$

or at $r=R$

$$\frac{E'(r)}{r}-\lambda\frac{E'(r)}{r^2}+\lambda\frac{E''(r)}{r}=0$$

$$\Rightarrow\left(1-\frac{\lambda}{r^2}\right)E'(r)=-\lambda E''(r)$$

$$\frac{h'(r)}{r}+\lambda\frac{h''(r)}{r}-\lambda\frac{h'(r)}{r^2}=0$$

$$\Rightarrow\left(1-\frac{\lambda}{r}\right)h'(r)=-\lambda h''(r)$$

or at $r = R$, $E'(R) = 0$ $H'(R) = 0$

we sumarize the boundary conditions as:

4.8 Boundary condition interms of E(r)

1 at $r = 0$ $\frac{E(r)}{r} = 0$ or $E(r) = 0$

2 at $r = 0$ $\frac{d}{dr}\left[\frac{1}{r}\frac{d}{dr}\right]E(r) = 0$

3 at $r = R$ $E(r) = R$

4 at $r = R$ $\left(1 - \frac{\lambda}{r}\right)E'(r) = -\lambda E''(r)$

4.9 Boundary conditions interms of H(r)

1 at $r = 0$;

$$\frac{d}{dr}\left[\frac{1}{r}\frac{d}{dr}\right]H(r) = 0$$

2 at $r = R$; $\left(1 - \frac{\lambda}{r}\right)H'(R) = -\lambda h''(r)$

4.10 Additional boundary condition for H(r)

at $z = 0$ $2\pi \int\limits_0^R r u_z(0, r) dr = Q_0$

at $z = 0$, $\int\limits_0^R \left[\left(b_0 z + \frac{1}{2}b_1 z^2\right)\frac{E'(r)}{r} + \frac{H'(r)}{r}\right] = \frac{Q_0}{2\pi}$

at $z = 0$, $\int\limits_0^R H'(r) dr = -\frac{Q_0}{2\pi}$

at $z = 0$, $[H(r)]_0^R = -\frac{Q_0}{2\pi}$

3 $H(R) - H(0) = -\frac{Q_0}{2\pi}$

4 $H(0) = 0$

4.11 Solution of the equation involving E(r)

$$\left(\frac{d^2}{dr^2}-\frac{1}{r}\frac{d}{dr}\right)\left(\frac{d^2}{dr^2}-\frac{1}{r}\frac{d}{dr}\right)E(r)=0$$

We note that

$$\left(\frac{d^2}{dr^2}-\frac{1}{r}\frac{d}{dr}\right)=r\frac{d}{dr}\left(\frac{1}{r}\frac{d}{dr}\right)$$

and wrtie this equation in alternate form

$$\left[r\frac{d}{dr}\left(\frac{1}{r}\frac{d}{dr}\right)\right]\left[r\frac{d}{dr}\left(\frac{1}{r}\frac{d}{dr}\right)\right]E(r)=0$$

multiplying by $\frac{1}{r}$ then we get,

$$\frac{d}{dr}\left(\frac{1}{r}\frac{d}{dr}\right)\left[r\frac{d}{dr}\left(\frac{1}{r}\frac{d}{dr}\right)\right]E(r)=0$$

Integrating once w.r.t r

$$\left(\frac{1}{r}\frac{d}{dr}\right)\left[r\frac{d}{dr}\left(\frac{1}{r}\frac{d}{dr}\right)\right]E(r)=D_1$$

or

$$\left(\frac{d}{dr}\right)\left[r\frac{d}{dr}\left(\frac{1}{r}\frac{d}{dr}\right)\right]E(r)=D_1r$$

Integrating again w.r.t "r"

$$\left[r\frac{d}{dr}\left(\frac{1}{r}\frac{d}{dr}\right)\right]E(r)=\frac{D_1r^2}{2}+D_2$$

dividing both sides by r

$$\left[\frac{d}{dr}\left(\frac{1}{r}\frac{d}{dr}\right)\right]E(r)=\frac{D_1r}{2}+\frac{D_2}{r}$$

before,

Integrating one more time w.r.t r ,we use boundary conditon B.C.2

or at $r=0$, $\frac{d}{dr}\left[\frac{1}{r}\frac{d}{dr}\right]E(r)=0$

We get $D_2=0$ (must)

thus,

$$\left[\frac{d}{dr}\left(\frac{1}{r}\frac{d}{dr}\right)\right]E(r)=\frac{D_1r}{2}$$

Now we integrate again

$$\left(\frac{1}{r}\frac{d}{dr}\right)E(r) = \frac{D_1 r^2}{4} + D_3$$

or

$$\frac{d}{dr}E(r) = \frac{D_1 r^3}{4} + D_3 r$$

Integrating fourth time w.r.t "r"

$$E(r) = \frac{D_1 r^4}{16} + \frac{D_3 r^2}{2} + D_4 \tag{4.31}$$

Using the boundary condition B.C.1 at $r = 0$,$E(r) = 0$

We find that $D_4 = 0$

Thus ,

$$E(r) = \frac{D_1 r^4}{16} + \frac{D_3 r^2}{2} \tag{4.32}$$

Applying the B.C. 3 and 4

at $r = R \quad E(R) = R$ we get,

$$R = \frac{D_1 R^4}{16} + \frac{D_3 R^2}{2}$$

$$1 = \frac{D_1 R^3}{16} + \frac{D_3 R}{2} \tag{4.33}$$

at $r = R \quad , \quad E'(R) = 0$ leads to,

$$0 = \frac{D_1 R^3}{16} + \frac{D_3 R}{2} \tag{4.34}$$

We solve (4.34) and (4.33) for D_1 and D_3

$$\frac{D_1 R^3}{16} + \frac{D_3 R}{2} = 1 \tag{4.35}$$

at $r = R$

$$\frac{E'(r)}{r} - \lambda\frac{E'(r)}{r^2} + \lambda\frac{E''(r)}{r} = 0 \tag{4.36}$$

$$E(r) = \frac{D_1 r^4}{16} + \frac{D_3 r^2}{2}$$

$$E'(r) = \frac{D_1 r^3}{4} + D_3 r$$
$$E''(r) = \frac{3}{4} D_1 r^2 + D_3$$

from(4.36)

$$\frac{1}{r}\left[\frac{D_1 r^3}{4} + D_3 r\right] - \frac{\lambda}{r^2}\left[\frac{D_1 r^3}{4} + D_3 r\right] + \frac{\lambda}{r}\left[\frac{D_1 r^3}{4} + D_3 r\right] = 0$$

at r=R

$$\frac{D_1 R^2}{4} + D_3 - \frac{\lambda D_1 R}{4} + \frac{\lambda D_3}{R} + \frac{3\lambda D_1 R}{4} + \frac{\lambda D_3}{R} = 0$$
$$D_1\left[\frac{R^2}{4} + \frac{\lambda R}{2}\right] + D_3\left[1 + \frac{2\lambda}{R}\right]$$

from(4.35)

$$\frac{D_1 R^3}{16} + \frac{D_3 R}{2} = 1$$
$$\frac{D_3 R}{2} = 1 - \frac{D_1 R^3}{16}$$
$$D_3 = \frac{1}{R}\left(2 - \frac{D_1 R^3}{8}\right)$$
$$D_1\left[\frac{R^2}{4} + \frac{3R}{4}\lambda - \frac{R}{4}\lambda\right] + D_3\left[1 + \frac{2\lambda}{R}\right] = 0$$
$$D_1\left[\frac{R^2}{4} + \frac{3R}{4}\lambda - \frac{R}{4}\lambda\right] + \frac{1}{R}\left(2 - \frac{D_1 R^3}{8}\right)\left[1 + \frac{2\lambda}{R}\right]$$
$$D_1\left[\frac{R^2}{4} + \frac{3R}{4}\lambda - \frac{R}{4}\lambda\right]\left(\frac{2}{R} - \frac{D_1 R^2}{8}\right)\left[1 + \frac{2\lambda}{R}\right]$$
$$D_1\left[\frac{R^2}{4} + \frac{3R}{4}\lambda - \frac{R}{4}\lambda\right] + \left[\frac{2}{R} + \frac{4\lambda}{R^2} - \frac{D_1 R^2}{8} - \frac{D_1 R\lambda}{4}\right]$$
$$D_1\left[\frac{R^2}{4} - \frac{R^2}{8} + \frac{3\lambda R}{4} - \frac{\lambda R}{4} - \frac{\lambda R}{4}\right] + \frac{2}{R} + \frac{4\lambda}{R^2} = 0$$
$$D_1\left[\frac{R^2}{8} + \frac{\lambda R}{4}\right] = -\frac{2}{R} - \frac{4\lambda}{R^2}$$
$$D_1 = \left(-\frac{2}{R} - \frac{4\lambda}{R^2}\right)\left(\frac{8}{R^2} + \frac{4}{\lambda R}\right)$$
$$D_1 = -\frac{32}{R^3} - \frac{8}{\lambda R} - \frac{32\lambda}{R^4}$$

$$D_3 = \frac{1}{R}\left(2 - \frac{D_1 R^3}{8}\right)$$

$$= \frac{2}{R} - \frac{D_1 R^2}{8}$$

$$= \frac{2}{R} - \frac{R^2}{8}\left[-\frac{32}{R^3} - \frac{8}{\lambda R^2} - \frac{32\lambda}{R^4}\right]$$

$$D_3 = \frac{2}{R} + \frac{4}{R} + \frac{1}{\lambda} + \frac{4\lambda}{R^2}$$

$$D_3 = \frac{6}{R} + \frac{1}{\lambda} + \frac{4\lambda}{R^2}$$

$$\left(\frac{1}{R} - \frac{\lambda}{R^2}\right) E'(R) = -\lambda \frac{E''(R)}{R}$$

$$\left(1 - \frac{\lambda}{R}\right) E'(R) = -\lambda E''(R)$$

$$\left(1 - \frac{\lambda}{R}\right)\left(\frac{D_1 R^3}{4} + D_3 R\right) = -\lambda\left[\frac{3D_1 R^2}{4} + D_3\right]$$

$$\frac{D_1 R^3}{4} + D_3 R - \frac{\lambda D_1 R^2}{4} - \lambda D_3 = -\frac{3D_1 \lambda R^2}{4} - \lambda D_3$$

$$\frac{D_1 R^3}{4} + \frac{D_1 \lambda R^2}{2} + D_3 R = 0$$

$$D_1\left[\frac{R^3}{4} + \frac{\lambda R^2}{2}\right] + D_3 R = 0 \tag{4.37}$$

$$\frac{D_1 R^3}{16} + \frac{D_3 R}{2} = 1 \tag{4.38}$$

subtract equation (4.38)from equation (4.37)

$$D_1\left[\frac{R^3}{16} + \frac{\lambda R^2}{4}\right] = -1$$

$$D_1 = -\frac{16}{R^3 + 4\lambda R^2}$$

$$\frac{D_1 R^3}{16} + \frac{D_3 R}{2} = 1$$

putt the value of D_1

$$\frac{R^3}{16}\left[\frac{-16}{R^3 + 4\lambda R^2} + \frac{D_3 R}{2}\right] = 1$$

$$\frac{D_3 R}{2} = 1 + \frac{R^3}{R^3 + 4\lambda R^2}$$
$$D_3 = \frac{2}{R} + \frac{2R^2}{R^3 + 4\lambda R^2}$$

now,

$$E(r) = \frac{D_1 r^4}{16} + \frac{D_3 r^2}{2}$$
$$E(r) = \frac{-r^4}{16}\left[\frac{-16}{R^3 + 4\lambda R^2} + \frac{r^2}{2}\right] + \left[\frac{2R^2}{R^3 + 4\lambda R^2} + \frac{2}{R}\right]$$
$$E(r) = -\frac{-r^4}{R^2}\frac{1}{4\lambda + R} + \frac{2r^2}{R}\frac{(2\lambda + R)}{(4\lambda + R)} \tag{4.39}$$

4.12 Solution of equation (4.6) for$H(r)$

$$\left[\left[\frac{d^2}{dr^2} - \frac{1}{r}\frac{d}{dr}\right]\left[\frac{d^2}{dr^2} - \frac{1}{r}\frac{d}{dr}\right]\right]H(r)$$
$$+2b_1[\left[\frac{d^2}{dr^2} - \frac{1}{r}\frac{d}{dr}\right]F(r) = 0$$

We know that,

$$E(r) = \frac{-r^4 + r^2 R^2}{R^3 + 4\lambda R^2} + \frac{r^2}{R}$$
$$\frac{dE(r)}{dr} = \frac{-4r^3 + 2R^2 r}{R^3 + 4\lambda R^2} + \frac{2r}{R}$$
$$\frac{d^2E(r)}{dr} = \frac{-12r^2 + 2R^2}{R^3 + 4\lambda R^2} + \frac{2}{R}$$

thus,

$$\frac{d^2E(r)}{dr^2} - \frac{1}{r}\frac{dE(r)}{dr} = \frac{-12r^2 + 2R^2}{R^3 + 4\lambda R^2} + \frac{2}{R} - \frac{1}{r}\left[\frac{-4r^3 + 2R^2 r}{R^3 + 4\lambda R^2} + \frac{2r}{R}\right]$$
$$\frac{d^2E(r)}{dr^2} - \frac{1}{r}\frac{dE(r)}{dr} = \frac{-8r^2}{R^3 + 4\lambda R^2} \tag{4.40}$$

Substituting (4.40) back in (4.6) we obtain

$$\left[\left[\frac{d^2}{dr^2} - \frac{1}{r}\frac{d}{dr}\right]\left[\frac{d^2}{dr^2} - \frac{1}{r}\frac{d}{dr}\right]\right]H(r) + 2b_1\left[-\frac{8r^2}{R^3 + 4\lambda R^2}\right] = 0$$

or

$$\left[\left[\frac{d^2}{dr^2} - \frac{1}{r}\frac{d}{dr}\right]\left[\frac{d^2}{dr^2} - \frac{1}{r}\frac{d}{dr}\right]\right]H(r) = 16b_1\frac{r^2}{R^3 + 4\lambda R^2} \tag{4.41}$$

Introducing an operator

$$\left(\frac{d^2}{dr^2} - \frac{1}{r}\frac{d}{dr}\right) = r\frac{d}{dr}\left(\frac{1}{r}\frac{d}{dr}\right)$$

We express(4.40) in operator form

$$\left[r\frac{d}{dr}\left(\frac{1}{r}\frac{d}{dr}\right)\right]\left[r\frac{d}{dr}\left(\frac{1}{r}\frac{d}{dr}\right)\right]H(r) = 16b_1\frac{r^2}{R^3 + 4\lambda R^2} \tag{4.42}$$

Dividing both sides by r,

$$\left[\frac{d}{dr}\left(\frac{1}{r}\frac{d}{dr}\right)\right]\left[r\frac{d}{dr}\left(\frac{1}{r}\frac{d}{dr}\right)\right]H(r) = 16b_1\frac{r}{R^3 + 4\lambda R^2}$$

Integrating with respect to r leads to,

$$\left(\frac{1}{r}\frac{d}{dr}\right)\left[r\frac{d}{dr}\left(\frac{1}{r}\frac{d}{dr}\right)\right]H(r) = 8b_1\frac{r^2}{R^3 + 4\lambda R^2} + B_1$$

Where$B_1$is a constant of integration

mulityplying r on bothsides ,

$$\left(\frac{d}{dr}\right)\left[r\frac{d}{dr}\left(\frac{1}{r}\frac{d}{dr}\right)\right]H(r) = 8b_1\frac{r^3}{R^3 + 4\lambda R^2} + B_1 r \tag{4.43}$$

now integrating,

$$\left[r\frac{d}{dr}\left(\frac{1}{r}\frac{d}{dr}\right)\right]H(r) = \frac{8b_1}{4}\frac{r^4}{R^3 + 4\lambda R^2} + \frac{B_1 r^2}{2} + B_2 \tag{4.44}$$

where B_2 is a constant of integration.

Dividing (4.44) by r on bothsides

$$\left[\frac{d}{dr}\left(\frac{1}{r}\frac{d}{dr}\right)\right]H(r) = 2b_1\frac{r^3}{R^3 + 4\lambda R^2} + \frac{B_1 r}{2} + \frac{B_2}{r} \tag{4.45}$$

When the boundary condition (4.9) is applied on (4.9)

at $r = 0$

$$\frac{d}{dr}\left[\frac{1}{r}\frac{d}{dr}\right]H(r) = 0$$

We notice that we must have

$$B_2 = 0$$

Therefore (4.45) reduces to,

$$\left[\frac{d}{dr}\left(\frac{1}{r}\frac{d}{dr}\right)\right]H(r) = 2b_1\frac{r^3}{R^3+4\lambda R^2}+\frac{B_1 r}{2} \tag{4.46}$$

Integrating of (4.9) with respect to r, gives us

$$\left(\frac{1}{r}\frac{d}{dr}\right)H(r) = 2b_1\frac{r^4}{4\left(R^3+4\lambda R^2\right)}+\frac{B_1 r^2}{4}+B_3 \tag{4.47}$$

where B_3 is an arbitrary constant

On multiplying bothsides of(4.47) by r, we get

$$\frac{dH(r)}{dr} = b_1\frac{r^5}{2\left(R^3+4\lambda R^2\right)}+\frac{B_1 r^3}{4}+B_3 r \tag{4.48}$$

finally,we get after integration of (4.48)

$$H(r) = b_1\frac{r^6}{12\left(R^3+4\lambda R^2\right)}+\frac{B_1 r^4}{16}+\frac{B_3 r^2}{2}+B_4$$

where B_4 is a constant

Now we use the boundary condition (3) at $z=0$; $H'(R)-H(0)=-\frac{Q_0}{2\pi}$

Then (4.48) becomes,

$$-\frac{Q_0}{2\pi} = \left[b_1\frac{R^6}{12\left(R^3+4\lambda R^2\right)}+\frac{B_1R^4}{16}+\frac{B_3R^2}{2}+B_4\right]-\left[b_1\frac{0^6}{12\left(R^3+4\lambda R^2\right)}+\frac{B_1 0^4}{16}+\frac{B_3 0^2}{2}+B_4\right]$$

or

$$-\frac{Q_0}{2\pi} = b_1\frac{R^6}{12\left(R^3+4\lambda R^2\right)}+\frac{B_1R^4}{16}+\frac{B_3R^2}{2} \tag{4.49}$$

$$-\frac{Q_0}{2\pi R^2} = b_1\frac{R^2}{12\left(R+4\lambda\right)}+\frac{B_1R^2}{16}+\frac{B_3}{2}$$

$$-\frac{Q_0}{2\pi R^2}-b_1\frac{R^2}{12\left(R+4\lambda\right)} = \frac{B_1R^2}{16}+\frac{B_3}{2} \tag{4.50}$$

Now we use the boundary condition,

$$\left(1-\frac{\lambda}{r}\right)H'(r) = -\lambda H''(r)$$

at r=R

$$\left(1-\frac{\lambda}{r}\right)\left[\frac{b_1R^5}{2R^2\left(R+4\lambda\right)}+\frac{B_1R^3}{4}+B_3R\right] = -\lambda\left[\frac{5b_1R^4}{2R^2\left(4\lambda+R\right)}+\frac{3B_1R^2}{4}+C_1\right]$$

$$\frac{b_1R^3}{2(4\lambda+R)}-\frac{\lambda b_1R^2}{2(4\lambda+R)}+\frac{B_1R^3}{4}-\frac{\lambda B_1R^2}{4}+B_3R-\lambda B_3=-\frac{5\lambda b_1R^2}{2(4\lambda+R)}-\frac{3\lambda B_1R^2}{4}-\lambda C_1$$

$$\frac{b_1R^3}{2(4\lambda+R)}-\frac{\lambda b_1R^2}{2(4\lambda+R)}+\frac{5\lambda b_1R^2}{2(4\lambda+R)}+\frac{B_1R^3}{4}-\frac{\lambda B_1R^2}{4}+\frac{3\lambda B_1R^2}{4}+B_3R=0$$

$$\frac{b_1R^3}{2(4\lambda+R)}+\frac{2\lambda b_1R^2}{2(4\lambda+R)}+\frac{B_1R^3}{4}\left[1+\frac{2\lambda}{R}\right]+B_3R=0$$

$$\frac{b_1R^3}{2(4\lambda+R)}\left[1+\frac{4\lambda}{R}\right]+\frac{B_1R^3}{4}\left[1+\frac{2\lambda}{R}\right]+B_3R=0$$

$$\frac{B_1R^3}{4}\left[\frac{2\lambda+R}{R}\right]+B_3R=\frac{-b_1R^2}{2}$$

$$\frac{B_1R}{4}(2\lambda+R)+B_3=\frac{-b_1R}{2} \tag{4.51}$$

multimlying equation (4.51) by $\frac{1}{2}$

$$\frac{B_1R}{8}(2\lambda+R)+\frac{B_3}{2}=-\frac{b_1R}{4} \tag{4.52}$$

subtracting equation(4.52) from (4.50)

$$\frac{B_1R}{8}(2\lambda+R)-\frac{B_1R^2}{16}=\frac{Q_0}{2\pi R^2}+\frac{b_1R^2}{12(4\lambda+R)}-\frac{b_1R}{4}$$

$$\frac{B_1R^2}{8}\frac{[(4\lambda+2R)-R]}{2R}=\frac{Q_0}{2\pi R^2}+\frac{b_1R}{4}\frac{[R-3(4\lambda+R)]}{3(4\lambda+R)}$$

$$\frac{B_1R}{16}(4\lambda+R)=\frac{Q_0}{2\pi R^2}+\frac{b_1}{6}\frac{R(-6\lambda-R)}{(4\lambda+R)}$$

$$B_1=\left[\frac{Q_0}{2\pi R^2}-\frac{b_1R(6\lambda+R)}{6(4\lambda+R)}\right]\frac{16}{R(4\lambda+R)}$$

$$B_1=8\left[\frac{Q_0}{\pi R^3(4\lambda+R)}-\frac{b_1(6\lambda+R)}{3(4\lambda+R)^2}\right]$$

$$B_3=-\frac{b_1R}{2}-\frac{B_1R}{4}(2\lambda+R)$$

using the value of B_3

$$B_3=-\frac{b_1R}{2}-\frac{R(2\lambda+R)}{4}\left(8\left[\frac{Q_0}{\pi R^3(4\lambda+R)}-\frac{b_1(6\lambda+R)}{3(4\lambda+R)^2}\right]\right)$$

$$B_3=-\frac{b_1R}{2}-\frac{2Q_0}{\pi R^2}\frac{(2\lambda+R)}{(4\lambda+R)}+\frac{2b_1R}{3}\frac{(6\lambda+R)(2\lambda+R)}{(4\lambda+R)^2}$$

$$-\frac{2Q_0}{\pi R^2}\frac{(2\lambda+R)}{(4\lambda+R)}+\frac{b_1 R}{6}\left[\frac{4\left(12\lambda^2+8\lambda R+R^2\right)-3\left(16\lambda^2+R^2+8\lambda R\right)}{\left(4\lambda+R^2\right)^2}\right]$$

$$B_3=-\frac{2Q_0}{\pi R^2}\frac{(2\lambda+R)}{(4\lambda+R)}+\frac{b_1 R}{6\,(4\lambda+R)}\left(R^2+8\lambda R\right)$$

Substituting the value of B_1 and B_3

$$H(r)=\frac{B_3 r^2}{2}+\frac{B_1 r^4}{16}+\frac{b_1 r^6}{12R^2}\frac{1}{(4\lambda+R)} \tag{4.53}$$

$$H(r)=r^2\left[\frac{b_1 R}{12\,(4\lambda+R)^2}\left(R^2+8\lambda R\right)-\frac{Q_0}{\pi R^2}\frac{(2\lambda+R)}{(4\lambda+R)}\right]+r^4\left[\frac{Q_0}{2\pi R^3}\frac{1}{(4\lambda+R)}-\frac{b_1}{6}\frac{(6\lambda+R)}{(4\lambda+R)^2}\right]$$
$$+\frac{b_1}{12}\frac{r^6}{R^2}\frac{1}{(4\lambda+R)}$$

Substituting the value of E(r) and H(r) in stream function.

$$\psi(r,z)=\left(b_0 z+\frac{1}{2}b_1 z^2\right)E(r)+H(r) \tag{4.54}$$

$$\psi(r,z)=\left(b_0 z+\frac{1}{2}b_1 z^2\right)\left[-\frac{-r^4}{R^2}\frac{1}{4\lambda+R}+\frac{2r^2}{R}\frac{(2\lambda+R)}{(4\lambda+R)}\right]$$
$$+r^2\left[\frac{b_1 R}{12\,(4\lambda+R)^2}\left(R^2+8\lambda R\right)-\frac{Q_0}{\pi R^2}\frac{(2\lambda+R)}{(4\lambda+R)}\right]+r^4\left[\frac{Q_0}{2\pi R^3}\frac{1}{(4\lambda+R)}-\frac{b_1}{6}\frac{(6\lambda+R)}{(4\lambda+R)^2}\right]$$
$$+\frac{b_1}{12}\frac{r^6}{R^2}\frac{1}{(4\lambda+R)} \tag{4.55}$$

4.13 Our radial component of velocity is defined through :

$$u_r=\frac{1}{r}\frac{\partial\psi}{\partial z}$$
$$u_r=[b_0+b_1 z]\left[\frac{2r}{R}\frac{(2\lambda+R)}{(4\lambda+R)}-\frac{r^3}{R^2}\frac{1}{4\lambda+R}\right]$$

4.14 Our axial component of velocity is defined through :

$$u_z = -\frac{1}{r}\frac{\partial \psi}{\partial r}$$

$$u_z = \left[b_0 + \frac{1}{2}b_1 z\right]\left[\frac{4}{R}\left(\frac{2\lambda + R}{4\lambda + R}\right) - \frac{4r^2}{R}\frac{1}{4\lambda + R}\right] - 2\left[\frac{b_1 R}{12}\frac{R^2 + 8\lambda R^2}{4\lambda + R} - \frac{Q_0}{\pi R^2}\frac{2\lambda + R}{4\lambda + R}\right]$$
$$-4r^2\left[\frac{Q_0}{2\pi R^3}\frac{1}{4\lambda + R} - \frac{b_1}{6}\frac{6\lambda + R^2}{4\lambda + R}\right] - \frac{b_1 r^4}{2R^2}\frac{1}{4\lambda + R}$$

4.15 Volume flow rate formula

$$Q(z) = \int_0^R 2\pi r u_z(r, z) dr \tag{4.56}$$

We recall that (4.5)

$$u_z = -\frac{1}{r}\frac{\partial \psi}{\partial r} = -\left[\frac{1}{r}E'(r)\left(b_0 z + \frac{1}{2}b_1 z^2\right) + \frac{H'(r)}{r}\right]$$

$$Q_z = -2\pi \int_0^R r\left[\left(b_0 z + \frac{1}{2}b_1 z^2\right)\frac{E'(r)}{r} + \frac{H'(r)}{r}\right] dr$$

Differentiating (4.56)

$$\frac{dQ(z)}{dz} = -2\pi \int_0^R r\frac{d}{dz}\left\{\frac{E'(r)}{r}\left(b_0 z + \frac{1}{2}b_1 z^2\right) + \frac{1}{r}H'(r)\right\} dr$$

or

$$\frac{dQ(z)}{dz} = -2\pi \int_0^R E'(r)\left(b_0 + b_1 z\right) dr$$

or

$$\frac{dQ(z)}{dz} = -2\pi\left(b_0 + b_1 z\right)\int_0^R E'(r) dr$$

or

$$\frac{dQ(z)}{dz} = -2\pi\left(b_0 + b_1 z\right)\left[E(r)\right]_0^R$$

$$\frac{dQ(z)}{dz} = -2\pi\left(b_0 + b_1 z\right)\left[E(R) - E(0)\right]$$

since,

$$\frac{dQ(z)}{dz} = -2\pi\left(b_0 + b_1 z\right)\left[\frac{2R^2}{R}\frac{(2\lambda + R)}{(4\lambda + R)} - \frac{R^4}{R^2}\frac{1}{(4\lambda + R)}\right] - 0$$

$$\frac{dQ(z)}{dz} = \frac{-2\pi\left(b_0 + b_1 z\right)}{(4\lambda + R)}\left[2R\left(2\lambda + R\right) - R^2\right]$$

$$\frac{dQ(z)}{dz} = -2\pi R\left(b_0 + b_1 z\right)$$

Integrating equation w.r.t "z"

$$\int_0^R \frac{dQ(z)}{dz}dz = -2\pi R\int_0^z \left(b_0 + b_1 z\right)dz$$

$$\left[Q(z)\right]_0^z = -2\pi R\left[b_0 z + \frac{b_1 z^2}{2}\right]$$

$$Q(z) - Q(0) = -\pi R\left[2b_0 z + b_1 z^2\right]$$

$$Q(z) = Q(0) - \pi R\left[2b_0 z + b_1 z^2\right]$$

4.16 Maximum radial velocity:

$$u_r = \left(b_0 + b_1 z\right)\left[\frac{2r}{R}\frac{(2\lambda + R)}{(4\lambda + R)} - \frac{r^3}{R^2}\frac{1}{(4\lambda + R)}\right]$$

diff w.r.t r

$$\frac{\partial u_r}{\partial r} = \left(b_0 + b_1 z\right)\left[\frac{2}{R}\frac{(2\lambda + R)}{(4\lambda + R)} - \frac{3r^2}{R^2}\frac{1}{(4\lambda + R)}\right]$$

$$\frac{\partial u_r}{\partial r} = 0$$

$$\left(b_0 + b_1 z\right)\left[\frac{2}{R}\frac{(2\lambda + R)}{(4\lambda + R)} - \frac{3r^2}{R^2}\frac{1}{(4\lambda + R)}\right] = 0$$

$$\frac{2}{R}\frac{(2\lambda + R)}{(4\lambda + R)} - \frac{3r^2}{R^2}\frac{1}{(4\lambda + R)} = 0$$

$$\frac{2}{R}\frac{(2\lambda + R)}{(4\lambda + R)} = \frac{3r^2}{R^2}\frac{1}{(4\lambda + R)}$$

$$\frac{2}{R}\left(2\lambda + R\right) = \frac{3r^2}{R^2}$$

$$\frac{r^2}{R^2} = \frac{2}{3R}(2\lambda + R)$$
$$\frac{r}{R} = \pm\sqrt{\frac{2}{3R}(2\lambda + R)}$$
$$\frac{\partial^2 u_r}{\partial r^2} = (b_0 + b_1 z)\left[-\frac{6r}{R^2}\frac{1}{(4\lambda + R)}\right]$$
$$= \frac{(b_0 + b_1 z)}{(4\lambda + R)}\left[-6\sqrt{\frac{2}{3R}(2\lambda + R)}\right]\frac{1}{R} < 0$$

4.17 Results and discussions

In order to observe the geometrical structure in various orientations of the pipe, we draw the following necessary graphs in Figs and to examine the creeping flow in the tube by depicting velocity profiles. In Figs. 1 to 4 we have tried to expose the characteristics of velocity, pressure and stress fields for flow of both viscous and creep fluids.

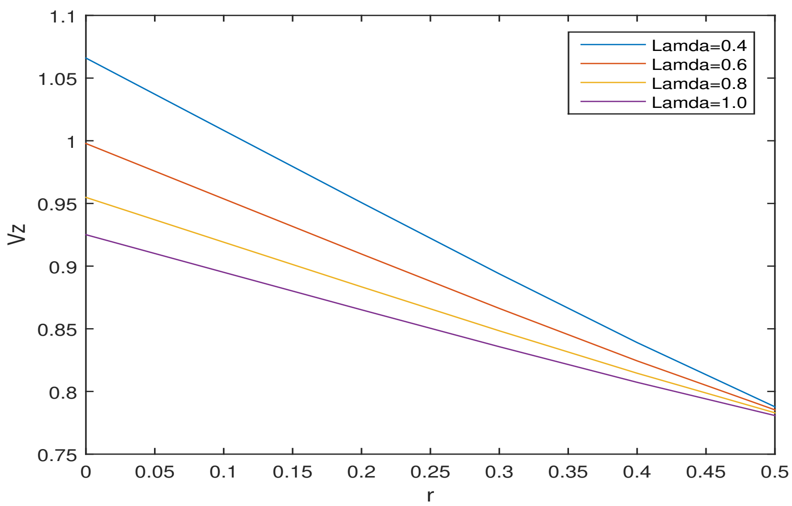

Figure 4.2: Effect of slip coefficient on axial component of velocity for tube flow

fig illustrates a plot of axial component of velocity uz vs radius "r" for slip coefficient $\lambda = 0.4, 0.6, 0.8, 1.00$.It is seen from the figure that the axial component of velocity decreases along radidus.we can see that as we increases the values of

λ its effects on axial component of velocity which decreases.The presence of slip velocity appears to reduce the wall shear and flattens the velocity profiles. We

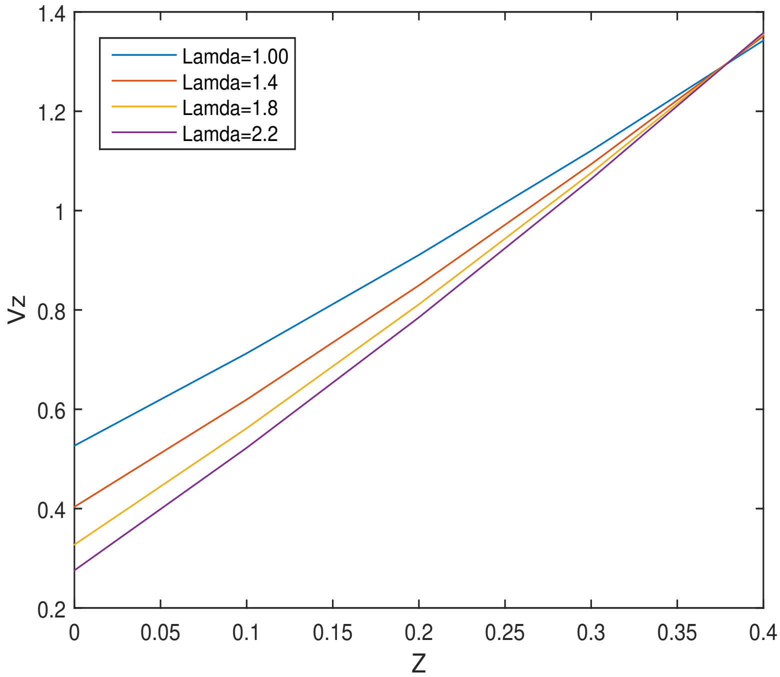

Figure 4.3: Effect of slip coefficient on axial component of velocity along its aixes

checked the axial component of velocity for different values ofλ .From the graph it is clear that axial components velocity increases along its aixes at specific values of $\lambda = 1.00, 1.4, 1.8, 2.2$ fig illustrates a plot of radial component of velocity vs z component.It is clear that radial component of velocity decreases as the value of λ increase.

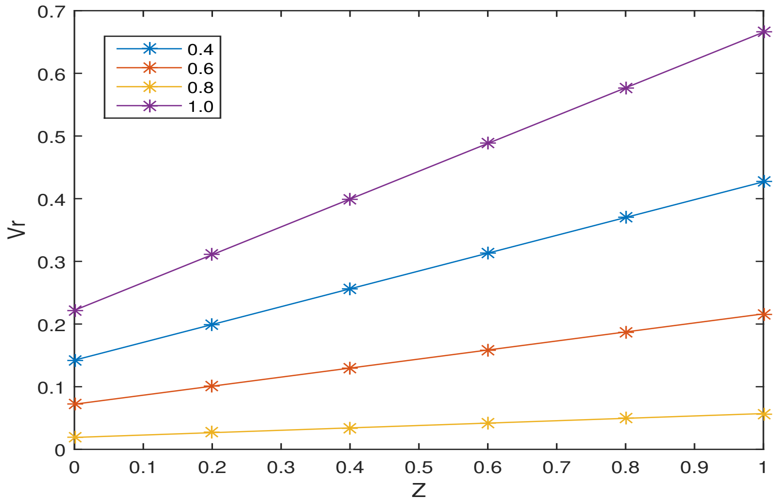

Figure 4.4: Effect of slip coefficien on radial component of velocity along aixes

We plot a graph of radial component of velocity vs radius "r" to check the effect of slip coefficient on radial velocity for different value of λ.Radial velocity suddenly decreases with the increase the value of λ.In the figure 4.4 we can see that abrupt change in the graph as we increase the value of λ.

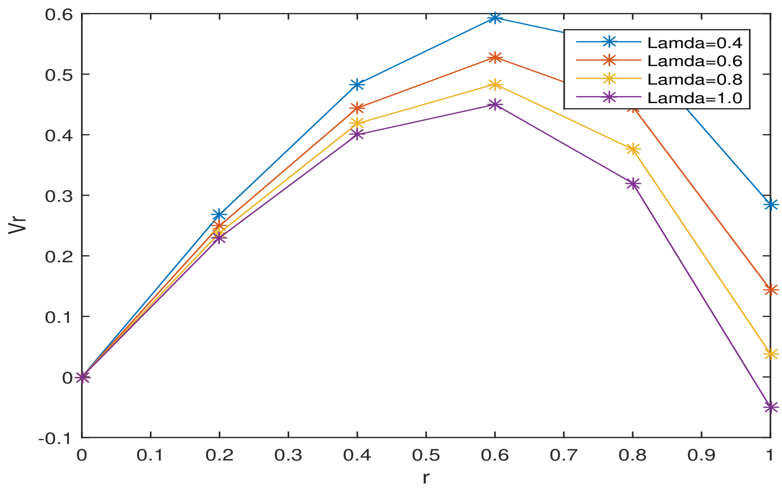

Figure 4.5: Effect of slip coefficient on radial component of velocity along radius

Chapter 5

Summary and Conclusion

This chapter gives the summary of the results obtained.In this thesis four chapters are presented. Chapter 1 contains literature related to research done by scholars on incompressible , steady and laminar flow through porous tube. Chapter 2 have the basic definitions of fluid characteristics and two basic equations which are Navier stokes equation and continuity equation used for fluid flow problem.Chapter 3 is the review of the paper.The solution is formed by stream function with the help of no-slip flow boundary condition for two-dimensional flow in renal tubules. we have been discussed Graphically the effect of velocity components and pressure distribution in this chapter.

In chapter 4 we presented the same model as in chapter 3 but for the slip-flow case.The solution of the parametric equations is formed by using the stream function technique. The graphical solutions of velocity profiles are also presented for different values of slip coefficient λ.The results indicated that axial and radial component of velocity decreases with the increase of slip coefficient λ and the presence of slip reduce the wall shear stress tending to flattens the velocity profiles.

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