Thermodynamics of Specific Black Holes

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DEDICATICATED TO

My Beloved Family
Especially
My Mother

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ABSTRACT

In this thesis, we modify the first law of thermodynamics for regular black hole of non-minimal Einstein-Yang-Mill theory with gauge field of magnetic Wu-Yang type and a regular black hole, which is associated with cosmological constant by the surface tensions. The corresponding Smarr relations are also satisfied for them. The cosmological constant is taken as constant as well as variable (which is correlated with pressure). We calculate the Gibbs free energy to discuss the global stability of the above mentioned regular black holes in general relativity. At first, the cosmological constant is adopted as fixed and then as a variable, which is correlated with the pressure. The first law of thermodynamics is modified for both regular black holes, mentioned above, when the cosmological constant is treated as fixed and as a variable.

Also, we investigate the thermodynamical properties and consider the logarithmic correction to entropy in order to analyze thermal fluctuations of two another regular black hole: 1) The AdS black hole in Born-Infeld massive gravity with a non-abelian hair. 2) The charged AdS black hole with a global monopole. We develop the many thermodynamical quantities such as entropy, temperature, pressure, heat capacity of a system at constant volume and pressure, ratio between the heat capacities at constant pressure and volume, the Gibbs free energy and the Helmholtz free energy for both black holes. We observe the critical behavior and phase transition and check the validity of the first law of thermodynamics. We observe the local stability and the global stability of the black holes in the grand canonical ensemble and canonical ensemble for the specific values of different parameters, such as, symmetry breaking parameter
η and massive parameter m.

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Chapter 1
Introduction

In theoretical physics, general relativity (GR) is one of the most successful theories. Einstein published his geometric theory of gravity, known as GR in **1915**. The theory is geometric because according to this theory gravity is not force, rather it is an effect of a space-time manifold curved by mass and energy. The most famous and central equations in GR are Einstein field equations, also known as Einstein equations, which compromise the set of 10 equations in GR that describes the fundamental interaction of gravitation. By using the tensor formalism, they can be written as

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi G T_{\mu\nu} \tag{1.0.1}$$

Karl Schwarzschild, in **1916** found the solutions to the Einstein equations, known as Schwarzschild solution, which describes geometry of the space around the exotic massive spherical object in a vacuum, that of black hole (BH).

In modern theoretical physics, BH is an interesting subject. At all times, BH is associated with a singularity inside it. Einstein's theory of space, time and gravity empowers the existence of singularities. According to the GR, a singularity is a definite point in the space-time where explicit quantities even the curvature of space-time is infinite. The problem of singularities is one of the exceptional problems and GR made prediction about singularities in interior of BHs. Firstly, it was anticipated that both of these singularities does not naturally exist but occurs as a result of preparative procedure of finding the harmony in the solutions. Penrose and Hawking's theorems of singularities lead us to believe that they are inevitable [1]-[6]. As the initial exact BH solutions in GR predicts the singularity in the interior of an event horizon.

Sakharov [7] and Gliner [8] proposed that for sources like matter which have a de-Sitter core in the heart of the space-time it's singularities could have been avoided and then Bardeen availed this theory to put forward initial solution of the static spherically symmetric regular black hole (RBH), called "Bardeen BH". Bronnikov and Melnikov particularize the detail of the many types of the RBH [9], explained geometries of RBH in a nutshell and revealed actuality of the black-universe (BU) solutions in GR in agreement with the minimally coupled phantom fields. By coupling GR to non-linear electrodynamics (NED) many different kinds of RBH solutions have been derived [10]-[14]. It is mention here that RBHs asymptotically behave like an ordinary charged Reissner-Nordstrom (RN) BH solutions and all fields and curvature invariants are regular everywhere instead of usual singularities at their center. These solutions and singularity theorems have no contradiction at all between them.

Ayon-Beato and Garcia, in GR, introduced a proposal of exact nonsingular BH solution fulfilling weak energy condition (WEC), which is produced by new NED combined to gravity [10]. Bambi and Modesto proposed the solutions for the rotating RBH [15]. The spherically symmetric radiating and rotating solutions have been proposed by using the Newman-Janis algorithm [16]. Fan and Wang, in the gravity model composed those solutions of BHs which bear magnetic charge on them. Strategy of formulating the solutions of the electrically charged BHs has been illustrated and thermodynamical possessions has been studied along with formulation of first law of thermodynamics. Also generalization of construction of AdS BH solutions has been done [17].

Hawking and Penrose, during late **1960s**, put into practical a new complicated mathematical model which led to both proposing and proving numerous theorems of singularity [3]. There is a parallelism between ordinary second law (OSL) of thermodynamics, which is stated as "the entropy of the closed system never decreases" and Hawking's theorem, whose statement is that "the surface area of BH never decreases" [18], and Bekenstein was the first to analyzed the resemblance between these two [19]. Then, in **1972**, Hawking-Bekenstein debate started until **1974**, when Hawking discovered the radiation of BH near event horizon produced by it's quantum effects and named it, Hawking's radiation [20].

The discovery about the area of event horizon is irreducible lead towards a successful conclusions in GR and made up an essential demonstration that BHs have thermodynamical properties. Thermodynamical properties of the space-times proved to be very important and grasped the worthwhile attentions [21]. They investigated the properties for the magnetically charged regular black hole (MCRBH). Phase transitions is another well-known phenomenon, which is based on the fundamental principles of thermodynamics. In the succeeding work, the phase transitions and the thermodynamical stabilities has been studied for many different BHs [22].

Quantum fluctuations in cosmology is one of the most fascinating topic in GR. Another aspect of quantum fluctuations: specifically, quantum fluctuations in cosmology appear, in almost all models, to lead to eternal inflation and an infinite multiverse. Sakharov gave the concept that maybe the origin of structure in universe is the quantum fluctuations [23].

In GR, BHs have shown many generative features of space-time such as the event horizon, the singularity, Hawking radiation and their connection with thermodynamics. Among all of the above phenomena, the strong support to the thermodynamical properties of BHs is the Hawking radiation. Classically, as nothing can escape from the inside of an event horizon of a BH, quantum mechanically it has been proved that the BHs can emit radiation, known as Hawking radiation [24]-[25] and has the Planck spectrum. This aspect comply with the thermodynamics of BHs. Discovery of Hawking radiation leads towards that the BHs have temperature, therefore, the concept which is proposed by Bekenstein about the entropy for BHs is no longer a mystery. Also, the incredible work of Hawking made the entropy quantitative which is related to the area of the event horizon as, $S = \frac{A}{4}$ [26]. As compared to other objects containing same volume as the BHs considered to have higher entropy [20] and [24]. The BHs are fluctuating, that is, the event horizon becomes fuzzy because of the quantized metric. Because of the quantum fluctuations, the need of making corrections to the maximum entropy of the BHs has been emerged, which leads towards the development of the holographic principle, which is an emerging new paradigm in quantum gravity [27]-[28]. Whereas, the holographic principle implies that the degrees of freedom in a spatial region can all be encoded on its boundary, with a density not exceeding one degree of freedom per planck cell [29].

Kastor et al., for anti de Sitter (AdS) BH, treated the cosmological constant associated with the pressure such as $P = -\frac{\Lambda}{8\pi}$, while treated it's conjugate quantity as thermodynamic volume. The expanded first law of thermodynamics presented which has the new term in format of effective volume times change in pressure, which indicates that mass of AdS BH should be seen as the enthalpy of BH [30]. Kubiznak and Mann acquired this concept and investigated thermodynamics of charged AdS BH in extended phase space [31]. In this way, investigation on thermodynamics of many BHs has been made and many significative critical phenomena were found in [32]-[49]. Hansen et al., modified the first law of thermodynamics by using the radial Einstein's equation at the Kerr BH by the surface tension [50]. Chen and Zeng, modify the first law of thermodynamics of the Schwarzschild dS-BH at BH horizon and cosmological horizon. They have also derived the apparent horizon of the Friedmann-Robertson-Walker cosmology by the surface tension [51]. In recent work, for the dS and AdS space-times, Chen et al., modified the first law of thermodynamics by the surface tension at the BH horizon and cosmological horizon [52].

Several approaches have been made practical and effective for evaluating the entropy corrections such as by using the non-pertubative quantum GR, in which the density of microstates for asymptotically flat BHs have been calculated, which construct the logarithmic correction to the standard Bekenstein entropy area relation [53]. Including, one can also use the methodologies based on, such as, generating logarithmic correction terms, for all those BHs whose microscopic degrees of freedom are explained by conformal field theory [54]-[55], Hamiltonian partition functions [56], loop quantum gravity [57], near horizon symmetries [58] and thermodynamic arguments [59]. Moreover, the corrections to the entropy of dilaton BHs are evaluated, which emerged to be the logarithmic corrections [60]. By using the brick wall method, quantum corrections to the entropy of a static, spherically symmetric BHglobal monopole system arising from the Dirac spinor field are investigated [61]. The effect of thermal fluctuations for the entropy of both neutral and charged black holes has been investigated [62].

Entropy corrections for Schwarzschild BHs has been found by putting it in the center of a spherical cavity of finite radius to achieve equilibrium with surroundings and conclude that there are two correction terms for Schwarzschild BHs [63]. In a massive theory of gravity, the effects of the quantum fluctuations has been analyzed for a charged BTZ BH in asymptotically AdS and dS space-times [64]. By taking into account the effects of statistical quantum fluctuations around the thermal equilibrium and via conformal field theory (CFT), the logarithmic corrections to the BH entropy product formula of outer and inner horizon has been examined recently [65]. The effects of thermal fluctuations on charged ADS BHs and modified Hayward BHs have been investigated in recent work [66]-[67].

The dRGT model, which is one of the most famous and interesting theories of massive gravity, introduced by the Rham et al. [68]-[69]. This theory, in the Einstein-Hilbert action, added a potential which included the mass in the graviton. On the basis of the reference metric, there are various modifications to the dRGT model but, Vegh [70] gives the most best one. His model undergoes the breaking of the translational symmetry. Zhang and Li proved that the this model is stable [71]. After that, the thermodynamical properties and phase transition of many BHs have been investigated [72]-[76]. On the other hand, because of the presence of few constraints in the Maxwell theory implies to reckon the NED [77]-[85].

NED theories are more valuable then the Maxwell theories. At the origin, for removing the divergency of self energy of a point-like charge, Born and Infeld introduced one of the best and interesting NED theories which, known as Born-Infeld (BI) theory [86]-[87]. Coupled to the BI NED, BH solutions and their Van der Waals kind of behavior in massive gravity has been investigated [88]. In the Maxwell field, one can assume the non-abelian Yang-Mills (YM) field coupled to gravity as a matter source. In Gauss-Bonnet-massive gravity, the thermodynamical properties of BHs and their phase transition in the existence of YM field have been investigated in [89]. In the presence of the YM and BI NED fields, the exact BH solutions of Einstein-Massive theory has been obtained by Hendi and Momennia[90].

In this thesis, we've presented modified first laws of thermodynamics at horizon for two RBHs: 1) The non-minimal MCRBH and 2) RBH with cosmological constant. Also, we've presented the effects of thermal fluctuations for two RBHs: 1) The AdS BH in BI massive gravity with a non-abelian hair. 2) The charged AdS BH with a global monopole. Rest of the thesis is organized as follows:

ii) In chapter **2**, we put some light on the basic definitions, which are necessary for developing an understanding for the thesis.

iii) Chapter **3**, contains the modification of first law of thermodynamics for non-minimal MCRBH and RBH with cosmological constant. Also, global stability for above mentioned RBHs has been discussed.

iv) Chapter **4**, contains the analysis and investigation on the effects of thermal fluctuations for AdS BH in BI massive gravity with a non-abelian hair and charged AdS BH with a global monopole. Also, Local and global stability in canonical ensemble and grand canonical ensemble has been discussed.

v) Chapter **5** contains the conclusion of the thesis.

vi) Last chapter contains the bibliography of the thesis.

Chapter 2
Preliminaries

This chapter having the basic concepts for understanding the thesis work.

2.1 Black Holes

When a star is under equilibrium, during nuclear reaction, the outward thermal pressure balances the inward gravity. The star gave out of hydrogen, gas pressure begin decreasing and gravitational pull gets stronger and star set out reducing. The core of star continue on reducing under the gravity. A stage comes, when the pressure isn't enough to overcome the gravitational pull and fall down onto itself. Because of the gravitational collapse, star reaches towards its endmost destiny, that is, a BH. BH is predestined result of Einstein's theory of GR, which inferred that matter wraps space-time.

All the matter bundle up in the vicinity of core of space-time, which inhibiting every thing, even radiation, inside it from getting away. Therefore, BH is a region in space where gravitational pull is so strong that nothing can get out from it after passing through it's event horizon, which is the boundary of BH. The reason of gravity being strong is that all matter has been collapsed into a point in space, which is known as singularity. Black hole is bounded by a well-defined surface called event horizon.

2.2 Event Horizon

Newton's law of gravity implies that an adequately dense body's escape speed can be equal to or greater than speed of light, determined by the body's mass and radius. Therefore, we can define the event horizon of a BH as the surface where escape speed equals the speed of light. We can distinguish the event horizon depending upon the three characteristics.

- Firstly, event horizon is a static limit, such that, nothing can stay fixed and steady due to the strong gravitational pull of BH. After passing the event horizon, body start falling into the singularity, which is specified as the future. Also, time turns out to be like space and space as timelike.

- Secondly, it's an infinite redshift surface, that is, far off observer experiences radiation wavelength to be larger as compared to the original wavelength observed by an observer near the event horizon of BH. Because an observer becomes closer to the BH horion, redshift becomes infinite and then a stage comes at which radiation cannot be seen at all, which implies invisibility of BH.

- Thirdly and last, it acts like a one-way membrane. Once an object crosses the event horizon, it can't escape from it. Space-time Curvature near the BH, shows appearance of tidal forces. As event horizon in space-time is tangential to a light cone, it can't be pass over again. Hence, it acts like a point of no return.

2.3 Types of Black Holes

The most simple BH is the one, which consists of only mass. It doesn't have any charge and angular momentum. This type of BH is generally known as Schwarzschild BH. It is the first exact solution to the Einstein field equations. In GR, Birkhoff's theorem states that any spherically symmetric solution of the vacuum field equations must be stationary and asymptotically flat. After that, many BH solutions of were discovered by other scientists. The RN solution presents a geometry of BH along with electric charge. Kerr discovered the geometry of rotating BH. After that Newmann generalized the Kerr solution and discovered that BH which has both spin and charge, which is known as Kerr-Newman metric. These general solutions converge to the Schwarzschild solution at distances that are large compared to the ratio of charge and angular momentum to mass.

2.4 Regular Black Holes

In 1968, a famous model, known as Bardeen's model was proposed by Bardeen [91]. Bardeen's model was the first static spherically symmetric RBH solution, which obeys the weak energy condition. Also, no known physical sources are associated with the RBH solutions. These BHs are solutions of Einstein field equations with no essential singularity, hence their metric is regular everywhere. The study of RBHs solutions is very important to understand the gravitational collapse. Bardeen model is strong for avoiding the singularities. The metric is given by the line element as

$$ds^2 = f(r)dt^2 - f^{-1}(r)dr^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2),$$

where $f(r)$ is given by

$$f(r) = 1 - \frac{2mr^2}{(r^2 + q^2)^{\frac{3}{2}}},$$

where q is the charge. Bardeen replaced the usual RN metric function with the above defined metric function. There exist an event horizon, if $q^2 < \frac{16}{27}m^2$. The RBHs doesn't carry any physical singularities and satisfies the null convergence. The metric function of the RBH reduces to the Schwarzschild metric by taking $q = 0$.

2.5 Schwarzschild BH

In all respect, the physical occurrence and phenomenon, such as geodesics which states the space-time and also, near the BH it gets modified. Because of the Newtonian gravity, the ring-shaped orbits of matter bodies surrounding a heavy gravitational mass, can have the existence at all radius. In 1916, Schwarzschild completed his work and presented the solutions to the incomplete Einstein's field equations [92]. The Schwarzschild solution, which is also known as Schwarzschild metric, explains that there exist a gravitational field in the exterior of the spherical mass by supposing that the there is no electric charge, angular momentum or cosmological constant. Therefore, a Schwarzschild BH, is a BH along with no electric charge or angular momentum. The Schwarzschild BH have the most stable inner orbit at the $\frac{6GM}{c^2}$, where G, M and c represents gravitational constant, mass and speed of light, respectively. Also, the orbit having the radius of $\frac{3GM}{c^2}$ describes the geodesic of light. The Schwarzschild BH have the most unstable orbits between the radius $\frac{6GM}{c^2}$ and $\frac{3GM}{c^2}$.

2.5.1 Schwarzschild Metric

The line element of the Schwarzschild metric, in Schwarzschild coordinates has the form

$$ds^2 = \Delta(r)dt^2 - \frac{1}{\Delta(r)}dr^2 - r^2(d\theta^2 + \sin^2\theta d\varphi^2), \qquad (2.5.1)$$

where $\Delta(r) = (1 - \frac{2GM}{r})$. By **Birkhoff's theorem**, it has the general spherical symmetric, static and non-rotating metric. Thus, the Schwarzschild BH is the static BH which can't be differentiated from other BH's disregarded only by it's mass.

2.5.2 Schwarzschild Radius

Schwarzschild radius is the distance between the BH singularity and it's event horizon. The Schwarzschild radius (r_s), which is the radial coordinate value, can be defined as

$$r_s = \frac{2GM}{c^2} = 2m, \tag{2.5.2}$$

where m is the geometric mass of the BH. Schwarzschild radius is that radius at which the escape speed equals the speed of light. If an object is not closer than the r_{s}, it can still escape the black hole.

The Schwarzschild curvature is gotten as, $R_{\mu\nu\rho\sigma}R^{\mu\nu\rho} = \frac{12r_{\mathrm{s}}^2}{r^6}$, the curvature is singular at $r = 0$ but nothing is singular at the horizon. Hence, $r = 0$ is the essential singularity.

2.6 Thermodynamics of Black Holes

2.6.1 Hawking effect

In the background of BH spacetime there exists vacuum fluctuations in the quantum field that pervades all of the spacetime. Therefore, in the empty space there is something happening all the time in the surrounding of the BH. In 1974, Stephen Hawking with the characteristic temperature distribution proved that the BHs aren't completely black but they also emit thermal radiations, by applying the quantum field theory on the static black holes, known as Hawking effect. In the quantum field due to the inherent fluctuations of the black holes cause to happen pair of virtual and real particles and so, those photons which have negative energy fall through the event horizon and those which have positive energy reaches to infinity. For the small BHs, the temperature, when photons reach infinity is $T = \frac{\hbar}{8\pi M}$, where $\hbar$ is the Planck's constant and M is the mass of the BH. Hawking proved, those photons which have a characteristic spectrum of a black body have the temperature.

2.6.2 Area Theorem

Hawking proved that the total area of the event horizons of any collection of classical BHs never reduces, even if they collide and swallow or merge into each other, that is, $\frac{dA}{dt} \geq 0$. The area of the event horizon of the Schwarzschild BH is

$$\begin{aligned} A &= 16\pi M^2. && (2.6.1) \\ dM &= \frac{1}{32\pi M} dA = \frac{1}{8\pi M} d(\frac{A}{4}), && (2.6.2) \end{aligned}$$

where A is the surface area, M is the mass, E is the total energy, dM represents the change in E and $\frac{1}{8\pi M}$ denotes the temperature of the Schwarzschild BH. Moreover, Eq.(2.6.1) can be written as $dE = TdS$ along with

$$S = \frac{A}{4}. \tag{2.6.3}$$

According to area theorem, the entropy S of the BH can never reduce. First and second laws of BH thermodynamics can be found in the Eq. (2.6.1) and Eq. (2.6.3). Therefore, in each and every aspect, BH act in accordance with a thermodynamic black body having temperature $\frac{1}{8\pi M}$ and entropy $\frac{A}{4}$. This correspondence came into view after the discovery of the area theorem but before that it was assumed that the BH have no temperature. Hawking effect fulfilled the incomplete relationship.

2.6.3 Black Hole Thermodynamics Laws

Standard laws of thermodynamics have a resemblance to the laws of BH mechanics. Laws of BH thermodynamics are as

Zeroth Law

The Temperature T_h, on event horizon of BH is constant. The relation $T_h \propto \kappa$ implies that everywhere, on surface of event horizon of BH, surface gravity κ of BH stays constant.

First Law

The first law of BH mechanics can be written as

$$dM = T_h dS_h + \omega_+ dJ,$$

where $dS_h \propto dA_h$, by using this relation, therefore, an entropy S_h, which is proportional to the surface area A_h, can be associated with the BH. Furthermore, more generally, for the charged BH, at an electrostatic potential Φ, the work done in attaining the charge dQ can also be added and the relation can be written as

$$dM = T_h dS_h + \omega_+ dJ + \Phi dQ,$$

where dS is change in entropy. Also, dJ represents change in angular momentum and dQ represents change in electric charge. This is first law of BH thermodynamics. When BH changes it's static state into another, it's mass also changes by the above expression.

Second Law

The area of the BH and therefore, it's entropy doesn't decreases in any process, that is, $\delta S \geq 0$. This is the direct translation of the area theorem.

Third Law

The BH's temperature, by a finite number of operations, cannot be reduced to absolute zero.

2.6.4 Generalized Second Law

In terms of quantum mechanics, the second law of BH thermodynamics does not hold due to the Hawking radiation. As Hawking radiation carry away the mass of the BH and so, the surface area of the BH decreases and so do, eventually, it's entropy. Therefore, the right statement of the second law of BH thermodynamics is the generalized form of the law, which states that, the total entropy of the universe, including all that of the BHs, cannot decreases.

The entropy of BH in universe decreases on no account. Also, the surface area of the BH can never reduces. But the entropy of the BH decreases when the matter falls into the BH and ceases to be visible, which is a contradiction to the second law of BH thermodynamics. As Hawking radiation carry away the mass of the BH and so, the surface area of the BH decreases and so do, eventually, it's entropy, which is also a contradiction to the area theorem. Therefore, a new entropy was presented as $S' = S + \frac{1}{4}A$, where A is total area of all BHs. Also, S is total entropy of all matter and radiation excluding all BHs, to refrain form circumstances like above. Thus, S and A can be changed separately, but new proposed entropy S' never reduces.

The surface area A of BH increases and entropy S decreases at the same time, when BH swallows a matter. Also, the surface area of the event horizon of the BH decreases due to the emission of the thermal radiation and on the other hand, the entropy of the universe increases. Particularly, neither the BH area theorem nor the second law of thermodynamics are satisfied. Bekenstein presented a correspondence between the second law of BH thermodynamics and area theorem by proposing the generalized second law of BH thermodynamics, which is $\Delta S' \geq 0$. In general, Bardeen et al., [93] gave proof of laws of BH mechanics. These are straight mathematical analogues of zeorth and first laws of thermodynamics, which are applicable to static BHs.

2.7 Thermal Stability

For studying stability conditions, sign (positive or negative) of heat capacity has to be considered. The bound point, which be representative of heat capacity, is a point like border, located between the physical BH's associated to the positive temperature and the non-physical BH's which are associated with the negative temperature. One thermal phase transition point, where the BH's pass over from one diverging point to the other one has been indicated by the both divergence points of the heat capacity. Therefore, the divergence point is like a separating point, located between the unstable BH's with negative heat capacity and stable BH's with the positive heat capacity, which are also known as metastable BH's. Hence, logically one can conclude that those BH's whose both heat capacity and temperature are positive, simultaneously, are located everywhere. For investigating the, thermodynamical stability, we utilize a famous relation

$$E = \int T dS, \tag{2.7.1}$$

by using the above relation, internal energy can be determined. We can conclude that the it's value reduces by a strikingly large extent because of the logarithmic corrections.

2.7.1 Thermal Fluctuations

The BH entropy is obtained by famous Bekenstein-Hawking BH entropy formula $S = \frac{A}{4}$, where A is the area of the BH, which is also the maximum entropy, which can be embodied by other objects of the same volume. The entropy of a BH scales with it's area has conducted the advancement in the holographic principle [94, 95]. Because of holographic principle, it causes degrees of freedom in a place of space to be equivalent in relation to degrees of freedom on surface of that specific place of space. Although, the holographic principle is regarded to be invaded near the Planck scale. This contravention of the holographic principle happen due to quantum fluctuations in BH. Therefore, the Bekenstein-Hawking BH entropy relation is supposed to be altered near the Planck scale.

In the BH thermodynamics, the thermal fluctuations are due to the quantum fluctuations in BH [96, 97]. For the large BHs, we can omit these thermal fluctuations, but due to the Hawking radiation, size of the BH decreases and thus, quantum fluctuations increases. Since, the size of the BHs has been decreased, therefore, the thermal fluctuations will start making modifications to the thermodynamics of the BH. Because of these modifications, we need to obtain the correction terms, which are the logarithmic functions of the original thermodynamical quantities, produced due to the thermal fluctuations.

2.7.2 Heat Capacity

In BH thermodynamics, quantity of heat which is necessary to customize temperature of BH, known as thermal capacity or heat capacity. There are two types of heat capacity corresponding to system. Specific heat, when heat is add up to system at the fixed pressure C_p. Also, one which measure the specific heat, when heat is add up to system at the fixed volume C_v. We can evaluate specific heat at fixed volume by utilizing following relation

$$C_v = T\left(\frac{\partial S}{\partial r}\right)\left(\frac{\partial r}{\partial T}\right). \tag{2.7.2}$$

2.7.3 Global Stability

Global stability is accompanied by that phase of a system, which is analogous to the global maximal of the total entropy. Suppose that a there is a system, which is at equilibrium state along a thermodynamic reservoir having temperature T and angular velocities Ω_i. Providing that, the interchange into system and reservoir, for mass ($\Delta M \neq 0$) and angular momentum ($\Delta J_i \neq 0$), the most suitable ensemble is grand canonical ensemble in order to explain the thermodynamics of system. Also, that phase, which lessen down Gibbs free energy in thermodynamic system. One can obtain Gibbs free energy, for given T and Ω_i by,

$$G = M - TS - \Omega_i J_i.$$

Thus, if there are two phases, which are feasible and satisfy $G_1(T, \Omega_i) >$

$G_2(T, \Omega_i)$, then phase 1 is the one which is unstable. The thermal fluctuations will begin from phase 1, by carrying phase transition to phase 2. According to the second law of thermodynamics, it is determined that the internal equilibrium won't be affected, due to the large size of reservoir. Therefore, for the reservoir, interchanges are capable of being restored: $\Delta M_{\text{res}} = T\Delta S_{\text{res}} + \Omega_i \Delta J_i\text{res}$. The exchanges, between mass and angular momentum needs $\Delta M_{\text{res}} = -\Delta M$ and $\Delta J_i\text{res} = -\Delta J_i$. Then, fluctuations obeys favored by the second law,

$$\Delta S + \Delta S_{res} \geq 0,$$

which implies that $\Delta G \leq 0$. If we put a limit on interchange, such that, $\Delta J_i = 0$ but $\Delta M \neq 0$, then the appropriate ensemble is the canonical. The best phase will incline towards the one, which reduces the Helmholtz free energy,

$$F = M - TS,$$

for the given T and J_i. The second law implies that the $\Delta F \leq 0$.

Chapter 3
Modification of the First Law of Thermodynamics

In this chapter, we consider two different RBHs and modify the laws of thermodynamics.

3.1 Modification of First Law of Thermodynamics for non-minimal RBH

The establishment of BH thermodynamics came into being in between 1960's to 1970's. The discoveries of Bekenstein and Hawking break the grounds for BH thermodynamics [20] and [98]. Recently, Astorino examined conserved charges and thermodynamics of accelerating RN-BH, where he used phase space technique for evaluating mass which satisfies standard first law of thermodynamics [99]. In the mean while, Chen et al. [100] make great efforts for modification of first law of thermodynamics with fixed and varied cosmological constant Λ for RN-AdS and dS BHs.

3.1.1 Fixed Cosmological Constant

Consider the line-element of non-minimal MCRBH whose new exact regular spherically symmetric solution was presented by Balakin et al. [101], which is given by

$$ds^2 = -\Delta dt^2 + \frac{1}{\Delta}dr^2 + r^2(d\theta^2 + \sin^2 d\Phi^2), \tag{3.1.1}$$

with the metric function $\Delta(r)$,

$$\Delta(r) = 1 + \left(\frac{r^4}{r^4 + 2Q_m^2\lambda}\right)\left(-\frac{2M}{r} + \frac{Q_m^2}{r^2} - \frac{\Lambda r^2}{3}\right), \tag{3.1.2}$$

where λ, Q_m, M, Λ and r are non-minimal parameter, magnetic charge, mass, cosmological constant and radius of MCRBH, respectively. The limiting case $\lambda = 0$ produces magnetized RN solution in relation to cosmological

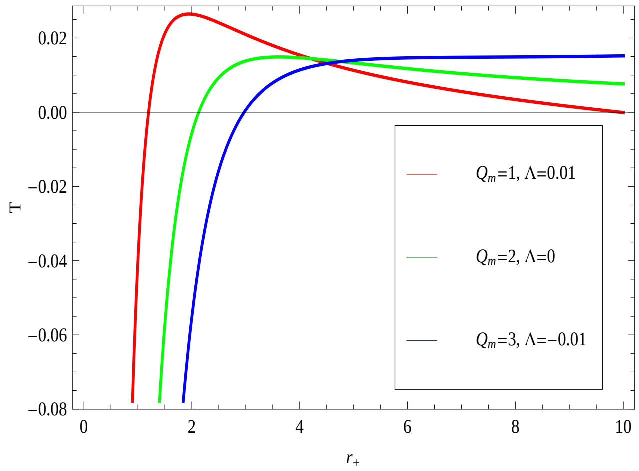


Figure 3.1: Plot of temperature versus horizon radius at the fixed values of magnetic charge.

constant as

$$\Delta(r) = 1 - \frac{2M}{r} + \frac{Q_m^2}{r_+^2} - \frac{\Lambda r_+^2}{3}. \tag{3.1.3}$$

Using Eq. (3.1.2), we can obtain the mass of the non-minimal MCRBH in horizon radius as follows

$$2M = r_+ + \frac{2Q_m^2\lambda}{r_+^3} + \frac{Q_m^2}{r_+} - \frac{\Lambda r_+^3}{3}, \tag{3.1.4}$$

The event horizon of the non-minimal MCRBH is obtained by setting $\Delta(r) = 0$ in Eq. (3.1.2), which is located at the outer horizon r_+.

In order to carry out the analysis of the thermodynamical properties of the non-minimal MCRBH, we use the definition of Misner-Sharp mass [102]. For the non-minimal MCRBH, the Misner-Sharp mass on the horizon can be obtained as $E = M - \frac{Q_m^2\lambda}{r_+^3} - \frac{Q_m^2}{2r_+} - \frac{r_+^3}{2l^2}$. By representing the mass M in E, we can find

$$E = \frac{r_+}{2}. \tag{3.1.5}$$

The non-minimal MCRBH entropy and temperature are

$$S = \frac{A}{4} = \pi r_+^2. \tag{3.1.6}$$

$$T = \frac{\Delta'(r_+)}{4\pi} = \frac{-\Lambda r_+^9 + 3Mr_+^6 + 6Q_m^4 q r_+^2 - 18MQ_m^2\lambda r_+^2 - (6\lambda\Lambda + 3)Q_m^2 r_+^5}{6\pi(r_+^4 + 2Q_m^2\lambda)^2}, \tag{3.1.7}$$

respectively, where $\Delta'(r_+) = \frac{\partial\Delta}{\partial r}|_{r=r_+}$. Temperature T is evaluated by r_+, Q_m and Λ. Figure **3.1** illustrates the behavior of the temperature T with respect to r_+ for specific values of Q_m and Λ. The temperature shows the fluctuations between negative and positive values with the passage of horizon radius. For example, when $\Lambda = 0.01$, the temperature exhibits the negative behavior for $0.55 \leq r_+ \leq 0.9$, however, it is positive and attains maximum value at $r_+ = 1.9$ while approaches to zero for higher values of horizon radius ($r_+ \geq 1$). When $\Lambda = 0$, the temperature is negative for the small values of horizon radius, positive for the higher values and approaches to $T = 0.01$. For $\Lambda = -0.01$, the temperature approaches to $T = 0.15$. We observe that the temperature is higher for positive cosmological constant as compare to the negative cosmological constant and $\Lambda = 0$.

From Eq.(3.1.7), we have the expression for physical mass as

$$M = \frac{-\Lambda r_+^6 + 3Q_m^2 r_+^2 + 3r_+^4 + 6Q_m^2\lambda}{6r_+^3}. \tag{3.1.8}$$

By using Eqs. (3.1.7) and (3.1.8), the temperature leads to

$$T = \frac{r_+^4 - \Lambda r_+^6 - Q_m^2 r_+^2 - 6Q_m^2\lambda}{4\pi r_+(r_+^4 + 2Q_m^2\lambda)}. \tag{3.1.9}$$

The concept of horizon thermodynamics emerged from the discovery that Einsteins equations on the BH horizon can be interpreted as a thermodynamical identity. The radial Einstein equation at the horizon is needed to be evaluated essentially in order to investigate the thermodynamics and the surface tension

$$G_r^r|_{r+} = 8\pi T_r^r|_{r+} = \frac{r_+\Delta'(r_+) - 1}{r_+^2}. \tag{3.1.10}$$

By inserting Eq. (3.1.10) into the Eq. (3.1.7), the non-minimal MCRBH temperature can be rewritten as follows

$$T = \frac{8\pi r_+^2 T_r^r|r_+ + 1}{4\pi r_+}. \tag{3.1.11}$$

Since the entropy of a non-minimal MCRBH which is related to the area of the BH horizon is $S = \pi r_+^2$. Furthermore, the partial derivative of S is $\delta S = 2\pi r_+ \delta r_+$. Multiplying the both sides of Eq. (**??**) by δS, we get

$$T\delta S = 2r_+ T|_r^r \delta S + \frac{\delta r_+}{2}. \tag{3.1.12}$$

The first term on the R.H.S in above equation is depend on the matter while the second term can be identified as the differential form of the Misner-Sharp mass. $T_r^r|_{r+}$ can be obtained from the Eq. (3.1.11) as,

$$T_r^r|_{r+} = \frac{1}{8\pi r_+^2}[\frac{(r_+^4 - \Lambda r_+^6 - Q_m^2 r_+^2 - 6Q_m^2\lambda)}{r_+^4 + 2Q_m^2\lambda} - 1]. \tag{3.1.13}$$

Also, Eq. (3.2.12) can also be written as,

$$\delta E = T\delta S - \sigma\delta A. \tag{3.1.14}$$

The above equation is the modified first law of thermodynamics at the non-minimal MCRBH horizon. Where $\sigma = \frac{r_+ T_r^r|_{r+}}{2} = \frac{1}{16\pi r_+}[\frac{(r_+^4 - \Lambda r_+^6 - Q_m^2 r_+^2 - 6Q_m^2\lambda)}{r_+^4 + 2Q_m^2\lambda} - 1]$ describes the surface tension at the horizon and δA can be identified as the differential form of the horizon area, whereas horizon area is $A = 4S$. The surface tension σ depends upon three factors such as magnetic charge Q_m, non-minimal parameter λ and cosmological constant Λ. The surface tension approaches to zero for $r_+^6 = -\frac{(r_+^2 - 8\lambda)Q_m}{\Lambda}$. If $r_+^6 > -\frac{(r_+^2 - 8\lambda)Q_m^2}{\Lambda}$ then it yields positive surface tension and if $r_+^6 < -\frac{(r_+^2 - 8\lambda)Q_m^2}{\Lambda}$ then the surface tension will be negative. The corresponding Smarr relation takes the following form

$$E = 2TS - 2\sigma A. \tag{3.1.15}$$

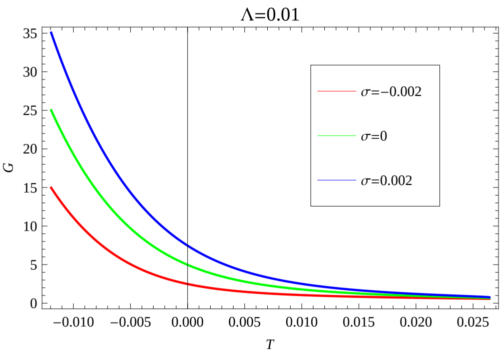


Figure 3.2: Plot of Gibbs free energy versus temperature in the presence of surface tension for first case when cosmological constant is positive and $Q_m = 1$.

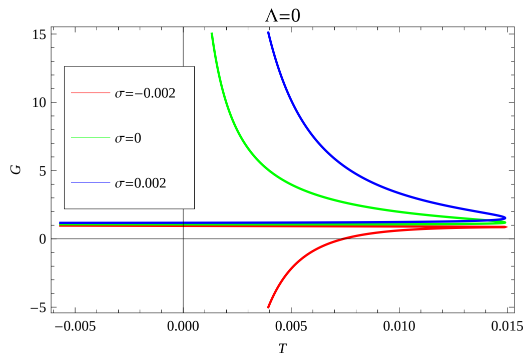


Figure 3.3: Plot of Gibbs free energy versus temperature in the presence of surface tension for second case when the cosmological constant is zero and $Q_m = 2$.

It is found that this relation is satisfied for the values of $E,\ T,\ S,\ A$. It is also mentioned here that the modified first law of thermodynamics in Eq. (3.1.14) is different from the Eq. (3.8) in [103]. Conventional thermodynamics analysis requires an application of the first law of thermodynamics, known as energy analysis. In Conventional thermodynamics, in accordance with the specifications of stable equilibrium, we calculate the thermodynamic variables, such as, the heat capacity at the constant pressure and volume and the Gibbs free energy in order to analyze the local and global thermodynamic stability of the BHs in GR. Moreover, the free energy in the grand canonical ensemble, also called Gibbs free energy. The expression for the Gibbs free energy is

$$G = E - TS + \sigma A, \tag{3.1.16}$$

$$G = \frac{r_+}{2} - \left(\frac{r_+^4 - \Lambda r_+^6 - Q_m^2 r_+^2 - 6Q_m^2\lambda}{4\pi r_+(r_+^4 + 2Q_m^2\lambda)}\right)\pi r_+^2 + \sigma 4\pi r_+^2. \tag{3.1.17}$$

We get $G = \frac{r_+}{4}$, using the expressions for $E,\ T,\ S,\ \sigma$ and A. Carrying out

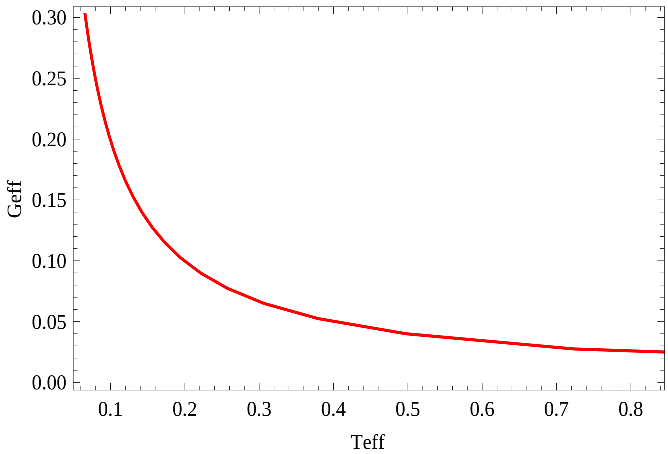


Figure 3.4: Plot of effective Gibbs free energy versus effective temperature.

the differential on the Gibbs free energy yields,

$$\delta G = -S\delta T + A\delta\sigma. \tag{3.1.18}$$

As the effective temperature represents the temperature of a black body that would emit the same total amount of radiation, consider the effective temperature be $T_{eff} = T - 4\sigma = \frac{1}{4\pi r_+}$, and using the effective temperature in Eq.(3.1.17) which leads to $\delta E = T_{eff}\delta S$. The Gibbs free energy in terms of effective temperature will be $G_{eff} = E - T_{eff}S$ which behaves in accordance with $\delta G_{eff} = -S\delta T_{eff}$. In Figure **3.2**, for $-0.002 \leq \sigma \leq 0.002$, we obtain the highest trajectory of the Gibbs free energy for positive cosmological constant. In Figure **3.3**, for the $\Lambda = 0$, the Gibbs free energy increases for positive values of temperature. At $T = 0.015$, the curve bounces back and the energies for $\sigma = 0$ and $\sigma = 0.002$ is positive but for $\sigma = -0.002$, the Gibbs free energy is negative. Figure **3.4** shows that the effective Gibbs free energy decreases with the increase of the effective temperature.

3.1.2 The Pressure

Here, we discuss the pressure at the horizon of MCRBH. Horizon thermodynamics takes a different approach to the problem of pressure in the BH thermodynamics. Since temperature with surface gravity yields $T = \frac{\kappa}{2\pi} = \frac{\Delta'(r_+)}{4\pi}$, horizon thermodynamics is based upon the approach that the energy-momentum tensor on the horizon is interpreted as

$$P = T^r_r|_{r+} = \frac{1}{8\pi r_+^2}[(\frac{r_+^4 - \Lambda r_+^6 - Q_m^2 r_+^2 - 6Q_m^2\lambda}{r_+^4 + 2Q_m^2\lambda}) - 1]. \tag{3.1.19}$$

The area and the volume are inter-linked as $V = \frac{Ar_+}{3}$ which generates $\sigma\delta A = P\delta V$. By using Eq.(3.1.19), the modified law of thermodynamics in the presence of pressure can be written as follows

$$\delta E = T\delta S - P\delta V. \tag{3.1.20}$$

The pressure and surface tensions are directly proportional to each other, thus if pressure is positive, negative or zero then it will be in accordance with positive, negative or zero surface tension, respectively. Using the pressure, $P = T^r_r|_{r+}$ in the Eq. (3.1.11), we get,

$$P = \frac{T}{2r_+} - \frac{1}{8\pi r_+^2}, \tag{3.1.21}$$

which leads to

$$P = \frac{r_+^4 - \Lambda r_+^6 - Q_m^2 r_+^2 - 6Q_m^2\lambda}{8\pi r_+^2(r_+^4 + 2Q_m^2\lambda)} - \frac{1}{8\pi r_+^2}. \tag{3.1.22}$$

The Van der Waals equation correlate with state variables the pressure P, the volume V, the number of particles N and the temperature T, which can be written as

$$P = \frac{kT}{\nu - b} - \frac{a}{\nu^2}, \tag{3.1.23}$$

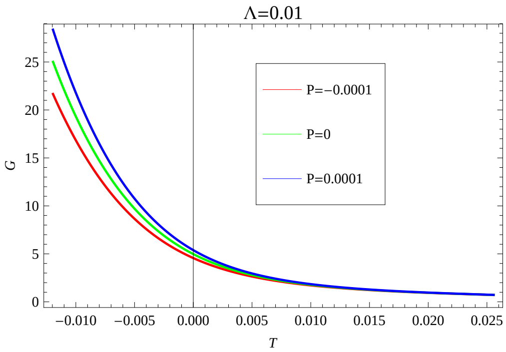


Figure 3.5: Plot of Gibbs free energy versus temperature for first case when $\Lambda = 0.01, Q_m = 1$

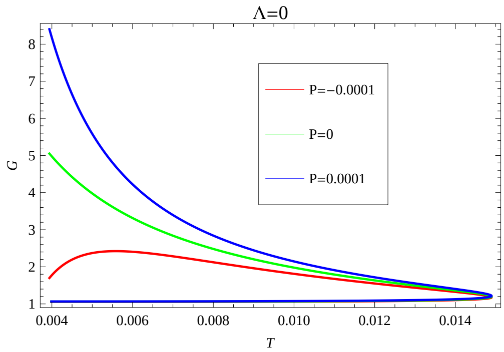


Figure 3.6: Plot of Gibbs free energy versus temperature for second case when $\Lambda = 0, Q_m = 2$

where k is the Boltzmann constant, ν is the specific volume, b is the size of molecules of the system and a is a constant which represents the attraction between the molecules of the system. By comparing the Eq. (3.1.21) and Eq. (3.1.23), we get $a = \frac{1}{2\pi}$, $b = 0$ and $\nu = 2r_+$. Hence the above expression of pressure takes the following form

$$P = \frac{T}{\nu} - \frac{1}{2\pi\nu^2}. \tag{3.1.24}$$

The Gibbs free energy is

$$G = E - TS + PV = \frac{r_+}{2} - \left(\frac{r_+^4 - \Lambda r_+^6 - Q_m^2 r_+^2 - 6Q_m^2\lambda}{4\pi r_+(r_+^4 + 2Q_m^2\lambda)}\right)\pi r_+^2 + P\left(\frac{4\pi r^3}{3}\right), \tag{3.1.25}$$

which behave in accordance with

$$\delta G = -S\delta T + V\delta P, \tag{3.1.26}$$

varying the horizon equation of state by treating the pressure P and temperature T as independent thermodynamics quantities then horizon equation of state established a new horizon first law in [104], (which is same as Eq.(29) of [104]) where temperature T, volume V and pressure P were specified. Also, horizon entropy S and Gibbs free energy G were the derived concepts.

By applying a degenerate Legendre transformation on Eq.(3.1.23), first law can be easily regained. This condition prevail over the mystery between the heat and the work terms. In Figure **3.5**, we observe the behavior of Gibbs free energy for varying quantities of temperature and fixed quantities of pressure. For first case, when the cosmological constant is positive, diagram illustrates that the Gibbs free energy values are lower for the high temperatures. As temperature decreases, Gibbs free energy increases for all fixed values of pressure. In Figure **3.6**, when $\Lambda = 0$, Gibbs free energy remains same with increase in temperature. When $T = 0.016$, curves bounce back and Gibbs free energy increases for lower values of temperature. For negative value of pressure, Gibbs free energy approaches to zero while for positive value of pressure and $P = 0$, Gibbs free energy increases.

3.1.3 Variable Cosmological Constant

Until now, the cosmological constant Λ has been seen as a constant correlated to the pressure P. But now we select the variable cosmological constant Λ and make extra attempts to examine the thermodynamics of the non-minimal MCRBH. In Eq. (3.2.12), the first term on the R.H.S can be rewritten as

$$2r_+T^r_r|_{r+}\delta S = (-\frac{Q_m^2}{4\pi r_+^3} - \frac{Q_m^2\lambda}{\pi r_+^5})\delta S + (-\frac{\Lambda r_+}{4\pi})\delta S. \tag{3.1.27}$$

By taking the differential on the entropy S, horizon area A and the volume V, we get,

$$2r_+T^r_r|_{r+}\delta S = (-\frac{Q_m^2}{16\pi r_+^3} - \frac{Q_m^2\lambda}{4\pi r_+^5})\delta A + (-\frac{\Lambda}{8\pi})\delta V. \tag{3.1.28}$$

which leads to

$$\delta E = T\delta S - \sigma_{eff}\delta A - P\delta V, \tag{3.1.29}$$

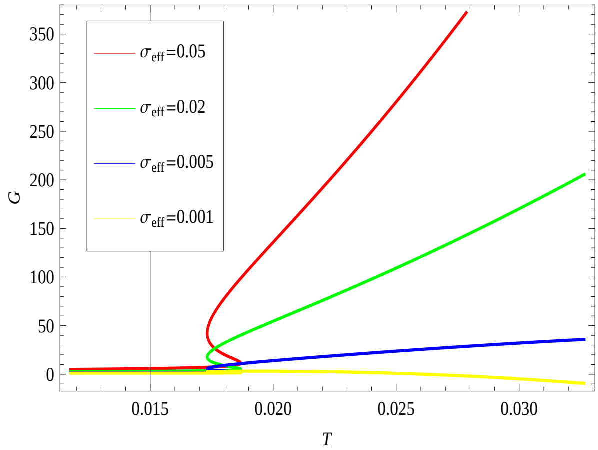


Figure 3.7: Plot of Gibbs free energy versus temperature. The $G-T$ diagram is depicted at the fixed values of $\sigma_{\rm eff}$ and $P = 0.005$.

here, the coefficient of δA depends upon the magnetic charge Q_m, $P = -\frac{\Lambda}{8\pi}$ and $\sigma_{eff} = -\frac{Q_m^2}{16\pi r_+^3} - \frac{Q_m^2\lambda}{4\pi r_+^5}$ (effective surface tension). Moreover, one can get enthalpy (E_o) as follows

$$E_o = E + PV = M - \frac{Q_m^2}{2r_+} - \frac{Q_m^2\lambda}{r_+^3}. \tag{3.1.30}$$

By using the transformation $P(\delta V) = \delta(PV) - V\delta P$ and differential of Eq. (3.1.29), we obtain

$$\delta E_o = T\delta S - \sigma_{eff}\delta A + V\delta P. \tag{3.1.31}$$

Thus, Eq. (3.1.31) is also the modified first law of thermodynamics of the non-minimal MCRBH. There is an additional term such as $V\delta P$ and the total energy which is attained as $M - \frac{Q_m^2}{2r_+} - \frac{Q_m^2\lambda}{r_+^3}$, because of the existence of the cosmological constant Λ as in variable form. The corresponding Smarr relation leads to

$$E_o = 2(TS - \sigma_{\rm eff}A - PV). \tag{3.1.32}$$

By using the expressions of pressure P and temperature T, we obtain

$$P = \frac{T}{2r_+} - \frac{1}{8\pi r_+^2} + \frac{Q^2}{8\pi r_+^4}. \tag{3.1.33}$$

The corresponding Gibbs free energy is $G = E - TS + \sigma_{\text{eff}} A + PV$. In Figure **3.7**, for $T < 0.017$, the Gibbs free energy have lower values for lower values of temperature. When $0.017 \leq T \leq 0.019$, at the different fixed values of effective surface tension, curves depict an interesting behavior. For $T = 0.019$, the curves bounce back and the Gibbs free energy increases for the decreasing values of temperature but when $T = 0.017$ the Gibbs free energy increases for the lower values of temperature. For $\sigma_{\text{eff}} = 0.001$, the Gibbs free energy decreases for the higher values of temperature. We conclude that for the higher positive values of effective surface tension, the Gibbs free energy increases and for lower values of effective surface tension it decreases.

$$\delta E_o = T_o \delta S + V \delta P, \tag{3.1.34}$$

which is the modified first law of thermodynamics for the non-minimal MCRBH and the corresponding Smarr relation turns out to be

$$E_o = 2(T_o S - PV), \tag{3.1.35}$$

where $T_o = T - 4\sigma_{\text{eff}} = \frac{8\lambda^2 Q_m^4 + 2\lambda Q_m^4 r_+^2 - 2\lambda Q_m^2 r_+^4 + r_+^8 - r_+^{10}\Lambda}{8\pi\lambda Q_m^2 r^5 + 4\pi r_+^9}$. Therefore, the new Gibbs free energy in terms of the effective temperature T_o is written as $G = E - T_o S + PV$. In Figure **3.8**, we observe the behavior of Gibbs free energy against T_o, where temperature is redefined as $T_o = T - 4\sigma_{\text{eff}}$. For lower values of temperature, the Gibbs free energy is maximum at higher values of pressure. When pressure is positive, the Gibbs free energy is higher as compared to when $P = 0$. We conclude that the Gibbs free energy decreases with increase in temperature.

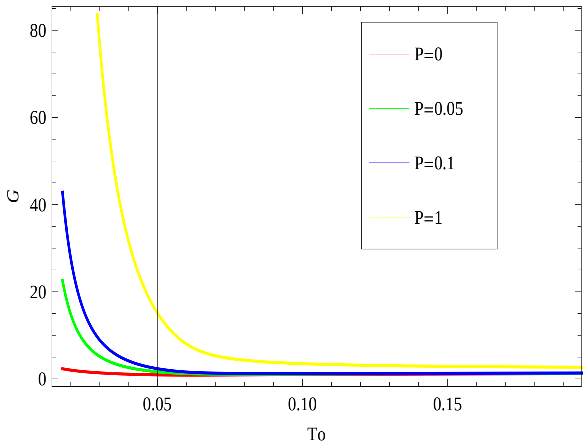


Figure 3.8: Plot of Gibbs free energy versus T_o.

3.2 RBH with Cosmological Constant

3.2.1 Modifying Laws for RBH with Cosmological Constant

Mo Wen-Juan et. al [105] presented a new kind of RBH solution associated with the cosmological constant in NED. The $\Delta(r)$ of this metric has the following form

$$\Delta(r_+) = 1 - 2\left(\frac{\frac{Mr_+^3}{(r_+^2+q^2)^{\frac{3}{2}}} - \frac{q^2r_+^3}{2(r_+^2+q^2)^2} + \frac{\Lambda r_+^3}{6}}{r_+}\right). \tag{3.2.1}$$

By using $\Delta(r) = 0$, we attain the BH horizon. The expression for mass M is ,

$$M = \frac{(r_+^2+q^2)^{\frac{3}{2}}}{2r_+^2} + \frac{q^2}{2(r_+^2+q^2)^{\frac{1}{2}}} - \frac{\Lambda(r_+^2+q^2)^{\frac{3}{2}}}{6}. \tag{3.2.2}$$

The q represent the electric charge of the given RBH. As we are considering the definition of the Misner-Sharp mass for investigating the thermodynamic properties of our RBH, therefore, the expression for mass is $E = \frac{Mr_+^3}{(r_+^2+q^2)^{\frac{3}{2}}} - \frac{q^2r_+^3}{2(r+^2+q^2)^2} + \frac{\Lambda r_+^3}{6}$. By using the expression of mass M from

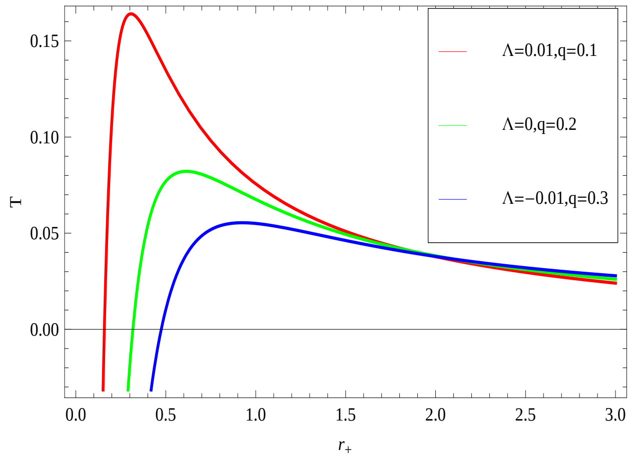


Figure 3.9: Plot of T versus r_+

Eq. (3.2.2) we have get a very brief expression as $E = \frac{r_+}{2}$. The temperature of this RBH with cosmological constant is,

$$T = -\frac{2q^6 + 3q^4 r_+^2 + q^2 r_+^4 - r_+^6 + r_+^4(q^2 + r_+^2)^2\Lambda}{4\pi r_+(q^2 + r_+^2)^3}. \tag{3.2.3}$$

Figure **3.9** illustrates the behavior of the temperature and horizon radius at the fixed values of q and Λ. We have considered the three cases of Λ in Figure **3.9**. For the positive cosmological constant, the temperature is increasing for $0.1 \leq r \leq 0.4$ but it starts decreasing when $r \geq 0.4$. In the absence of cosmological constant, the temperature increases for $0.3 \leq r \leq 0.5$ but when it reaches at $r = 0.5$ it bounces back and starts decreasing. For negative cosmological constant, the temperature is increasing for the small values of horizon radius while it starts decreasing when $r \geq 1.0$. The temperatures almost approach to the same value when $r = 2.0$ but not for $r \geq 2.0$.

When we rewrite Eq.(3.2.12) as Eq.(3.1.20), which is the modification of the first law of thermodynamics at horizon of the RBH with cosmological constant, the surface tension at the horizon becomes

$$\sigma = -\frac{3q^6 + 6q^4 r_+^2 + 4q^2 r_+^4 + r_+^4(q^2 + r_+^2)^2\Lambda}{16\pi r_+(q^2 + r_+^2)^3}. \tag{3.2.4}$$

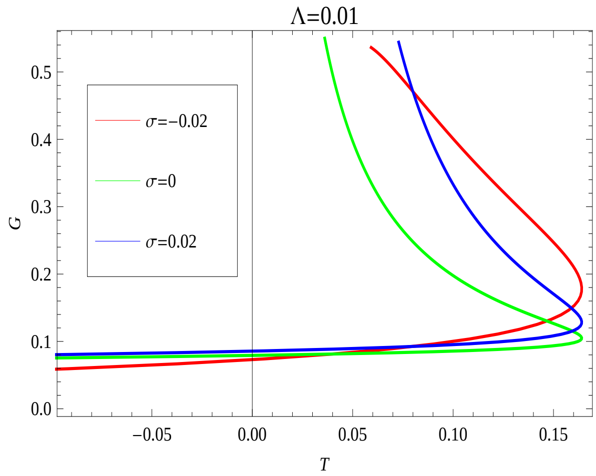


Figure 3.10: Plot of G versus T

The surface tension σ depends on the electric charge q and the cosmological constant Λ. When $r_+^4 = -\frac{(3q^6+6q^4r_+^2+4q^2r_+^4)l^2}{\Lambda(q^2+r_+^2)^2}$, the surface tension is zero, $r_+^4 > -\frac{(3q^6+6q^4r_+^2+4q^2r_+^4)l^2}{\Lambda(q^2+r_+^2)^2}$, the surface tension is positive while negative surface tension is obtained for $r_+^4 < -\frac{(3q^6+6q^4r_+^2+4q^2r_+^4)l^2}{\Lambda(r_+^2+q^2)^2}$. The corresponding Smarr relation is same as Eq. (3.1.15). The Gibbs free energy turns out to be

$$\begin{aligned} G &= E - TS + \sigma A \\ &= \frac{r_+}{2} + \left(2q^6 + 3q^4r_+^2 + q^2r_+^4 - r_+^6 + r_+^4(q^2+r_+^2)^2\Lambda\right) \\ &\times \frac{\pi r_+^2}{4\pi r_+(q^2+r_+^2)^3} + 4\pi r_+^2\sigma. \end{aligned} \tag{3.2.5}$$

Figure **3.10** represents the behavior of Gibbs free energy for the different values of surface tension, three cases will be discussed, i.e. the surface tension is positive, negative and zero. When the temperature is low, Gibbs free energy is increasing at slow rate. When $T = 0.16$, the Gibbs free energy increasing rapidly and temperature bounces back for the $-0.02 \leq \sigma \leq 0.02$. The effective temperature for the RBH with cosmological constant is also found to be $T_{eff} = T - 4\sigma = \frac{1}{4\pi r_+}$. Now we will discuss the pressure for the

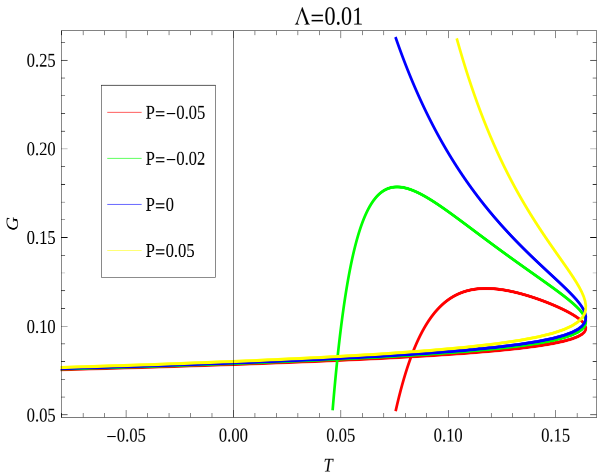


Figure 3.11: Plot of G versus T

RBH with cosmological constant at the horizon. Since the thermodynamic pressure is $P = T_r^r|r_+$, therefore, the expression for the pressure is evaluated as

$$P = -\frac{3q^6 + 6q^4r_+^2 + 4q^2r_+^4 + r_+^4(q^2+r_+^2)^2\Lambda}{8\pi r_+^2(q^2+r_+^2)^3}. \tag{3.2.6}$$

In the presence of pressure, the Gibbs free energy is,

$$\begin{aligned} G &= E - TS + PV \\ &= \frac{r_+}{2} + \left(2q^6 + 3q^4r_+^2 + q^2r_+^4 - r_+^6 + r_+^4(q^2+r_+^2)^2\Lambda\right) \\ &\times \frac{\pi r_+^2}{4\pi r_+(q^2+r_+^2)^3} + P\left(\frac{4\pi r^3}{3}\right). \end{aligned} \tag{3.2.7}$$

Figure **3.11** represents the behavior of the Gibbs free energy varying along with the temperature at the fixed values of pressures. We observe that when $P < 0$, the Gibbs free energy increases for higher values of temperature, after certain value it bounces back and temperature decreases. Once the Gibbs free energy reaches its maximum value, it starts decreasing. Both red and green curves follows this behavior. But when $P \geq 0$, the Gibbs free energy increase for the different values of temperature.

Recently, the cosmological constant has been seen as a varied constant which is correlated with the pressure. We will consider the varied cosmological constant and investigate the thermodynamics of the RBH with the cosmological constant. The first term on the R.H.S of Eq.(3.2.12) can be rewritten as

$$2r_+T^r_r|_{r+}\delta S = \Big(- \frac{3q^6+6q^4r_+^2+4q^2r_+^4}{4\pi r_+(q^2+r_+^2)^3}\Big)\delta S + \frac{-r_+^3\Lambda}{4\pi(q^2+r_+^2)}\delta S, \quad (3.2.8)$$

$$2r_+T^r_r|_{r+}\delta S = \Big(- \frac{3q^6+6q^4r_+^2+4q^2r_+^4}{16\pi r_+(q^2+r_+^2)^3}\Big)\delta A + \frac{-r_+^2\Lambda}{8\pi(q^2+r_+^2)}\delta V. \quad (3.2.9)$$

By using the relations between the entropy, horizon area and volume, we obtain the equal sign in the above equation. It is clear that the first term depends on the electric charge and second term depends on the cosmological constant. Let $\sigma_{\text{eff}} = -\frac{3q^6+6q^4r_+^2+4q^2r_+^4}{16\pi r_+(q^2+r_+^2)^3}$ and $P = \frac{-r_+^2\Lambda}{8\pi(q^2+r_+^2)}$, then also for RBH with cosmological constant, Eq.(3.2.12) takes the form of the Eq.(3.1.29), which further can be rewritten as Eq.(3.1.31). In these equations $E_o = E + PV$ is identified as the enthalpy and its expression is evaluated as

$$E_o = \frac{Mr_+^3}{(r_+^2+q^2)^{\frac{3}{2}}} - \frac{r_+\big(3q^6+9q^4r_+^2+7q^2r_+^4+r_+^2(q^2+r_+^2)^2(2q^2+3r_+^2)\Lambda\big)}{6(q^2+r_+^2)^3}. \quad (3.2.10)$$

After making analysis on the E_o, we have seen that it is the Misner-Sharp mass for the RBH with cosmological constant and by taking into account the expressions of $T,\ S,\ \sigma,\ A,\ P$ and V the corresponding Smarr relation is same as the Eq.(3.1.32), that we have obtained for the non-minimal RBH.

We obtain the expression for the corresponding Gibbs free energy as

$$\begin{aligned} G &= E - TS + \sigma_{\text{eff}}A + PV \\ &= \frac{r_+}{2} + \Big(\frac{2q^6+3q^4r_+^2+q^2r_+^4-r_+^6-8\pi r_+^4(q^2+r_+^2)^2P}{4\pi r_+(q^2+r_+^2)^3}\Big) \\ &\times \pi r_+^2 + 4\pi r_+^2\sigma_{\text{eff}} + \frac{4\pi r_+^3 P}{3}. \end{aligned} \quad (3.2.11)$$

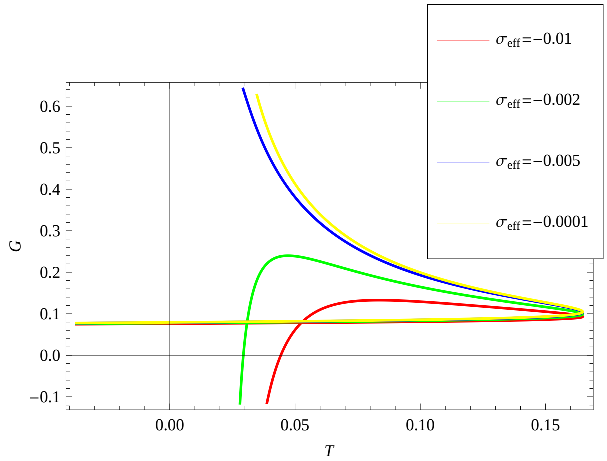


Figure 3.12: Plot of G versus T

In Fig. **3.12**, we observe the behavior of Gibbs free energy at different temperatures and fixed values of σ_{eff}. When $0.01 \leq T \leq 0.17$, the Gibbs free energy increases along with the increase in temperature and give us a characteristic cusp for the fixed values of surface tension. At $T = 0.17$ the curves bounces back, and Gibbs free energy increases for the lower values of temperatures. For $\sigma_{\text{eff}} = -0.01$ the Gibbs free energy start decreasing for the certain value of temperature and becomes negative. When the surface tension $\sigma_{\text{eff}} \geq -0.0001$, the Gibbs free energy have higher values. Eq. (3.1.34) and Eq. (3.1.35) are found to be same for the RBH with cosmological constant. Thus, for the lower negative values of surface tension, the Gibbs free energy have positive higher values. But T_o is,

$$T_o = T - 4\sigma_{\text{eff}} = -\frac{5q^6 + 9q^4 r_+^2 + 5q^2 r_+^4 - r_+^6 + r_+^4 (q^2 + r_+^2)^2 \Lambda}{4\pi r_+ (q^2 + r_+^2)^3}. \quad (3.2.12)$$

Eq. (3.1.34) can also be seen as the modified first law of thermodynamics for the RBH with cosmological constant. The corresponding Gibbs free

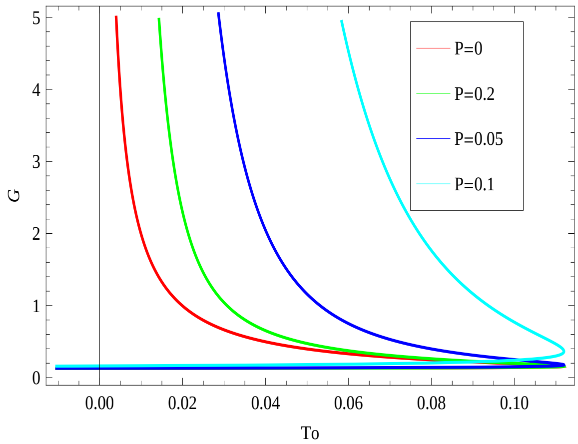


Figure 3.13: Plot of G versus T_o

energy is evaluated as,

$$\begin{aligned} G &= E - T_o S + PV \\ &= \frac{r_+}{2} + \left(- \frac{5q^6 + 9q^4 r_+^2 + 5q^2 r_+^4 - r_+^6 - 8\pi r_+^4 (q^2 + r_+^2)^2 P}{4\pi r_+ (q^2 + r_+^2)^3} \right) \\ &\times \ \pi r_+^2 + P\left(\frac{4\pi r_+^3}{3}\right). \end{aligned} \tag{3.2.13}$$

Fig. **3.13**. illustrate the behavior of the Gibbs free energy at different values of T_o and fixed pressure. When $T_o = 0.112$, the curve bounces back and the Gibbs free energy increases along with the decrease in temperature at fixed values of pressure. When $P = 0$, the Gibbs free energy increases. We conclude that when $0 \leq P \leq 0.5$, the Gibbs free energy increase for the positive values of temperature.

Chapter 4
AdS BH in BI Massive Gravity with a Non-Abelain Hair

4.1 AdS BH in BI Massive Gravity with a Non-Abelain Hair

The exact AdS BH solutions of Einstein-Massive theory in the presence of YM and BI NED is being developed by Hendi and Momennia [90]. Considering whose line element is given by

$$ds^2 = -f(r)dt^2 + \frac{1}{f(r)}dr^2 + r^2(d\theta^2 + \sin^2 d\phi^2), \tag{4.1.1}$$

where the metric function is

$$f(r) = 1 - \frac{m_o}{r} - \frac{\Lambda r^2}{3} + \frac{\upsilon^2}{r^2} + \frac{m^2}{2r}(cc_1 r^2 + 2c^2 c_2 r) + \frac{2\beta^2 r^2}{3}(1 - \chi_1), \tag{4.1.2}$$

where

$$\chi_1 =_2 F_1(\frac{-1}{2}, \frac{-3}{4}; \frac{1}{4}; \frac{-q^2}{\beta^2 r^4}), \tag{4.1.3}$$

which is a hypergeometric function, Λ is the cosmological constant, m is the massless graviton, q is the integration constant which is related to the total electric charge of BH, ν is the magnetic parameter, c, c_1 and c_2 are free constants. The m_o is the integration constant which relates with the total mass of BH. The obtained solution possess a Coulomb charge, massive term and a non-abelian hair.

By taking into consideration the obtained $f(r)$, the fourth term $\frac{\nu^2}{r^2}$ is related to the magnetic charge with non-abelian hair, the fifth term is related to the massive gravitons and the last term is related to the non-linearity of electric charge. For the massless graviton such as $m = 0$ and for the linear electrodynamics $\beta \longrightarrow \infty$, the $f(r)$ reduces to the EYM solution with Maxwell field. By setting $f(r = r_+) = 0$, we have

$$m_o = r - \frac{\Lambda r^3}{3} + \frac{\nu^2}{r} + \frac{m^2(cc_1 r^2 + 2c^2 c_2 r)}{2} + \frac{2\beta^2 r^3(1 - \chi_1)}{3}. \tag{4.1.4}$$

By using the Hamiltonian approach, it was introduced that the total mass M of BH can be obtained by the massive gravity as [107]

$$M = \frac{m_o}{2}. \tag{4.1.5}$$

Hence, the total mass of BH reduces to

$$M = \frac{r}{2} - \frac{\Lambda r^3}{6} + \frac{\nu^2}{2r} + \frac{m^2(cc_1 r^2 + 2c^2 c_2 r)}{4} + \frac{\beta^2 r^3 (1 - \chi_1)}{3}, \tag{4.1.6}$$

which implies that outer horizon $r_+ \neq 0$. The entropy and volume is related to BH horizon, and are given by

$$S_o = \pi r_+^2, V = \frac{4\pi r_+^3}{3}. \tag{4.1.7}$$

In order to study the thermal fluctuations, thermal stability and phase transition of BH solutions, we examine the conserved and thermodynamic quantities such as pressure, entropy, specific heats, Gibbs free energy and Helmholtz free energy of BH's by using the logarithmic correction terms. The temperature of AdS BH in BI massive with a non-abelian hair can be written as

$$T = \frac{f'(r)}{4\pi} |_{r=r_+}= \tag{4.1.8}$$

$$T = \frac{1}{4\pi r_+}\left(1 - \Lambda r_+^2 - \frac{\nu^2}{r_+^2} + m^2(cc_1 r_+ + c^2 c_2) + 2\beta^2 r_+^2 \left(1 - \sqrt{1 + \frac{q^2}{\beta^2 r_+^2}}\right)\right). \tag{4.1.9}$$

By taking differential of Eq. (4.1.9), we get

$$\frac{\partial T}{\partial r} = -\frac{1}{4\pi}\left(\frac{-4q^2}{r^4\sqrt{1+\frac{q^2}{\beta^2 r^4}}} - 2\beta^2 + 2\beta^2\sqrt{1 + \frac{q^2}{\beta^2 r^4}} + \Lambda - \frac{3\nu}{r^4} + \frac{1 + c^2 m^2 c_2}{r^2}\right). \tag{4.1.10}$$

Now we use the logarithmic correction terms for the entropy in order to discuss the thermal fluctuations. The corrected term for the entropy is

$$S = S_o - \frac{b}{2}\log[S_o T^2], \tag{4.1.11}$$

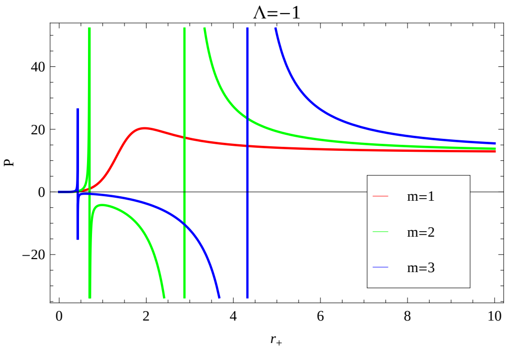


Figure 4.1: Plot of P versus r_+ for AdS BH in BI massive gravity with a non-abelain hair for negative cosmological constant. Specific values of different parameters are $\beta = 1,\ \nu = 1,\ c = 1$ and $c_2 = 2$.

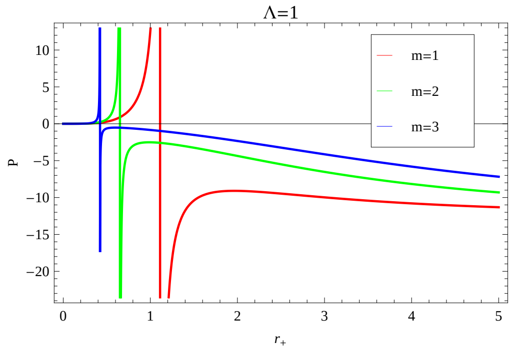


Figure 4.2: Plot of P versus r_+ for AdS BH in BI massive gravity with a non-abelain hair for positive cosmological constant. Specific values of different parameters are $\beta = 1,\ \nu = 1,\ c = 1$ and $c_2 = 2$.

where b is a constant parameter, which is added in order to handle the logarithmic correction terms producing because of the thermal fluctuations. By setting $b = 0$, entropy can be recovered without any correction term. For large BHs, temperature is very small, one can consider $b = 0$ and for small BHs, temperature is very large, one can consider $b = 1$. Using Eqs. (4.1.7) and (4.1.9) in Eq. (4.1.11), we obtain

$$S = \pi r^2 - \frac{b}{2}\log\left[\frac{\left(1 - \Lambda r_+^2 - \frac{\nu^2}{r_+^2} + m^2(cc_1 r_+ + c^2 c_2) + 2\beta^2 r_+^2\left(1 - \sqrt{1 + \frac{q^2}{\beta^2 r_+^2}}\right)\right)^2}{16\pi}\right]. \tag{4.1.12}$$

Now, we calculate pressure by using the following relation:

$$P = T\left(\frac{\partial S}{\partial r}\right)\left(\frac{\partial r}{\partial V}\right) \tag{4.1.13}$$

Using Eqs. (4.1.7), (4.1.9) and (4.1.12), one can be obtain

$$P = -\frac{4\pi}{-\frac{4q^2}{r^4\sqrt{1+\frac{q^2}{\beta^2 r^4}}} - 2\beta^2 + 2\beta^2\sqrt{1+\frac{q^2}{\beta^2 r^4}} + \Lambda - \frac{3\nu}{r^4} + \frac{1+c^2m^2c_2}{r^2}}. \tag{4.1.14}$$

Figures. **4.1** and **4.2**, represent the behavior of pressure for negative and positive cosmological constant, respectively, for fixed values of m. In Fig. **4.1**, when $m = 1$, the pressure is positive and it is highest for $r_+ = 1.8$. For $m = 3$, there are two divergent points, when $0 \leq r_+ \leq 0.49,\ 0.5 \leq r_+ \leq 3.85$ and $r_+ \geq 4.45$ the pressure is positive, negative and positive, respectively. The value of pressure is higher for $m = 3$ as compared to $m = 1$ and $m = 2$. In Fig. **4.2**, for the small horizon radius, the pressure is positive but for the large horizon radius, pressure is negative. Hence, we conclude that for the positive cosmological constant, the pressure decreases but for the higher values of m it increases. The first law of thermodynamics is defined as [108]-[109]

$$dM = TdS + VdP + \cdots. \tag{4.1.15}$$

One can easily examine the validity of the above relation and see that the its not valid. We will use the logarithmic corrections in order to obtain the thermodynamic quantities which have to satisfy the above relation.

For investigating the first law of thermodynamics, Eq. (4.1.15) can be re-written as follows:

$$dM - TdS - VdP = 0. \tag{4.1.16}$$

4.1.1 Thermal Stability

Here, we will investigate the thermal stability of the AdS BH in BI massive gravity with a non-abelian hair.

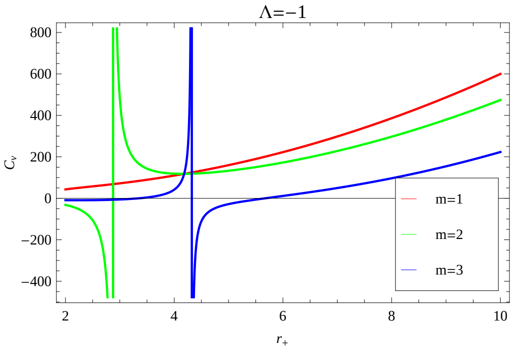


Figure 4.3: Plot of specific heat versus horizon radius for negative cosmological constant. Specific values of parameters are $\beta = 1,\ \nu = 1,\ c = 1, c_1 = 1, c_2 = 1$ and $b = 0$

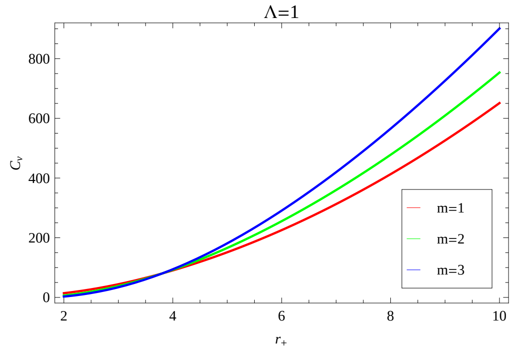


Figure 4.4: Plot of specific heat versus horizon radius for positive cosmological constant. Specific values of parameters are $\beta = 1,\ \nu = 1,\ c = 1, c_1 = 1, c_2 = 1$ and $b = 0$

In BH thermodynamics, the amount of heat that is required to change the temperature of a BH, is known as thermal capacity or heat capacity. We obtain the specific heat at the constant volume by using the following relation

$$C_v = T(\frac{\partial S}{\partial r})(\frac{\partial r}{\partial T}). \tag{4.1.17}$$

Using Eqs. (4.1.9), (4.1.10) and (4.1.12), we obtain

$$\begin{aligned} C_v &= \frac{-1}{r\Big(-\frac{4q^2}{r^4\sqrt{1+\frac{q^2}{\beta^2r^4}}} - 2\beta^2 + 2\beta^2\sqrt{1+\frac{q^2}{\beta^2r^4}} + \Lambda - \frac{3\nu}{r^4} + \frac{1+c^2m^2c_2}{r^2}\Big)} \\ &\times \Big(1 - 2r^2\Big(-1 + \sqrt{1+\frac{q^2}{\beta^2r^4}}\beta^2 - r^2\Lambda - \frac{\nu}{r^2} + cm^2(rc_1+cc_2)\Big)\Big) \\ &\times \Big(2\pi r - \Big(b\Big(4r\beta^2 - \frac{4r\beta^2}{\sqrt{1+\frac{q^2}{\beta^2r^4}}} - 2r\Lambda + \frac{2\nu}{r^3} + cm^2c_1\Big)\Big)\Big(1 \\ &- 2r^2\Big(-1 + \sqrt{1+\frac{q^2}{\beta^2r^4}}\beta^2 - r^2\Lambda - \frac{\nu}{r^2} + cm^2(rc_1+cc_2)\Big)\Big)^{-1}\Big) \end{aligned} \tag{4.1.18}$$

The behaviors of heat capacity at the constant volume for the AdS BH in BI massive gravity with a non-abelian hair is displayed in Figs. **4.3** and

4.4. In order to check the local stability of the BH we discuss two cases for different fixed values of m. When $\Lambda = -1$, $m = 1$ the heat capacity always remain positive and indicates that the BH is locally stable and no phase transition take place. For $m = 2$, we can observe that the graph displays three regions divided by two divergent points. The intermediate BH (IBH) and the large BH (LBH) have positive heat capacity and the BH is thermodynamically stable in their regions. But for the small BH (SBH), the heat capacity is negative implying that BH is not stable in this region. When $m = 3$, the phase transition takes place between SBH and LBH. For $\Lambda = 1$ and all fixed values of m, heat capacity is positive implying that no phase transition will take place and thus, the BH is locally stable for $\Lambda = 1$.

Furthermore, the specific heat at the constant pressure can be evaluated by the following relation

$$C_p = \frac{(\frac{\partial M}{\partial r})}{(\frac{\partial T}{\partial r})}. \tag{4.1.19}$$

Using Eqs. (4.1.6) and (4.1.10), we have

$$C_p = \frac{-2\pi\left(r^2 - r^4(2\beta^2(-1+\sqrt{1-\frac{q^2}{\beta^2 r^4}}) + \Lambda) - \nu^2 + cm^2r^2(rc_1 + cc_2)\right)}{r^2\left(-\frac{4q^2}{r^4\sqrt{1+\frac{q^2}{\beta^2 r^4}}} - 2\beta^2 + 2\beta^2\sqrt{1+\frac{q^2}{\beta^2 r^4}} + \Lambda - \frac{3\nu}{r^4} + \frac{1+c^2m^2c_2}{r^2}\right)}. \tag{4.1.20}$$

The ratio of the above two heat capacities is also investigated which can be expressed as

$$\gamma = \frac{C_p}{C_v} \tag{4.1.21}$$

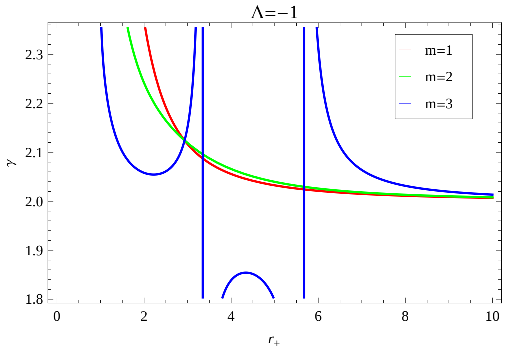


Figure 4.5: Plot of γ versus r_+ for AdS BH in BI massive gravity with a non-abelian hair for negative cosmological constant. Specific values of different parameters are $\beta = 1,\ \nu = 1,\ c = 1,\ c_1 = -1,\ c_2 = 2$ and $b = 0$

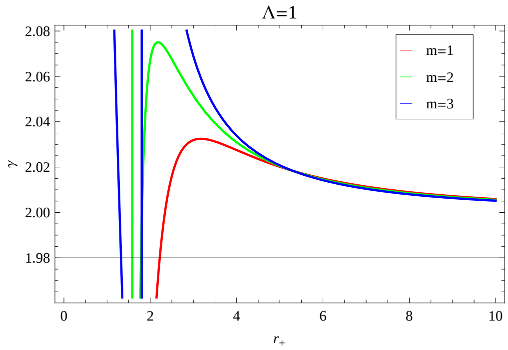


Figure 4.6: Plot of γ versus r_+ for AdS BH in BI massive gravity with a non-abelian hair for positive cosmological constant. Specific values of parameters are $\beta = 1,\ \nu = 1,\ c = 1,\ c_1 = -1,\ c_2 = 2$ and $b = 0$

Using Eqs. (4.1.18) and (4.1.20), we obtain

$$\begin{aligned}\gamma &= \Bigg(4\pi r^2(q^2+r^4\beta^2)\big(-r^2+r^4(2\beta^2(-1+\sqrt{1-\frac{q^2}{\beta^2 r^4}})+\Lambda)+\nu^2\\ &- cm^2r^2(rc_1+cc_2)\big)\Bigg)\Bigg(-4b\beta^4r^8\sqrt{1+\frac{q^2}{\beta^2r^4}}+2b(q^2+r^4\beta^2)\\ &\times (r^4(2\beta^2-\Lambda)+\nu)+2\pi r^2(q^2+r^4+\beta^2)\big(-r^2+r^4\big(2\beta^2\big(-1\\ &+ \sqrt{1+\frac{q^2}{\beta^2r^4}}\big)+\Lambda\big)+\nu\big)+cm^2r^3(q^2+r^4\beta^2)\big((b-2\pi r^2)c_1\\ &- 2c\pi rc_2\big)\Bigg)^{-1}. \end{aligned} \tag{4.1.22}$$

Figs. **4.5** and **4.6** illustrate the behavior of γ for $\Lambda = -1$ and $\Lambda = 1$, respectively. Because of the logarithmic correction to the entropy, the value of the γ increases for $\Lambda = 1$ and decreases for $\Lambda = -1$. When $\Lambda = -1$, the value of γ decreases and shows a stable behavior for $m = 1$ and $m = 2$. But when $m = 3$, the values of γ shows an unstable behavior for larger horizon it becomes stable. When $\Lambda = 1$, the values of γ demonstrate a stable behavior for all different fixed value of m and approaches to the same value of γ. Hence, we can conclude that the for the positive cosmological constant, γ represents a stable behavior for all values of horizon radius but for the negative cosmological constant, represents an unstable behavior for small values of horizon radius and higher values of massless graviton but, for the larger values of horizon radius it represents a stable behavior.

4.1.2 The Gibbs free energy

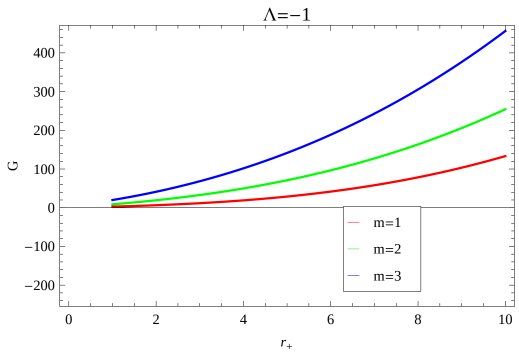
In order to analyze and discuss the global stability, we now treat the AdS BH in BI massive gravity with a non-abelian hair as a thermodynamical object by considering it as a grand canonical ensemble. In the grand canonical ensemble, free energy is also known as the Gibbs free energy. For further discussion, the Gibbs free energy can be defined as

$$G = M - TS. \tag{4.1.23}$$

Using Eqs. (4.1.6), (4.1.9) and (4.1.12), we get

$$\begin{aligned} G &= r - \frac{r^3\Lambda}{3} + \frac{\nu^2}{r} - \frac{2r^3\beta^2}{3}\Big(-1 +_2 F_1(\frac{-1}{2}, \frac{-3}{4}; \frac{1}{4}; \frac{-q^2}{\beta^2 r^4})\Big) + \frac{1}{2}cm^2 r\big(rc_1 + 2cc_2\big) \\ &+ \Big(\frac{1}{8\pi r^3}\Big) \times \Big(2\pi r^2 + b\log[16\pi] - b\log[\Big(-1 + 2r^2\beta^2\Big(-1 + \sqrt{1 + \frac{q^2}{r^4\beta^2}}\Big) \\ &+ r^2\Lambda + \frac{\nu}{r^2} - cm^2(rc_1 + cc_2)\Big)^2]\Big)\Big(-r^2 + r^4\Big(2\beta^2\Big(-1 + \sqrt{1 + \frac{q^2}{r^4\beta^2}}\Big) + \Lambda\Big) \\ &+ \nu - cm^2 r^2(rc_1 + cc_2)\Big) \end{aligned} \tag{4.1.24}$$

The graphical representation of the Gibbs free energy and the horizon radius for the fixed values of massless graviton and cosmological constant takes place in the Figs. **4.7** and **4.8**. For $\Lambda = -1$, the Gibbs free energy is positive for the larger horizon radius. The value of Gibbs free energy is higher for $m = 3$ as compared to $m = 1$. Thus, we conclude that the



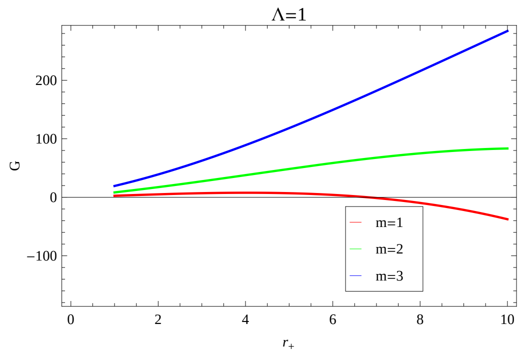


Figure 4.7: Plot of Gibbs free energy versus r_+ for AdS BH in BI massive gravity with a non-abelian hair for positive cosmological constant. Specific values of parameters are $\beta = 1$, $\nu = 1$, $c = 1$, $c_1 = 1$, $c_2 = 2$ and $b = 0$

Figure 4.8: Plot of Gibbs free energy versus r_+ for AdS BH in BI massive gravity with a non-abelian hair for positive cosmological constant. Specific values of parameters are $\beta = 1$, $\nu = 1$, $c = 1$, $c_1 = 1$, $c_2 = 2$ and $b = 0$

Gibbs free energy of AdS BH in BI massive gravity with a non-abelian hair is higher for the larger horizon and higher values of massless graviton for negative cosmological constant. In Fig. **4.8**, when $\Lambda = 1$, the Gibbs free energy is positive and higher for $m = 2$ and $m = 3$. For $m = 1$, the Gibbs free energy approaches to zero, when $r_+ = 7$ and it becomes negative for $r_+ > 7$. We conclude that, for the positive cosmological constant, the Gibbs free energy is negative for lower values of m and positive for higher values of m.

4.1.3 Canonical ensemble

If the transference of the charge on the BH is restrained then the BH could be regarded as a thermodynamically closed system such as a conical ensemble. When the charge is fixed, the free energy is known as the Helmholtz free energy in the canonical ensemble and its expression can be derived by using the following relation

$$F = G - PV. \tag{4.1.25}$$

By using the Eqs. (4.1.7), (4.1.14) and (4.1.24), the Helmholtz free energy for AdS BH in BI massive gravity with a non-abelain hair can be obtained as

$$\begin{aligned}
F &= r - \frac{r^3\Lambda}{3} + \frac{\nu^2}{r} - \frac{2r^3\beta^2}{3}\Big(-1 +{}_2F_1(\frac{-1}{2}, \frac{-3}{4}; \frac{1}{4}; \frac{-q^2}{\beta^2 r^4})\Big) \\
&+ \frac{1}{2}cm^2 r\big(rc_1 + 2cc_2\big) + \Big(\frac{1}{8\pi r^3}\Big)\Big(2\pi r^2 + b\log[16\pi] - b\log\Big[\Big(-1 \\
&+ 2r^2\beta^2\Big(-1 + \sqrt{1 + \frac{q^2}{r^4\beta^2}}\Big) + r^2\Lambda + \frac{\nu}{r^2} - cm^2(rc_1 + cc_2)\Big)^2\Big]\Big) \\
&\times \Big(-r^2 + r^4\Big(2\beta^2\Big(-1 + \sqrt{1 + \frac{q^2}{r^4\beta^2}}\Big) + \Lambda\Big) + \nu - cm^2r^2(rc_1 + cc_2)\Big) \\
&+ (16\pi^2 r^3)\Big(3\Big(- \frac{4q^2}{r^4\sqrt{1 + \frac{q^2}{\beta^2 r^4}}} - 2\beta^2 + 2\beta^2\sqrt{1 + \frac{q^2}{\beta^2 r^4}} + \Lambda - \frac{3\nu}{r^4} \\
&+ \frac{1 + c^2m^2c_2}{r^2}\Big)\Big)^{-1}
\end{aligned} \tag{4.1.26}$$

Behavior of the Helmholtz free energy F against the horizon radius r_+ is represented in the Figs. **4.9** and **4.10** for both positive and negative cosmological constant at fixed values of massless graviton. When $\Lambda = -1$, for $m = 1$ the Helmholtz free energy decreases and becomes negative for the larger horizon. When $m = 2$, the Helmholtz free energy is positive for SBH but becomes negative when $r_+ \geq 2.8$. When $m = 3$, the Helmholtz free energy becomes negative for $r_+ \geq 4.3$. We conclude that the Helmholtz free energy is negative for the larger horizon and higher values of massless graviton for the negative cosmological constant. When $\Lambda = 1$, for $m = 1$ the Helmholtz free energy is negative for SBH but, when $r \geq 1.1$ Helmholtz free energy is positive. When $m = 2$ and $m = 3$, the Helmholtz free energy is

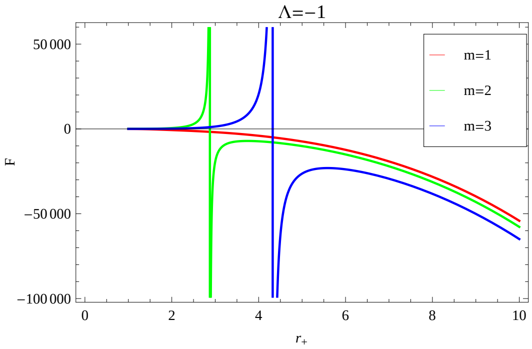


Figure 4.9: Plot of Helmholtz free energy versus r_+ for AdS BH in BI massive gravity with a non-abelian hair for positive cosmological constant. Specific values of parameters are $\beta = 1,\ \nu = 1,\ c = 1,\ c_1 = -1,\ c_2 = 2$ and $b = 0$

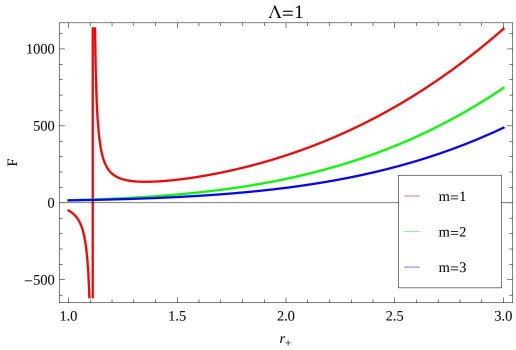


Figure 4.10: Plot of Helmholtz free energy versus r_+ for AdS BH in BI massive gravity with a non-abelian hair for positive cosmological constant. Specific values of parameters are $\beta = 1,\ \nu = 1,\ c = 1,\ c_1 = -1,\ c_2 = 2$ and $b = 0$

positive and increases for the larger values of horizon radius. We conclude that, the Helmholtz free energy is higher for $m = 1$ as compared with $m = 2$ and $m = 3$. Also the Helmholtz free energy is positive, which leads to stability for the larger horizon radius and positive cosmological constant.

4.2 Charged AdS BH With a Global Monopole

The gravitational field of a global monopole, which is an approximate solution for the metric outside a monopole resulting from the breaking of a $O(3)$ symmetry by Barriola and Vilenkin [110]. When a charged AdS BHs swallows a global monopole, the general static spherically symmetric metric and gauged potential can be given by [111]

$$d\tilde{s}^2 = -\tilde{f}(\tilde{r})d\tilde{t}^2 + \frac{1}{\tilde{h}(\tilde{r})} + \tilde{r}^2(d\theta^2 + sin^2\theta d\phi^2), \tag{4.2.1}$$

$$\tilde{A} = \frac{\tilde{q}}{\tilde{r}}d\tilde{t}, \tilde{f}(\tilde{r}) = \tilde{h}(\tilde{r}) = 1 - 8\pi\eta_o^2 - \frac{2\tilde{m}}{\tilde{r}} + \frac{\tilde{q}^2}{\tilde{r}^2} + \frac{\tilde{r}^2}{l^2}, \tag{4.2.2}$$

where $\tilde{m}$ is the mass parameter and $\tilde{q}$ is an electric charge parameter. Also, l is associated with the cosmological constant as $\Lambda = -\frac{3}{l^2}$. By using the co-ordinate transformations as follows

$$\tilde{t} = (1 - 8\pi\eta_o^2)^{-\frac{1}{2}}t, \tilde{r} = (1 - 8\pi\eta_o^2)^{\frac{1}{2}}r. \tag{4.2.3}$$

New parameters can be defined as follows

$$m = (1 - 8\pi\eta_o^2)^{-\frac{3}{2}}\tilde{m}, q = (1 - 8\pi\eta_o^2)^{-1}\tilde{q}, \eta^2 = 8\pi\eta_o^2, \tag{4.2.4}$$

where η is the symmetry breaking parameter. In view of Eqs. (4.2.1) and (4.2.3), Eq. (4.2.4) can be written as,

$$ds^2 = -f(r)dt^2 + \frac{1}{h(r)}dr^2 + (1 - \eta^2)r^2(d\theta^2 + sin^2\theta d\phi^2), \tag{4.2.5}$$

particularly, the solution will take the form of the four-dimensional Reissner-Nordstöm (RN) AdS BH, if we set $\eta = 0$, where

$$A = \frac{q}{r}dt, f(r) = h(r) = 1 - \frac{2m}{r} + \frac{q^2}{r^2} + \frac{r^2}{l^2}, \tag{4.2.6}$$

by using $l^2 = -\frac{3}{\Lambda}$, we get

$$f(r) = 1 - \frac{2m}{r} + \frac{q^2}{r^2} - \frac{\Lambda r^2}{3}, \tag{4.2.7}$$

In order to calculate the thermodynamical quantities of charged AdS BH with global monopole, we begin with the event horizon of the BH situated at $r = r_h$, which is the largest root of the $f(r) = h(r) = 0$. At infinity, the electric charge q along with the potential can be evaluated by

$$Q = \frac{1}{4\pi}\oint F^{\mu\omega}d^2\sum \mu\omega = (1 - \eta^2)q, \tag{4.2.8}$$

$$\Phi = \xi^\mu A_\mu|_{r=r_h} = \frac{q}{r_h}, \tag{4.2.9}$$

where ξ^{μ} is the timelike killing vector $(\partial_t)^{\mu}$ [112]. The Arnowitt-Deser-Misner (ADM) mass M of the system can be evaluated by the Komar integral

$$M = \frac{1}{8\pi}\oint \xi^{\mu;\nu}_{(t)} d^2 \sum_{\mu\nu} \tag{4.2.10}$$

$$M = (1-\eta^2)m. \tag{4.2.11}$$

In order to find the expression for mass, put $f(r) = 0$ in Eq. (4.2.7), we obtain

$$m = \frac{r}{2} + \frac{q^2}{2r} - \frac{\Lambda r^3}{6}. \tag{4.2.12}$$

By using Eqs. (4.2.8), (4.2.11) and (4.2.12), we get

$$M = (1-\eta^2)\left(\frac{r}{2} + \frac{Q^2}{(1-\eta^2)^2 2r} - \frac{\Lambda r^3}{6}\right). \tag{4.2.13}$$

The Hawking temperature T of the charged AdS BH with global monopole is given by

$$T = \frac{f'(r)}{4\pi} = \frac{1}{4\pi r_h}\left(1 - \Lambda r_h^2 - \frac{Q^2}{(1-\eta^2)^2 r_h^2}\right). \tag{4.2.14}$$

The area A at the horizon of the charged AdS BH with golobal monopole can be obtained as

$$A = \int_{r=r_h} \sqrt{g_{\theta\theta}g_{\phi\phi}}\, d\theta d\phi = 4\pi(1-\eta^2)r_h^2. \tag{4.2.15}$$

The entropy S_o can be evaluated by using the above equation, we get

$$S_o = \frac{A}{4} = \pi(1-\eta^2)r_h^2. \tag{4.2.16}$$

The logarithmic correction term for the entropy of charged AdS BH with global monopole can obtained by using Eq. (4.1.11) as

$$S = \pi(1-\eta^2)r^2 - \frac{b}{2}\log\left[\frac{(1-\eta^2)\left(1 + \frac{Q^2}{r^2(-1+\eta^2)^2} - r^2\Lambda\right)^2}{16\pi}\right]. \tag{4.2.17}$$

The cosmological constant can be interpreted as the thermodynamic pressure as [113]

$$P = -\frac{\Lambda}{8\pi}, \tag{4.2.18}$$

also, the associated conjugate thermodynamic volume $V = (\frac{\partial M}{\partial P})$ can be given as

$$V = \frac{4\pi(1-\eta^2)r_h^3}{3}. \tag{4.2.19}$$

From above results, it is clear that the thermodynamical quantities such as, electric charge Q, ADM mass M, area at horizon of BH A, thermodynamic volume V, corrected term of entropy S and temperature T are associated in relation to symmetry breaking parameter η. First law of thermodynamics holds for above thermodynamical quantities and can be verified easily

$$dM = TdS + \Phi dQ + VdP, \tag{4.2.20}$$

and obey Smarr relation

$$M = 2(TS - VP) + \Phi Q \tag{4.2.21}$$

Hence, we conclude that 1^{st} law of thermodynamics and corresponding Smarr relation has no influence because of global monopole. As compared to RN AdS BH [114], the thermodynamical quantities of the charged AdS BH with global monopole shows a remarkable dependence on the internal global monopole.

We step forward, in order to analyze and discuss the thermal stability, phase transition, local and global stability in view of grand canonical and canonical ensemble. We have defined the Hawking temperature T and the thermodynamic volume V as before and we can evaluate the pressure P by

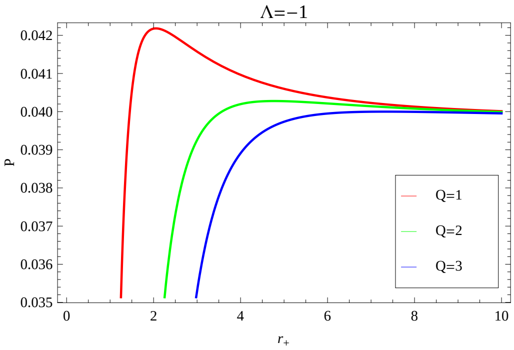


Figure 4.11: Plot of P versus r_+ for charged AdS BH with a global monopole for negative cosmological constant.

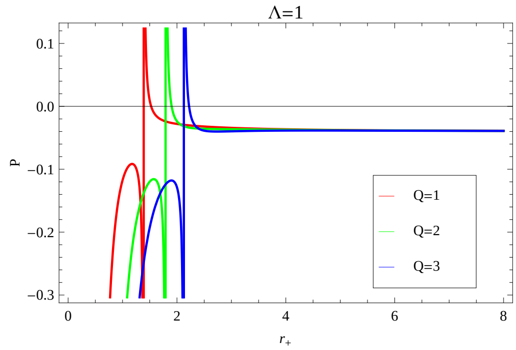


Figure 4.12: Plot of P versus r_+ for charged AdS BH with a global monopole for positive cosmological constant.

using the following relation such as

$$P = T(\frac{\partial S}{\partial r})(\frac{\partial r}{\partial V}), \tag{4.2.22}$$

$$P = \frac{\left(1 - \frac{Q^2}{r^2(-1+\eta^2)^2} - r^2\Lambda\right)\left(-2\pi r(-1+\eta^2) + \frac{2b(Q^2+r^4(-1+\eta^2)^2\Lambda)}{r(Q^2-r^2(-1+\eta^2)^2(-1+r^2\Lambda))}\right)}{16\pi^2 r^3(1-\eta^2)} \tag{4.2.23}$$

In Figs. **4.11** and **4.12**, we analyze pressure for various values of Q, when Λ is positive and negative, respectively. In Fig. **4.11**, when $\Lambda = -1$, we can see that, in the presence of logarithmic correction the value of pressure is higher for the small horizon radius and small value of electric charge. The pressure is lower for increasing values of electric charge and horizon radius. When $Q = 1$, $Q = 2$ and $Q = 3$, the pressure is at it's peak value for $r_+ = 2$, $r_+ = 3.75$ and $r_+ = 6$, respectively and approaches to the specific value. In Fig. **4.12**, when $\Lambda = -1$, the pressure shows an interesting behavior for different values of Q. Pressure is positive only when $1.6 \geq r_+ \geq 1.4$, $2 \geq r_+ \geq 1.8$ and $2.3 \geq r_+ \geq 2.1$ for $Q = 1$, $Q = 2$ and $Q = 3$, respectively. The values of pressure stays negative and constant

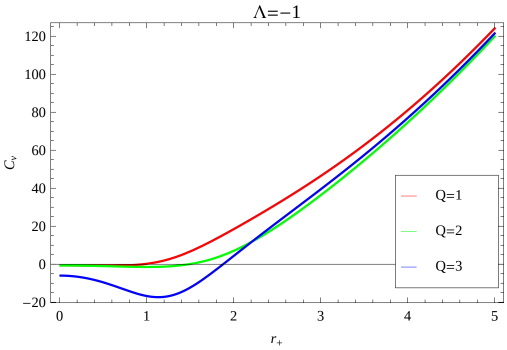


Figure 4.13: Plot of C_v versus $r+$ for charged AdS BH with a global monopole for negative cosmological constant.

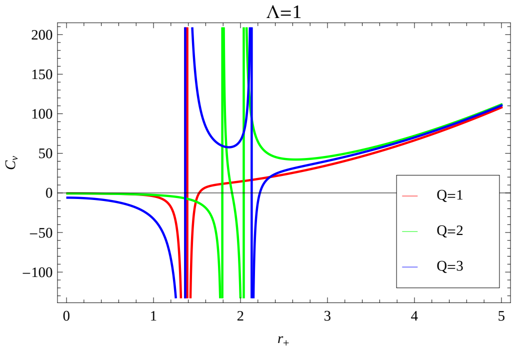


Figure 4.14: Plot of C_v versus r_+ for charged AdS BH with a global monopole for positive cosmological constant.

when $r_+ \geq 5$, for all considered specific values of Q.

Further, to analyze and study the phase transition of the charged AdS BH with a global monopole, we move to evaluate the important thermodynamical quantities in the regarding matter, such as, the heat capacities C_p and C_v. Generally, when the heat capacity is positive it implies that the BH is stable but when it's negative then it implies that the BH is unstable. Instability of a BH means that the BH cannot bear not even a small perturbation and proceed to disappear. Heat capacity, as heat is add up in system at fixed C_v can be evaluated by using relation Eq. (4.1.17) as

$$\begin{aligned} C_v &= \frac{1}{3Q^2 - r^2(-1+\eta^2)^2(1+r^2\Lambda)} \times \Bigg(r^3(-1+\eta^2)^2\big(1 - \frac{Q^2}{r^2(-1+\eta^2)^2} \\ &- r^2\Lambda\big)\big(-2\pi r(-1+\eta^2) + \frac{2b(Q^2 + r^4(-1+\eta^2)^2\Lambda)}{r(Q^2 - r^2(-1+\eta^2)^2(-1+r^2\Lambda))}\big)\Bigg) \end{aligned} \tag{4.2.24}$$

The graphical analysis of heat capacity at constant volume is represented in Fig. **4.13** and **4.14** for negative and positive cosmological constant, respectively. In Fig. **4.13**, when $\Lambda = -1$, the C_v is negative for $0 \leq r_+ \leq 0.1$, $0 \leq r_+ \leq 1.5$ and $0 \leq r_+ \leq 1.89$ for $Q = 1$, $Q = 2$ and

$Q = 3$, respectively. The heat capacity is positive for higher values of horizon radius. We conclude that, for negative cosmological constant, phase transition takes place and our BH is unstable for lower values of horizon radius but stable for larger values of horizon radius. In Fig. **4.14**, when $Q = 1$, the curve has two regions with one divergent point. The small radius region has negative heat capacity therefore, BH is unstable in this region while, the large radius region has positive heat capacity therefore, it is stable. When $Q = 2$, the curve has four regions with three divergent points. For SBH, thermal capacity is negative and thus unstable. For initial part of IBH, the heat capacity is positive and BH is stable in this region. But in the second part of the IBH, the heat capacity is negative and so BH is unstable. For the LBH, the BH has positive heat capacity and therefore, BH is stable in this region. The phase transition takes place in between the SBH, first and second part of the IBH and the LBH. When $Q = 3$, the curve has two divergent points in three regions. The phase transition takes place again as shown in the figure.

Now, heat capacity, as heat is add up in system at fixed pressure C_p can be evaluated by using relation Eq. (4.1.19) as

$$C_p = -\frac{2\pi r^2(-1+\eta^2)(Q^2+r^2(-1+\eta^2)^2(-1+r^2\Lambda))}{-3Q^2+r^2(-1+\eta^2)^2(1+r^2\Lambda)} \tag{4.2.25}$$

Also, γ turns out to be

$$\begin{aligned}\gamma &= \Big(\pi r^2(-1+\eta^2)(-Q^2+r^2(-1+\eta^2)^2(-1+r^2\Lambda))\Big)\Big(b(Q^2\\ &+ r^4(-1+\eta^2)^2\Lambda) - \pi r^2(-1+\eta^2)(Q^2+r^2(-1+\eta^2)^2(1-r^2\Lambda))\Big)^{-1}\end{aligned} \tag{4.2.26}$$

Figs. **4.15** and **4.16** represent the graphical analysis on γ versus r_+ for negative and positive cosmological constant, respectively. In Fig. **4.15** for

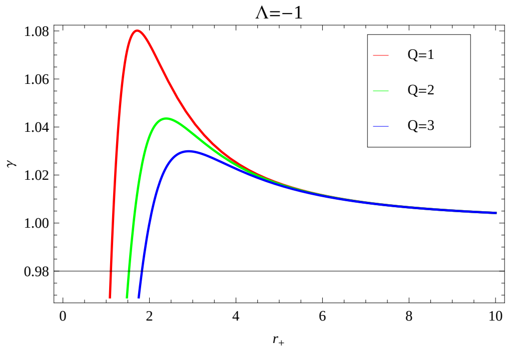


Figure 4.15: Plot of γ versus r_+ for charged AdS BH with a global monopole for negative cosmological constant.

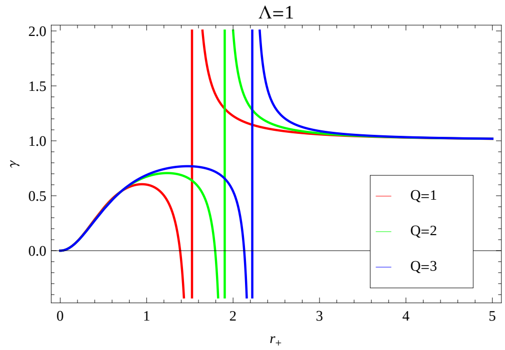


Figure 4.16: Plot of γ versus r_+ for charged AdS BH with a global monopole for positive cosmological constant.

$\Lambda = -1$, the γ is higher for the lower values of Q. For $Q = 1$, $Q = 2$ and $Q = 3$, the value of γ is higher when $r_+ = 1.6$, $r_+ = 2.5$ and $r_+ = 3$, respectively. For $r_+ \geq 5$, the value of γ starts approaching to the same value for all different values of Q. We conclude that for the negative cosmological constant and all different values of Q, the values of γ is positive. In Fig. **4.16**, we analyze the graph for three different values of Q and discuss the case for $\Lambda = 1$. When $Q = 1$, for the small radius region, the γ is positive and but for the first part of the intermediate region γ is negative and again for the second part of intermediate region and large radius region, the γ is positive and thus, the BH is stable. When $Q = 1$ and $Q = 2$, the both curves shows similar behavior as per $Q = 1$ but, after some intervals. Hence, we conclude that the for the smaller values of radius, the γ is negative and the charged AdS BH with a global monopole experiences a unstable behavior while for the larger values of the radius, the γ is positive and the charged AdS BH with a global monopole experiences a stable behavior and the curve approaches to the same value of γ.

Until now, we have analyzed and discussed the local thermodynamical properties of the charged AdS BH with a global monopole. We have find that the charged AdS BH with a global monopole experiences a SBH and LBH phase transition, which is interestingly remarkable. If we compare our results with RN AdS BH, we can easily see that the existence of the global monopole has influence on the critical points but the law of corresponding states remain unchanged and constant.

Now, we investigate global stability of charged AdS BH with a global monopole by using Gibbs free energy, which can be derived by using Eq. (4.1.23). It's expression for charged AdS BH with global monopole is

$$\begin{aligned} G &= \frac{-1}{24r^3(-1+\eta^2)^2}\Big(18Q^2r^2(-1+\eta^2)+2r^4(-1+\eta^2)^3(3+r^2\Lambda) \\ &+ \frac{1}{\pi}\times\big(3b(Q^2+r^2(-1+\eta^2)^2(-1+r^2\Lambda)) \\ &\times \log[\frac{(1-\eta^2)(-1+\frac{Q^2}{r^2(-1+\eta^2)^2}+r^2\Lambda)^2}{16\pi}]\big)\Big) \end{aligned} \tag{4.2.27}$$

Figs. **4.17** and **4.18** depict the graph between the Gibbs free energy G against the radius of the BH at the horizon r_+ at three different values of Q. In Fig. **4.17**, when $\Lambda=-1$, apparently there are no divergent points. In the small region radius, the Gibbs free energy is negative for all three different values of Q. But the Gibbs free energy is positive and displays a smooth curve in the IBH and again its negative in the LBH. Thus, we conclude that the charged AdS BH with a global monopole is unstable in the SBH and LBH but, its stable in the IBH. In Fig. **4.18**, for all three different values of Q, the Gibbs free energy is negative for the SBH and positive for the LBH. When $Q=3$, the Gibbs free energy is highest as compared to for the $Q=1$ and $Q=2$. Thus, the Gibbs free energy is higher for the higher the value of Q. We conclude that the charged AdS BH with a global monopole is globally stable for the larger horizon radius.

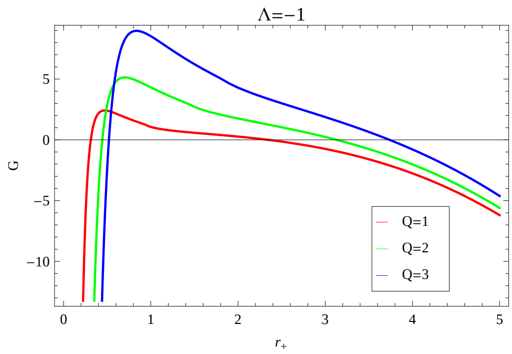


Figure 4.17: Plot of G versus r_+ for charged AdS BH with a global monopole for negative cosmological constant.

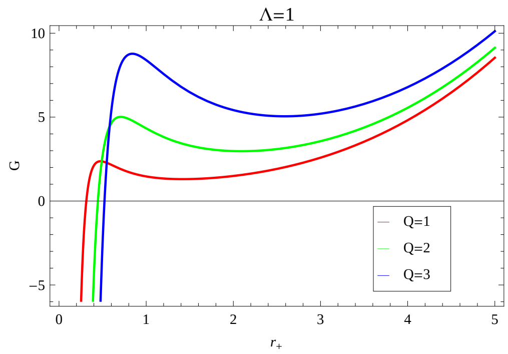


Figure 4.18: Plot of G versus r_+ for charged AdS BH with a global monopole for positive cosmological constant.

Now, for investing the thermodynamically global stability in the canonical ensemble, we will consider the charged AdS BH with a global monopole in the canonical ensemble. The free energy or the Helmholtz free energy for charged AdS BH with a global monopole can be derived by using the Eq. (4.1.25) as

$$\begin{aligned} F &= \frac{1}{24}\Bigg(\frac{4(-1+\eta^2)(-3(q^2+r^2)+r^4\Lambda)}{r} - \frac{1}{\pi}\Bigg(2(1-\frac{Q^2}{r^2(-1+\eta^2)^2}-r^2\Lambda) \\ &\times \quad (-2\pi r(-1+\eta^2)+\frac{2b(Q^2+r^4(-1+\eta^2)^2\Lambda)}{r(Q^2-r^2(-1+\eta^2)^2(-1+r^2\Lambda))})\Bigg) \\ &- \quad \frac{1}{\pi r^3(-1+\eta^2)^2}\Bigg(3(Q^2+r^2(-1+\eta^2)^2(-1+r^2\Lambda))(2\pi r^2(-1+\eta^2) \\ &+ \quad b\log[\frac{(1-\eta^2)(-1+\frac{Q^2}{r^2(-1+\eta^2)^2}+r^2\Lambda)}{16\pi}])\Bigg)\Bigg) \end{aligned} \tag{4.2.28}$$

The graphical analysis of Helmholtz free energy for charged AdS BH with a global monopole for specific values of different parameters is represented in the Fig. **4.19** and **4.20**. In Fig. **4.19**, for $Q=1$, $Q=2$ and $Q=3$, when cosmological constant is negative, Helmholtz free energy is positive for $0.25 \leq r_+ \leq 1.5$, $0.6 \leq r_+ \leq 2.25$ and $0.65 \leq r_+ \leq 2.8$, respectively. The Helmholtz free energy is negative for the larger horizon radius and

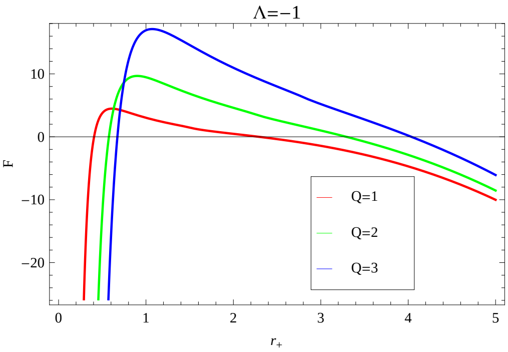


Figure 4.19: Plot of F versus r_+ for charged AdS BH with a global monopole for negative cosmological constant.

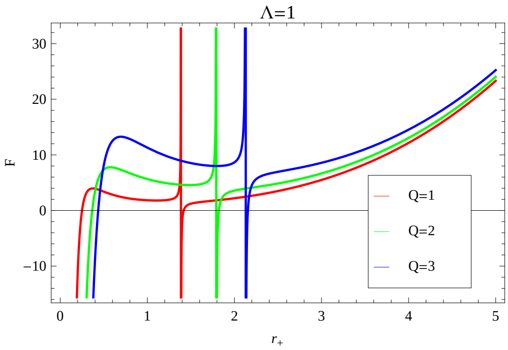


Figure 4.20: Plot of F versus r_+ for charged AdS BH with a global monopole for positive cosmological constant.

thus, the charged AdS BH with a global monopole is unstable for LBH. In Fig. **4.20**, for $\Lambda = 1$, Helmholtz free energy displays an interesting and yet complicated behavior. For $Q = 1$, the Helmholtz free energy is negative for the SBH but its positive for the IBH and LBH. Similarly, for $Q = 2$ and $Q = 3$, the Helmholtz free energy is negative for the SBH and positive for the IBH and LBH. The Helmholtz free energy is increasing and stays positive for the larger horizon radius. The Helmholtz free energy is higher for the higher values of Q. Thus, we conclude that, for the lower horizon radius, the charged AdS BH with a global monopole is globally unstable and for the higher horizon radius, its globally stable.

Chapter 5
Summary

In this thesis, we have presented modified first laws of thermodynamics at horizon for two RBHs: 1) The non-minimal MCRBH and 2) RBH with cosmological constant. Also, we've presented the effects of thermal fluctuations for two RBHs: 1) The AdS BH in BI massive gravity with a non-abelian hair. 2) The charged AdS BH with a global monopole. The summary is given below:

- **For The non-minimal MCRBH**

  For the non-minimal MCRBH, we've modified the first law of thermodynamics by using entropy. Also, utilized radial Einstein's equation at BH horizon by surface tension. Also, we've treated cosmological constant Λ in two ways, firstly it has been adopted as fixed and then secondly, as the variable which is correlated to the pressure. The cosmological constant has been treated as the thermodynamic pressure such as $P = -\frac{\Lambda}{8\pi}$ which has been accounted for exploring the thermodynamics. We have obtained the effective surface tension, temperature and the Gibss free energy.

- **For The RBH with cosmological constant**

  For RBH with cosmological constant Λ, we've taken cosmological constant as pressure $P = \frac{-r_+^2\Lambda}{8\pi(q^2+r_+^2)}$. Modified 1^{st} law of thermodynamics at horizon is obtained by surface tension. Modified 1^{st} law of thermodynamics by effective surface tension and pressure has been obtained, which has been rewritten as Eq.(3.1.31) for RBH with cosmological constant. We have obtained the effective surface tension and the discussion about the relation between the temperature and the Gibss free energy has been done.

- **For The AdS BH in BI massive gravity with a non-abelian hair**

  We've analyzed the effects of thermal fluctuations for the AdS BH in the massive gravity with a non-abelian hair by using the logarithmic correction terms to the entropy. We've used the Einstein-Massive BH solutions in the presence of YM and BI NED fields and shown that the first law thermodynamics is valid for the modify solutions. Furthermore, we have observed that the for the AdS BH in BI massive gravity with a non-abelian hair the pressure decreases for the negative cosmological constant but for the increasing values of massless graviton, pressure increases.

  Also, we have analyzed the local stability for the AdS BH in BI massive gravity with a non-abelian hair. Moreover, we've studied the thermal stability for the different parameters. We've investigated the behavior of γ for negative and positive cosmological constant. The critical behaviors and and phase structure has been discussed. The AdS BH in BI massive gravity with a non-abelian hair experiences a phase transition between the SBH and LBH. Also, studied effects of the many parameters such as, β and ν on critical points. The first BH is thermodynamically globally stable for both negative and positive cosmological constant in grand canonical ensemble. In canonical ensembble, for negative cosmological constant first BH is unstable for larger horizon radius.

- **For The charged AdS BH with a global monopole**

We've analyzed all the above properties for charged AdS BH with a global monopole. We calculated the thermodynamical quantities and have found out that due to the existence of a global monopole, the thermodynamical quantities are very much close and associated along symmetry breaking parameter η. In presence of symmetry breaking parameter η, we have shown that first law of thermodynamics and the corresponding Smarr relation is also verified. The local stability of the charged AdS BH with a global monopole in the existence of the logarithmic correction to the entropy is also analyzed.

The value of the heat capacity increase in the presence of logarithmic correction. The charged AdS BH with global monopole along with logarithmic correction to the entropy is stable for the LBH. In the grand canonical ensemble, when $\Lambda = -1$, the charged AdS BH with global monopole along with logarithmic correction to the entropy is thermodynamically globally stable for the IBH whereas, $\Lambda = 1$, globally stable for the LBH. We have also analyzed the global stability of the the charged AdS BH with global monopole along with logarithmic correction to the entropy in the canonical ensemble.

Chapter 6
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