Analysis of Integral Equations of Ahlfors Map for Multiply Connected Regions

By

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CIIT/FA16-RMT-025/LHR

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In partial fulfillment

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A Post Graduate Thesis submitted to the Department of Mathematics as partial fulfillment of the requirement for the award of Degree of MS in Mathematics.

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Analysis of Integral Equations of Ahlfors Map for Multiply Connected Regions

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Declaration

I Razi Ahmad registration number **CIIT/FA16-RMT-025/LHR** hereby declare that I have produced the work presented in this thesis, during the scheduled period of study. I also declare that I have not taken any material from any source except referred to wherever due, that amount of plagiarism is within acceptable range. If a violation of HEC rules on research has occurred in this thesis, I shall be liable to punishable action under the plagiarism rules of the HEC.

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Certificate

It is certified that **<u>Razi Ahmad (CIIT/FA16-RMT-025/LHR)</u>** has carried out all the work related to this report under my supervision at the Department of Mathematics, COMSATS University Islamabad, Lahore Campus and the work fulfills the requirement for award of MS Mathematics degree.

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DEDICATION

To

My Parents, family and all those who help me in every aspect of life.

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First, I recognize the creator and finisher of my faith and in memorial of my life, I say thank ALLAH Almighty and Prophet Muhammad (S.A.W) for everything. I am grateful to Almighty ALLAH, without Whose endowments it was difficult to finish this proposal.

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ABSTRACT

This research is to analyze the boundary integral equation of Ahlfors map of multiply connected region also the derivation of a new boundary integral equation related to Ahlfors map. These integral equations are then parameterized and discretized into the system of equations by using the Nystrom method with trapezoidal rule.

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Chapter 1

Introduction

1.1 Introduction

The dynamical role in complex analysis is of Conformal mapping which proved to be an important mathematical tool for science and engineering. Regions having complicated boundaries can be changed into simple and manageable configurations by means of complex variable functions. The angles between the curves remain same in respect of magnitude and the direction, in conformal mapping. By using functions of conformal mapping, many problems of different sciences can be easily solved as complicated regions can be changed into some standard canonical regions in which finding the results become easy. Then by means of inverse mapping properties the results can be altered back into the original region to get desired results. This method has been used to solve many problems in electrostatics, heat conduction, mechanics, image processing and aerodynamics [1] .

Reimann mapping theorem is an important theorem for conformal mapping in complex analysis. This theorem provides the necessary conditions for uniqueness and existance of a conformal mapping for a simply connected region onto the unit disc.

Reimann map being a one−to−one function converts a simply connected space onto the unit disc, but Ahlfors map being many−to−one maps a multi connected region onto the unit disc. If conformal map is simply connected, then Ahlfors map becomes Reimann Map.

1.2 Background of the Problem

In Conformal mapping the exact mapping functions are still unknown except for some particular regions while other functions can be find out numerically. In literature a bounded multiply connected region can be transformed conformally onto the five canonical regions. These regions have been discussed in Nehari [2], Bergmann [3] and Cohn [4].

Different numerical methods have been purposed for computing Riemann conformal mapping function like integral equation method, expansion method, osculation methods, iterative methods, charge simulation method and Cauchy Reimann equation methods by Henrici [5], Kythe [6], Murid [7], Trefethen [8], Wegmann [9], Wen [10] and Nazar [11].

Out of these methods, Integral equation method is the most suitable method for computing the conformal map. Many Authors like Kerzmann et al. [12, 13], Lee et al. [14], Nasser [15, 16, 17, 18] have used integral equation method to find one−to−one conformal map from multiply connected regions onto some standard regions. Murid and Razali [19] have found boundary integral equations for Ahlfors map of doubly connected regions. As these are not the Fredholm integral equations therefore, not a single one numerical experiment is reported. Murid and Mohamed[20], Mohamed and Murid [21], Mohamad [22] have discussed some numerical methods to solve the integral equations based upon Neumann and Kerzmann−Stein kernel of conformal mapping for doubly connected regions onto an annulus.

Murid and Hu [23, 24] have given an integral equation method for conformal map of bounded as well as unbounded multiply connected regions onto a slit and annulus having circular slits respectively by the Neumann kernel. However, kernels of boundary integral equations contain unknown circular radii. Sangawi [25] derived some integral equations related to the conformal mapping of bounded multiply connected region having smooth boundaries onto the five canonical slit regions.

As Reimann map is $1-1$ so, it has only one zero which is known. However Ahlfors map from $n-$connected regions onto unit disk have n zeros which are still unknown. Nazar [11] have derived six new integral equations using the approach presented in Sangawi [25]. These integral equations were derived for finding the unknown zeros of Ahlfors map. Out of these integral equations, one was verified but other integral equations are still unverified.

1.3 Statement of the Problem

This research wish to analyze one of the integral equation derived by Nazar [11] as well as the derivation and analysis of a new integral equation for conformal mapping of multiply connected regions onto the unit disc by using the known values of zeros of Ahlfors map.

1.4 Research Objectives

Following are the main objectives of this research work:

(i) To analyze the integral equation of Ahlfor's map for multiply connected regions onto the unit disc.

(ii) Formation of new integral equation of Ahlfor's map for multiply connected regions using the same appraoch as in [11].

(iii) Parameterization of the integral equations of Ahlfor's map.

(iv) Discretization of the integral equations of Ahlfor's map.

Chapter 2

Background And Literature Review

2.1 Introduction

An overview and introduction about the conformal mapping, Riemann mapping function and some literature on Ahlfors mapping function for multiply connected regions and zeros of Ahlfors map is represented in this chapter. Conformal mapping has an important part in many fields of science as well as engineering. In literature many researchers have applied various numerical methods for computing conformal mapping like integral equation methods, expansion methods and iterative methods [26]

Some initial work for multiply connected regions under conformal mapping are given by Ellacott [27], Papamicheal and Warby [28], Symm [29], Papamicheal and Kakkinos [30], Reichel [31], Kokkinos et al.[32], Murid and Razali [33], Okano et al.[34, 35], Rabinowitz [36], Bergman [3] and Levin, Papamichael, and Sideridis [37]. For conformal mapping, the well known canonical regions are annulus with circular slits region, disk with circular slits region, circular slits region, redial slits region and parallel slits region.

This chapter consists of the following sections. Section 2.2 some ideas on conformal mapping. Section 2.3 Reimann mapping theorem. Section 2.4 Canonical regions. Section 2.5 and 2.6 gives an idea about Ahlfors mapping function and its zeros respectively. Section 2.7 gives an important theorem.

2.2 Conformal Mapping

The graphical representation of a real valued function has always been very helpful to understand the basic ideas. A real valued function $h = f(v)$ can generally be represented as two-dimensional graph. However to draw the graph of a complex function needs to display both the domain and range values in the 2-dimensions. It is important to visualize both the domain region and range values in 2-dimensions. To over come this difficulty we represent domain and range value of a complex value function in separate planes. In complex analysis, a complicated region is transformed into a simple region where the angles between the curves are preserved.

"A mapping of complex-valued function $w = g(\mu)$ is conformal if and only if it preserves the local angles."

Theorem 2.2.1. *Let $g(\mu)$ is an analytic function of region Ω and $g'(\mu) \neq 0$, then the mapping $w = g(\mu)$ is conformal at every point $\mu \in \Omega$.*

This theorem gives a relationship between conformal mapping and analytic function.

2.3 Riemann Mapping Theorem

Bernhard Riemann (1826-1866) a German mathematician gave the idea of Riemann Mapping Theorem.

According to this theorem, for any simply connected space of a complex plane, there exist one and only one conformal map onto the unit disc.

So, by Riemann mapping theorem, one can say any two simply connected boundaries are equivalent. In Henrici [5] the Riemann mapping theorem is given as

Theorem 2.3.1. *"Let Ω be a simply connected region which is not the whole plane and let $b \in \Omega$. Then there exist a unique analytic function R satisfying the conditions $R(b) = 0, \quad R'(b) > 0$, and assuming every point in the unit disc $R : |\omega| < 1$ exists exactly once."*

The function R is known as Riemann mapping function. If boundary Λ of the curve Ω is parameterized by $\mu(v)$, $0 \leq v \leq 2\pi$ in counterclockwise direction then $\mu(v)$ describes the unit circle.

2.4 Canonical Regions

Canonical regions have much importance in the theory of conformal mapping for simply as well as multiply connected regions in the complex plane. Many regions can be mapped conformally onto the canonical regions. For simply connected regions the interior and exterior of the unit disc are common canonical regions.

In literature Koebe's [38] has given the concept of various canonical regions as a mixture of circular and radial slits, domains containing linear slits and domains containing logarithmic spiral slits.

2.5 Ahlfors Mapping Function

"A conformal map from a multiply connected region onto the unit disk is known as Ahlfors Map." If the region is simply connected then the Ahlfors map reduces to the Reimann map. Many geometrical structures of a Reimann map are common with Ahlfors map. An analytic and unique function which maps a simply connected region onto the unit disk is known as Reimann mapping function. So, it is unique conformal, one−to−one and onto map. In Ahlfors map any multiply connected space can be mapped conformally onto the unit disk. It is also unique and analytic. Since Ahlfors map maps multi connected region on the unit disk in one-to-one ways therefore, in the multiply connected setting, Ahlfors map is a generalization of Reimann Mapping function.

Ahlfors map are very helpful in many applied sciences. For example in fluid mechanics there arises various problem in transonic flow through an obstacle in the planar region. These problems can be solved easily by means of conformal map to make numerical computations.

2.6 Ahlfors Mapping Function and its Zeros

In [39, 40, 41, 42, 43, 44, 45], a lot of work have been done regarding Ahlfors map. Nazar ,et al [46] find the zeros of Ahlfors map for doubly connected regions. In [47] Nazar, Ali H.M.Murid and Ali W.K. Sanawi compute zeros of Ahlfors map for multiply connected regions. Being one-to-one map, Reimann map has only one zero which can be choosen freely. So, there arises no problem regarding choosing the zeros of Reimann map in finding out the conformal transformations under Reimann map. However, the Ahlfors map has n zeros which are generally unknown.

Murid and Razali [19] have computed an integral equation satisfying a boundary relationship in doubly connected region and also gave the applications to Ahlfors map. However the integral equation is not able to find out the Ahlfors map as it involves unknown zeros of Ahlfors map.

By [48] the Ahlfors map, Szegö kernel $S(\mu, b_0)$ and the Garabedian kernel $L(\mu, b_0)$ are related as

$$g(\mu) = \frac{S(\mu, b_0)}{L(\mu, b_0)}, \quad \mu \in \Omega \cup \lambda. \tag{2.1}$$

The Szegö kernel $S(\mu, b_0)$ and Garabedian kernel $L(\mu, b_0)$ are related on Λ as

$$S(b_0, \mu) ds_\mu = \frac{1}{\mathrm{i}} L(\mu, b_0) d\mu.$$

Using the fact that $ds_\mu = |d\mu|$ and $\tau(\mu) = \frac{d\mu}{|d\mu|}$, gives

$$S(b_0, \mu) = \frac{1}{\mathrm{i}} L(\mu, b_0)\tau(\mu).$$

Taking the conjugate and using $\overline{S(b_0, \mu)} = S(\mu, b_0)$,we have

$$L(\mu, b_0) = \mathrm{i}\overline{\tau(\mu) S(\mu, b_0)}, \quad \mu \in \Lambda.$$

Thus (2.1) becomes

$$g(\mu(v)) = \frac{1}{\mathrm{i}} \frac{S(\mu(v), b_0)\tau(\mu(v))}{\overline{S(\mu(v), b_0)}}, \quad \mu(v) \in \Lambda. \tag{2.2}$$

Thus from (2.2) the boundary values of the Ahlfors map can be computed from the boundary values of the szegö kernel. The Szegö kernel satisfy the integral equation [48]

$$S(\mu, b_0) + \int_{\Lambda} \kappa(\mu, w) S(w, b_0) |dw| = x(\mu), \quad u \in \Lambda, \tag{2.3}$$

where

$$\begin{aligned} x(\mu(v)) &= -\frac{1}{2\pi \mathrm{i}} \frac{\overline{\tau(\mu(v))}}{\overline{\mu(v) - b_0}}, \\ \kappa(\mu, w) &= \begin{cases} \dfrac{1}{2\pi \mathrm{i}} \left[\dfrac{\tau(w)}{\mu - w} - \dfrac{\overline{\tau(\mu)}}{\overline{\mu - w}} \right], & \mu \neq w, \\ 0, & \mu = w. \end{cases} \end{aligned} \tag{2.4}$$

The kernel $\kappa(\mu, w)$ is also known as Kerzman-Stein kernel.

In [49] Bell et al. briefly discussed about zeros of the Ahlfors map for some specific regions of doubly−connected regions in Bell domain. By using the both Szegö kernel and Garabedian kernel Tegtmeyer et al. [50] computed the Ahlfors map for annulus. By using these kernel they are able to fined out two zeros b_0 and $b_1 = \frac{-r}{b_0}$ for Ahlfors map respectively, where r is the radius of the inner circle.

Nazar [11] has derived the six integral equations for Ahlfors map of multi connected region by using the following theorem.

Theorem 2.6.1. *If a complex valued function $R(\mu)$ satisfies the interior relationship*

$$R(\mu) = \frac{B(\mu)\overline{\tau(\mu)}}{\overline{E(\mu)}} \overline{R(\mu)} + \overline{h(\mu)},$$

or

$$E(\mu) = \overline{B(\mu)} \tau(\mu) \frac{R(\mu)^2}{|R(\mu)|^2} + \frac{E(\mu) h(\mu)}{\overline{R(\mu)}}, \tag{2.5}$$

then

$$\tau(\mu) R(\mu) + \int_{\Lambda} \kappa(\mu, w) \tau(w) R(w) |dw| \\ + B(\mu) \left[\Sigma_{b_i \, inside \Lambda} Res_{w=b_i} \frac{R(w)}{(w-\mu) E(w)} \right]^{-} = -\overline{L_F^{-}(\mu)} \tag{2.6}$$

where

$$\kappa(\mu, w) = \begin{cases} \dfrac{1}{2\pi \mathrm{i}}\left[\dfrac{\tau(\mu)}{\mu - w} - \dfrac{B(\mu)\overline{\tau(w)}}{B(w)(\overline{\mu} - \overline{w})}\right], & \mu \neq w \in \Lambda, \\ -\dfrac{1}{2\pi \mathrm{i}|\mu'(v)|}\dfrac{p'(v)}{p(v)}, \quad p(v) = B(\mu(v)) & \mu = w \in \Lambda, \end{cases} \tag{2.7}$$

and

$$L_F^-(\mu) = \frac{-1}{2}\frac{h(\mu)}{\tau(\mu)} + PV\frac{1}{2\pi \mathrm{i}}\int_\Lambda \frac{\overline{B(\mu)}h(w)}{\overline{B(w)}(w-\mu)\tau(w)}dw. \tag{2.8}$$

The sign "-" in (2.8) represents the complex conjugate and the sum is over all zeros of E lie inside Ω. If the function E having no zeros in Ω then no residue term occurs. This theorem will be used in next chapter to find out a new integral equation related to the Ahlfors map.

Chapter 3

New Integral Equation For Ahlfors Map

In this chapter a new integral equation for Ahlfors map is presented, which is derived using the same approach as in Nazar et al. [11], also an overview of the integral equations of Ahlfor map for multiply connected regions derived here is presented. Out of these integral equations, one integral equation is analyzed.

3.1 Auxiliary Material

Consider a bounded multiply connected region as Ω with connectivity $N+1$. The boundary curves $\Lambda_0, \Lambda_1, \ldots, \Lambda_N$ are such that the curves $\Lambda_1, \ldots, \Lambda_N$ lie in the interior of Λ_0, with Λ_0 in counterclockwise orientation and inner curves $\Lambda_1, \ldots, \Lambda_N$ have clockwise orientation. The positive direction of the contour is for Ω to be on the left as one traces the boundary as shown in Figure 3.1.

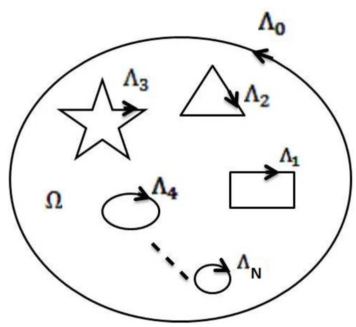

Figure 3.1: Multiply Connected Region.

Consider $g(\mu)$ as Ahlfor function which maps the region Ω onto the unit disc conformally. Due to the canonical region as unit disk, the function $g(\mu)$ is determined up to a factor of modulus one. The uniqueness condition of function $g(\mu)$ as in [51]

$$g(b_d) = 0, \quad g'(b_0) > 0, \quad d = 0, 1, 2, ..., N,$$

where $b_d \in \Omega$ are the zeros of Ahlfors map. The function g is represented as

$$g(\mu_d(v)) = e^{\mathrm{i}\theta_d(v)}, \quad \Lambda_d : \mu = \mu_d(v), \quad 0 \leq v \leq 2\pi, \tag{3.1}$$

In above equation (3.1) the boundary correspondence functions of Λ_d are $\theta_d(v)$, where $d = 0, 1, \ldots, N$ The unit tangent to Λ at $\mu(v)$ is denoted by $\tau(\mu(v)) = \mu'(v)/|\mu'(v)|$.

Diffrentiating (3.1)

$$g'(\mu_d(v))\mu_d'(v) = e^{\mathrm{i}\theta_d(v)}\mathrm{i}\theta_d'(v) = g(\mu_d(v))\mathrm{i}\theta_d'(v) \tag{3.2}$$

Taking modulus of (3.2) we have

$$|g'(\mu_d(v))||\mu_d'(v)| = |\theta_d'(v)|. \tag{3.3}$$

By Dividing (3.2) and (3.3) we have

$$\frac{g'(\mu_d(v))\mu_d'(v)}{|g'(\mu_d(v))||\mu_d'(v)|} = \frac{g(\mu_d(v))\mathrm{i}\theta_d'(v)}{|\theta_d'(v)|}. \tag{3.4}$$

The above expression (3.4) as the equivalent form

$$g(\mu_d(v)) = \frac{1}{\mathrm{i}}\tau(\mu_d(v))\frac{|\theta_d'(v)|}{\theta_d'(v)}\frac{g'(\mu_d(v))}{|g'(\mu_d(v))|}, \qquad \mu_d \in \Lambda_d. \tag{3.5}$$

As the mapping is conformal so, the bondary relationship (3.5) can be represented as

$$g(\mu_d(v)) = \mathrm{sign}(\theta_d'(v))\frac{1}{\mathrm{i}}\tau(\mu_d(v))\frac{g'(\mu_d(v))}{|g'(\mu_d(v))|}, \quad \mu_d(v) \in \Lambda_d.$$

where

$$\frac{|\theta_d'(v)|}{\theta_d'(v)} = \mathrm{sign}(\theta_d'(v)) = \begin{cases} +1, & d = 0 \\ -1 & d = 1, 2, \ldots, N. \end{cases}$$

In general,

$$g(\mu(v)) = \mathrm{sign}(\theta'(v))\frac{1}{\mathrm{i}}\tau(\mu(v))\frac{g'(\mu(v))}{|g'(\mu(v))|}, \quad \mu(v) \in \Lambda. \tag{3.6}$$

Since Ahlfors map for multiply connected region is defined as

$$g(\mu) = \prod_{c=0}^{N}(\mu - b_c)f(\mu), \quad \mu \in \Omega \cup \Lambda, \tag{3.7}$$

By using the equations (3.6), (3.7) and the theorem (2.6.1) , Nazar [11, 51] has derived the six integral equations of Ahlfors map for multiply connected regions with Neumann type kerrnel, Neumann−Stein type kernel and kerzman−Stein type kernel. In these equations only one integral equation is numerically verified while others have to be verified by using the known values of zeros of Ahlfors map. In next sections we will try to parametrize and discritize one of the integral equation from Nazar [11] and also make a new integral equation of Ahlfors map of multi connected regions using the similar approach in Nazar [11].

3.2 An Integral Equation

In this section an integral equation derived by Nazar [11, 51] is presented.

$$g'(\mu)g'(b_0)\tau(\mu) + \int_{\Lambda} \kappa(\mu,w)g'(w)g'(b_0)\tau(w)|dw| = \overline{A(\mu)}, \quad \mu \in \Lambda, \tag{3.8}$$

where

$$A(\mu) = \frac{2\tau(\mu)\sum_{c=1}^{N-1}\left[\frac{\prod_{d=1}^{N-1}(b_0-b_d)^2}{b_0-b_c}\right](b_0-\mu) - \tau(\mu)\prod_{d=1}^{N-1}(b_0-b_d)^2}{(b_0-\mu)^2\prod_{d=1}^{N}(\mu-b_d)^2}.$$

and $\kappa(\mu, w)$ is the Neumann-type kernel and defined as

$$\kappa(\mu,w) = \begin{cases} \dfrac{1}{2\pi\mathrm{i}}\left[\dfrac{\tau(\mu)}{\mu-w} - \dfrac{\prod_{c=1}^{N}(\overline{w-b_c})^2}{\prod_{c=1}^{N}(\overline{\mu-b_c})^2}\dfrac{\overline{\tau(\mu)}}{(\overline{\mu}-\overline{w})}\right], & \mu \neq w \in \Lambda, \\ \dfrac{1}{2\pi\mathrm{i}|\mu'(\mathrm{t})|}\left[\mathrm{iIm}\left(\dfrac{\mu''(\mathrm{v})}{\mu'(\mathrm{v})}\right) + 2\displaystyle\sum_{\mathrm{c}=1}^{\mathrm{N}}\dfrac{\overline{\mu'(\mathrm{v})}}{\overline{\mu(\mathrm{v})-\mathrm{b_c}}}\right], & \mu = w \in \Lambda. \end{cases}$$

The equation (3.8), $A(\mu)$ and $\kappa(\mu, w)$ contains the zeros of Ahlfors map.

3.3 Derivation of New Integral Equation of Ahlfors Map

Taking square of equation (3.6) we have

$$g(\mu)^2 = -(\tau(\mu))^2\frac{g'(\mu)^2}{|g'(\mu)|^2}, \quad \mu \in \Lambda. \tag{3.9}$$

Dividing (3.9) by (3.7) we get

$$g(\mu) = -\frac{\tau(\mu)^2}{\prod_{c=0}^{N}(\mu-b_c)f(\mu)}\frac{g'(\mu)^2}{|g'(\mu)|^2}, \quad \mu \in \Lambda \tag{3.10}$$

Above equation can be written as

$$g(\mu)f(\mu)(\mu-b_0) = -\frac{\tau(\mu)^2}{\prod_{c=1}^{N}(\mu-b_c)}\frac{g'(\mu)^2}{|g'(\mu)|^2}, \quad \mu \in \Lambda \tag{3.11}$$

By comparing (3.11) with (2.5), we have the following assignments

$$E(\mu) = g(\mu)f(\mu)(\mu-b_0), \quad B(\mu) = -\frac{\overline{\tau(\mu)}}{\prod_{c=1}^{N}\overline{(\mu-b_c)}}, \quad R(\mu) = g'(\mu), \quad h(\mu) = 0. \tag{3.12}$$

Using these values in theorem (2.6.1), the integral equation associated with (3.11) is

$$\tau(\mu)g'(\mu)+\int_{\Lambda}\kappa_1(\mu,w)\tau(w)g'(w)|dw| =$$
$$\frac{\overline{\tau(\mu)}}{\prod_{c=1}^{N}\overline{(\mu-b_c)}}\left[\Sigma_{b_d\text{inside}\Lambda}Res_{w=b_d}\frac{g'(w)}{(w-\mu)(w-b_0)g(w)f(w)}\right]^{-},\quad \mu\in\Lambda \quad (3.13)$$

where

$$\kappa_1(\mu,w)=\frac{1}{2\pi\mathrm{i}}\left[\frac{\tau(\mu)}{\mu-w}-\frac{\prod_{c=1}^{N}\overline{(w-b_c)}}{\prod_{c=1}^{N}\overline{(\mu-b_c)}}\frac{\overline{\tau(\mu)}}{\overline{\mu}-\overline{w}}\right],\quad \mu\neq w\in\Lambda. \quad (3.14)$$

From (2.7) $\kappa_1(\mu,w)$ for $\mu=w\in\Lambda$ is determined as

$$\kappa_1(\mu,w)=-\frac{1}{2\pi\mathrm{i}|\mu'(v)|}\frac{q'(v)}{q(v)},\quad q(v)=B(\mu(v)),\quad \mu=w\in\Lambda. \quad (3.15)$$

As from (3.12)

$$B(\mu(v))=-\frac{\overline{\tau(\mu(v))}}{\prod_{c=1}^{N}\overline{(\mu(v)-b_c)}},$$

above equation can be written as

$$-q(v)=\frac{\overline{\mu'(v)}}{|\mu'(v)|\overline{(\mu(v)-b_1)}\dots\overline{(\mu(v)-b_N)}}$$

Applying log on both sides

$$\log(-q(v))=\log(\overline{\mu'(v)})-\log(|\mu'(v)|)-\log(\overline{\mu(v)-b_1})-\cdots-\log(\overline{\mu(v)-b_N})$$

$$=\log(\overline{\mu'(v)})-\frac{1}{2}\log(|\mu'(v)|^2)-\log(\overline{\mu(v)-b_1})-\cdots-\log(\overline{\mu(v)-b_N}).$$

By differentiating both sides with respect to v, and using the fact $|\mu|^2=\mu\overline{\mu}$,we get

$$\frac{q'(v)}{q(v)}=\frac{\overline{\mu''(v)}}{\overline{\mu'(v)}}-\frac{1}{2}\left[\frac{\mu''(v)}{\mu'(v)}+\frac{\overline{\mu''(v)}}{\overline{\mu'(v)}}\right]-\frac{\overline{\mu'(v)}}{\overline{\mu(v)-b_1}}-\cdots-\frac{\overline{\mu'(t)}}{\overline{\mu(v)-b_N}}$$

$$\frac{q'(v)}{q(v)}=\frac{1}{2}\left[-2\mathrm{iIm}\left(\frac{\mu''(\mathrm{v})}{\mu'(\mathrm{v})}\right)\right]-\frac{\overline{\mu'(\mathrm{v})}}{\overline{\mu(\mathrm{v})-\mathrm{b}_1}}-\cdots-\frac{\overline{\mu'(\mathrm{v})}}{\overline{\mu(\mathrm{v})-\mathrm{b}_\mathrm{N}}},$$

so we have

$$\frac{q'(v)}{q(v)}=-\left[\mathrm{iIm}\left(\frac{\mu''(\mathrm{v})}{\mu'(\mathrm{v})}\right)+\sum_{\mathrm{c}=1}^{\mathrm{N}}\frac{\overline{\mu'(\mathrm{v})}}{\overline{\mu(\mathrm{v})-\mathrm{b}_\mathrm{c}}}\right].$$

$$\kappa_1(\mu,w)=\frac{1}{2\pi\mathrm{i}|\mu'(\mathrm{t})|}\left[\mathrm{iIm}\left(\frac{\mu''(\mathrm{v})}{\mu'(\mathrm{v})}\right)+\sum_{\mathrm{c}=1}^{\mathrm{N}}\frac{\overline{\mu'(\mathrm{v})}}{\overline{\mu(\mathrm{v})-\mathrm{b}_\mathrm{c}}}\right].$$

Therefore

$$\kappa_1(\mu,w)=\begin{cases}\dfrac{1}{2\pi\mathrm{i}}\left[\dfrac{\tau(\mu)}{\mu-w}-\dfrac{\prod_{c=1}^{N}(\overline{w-b_c})}{\prod_{c=1}^{N}(\overline{\mu-b_c})}\dfrac{\overline{\tau(\mu)}}{\overline{\mu}-\overline{w}}\right], & \mu\neq w\in\Lambda.\\ \dfrac{1}{2\pi\mathrm{i}|\mu'(\mathrm{v})|}\left[\mathrm{iIm}\left(\dfrac{\mu''(\mathrm{v})}{\mu'(\mathrm{v})}\right)+\displaystyle\sum_{\mathrm{c}=1}^{\mathrm{N}}\dfrac{\overline{\mu'(\mathrm{v})}}{\overline{\mu(\mathrm{v})-\mathrm{b_c}}}\right], & \mu=\mathrm{w}\in\Lambda.\end{cases} \tag{3.16}$$

As the residue in (3.13) involves a simple pole then by [52]

$$Res_{w=b}S(w)=\frac{x(b)}{y'(b)}. \tag{3.17}$$

with $x(w)=\frac{g'(w)}{(w-\mu)}$ and $y(w)=(w-b_0)g(w)f(w)$, implies that

$$x(b_0)=\frac{g'(b_0)}{(b_0-\mu)}\neq 0,\quad y(b_0)=0,$$

$$y'(b_0)=g(b_0)f(b_0). \tag{3.18}$$

by putting (3.18) in (3.17) we have

$$Res_{w=b_0}\frac{g'(w)}{(w-\mu)(w-b_0)f(w)g(w)}=\frac{g'(b_0)}{(b_0-\mu)g(b_0)f(b_0)} \tag{3.19}$$

For doubly connected region, the Ahlfors map is given as

$$g(\mu)=(\mu-b_0)(\mu-b_1)f(\mu)$$

$$g'(\mu)=(\mu-b_0)\left[(\mu-b_1)f'(\mu)+f(\mu)(1)\right]+(\mu-b_1)f(\mu)(1),$$

$$g'(b_0)=(b_0-b_1)f(b_0). \tag{3.20}$$

Thus (3.19) becomes

$$Res_{w=b_0}\frac{g'(w)}{(w-\mu)(w-b_0)f(w)g(w)}=\frac{(b_0-b_1)}{(b_0-\mu)g(b_0)}. \tag{3.21}$$

For triply connected region, the Ahlfors map is defined as

$$g(\mu)=(\mu-b_0)(\mu-b_1)(\mu-b_2)f(\mu)$$

$$g'(\mu)=(\mu-b_0)(\mu-b_1)\left[(\mu-b_2)f'(\mu)+f(\mu)(1)\right]+(\mu-b_2)f(\mu)\left[(\mu-b_0)(1)+(\mu-b_1)(1)\right],$$

$$g'(b_0)=(b_0-b_1)(b_0-b_2)f(b_0) \tag{3.22}$$

using (3.22), (3.19) becomes

$$Res_{w=b_0}\frac{g'(w)}{(w-\mu)(w-b_0)f(w)g(w)} = \frac{(b_0-b_1)(b_0-b_2)}{(b_0-\mu)g(b_0)}. \tag{3.23}$$

Similarly for N-connected regions. the residue of (3.13) becomes

$$Res_{w=b_0}\frac{g'(w)}{(w-\mu)(w-b_0)f(w)g(w)} = \frac{\prod_{c=1}^{N}(b_0-b_c)}{(b_0-\mu)g(b_0)}$$

By putting the values and multiplying (3.13) with $g'(b_o)$ we get

$$g'(\mu)g'(b_0)\tau(\mu) + \int_{\Lambda}\kappa_1(\mu,w)g'(w)g'(b_0)\tau(w)|dw| = \overline{M(\mu)} \tag{3.24}$$

Where

$$M(\mu) = \overline{\tau(\mu)}\frac{\prod_{c=1}^{N}(b_0-b_c)}{\prod_{c=1}^{N}\overline{(\mu-b_c)}}\frac{g'(b_0)}{g(b_0)}. \tag{3.25}$$

and

$$\kappa_1 = \begin{cases} \dfrac{1}{2\pi \mathrm{i}}\left[\dfrac{\tau(\mu)}{\mu-w} - \dfrac{\prod_{c=1}^{N}(\overline{w-b_c})}{\prod_{c=1}^{N}(\overline{\mu-b_c})}\dfrac{\overline{\tau(\mu)}}{\overline{\mu}-\overline{w}}\right], & \mu \neq w \in \Lambda. \\ \dfrac{1}{2\pi \mathrm{i}|\mu'(\mathrm{v})|}\left[\mathrm{iIm}\left(\dfrac{\mu''(\mathrm{v})}{\mu'(\mathrm{v})}\right) + \displaystyle\sum_{\mathrm{c}=1}^{\mathrm{N}}\dfrac{\overline{\mu'(\mathrm{v})}}{\overline{\mu(\mathrm{v})-\mathrm{b_c}}}\right], & \mu = \mathrm{w} \in \Lambda. \end{cases}$$

Finaly we have derived a new boundary integral equation of Ahlfors map for bounded multiply connected regions and the non-homogeneous part M and the κ_1 contains the zeros of Ahlfors map.

Chapter 4

Parameterization And Discretization Of Integral Equation

This chapter contains the parameterization and discretization of the integral equations (3.8) and (3.24) respectively. For parameterizing, the integral equations (3.8) and (3.24) are separated into a system of two integral equations. After parameterization, the integral equations are discretized with the help of Nyström method with trapezoidal rule.

4.1 Parameterization of Integral Equations

By letting $G_1(\mu) = g'(\mu)\tau(\mu)$ the boundary integral equation (3.8) has the equivalent form

$$G_1(\mu)g'(b_0) + \int_\Lambda \kappa(\mu, w)g'(b_0)G_1(w)|dw| = \overline{A(\mu)}, \tag{4.1}$$

where

$$\kappa(\mu, w) = \begin{cases} \dfrac{1}{2\pi \mathrm{i}}\left[\dfrac{\tau(\mu)}{\mu - w} - \dfrac{\prod_{c=1}^{N}(\overline{w - b_c})^2}{\prod_{c=1}^{N}(\overline{\mu - b_c})^2}\dfrac{\overline{\tau(\mu)}}{(\overline{\mu} - \overline{w})}\right], & \mu \neq w \in \Lambda, \\ \dfrac{1}{2\pi \mathrm{i}|\mu'(\mathrm{t})|}\left[\mathrm{iIm}\left(\dfrac{\mu''(\mathrm{v})}{\mu'(\mathrm{v})}\right) + 2\displaystyle\sum_{\mathrm{c}=1}^{\mathrm{N}}\dfrac{\overline{\mu'(\mathrm{v})}}{\overline{\mu(\mathrm{v}) - \mathrm{b_c}}}\right], & \mu = w \in \Lambda. \end{cases}$$

$$A(\mu) = \frac{2\tau(\mu)\sum_{c=1}^{N-1}\left[\frac{\prod_{d=1}^{N-1}(b_0 - b_d)^2}{b_0 - b_c}\right](b_0 - \mu) - \tau(\mu)\prod_{d=1}^{N-1}(b_0 - b_d)^2}{(b_0 - \mu)^2\prod_{d=1}^{N}(\mu - b_d)^2}.$$

The equation (4.1) can be separated into the system of two integral equation and then parameterize it. As $\Lambda = \Lambda_0 \cup \Lambda_1$ So, the intgral equation (3.8) can be separated into

$$G_1(\mu_0)g'(b_0) + \int_{\Lambda_0} \kappa(\mu_0, w)g'(b_0)G_1(w)|dw| - \int_{\Lambda_1} \kappa(\mu_0, w)g'(b_0)G_1(w)|dw| = A(\mu_0), \quad \mu_0 \in \Lambda_0 \tag{4.2}$$

$$G_1(\mu_1)g'(b_0) + \int_{\Lambda_0} \kappa(\mu_1, w)g'(b_0)G_1(w)|dw| - \int_{\Lambda_1} \kappa(\mu_1, w)g'(b_0)G_1(w)|dw| = A(\mu_1), \quad \mu_1 \in \Lambda_1 \tag{4.3}$$

Now we parameterize both equations (4.2)and (4.3) by using parametric representations $\mu_0(v)$ of Λ_0 for $v : 0 \leq v \leq 2\pi$ and $\mu_1(v)$ of Λ_1 for $v : 0 \leq v \leq 2\pi$ .Let $w = \mu(s)$ and using the fact $dw = \mu'(s)ds$, equations (4.2) and (4.3) becomes

$$G_1(\mu_0(v))g'(b_0) + \int_0^{2\pi} \kappa(\mu_0(v), \mu_0(s))g'(b_0)G_1(\mu_0(s))|\mu_0'(s)|ds$$
$$- \int_0^{2\pi} \kappa(\mu_0(v), \mu_1(s))g'(b_0)G_1(\mu_1(s))|\mu_1'(s)|ds = A(\mu_0(v)), \quad \mu_0(v) \in \Lambda_0 \quad (4.4)$$

$$G_1(\mu_1(v))g'(b_0) + \int_0^{2\pi} \kappa(\mu_1(v), \mu_0(s))g'(b_0)G_1(\mu_0(s))|\mu_0'(s)|ds$$
$$- \int_0^{2\pi} \kappa(\mu_1(v), \mu_1(s))g'(b_0)G_1(\mu_1(s))|\mu_1'(s)|ds = A(\mu_1(v)), \quad \mu_1(v) \in \Lambda_1 \quad (4.5)$$

Now multiplying (4.4) and (4.5) by $|\mu_0'(v)|$ and $|\mu_1'(v)|$ respectively we get

$$|\mu_0'(v)|G_1(\mu_0(v))g'(b_0) + \int_0^{2\pi} |\mu_0'(v)|\kappa(\mu_0(v), \mu_0(s))g'(b_0)G_1(\mu_0(s))|\mu_0'(s)|ds$$
$$-\int_0^{2\pi} |\mu_0'(v)|\kappa(\mu_0(v), \mu_1(s))g'(b_0)G_1(\mu_1(s))|\mu_1'(s)|ds = |\mu_0'(v)|A(\mu_0(v)), \quad \mu_0(v) \in \Lambda_0 \quad (4.6)$$

$$|\mu_1'(v)|G_1(\mu_1(v))g'(b_0) + \int_0^{2\pi} |\mu_1'(v)|\kappa(\mu_1(v), \mu_0(s))g'(b_0)G_1(\mu_0(s))|\mu_0'(s)|ds$$
$$-\int_0^{2\pi} |\mu_1'(v)|\kappa(\mu_1(v), \mu_1(s))g'(b_0)G_1(\mu_1(s))|\mu_1'(s)|ds = |\mu_1'(v)|A(\mu_1(v)), \quad \mu_1(v) \in \Lambda_1 \quad (4.7)$$

To simplify (4.6) and (4.7), let us define

$$\zeta_0(v) = |\mu_0'(v)|G_1(\mu_0(v)),$$

$$\zeta_1(v) = |\mu_1'(v)|G_1(\mu_1(v)),$$

$$\sigma_0(v) = |\mu'(v)|A(\mu_0(v)),$$

$$\sigma_1(v) = |\mu_1'(v)|A(\mu_1(v)),$$

$$\kappa_{00}(v_o, s_0) = |\mu_0'(v)|\kappa(\mu_0(v), \mu_0(s)),$$

$$\kappa_{01}(v_0, s_1) = |\mu_0'(v)|\kappa(\mu_0(v), \mu_1(s)),$$

$$\kappa_{10}(v_1, s_0) = |\mu_1'(v)|\kappa(\mu_1(v), \mu_0(s)),$$

$$\kappa_{11}(v_1, s_1) = |\mu_1'(v)|\kappa(\mu_1(v), \mu_1(s)),$$

So the system of equations (4.6) and (4.7) can be written as

$$\zeta_0(v)g'(b_0) + \int_0^{2\pi} \kappa_{00}(v_0, s_0)\zeta_0(s)g'(b_0)ds \\ - \int_0^{2\pi} \kappa_{01}(v_0, s_1)\zeta_1(s)g'(b_0)ds = \sigma_0(v), \quad (4.8)$$

$$\zeta_1(v)g'(b_0) + \int_0^{2\pi} \kappa_{10}(v_1, s_0)\zeta_0(s)g'(b_0)ds \\ - \int_0^{2\pi} \kappa_{11}(v_1, s_1)\zeta_1(s)g'(b_0)ds = \sigma_1(v), \quad (4.9)$$

Finally we have parameterized the boundary integral equation(3.8) for Ahlfors map of bounded multiply connected region.

Now we will parameterized the boundary integral equation (3.24) by letting $G_2(\mu) = g'(\mu)\tau(\mu)$. So, (3.24) becomes

$$G_2(\mu)g'(b_0) + \int_\Lambda \kappa_1(\mu, w)G_2(w)g'(b_0)|dw| = \overline{M(\mu)} \quad (4.10)$$

Where

$$M(\mu) = \overline{\tau(\mu)}\frac{\prod_{c=1}^{N}(b_0 - b_c)}{\prod_{c=1}^{N}\overline{(\mu - b_c)}}\frac{g'(b_0)}{g(b_0)}.$$

and

$$\kappa_1 = \begin{cases} \dfrac{1}{2\pi\mathrm{i}}\left[\dfrac{\tau(\mu)}{\mu - w} - \dfrac{\prod_{c=1}^{N}\overline{(w - b_c)}}{\prod_{c=1}^{N}\overline{(\mu - b_c)}}\dfrac{\overline{\tau(\mu)}}{\overline{\mu} - \overline{w}}\right], & \mu \neq w \in \Lambda. \\ \dfrac{1}{2\pi\mathrm{i}|\mu'(\mathrm{t})|}\left[\mathrm{iIm}\left(\dfrac{\mu''(\mathrm{v})}{\mu'(\mathrm{v})}\right) + \displaystyle\sum_{\mathrm{c}=1}^{\mathrm{N}}\dfrac{\overline{\mu'(\mathrm{v})}}{\overline{\mu(\mathrm{v}) - \mathrm{b_c}}}\right], & \mu = w \in \Lambda. \end{cases}$$

The equation (4.10) can be separated into the system of two integral equation and then parameterize it. Since $\Lambda = \Lambda_0 \cup \Lambda_1$, we can separate the intgral equation (3.24) into

$$G_2(\mu_0)g'(b_0) + \int_{\Lambda_0} \kappa_1(\mu_0, w)g'(b_0)G_2(w)|dw| \\ - \int_{\Lambda_1} \kappa_1(\mu_0, w)g'(b_0)G_2(w)|dw| = M(\mu_0), \quad \mu_0 \in \Lambda_0 \quad (4.11)$$

$$G_2(\mu_1)g'(b_0) + \int_{\Lambda_0} \kappa_1(\mu_1, w)g'(b_0)G_2(w)|dw| \\ - \int_{\Lambda_1} \kappa_1(\mu_1, w)g'(b_0)G_1(w)|dw| = M(\mu_1), \quad \mu_1 \in \Lambda_1 \quad (4.12)$$

Now we parameterize both equations (4.11)and (4.12) by using parametric representations $\mu_0(v)$ of Λ_0 for $v : 0 \leq v \leq 2\pi$ and $\mu_1(v)$ of Λ_1 for $v : 0 \leq v \leq 2\pi$ .Let $w = \mu(s)$ and using the fact $dw = \mu'(s)ds$, equations (4.11) and (4.12) becomes

$$G_2(\mu_0(v))g'(b_0) + \int_0^{2\pi} \kappa_1(\mu_0(v), \mu_0(s))g'(b_0)G_2(\mu_0(s))|\mu_0'(s)|ds \\ - \int_0^{2\pi} \kappa_1(\mu_0(v), \mu_1(s))g'(b_0)G_2(\mu_1(s))|\mu_1'(s)|ds = M(\mu_0(v)), \quad \mu_0(t) \in \Lambda_0 \quad (4.13)$$

$$G_2(\mu_1(v))g'(b_0) + \int_0^{2\pi} \kappa_1(\mu_1(v), \mu_0(s))g'(b_0)G_2(\mu_0(s))|\mu_0'(s)|ds \\ - \int_0^{2\pi} \kappa_1(\mu_1(v), \mu_1(s))g'(b_0)G_2(\mu_1(s))|\mu_1'(s)|ds = M(\mu_1(v)), \quad \mu_1(t) \in \Lambda_1 \quad (4.14)$$

Now multiplying (4.13) and (4.14) by $|\mu_0'(v)|$ and $|\mu_1'(v)|$ respectively we get

$$|\mu_0'(v)|G_2(\mu_0(v))g'(b_0) + \int_0^{2\pi} |\mu_0'(v)|\kappa_1(\mu_0(v), \mu_0(s))g'(b_0)G_2(\mu_0(s))|\mu_0'(s)|ds \\ -\int_0^{2\pi} |\mu_0'(v)|\kappa_1(\mu_0(v), \mu_1(s))g'(b_0)G_2(\mu_1(s))|\mu_1'(s)|ds = |\mu_0'(v)|M(\mu_0(v)), \quad \mu_0(v) \in \Lambda_0 \quad (4.15)$$

$$|\mu_1'(v)|G_2(\mu_1(v))g'(b_0) + \int_0^{2\pi} |\mu_1'(v)|\kappa_1(\mu_1(v), u_0(s))g'(b_0)G_2(\mu_0(s))|\mu_0'(s)|ds \\ -\int_0^{2\pi} |\mu_1'(v)|\kappa_1(\mu_1(v), \mu_1(s))g'(b_0)G_2(\mu_1(s))|\mu_1'(s)|ds = |\mu_1'(v)|M(\mu_1(v)), \quad \mu_1(v) \in \Lambda_1 \quad (4.16)$$

To simplify (4.15) and (4.16), let us define

$$\zeta_0(v) = |\mu_0'(v)|G_2(\mu_0(v)),$$

$$\zeta_1(v) = |\mu_1'(v)|G_2(\mu_1(v)),$$

$$\sigma_0(v) = |\mu'(v)|M(\mu_0(v)),$$

$$\sigma_1(v) = |\mu_1'(v)|M(\mu_1(v)),$$

$$\kappa_{00}(v_o, s_0) = |\mu_0'(v)|\kappa_1(\mu_0(v), \mu_0(s)),$$

$$\kappa_{01}(v_0, s_1) = |\mu_0'(v)|\kappa_1(\mu_0(v), \mu_1(s)),$$

$$\kappa_{10}(v_1, s_0) = |\mu_1'(v)|\kappa_1(\mu_1(v), \mu_0(s)),$$

$$\kappa_{11}(v_1, s_1) = |\mu_1'(v)|\kappa_1(\mu_1(v), \mu_1(s)),$$

So the system of equations (4.15) and (4.16) can be written as

$$\zeta_0(v)g'(b_0) + \int_0^{2\pi} \kappa_{00}(v_0, s_0)\zeta_0(s)g'(b_0)ds - \int_0^{2\pi} \kappa_{01}(v_0, s_1)\zeta_1(s)g'(b_0)ds = \sigma_0(v), \quad (4.17)$$

$$\zeta_1(v)g'(b_0) + \int_0^{2\pi} \kappa_{10}(v_1, s_0)\zeta_0(s)g'(b_0)ds - \int_0^{2\pi} \kappa_{11}(v_1, s_1)\zeta_1(s)g'(b_0)ds = \sigma_1(v), \quad (4.18)$$

Finally we have parameterized the boundary integral equation(3.24) for Ahlfors map of bounded multiply connected region.

4.2 Discretization

In this section we will discretize the (4.8), (4.9), (4.17) and (4.18) as parameterized in last section. As the function ς, σ and κ are periodic therefore we will solve (4.8), (4.9), (4.17) and (4.18) numerically by using Nyström method with trapezoidal rule. For discretization of the system of integral equations we have consider the following conditions.

(i) For n equidistant points, $v_d = \frac{(d-1)2\pi}{n}, 1 \leq d \leq n$ on Λ_0
(ii) For m equidistant points, $v_{\widetilde{d}} = \frac{(\widetilde{d}-1)2\pi}{m}, 1 \leq \widetilde{d} \leq m$ on Λ_1

Using these condition and applying Nyström method with trapezidal rule to discretize (4.8), (4.9), (4.17) and (4.18), we get

$$\zeta_0(v_d)g'(b_0) + \frac{2\pi}{n}\sum_{c=1}^{n}\kappa_{00}(v_d, v_c)\zeta_0(v_c)g'(b_0)ds \\ - \frac{2\pi}{m}\sum_{c=1}^{m}\kappa_{01}(v_d, v_c)\zeta_1(v_c)g'(b_0)ds = \sigma_0(v_d), \quad (4.19)$$

$$\zeta_1(v_d)g'(b_0) + \frac{2\pi}{n}\sum_{c=1}^{n}\kappa_{10}(v_d, v_c)\zeta_0(v_c)g'(b_0)ds \\ - \frac{2\pi}{m}\sum_{c=1}^{m}\kappa_{11}(v_d, v_c)\zeta_1(v_c)g'(b_0)ds = \sigma_1(v_d), \quad (4.20)$$

Equation (4.19) and (4.20) can be verified numerically by using the known values of zeros of Ahlfors map. After finding the results we will find the difference of numerical values of both sides of (4.19) and (4.20). We can expect the difference is nearly close to zero. So, equations (4.19) and (4.20) can be written as

$$e_{0d} = \zeta_0(v_d)g'(b_0) + \frac{2\pi}{n}\sum_{c=1}^{n}\kappa_{00}(v_d, v_c)\zeta_0(v_c)g'(b_0)ds \\ - \frac{2\pi}{m}\sum_{c=1}^{m}\kappa_{01}(v_d, v_c)\zeta_1(v_c)g'(b_0)ds - \sigma_0(v_d) = 0, \quad (4.21)$$

$$e_{1d} = \zeta_1(v_d)g'(b_0) + \frac{2\pi}{n}\sum_{c=1}^{n}\kappa_{10}(v_d, v_c)\zeta_0(v_c)g'(b_0)ds \\ - \frac{2\pi}{m}\sum_{c=1}^{m}\kappa_{11}(v_d, v_c)\zeta_1(v_c)g'(b_0)ds - \sigma_1(v_d) = 0, \quad (4.22)$$

If numerical values of $\{e_{0d}, e_{1d}\}$ are nearly closed to zero, then the boundary integral equations (3.8) and (3.24) are numerically verified.

4.3 Conclusion

In this work we have analyzed one of the integral equation derived by Nazar [11] as well as the derivation and analysis of a new integral equation for conformal mapping of multiply connected regions onto the unit disc by using the known values of zeros of Ahlfors map. Then these equations are parametrized and discritized with the help of Nyström method with trapezidal rule.

Chapter 5

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