Two Population Neural Field Model With Constant External Input

By: Ammara Basharat

Registration ID :CIIT/FA16-RMT-032/LHR

MS Thesis

In

Master of Science in Mathematics

COMSATS University Islamabad,Lahore Campus

Spring, 2018

COMSATS University Islamabad, Lahore Campus

Two Population Neural Field Model With Constant External Input

A Thesis Presented to

COMSATS University Islamabad

In partial fulfillment

of the requirement for the degree of

MS Mathematics

By

Ammara Basharat

CIIT/FA16-RMT-032/LHR

Spring,2018

Two Population Neural Field Model With Constant External Input

A Post Graduate Thesis submitted to the name of Department of Mathematics as partial fulfillment of the requirement for the award of degree of MS Mathematics

Name	Registration ID
Ammara Basharat	CIIT/FA16-RMT-032/LHR

Supervisor

Dr. Muhammad Yousaf

Assistant Professor Department of Mathematics

COMSATS University Islamabad

Lahore Campus

june, 2018

Final Approval

This thesis titled

Two Population Neural Field Model With Constant External Input

By

Ammara Basharat
CIIT/FA16-RMT-032/LHR

Has been approved
For the COMSATS University Islamabad,Lahore Campus

External Examiner:______________________________

Dr. Imran Javed
Associate Professor,Department of Mathematics,
BZU,Multan

Supervisor:______________________________________

Dr. Muhammad Yousaf
Assistant Professor,Department of Mathematics,
COMSATS University Islamabad,Lahore Campus

HoD:____

Dr.Sarfraz Ahmad
Head, Department of Mathematics,
COMSATS University Islamabad,Lahore Campus.

Declaration

I Ammara Basharat, Registration ID CIIT/FA16-RMT-032/LHR hereby declare that I have produced the work presented in this thesis, during the scheduled period of study. I also declare that I have not taken any material from any source except referred to wherever that amount of plagiarism is within acceptable range. If a violation of HEC rules on research has occurred in this thesis, I shall be liable to punishable action under the plagiarism rules of the HEC

Date:_______________

Ammara Basharat
CIIT/FA16-RMT-032/LHR

Certificate

It is certified that Ammara Basharat, Registration ID CIIT/FA16-RMT-032/LHR has carried out all the work related to this thesis under my supervision at the Department of Mathematics, COMSATS University Islamabad,Lahore Campus and the work fulfills the requirement for award of MS degree

Date:______________

Head of Department: Supervisor:

___________________________ ___________________________

Dr. Sarfraz Ahmad
Head, Department of Mathematics
CUI, Lahore Campus

Dr. Muhammad Yousaf
Assistant Professor
CUI, Lahore Campus

DEDICATION TO

I would like to dedicate my dissertation to my beloved Parents, without their prayers it would not have been possible for me to complete my MS. I would also like to bestow this piece of work to my teacher Dr. Muhammad Yousaf Bhatti. He always encouraged me to work hard and Guided me towards the right way and destination.

IN THE NAME OF

ALLAH

THE MOST GRACIOUS COMPASSIONATE

THE MOST MERCIFUL AND BENEFICENT

WHOSE HELP AND GUIDANCE ALWAYS

SOLICIT AT EVERY STEP AND MOMENT.

ACKNOWLEDGMENTS

Praise to be ALLAH, the Cherisher and Lord of the World, Most gracious and Most Merciful

On completion of my MS project,first of all, I would like to thank my ALLAH almighty, The most merciful and beneficent, who gave me the health, courage and energy to accomplish my work in due time and without whose help this study which required untiring effort would not have been possible to complete within the time limit.

Secondly, I am deeply indebted to my supervisor Dr. Muhammad Yousaf Bhatti for his technical guidance, help and support throughout this project. He always provided me with valuable suggestions and his commitment in particular is solely responsible to make this project successful. He also helped me in keeping my spirits high during the project time frame.

Immerse thanks to my parents and other family members in boosting up my courage and providing me with zeal and fervour to accomplish this task. Furthermore their prayers and well wishes had never left me alone and pampered me well during all the phases of this dissertation.

Ammara Basharat
CIIT/FA16-RMT-032/LHR

ABSTRACT

In this work, We have study the two-population neural field Model under the effect of constant external inputs. Earlier, Blomquist et al.[1] investigated the stationary symmetric solution of two population model with no external input.They found that Amari stability analysis and linearized stability analysis produces same results. Later on Yousaf et al. [2] investigated the same model under the influence of spatial dependent external input and proved that the Amari technique and linearized stability technique does not produce same results. They also investigate the number of bumps as a function of amplitude parameter of external input and found at most four bump pairs. In our research work, We have to explore the two population neural field model under the effect of constant external input on bump solutions. We further, have to investigate the stability properties of stationary symmetric solutions by using two approaches (i) Amari approach (ii) standard linearized approach.
Numerically, We will investigate the number of bumps as a function of width and amplitude parameter of external input.

TABLE OF CONTENTS

Contents

1 Introduction **13**
1.1 Human nervous system . . . 13
1.1.1 Human Brain . . . 13
1.1.2 Brain Cells . . . 15
1.2 Neuroglia . . . 15
1.3 Neurons . . . 16
1.4 Modeling Approaches . . . 17
1.5 Compartmental modeling . . . 17
1.5.1 Point Neuron Models . . . 18
1.5.2 Firing Rate Models . . . 19

2 Two Population Model with Constant External Input **21**
2.1 Boundedness of the solutions . . . 22
2.2 Uniqueness and Existence of Stationary Symmetric Solution . . . 23
2.2.1 Existence of Stationary Symmetric Solution . . . 23
2.2.2 Geometrical interpretation of the mapping $\underline{F}$ . . . 25
2.2.3 Uniqueness . . . 26

3 Comparison of Two Stability Approaches **28**
3.1 Amari Stability Analysis . . . 28
3.2 Linearized Stability Analysis . . . 31
3.3 Amari Stability versus Linearized stability Analysis . . . 40

4 Effect of Spatial Dependent External Input **42**
4.1 What are Bumps: . . . 42
4.2 Numerical Results . . . 44
4.2.1 Numerical example on six bumps pair when amplitude parameters are fixed: . . . 44
4.2.2 Numerical example on six bumps pair when width parameters are fixed: . . . 45

5 Conclusion and Discussion **47**

Refrences **49**

Chapter 1
Introduction

1 Introduction

1.1 Human nervous system

There are numerous elements in human body that reacts of produced stimuli through nervous system. The human nervous system is the body's conductive system that consist of billion of nerve cells to transfer signals from one part of body to another. Human nervous system consists of two major classes i.e central nervous system(CNS) and peripheral nervous system(PNS). [49][51] These two systems collaborate and collect information from inside and outside of the environment.[50]

1.1.1 Human Brain

The human brain is the most complex part of human nervous system. It is located in head and covered with skull. It is similar to the brain of mammals but cerebral cortex is highly developed in humans. [9] It contains so many different regions which are highly important in their respective functions. The human brain mainly divided into three parts which are the following [13]

- **Forebrain**
- **Midbrain**
- **Hindbrain**

1- The Forebrain

The forebrain contains cerebral cortex which surrounds mid brain and brain stem. Cerebral cortex is the most important and complex structure in our brain. It is grayish in colour because the nerves present in this area have short of insulation due to which most of the other parts appears white. It is densely packed with neurons. It is larger in size and have so many wrinkles. [19]
Cerebral cortex is the outer surface that covers human brain. The outer surface of cortex is about 2 to 4 millimeters thick and it is surrounding round about 1.5 square meter surface area. It work is associated with thought, perception, awareness, realization and remembrance. [37] [19] The cerebral cortex is composed of four lobes that are given as

- **Frontal lobe**
- **Perital lobe**
- **Temporal lobe**
- **Occipital lobe**

- **Frontal lobe**

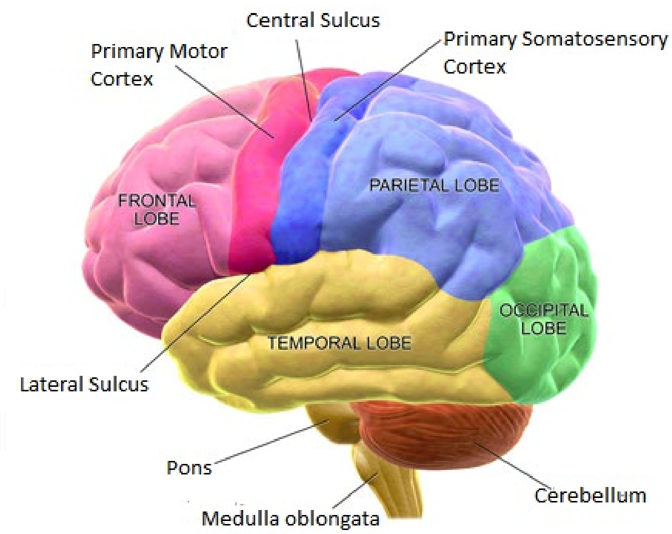

Figure 1: Cerebral Cortex [11]

The Frontal lobe is more developed and larger part in humans. It controls important human skills like language comprehension, memory, decision making etc. Its work is associated with tasks, short term memory, motivation, planing etc. It is the last lobe to mature due to which children do not effectively take decisions. The frontal lobe is the most common part to happen brain injury due to which a person loses his personality and faces difficulty in taking decisions. [30]

- **Perital lobe**

The Perital lobe is major lobe of cerebral cortex present in middle of all lobes. It is a part of cortex due to which it has many responsibilities. It convey message to other parts within seconds. It is responsible for many functions such as makes sense of the world, perceptions, taste, touch or feel etc. When this lobe damaged, a man becomes unable to feel any sensation of touch. [12]

- **Temporal lobe**

The temporal lobe is located behind the left side of other lobes and in bottom middle part of cortex. Its work is purely associated with hearing, listening and understanding sounds having different pitches. A person without temporal lobe cannot understand a meaningful talk. So this is very important lobe of human brain due to which we are able to listen, understand and talk about everything.[29]

- **Occipital lobe**

The last one is occipital lobe that is present in the back of head. It makes a man to being able to correctly understand and seeing things. Similar to the temporal lobe, makes sense of aural information, this makes sense of visual information.[31]

2- The Midbrain

The mid brain is located at the center of brain between fore brain and hind brain. It is composed of brain stem which is base of the brain connected to spinal cord. It plays important role in conveying message from brain to all other parts of body. It controls most important functions such as breathing, temperature, blood pressure, listening and viewing. [21] [18]

3- The Hindbrain

The hind brain is the important part of central nervous system. It consist of medulla oblongata,pons and cerebellum. Medulla works as involuntary function. The pons is the connection between cortex and cerebellum. The transmission of messages in all around body is possible due to pons. Although it is very small in size but without pons, all body movements would not be possible.[32] [33]
The cerebellum is the smaller structure (little brain) at the back of the brain. It is small portion of brain and round about ten percent of total weight. It receives sensory information through spinal cord and then regulate these movements. It is responsible for balance and coordination. Any damage to cerebellum cause serious problems such as slower association, lack of stability etc. [34]
Infect Our body movements is all due to brain. It controls our thoughts, plans, actions even our body temperature and breathing. Its work is associated with communication of signals, comprehension that and after then decision making. [2]

1.1.2 Brain Cells

The main organ of human nervous system is brain and spinal cord. Brain can be divided into two parts:

a. Neuroglia

b. Neurons

1.2 Neuroglia

Ependymal cells, astrocytes, microglia and oligodendrocytes are four important functions of neuroglia is to hold and surround neurons, supply nutrients and oxygen to neurons, control the ionic composition of extracellular space and to play the role of insulator between neurons and remove the dead neurons.[2]

1.3 Neurons

The human brain is the main communication center and this communication is only possible due to nerve cells that are called Brain cells or neurons. A human brain consist of billions nerve cells that have a work in conveying messages from one part of body to another. Some are the sensory neurons whose work is associated with touch, sound and light. They responds such stimuli and send signals to brain through spinal cord [35]. The basic unit of brain is neuron [49][51]. A neuron can be divided into three segments:

a. Dendrites

b. Soma(cell body)

c. Axon

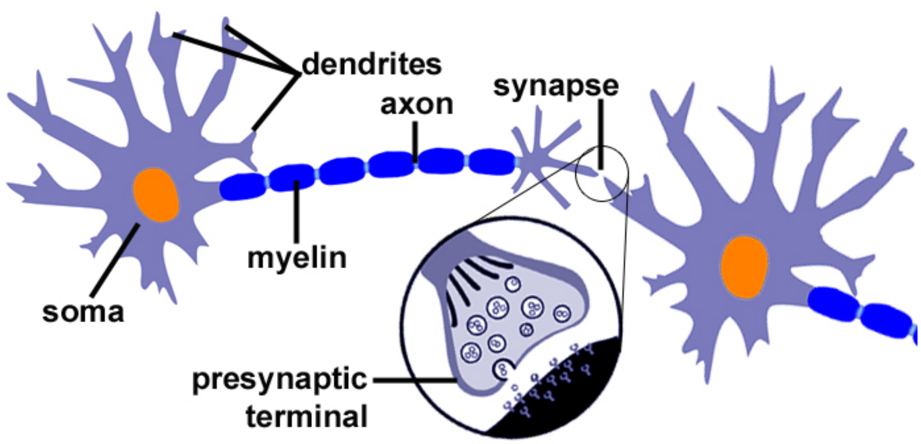

Figure 2: Neuron cell [11]

The dendrites collect signals from nearby neurons and transmit them to soma. Soma is the main cell body. It is also called processing unit. If the collected electrical potential in soma overshoots threshold value, it fires or generates an electrical signal (called spike), which then moves along the axon to a connected target neurons.[3]

Neurons convey message to other part of body through junctions called synapses. synapse is a small gap contained three parts. The first one is presynaptic terminal which contains neurotransmitter, that produces energy in the cell and cell bodies called organelles. The second part is postsyanaptic terminal that contains receptors for neurotransmitter. The last one is the space between presynaptic and postsyanaptic terminal. Thus the presynaptic terminal of one neuron is connected to postsyanaptic terminal of other neuron through this synaptic gap. [23]

Myelin is the necessary coating surrounded axon and helps neuron in conveying signals quickly through long distances. [39]

Synapses have two forms i.e excitatory and inhibitory. Excitatory synapses brings positive action in whole body while inhibitory decrease the firing action and cause negative action in cells. Excitatory neurons are make up of round about 80% of our settlement and try to bring the affected neurons closed to the action potential while the remaining 20% are inhibitory and they take away neurons from action potential. [35]

1.4 Modeling Approaches

Brain transfers information from one part of body to another part. This information is translated by different mathematical models. A model is a mathematical state designed to predict and explain the basic biological process. A mathematical model express the signal transformation of neuron.Use of mathematical methods with large computing power of computer in exploring the biological systems.

1.5 Compartmental modeling

Compartmental modeling is frequently used to describe the transformation of material in biological system. It is the system which contains many different compartments through which material transformed under certain rules. Neuron can be divided into different small compartments. Each compartment behaves like a small circuit, in which current is carried by the ions. The current moving through the compartments (the ion channels) is proportional to the difference in voltage times the conductance. The following Fig.shows construction of compartmental modeling.

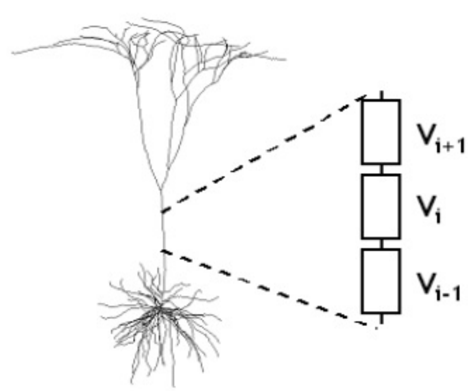

Figure 3: Construction of compartmental model for a neuron [2]

Mathematical equation for *n*th compartments is given from Kirchhoff's law of current as

$$\mathbf{g_{n,n+1}}(\mathbf{V_{n+1}} - \mathbf{V_n}) - \mathbf{g_{n-1,n}}(\mathbf{V_n} - \mathbf{V_{n-1}}) = \mathbf{k_n}\frac{\mathbf{dV_n}}{dt} + \sum_t \mathbf{I}_n^t + \sum_j \mathbf{I}_n^j \tag{1}$$

Where the left term of equation is due to the ohmic current between nth and the other two neighboring sections. The first right term gives the membrane capacitance while the other term shows the current injected from the pre-synaptic neurons and the last term is due to the leak ion channels. From end to end these channels current depends on the absorption of certain ions within and outer of the membrane. [40] [14]

1.5.1 Point Neuron Models

Compartmental modeling have many disadvantages due to use of no. of parameters and pricey calculations. This made difficult to compute and run these parameters. Then to overcome this intricacy, the model is reduced to a single compartment by removing a no. of compartments. In this way dendrites and soma malformed into a single point and neuron becomes a one dimensionless component. [40] [15]

It is a simple form for repetitive firing in neurons. It is just a point and when threshold is approached it fires spike of constant width and amplitude. [15]

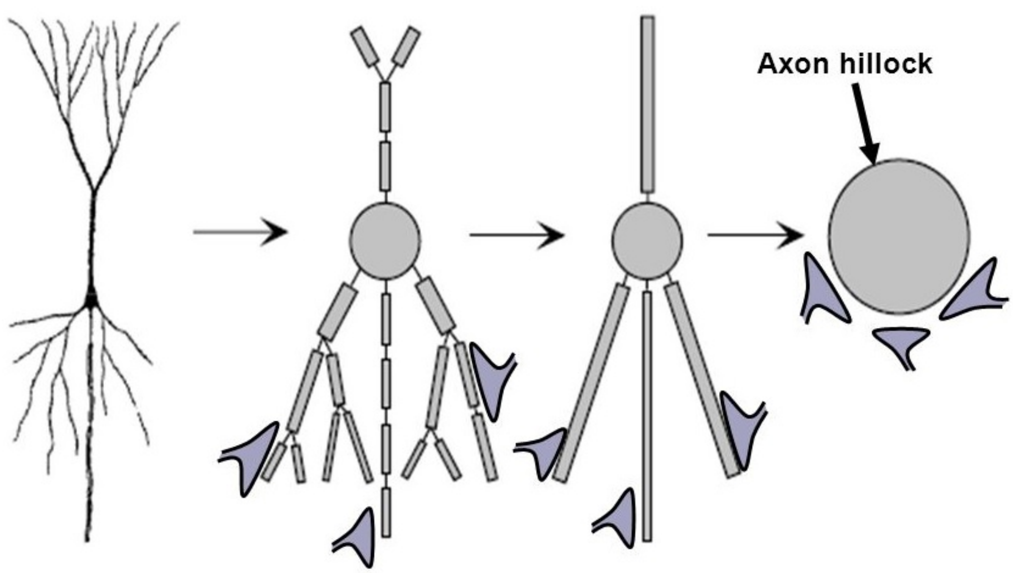

Figure 4: Conversion from compartmental models to point neurons [16]

Point neuron model is the simple point model that focus on the sub threshold action and output of neuron. Since all action potentials of the neuron have same behavior with almost same amplitude and the time. [4].

The sub-threshold action and output of the neuron are focal point of this model. In a single section model, action potential sub-threshold dynamics of the membrane is given as [3]

$$\mathbf{k}\frac{\mathbf{dV}}{dt} = -\mathbf{g_L}(\mathbf{V}-\mathbf{E_L}) - \sum_t \mathbf{I}^t \tag{2}$$

Here gL and EL represents conductance and resting potential of the neuron, when input current is absent. In this model currents omitted between sections of a single neuron, instead we mull over only the Ohmic leak current. But after converting compartmental modeling into point neuron, it is still not easy to come across critical result for the full dynamics of system for such simple spiking neurons, or even just the settlement rate dynamics. [40][43]

1.5.2 Firing Rate Models

To form a large system of neurons, one of the most general method is to apply a generalization called a firing rate model. Since these models can represents the whole population of neurons then a single neuron, so they are known as two population models. [17]

A neural tissue has a large dynamics which frequently studied by a firing rate models. Firing rate model is actually explains that how group of neurons give firing response to the input signals and what effect produced to other neurons after this response. These models have been commonly utilized in the investigation of the network properties of the strongly interlinked cortical networks. [5].

1.) These models are developed in form of integro-differential and equations. It makes easier to study the the properties of the solutions by using different mathematical techniques developed in applied mathematics.

2.) These models have less free parameters than spiking models, appropriate tuning of these free parameters parameters can be difficult.

3.) It is easier to work with these models, because in experiments spike trains vary.

Using these models make mathematical analysis of network behavior transparent and easier. [5].

Chapter 2
Two Population Model with Constant External Input

2 Two Population Model with Constant External Input

Neurons can be classified as excitatory and inhibitory neurons. If the post synaptic neuron gets closer to fire an action potential, then the post synaptic neuron is called excitatory neuron and if it gets away to fire action potential then it is inhibitory neuron. Blomquist et al [1] considers a two population model (3) similar to Wilson-Cowan type model [8] on the form

$$\frac{\partial v_e}{\partial t} = -v_e + \int_{-\infty}^{\infty} \omega_{ee}(x-x')Q_e(v_e(x',t)-\theta_e)dx' - \int_{-\infty}^{\infty} \omega_{ie}(x-x')Q_i(v_i(x',t)-\theta_i)dx' \quad (3a)$$

$$\tau\frac{\partial v_i}{\partial t} = -v_i + \int_{-\infty}^{\infty} \omega_{ei}(x-x')Q_e(v_e(x',t)-\theta_e)dx' - \int_{-\infty}^{\infty} \omega_{ii}(x-x')Q_i(v_i(x',t)-\theta_i)dx' \quad (3b)$$

Here the parameter τ is the ratio between inhibitory and excitatory time constants and is therefore named *relative inhibition time*. The parameter θ_e and θ_i represents threshold value for excitatory and inhibitory population. The parameters $v_e(x,t)$ and $v_i(x,t)$ denotes activity levels of excitatory and inhibitory populations, respectively at the time t and position x. Q_e and Q_i are the firing rate function for inhibitory and excitatory population.

$\omega_{m,n}(m=e, n=i)$ represents the connectivity function. Which are supposed to be real valued, bounded, symmetric and fulfill condition of normalization ($\int_{-\infty}^{\infty}\omega_{mn} = 1$).

In the present work, the model studied by Blomquist et al [1] with constant external inputs, Schematically the whole process is shown in the Fig. .5 The actual model reads

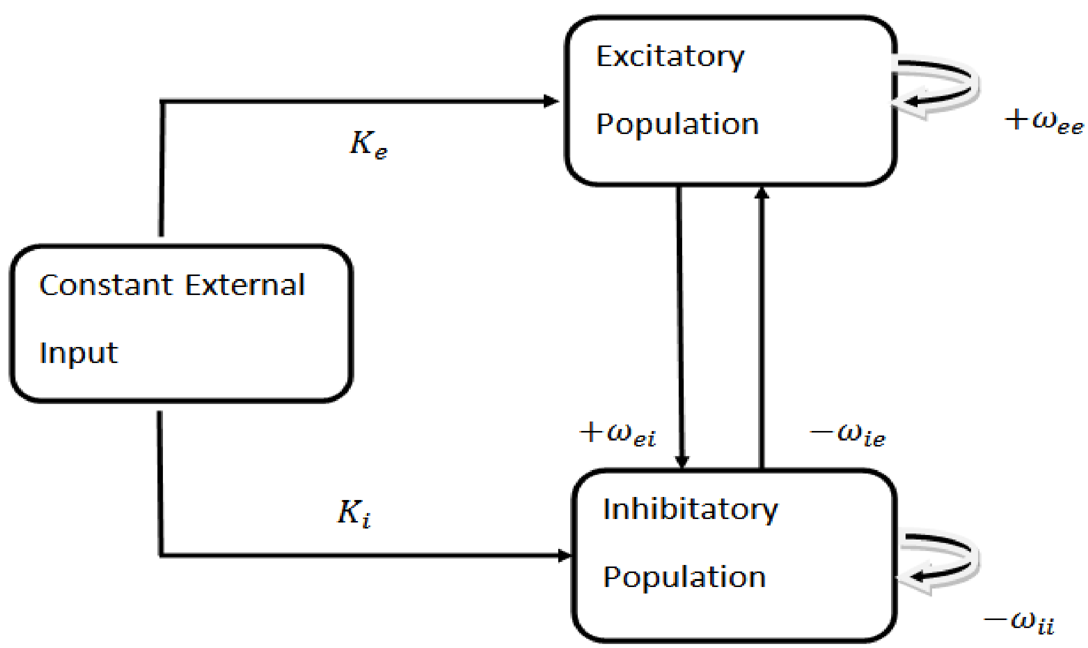

Figure 5: Two population model

$$\frac{\partial v_e}{\partial t} = -v_e + \int_{-\infty}^{\infty} \omega_{ee}(x-x')Q_e(v_e(x',t)-\theta_e)dx' - \int_{-\infty}^{\infty} \omega_{ie}(x-x')Q_i(v_i(x',t)-\theta_i)dx' + K_e \tag{4a}$$

$$\tau\frac{\partial v_i}{\partial t} = -v_i + \int_{-\infty}^{\infty} \omega_{ei}(x-x')Q_e(v_e(x',t)-\theta_e)dx' - \int_{-\infty}^{\infty} \omega_{ii}(x-x')Q_i(v_i(x',t)-\theta_i)dx' + K_i \tag{4b}$$

Here K_e, K_i are the constant external inputs to the excitatory and inhibitory populations respectively, while the remaining variables are as defined previously.

Yousaf et al [2] talk about the two population model (32) under the spatial external input and find the stability approaches by using Amari's approach and the linearized stability analysis. In this work, we will study the same problems within the framework of the model (32) under the condition of constant external input functions K_e, K_i. Blomquist et al [1] and Yousaf et al [2] talk about the problem of uniqueness and existence of stationary symmetric solutions and their stability properties by using both stability analysis. In this work, we will study the same problems within the framework of the model (32) under the condition of constant external input functions K_e, K_i.

2.1 Boundedness of the solutions

Let us now suppose that the constant external inputs K_e and K_i are bounded. Introduce new variables

$$V_e(x,t) = v_e(x,t) - K_e \tag{5a}$$

$$V_i(x,t) = v_i(x,t) - K_i \tag{5b}$$

Here the evolution equations for V_e and V_i are then given by

$$\frac{\partial V_e}{\partial t} = -v_e + \int_{-\infty}^{\infty} \omega_{ee}(x-x')Q_e(v_e(x',t)-\tilde{\theta}_e)dx' - \int_{-\infty}^{\infty} \omega_{ie}(x-x')Q_i(v_i(x',t)-\tilde{\theta}_i)dx' \tag{6a}$$

$$\tau\frac{\partial V_i}{\partial t} = -v_i + \int_{-\infty}^{\infty} \omega_{ei}(x-x')Q_e(v_e(x',t)-\tilde{\theta}_e)dx' - \int_{-\infty}^{\infty} \omega_{ii}(x-x')Q_i(v_i(x',t)-\tilde{\theta}_i)dx' \tag{6b}$$

where $\tilde{\theta}_e$ and $\tilde{\theta}_i$ threshold values defined by

$$\tilde{\theta}_e(x) = \theta_e - K_e\ ,\ \tilde{\theta}_i(x) = \theta_i - K_i \tag{7}$$

Hence, normalization condition applied on ω_{mn} and the assumption imposed on the firing rate function Q_m implies the uniform bounds

$$0 \leq [\int_{-\infty}^{\infty} \omega_{mn}Q_m(v_m - \tilde{\theta}_m](x,t) \leq 1 \tag{8}$$

for all values of x and t.so we get the explicit bounds for the solution v_e and v_i of (6), which can be given as

$$(\tilde{V}_e(x)+1)e^{-t-1} \leq v_e(x,t) \leq ((\tilde{V}_e(x)-1)e^{-t+1} \tag{9a}$$

$$(\tilde{V}_i(x)+1)e^{(-t/\tau)-1} \leq v_i(x,t)) \leq ((\tilde{V}_i(x)-1)e^{(-t/\tau)+1} \tag{9b}$$

by proceeding in the same way as in Blomquist et al [1]. Here

$$\tilde{V}_e(x) = V_e(x) - K_e \ , \ \tilde{V}_i(x) = V_i(x) - K_i \tag{10}$$

are the initial conditions for the system (6). Then, by replacing to the original variables, we will get the bounds

$$(V_e(x) - K_e + 1)e^{(-t)-1} + K_e \leq v_e(x,t) \leq (V_e(x) - K_e - 1)e^{(-t)+1} + K_e \tag{11a}$$

$$(V_i(x) - K_i + 1)e^{(-t/\tau)-1} + K_i \leq v_i(x,t) \leq (V_i(x) - K_i - 1)e^{(-t/\tau)+1} + K_i \tag{11b}$$

for solutions v_e and v_i. Boundedness of K_e,K_i implies the boundedness of the components v_e and v_i. We also observe that if

$$K_m - 1 \leq V_m(x) \leq K_m + 1, m = e,i \tag{12}$$

for all x, then

$$K_m - 1 \leq v_m(x,t) \leq K_m + 1, m = e,i \tag{13}$$

uniformity for all t.

2.2 Uniqueness and Existence of Stationary Symmetric Solution

In this section we will explore the existence and uniqueness of constant bump solutions of the system (32) by generalizing the arguments presented by Blomquist et al [1], Yousaf et al [2]. For the existence of constant bumps, we will first investigate the threshold values for the system (32) have constant solutions (*bumps*). After proving the existence of bumps, the uniqueness issue can also be designed.

2.2.1 Existence of Stationary Symmetric Solution

Here we will investigate the possibility of having constant bump solutions of (32) by following the similar approach adopted by Blomquist et al [1], Yousaf et al [2]. We proceed as follows

1. The constant external inputs K_m $(m = e,i)$ are assumed to be time and space independent, i.e $K_m(t) = K_m$ and modeled by means of Gaussian functions

   $$K_m(x) = A_m \ (m = e,i) \tag{14}$$

   Where, $A_m, (m = e,i)$.
   denotes the width of the constant external inputs respectively.

2. The solutions of the system (32) are presume to be the time and space independent, i.e $v_m(x,t) = V_m(x)$. Here $V_e(x)$ and $V_i(x)$ are bounded and smooth functions satisfying the following conditions

   - $V_m(x) = V_m(-x), \quad m = e,i$

   - $\lim_{|x|\to\infty} V_m(x) = \lim_{|x|\to\infty} K_m(x) = K_m$
     where $K_m,\ m = e,i$ is constant

   - There exist unique points $a,b > 0$, such that $V_e(\pm a) = \theta_e$ and $V_i(\pm b) = \theta_i$ as

$$K_m(x) = A_m \ (m = e,i) \tag{15}$$

Above conditions shows that we are only considering constant bump solutions. The parameters c,d denote the widths of the excitatory and inhibitory pulses respectively. According to the conditions $1-3$, expressions for $V_e(x)$ and $V_i(x)$ are given as

$$V_e(x) = W_{ee}(x+c) - W_{ee}(x-c) - W_{ie}(x+d) + W_{ie}(x-d) + K_e \tag{16a}$$

$$V_i(x) = W_{ei}(x+c) - W_{ei}(x-c) - W_{ii}(x+d) + W_{ii}(x-d) + K_i \tag{16b}$$

Here $W_{mn}(y)$ is the integral defined as

$$W_{mn}(y) = \int_0^y \omega_{mn}(y)dy \tag{17}$$

Notice that W_{mn} is an odd function. The conditions for a stationary symmetric solution are

$$q_e(c,d) = \theta_e \quad q_i(c,d) = \theta_i \tag{18}$$

where q_e and q_i are given as

$$q_e(c,d) = W_{ee}(2c) - W_{ie}(c+d) + W_{ie}(c-d) + K_e \tag{19}$$

$$q_i(c,d) = W_{ei}(c+d) - W_{ei}(d-c) - W_{ii}(2d) + K_i \tag{20}$$

To show the existence of stationary symmetric solution, we want to get $\theta_m (m = e,i)$ in interval $(0,1]$, for the system (32) possesses solution. We introducing two subsets Σ of $\mathbf{R}^2$ and interval I in (θ_e, θ_i) are determine as

$$I = (0,1] \times (0,1] \tag{21}$$

$$\Sigma = \{(c,d) | a \geq 0, b \geq 0\} \tag{22}$$

Where I be the *threshold value set* and Σ denotes the *pulse width set*.
The theorem formulated by Yousaf et al[2] and Blomquist et al[1] on existence of bumps also holds true in our case for constant external input. The theorem reads as

Theorem *Suppose that the ω_{mn} are continuous almost everywhere. Introduce the vector field $\underline{F} : \mathbf{R}^2 \to \mathbf{R}^2$ defined as*

$$\underline{F} = \begin{bmatrix} q_e \\ q_i \end{bmatrix} \tag{23}$$

where q_e and q_i are given by expressions (19) *and* (20). *Assume that $(A_e, A_i) \in I$. Then We have following results:*

- *The set $\underline{F}(\Sigma)$ is bounded*

- *The non-empty intersection $\underline{F}(\Sigma) \cap$ I.*

Proof Firstly, we will prove that $\underline{F}(\Sigma)$ is bounded. To apply the condition of symmetric and normalization on ω_{mn}, we find that We also notice that

$$|K_e| \leq A_e \qquad |K_i| \leq A_i \tag{24}$$

Since by the triangle inequality

$$|q_e(c,d)| \leq |W_{ee}(2c)| + |W_{ie}(c+d)| + |W_{ie}(c-d)| + |K_e| \tag{25}$$
$$|q_i(c,d)| \leq |W_{ei}(c+d)| + |W_{ei}(d-c)| + |W_{ii}(2d)| + |K_i| \tag{26}$$

We find that

$$|q_e(c,d)| \leq \frac{3}{2} + |A_e|, \quad |q_i(c,d)| \leq \frac{3}{2} + |A_i| \tag{27}$$

Which shows boundedness of $\underline{F}(\Sigma)$.

Next, let us prove that $\underline{F}(\Sigma) \cap I$ is non-empty. The presupposition imposed on the connectivity functions ω_{mn} imply that the anti derivatives W_{mn} are continuous functions of x. Therefore the mapping of vector field $\underline{F} : \mathbf{R}^2 \to \mathbf{R}^2$ are continuous. so $\underline{F}(0,0) = (A_e, A_i)$. Since by supposition $(A_e, A_i) \in I$, we get $\underline{F}(0,0) \in I$. In the threshold value plane,the Continuity of vector field $\underline{F}$ implies that there is an open set, say $\Sigma_\circ$ about $(0,0)$, that is mapped bijectively to an open disc $\underline{F}(\Sigma_\circ)$ about (A_e, A_i). The intersection between $\underline{F}(\Sigma_\circ)$ and the set I must be non-empty.
Hence, there exists a subset of the threshold values if $(A_e, A_i) \in I$, for which the system 34 (with K_e and K_i given by (24))possesses constant bump solutions given by $(16) - (20)$. So the (θ_e, θ_i) *threshold values* has solution for the system (32).
For the case, where the external input functions are modeled by continuous and symmetric functions which approach to a constant at infinity the existence theorem for bumps also holds true. Thus we are not constricted to the Gaussian external input functions 33.

2.2.2 Geometrical interpretation of the mapping $\underline{\mathbf{F}}$

Notice that the vector field $\underline{F}$ can be written as $\underline{F} = \underline{F_0} + \underline{K}$, where

$$\underline{F}_0 = \begin{bmatrix} W_{ee}(2c) - W_{ie}(c+d) + W_{ie}(c-d) \\ W_{ei}(c+d) - W_{ei}(d-c) - W_{ii}(2d) \end{bmatrix} \tag{28}$$

$$\underline{K} = \begin{bmatrix} K_e \\ K_i \end{bmatrix} \tag{29}$$

When $\underline{F} = \underline{F}_0 (\Leftrightarrow \underline{K} \equiv 0)$ we have the situation without external input, which has been studied in Blomquist et al [1]. A non zero $\underline{K}$ represents the effect of external inputs.The mapping $\underline{F}$ has a very simple geometrical interpretation when $\underline{K}$ =constant.
First we map pulse width set Σ to the threshold value plane I by the mapping $\underline{F}_0$. This produces the image set $\underline{F}_0(\Sigma)$. The image set $\underline{F}(\Sigma)$ then emerges as a rigid translation of $\underline{F}_0(\Sigma)$ in the threshold value plane.
This means the components $\tilde{V}_m(x) = V_m(x) - K_m$ have the same shape as without external input, i.e. the same situation as in Blomquist et al [1].

2.2.3 Uniqueness

Now suppose that the (θ_e, θ_i) associate with the set of admissible threshold values and (c_{eq}, d_{eq}) be the relative solution of the system (32). We suppose that W_{mn} are continuous and (q_e) and (q_i) refer by (19) and (20) are continuously differentiable. $\underline{F}$ is one-to-one and onto, if the Jacobian of $\underline{F}$ at the point $\underline{c}_{eq} = (c_{eq}, d_{eq})$ is non singular,(By inverse function theorem) i.e

$$det[\frac{\partial \underline{F}}{\partial \underline{c}}](\underline{c}_{eq}) \neq 0 \tag{30}$$

If the Jacobian of $\underline{F}$ at $\underline{c}_{eq}$ is singular, then this vector field will not guarantee a one-to-one correspondence. There will be an open neighborhood of $\underline{c}_{eq}$, that is bijectively mapped to the open neighborhood of (θ_e, θ_i) (By inverse function theorem) . It means for each point in (θ_e, θ_i) there exist a unique pulse pair.
The condition (30) is called the transversality condition. Notice that the generic situation consists of two pulse pairs for admissible threshold values just as for the case of no external inputs. We will demonstrate numerically in the next section.

Chapter 3
Comparison of Two Stability Approaches

3 Comparison of Two Stability Approaches

In this chapter we will explore the stability properties of constant bumps by using Amari approach and linearized stability analysis.

3.1 Amari Stability Analysis

$(V_e(x), V_i(x))$ be the stationary symmetric solution and (c,d) is the intersection point between the level curves

$$q_e(c,d) = \theta_e \ , \quad q_i(c,d) = \theta_i \tag{31}$$

So 31 be the starting point of the simplified stability analysis [1]. We will investigate the stability properties of constant bumps, First we assume that $(\theta_e,\ \theta_i)$ belongs to the admissible threshold value set, in which the level curves will intersect at least at one point (c_{eq}, d_{eq}) in pulse width set $\sum$. By following earlier approaches expand by Amari [6] and lately used by Bloomquist, Wyller and Einevoll [1], We make the following assumptions

1. The pulse width coordinates of the perturbed pulses are time dependent.

2. The perturbed pulses and stationary pulses satisfy the same threshold values, i.e,

$$v_e(c(t),t) = \theta_e \tag{32a}$$

$$v_i(d(t),t) = \theta_i \tag{32b}$$

3. The slopes of the perturbed pulses approximated by the slopes of the bumps at the points (c_{eq}, d_{eq}), i.e

$$\partial_x v_e(c,t) \approx V_e'(c_{eq}) \ \ , \ \ \partial_x v_i(d,t) \approx V_i'(d_{eq}) \tag{33}$$

Since $V_e'(c_{eq}) < 0$ and $V_i'(d_{eq}) < 0$, We have

$$\partial_x v_e(c,t) \approx -|V_e'(c_{eq})| \ \ , \ \ \partial_x u_i(d,t) \approx -|V_i'(d_{eq})| \tag{34}$$

By differentiating (32a) and (32b) with respect to t and using (32), We get

$$\begin{aligned} |\partial_x v_e(c,t)|\frac{dc}{dt} &= -v_e(c,t) + \int_{-\infty}^{\infty} \omega_{ee}(c-y)H_e(v_e(y,t)-\theta_e)dy \\ &- \int_{-\infty}^{\infty} \omega_{ie}(c-y)H_i(v_i(y,t)-\theta_i)dy \\ &= q_e(c,d) - \theta_e \end{aligned} \tag{35}$$

$$\begin{aligned} \tau|\partial_x v_i(d,t)|\frac{dd}{dt} &= -v_i(d,t) + \int_{-\infty}^{\infty} \omega_{ei}(d-y)H_e(v_e(y,t)-\theta_e) \\ &- \int_{-\infty}^{\infty} \omega_{ii}(d-y)H_i(v_i(y,t)-\theta_i)dy \\ &= q_i(c,d) - \theta_i \end{aligned} \tag{36}$$

Here q_e and q_i are given by (19) and (20). By taking into account the approximations (33) and (34), the dynamical system (35) and (36) can be rewritten as

$$|V_e'(c_{eq})|\frac{dc}{dt} = q_e(c,d) - \theta_e \tag{37}$$

$$\tau|V_i'(d_{eq})|\frac{dd}{dt} = q_i(c,d) - \theta_i \tag{38}$$

The system (37) and (38) constitutes an autonomous system defined by the vector field

$$\tilde{Q} : \Sigma \to R^2, \; \tilde{Q} = \begin{bmatrix} \tilde{Q}_e \\ \tilde{Q}_i \end{bmatrix} \tag{39}$$

where

$$\tilde{Q}_e(c,d) = \frac{1}{|V_e'(c_{eq})|}\{f_e(c,d) - \theta_e\} \tag{40}$$

$$\tilde{Q}_i(c,d) = \frac{1}{|V_i'(d_{eq})|}\{f_i(c,d) - \theta_i\} \tag{41}$$

Let $\underline{c}_{eq} = (c_{eq}, d_{eq})^t$ denotes the equilibrium point of (37) and (38). In order to find the stability of this point, We proceed by finding the Jacobian of the vector field $\tilde{Q}$ evaluated at $\underline{c}_{eq}$ and then calculating the eigenvalues: The Jacobian is given by

$$J_A = \frac{\partial \tilde{Q}}{\partial \underline{a}}(\underline{c}_{eq}) = \begin{bmatrix} \frac{\partial \tilde{Q}_e}{\partial \underline{c}} & \frac{\partial \tilde{Q}_e}{\partial \underline{d}} \\ \frac{\partial \tilde{Q}_i}{\partial c} & \frac{\partial \tilde{Q}_i}{\partial d} \end{bmatrix} \tag{42}$$

=

$$\begin{bmatrix} \eta_A & -\nu_A \\ \frac{1}{\tau}\kappa_A & -\frac{1}{\tau}\beta_A \end{bmatrix} \tag{43}$$

Where η_A, ν_A, κ_A and β_A are defined as

$$\eta_A = \frac{1}{|V_e'(c_{eq})|}\{2\ \omega_{ee}(2c_{eq}) - \omega_{ie}(c_{eq}+d_{eq}) + \omega_{ie}(c_{eq}-d_{eq}) \tag{44}$$

$$\nu_A = \frac{1}{|V_e'(c_{eq})|}\{\omega_{ie}(c_{eq}+d_{eq}) + \omega_{ie}(c_{eq}-d_{eq})\} \tag{45}$$

$$\kappa_A = \tau\frac{1}{|V_i'(c_{eq})|}\{\omega_{ei}(c_{eq}+d_{eq}) + \omega_{ei}(d_{eq}-c_{eq})\} \tag{46}$$

$$\beta_A = \tau\frac{1}{|V_i'(d_{eq})|}\{2\ \omega_{ii}(2d_{eq}) - \omega_{ei}(c_{eq}+d_{eq}) + \omega_{ei}(d_{eq}-c_{eq}) \tag{47}$$

So, Λ is the eigenvalues of Jacobian J_A which satisfy the quadratic equation

$$\tau\Lambda^2 + \Lambda(\beta_A - \tau\eta_A) + \gamma_A = 0 \tag{48}$$

Where,

$$\gamma_A = \kappa_A \nu_A - \beta_A \eta_A \tag{49}$$

The stability depends on the invariants of the $J_{\mathbf{A}}$ i.e the determinant D and trace T , i.e

$$\mathbf{T} \quad = tr(J_A) = \eta_A - \frac{\beta_A}{\tau} \;\;,\;\; D = det(J_A) = \frac{\gamma_A}{\tau} \tag{50}$$

If $\gamma_A < 0$, the $detJ_A < 0$, which indicates to a saddle point instability. When $\gamma_A > 0$ ($\Leftrightarrow$ $detJ_A > 0$), We have the following situations:

We have stability whenever $T < 0$. This disscussed in the following three cases

1.) For all values of τ , $\beta_A > 0$ and $\eta_A < 0$

2.) For $\tau < \tau_{cr}$, $\beta_A > 0$ and $\eta_A > 0$,

3.) For $\tau > \tau_{cr}$, $\beta_A < 0$ and $\eta_A < 0$, where $\tau_{cr} = |\beta_A|/|\eta_A|$

The situations with $T > 0$, $detJ_A > 0$ corresponds to instability, which occurs in one of the following three situations:

1.) $\beta_A < 0$ and $\eta_A > 0$ for all values of τ

2.) $\beta_A > 0$ and $\eta_A > 0$, for $\tau > \tau_{cr}$

3.) $\beta_A < 0$ and $\eta_A < 0$ for $\tau < \tau_{cr}$ with $\tau_{cr} = |\beta_A|/|\eta_A|$

The Amari stability is summarized in Table 1

Table 1: Stability results by Amari approach ($\tau_{cr} = |\beta_A|/|\eta_A|$)

$\gamma_A < 0$			saddle point instability
$\gamma_A > 0$	$\beta_A > 0$	$\eta_A < 0$	stability for all τ
	$\beta_A > 0$	$\eta_A > 0$	stability if $\tau < \tau_{cr}$, instability for $\tau > \tau_{cr}$
	$\beta_A < 0$	$\eta_A < 0$	stability if $\tau > \tau_{cr}$, instability if $\tau < \tau_{cr}$
	$\beta_A < 0$	$\eta_A > 0$	instability

The parameters β_A, $\eta_A > 0$ for constant external inputs, and the connectivity functions $\omega_{ei}(x)$ and $\omega_{ie}(x)$ being monotonically decreasing functions for $x > 0$. The parameters β_A and η_A may become negative for spatial external inputs.

3.2 Linearized Stability Analysis

In this section we will discuss the stability of bumps by using standard linearization procedure in a way similar to the one used by Blomquist et al [1]. We will compare the results of this discussion with the predictions obtained by using the Amari approach. Let $V_e(x)$ and $V_i(x)$ be the stationary symmetric and time independent solutions of (32), i.e let us suppose the perturbed state

$$v_e(x,t) = V_e(x) + \xi(x,t) \tag{51a}$$

$$v_i(x,t) = V_i(x) + \chi(x,t) \tag{51b}$$

We expand the above equation by Taylor expansion of Heaviside function H yields,

$$H(V_e - \theta_e + \ \xi) = H(V_e - \theta_e) + \xi \ \delta(V_e - \theta_e) + \cdots \tag{52a}$$

$$H(V_i - \theta_i + \ \chi) = H(V_i - \theta_i) + \chi \ \delta(V_i - \theta_i) + \cdots \tag{52b}$$

Where δ represents the Dirac delta function. So $|\xi| \ll |V_e - \theta_e|$ and $|\chi| \ll |V_i - \theta_i|$ by assumption, these two terms in the expansions are left because of lowest order terms.

By using (52) and (51) in (32), We will get the linearized perturbed system,

$$\xi_t = -\xi + \int_{-\infty}^{\infty} \omega_{ee}(x-x')\delta(\ (V_e - \theta_e\ )\ \xi)dx' - \int_{-\infty}^{\infty} \omega_{ie}(x-x')\delta(\ (V_i - \theta_i\ )\chi)dx' \tag{53a}$$

$$\tau\chi_t = -\chi + \int_{-\infty}^{\infty} \omega_{ei}(x-x')\delta(\ (V_e - \theta_e\ )\xi)dx' - \int_{-\infty}^{\infty} \omega_{ii}(x-x')\delta(\ (V_i - \theta_i\ )\chi)dx' \tag{53b}$$

The solutions of the perturbed system of the form

$$\xi(x,t) = e^{\Lambda t}\xi_1(x) \quad \chi(x,t) = e^{\Lambda t}\chi_1(x) \tag{54}$$

Using (54), the system (53) becomes

$$(\Lambda+1)\xi_1(x) = \int_{-\infty}^{\infty} \omega_{ee}(x-x')\delta((V_e - \theta_e)\xi_1)dx' - \int_{-\infty}^{\infty} \omega_{ie}(x-x')\delta((V_i - \theta_i)\chi_1)dx' \tag{55a}$$

$$(\tau\Lambda+1)\chi_1(x) = \int_{-\infty}^{\infty} \omega_{ei}(x-x')\delta((V_e - \theta_e)\xi_1)dx' - \int_{-\infty}^{\infty} \omega_{ii}(x-x')\delta((V_i - \theta_i)\chi_1)dx' \tag{55b}$$

The integrals used in (55) are by definition given as

$$\int_{-\infty}^{\infty} \omega(\delta(v-\theta)z)](x) = \int_{-\infty}^{\infty} \omega(x'-x)\delta(v(x')-\theta)z(x')dx' \tag{56}$$

Here ω is connectivity function defined by **??**, v stands for activity levels and z represents the disturbances imposed on bumps. Splitting the above integral in to two parts

$$[\int_{-\infty}^{\infty} \omega(\delta(v-\theta)z)](x) = \int_{-\infty}^{0} \omega(x'-x)\delta(v(x')-\theta)z(x')dx' + \int_{0}^{\infty} \omega(x'-x)\delta(v(x')-\theta)z(x')dx' \tag{3}$$

We assume that the function v satisfies the following conditions:

1.) $v(x) = v(-x)$ i.e u is spatially symmetric.

2.) $v(x) < 0$ for $0 < x < x_1$ where $v'(x_1) = 0$

3.) $v'(x) > 0$ for $x > x_1$

4.) $y = v(x'), x' = v^{-1}(y),\ dy = v'(x')dx'$

5.) $v(0) = y_0, v(x_1) = y_1, v(\infty) = P$, $y_1 < P < y_0$

6.) $v(c) = \theta, 0 < \theta < y_0$

The integral defined by (56), can be evaluated in two steps

1.) In this step we will evaluate the second term on the right hand side of (3), we proceed as follows

$$\int_0^\infty \omega(x'-x) \quad \delta(v(x')-\theta)z(x')dx' = \textstyle\int_0^{x_1} \omega(x'-x)\delta(v(x')-\theta)z(x')dx' \\ + \textstyle\int_{x_1}^\infty \omega(x'-x)\delta(v(x')-\theta)z(x')dx' \tag{4}$$

Using the conditions $(1. \to 6.)$, the integrals of right hand side of (4) can be expressed as

$$= \int_{y_0}^{y_1} \omega(v^{-1}(y)-x)\delta(y-\theta)z(v^{-1}(y))\frac{1}{v'(v^{-1}(y))}dy \\ + \int_{y_1}^{P} \omega(v^{-1}(y)-x)z(v^{-1}(y))\delta(y-\theta)\frac{1}{v'(v^{-1}(y))}dy \tag{5}$$

By using the properties of the δ function, we find that

$$\int_{y_1}^{y_0} \omega(v^{-1}(y)-x)z(v^{-1}(y))\delta(y-\theta)\frac{1}{v'(v^{-1}(y))}dy = \omega(x-a)z(a)|v'(a)^{-1}| \tag{6}$$

Since $y_1 < 0 < y_0$ and $y_0 > \theta > O$, we find that

$$\int_{y_1}^{O} \omega(v^{-1}(y)-x)z(v^{-1}(y))\delta(y-\theta)\frac{1}{v'(v^{-1}(y))}dy = 0 \tag{7}$$

2.) Using the similar reasoning the first part of the RHS of (3) yields

$$\int_{-\infty}^{0} \omega(x'-x)\delta(v(x')-\theta)z(x')dx' = \omega(x+a)z(-a)|v'(a)^{-1}| \tag{8}$$

Hence

$$[\int_{-\infty}^{\infty} \omega(\delta(v-\theta)z)](x) = \omega(x+c)z(-c)|v'(c)^{-1}| + \omega(x-c)z(c)|v'(c)^{-1}| \tag{9}$$

Now we suppose that there is one-to-one correspondence between the pulse width coordinates (c_{eq}, d_{eq}) and the bumps $(V_e(x), V_i(x))$ through the relation

$$V_e(c_{eq}) = \theta_e \ , \ V_i(d_{eq}) = \theta_i \tag{10}$$

By using these assumptions, we can evaluate the integrals appearing in (55a) and (55). We end up with the system

$$(\Lambda+1)\xi_1(x) = \frac{1}{|V_e'(c_{eq})|}[\omega_{ee}(x+c_{eq})\xi_1(-c_{eq})+\omega_{ee}(x-c_{eq})\xi_1(c_{eq})] \\ -\frac{1}{|V_i'(d_{eq})|}[\omega_{ie}(x+d_{eq})\chi_1(-d_{eq})+\omega_{ie}(x-d_{eq})\chi_1(d_{eq})] \tag{11a}$$

$$(\Lambda\tau+1)\chi_1(x) = \frac{1}{|V_e'(c_{eq})|}[\omega_{ei}(x+c_{eq})\xi_1(-c_{eq})+\omega_{ei}(x-c_{eq})\xi_1(c_{eq})] \\ -\frac{1}{|V_i'(d_{eq})|}[\omega_{ii}(x+d_{eq})\chi_1(-d_{eq})+\omega_{ii}(x-d_{eq})\chi_1(d_{eq})] \tag{11b}$$

Here,

$$|V_e'(c_{eq})| = \omega_{ee}(0)-\omega_{ee}(2c)+\omega_{ie}(c+d)-\omega_{ie}(c-d) \tag{12a}$$

$$|V_i'(d_{eq})| = \omega_{ei}(d-c)-\omega_{ei}(a+c)+\omega_{ii}(2d)-\omega_{ii}(0) \tag{12b}$$

The system equations (11) equivalence to,

$$\xi_1(c_{eq}) = \xi_1(-c_{eq}) = \chi_1(d_{eq}) = \chi_1(-d_{eq}) \equiv 0 \\ \Leftrightarrow \xi_1(x) = \chi_1(x) \equiv 0 \tag{13}$$

For non-trivial spatial perturbations, suppose that

$$\underline{X} = \begin{bmatrix} \xi_1(c_{eq}) \\ \xi_1(-c_{eq}) \\ \chi_1(d_{eq}) \\ \chi_1(-d_{eq}) \end{bmatrix} \neq 0 \tag{14}$$

The problem relate to finding the Λ for which (14) holds true.further, proceed as follows: Let $x=\pm c_{eq}$ and $x=\pm b_{eq}$ in (11) , we find that

$$(\Lambda+1)\xi_1(c_{eq}) = \frac{1}{|U_e'(c_{eq})|}[\omega_{ee}(2c_{eq})\xi_1(-c_{eq})+\omega_{ee}(0)\xi_1(c_{eq})] \\ -\frac{1}{|U_i'(d_{eq})|}[\omega_{ie}(c_{eq}+d_{eq})\chi_1(-d_{eq})+\omega_{ie}(c_{eq}-d_{eq})\chi_1(d_{eq})] \tag{15a}$$

$$(\Lambda+1)\xi_1(-c_{eq}) = \frac{1}{|V_e'(c_{eq})|}[\omega_{ee}(0)\xi_1(-c_{eq})+\omega_{ee}(2c_{eq})\xi_1(C_{eq})] \\ -\frac{1}{|V_i'(d_{eq})|}[\omega_{ie}(C_{eq}-d_{eq})\chi_1(-d_{eq})+\omega_{ie}(c_{eq}+d_{eq})\chi_1(d_{eq})] \tag{15b}$$

$$(\Lambda\tau+1)\chi_1(d_{eq}) = \frac{1}{|V_e'(c_{eq})|}[\omega_{ei}(c_{eq}+(d_{eq})\xi_1(-c_{eq})+\omega_{ei}((c_{eq}-d_{eq})\xi_1(c_{eq})] - \frac{1}{|V_i'(d_{eq})|}[\omega_{ii}(2d_{eq})\chi_1(-d_{eq})+\omega_{ii}(0)\chi_1(d_{eq})] \tag{15c}$$

$$(\Lambda\tau+1)\chi_1(-d_{eq}) = \frac{1}{|V_e'(c_{eq})|}[\omega_{ei}(c_{eq}-(d_{eq})\xi_1(-c_{eq})+\omega_{ei}((c_{eq}+d_{eq})\xi_1(c_{eq})] - \frac{1}{|V_i'(d_{eq})|}[\omega_{ii}(0)\chi_1(-d_{eq})+\omega_{ii}(2d_{eq})\chi_1(d_{eq})] \tag{15d}$$

The above system (15) is of four linear homogenous equations in which the four unknowns can be expressed in the matrix form as,

$$\mathbf{Z}\,.\;\underline{\mathbf{X}} \;=\; \underline{\mathbf{O}} \tag{16}$$

Here $\underline{\mathbf{X}}$ is defined by (14), while the matrix $\mathbf{Z}$ is given as

$$\mathbf{Z} = \begin{bmatrix} z_1-(\Lambda+1) & z_2 & -z_3 & -z_4 \\ z_2 & z_1-(\Lambda+1) & -z_4 & -z_3 \\ y_1 & y_2 & -y_3-(\Lambda\tau+1) & -y_4 \\ y_2 & y_1 & -y_4 & -y_3-(\Lambda\tau+1) \end{bmatrix} \tag{17}$$

Where,

$$z_1 = \frac{\omega_{ee}(0)}{|V_e'(c_{eq})|}, \qquad z_2 = \frac{\omega_{ee}(2c_{eq})}{|V_e'(c_{eq})|} \tag{18a}$$

$$z_3 = \frac{\omega_{ie}(c_{eq}-d_{eq})}{|V_e'(c_{eq})|}, \;\; z_4 = \frac{\omega_{ie}(c_{eq}+d_{eq})}{|V_e'(c_{eq})|} \tag{18b}$$

$$y_1 = \frac{\omega_{ei}(c_{eq}-d_{eq})}{|V_i'(d_{eq})|}, \;\; y_2 = \frac{\omega_{ei}(a_{eq}+d_{eq})}{|V_i'(d_{eq})|} \tag{18c}$$

$$y_3 = \frac{\omega_{ii}(0)}{|V_i'(d_{eq})|}, \qquad y_4 = \frac{\omega_{ii}(2d_{eq})}{|V_i'(d_{eq})|} \tag{18d}$$

Now, consider the matrix $\mathbf{G}$

$$\mathbf{G} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{2} & -\frac{1}{2} \end{bmatrix} \tag{19}$$

and introduce the substitution

$$\underline{\mathbf{Y}} = \mathbf{G}\cdot\underline{\mathbf{X}} \tag{20}$$

We readily find that

$$\underline{\mathbf{Y}} = \begin{bmatrix} \frac{1}{2}\{\xi_1(c_{eq})+\xi_1(-c_{eq})\} \\ \frac{1}{2}\{\chi_1(d_{eq})+\chi_1(-d_{eq})\} \\ \frac{1}{2}\{\xi_1(c_{eq})-\xi_1(-c_{eq}) \\ \frac{1}{2}\{\chi_1(d_{eq})-\chi_1(-d_{eq})\} \end{bmatrix} = \begin{bmatrix} \xi_e(c_{eq}) \\ \chi_e(d_{eq}) \\ \xi_o(c_{eq}) \\ \chi_o(d_{eq}) \end{bmatrix} \tag{21}$$

Here $\underline{\mathbf{Y}}$ is the vector having even and odd parts of of ξ_1 *and* χ_1 evaluated at c_{eq} and d_{eq}. Here the subscripts $'e'$ and $'o'$ stands for even and odd respectively. The similarity transformation from (16) to (20) transforms the matrix $\mathbf{Z}$ to the block diagonal matrix $\mathbf{PZP}^{-1}$, which can be written in a simplified form as

$$\mathbf{PZP}^{-1} = \begin{bmatrix} \mathbf{M}_1 & O \\ O & \mathbf{M}_2 \end{bmatrix} \tag{22}$$

where $\mathbf{M}_1$ and $\mathbf{M}_2$ are given as

$$\mathbf{M}_1 = \begin{bmatrix} z_1 + z_2 - \Lambda - 1 & -z_3 - z_4 \\ & \\ y_1 + y_2 & -y_3 - y_4 - \Lambda\tau - 1 \end{bmatrix} \tag{23}$$

$$\mathbf{M}_2 = \begin{bmatrix} z_1 - z_2 - \Lambda - 1 & -z_3 + z_4 \\ & \\ y_1 - y_2 & -y_3 + y_4 - \Lambda\tau - 1 \end{bmatrix} \tag{24}$$

The determinant of $\mathbf{M}_1$ is

$$|\mathbf{M}_1| = \tau\Lambda^2 + (\beta_L - \eta_L\tau)\Lambda + \gamma_L \tag{25}$$

$$\beta_L = y_3 + y_4 + 1 \tag{26a}$$

$$\eta_L = z_1 + z_2 - 1 \tag{26b}$$

$$\gamma_L = (y_1 + y_2).(z_3 + z_4) - (y_3 + y_4 + 1)(z_1 + z_2 - 1) \tag{26c}$$

Likewise, the determinant of $\mathbf{M}_2$ can be calculated as

$$|\mathbf{M}_2| = \tau\Lambda^2 + (\beta'_L - \eta'_L\tau)\Lambda + \gamma'_L \tag{27}$$

Here

$$\beta'_L = y_3 - y_4 + 1 \tag{28a}$$

$$\eta'_L = z_1 - z_2 - 1 \tag{28b}$$

$$\gamma'_L = (y_1 - y_2).(z_3 - z_4) - (y_3 - y_4 + 1)(z_1 - z_2 - 1) \tag{28c}$$

So the determinant of matrix $\mathbf{PZP}^{-1}$ factorizes as

$$\det(\mathbf{PZP}^{-1}) = (\tau\Lambda^2 + (\beta_L - \eta_L\tau)\Lambda + \gamma_L) \cdot (\tau\Lambda^2 + (\beta'_L - \eta'_L\tau)\Lambda + \gamma'_L) \tag{29}$$

where the subscript L show that stability analysis is linearized. For (14) to hold true $det(\mathbf{PZP}^{-1}) = 0$, which gives two characteristic equations

$$\tau\Lambda^2 + (\beta_L - \eta_L\tau)\Lambda + \gamma_L = 0 \tag{30}$$

$$\tau\Lambda^2 + (\beta'_L - \eta'_L\tau)\Lambda + \gamma'_L = 0 \tag{31}$$

If all eigenvalues of the characteristic equations have negative real part then they are stable and if at least one of the eigenvalues has non-negative real part then we have instability. Λ can be expressed in terms of the traces $(tr(\mathbf{M}_i))$ and the determinant $(det(\mathbf{M}_i))$. let us introduce the notations

$$T_1 = tr(\mathbf{M}_1) = \eta_L - \frac{\beta_L}{\tau}, \quad \gamma_L = \tau\, det(\mathbf{M}_1) \tag{32}$$

$$T_2 = tr(\mathbf{M}_2) = \eta_L' - \frac{\beta_L'}{\tau}, \quad \gamma_L' = \tau\, det(\mathbf{M}_2) \tag{33}$$

The stability of the bumps can now be analyzed by means of the parameters T_1 ,T_2 , ,γ_L and γ_L'. Notice that $\beta_L > 0$. Moreover, if ω_{ii} is a decreasing function of positive x then $\eta_L' > 0$. In what follows we assume that $\beta_L' > 0$. Hence the critical time constants T_{cr} and T_{cr}' defined by

$$\tau_{cr} = \frac{\beta_L}{|\eta_L|} \;\; , \;\; \tau_{cr'} = \frac{\beta_L'}{|\eta_L'|} \tag{34}$$

Let $\Lambda_\pm$ and $\Lambda_\pm'$ denote the solutions of (30) and (31) respectively.
Here we will prove the identities

1.) $\beta_L = \beta_A$

2.) $\eta_L = \eta_A$

3.) $\gamma_A = \gamma_L$

4.) $\gamma_L' = 0$

Here β_L,η_L,γ_L,β_A,η_A,and γ_A are defined by the equations (26), (44, and (49) respectively. The proofs proceed as follows. From **??**

$$V_e(x) = \int_{x-c}^{x+c} \omega_{ee}(y)dy - \int_{x-d}^{x+d} \omega_{ie}(y)dy + K_e \tag{35}$$

$$V_i(x) = \int_{x-c}^{x+c} \omega_{ei}(y)dy - \int_{x-d}^{x+d} \omega_{ii}(y)dy + K_i \tag{36}$$

Differentiating (35) and (36) with respect to x, we get

$$V_e'(x) = \omega_{ee}(x+c) - \omega_{ee}(x-c) - \omega_{ie}(x+d) + \omega_{ie}(x-d) \tag{37}$$

$$V_i'(x) = \omega_{ei}(x+c) - \omega_{ei}(x-c) - \omega_{ii}(x+d) + \omega_{ii}(x-d) \tag{38}$$

Hence the expressions for $V_e'(c)$ and $V_i'(d)$ are given as

$$V_e'(c_{eq}) = \omega_{ee}(2c) - \omega_{ee}(0) - \omega_{ie}(c+d) + \omega_{ie}(c-d) \tag{39}$$

$$V_i'(d_{eq}) = \omega_{ei}(c+d) - \omega_{ei}(d-c) - \omega_{ii}(2d) + \omega_{ii}(0) \tag{40}$$

The parameters β_A,η_A,γ_A and β_L,η_L,γ_L are defined. For simplicity we introduce the new parameters

$$z_1 = \frac{\omega_{ee}(0)}{|V_e'(c_{eq})|},\ z_2 = \frac{\omega_{ee}(2c_{eq})}{|V_e'(c_{eq})|},\ z_3 = \frac{\omega_{ie}(c_{eq}-d_{eq})}{|V_e'(c_{eq})|} \tag{41a}$$

$$z_4 = \frac{\omega_{ie}(c_{eq}+d_{eq})}{|V_e'(c_{eq})|},\ y_1 = \frac{\omega_{ei}(c_{eq}-d_{eq})}{|V_i'(d_{eq})|},\ y_2 = \frac{\omega_{ei}(c_{eq}+d_{eq})}{|V_i'(d_{eq})|} \tag{41b}$$

$$y_3 = \frac{\omega_{ii}(0)}{|V_i'(d_{eq})|},\ y_4 = \frac{\omega_{ii}(2d_{eq})}{|V_i'(d_{eq})|} \tag{41c}$$

1.) $\beta_L = \beta_A$

The expressions for β_L and β_A are given as

$$\beta_L \quad = y_3 + y_4 + 1 = \frac{\omega_{ii}(0)+\omega_{ii}(2d_{eq})+|V_i'(d_{eq})|}{|V_i'(d_{eq})|} \tag{42}$$

$$\beta_A \quad = \frac{1}{|V_i'(d_{eq})|}\{2\ \omega_{ii}(2d_{eq}) - \omega_{ei}(c_{eq}+d_{eq}) + \omega_{ei}(d_{eq}-c_{eq}) \tag{43}$$

Since by assumption the slope of the pulse at the equilibrium point is negative, using (40) we find that

$$|V_i'(d_{eq})| = \omega_{ei}(d-c) - \omega_{ei}(c+d) + \omega_{ii}(2d) - \omega_{ii}(0) \tag{44}$$

Hence the expression (42) can be simplified to

$$\beta_L = \frac{2\omega_{ii}(2d_{eq}) + \omega_{ei}(d-c) - \omega_{ei}(c+d)}{|V_i'(d_{eq})|} \tag{45}$$

$$= \frac{2\omega_{ii}(2d_{eq}) + \omega_{ei}(d-c) - \omega_{ei}(c+d)}{|V_i'(d_{eq})|} \tag{46}$$

comparing (43) and (46)

$$\beta_L = \beta_A \tag{47}$$

2.) $\eta_L = \eta_A$

The expressions for η_L *and* η_A are

$$\eta_L = z_1 + z_2 - 1 = \frac{\omega_{ee}(0) + \omega_{ee}(2c) - |V_e'(c)|}{|V_e'(c)|} \tag{48}$$

$$\eta_A = \frac{1}{|V_e'(c)|}\{2\ \omega_{ee}(2c) - \omega_{ie}(c+d) + \omega_{ie}(c-d)\} \tag{49}$$

Since by assumption, the slope of the pulse at the equilibrium point is negative, we get

$$|V_e'(c)| = -\omega_{ee}(2c) + \omega_{ee}(0) + \omega_{ie}(c+d) - \omega_{ie}(c-d) \tag{50}$$

Hence expression (48) can be rewritten as

$$\eta_L = \frac{2\omega_{ee}(2c) + \omega_{ie}(c-d) - \omega_{ie}(c+d)}{|V_i'(c)|} \tag{51}$$

then by comparing (49) and (51) we get

$$\eta_L = \eta_A \tag{52}$$

3.) $\gamma_A = \gamma_L$

The expressions for γ_L and γ_A are given as

$$\gamma_L = (y_1 + y_2).(z_3 + z_4) - (y_3 + y_4 + 1)(z_1 + z_2 - 1) \tag{53}$$

$$\gamma_A = \left(\frac{\omega_{ei}(c+d) + \omega_{ei}(d-c)}{|V_i'(d_{eq})|}\right)\left(\frac{\omega_{ie}(c+d) + \omega_{ie}(c-d)}{|V_e'(c_{eq})|}\right) - \beta_A \eta_A \tag{54}$$

Then by simplifying γ_L using (39), (40) and (41), we obtain

$$\begin{aligned}\gamma_L = 2z_2y_2 - 42z_2y_4 - 2z_2y_1 + 2z_4y_4 + 2z_4y_1 - 2z_3y_4 \\ +2A_3C_2\end{aligned} \tag{18}$$

then by using the assumptions (41), β_A and η_A can be written as

$$\beta_A = 2y_4 - y_2 + y_1 \tag{19}$$

$$\eta_A = 2z_2 - z_4 + z_3 \tag{20}$$

Hence

$$\begin{aligned}\gamma_A = 2z_2y_2 - 42z_2y_4 - 2z_2y_1 + 2z_4y_4 + 2z_4y_1 - 2z_3y_4 \\ +2z_3y_2\end{aligned} \tag{21}$$

Comparing (18) and (21) we get

$$\gamma_A \quad = \gamma_L \tag{22}$$

4.) $\gamma_L' = 0$

The expression for γ_L' is given as

$$\gamma_L' = (y_1 - y_2).(z_3 - z_4) - (y_3 - y_4 + 1)(z_1 - z_2 - 1) \tag{23}$$

Using (41)

$$\gamma_L' = 1 - L_1 + L_2 + L_3 + L_4 \tag{24}$$

where

$$L_1 = (\frac{\omega_{ee}(2c) - \omega_{ee}(0)}{|V_e'(c)|}), \;\; L_2 = (\frac{\omega_{ii}(0) - \omega_{ii}(2b)}{|V_i'(d)|} \tag{25}$$

$$L_3 = (\frac{\omega_{ee}(2c) - \omega_{ee}(0)}{|V_e'(c)|})(\frac{\omega_{ii}(0) - \omega_{ii}(2c)}{|V_i'(d)|}) \tag{26}$$

$$L_4 = (\frac{\omega_{ie}(c-d) - \omega_{ie}(c+d)}{|V_e'(c)|})(\frac{\omega_{ei}(c-d) - \omega_{ei}(a+b)}{|V_i'(d)|}) \tag{27}$$

Hence (23) can be written as

$$\gamma_L' = 1 + \frac{d_2' d_3 + d_5' d_6'}{|V_e'(c)||V_i'(d)|} + \frac{d_2'}{|V_e'(c)|} + \frac{d_3'}{|V_i'(d)|} \tag{28}$$

Here

$$d_2' = \omega_{ee}(2c) - \omega_{ee}(0)\,, \quad d_3' = \omega_{ii}(0) - \omega_{ii}(2d) \tag{29}$$

$$d_5' = \omega_{ie}(c-d) - \omega_{ie}(c+d)\,, \quad d_6' = \omega_{ei}(a-b) - \omega_{ei}(c+d) \tag{30}$$

Now by using (44) and (50)

$$|V_e'(a)||V_i'(b)| = d_2' d_3' - d_5' d_6' + d_3' d_5' - d_2' d_6' \tag{31}$$

(32)

Then by tedious algebraic calculations

$\gamma_L' = 0$

In above it is proved that $\gamma_L' = 0$, when the external inputs are constant. We also Notice that $\beta_L = \beta_L'$ and $\gamma_L = \gamma_L'$, which shows that both stabilities analysis gives same results in this case.

We get the following two classification schemes for the stability

The table 2 and table 3 shows the relationship between the parameters involved and the properties of the eigenvalues $\Lambda_\pm$, $\Lambda_\pm'$ of (30) and (31) These tables serve as a basis for the stability discussion. If at least one of the parameters γ_L and γ_L' is negative we will get saddle point instability. We get the following results

$\gamma_L < 0$		$\Lambda_- < 0 < \Lambda_+$
$\gamma_L > 0$	$\eta_L < 0$	$Re(\Lambda_\pm) < 0$ for all τ
	$\eta_L > 0$	$Re(\Lambda_\pm) < 0$ for $\tau < \tau_{cr}$
		$Re(\Lambda_\pm) > 0$ for $\tau > \tau_{cr}$

Table 2: classification scheme-1 for linearized stability analysis ($\tau_{cr} = \frac{\beta_L}{|\eta_L|}$)

$\gamma_L' < 0$		$\lambda_-' < 0 < \Lambda_+'$
$\gamma_L' > 0$	$\eta_L' < 0$	$Re(\Lambda_\pm') < 0$ for all τ
	$\eta_L' > 0$	$Re(\Lambda_\pm') < 0$ for $\tau < \tau_{cr}'$
		$Re(\Lambda_\pm') > 0$ for $\tau > \tau_{cr}'$

Table 3: classification scheme-2 for linearized stability analysis ($\tau_{cr'} = \frac{\beta_L'}{|\eta_L'|}$)

1.) If $\eta_L, \eta_L' < 0$,we have stability for all τ

2.) If $\eta_L > 0, \eta_L' < 0$, we have stability $(instability)$ for $\tau < \tau_{cr}(\tau > \tau_{cr})$

3.) If $\eta_L' > 0, \eta_L < 0$, we have stability $(instability)$ for $\tau < \tau_{cr}'(\tau > \tau_{cr}')$

4.) If $\eta_L, \eta_L' > 0$, we have stability $(instability)$ for $\tau < \tau_{cr}''(\tau > \tau_{cr}'')$

3.3 Amari Stability versus Linearized stability Analysis

In above, it is shown that,

$$\beta_L = \beta_A \tag{31a}$$

$$\eta_L = \eta_A \tag{31b}$$

$$\gamma_A = \gamma_L \tag{31c}$$

$$\gamma_L' = 0 \tag{31d}$$

In Blomquist et al [1], the Amari approach and linearized stability approach yield the same stability results with one notable exception. But in Yousaf et al [2] Amari approach and linearized stability approach does nt give the same stability results. In our case (model with constant external inputs) the identities (31) show that the Amari approach and linearized approach gives the same stability predictions, when we have constant external inputs.

Chapter 4
Effect of Spatial Dependent External Input

4 Effect of Spatial Dependent External Input

In this chapter we will consider the same two population model (32). Previously Blomquist et al discussed the number of bumps as a function of amplitude parameter with no external input and found only two bumps pair by fixed the width parameter[1]. Later on Yousaf et al also investigated the number of bump pairs as a function of amplitude parameter with spatial external input and they found at most four bump pairs by fixed the width parameter[2].
In our work we will investigate the number of bump pairs as a function of width parameter with spatial dependent external input. The model read as

$$\frac{\partial v_e}{\partial t} = -v_e + \omega_{ee} \otimes Q_e(v_e - \theta_e) - \omega_{ie} \otimes Q_i(v_i - \theta_i) + K_e(x) \tag{32a}$$

$$\tau \frac{\partial v_i}{\partial t} = -v_i + \omega_{ei} \otimes Q_e(v_e - \theta_e) - \omega_{ii} \otimes Q_i(v_i - \theta_i) + K_i(x) \tag{32b}$$

Here $K_e(x)$, $K_i(x)$ are the spatial dependent external inputs to the excitatory and inhibitory populations respectively, while the remaining variables are as defined previously. The modeled 32 by means of Gaussian functions

$$K_m(x) = A_m \ \exp[-(\frac{x}{\rho_m})^2] \tag{33}$$

Here $A_m, \rho_m (m = e, i)$ denotes the amplitude and width of the external inputs respectively.

In this chapter we will investigate the number of bumps as a function of width parameter of external input and fixed the amplitude parameter and found that there is at most six number of bump pairs exists. For justification we reduce the width parameter and find number of bump pair as a function of amplitude parameter and found six number of bumps pair.

4.1 What are Bumps:

We will find the number of bumps by intersection of two level curves as shown in Figure 6

1. Stationary symmetric solution of two population neural field model are called bumps.
2. Bumps serves as working memeory.

Numerically, bump pair generate by intersection of two level curves.

$$q_e(c,d) = \theta_e \qquad q_i(c,d) = \theta_i \tag{34}$$

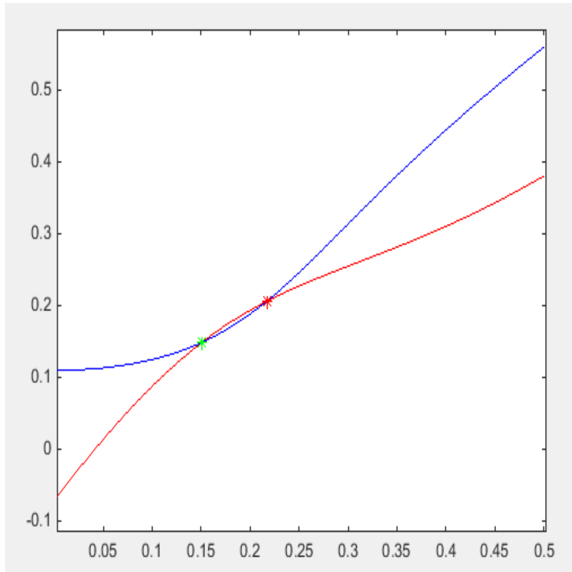

Figure 6: (a)intersection of level curves corresponding to solution of $q_e = \theta_e$, $q_i = \theta_i$ (b)first point belong to narrow BP and second point belong to broad BP (c)corresponding Bumps are shown in figure 7

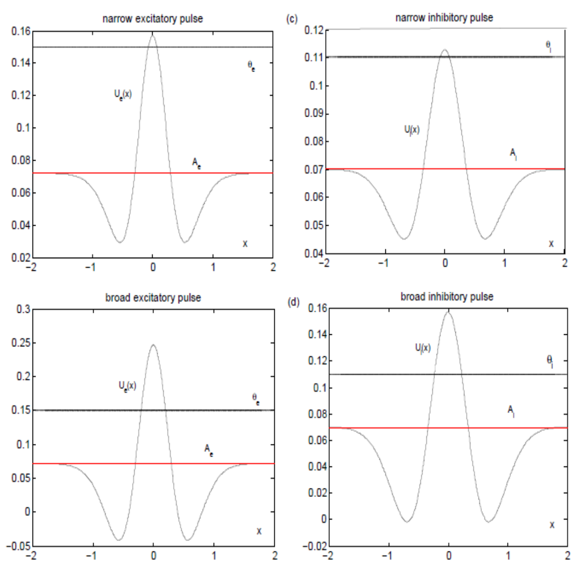

Figure 7: (a)corresponding threshold values θ_e,θ_i for excitatory and inhibitory BP (b)first excitatory column and second inhibitory column for narrow and broad BP.(c) U_e,U_i are the stationary symmetric solutions

4.2 Numerical Results

Earlier, Yousaf et al[2] investigated four number of bump pair as a function of amplitude parameter for each set of threshold values (the so called generic case). Proceeding same arguments we found six number of bump pair as a function of width parameter of external input and amplitude parameter of external input.

4.2.1 Numerical example on six bumps pair when amplitude parameters are fixed:

Let us consider the case with constant external inputs, we fixed the amplitude parameters and find the number of bump pair as a function width parameter of external input. We fix the synaptic footprints are given to be

$$\sigma_{ee} = 0.35 \,, \sigma_{ie} = 0.60 \,, \sigma_{ei} = 0.48 \,, \sigma_{ii} = 0.69 \tag{35}$$

and the threshold values as

$$\theta_e = 0.12 \quad , \quad \theta_i = 0.08 \tag{36}$$

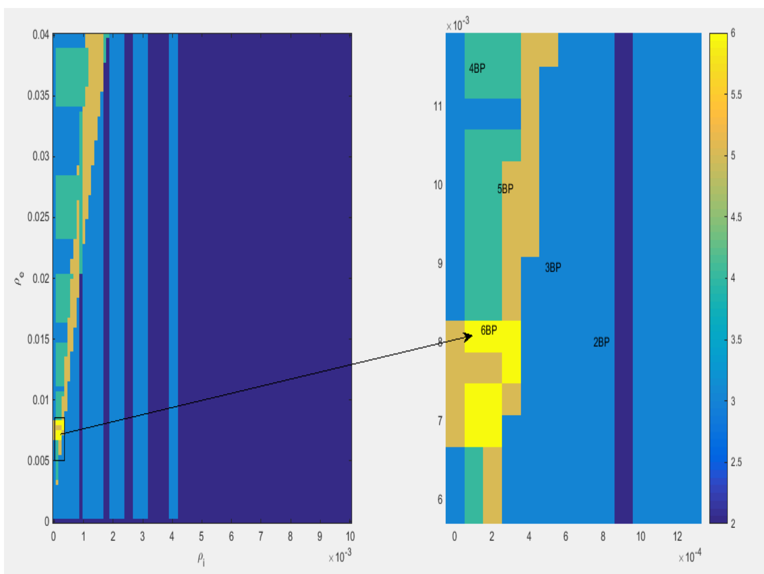

Figure 8: (a) Stationary symmetric solution Dependence on the width *rhoi*, *rhoe* of the constant external input.(b) six BP scenario.(c) Magnified view

.

The amplitude parameters of the external input are fixed

$$A_e = 0.19 \ , \ A_i = 0.7 \tag{37}$$

The corresponding condition (34) produces six bump pair of solutions

$$(a_2, b_2) = (0.0009091, 0.007677) \ , \ (a_3, b_3) = (0.0006061, 0.008485) \tag{38}$$

$$(a_4, b_4) = (0.000202, 0.01172) \ , \ (a_5, b_5) = (0.000404, 0.009697) \tag{39}$$

$$(a_6, b_6) = (0.000303, 0.007677) \tag{40}$$

Which corresponds to the crossings of the level curves $q_e = \theta_e$ and $q_i = \theta_i$ for a constant external input and is shown in Figure.8. If we reduce the amplitude parameter probably we can get the zero bump pair.

4.2.2 Numerical example on six bumps pair when width parameters are fixed:

In this section we will give a numerical example with six bump pair solution when the width paramerters are fixed and find the number of bump pair as a function of amplitude parameter of external input. We fix the synaptic footprints are given to be

$$\sigma_{ee} = 0.35 \, , \sigma_{ie} = 0.60 \, , \sigma_{ei} = 0.48 \, , \sigma_{ii} = 0.69 \tag{41}$$

and the threshold values as

$$\theta_e = 0.12 \quad , \quad \theta_i = 0.08 \tag{42}$$

The amplitude and width parameters of the external input are given as

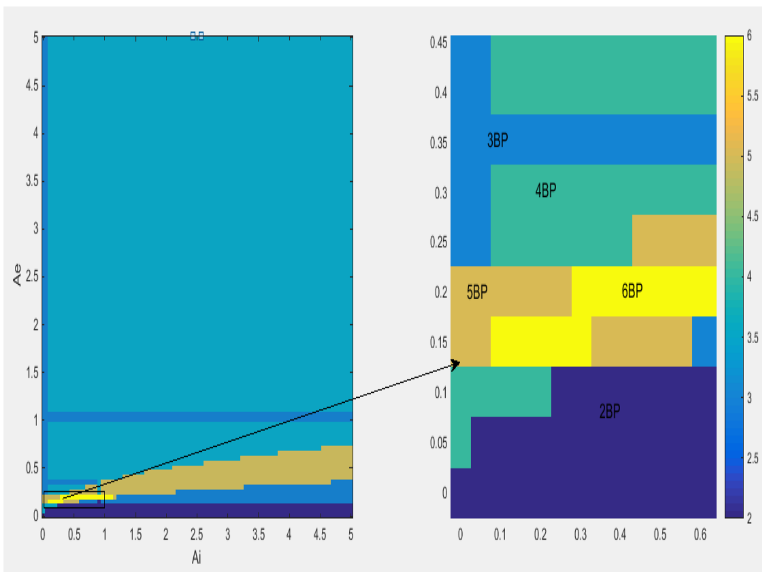

Figure 9: (a) six BP scenario.(b) Magnified view

.

$$A_e = 0.2525 \ , \ A_i = 0.7576 \ , \ \rho_e = 2.068965517241379e-03 \ , \ \rho_i = 1.379310344827586e-03 \tag{43}$$

The corresponding condition (34) produces six bump pair of solutions when the width parameter are fixed

$$(a_2, b_2) = (0.303, 0.101 \ , \ (a_3, b_3) = (0.05051, 0.3535) \tag{44}$$

$$(a_4, b_4) = (0.0.1515, 0.303 \ , \ (a_5, b_5) = (0.1515, 0.202) \tag{45}$$

$$(a_6, b_6) = (0.5051, 0.202) \tag{46}$$

which corresponds to the crossings of the level curves $q_e = \theta_e$ and $q_i = \theta_i$ for a constant external input and is shown in Figure.9

Chapter 5
Conclusion and Discussion

5 Conclusion and Discussion

We have formulated a Wilson-Cowan [8] type of model (32) where we have taken into account constant external inputs. Earlier,Blomquist et al[1] investigaed the two population model with no external input and they found that Amari's stability analysis and linearized stability analysis produces same result. Later on, Yousaf et al[2] investigated same population model with spatial external input and proved that both stability approaches does not produce same result. They also investigated the number of bumps as a function of amplitude parameter of external input and found at most four bump pair. In present work we studied the two population neural field model under the effect of constant external input on bump solutions. In the first part we investigated the stability properties by using Amari's stability analysis and linearized stability analysis and found that both stability approaches gives same results. In the second step we investigated number of bumps as a function of width parameter of external input and also found number of bumps as a function of amplitude parameter of external input and found at most six number of bump pairs.

The conclusions of the present work can be summarized as follows.

1. We investigated the two population neural field model (32), under the effect of constant external input.

2. Mathematically, we investigated the stability properties of bumps by using Amari and linearized stability approaches.Both stability approaches gives same results.

3. Numerically,we Investigated number of bumps as a function of width and amplitude parameter of external input and found at most six number of bump pair in Figure 8, 9.

Future Work

The number of bump pair as a function of amplitude parameter of external input can be achieved, for different values of width parameter. The number of bump pair as a function of width parameter of external input can be achieved, for different values of amplitude parameter. In future more than six bump pairs can be achieved.

Chapter 6
References

References

[1] P. Blomquist, J. Wyller, G. T. Einevoll, Localized activity patterns in two population neuronal networks, Physica D 206(2005)180 − 212.

[2] Muhammad Yousaf, J. Wyller, T. Tetzlaff, G. T. Einevoll, Effect of localized input on bump solutions in a two-population neural-field model, Nonlinear Analysis: Real World Application 14(2013)997 − 1025.

[3] P. Dayan, L. F Abbott. Theoretical neuroscience. MIT press, Cambridge, MA 2001.

[4] Eugene M. Ezhikevitch, Dynamical Systems in Neuroscience. MIT press, Cambridge 2007.

[5] P. Blomquist et al, Gaute. T. Einevoll and John Wyller. Biophysical Reviews and Letters on the origin and Properties of Two population neural field models.

[6] S. Amari, Dynamics of pattern formation in lateral-inhibition type neural fields, Bio. Cybernet. 27(1977)7787.

[7] D.J. Pinto, G.B. Ermentrout, Spatially structured activity in synaptically coupled neuronal networks: II. Lateral inhibition and standing pulses, SIAM J. Appl. Math. 62(1)(2001)226243.

[8] H.R. Wilson, J.D. Cowan, A mathematical theory of the functional dynamics of cortical and thalamic nervous tissue, Kybernetik 13(1973)5580.

[9] $://en.wikipedia.org/wiki/Human_brain$

[10] https://en.wikipedia.org/wiki/Linearity

[11] https://www.google.com/search?$q = human + brain + in + pdf$

[12] http://brainmadesimple.com/parietal-lobe.html

[13] http://www.news-medical.net/health/Human-Brain-Structure.aspx
http://www.healthline.com/human-body-maps/pons

[14] W. Gerstner, W M. Kistler, Spiking Neuron Models, Cambridge University Press, 2002

[15] http://www.msevans.com/neuralmodels/Ptnrn10.html

[16] https://www.google.com/search?q=point+neuron+model

[17] http://link.springer.com/chapter/10.1007/$978 - 0 - 387 - 87708 - 2_1 1$

[18] $http://www.medicinenet.com/script/main/art.asp?articlekey = 2517$

[19] http://biology.about.com/od/anatomy/a/aa032505a.htm

[20] $http://www.appsychology.com/Book/Biological/cerebral_cortex.htm$

[21] http://www.news-medical.net/health/Human-Brain-Structure.aspx

[22] www.studyinukraine.eu

[23] http://www.slideshare.net/neurosciust/the-synapse-and-the-presynaptic-and-postsynaptic-terminals

[24] S.K.Leong,An introduction to the human nervous system.

[25] http://link.springer.com/article/10.1007

[26] Edwards, Harold M. (1995). Linear Algebra. Springer. p. 78. ISBN 9780817637316.

[27] J. Mawhin,Bounded solutions of some second order difference equations,Georgian Math. J.14(2007), 315-324.

[28] J. Mawhin and J.R. Ward Jr.,Bounded solutions of some second order nonlinear differential equations,J. London Math. Soc. (2)58(1998), 733-747.

[29] http://brainmadesimple.com/temporal-lobe.html

[30] $http://www.healthline.com/human-body-maps/frontal-lobe/male$

[31] http://www.healthline.com/human-body-maps/occipital lobe

[32] $http://brainmadesimple.com/pons.html$

[33] http://www.healthline.com/human-body-maps/pons

[34] http://www.healthline.com/human-body-maps/cerebellum

[35] E. Bruce GoldStein(2009). In: Sensation and Perception, Edition eighth,

[36] ://www.mayfieldclinic.com/PDF/PE-AnatBrain.pdf

[37] Kandel, Eric R.; Schwartz, James H.; Jessell, Thomas M. (2000). Principles of Neural Science Fourth Edition. United State of America: McGraw-Hill. p. 324. ISBN 0-8385-7701-6

[38] ://www.icuf.org/newdevelopment/wp-content/uploads/2012/07/13-L.14.26-The-Brain-Power-Point.pdf

[39] readingroom.mindspec.org/?

[40] J. Wyller, P. Blomquist, G. T. Einevoll, On the origin and properties of two-population neural field models, a tutorial introduction, Biophysical Reviews and Letters 2 (2007) 79-98.

[41] J. M. Bower, D. Beeman, The Book of Genesis: Exploring Realistic Neural Models with the General Neural Simulation System, 2nd edn. Springer, New York, NY, 1998.

[42] C. Koch, I. Segev, Methods in Neuronal Modeling, 2nd edn. MIT Press, Cambridge, MA, 1998. http://www.healthline.com/human-body-maps/occipital lobe

[43] E. M. Ezhikevitch, Dynamical Systems in Neuroscience, MIT press, Cambridge 2007.

[44] $: //en.wikipedia.org/wiki/Jacobian_matrix_and_determinant$

[45] ://www.google.com/search?q=pdf+structure+of+neuron

[46] $http : //www.lschs.org/uploaded/pdfs/parts-of-the-brain-and-brain-function.pdf$

[47] ://www.icuf.org/newdevelopment/wp-content/uploads/2012/07/13-L.14.26-The-Brain-Power-Point.pdf

[48] Saddler, David; Shea, Julia; Ward, Derek (2011), ”12 B Stationary Points and Turning Points”, Cambridge 2 Unit Mathematics Year 11, Cambridge University Press, p. 318, ISBN 9781107679573

[49] $https : //en.wikipedia.org/wiki/Nervous_system$

[50] S.K.Leong,An introduction to the human nervous system.

[51] $: //www.britannica.com/science/human-nervous-system$