Intrinsic Nature of Gaussian Curvature

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Declaration

I Faraz William , CIIT/FA21-RMT-102/LHR, hereby state that my MS thesis titled "Intrinsic Nature of Gaussian Curvature" is my own work and has not been submitted previously by me for taking any degree from this University "COMSATS University Islamabad, Lahore Campus" or anywhere else in the country/world. At any time if my statement is found to be incorrect even after my Graduate the university has the right to withdraw my MS degree.

Date:______________ ____________________________

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Certificate

It is certified that Faraz William, CIIT/FA21-RMT-102/LHR has carried out all the work related to this thesis under my supervision at the Department of Mathematics, COMSATS University Islamabad, Lahore Campus and the work fulfills the requirement for award of MS degree.

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DEDICATION

To My Parents and All Family

ACKNOWLEDGEMENTS

Praise to be ALLAH, the Cherisher and Lord of the World, Most gracious and Most Merciful

First of all, I am thankful to God for giving me the opportunity to pursue higher studies. It is His mercy that I got this opportunity and I am finally finishing my degree. I am thankful to God for such a supporting family who has always motivated and appreciated me.

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ABSTRACT

Intrinsic Nature of Gaussian Curvature

We will define the notion of a regular surface. We will meaningfully define different types of curvature of a surface namely, normal curvature, principal curvature, mean curvature and the Gaussian Curvature. By using the tools of Differential Geometry we will prove that the Gaussian Curvature is the intrinsic curvature of a surface meaning that, it is independent of the embedding of the surface in the ambient space.

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Chapter 1

Introduction

The study of curvature is fundamental in differential geometry. Curvature of surface can be defined in many different ways. In this thesis we will define 4 types of curvatures namely normal curvature, principal curvature, mean curvature and Gaussian Curvature. Each of these curvatures serves a distinct purpose but only one of them intrinsic to the surface i.e Gaussian Curvature.

In order to demonstrate the intrinsic nature of Gaussian Curvature we begin with defining some fundamental concepts in differential geometry. We will give basic definitions like local parametrization, atlas and tangent space etc. The standard inner product on $\mathbb{R}^3$ induces an inner product on the tangent space T_pS of each point p belonging to the surface, enabling us to define the notion of first fundamental form of tangent space T_pS as $I_p(v) = < v, v >$, where v belongs to T_pS. From the first fundamental form we get the coefficients $E(p), F(p)$ and $G(p)$ called as the metric coefficients. We will prove a remarkable result in Chapter 2 namely lemma 2.4.3 which will prove that, the metric coefficients are preserved by local isometry under suitable parametrizations.
In section 2.5 we will define the notion of an orientable surface. We will observe the that the Gauss map $N(p) : S \to S^2$ can be defined for all the orientable surfaces. The differential of the Gauss map $dN_p : T_pS \to T_{N(p)}S^2$ will play a crucial role in defining many of the important concepts. For instance, the Gaussian curvature, mean curvature, principal curvature are all defined with the help of the dN_p map. Moreover, the properties like linearity and symmetricity make it even more useful.
The second fundamental form is also defined with the aid of the dN_p map. It is defined as a function Q_p from tangent space T_pS to $\mathbb{R}$. In lemma 2.6.3 it will be shown that the second fundamental form evaluated at tangent vector $v \in T_pS$ yields the normal curvature. While the coefficients of the second fundamental form are not intrinsic to the surface but they do play a crucial role in proving that the Gaussian Curvature is intrinsic to the surface. The matrix form of differential of the Gauss map will help us find explicit expression for principal, mean and Gaussian Curvature.
In the section 4 we will introduce the Christoffel Symbols. We will observe that the Christoffel Symbols depend only on the coefficients of the first fundamental form. We will eventually express the Gaussian Curvature in terms of the Christoffel Symbols in sec-

tion 4.1. The Christoffel Symbols only depend on the coefficients of the first fundamental form and from lemma 2.4.3 we know that the coefficients of the first fundamental form are intrinsic to the surface. Therefore the Christoffel Symbols must be intrinsic.

The intrinsic nature of the Gaussian Curvature is extremely surprising because the definition of the Gaussian Curvature explicitly involves the dN_p map. Therefore, proving that the Gaussian Curvature is independent of the embedding of the surface in $\mathbb{R}^3$ is an utterly non-trivial result. The importance of the intrinsic nature of the Gaussian Curvature is evident from the fact that we can prove, for example, the Earth is not flat without having to leave our planet.

Chapter 2

Some preliminaries

2.1 Some Basic Defintions in Differential Geometry

Definition 1. *A parametrized surface S is given by a map of class C^∞ from an open set $V \subseteq \mathbb{R}^2$ to $\mathbb{R}^3$ i.e $\psi : V \to \mathbb{R}^3$ such that the differential map of ψ at every point is injective i.e $d\psi_p$ is injective at every point $p \in S$ and therefore has rank 2.*

The differential of the map $\psi(x_1,x_2) = (\psi_1, \psi_2, \psi_3)$ has the following form:

$$d\psi_p = \begin{bmatrix} \frac{\partial \psi_1}{\partial x_1} & \frac{\partial \psi_1}{\partial x_2} \\ \frac{\partial \psi_2}{\partial x_1} & \frac{\partial \psi_2}{\partial x_2} \\ \frac{\partial \psi_3}{\partial x_1} & \frac{\psi_3}{\partial x_2} \end{bmatrix}.$$

Definition 2. *Consider $S \subseteq \mathbb{R}^3$ and suppose S is connected. Then S is called a regular surface if for every p in S there exists an open set $V \subseteq \mathbb{R}^2$ and a parametrization $\psi : V \to S$ of class C^∞ which meets the following criteria: ‘*

1. *$\psi(V) \subseteq S$ is an open neighbourhood of p in the surface.*
2. *ψ is homeomorphic to its image.*
3. *The differential map of ψ i.e $d\psi_p$ at each point has rank 2 i.e $d\psi_p$ is injective at each point.*

Recall that the tangent vectors are fundamental in the study of the curves in $\mathbb{R}^3$. For example the unit normal and unit binormal vectors are defined with the help of the tangent vectors. Therefore, it is natural to extend the concept of the tangent vectors to surfaces as well.

Definition 3. *Consider a regular surface S and suppose $p \in S$. Let $\gamma : (-\delta, \delta) \to S$ be a curve whose image lives inside S and $\gamma(0) = p$. A vector of the form $\gamma'(0)$ is called as the tangent vector of S at p. The space of all such vectors is called as tangent space at point p, denoted by T_pS.*

Definition 4. *The set $\{\psi_\alpha : V_\alpha \to S \quad | \quad V_\alpha \subseteq \mathbb{R}^2\}$ of the local parametrizations of a regular surface $S \subseteq \mathbb{R}^3$ such that $S = \bigcup_\alpha \psi_\alpha(V_\alpha)$ is called an atlas.*

2.2 First Fundamental Form

We introduce the first fundamental form as a function from T_pS to $\mathbb{R}$ i.e $I_p : T_pS \to \mathbb{R}$ defined as:

$$I_p(v) =< v, v >,$$

where $< -,- >$ is the standard inner product in $\mathbb{R}^3$ and $v \in T_pS$.

Moreover, the scalar product between any two vectors in T_pS can be calculated using the first fundamental form as:

$$< u, v >= \frac{1}{4}\left[I_p(u+v) - I_p(u-v)\right].$$

Consider a parametrized surface S and suppose $\psi : V \to S$ be the local parametrization of the parametrized surface. Let $\{\partial_1, \partial_2\} \subseteq T_pS$ be the basis of T_pS induced by ψ. Then we have: $u = u_1\partial_1 + u_2\partial_2$ and $v = v_1\partial_1 + v_2\partial_2$. So, we have:

$$\begin{aligned} < u, v > &= < u_1\partial_1 + u_2\partial_2, v_1\partial_1 + v_2\partial_2 > \\ &= (u_1v_1) < \partial_1, \partial_1 > + (u_1v_2 + u_2v_1) < \partial_1, \partial_2 > + (u_2v_2) < \partial_2, \partial_2 > . \end{aligned}$$

So, we define the coefficients of the first fundamental form (also known as metric coefficients) as the following:

$\forall p \in S \qquad E(p) =< \partial_1, \partial_1 > \qquad F(p) =< \partial_1, \partial_2 > \qquad G(p) =< \partial_2, \partial_2 > .$

So for any $v \in T_pS$ we have:

$$\begin{aligned} I_p(v) &= v_1^2E(p) + 2v_1v_2F(p) + v_2^2G(p) \\ &= \begin{bmatrix} v_1 & v_2 \end{bmatrix} \begin{bmatrix} E(p) & F(p) \\ F(p) & G(p) \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}. \end{aligned} \tag{2.2.1}$$

Morevoer, the inner product between any two vectors $u, v \in T_pS$ can be expressed as:

$$< u, v >= u_1v_1E(p) + (u_1v_2 + u_2v_1)F(p) + u_2v_2G(p). \tag{2.2.2}$$

In matrix form:

$$< u, v > = \begin{bmatrix} u_1 & u_2 \end{bmatrix} \begin{bmatrix} E & F \\ F & G \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}. \tag{2.2.3}$$

Remark: The value of the metric coefficients is subject to the chosen parametrization. The following examples illustrate the above remark.

Example 1.

Consider the local parametrization of the sphere as $\psi(x_1, x_2) = \left(x_1, x_2, \sqrt{1 - x_1^2 - x_2^2}\right)$ Then we have:

$$\partial_1 = \frac{\partial \psi}{\partial x_1} = \left(1, 0, \frac{-x_1}{\sqrt{1 - x_1^2 - x_2^2}}\right),$$

$$\partial_2 = \frac{\partial \psi}{\partial x_2} = \left(0, 1, \frac{-x_2}{\sqrt{1 - x_1^2 - x_2^2}}\right).$$

The metric coefficients can be calculated as:

$$E = < \partial_1, \partial_1 > = 1 + 0 + \frac{x_1^2}{1 - x_1^2 - x_2^2} = \frac{1 - x_2^2}{1 - x_1^2 - x_2^2}$$

$$F = < \partial_1, \partial_2 > = \frac{x_1 x_2}{1 - x_1^2 - x_2^2}$$

$$G = < \partial_2, \partial_2 > = \frac{1 - x_1^2}{1 - x_1^2 - x_2^2}.$$

Let us change the local parametrization and observe its effect on metric coefficients. Consider: $\tilde{\psi}(\theta, \alpha) = (\sin\theta\cos\alpha, \sin\theta\sin\alpha, \cos\theta)$.

$$\partial_1' = \frac{\partial \tilde{\psi}}{\partial \theta} = (\cos\theta\cos\alpha, \cos\theta\sin\alpha, -\sin\theta)$$

$$\partial_2' = \frac{\partial \tilde{\psi}}{\partial \alpha} = (-\sin\theta\sin\alpha, \sin\theta\cos\alpha, 0).$$

So, the metric coefficients can be calculated as follows:

$$\tilde{E} = < \partial_1', \partial_1' > = \cos^2\theta(\cos^2\alpha + \sin^2\alpha) + \sin^2\theta = 1,$$

$$\tilde{F} = <\partial_1', \partial_2' >= -\sin\theta\cos\theta\cos\alpha\sin\alpha + \sin\theta\cos\theta\cos\alpha\sin\alpha = 0,$$

$$\tilde{G} = <\partial_2', \partial_2' >= \sin^2\theta(\sin^2\alpha + \cos^2\alpha) = \sin^2\theta.$$

Hence, it is evident from the above example that changing the local parametrization alters the metric coefficients.

Let us see some examples in which we explicitly calculate the metric coefficients.

Example 2.

Consider a regular surface S with the following parametrization $\psi : V \to \mathbb{R}^3$ where $V \subseteq \mathbb{R}^2$ and g is a smooth function.

$$\psi(x_1, x_2) = (x_1, x_2, g(x_1, x_2)). \tag{2.2.4}$$

Differentiate (2.2.4) w.r.t x_1 to get:

$$\partial_1 = \frac{\partial\psi}{\partial x_1} = \left(1, 0, \frac{\partial g}{\partial x_1}\right).$$

Similarly, differentiate (2.2.4) w.r.t x_2 to get:

$$\partial_2 = \left(0, 1, \frac{\partial g}{\partial x_2}\right).$$

So, we have:

$$\begin{aligned} E &= <\partial_1, \partial_1 >= \left\langle \left(1, 0, \frac{\partial g}{\partial x_1}\right), \left(1, 0, \frac{\partial g}{\partial x_1}\right) \right\rangle \\ &= 1 + \left(\frac{\partial g}{\partial x_1}\right)^2. \end{aligned}$$

Now,

$$\begin{aligned} F &= <\partial_1, \partial_2 >= \left\langle \left(1, 0, \frac{\partial g}{\partial x_1}\right), \left(0, 1, \frac{\partial g}{\partial x_2}\right) \right\rangle \\ &= \frac{\partial g}{\partial x_1} \cdot \frac{\partial g}{\partial x_2}. \end{aligned}$$

Similarly,

$$\begin{aligned} G &= <\partial_2, \partial_2>= \left\langle \left(0, 1, \frac{\partial g}{\partial x_2}\right), \left(0, 1, \frac{\partial g}{\partial x_2}\right) \right\rangle \\ &= 1 + \left(\frac{\partial g}{\partial x_2}\right)^2. \end{aligned}$$

Example 3.

Consider the plane:

$$\psi(x_1, x_2) = p + x_1 w_1 + x_2 w_2. \tag{2.2.5}$$

Let us calculate the metric coefficients:

Differentiate (2.2.5) w.r.t x_1 to get:

$$\partial_1 = \frac{\partial \psi}{\partial x_1} = w_1.$$

Similarly, differentiate (2.2.5) w.r.t x_2 to get:

$$\partial_2 = w_2.$$

Let us calculate the metric coefficients:

$$\begin{aligned} E &= <\partial_1, \partial_1>=<w_1, w_1> \\ E &= ||w_1||^2. \end{aligned}$$

Similarly,

$$F =<\partial_1, \partial_2>=<w_1, w_2> \quad ; \quad G =<\partial_2, \partial_2>= ||w_2||^2.$$

Example 4.

Suppose we have a right circular cylinder of radius 1 with the following parametrization:

$$\psi(x_1, x_2) = (\cos x_1, \sin x_1, x_2). \tag{2.2.6}$$

Let us calculate the metric coefficients:

Differentiate (2.2.6) to get:

$$\partial_1 = \frac{\partial \psi}{\partial x_1} \quad = \quad (-\sin x_1, \cos x_1, 0).$$

Similarly, differentiate (2.2.6) w.r.t x_2 to get:

$$\partial_2 = (0, 0, 1).$$

So, the metric coefficients can be calculated as:

$$\begin{aligned} E &= <\partial_1, \partial_1> = <(-\sin x_1, \cos x_1, 0), (-\sin x_1, \cos(x_1), 0)> \equiv 1, \\ F &= <\partial_1, \partial_2> = <(-\sin x_1, \cos x_1, 0), (0, 0, 1)> \equiv 0 \\ G &= <\partial_2, \partial_2> \equiv 1. \end{aligned}$$

2.3 Smooth Maps

In this section, we are going to define some basic notions about smooth maps on surfaces and smooth maps between surfaces. We are also going to state some theorems related to smooth functions.

Definition 5. *Consider a regular surface S and let $f : S \to \mathbb{R}$ be a function. Then f is called smooth (of class C^∞) on S if for every $p \in S$ there exists a local parametrization $\psi_1 : V_1 \to S$ and a neighbourhood $U_1 \subseteq V_1$ of $\psi_1^{-1}(p)$ such that $f \circ \psi_1 : V_1 \to \mathbb{R}$ is a smooth function in the neighbourhood U_1 of $\psi_1^{-1}(p)$.*

A priori it may happen that the function f is C^∞ at point p with respect to the local parametrization $\psi_1 : V_1 \to S$ but it is not smooth with respect to another local parametrization $\psi_2 : V_2 \to S$. The theorem given below will show that this cannot happen.

Theorem 2.3.1. *Consider a regular surface S and suppose $\psi_1 : V_1 \to S$ and $\psi_2 : V_2 \to S$ be* *two local parametrizations of S and $\eta = \psi_1(V_1) \bigcap \psi_2(V_2) \neq \emptyset$ then then function:*

$$h = \psi_2^{-1} \circ \psi_1|_{\psi_1^{-1}(\eta)} : \psi_1^{-1}(\eta) \to \psi_2^{-1}(\eta).$$

is a diffeomorphism.

Definition 6. *Suppose S_1 and S_2 be two regular surfaces. Let $H : S_1 \to S_2$ be a function. Then H is called a smooth function or a function of class C^∞ if for every point $p \in S_1$ there exist local parametrizations $\psi_1 : V_1 \to S_1$ in p and $\psi_2 : V_2 \to S_2$ in $H(p)$ such that the function $\psi_2^{-1} \circ H \circ \psi_1 : V_1 \to V_2$ is a smooth function wherever defined.*

Let us define differential map H between two surfaces, which is going to play an important role in later sections. Suppose S_1 and S_2 are two regular surfaces and $H : S_1 \to S_2$ be a smooth function between them. Suppose $p \in S_1$. Consider the parametrized curve $\gamma(t) : I \subseteq \mathbb{R} \to S_1$ whose support lies inside the surface S_1. It is centered at p and it is such that $\gamma'(0) = v \in T_pS_1$. Consider the map $H : S_1 \to S_2$ then the differential of H at p i.e $dH_p : T_pS_1 \to T_{H(p)}S_2$ is defined as:

$$dH_p(v) = (H \circ \gamma)'(0),$$

There is one important lemma about dH_p map which we are going to use in our later sections. We are just going to state the lemma and not prove it.

Lemma 2.3.2. *Let $H : S_1 \to S_2$ be a smooth function S_1 and S_2. $\psi(x_1, x_2) : V \to S_1$ be a local parametrization of S_1. Let $p \in S_1$ and $\{\partial_1 = \frac{\partial \psi}{\partial x_1}, \partial_2 = \frac{\partial \psi}{\partial x_2}\}$ be basis of T_pS_1 then :*

$$dH_p(\partial_i) = \frac{\partial (H \circ \psi)}{\partial x_i}.$$

2.4 Length of Curves and First Fundamental Form

Suppose we have a curve whose support lives completely inside the surface. The length of such a curve can be calculated using the first fundamental form. Consider a curve $\gamma(t)$:

$[\alpha', \beta'] \to S$. The following formula gives the length of the curve:

$$K(\gamma) = \int_{\alpha'}^{\beta'} \sqrt{I_{\gamma(t)}(\gamma'(t))}dt.$$

where $I_{\gamma(t)}(\gamma'(t))$ is the first fundamental form at the point $\gamma(t)$ calculated at the tangent vector $\gamma'(t)$.

Proposition 2.4.1. *Consider a regular surface S. Suppose we know the length of all the curves whose support is contained in S. Then we can recover the first fundamental form for this surface.*

Proof. Suppose we know the length of all the curves whose support live inside the surface. Consider the curve $\tilde{\gamma}(t)$ whose support lives inside the surface S. Consider the following function:

$$k(t) = K(\tilde{\gamma}_{[0,t]}(t)).$$

Differentiate both sides w.r.t t to get:

$$\begin{aligned} \frac{dk}{dt} &= \frac{d}{dt}\left(K(\tilde{\gamma}_{[0.t]}(t)\right) \\ \left|\frac{dk}{dt}\right| &= \left|\frac{d}{dt}\left(\int_0^t \sqrt{I_{\tilde{\gamma}(t)}(\tilde{\gamma}'(t))}\right)\right| \\ &= \sqrt{I_{\tilde{\gamma}(t)}(\tilde{\gamma}'(t))} \\ \left|\frac{dk}{dt}\right|^2 &= I_{\tilde{\gamma}(t)}(\tilde{\gamma}'(t)). \end{aligned}$$

Calculate the value at $t = 0$

$$\left|\frac{dk}{dt}\right|^2 (0) = I_p(v).$$

□

This shows that if we know the length of all the curves contained in a surface we can calculate the first fundamental form at any tangent vector of any curve passing through that point.

So, this means that the first fundamental form is associated with the intrinsic metric properties.

Definition 7. *Let ϕ be a smooth function between S_1 and S_2 i.e $\phi : S_1 \to S_2$ then ϕ is called isometry at any point p on S_1 if for every tangent vector $v \in T_pS_1$ the following condition is satisfied:*

$$I_{\phi(p)}(d\phi_p(v)) = I_p(v). \tag{2.4.1}$$

Proposition 2.4.2. *Let S_1 and S_2 be two regular surfaces and ϕ be an isometry between them. Let $p \in S_1$, then for every $u, v \in T_pS_1$ the following holds:*

$$< u, v >=< d\phi_p(u), d\phi_p(v) > .$$

Proof. From the previous section we have:

$$< u, v >= \frac{1}{4}\left[I_p(u+v) - I_p(u-v)\right] \tag{2.4.2}$$

Similarly,

$$\begin{aligned}
< d\phi_p(u), d\phi_p(v) > &= \frac{1}{4}\left[I_{\phi(p)}(d\phi_p(u) + d\phi_p(v)) - I_{\phi(p)}(d\phi_p(u) - d\phi_p(v))\right] \\
< d\phi_p(u), d\phi_p(v) > &= \frac{1}{4}\left[I_{\phi(p)}(d\phi_p(u+v)) - I_{\phi_p}(d\phi_p(u-v)\right] \quad linearity \quad of \quad d\phi_p \\
&= \frac{1}{4}\left[I_p(u+v) - I_p(u-v)\right] \qquad using \quad (2.4.1) \\
< d\phi_p(u), d\phi_p(v) > &= < u, v > . \quad using (2.4.2)
\end{aligned}$$

□

Definition 8. *ϕ is called a local isometry at p if there exists a neighbourhood V of p such that ϕ is an isometry at each point of V. ϕ is called a local isometry if this condition is satisfied at every point of the surface.*

The following result will prove that for any two locally isometric surfaces there exist suitable local parametrizations such that their metric coefficients form will coincide.

Lemma 2.4.3. *Suppose we have S_1 and S_2 two surfaces in $\mathbb{R}^3$ then S_1 is locally isometric to S_2 iff for every point p_1 in S_1 there exist p_2 in S_2 and an open set $V \subseteq \mathbb{R}^2$, a local parametrization $\psi_1 : V \to S_1$ of S_1 such that $\psi_1(0) = p_1$, and a local parametrization $\psi_2 : V \to S_2$ of S_2 such that $\psi_2(0) = p_2$ such that $E_1 \equiv E_2$, $F_1 \equiv F_2$ and $G_1 \equiv G_2$.*

Proof. Suppose the surfaces are locally isometric. Then for every point in S_1 we can find a neighbourhod V of p and a isometry $\phi : U \to \phi(U) \subseteq S_2$. Consider a local parametrization $\psi_1 : V \to S_1$ of S_1

Consider the following local parametrization ψ_2 of S_2:

$$\psi_2 = \phi \circ \psi_1 : V \to S_2.$$

The claim is that ψ_2 satisfies the required conditions i.e $E_2 \equiv E_1$, $F_2 \equiv F_1$ and $G_2 \equiv G_1$. Differentiating ψ_2 with respect to x_1, we get:

$$\frac{\partial \psi_2}{\partial x_1} = \frac{\partial}{\partial x_1}(\phi \circ \psi_1).$$

Since ϕ is a C^∞ function by definition we have $\frac{\partial}{\partial x_1}(\phi \circ \psi_1) = d\phi_p(\partial_1)$ hence,

$$\partial_1' = \frac{\partial \psi_2}{\partial x_1} = d\phi_p(\partial_1).$$

Similarly by differentiating ψ_2 with respect to x_2 we get

$$\partial_2' = \frac{\partial \psi_2}{\partial x_2} = d\phi_p(\partial_2).$$

Now, let us calculate the metric coefficients:

$$E_2 =< \partial_1', \partial_1' >=< d\phi_p(\partial_1), d\phi_p(\partial_1) > .$$

Since ϕ is an isometry so $< d\phi_p(\partial_1), d\phi_p(\partial_1) =< \partial_1, \partial_1 >$ so,

$$E_2 =< \partial_1, \partial_1 >= E_1$$

$$F_2 =< \partial_1', \partial_2' >=< d\phi_p(\partial_1), d\phi_p(\partial_2) > .$$

Since ϕ is an isometry so,

$$F_2 =< \partial_1, \partial_2 >= F_1$$

$$G_2 =< d\phi_p(\partial_2), d\phi_p(\partial_2) > .$$

Since ϕ is an isometry

$$\implies G_2 \equiv G_1.$$

Conversely, suppose we have two parametrizations ψ_1 and ψ_2 of surfaces S_1 and S_2 which satisfy the given conditions i.e $E_1 \equiv E_2$, $F_1 \equiv F_2$ and $G_1 \equiv G_2$

Consider the following function: $\phi = \psi_2 \circ \psi_1^{-1} : \psi_1(V) \to \psi_2(V)$ of class C^∞. We claim that ϕ is the required isometry between S_1 and S_2. Let us take $p \in \psi_1(V)$ and $v \in T_pS_1$. Then $v = v_1\partial_1 + v_2\partial_2$ where $v_1, v_2 \in \mathbb{R}$. Let us apply the map $d\phi_p : T_pS_1 \to T_{\phi(p)}S_2$ on the vector $v \in T_pS_1$. So, $d\phi_p(v) = v_1 d\phi p(\partial_1) + v_2 d\phi_p(\partial_2) = v_1\partial_1' + v_2\partial_2'$, where $\{\partial_1', \partial_2'\}$ is a basis of $T_{\phi(p)}S_2$. Therefore, we have:

$$\begin{aligned}
I_{\phi(p)}(d\phi_p(v)) &= < d\phi_p(v), d\phi_p(v) > \\
&= < v_1\partial_1' + v_2\partial_2', v_1\partial_1' + v_2\partial_2' > \\
&= v_1^2 < \partial_1', \partial_1' > +2v_1v_2 < \partial_1', \partial_2' > +v_2^2 < \partial_2', \partial_2' > \\
&= v_1^2E_2 + 2v_1v_2F_2 + v_2^2G_2 \\
&= v_1^2E_1 + 2v_1v_2F_1 + v_2^2G_1 \qquad \textit{since} \quad E_1 \equiv E_2, \quad F_1 \equiv F_2 \quad G_1 \equiv G_2 \\
&= < v, v > \\
I_{\phi(p)}(d\phi_p(v)) &= I_p(v).
\end{aligned}$$

This shows that ϕ is an isometry. □

Now, let us discuss the notion of orientibility.

2.5 Orientable Surface

Intuitively, orientability means we can distinguish between internal and external face of the surface. In case of planes, we can orient a plane by picking a basis for the plane. So, if we have two bases and we want to check if they correspond to same orientation we will calculate the change of basis matrix. If the sign of the determinant of this matrix is positive then both the bases correspond to the same orientation. In case of surfaces, a surface is said to be orientable if we can orient all of its tangent planes uniformly.

Definition 9. *Let S be a surface and let $\psi_1 : V_1 \to S$ and $\psi_2 : V_2 \to S$ be two local parametrizations of S. The local parametrizations ψ_1 and ψ_2 correspond to same orientation (i.e they are equioriented) if either $\psi_1(V_1) \cap \psi_2(V_2) = \emptyset$ or $det(Jac(\psi_2^{-1} \circ \psi_1)) > 0$ on $\psi_1^{-1}(\psi_1(V_1) \cap \psi_2(V_2)$ If $det(Jac(\psi_2^{-1} \circ \psi_1)) < 0$ then the surfaces are oppositely oriented. S is called orientable if there exists an atlas containing local parametrizations which are pairwise equioriented.*

Examples of orientable surfaces include the sphere and plane.

Remark: We can cannot determine the orientability of a surface just by looking at one local parametrization. Moreover, a single local parametrization will always produce an orientable image. The only hindrance (if any) in orientability is the way these local parametrizations are adhered together.

As mentioned earlier, the orientability of a surface is associated with the ability to distinguish between interior and exterior of the surface. Recall that in case of curves, the unit normal vector helps us find the concave side of the curve.

Definition 10. *A normal vector field S is a smooth map from S to $\mathbb{R}^3$ i.e $N :\to \mathbb{R}^3$ such that $< N(p), v >= 0$ for every v in the tangent space.*

If we take N as a unit vector then we will refer to it as normal versor field. The normal versor field is also known as the Gauss map.

Proposition 2.5.1. *Consider a surface S. The Gauss map $N(p)$ exists if and only if the surface is orientable.*

We will not prove the above proposition. However, one of the important consequence of this proposition is following definition.

Definition 11. *Consider an oriented surface S. Let β be the atlas through which the surface is oriented and ψ be the local parametrization. The Gauss map for the surface S is given by:*

$$N = \frac{\partial_1 \times \partial_2}{||\partial_1 \times \partial_2||}$$

Notice that $N(p)$ is a C^∞ map. In section 2.3 we have discussed how the differential of a smooth map is defined. Let us use that to give the following definition.

Definition 12. *Consider an oriented surface S and suppose N be the Gauss map. Let γ be curve whose support lives inside S. Then the differential of the N at p is a function $dN_p : T_pS \to T_{N(p)}S^2$ defined as:*

$$dN_p(v) = (N \circ \gamma)'(0)$$

Moreover, by the lemma 2.3.2 the we have the following result:

$$dN_p(\partial_i) = \frac{\partial(N \circ \gamma)}{\partial x_i}. \tag{2.5.1}$$

2.6 Normal Curvature and Second Fundamental Form

Recall that ,for a curve, the unit tangent vector helps us define the curvature. Defining the curvature of a curve is easy because we have a one dimensional tangent space. In case of surfaces, at each point p we have a two dimensional tangent space T_pS. So, the surface might curve differently in different tangent directions.

Therefore, for defining the curvature, we need a pick a curve passing through p and calculate its curvature. The problem we face while doing so is that the curvature might be influenced by some characteristic of the chosen curve. Therefore, we need to fix a special curve for which the curvature is only determined by the geometry of S and the tangent direction (vector) and not by some property specific to the chosen curve.

Lemma 2.6.1. *Consider an oriented surface S and suppose $p \in S$. Let $N(p)$ be the Gauss map. Given any tangent vector $v \in T_pS$, let P_v be the plane spanned by v and $N(p)$. The intersection of the plane P_v and S will be a regular curve.*

Proof. A point $x \in P_v$ if and only if:

$$< x - p, v \times N >= 0.$$

Suppose ψ parametrizes S such that $\psi(0) = p$. A point $\psi(y)$ belongs to $P_v \cap S$ iff

$$< \psi(y) - p, v \times N >= 0.$$

Define the following function $g : V \to \mathbb{R}$:

$$g(y) =< \psi(y) - p, v \times N > .$$

We have the set $D = \{y \in V | g(y) = 0\} \subseteq V$

If we shaw that D is a regular curve 0 then we can conclude that $\psi(D)$ is also a curve because ψ is homeomorphic to its image.

Now, $y = (y_1, y_2)$

Differentiate g with respect to y_i ; where $i = 1, 2$

$$\frac{\partial g}{\partial y_i} = \left\langle \frac{\partial \psi}{\partial y_i}, v \times N \right\rangle + \langle \psi(y) - p, 0 \rangle$$

$$\frac{\partial g}{\partial y_i} = \langle \partial_i, v \times N \rangle$$

$$\frac{\partial g}{\partial y_i}(0) = \left\langle \partial_i|_p, v \times N \right\rangle .$$

We have to show that 0 is not a critical point of g. Suppose on the contrary 0 is a critical point of g. Then,

$$v \times N(p) \perp \partial_i|_p.$$

$\implies v \times N(p)$ *is parallel to* $N(p)$ which is a contradiction. So, this means that our assumption that 0 is a critical point of g is wrong. Therefore by the implicit function theorem we know that D is graph in the neighbourhood of 0. Since it is a graph in 2 dimensions so it is a curve. □

The above result helps us give the following definition:

Definition 13. *Consider a regular surface S and suppose $p \in S$. Let $v \in T_pS$ and P_v be the plane defined above. $P_v \cap S$ is a regular curve called the normal section of S at p along v.*

The tangent vector of the normal section of S along v at point p is $\pm v$ because $\text{Span}\{v, N(p)\} \cap T_pS = \mathbb{R}v$. In our case we will orient normal section along v in such a way that $\gamma'(0) = v$. Let us define a special curvature for a curve whose support lives completely inside the surface S.

Definition 14. *Consider an arc length parametrized curve $\gamma(s)$ whose support lives completely inside the surface S and $\gamma(0) = p$ and $\gamma'(0) = v \in T_pS$. The normal curvature κ_n of γ is the following:*

$$\kappa_n =< \gamma''(0), N \circ \gamma(0) >= \kappa < n(0), N \circ \gamma(0) >,$$

Our next target is to find an effective to calculate the normal curvature. For the purpsose, let us recall the map dN_p introduced in section 2.5. We are going to look at some important properties of the dN_p. The following examples illustrate the calculation of the dN_p map.

Example 5.

Let N be the normal versor field of a plane P spanned by vectors v_1 and v_2.

$$N = \frac{v_1 \times v_2}{||v_1 \times v_2||}.$$

Let us calculate the dN_p map for any point p of P.

Suppose we have a curve $\gamma(t)$ in the plane. Let v be a tangent vector at p. So:

$$dN_p : T_pP \to T_{N(p)}S^2.$$

By definition of dN_p we can compute:

$$dN_p(v) = (N \circ \gamma)'(0).$$

Since N is a fixed constant vector so its derivative is zero. $\implies dN_p \equiv 0$ for every point p in P. Since we can do the same process for any curve and any point in the plane so this mean that

$$dN \equiv 0.$$

So, in some sense, dN_p tells us, to what extent the surface is flat.

Example 6.

Let S^2 be the unit sphere. Let us calculate the map dN_p of the unit sphere. Consider the following parametrization:

$$\psi(x,y) = (x, y, \sqrt{1 - x^2 - y^2}).$$

Then the normal versor field of sphere is the following:

$$N : S^2 \to S^2$$

$$N(p) = p \qquad \forall p \in S^2.$$

Suppose we have a curve $\gamma(t)$ whose support lives inside the unit sphere and is parametrized such that $\gamma(0) = p$ and let $\gamma'(0) = v$ Then by definition of dN_p we can write:

$$dN_p(v) = (N \circ \gamma)'(0)$$

Since $N(p) = p$ for every $p \in S^2 \implies N(\gamma(t)) = \gamma(t) \quad \forall t \in \mathbb{R}$.
So,

$$\begin{aligned} dN_p(v) &= \gamma'(0) \\ dN_p(v) &= v. \end{aligned}$$

Since we can do the same process for every curve whose support lives inside S^2 therefore we obtain:

$$dN_p \equiv id \quad \forall p \in S^2.$$

There is one important thing to note in above two examples: the image of the map dN_p is contained in T_pS in both cases i.e in the first example dN_p is identically zero. In the second example dN_p is the identity map.
This is not a coincidence that the image of dN_p is contained in T_pS. By definition dN_p is a map from T_pS to $T_{N(p)}S^2$ but we know that the in sphere the tangent space of each point is orthogonal to the point i.e $T_{N(p)}S^2 \perp N(p)$ but $N(p)$ is the normal vector of T_pS. so this means that $N(p) \perp T_pS$ and $N(p) \perp T_{N(p)}S^2$ therefore $T_{N(p)}S^2$ coincides with T_pS therefore $dN_p : T_pS \to T_pS$.

Lemma 2.6.2. *Consider an oriented surface S with the parametrization ψ and suppose N be its Gauss map. The map dN_p is a symmetric ($< dN_p(u), v >=< u, dN_p(v) >$) linear endomorphism.*

Proof. From the the above remark the map dN_p is a linear endomorphism because it is linear and it is a function from T_pS to T_pS. The only thing left to prove it that dN_p is symmetric with respect to the scalar product $< \cdot, \cdot >$ on $\mathbb{R}^3$.
Since $N \circ \psi$ is orthogonal to every vector in the T_pS, it is also orthogonal to the basis vectors ∂_1 and ∂_2.
So, we have:

$$< N \circ \psi, \partial_1 > \equiv 0.$$

By Differentiating with respect to x_2 we get:

$$< \frac{\partial(N \circ \psi)}{\partial x_2}, \partial_1 > + < N \circ \psi, \frac{\partial^2 \psi}{\partial x_2 \partial x_1} >= 0.$$

Using the remark (2.5.1) $\frac{\partial(N \circ \psi)}{\partial x_2} = dN_p(\partial_2)$ we have:

$$< dN_p(\partial_2), \partial_1 >= - < N \circ \psi, \frac{\partial^2 \psi}{\partial x_2 \partial x_1} > . \tag{2.6.1}$$

Similarly, $< N \circ \psi, \partial_2 >\equiv 0$ so, by differentiating it w.r.t x_1 and again using the remark (2.5.1) we get the following result:

$$< dN_p(\partial_1), \partial_2 >= - < N \circ \psi, \frac{\partial^2 \psi}{\partial x_1 \partial x_2} > . \tag{2.6.2}$$

Since ψ is local parametrization and hence ψ is of class C^∞ we have:

$$\frac{\partial^2 \psi}{\partial x_1 \partial x_2} = \frac{\partial^2 \psi}{\partial x_2 \partial x_1}$$

Therefore:

$$< dN_p(\partial_1), \partial_2 >=< \partial_1, dN_p(\partial_2) > .$$

So, the dN_p map is symmetric on the basis of the T_pS. Therefore it is symmetric for all the vectos in T_pS. □

Definition 15. *Consider an oriented surface S and suppose $N : S \to S^2$ is the Gauss map. Then the second fundamental form is the function $Q_p(w) : T_pS \to \mathbb{R}$ given by :*

$$Q_p(w) = - < dN_p(w), w > .$$

Let us find an effective way to calculate the second fundamental form.Let $S \subseteq \mathbb{R}^3$ be a regular surface and ψ be local patametrization of S. Let $p \in S$ and $v \in T_pS$. Suppose

$\{\partial_1, \partial_2\}$ be the basis of T_pS with respect to ψ then we have:

$$\begin{aligned}
Q_p(v) &= -< dN_p(v_1\partial_1 + v_2\partial_2), v_1\partial_1 + v_2\partial_2 > \\
&= -< v_1 dN_p(\partial_1) + v_2 dN_p(\partial_2), v_1\partial_1 + v_2\partial_2 > \qquad using \quad linearity \quad of \quad dN_p \\
&= -[v_1^2 < dN_p(\partial_1), \partial_1 > + v_1 v_2 < dN_p(\partial_1), \partial_2 > \\
&+ \; v_1 v_2 < dN_p(\partial_2), \partial_1 > + v_2^2 < dN_p(\partial_2), \partial_2 >].
\end{aligned}$$

Since dN_p is symmetric so $< dN_p(\partial_1), \partial_2 >=< \partial_1, dN_p(\partial_2) >$ therefore:

$$\begin{aligned}
Q_p(v) &= -[v_1^2 < dN_p(\partial_1), \partial_1 > 2v_1 v_2 < dN_p(\partial_1), \partial_2 > + v_2^2 < dN_p(\partial_2), \partial_2 > \\
&= v_1^2 e(p) + 2v_1 v_2 f(p) + v_2^2 g(p).
\end{aligned}$$

In matrix form:

$$\begin{bmatrix} v_1 & v_2 \end{bmatrix} \begin{bmatrix} e & f \\ f & g \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}. \tag{2.6.3}$$

To Calculate $e(p)$ we have the following:

Consider a parametrization ψ and suppose $N(p)$ be Gauss map at point p. We know that

$$< N \circ \psi, \partial_1 >= 0.$$

Differentiating both sides with respect to x_1

$$\begin{aligned}
\frac{\partial}{\partial x_1} < N \circ \psi, \partial_1 > &= < \frac{\partial}{\partial x_1}(N \circ \psi), \partial_1 > + < N \circ \psi, \frac{\partial^2 \psi}{\partial x_1^2} >= 0 \\
&= < dN_p(\partial_1), \partial_1 > + < N \circ \psi, \frac{\partial^2 \psi}{\partial x_1^2} >= 0 \quad since \quad dN_p(\partial_1) = \frac{\partial}{\partial x_1}(N \circ \psi).
\end{aligned}$$

So this means that:

$$e(p) = - < N \circ \psi, \frac{\partial^2 \psi}{\partial x_1^2}. > \tag{2.6.4}$$

Similarly we have:

$$f(p) = - < N \circ \psi, \frac{\partial^2 \psi}{\partial x_1 \partial x_2} > \qquad , \quad g(p) = - < N \circ \psi, \frac{\partial^2 \psi}{\partial x_2^2} > . \tag{2.6.5}$$

The following lemma is an excellent application of second fundamental form.

Lemma 2.6.3. *Consider an oriented surface S with the Gauss map N and $p \in S$ Then*

1. *Two curves passing through point p having same tangent at p have the same normal curvature.*
2. *The value of second fundamental form at any vector v gives us the normal curvature along that vector.*

Proof. Let we have a curveγ parameterized by arc length whose support is completely contained in the surface S i.e $\gamma(s) \subseteq S$ such that $\gamma(0) = p$ and $\gamma'(0) = v \in T_pS$. Since the normal versor field (Gauss map) is orthogonal to tangent space of each point of surface. So, the $N \circ \gamma(s)$ is orthogonal to each tangent vector of the curve
so we have:

$$< N \circ \gamma, \gamma'(s) >= 0.$$

By Differentiating:

$$\left\langle \frac{\partial(N \circ \gamma)}{\partial s}(s), \gamma'(s) \right\rangle + \langle N \circ \gamma, \gamma''(s) \rangle = 0.$$

put $s = 0$

$$\left\langle \frac{\partial(N \circ \gamma)}{\partial s}(0), \gamma'(0) \right\rangle + \langle N(p), \gamma''(0) \rangle = 0$$

We know that $\gamma(0) = p$ and $\gamma'(0) = v \in T_pS$ By definition of dN_p we have:

$$dN_p(v) = (N \circ \gamma)'(0) = \frac{\partial(N \circ \gamma)}{\partial s}(0).$$

Therefore,

$$\langle dN_p(v), v \rangle = -\langle N, \gamma''(0) \rangle.$$

Moreover, if the γ is biregular then $\gamma''(0) = \tilde{\kappa} n$. So,

$$-< dN_p(v), v >=< N(p), \gamma''(0) >= \tilde{\kappa} < N(p), n(0) > .$$

which is exactly the value of normal curvature.
Note that the second equality holds only when the curve is biregular.

$$Q_p(v) =< N(p), \gamma''(0) >= \tilde{\kappa} < N(p), n(0) > . \tag{2.6.6}$$

Moreover, suppose we have an other curve $\tilde{\gamma}$ such that $\tilde{\gamma}(0) = p$ and $\gamma'(0) = v \in T_pS$. Since both γ and $\tilde{\gamma}$ have same tangent vectors at point p therefore the value of second fundamental form on tangent vectors of both curves at point is same. Since $Q_p(v)$ measures the normal curvature along vector in T_pS therefore the normal curvature of both curves is same.

$\square$

Chapter 3

Curvatures

In this section, we are going to use the dN_p map to introduce some new types of curvatures. Recall that, dN_p is a symmetric linear endomorphism from T_pS to T_pS. Since dN_p is symmetric, therefore we can apply spectral theorem which is stated as:

Theorem 3.0.1. *Suppose we have a finite dimenssional vector space endowed with a positive definite scalar product. Then every symmetric linear operator is diagonalizable. Moreover, there exists an orthonormal basis of V consisting of eigenvectors of L.*

The above result implies that there exists a basis of T_pS made solely of eigenvectors of dN_p. So, in particular, it means that the eigenvectors of dN_p live inside T_pS. All these nice properties to of dN_p are

Definition 16. *Consider an oriented surface S and suppose $N : S \to S^2$ be the Gauss map. The eigenvector of length one of dN_p is called as principal direction of S at p. The eigenvalue corresponding to principal direction is called as principal curvature.*

Since we know that the T_pS contains the principal direction of dN_p so suppose $v \in T_pS$ is the principal direction of S at p and let λ be the corresponding principal curvature Then lets calculate the second fundamental form at this principal direction.

$$\begin{aligned} Q_p(w) &= -< dN_p(w), w > \\ &= -< -\lambda w, w > \\ &= \lambda < w, w > \\ Q_p(v) &= \lambda. \end{aligned}$$

Which shows that principal curvature at point p is a normal curvature at point p.

Proposition 3.0.2. *Consider a regular surface S and suppose $N : S \to S^2$ is the Gauss map. Then we can find principal directions $v_1, v_2 \in T_pS$ with the principal curvatures λ_1 and λ_2 respectively such that $\lambda_1 \leq \lambda_2$ satisfying the following conditions:*

1. *$\{v_1, v_2\}$ is an orthonormal basis of T_pS*

2. *Suppose we have a unit vector* $v \in T_pS$ *and an angle* $\theta \in (-\pi, \pi]$ *such that* $\cos\theta =< v, v_1 >_p$ *and* $\sin\theta =< v, v_2 >$ *then*

$$Q_p(v) = \lambda_1 \cos^2\theta + \lambda_2 \sin^2\theta.$$

3. $[\lambda_1, \lambda_2]$ *is the interval of all possible normal curvatures of the surface at point p.*

Proof. As we know from (2.6.2) that $dN_p : T_pS \to T_pS$ is a symmetric linear operator. Therefore, the spectral theorem implies that there exist an orthonormal basis of T_pS consisting of eigenvectors of dN_p i.e $\exists \quad v_1, v_2 \in T_pS$ such that $< v_1, v_2 >= 0$ and $||v_1|| = ||v_2|| = 1$ and $\{v_1, v_2\}$ is a basis of T_pS.
Since we have established that v_1, v_2 form a basis of T_pS and we are given a unit vector $v \in T_pS$ and an angle θ such that $< v_1, v >= \cos\theta$ and $< v_2, v >= \sin\theta$ we have:

$$v = \cos\theta v_1 + \sin\theta v_2.$$

Now,

$$\begin{aligned} Q_p(v) &= -< dN_p(v, v > \\ &= -< \cos\theta dN_p(v_1) + \sin\theta dN_p(v_2), \cos\theta v_1 + \sin\theta v_2 > . \end{aligned}$$

Since v_1 and v_2 are the eigenvectors of dN_p with eigenvalues $-\lambda_1$ and $-\lambda_2$ respectively so:

$$Q_p(v) \quad = \quad \lambda_1 cos^2\theta + \lambda_2 sin^2\theta.$$

Which proves (2). Now by using the fact that $\cos^2\theta + \sin^2\theta = 1$ we have:

$$\implies Q_p(v) = \lambda_1 + (\lambda_2 - \lambda_1)\sin^2\theta.$$

In order to maximize $Q_p(v$ we need to maximize $\sin^2\theta$ so at $\theta = \pm\frac{\pi}{2}$ we have $\sin^2\theta = 1$ So, we get:

$$Q_p(v) = \lambda_2.$$

$\implies \lambda_2$ is the maximum value of $Q_p(v)$

Similarly in order to minimize $Q_p(v)$ we need to minimize $\sin\theta$ so when $\theta = \pi$ or 0 $\sin\theta = 0$ So, we get

$$Q_p(v) = \lambda_1.$$

So, λ_1 is the minimum value of $Q_p(v)$ and clearly $Q_p(v) = \lambda_1 \cos^2\theta + (\lambda_2 - \lambda_1)\sin^2\theta$ is continuous function. So, by intermediate value theorem the only possible value of normal curvature are all the values in interval $[\lambda_1, \lambda_2]$. □

Now,we learned about two types of curvatures of a surface i.e Normal Curvature and Principal Curvature but these are not the only curvatures of surface. Let us define two new types of curvatures.

Definition 17. *Consider an oriented surface S and suppose $N : S \to S^2$ be the Gauss map. Then the Gaussian Curvature κ of the surface S at any point p on the surface is the following:*

$$\kappa(p) = det(dN_p) \qquad \forall p \in S.$$

Since λ_1 and λ_2 are the principal curvatures then the Gaussian Curvature can be expressed as:

$$\kappa(p) = \lambda_1\lambda_2.$$

The Gaussian Curvature helps us classify the points on a surface. The points on the surface can be classified into the following categories:

1. If the κ is positive at p then it is an elliptic point.
2. If the κ is negative on p then p is hyperbolic point.
3. If κ is 0 at p and $dN_p \neq 0$ then p is called parabolic point.
4. If the κ at p is zero and $dN_p = 0$ then p is a planar point.

Definition 18. *Suppose we have orientable surface S with N as its Gauss map. Then the mean curvature of the surface S at point p is given by:*

$$H(p) = \frac{1}{2} tr(dN_p).$$

Where $tr(dN_p)$ is trace of the matrix form of dN_p.

Our next target is to find an effective way to calculate the principal, Gaussian and the mean curvature. The differential of the Gauss map dN_p map has played an important role in defining all the previously mentioned curvatures. So, if we find way to express the dN_p map in a matrix form then we can calculate all the curvatures defined previously.

Fora a Surface S. Let $v, w \in T_pS$ and $\{\partial_1, \partial_2\}$ be the basis of T_pS. Let us calculate the following:

$$\begin{aligned}
-< dN_p(u), v > &= -< u_1 dN_p(\partial_1) + u_2 dN_p(\partial_2), v_1\partial_1 + v_2\partial_2 > \\
&= -(u_1v_1 < dN_p(\partial_1), \partial_1 > + (u_1v_2 + u_2v_1) < dN_p(\partial_2), \partial_1 > \\
&\quad + u_2v_2 < dN_p(\partial_2), \partial_2)) \\
&= u_1v_1e(p) + (u_1v_2 + u_2v_1)f(p) + u_2v_2g(p).
\end{aligned}$$

In matrix form:

$$-< dN_p(u), v >= \begin{bmatrix} u_1 & u_2 \end{bmatrix} \begin{bmatrix} e(p) & f(p) \\ f(p) & g(p) \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}. \tag{3.0.1}$$

Let M be the matrix representing the dN_p map then by using the result in (2.2.3) we have:

$$-< dN_p(u), v >= -M \begin{bmatrix} u_1 & u_2 \end{bmatrix} \begin{bmatrix} E(p) & F(p) \\ F(p) & G(p) \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}. \tag{3.0.2}$$

Combining the above two mentioned results we have:

$$-< dN_p(u), v >= \begin{bmatrix} u_1 & u_2 \end{bmatrix} \begin{bmatrix} e & f \\ f & g \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = -M \begin{bmatrix} u_1 & u_2 \end{bmatrix} \begin{bmatrix} E(p) & F(p) \\ F(p) & G(p) \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}.$$

Since $< dN_p(u), v >=< v, dN_p(u) >$

$$\begin{bmatrix} v_1 & v_2 \end{bmatrix} \begin{bmatrix} e & f \\ f & g \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = - \begin{bmatrix} v_1 & v_2 \end{bmatrix} \begin{bmatrix} E & F \\ F & G \end{bmatrix} M \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}.$$

By comparing both sides we get:

$$\begin{bmatrix} e(p) & f(p) \\ f(p) & g(p) \end{bmatrix} = - \begin{bmatrix} E(p) & F(p) \\ F(p) & G(p) \end{bmatrix} M$$

$$\implies M = - \begin{bmatrix} E(p) & F(p) \\ F(p) & G(p) \end{bmatrix}^{-1} \begin{bmatrix} e(p) & f(p) \\ f(p) & g(p) \end{bmatrix}.$$

Now,

$$\begin{bmatrix} E(p) & F(p) \\ F(p) & G(p) \end{bmatrix}^{-1} = \frac{1}{EG-F^2} \begin{bmatrix} G(p) & -F(p) \\ -F(p) & E(p) \end{bmatrix}$$

Therefore,

$$M = \frac{-1}{EG-F^2} \begin{bmatrix} G(p) & -F(p) \\ -F(p) & E(p) \end{bmatrix} \begin{bmatrix} e(p) & f(p) \\ f(p) & g(p) \end{bmatrix}.$$

So,

$$M = \frac{-1}{EG-F^2} \begin{bmatrix} Ge-Ff(p) & Gf-Fg(p) \\ Ef-Fe(p) & Eg-Ff(p) \end{bmatrix}. \tag{3.0.3}$$

So, M is the matrix that represents the dN_p map. We can use this matrix form of the dN_p map to calculate the principal, Gaussian and mean curvatures.

So, the Gaussian Curvature κ can be calculated as follows:

$$\kappa = det(dN_p) = det(M).$$

So,

$$\begin{aligned} det(M) &= \frac{1}{(EG-F^2)^2}[(Ge-Ff)(Eg-Ff)-(Gf-Fg)(Ef-Fe)] \\ &= \frac{(eg-f^2)(EG-F^2)}{(EG-F^2)^2} \\ Hence, & \\ \kappa &= \frac{eg-f^2}{EG-F^2}. \end{aligned}$$

Now, let us calculate the mean curvature H using the matrix M that represents dN_p:

$$H = \frac{1}{2}tr(dN_p) = \frac{1}{2}tr(M).$$

So,

Trace of the matrix $M = tr(M) = \frac{Ge-Ff+Eg-Ff}{EG-F^2} = \frac{Ge-2Ff+Eg}{EG-F^2}$.

Therefore,

$$H = \frac{Ge-2Ff+Eg}{2(EG-F^2)}.$$

Finally, we calculate the principal curvature of the surface. Let λ_1 and λ_2 be the eigenvalues of matrix M. We can write:

$$\kappa = \lambda_1\lambda_2 \qquad ; \quad H = \frac{1}{2}(\lambda_1+\lambda_2).$$

Since $H = \frac{1}{2}(\lambda_1+\lambda_2) \implies 2H - \lambda_2 = \lambda_1$ So,

$$\begin{aligned} \kappa &= (2H-\lambda_2)\lambda_2 \\ \kappa &= 2H\lambda_2 - \lambda_2^2. \end{aligned}$$

So, we have the following quadratic equation:

$$\lambda_2^2 - 2H\lambda_2 + \kappa = 0.$$

Using quadratic formula:

$$\lambda_2 = \frac{2H \pm \sqrt{4H^2-4\kappa}}{2} = H \pm \sqrt{H^2-\kappa}. \tag{3.0.4}$$

λ_2 gives us two eigenvalues of M and since M has order 2×2 so M can only have two eigenvalues hence λ_2 gives us both the eigen values of M. Moreover, even if we solve the systems of linear equations completely to find λ_1 we will get the exact same values for λ_1.

So, the principal curvatures of S are given by:

$$\lambda = -\lambda_2 = H \mp \sqrt{H^2 - \kappa}.$$

Where λ is the principal curvature.

Let us see some examples in which we compute the principal, Gaussian and mean curvatures.

Example 7.

Consider the following parametrization $\psi(x,y) = (x,y,h(x,y))$ where $\psi : \mathrm{V} \to R^3$ such that $\mathrm{V} \subseteq R^2$ and $\psi \in C^\infty$ (V). For simplicity we write $\frac{\partial h}{\partial x} = h_x \quad ; \quad \frac{\partial h}{\partial y} = h_y$, so we can write:

$$\partial_1 = \frac{\partial \psi}{\partial x} = (1,0,h_x), \qquad \partial_2 = \frac{\partial \psi}{\partial y} = (0,1,h_y).$$

Now,

$$\partial_1 \times \partial_2 = \begin{vmatrix} i & j & k \\ 1 & 0 & h_x \\ 0 & 1 & h_y \end{vmatrix}$$

After expanding the determinant we get:

$$\partial_1 \times \partial_2 = (-h_x, -h_y, 1)$$

$$||\partial_1 \times \partial_2|| = \sqrt{(h_x)^2 + (h_y)^2 + 1}.$$

We then get:

$$N = \frac{(-h_x, -h_y, 1)}{\sqrt{1 + ||\nabla h||^2}}.$$

Now Let us calculate metric coefficients:

$$E = 1 + h_x^2 \qquad F = h_x h_y \qquad G = 1 + h_y^2.$$

Moreover, from given parametrization:

$$\frac{\partial^2 \psi}{\partial x^2} = \left(0, 0, \frac{\partial^2 h}{\partial x^2}\right)$$

$$\frac{\partial^2 \psi}{\partial y \partial x} = \left(0, 0, \frac{\partial^2 h}{\partial y \partial x}\right)$$

$$\frac{\partial^2 \psi}{\partial y^2} = \left(0, 0, \frac{\partial^2 h}{\partial y^2}\right).$$

We have $N = \frac{\left(-h_x, -h_y, 1\right)}{\sqrt{1+||\nabla h||^2}}$
For simplicity we write:

$$\frac{\partial^2 h}{\partial x^2} = h_{xx}, \quad \frac{\partial^2 h}{\partial x \partial y} = h_{yx}, \quad \frac{\partial^2 h}{\partial y^2} = h_{yy}.$$

Now let us calculate the form coefficients :

$$e = < N, \frac{\partial^2 \psi}{\partial x^2} > = \frac{h_{xx}}{\sqrt{1+||\nabla h||^2}}$$

$$f = < N, \frac{\partial^2 \psi}{\partial y \partial x} > = \frac{h_{xy}}{1+||\nabla h||^2}$$

$$g = < N, \frac{\partial^2 \psi}{\partial y^2} > = \frac{h_{yy}}{\sqrt{1+||\nabla h||^2}}.$$

Using all the values calculated above we have:

$$EG - F^2 = \left(1+h_x^2\right)\left(\left(1+h_y^2\right) - \left(h_x^2 h_y^2\right) = 1+||\nabla h||^2\right.$$

$$eg - f^2 = \frac{1}{1+||\nabla h||^2}\left(h_{xx}h_{yy} - h_{xy}^2\right).$$

So the Gausssian Curvature is given by:

$$\kappa = \frac{eg - f^2}{EG - F^2} = \frac{h_{xx}h_{yy} - h_{xy}}{1+||\nabla h||^2}.$$

For the Mean Curvature we have:

$$eG = h_{xx}\frac{(1+h_y^2)}{\sqrt{1+||\nabla h||^2}} \qquad 2fF = 2\frac{h_{xy}\ h_x\ h_y}{\sqrt{1+||\nabla h||^2}} \qquad gE = \frac{h_{yy}(1+h_x^2)}{\sqrt{1+||\nabla h||^2}}.$$

So that we have Mean Curvature as:

$$H = \frac{1}{2}\frac{eG - 2fF + gE}{EG - F^2} = \frac{h_{xx}(1+h_y^2 - 2h_{xy}h_xh_y + h_{yy}(1+h_x^2)}{(1+||\nabla h||^2)^{3/2}}.$$

Example 8.

Consider the parametrization of helicoid: $\psi(x,y) = (y\cos x, y\sin x, ax)$. So, we have:

$$\frac{\partial \psi}{\partial x} = \partial_1 = (-y\sin x, y\cos x, a) \quad \frac{\partial \psi}{\partial y} = \partial_2 = (\cos x, \sin x, 0).$$

So, we have:

$$\begin{aligned} E =< \partial_1, \partial_1 > &= y^2\sin^2 x + y^2\cos^2 x + a^2 = y^2 + a^2 \\ F =< \partial_1, \partial_2 > &= -y\sin x\cos x + y\sin x cosx = 0 \\ G =< \partial_2, \partial_2 > &= \cos^2 x + \sin^2 x + 0 = 1 \\ So, \qquad EG - F^2 &= y^2 + a^2. \end{aligned}$$

Now, the normal versor field $N = \frac{\partial_1 \times \partial_2}{||\partial_1 \times \partial_2||}$ is given by:

$$N = \frac{(-asinx, acosx, -y)}{a^2 + y^2}.$$

From the given parametrization we have:

$$\frac{\partial^2 \psi}{\partial x^2} = (-y\cos x, -y\sin x, 0), \quad \frac{\partial^2 \psi}{\partial y^2} = (0,0,0), \quad \frac{\partial^2 \psi}{\partial y \partial x} = (-\sin x, \cos x, 0).$$

Moreover, we have:

$$e =< \frac{\partial^2 \psi}{\partial x^2}, N >= \frac{ay\cos x \sin x - ay\cos x \sin x + 0}{a^2 + y^2} = 0$$

$$f =< \frac{\partial^2 \psi}{\partial x \partial y}, N >= \frac{a}{\sqrt{a^2 + y^2}}$$

$$g =< \frac{\partial^2 \psi}{\partial y^2}, N >=< 0, N >= 0.$$

So, the Gaussaian Curvauture is:

$$\kappa = \frac{eg - f^2}{EG - F^2} = \frac{-a^2}{(a^2 + y^2)^2}.$$

For the Mean Curvature we have:

$$H = \frac{1}{2} \frac{eG - 2fF + gE}{EG - F^2} = 0.$$

We then have principal curvature: $\lambda_{1,2} = H \pm \sqrt{H^2 - \kappa}$

$$= 0 \pm \sqrt{0 - \frac{-a^2}{a^2 + y^2}} = \pm \frac{a}{\sqrt{a^2 + y^2}}.$$

Example 9.

Consider the parametrization of cylinder: $\psi(x,y) = (cosx, sinx, y)$.

Then we have:

$$\partial_1 = \frac{\partial \psi}{\partial x} = (-\sin x, \cos x, 0) \qquad \partial_2 = \frac{\partial \psi}{\partial y} = (0, 0, 1).$$

Now, we calculate the metric coefficients:

$E =< \partial_1, \partial_1 >= \sin^2 x + \cos^2 x + 0 = 1 \quad F =< \partial_1, \partial_2 >= 0 \quad G = \partial_2, \partial_2 >= 1.$

Now we find the normal versor field:

$$N = \frac{\partial_1 \times \partial_2}{||\partial_1 \times \partial_2||} = (\cos x, \sin x, 0)$$

So,

$$\frac{\partial^2 \psi}{\partial x^2} = (-\cos x, -\sin x, 0)$$

$$\frac{\partial^2 \psi}{\partial y^2} = (0,0,0)$$

$$\frac{\partial^2 \psi}{\partial y \partial x} = (0,0,0).$$

We then calculate the form coefficients:

$$e =< N, \frac{\partial^2 \psi}{\partial x^2} >= -1 \quad f =< N, \frac{\partial^2 \psi}{\partial x \partial y} >= 0 \quad g =< N, \frac{\partial^2 \psi}{\partial y^2} >= 0.$$

The κ can be calculated as:

$$\kappa = \frac{eg - f^2}{EG - F^2} = \frac{0}{EG - F^2} = 0.$$

Now we calculate the Mean Curvature:

$$H = \frac{1}{2} \frac{eG - 2fF + gE}{EG - F^2} = \frac{-1}{2}.$$

Finally, the principal curvatures are the following:

$$\lambda_{1,2} = \frac{-1}{2} \pm \sqrt{\frac{1}{4}} = -1, 0.$$

Example 10.

Consider the following parametrization of catenoid:

$$\psi(x,y) = (acoshxcosy, acoshxsiny, ax)$$

where $\psi : U \to \mathbb{R}^3$ and $\psi \in C^\infty(V)$. We have:

$$\partial_1 = \frac{\partial \psi}{\partial x} = (a\sinh x \cos y, a\sinh x \sin y, a)$$

$$\partial_2 = \frac{\partial \psi}{\partial y} = (-a\cosh x \sin y, a\cosh x \cos y, 0)$$

Also,

$$\begin{aligned} \partial_1 \times \partial_2 &= \begin{vmatrix} i & j & k \\ a\sinh x \cos y & a\sinh x \sin y & a \\ -a\cosh x \sin y & a\cosh x \cos y & 0 \end{vmatrix} \\ &= (-a^2\cosh x \cos y, -a^2\cosh x \sin y, a^2\cosh x \sinh x) \end{aligned}$$

and

$$||\partial_1 \times \partial_2|| = \sqrt{a^4\cosh^2 x \cos^2 y + a^4\cosh^2 x \sin^2 y + a^4\cosh^2 x \sinh^2 x}$$

which simplifies to $||\partial_1 \times \partial_2|| = a^2\cosh^2 x$.

The normal versor is

$$N = \frac{\partial_1 \times \partial_2}{||\partial_1 \times \partial_2||} = \frac{1}{\cosh x}(-\cos y, -\sin y, \sinh x).$$

Now, let us calculate the metric coefficients:

$$E = a^2\cosh^2 x, \qquad F \equiv 0, \qquad ; \qquad G = a^2\cosh^2 x.$$

We then get

$$EG - F^2 = a^4\cosh^4 x.$$

Now,

$$\frac{\partial^2 \psi}{\partial x^2} = (a\cosh x \cos y, a\cosh x \sin y, 0)$$

$$\frac{\partial^2\psi}{\partial x\partial y} = (-a\sinh x\sin y, a\sinh x\cos y, 0)$$

$$\frac{\partial^2\psi}{\partial y^2} = (-a\cosh x\cos y, -a\cosh x\sin y, 0).$$

The form coefficients can be calculated as follows:

$$e =< N, \frac{\partial^2\psi}{\partial x^2} >= -a[\cos^2 y + \sin^2 y] = -a$$

$$f =< N, \frac{\partial^2\psi}{\partial x\partial y} >= 0$$

$$g =< N, \frac{\partial^2\psi}{\partial y^2} >= a.$$

There Gaussian Curvature can be calculated as follows:

$$\kappa = \frac{eg - f^2}{EG - F^2} = \frac{-a^2}{a^4\cosh^4 x} = \frac{-1}{a^2\cosh^4 x}.$$

and,

$$H = \frac{1}{2}\frac{eG - 2fF + gE}{EG - F^2} = \frac{-a^3\cosh^2 x + 0 + a^3\cosh^2 x}{a^4\cosh^4 x} = 0.$$

Finally,

$$\lambda_{1,2} = H \pm \sqrt{H^2 - \kappa} = 0 \pm \sqrt{0 - \frac{-1}{a^2\cosh^4 x}} = \pm\frac{1}{a\cosh^2 x}.$$

Example 11.

Consider the parametrization:

$$\psi(t,\theta) = (\alpha(t)cos\theta, \alpha(t)sin\theta, \beta(t)).$$

Then we have:

$$\partial_1 = \frac{\partial\psi}{\partial t} = (\alpha'\cos\theta, \alpha'\sin\theta, \beta') \qquad \partial_2 = \frac{\partial\psi}{\partial\theta} = (-\alpha\sin\theta, \alpha\cos\theta, 0).$$

Now, we calculate the metric coefficients:

$$E =< \partial_1, \partial_1 >= (\alpha')^2(\cos^2\theta + \sin^2\theta) + (\beta')^2 = (\alpha')^2 + (\beta')^2$$

$$F =< \partial_1, \partial_2 >= -\alpha'\alpha\cos\theta\sin\theta + \alpha\alpha'\cos\theta\sin\theta = 0$$

$$G =< \partial_2, \partial_2 >= \alpha^2(t)\sin^2\theta + \alpha^2\cos^2\theta = \alpha^2.$$

Now,

$$\begin{aligned} \partial_1 \times \partial_2 &= \begin{vmatrix} i & j & k \\ \alpha'\cos\theta & \alpha'\sin\theta & \beta' \\ -\alpha\sin\theta & \alpha\cos\theta & 0 \end{vmatrix} \\ &= (-\alpha\beta'\cos\theta, -\alpha\beta'\sin\theta, \alpha\alpha') \end{aligned}$$

And $\quad ||\partial_1 \times \partial_2|| = \sqrt{(-\alpha\beta'\cos\theta)^2 + (-\alpha\beta'\sin\theta)^2 + (\alpha\alpha')^2} = \sqrt{(\alpha\beta')^2 + (\alpha\alpha')^2}$
$\implies ||\partial_1 \times \partial_2|| = \alpha\sqrt{(\beta')^2 + (\alpha')^2}.$

So the normal versor field is the following:

$$N = \frac{(-\beta'\cos\theta, -\beta'\sin\theta, \alpha')}{\sqrt{(\alpha')^2 + (\beta')^2}}.$$

From the given parametrization we have:

$$\frac{\partial^2\psi}{\partial t^2} = (\alpha''\cos\theta, \alpha''\sin\theta, \beta'')$$

$$\frac{\partial^2\psi}{\partial t\partial\theta} = (\alpha'\sin\theta, \alpha'\cos\theta, 0)$$

$$\frac{\partial^2\psi}{\partial\theta^2} = (\alpha\cos\theta, -\alpha\sin\theta, 0).$$

Let us calculate the form coefficients:

$$e =< N, \frac{\partial^2\psi}{\partial t^2} >= \frac{1}{\sqrt{(\alpha')^2+(\beta')^2}}[-\alpha''\cos^2\theta\beta' - \alpha''\sin^2\theta\beta' + \alpha'\beta'']$$

$$= \frac{\alpha'\beta'' - \alpha''\beta'}{\sqrt{(\alpha')^2+(\beta')^2}}$$

$$f =< N, \frac{\partial^2\psi}{\partial t\partial\theta} >= \frac{\alpha'\beta'\cos\theta\sin\theta - \alpha'\beta'\cos\theta\sin\theta}{\sqrt{(\alpha')^2+(\beta')^2}} \equiv 0$$

$$g =< N, \frac{\partial^2\psi}{\partial\theta^2} >= \frac{\alpha\beta'\cos^2\theta + \alpha\beta'\sin^2\theta}{\sqrt{(\alpha')^2+(\beta')^2}}$$

$$= \frac{\alpha\beta'}{(\sqrt{\alpha')^2+(\beta')^2}}.$$

Now, the Guassian Curvature is given by: $\kappa = \frac{eg-f^2}{EG-F^2}$

$$= \frac{\alpha\beta'' - \alpha''\beta'}{\alpha^2((\alpha')^2+(\beta')^2)^2}.$$

Now, let us calculate the mean curvature:

$$\begin{aligned} H &= \frac{1}{2}\frac{eG-2fF+gE}{EG-F^2} \\ &= \frac{\alpha(\alpha\beta''-\alpha''\beta')+\beta'((\alpha')^2+(\beta')^2)}{2\alpha((\alpha')^2+(\beta')^2)^{3/2}}. \end{aligned}$$

Chapter 4

Egregium Theorem

In this section, the Christoffel Symbols will be introduced. We will prove that the Christoffel Symbols are solely determined by the metric coefficients. Our target is to express Gaussian Curvature in terms of Christoffel Symbols. That would mean that that the Gaiussian Curvature is entirely determined by the metric coefficients. Therefore, by 2.4.3, the Gaussian Curvature is invariant under local isometry.

Recall that in case of curves in $\mathbb{R}^3$ at every point, there exist 3 linearly independent vectors namely tangent vector, normal vector and binormal vector associated to p such that $\{t,n,b\}$ form a basis of $\mathbb{R}^3$. We intend to replicate this construction for the surfaces. Consider a regular surface S. For every $p \in S$ the vectors ∂_1, ∂_2 and $N(p)$ for some given local parametrization $\psi(x_1,x_2)$ form a basis of ambient space ($\mathbb{R}^3$). There exist smooth functions $\Gamma^h_{ij}, h_{ij}, a_{ij}$ satisfying:

$$\frac{\partial^2 \psi}{\partial x_i x_j} = \Gamma^1_{ij}\partial_1 + \Gamma^2_{ij}\partial_2 + h_{ij}N \tag{4.0.1}$$

$$\frac{\partial(N \circ \psi)}{\partial x_j} = a_{1j}\partial_1 + a_{2j}\partial_2. \tag{4.0.2}$$

Where $i,j \in \{1,2\}$.

Since ψ is a C^∞ map so,

$$\frac{\partial^2 \psi}{\partial x_i \partial x_j} = \frac{\partial^2 \psi}{\partial x_j \partial x_i}.$$

Note that in (4.0.2) the coefficient of the normal vector N is 0 because N has magnitude 1. i.e

$$< N, N > \equiv 1.$$

Let us differentiate w.r.t x_j

$$\begin{aligned} 2 < \frac{\partial N}{\partial x_j}, N > &= 0 \\ So, \quad < \frac{\partial N}{\partial x_j}, N > &= 0. \end{aligned}$$

Therefore the coefficient of N in (4.0.2) is 0. Let us calculate the quantities a_{1j} and a_{2j} in

(4.0.2). Recall that

$$\frac{\partial(N\circ\psi)}{\partial x_j} = dN_p(\partial_j).$$

Recall the Matrix form of the map dN_p from (3.0.3).

$$dN_p = M = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = -\frac{1}{EG-F^2}\begin{bmatrix} eG-fF & fG-gF \\ fE-eF & gE-fF \end{bmatrix}.$$

So, dN_p calculated at ∂_j is the j^{th} column of the the matrix from of dN_p map. Therefore the coefficients a_{1j} and a_{2j} are the entries of the matrix form of the dN_p map. The quantities h_{ij} in (4.0.1) can also be calculated as follows:

$$\frac{\partial^2\psi}{\partial x_i \partial x_j} = \Gamma^1_{ij}\partial_1 + \Gamma^2_{ij}\partial_2 + h_{ij}.$$

Since N is orthogonal to ∂_1 and ∂_2 so, we have:

$$< N, \frac{\partial^2\psi}{\partial x_i x_j} >= h_{ij},$$

where $i,j \in \{1,2\}$. Clearly, for i and j belonging to set $\{1,2\}$ h_{ij} are form coefficients i.e $h_{11} = e(p)$, $h_{12} = h_{21} = f(p)$ and $h_{22} = g(p)$. So, only thing left to calculate are the Γ^1_{ij} and Γ^2_{ij}.

Definition 19. *The functions Γ^1_{ij} and Γ^2_{ij} in* (4.0.1) *are called as the Christoffel Symbols.*

Proposition 4.0.1. *The Christoffel Symbols are solely dependent on the metric coefficients.*

Proof. To calculate the Christoffel Symbols take inner product with ∂_1 on both sides in (4.0.1).

$$\left\langle \frac{\partial^2\psi}{\partial x_i \partial x_j}, \partial_1 \right\rangle = \Gamma^1_{ij} < \partial_1, \partial_1 > + \Gamma^2_{ij} < \partial_2, \partial_1 > \tag{4.0.3}$$

$$\left\langle \frac{\partial^2\psi}{\partial x_i x_j}, \partial_1 \right\rangle = \Gamma^1_{ij}E + \Gamma^2_{ij}F \tag{4.0.4}$$

Similarly, by taking inner product with ∂_2 in (4.0.1) we get:

$$\left\langle \frac{\partial^2 \psi}{\partial x_i \partial x_j}, \partial_2 \right\rangle = \Gamma^1_{ij} F + \Gamma^2_{ij} G \tag{4.0.5}$$

Putting $i = j = 1$ in (4.0.4) and (4.0.5) we get:

$$\left\langle \frac{\partial^2 \psi}{\partial x_1^2}, \partial_1 \right\rangle = \Gamma^1_{11} E + \Gamma^2_{11} F \tag{4.0.6}$$

$$\left\langle \frac{\partial^2 \psi}{\partial x_1^2}, \partial_2 \right\rangle = \Gamma^1_{11} F + \Gamma^2_{11} G \tag{4.0.7}$$

Moreover,

$$\frac{\partial^2 \psi}{\partial x_1^2} = \frac{1}{2} \frac{\partial}{\partial x_1} < \partial_1, \partial_1 >$$

and,

$$\frac{\partial^2 \psi}{\partial x_1^2} = \frac{\partial}{\partial x_1} < \partial_1, \partial_2 > - \left\langle \partial_1, \frac{\partial^2 \psi}{\partial x_1 \partial x_2} \right\rangle$$

Putting values in (4.0.6) and (4.0.7) we get:

$$\Gamma^1_{11} E + \Gamma^2_{11} F = \frac{1}{2} \frac{\partial}{\partial x_1} < \partial_1, \partial_1 > = \frac{1}{2} \frac{\partial E}{\partial x_1} \tag{4.0.8}$$

$$\Gamma^1_{11} F + \Gamma^2_{11} G = \frac{\partial}{\partial x_1} < \partial_1, \partial_2 > - \left\langle \partial_1, \frac{\partial^2 \psi}{\partial x_1 x_2} \right\rangle = \frac{\partial F}{\partial x_1} - \frac{1}{2} \frac{\partial E}{\partial x_1} \tag{4.0.9}$$

By similar calculations we get the following equations for $i = 1$ and $j = 2$:

$$\Gamma^1_{12} E + \Gamma^2_{12} F = \frac{1}{2} \frac{\partial E}{\partial x_2} \tag{4.0.10}$$

$$\Gamma^1_{12} F + \Gamma^2_{12} G = \frac{1}{2} \frac{\partial G}{\partial x_1} \tag{4.0.11}$$

Similarly for $i = j = 2$

$$\Gamma^1_{22} E + \Gamma^2_{22} F = \frac{\partial F}{\partial x_2} - \frac{1}{2} \frac{\partial G}{\partial x_1} \tag{4.0.12}$$

$$\Gamma^1_{22} F + \Gamma^2_{22} G = \frac{1}{2} \frac{\partial G}{\partial x_2} \tag{4.0.13}$$

The Christoffel Symbols are solutions of the above system of linear equations. The solution of the above system is solely dependent on the metric coefficients.

□

Let us do some examples in which we explicily compute the Christoffel Symbols to have a better understanding.

Example 12.

Consider the parametrization of the plane:

$$\psi(x_1,x_2) = (x_1,x_2,0).$$

then

$$\frac{\partial \psi}{\partial x_1} = \partial_1 = (1,0,0) \qquad \frac{\partial \psi}{\partial x_2} = \partial_2 = (0,1,0).$$

So, the metric coeffients are the following:

$$E \equiv 1 \qquad F \equiv 0 \qquad G \equiv 1.$$

Since E,F and G are constants so we have: Therefore we have:

$$\Gamma^1_{11} = \frac{1}{2E}\frac{\partial E}{\partial x_1} = 0 \quad ; \quad \Gamma^2_{11} = \frac{-1}{2G}\frac{\partial E}{\partial x_2} = 0 \quad ; \quad \Gamma^1_{12} = \frac{1}{2E}\frac{\partial E}{\partial x_2} = 0,$$

$$\Gamma^2_{12} = \frac{1}{2G}\frac{\partial G}{\partial x_1} = 0 \quad ; \quad \Gamma^1_{22} = \frac{-1}{2E}\frac{\partial G}{\partial x_1} = 0 \quad ; \quad \Gamma^2_{22} = \frac{1}{2G}\frac{\partial G}{\partial x_2} = 0.$$

Example 13.

From example (10), for a catenoid, we can calculate the following:

$$\frac{\partial E}{\partial x} = 2a^2 \cosh x \sinh x \qquad \frac{\partial E}{\partial y} = 0 \qquad \frac{\partial F}{\partial x} = \frac{\partial F}{\partial y} = 0$$

$$\frac{\partial G}{\partial x} = 2a^2 \cosh x \sinh x \qquad \frac{\partial G}{\partial y} = 0.$$

Since the parametrization is orthogonal i.e $F = 0$ so, the equations for the Christoffel symbols simplify to:

$$\begin{aligned}\Gamma_{11}^{1} &= \frac{1}{2E}\frac{\partial E}{\partial x} \\ &= \frac{2a^2\cosh x \sinh x}{2a^2\cosh^2 x} \\ \Gamma_{11}^{1} &= \frac{\sinh x}{\cosh x}.\end{aligned}$$

Since $\frac{\partial E}{\partial y} = 0$. So,

$$\Gamma_{11}^{2} = \frac{-1}{2G}\frac{\partial E}{\partial y} = 0$$

$$\Gamma_{12}^{1} = \frac{1}{2E}\frac{\partial E}{\partial y} = 0.$$

Now, $\frac{\partial G}{\partial x} = 2a^2\cosh x \sinh x$ Therefore:

$$\Gamma_{12}^{2} = \frac{1}{2G}\frac{\partial G}{\partial x} = \frac{1}{2a^2\cosh^2 x}(2a^2\cosh x\sinh x) = \frac{\sinh x}{\cosh x}$$

$$\Gamma_{22}^{1} = \frac{-1}{2E}\frac{\partial G}{\partial x} = \frac{-1}{2a^2\cosh^2 x}(2a^2\cosh x\sinh x) = \frac{-\sinh x}{\cosh x}.$$

Similarly $\frac{\partial G}{\partial y} = 0$. So,

$$\Gamma_{22}^{2} = \frac{1}{2G}\frac{\partial G}{\partial y} = 0.$$

Example 14.

From example (11), for a surface of revolution, we can calculate the folowing:

$$\frac{\partial E}{\partial t} = 2\alpha'\alpha'' + 2\beta'\beta'' \qquad \frac{\partial E}{\partial \theta} = 0 \qquad \frac{\partial F}{\partial t} = \frac{\partial F}{\partial \theta} = 0$$

$$\frac{\partial G}{\partial t} = 2\alpha\alpha' \qquad \frac{\partial G}{\partial \theta} = 0.$$

Now, since the parametrization is orthogonal i.e F=0 therefore the equations for the Christoffel Symbols reduce to:

$$\begin{aligned}\Gamma^1_{11} &= \frac{1}{2E}\frac{\partial E}{\partial t}\\ \Gamma^1_{11} &= \frac{(\alpha'\alpha''+\beta'\beta'')}{(\alpha')^2+(\beta')^2} \qquad\qquad using \quad \frac{\partial E}{\partial t} = 2\alpha'\alpha''+2\beta'\beta''\end{aligned}$$

Since $\frac{\partial E}{\partial \theta} = 0$. So,

$$\Gamma^2_{11} = \frac{-1}{2G}\frac{\partial E}{\partial \theta} = 0 \qquad \Gamma^1_{12} = \frac{1}{2E}\frac{\partial E}{\partial \theta} = 0.$$

Also, since $\frac{\partial G}{\partial t} = 2\alpha\alpha'$

$$\Gamma^2_{12} = \frac{1}{2G}\frac{\partial G}{\partial t} = \frac{1}{2(\alpha)^2}2\alpha\alpha' = \frac{\alpha'}{\alpha} \qquad \Gamma^1_{22} = \frac{-1}{2E}\frac{\partial G}{\partial t} = \frac{-1}{(\alpha')^2+(\beta')^2}\alpha\alpha'.$$

And $\frac{\partial G}{\partial \theta} = 0$. So,

$$\Gamma^2_{22} = \frac{1}{2G}\frac{\partial G}{\partial \theta} = 0.$$

Example 15.

Consider the following parametrization:

$$\psi(x,y) = (cosx, sinx, y).$$

Calculating the basis of T_pS:

$$\partial_1 = (-sinx, cosx, 0) \qquad \partial_2 = (0,0,1).$$

Therefore, we have:

$$E = \sin^2 x + \cos^2 x = 1 \qquad F \equiv 0 \qquad\qquad G \equiv 1.$$

Since E, F and G are constants so $\frac{\partial E}{\partial x} = \frac{\partial E}{\partial y} = \frac{\partial F}{\partial x} = \frac{\partial F}{\partial y} = \frac{\partial G}{\partial x} = \frac{\partial G}{\partial y} = 0$ and since $F \equiv 0$ so the equations of the Christoffel symbols reduce to the following:

$$\Gamma_{11}^1 = \frac{1}{2E}\frac{\partial E}{\partial x} = 0 \qquad \Gamma_{11}^2 = \frac{-1}{2G}\frac{\partial E}{\partial y} = 0 \qquad \Gamma_{12}^1 = \frac{1}{2E}\frac{\partial E}{\partial y} = 0$$

$$\Gamma_{12}^2 = \frac{1}{2G}\frac{\partial G}{\partial x} = 0 \qquad \Gamma_{22}^1 = \frac{-1}{2E}\frac{\partial G}{\partial x} = 0 \qquad \Gamma_{22}^2 = \frac{1}{2G}\frac{\partial G}{\partial y} = 0.$$

Example 16.

Consider the parametrization of helicoid:

$$\psi(x,y) = (ycosx, ysinx, ax) \qquad Where \quad a \neq 0.$$

$$\partial_1 = \frac{\partial \psi}{\partial x} = (-ysinx, ycosx, a) \qquad \partial_2 = \frac{\partial \psi}{\partial y} = (cosx, sinx, 0).$$

So, we have:

$$E = y^2 sin^2 x + y^2 cos^2 x + a = y^2 + a \quad F = -ysinxcosx + ycosxsinx = 0$$

$$G = cos^2 x + sin^2 x + 0 = 1.$$

Since $F \equiv 0$ so the simplified equations of the Christofel symbols are given by:

$$\Gamma_{11}^1 = \frac{1}{2E}\frac{\partial E}{\partial x} \qquad \Gamma_{11}^2 = \frac{-1}{2G}\frac{\partial E}{\partial y} \qquad \Gamma_{12}^1 = \frac{1}{2E}\frac{\partial E}{\partial y}$$
$$\Gamma_{12}^2 = \frac{1}{2G}\frac{\partial G}{\partial x} \qquad \Gamma_{22}^1 = \frac{-1}{2E}\frac{\partial G}{\partial x} \qquad \Gamma_{22}^2 = \frac{1}{2G}\frac{\partial G}{\partial y}.$$

Since F is constant so $\dfrac{\partial G}{\partial x} = 0$

$$\Gamma_{12}^2 = 0 \qquad \Gamma_{22}^1 = 0 \qquad \Gamma_{22}^2 = 0.$$

Moreover, since E does not depend on x so, $\frac{\partial E}{\partial x} = 0$ hence:

$$\Gamma_{11}^1 = 0.$$

Now,

$$E = y^2 + a^2 \implies \frac{\partial E}{\partial y} = 2y$$

So,

$$\Gamma_{11}^{2} = \frac{-1}{2G}\frac{\partial E}{\partial y} = \frac{-1}{2}(2y) = -y$$

$$\Gamma_{12}^{1} = \frac{1}{2E}\frac{\partial E}{\partial y} = \frac{y}{y^2 + a}.$$

4.1 Gauss's Theorema Egregium

Finally, we are going to prove our main result. We are going to express Gaussian Curvature in terms of Christoffel Symbols. The result will prove that the Gaussian Curvature is invariant under local isometry and therefore we can measure Gaussian Curvature while staying in the surface forgetting that there is an ambient space in which our surface is embedded. Recall that if $\psi(x_1, x_2)$ is a local parametrization then $\{\partial_1, \partial_2\ N\}$ form a basis for $\mathbb{R}^3$ where $\partial_1 = \frac{\partial \psi}{\partial x_1}$, $\partial_2 = \frac{\partial \psi}{\partial x_2}$ and $N = \frac{\partial_1 \times \partial_2}{|\partial_1 \times \partial_2|}$ Since ∂_1 ∂_2 and N form a basis for $\mathbb{R}^3$. For strategic purposes we are going to replace i by j and j by k in equation (4.0.1) where $i, j, k \in \{1, 2\}$. So, we can write:

$$\frac{\partial^2 \psi}{\partial x_j \partial x_k} = \Gamma_{jk}^1 \partial_1 + \Gamma_{jk}^2 \partial_2 + h_{jk} N. \tag{4.1.1}$$

Moreover let us replace j in equation (4.0.2) by i. So, we have:

$$\frac{\partial N}{\partial x_i} = a_{1i} \partial_1 + a_{2i} \partial_2. \tag{4.1.2}$$

Where h_{jk} are coefficients of the second fundamental form and a_{1i} and a_{2i} are the entries of the matrix form of the map dN_p.
Now, we want to calculate $\frac{\partial^3 \psi}{\partial x_i \partial x_j \partial x_k}$ and we know that ∂_1, ∂_2 and N form a basis for $\mathbb{R}^3$ so there must exist A_{ijk}^1, A_{ijk}^2 and B_{ijk} so that the following holds:

$$\frac{\partial^3 \psi}{\partial x_i \partial x j \partial x k} = A_{ijk}^1 \partial_1 + A_{ijk}^2 \partial_2 + B_{ijk} N.$$

Now to calculate A_{ijk}^1 , A_{ijk}^2 and B_{ijk} we differentiate the equation (4.1.1) with respect to x_i.

$$\begin{aligned} \frac{\partial^3 \psi}{\partial x_i \partial x_j \partial x_k} &= \partial_1 \frac{\partial}{\partial x_i}\left(\Gamma_{jk}^1\right) + \Gamma_{jk}^1 \frac{\partial^2 \psi}{\partial x_i \partial x_1} + \partial_2 \frac{\partial}{\partial x_i}\left(\Gamma_{jk}^2\right) \\ &+ \Gamma_{jk}^2 \frac{\partial^2 \psi}{\partial x_i \partial x_2} + h_{jk} \frac{\partial N}{\partial x_i} + N \frac{\partial (h_{jk})}{\partial x_i}. \end{aligned} \tag{4.1.3}$$

From (4.1.1) we can write:

$$\frac{\partial^2 \psi}{\partial x_i \partial x_1} = \Gamma^1_{i1}\partial_1 + \Gamma^2_{i1}\partial_2 + h_{i1}N$$

$$\frac{\partial^2 \psi}{\partial x_i x_2} = \Gamma^1_{i2}\partial_1 + \Gamma^2_{i2}\partial_2 + h_{i2}N.$$

From (4.1.2) we can have:

$$\frac{\partial N}{\partial x_i} = a_{1i}\partial_1 + a_{2i}\partial_2.$$

Put all these values in (4.1.3)

$$\begin{aligned}\frac{\partial^3 \psi}{\partial x_i \partial x_j \partial x_k} &= \partial_1 \frac{\partial(\Gamma^1_{jk})}{\partial x_i} + \Gamma^1_{jk}\left[\Gamma^1_{i1}\partial_1 + \Gamma^2_{i1}\partial_2 + h_{i1}N\right] + \partial_2 \frac{\partial(\Gamma^2_{jk})}{\partial x_i} + \Gamma^2_{jk}\left[\Gamma^1_{i2}\partial_1 + \Gamma^2_{i2}\partial_2 + h_{i2}N\right] \\ &\quad + h_{jk}\left[a_{1i}\partial_1 + a_{2i}\partial_2\right] + \frac{\partial h_{jk}}{\partial x_i}N\end{aligned}$$

$$\begin{aligned}\frac{\partial^3 \psi}{\partial x_i \partial x_j \partial x_k} &= \left[\frac{\partial \Gamma^1_{jk}}{\partial x_i} + \Gamma^1_{jk}\Gamma^1_{i1} + \Gamma^2_{jk}\Gamma^1_{i2} + h_{jk}a_{1i}\right]\partial_1 + \left[\frac{\partial \Gamma^2_{jk}}{\partial x_i} + \Gamma^1_{jk}\Gamma^2_{i1} + \Gamma^2_{jk}\Gamma^2_{i2} + h_{jk}a_{2i}\right]\partial_2 \\ &\quad + \left[\Gamma^1_{jk}h_{i1} + \Gamma^2_{jk}h_{i2} + \frac{\partial h_{jk}}{\partial x_i}\right]N.\end{aligned}$$

So we have:

$$\begin{aligned}A^1_{ijk} &= \frac{\partial \Gamma^1_{jk}}{\partial x_i} + \Gamma^1_{jk}\Gamma^1_{i1} + \Gamma^2_{jk}\Gamma^1_{i2} + h_{jk}a_{1i} \\ A^2_{ijk} &= \frac{\partial \Gamma^2_{jk}}{\partial x_i} + \Gamma^1_{jk}\Gamma^2_{i1} + \Gamma^2_{jk}\Gamma^2_{i2} + h_{jk}a_{2i} \\ B^2_{ijk} &= \Gamma^1_{jk}h_{i1} + \Gamma^2_{jk}h_{i2} + \frac{\partial h_{jk}}{\partial x_i}.\end{aligned}$$

Now, we know that the derivatives of the smooth functions commute so we know that $A^r_{ijk} = A^r_{jik}$

$$A^r_{ijk} = \frac{\partial \Gamma^r_{jk}}{\partial x_i} + \Gamma^1_{jk}\Gamma^r_{i1} + \Gamma^2_{jk}\Gamma^r_{i2} + h_{jk}a_{ri}.$$

where $r = 1 \quad or \quad 2$.

So we have:

$$A^r_{jik} = \frac{\partial \Gamma^r_{ik}}{\partial x_j} + \Gamma^1_{ik}\Gamma^r_{j1} + \Gamma^2_{ik}\Gamma^r_{j2} + h_{ik}a_{rj}.$$

Now we know that $A^r_{ijk} - A^r_{jik} = 0$ So,

$$\begin{aligned} \frac{\partial \Gamma^r_{jk}}{\partial x_i} - \frac{\partial \Gamma^r_{ik}}{\partial x_j} + \Gamma^1_{jk}\Gamma^r_{i1} - \Gamma^1_{ik}\Gamma^r_{j1} + \Gamma^2_{jk}\Gamma^r_{i2} - \Gamma^2_{ik}\Gamma^r_{j2} + h_{jk}a_{ri} - h_{ik}a_{rj} &= 0 \\ \implies \frac{\partial \Gamma^r_{jk}}{\partial x_i} - \frac{\partial \Gamma^r_{ik}}{\partial x_j} + \Gamma^1_{jk}\Gamma^r_{i1} - \Gamma^1_{ik}\Gamma^r_{j1} + \Gamma^2_{jk}\Gamma^r_{i2} - \Gamma^2_{ik}\Gamma^r_{j2} &= -(h_{jk}a_{ri} - h_{ik}a_{rj}). \end{aligned}$$

Put $i = r = 1$ and $j = k = 2$

$$\frac{\partial \Gamma^1_{22}}{\partial x_1} - \frac{\partial \Gamma^1_{12}}{\partial x_2} + \Gamma^1_{22}\Gamma^1_{11} - (\Gamma^1_{12})^2 + \Gamma^2_{22}\Gamma^1_{12} - \Gamma^2_{12}\Gamma^1_{22} = -(h_{22}a_{11} - h_{12}a_{12}) \tag{4.1.4}$$

$$\frac{\partial \Gamma^1_{22}}{\partial x_1} - \frac{\partial \Gamma^1_{12}}{\partial x_2} + \sum_{s=1}^{2} \Gamma^s_{22}\Gamma^1_{1s} - \Gamma^s_{12}\Gamma^1_{2s} = -(h_{22}a_{11} - h_{12}a_{12}). \tag{4.1.5}$$

Recall that, $h_{22} = g$ and $h_{12} = f$ and a_{ri} and a_{rj} are the entries of the matrix form of the map dN_p

From (3.0.3) we have:

$$a_{11} = \frac{eG - fF}{-(EG - F^2)} \quad ; \quad a_{12} = \frac{fG - gF}{-(EG - F^2)}$$

So, the equation (4.1.5) becomes:

$$\begin{aligned} \frac{\partial \Gamma^1_{22}}{\partial x_1} - \frac{\partial \Gamma^1_{12}}{\partial x_2} + \sum_{s=1}^{2} \Gamma^s_{22}\Gamma^1_{1s} - \Gamma^s_{12}\Gamma^1_{2s} &= -\left(\frac{-geG + gfF}{EG - F^2} - \frac{-f^2G + gfF}{EG - F^2}\right) && (4.1.6) \\ &= \frac{G(eg - f^2)}{EG - F^2} && (4.1.7) \end{aligned}$$

But recall that the Gaussian Curvature $\kappa = \frac{eg - f^2}{EG - F^2}$ So we have:

$$\frac{\partial \Gamma^1_{22}}{\partial x_1} - \frac{\partial \Gamma^1_{12}}{\partial x_2} + \sum_{s=1}^{2} \Gamma^s_{22}\Gamma^1_{1s} - \Gamma^s_{12}\Gamma^1_{2s} = GK$$

$$\kappa = \frac{1}{G}\left[\frac{\partial \Gamma^1_{22}}{\partial x_1} - \frac{\partial \Gamma^1_{12}}{\partial x_2} + \sum_{s=1}^{2} \Gamma^s_{22}\Gamma^1_{1s} - \Gamma^s_{12}\Gamma^1_{2s}\right].$$

which proves the Gauss's Theorem Egregium.

Lemma 4.1.1. *For any surface S with an orthogonal parametrization (i.e $F \equiv 0$) the Gaussian Curvature is the following:*

$$\kappa = \frac{-1}{2\sqrt{EG}}\left[\frac{\partial}{\partial x_1}\left(\frac{1}{\sqrt{EG}}\frac{\partial G}{\partial x_1}\right) + \frac{\partial}{\partial x_2}\left(\frac{1}{\sqrt{EG}}\frac{\partial E}{\partial x_2}\right)\right]$$

Proof. Recall that the equations of the Christoffel Symbols are following:

$$\begin{aligned}
E\Gamma^1_{11} + F\Gamma^2_{11} &= \frac{1}{2}\frac{\partial E}{\partial x_1} \\
F\Gamma^1_{11} + G\Gamma^2_{11} &= \frac{\partial F}{\partial x_1} - \frac{1}{2}\frac{\partial E}{\partial x_2} \\
E\Gamma^1_{12} + F\Gamma^2_{12} &= \frac{1}{2}\frac{\partial E}{\partial x_2} \\
F\Gamma^1_{12} + G\Gamma^2_{12} &= \frac{1}{2}\frac{\partial G}{\partial x_1} \\
E\Gamma^1_{22} + F\Gamma^2_{22} &= \frac{\partial F}{\partial x_2} - \frac{1}{2}\frac{\partial G}{\partial x_1} \\
F\Gamma^1_{22} + G\Gamma^2_{22} &= \frac{1}{2}\frac{\partial G}{\partial x_2}
\end{aligned}$$

Suppose we have an orthogonal parametrization that is, F=0 then the equations of the Christoffel Symbols reduce to the following:

$$\Gamma^1_{11} = \frac{1}{2E}\frac{\partial E}{\partial x_1} \quad ; \quad \Gamma^2_{11} = \frac{-1}{2G}\frac{\partial E}{\partial x_2}$$

$$\Gamma^1_{12} = \frac{1}{2E}\frac{\partial E}{\partial x_2} \quad ; \quad \Gamma^2_{12} = \frac{1}{2G}\frac{\partial G}{\partial x_1}$$

$$\Gamma^1_{22} = \frac{-1}{2E}\frac{\partial G}{\partial x_1} \quad ; \quad \Gamma^2_{22} = \frac{1}{2G}\frac{\partial G}{\partial x_2}$$

Now, Consider the equation of the Gaussian Curvature which we obtain by Theorem Egregium:

$$\kappa = \frac{1}{G}\left[\frac{\partial \Gamma^1_{22}}{\partial x_1} - \frac{\partial \Gamma^1_{12}}{\partial x_2} + \Gamma^1_{22}\Gamma^1_{11} - \Gamma^1_{12}\Gamma^1_{21} + \Gamma^2_{22}\Gamma^1_{12} - \Gamma^2_{12}\Gamma^1_{22}\right] \tag{4.1.8}$$

Since we have an orthogonal parametrization then we can put the expressions of the Christoffel Symbols in (4.1.8) So,

$$\begin{aligned}\kappa = \frac{1}{G}\Bigg[&\frac{\partial}{\partial x_1}\left(\frac{-1}{2E}\frac{\partial G}{\partial x_1}\right) - \frac{\partial}{\partial x_2}\left(\frac{1}{2E}\frac{\partial E}{\partial x_2}\right) - \frac{1}{2E}\frac{\partial G}{\partial x_1}\frac{1}{2E}\frac{\partial E}{\partial x_1} - \left(\frac{1}{2E}\frac{\partial E}{\partial x_2}\right)^2 \\ &+ \frac{1}{2G}\frac{\partial G}{\partial x_2}\frac{1}{2E}\frac{\partial E}{\partial x_2} + \frac{1}{2G}\frac{\partial G}{\partial x_1}\left(\frac{-1}{2E}\frac{\partial G}{\partial x_1}\right)\Bigg]\end{aligned}$$

$$\begin{aligned}&= \frac{-1}{2EG}\frac{\partial^2 G}{\partial x_1^2} + \frac{1}{2E^2G}\frac{\partial G}{\partial x_1}\frac{\partial E}{\partial x_1} - \frac{1}{4E^2G}\frac{\partial G}{\partial x_1}\frac{\partial E}{\partial x_1} - \frac{1}{4EG^2}\left(\frac{\partial G}{\partial x_1}\right)^2 \\ &\quad - \frac{1}{2EG}\frac{\partial^2 E}{\partial x_2^2} + \frac{1}{2E^2G}\left(\frac{\partial E}{\partial x_2}\right)^2 - \frac{1}{4E^2G}\left(\frac{\partial E}{\partial x_2}\right)^2 + \frac{1}{4EG^2}\frac{\partial G}{\partial x_2}\frac{\partial E}{\partial x_2}\end{aligned}$$

$$= \frac{-1}{2EG}\frac{\partial^2 G}{\partial x_1^2} + \frac{1}{4E^2G^2}\left(G\frac{\partial E}{\partial x_1} + E\frac{\partial G}{\partial x_1}\right)\frac{\partial G}{\partial x_1} - \frac{1}{2EG}\frac{\partial^2 E}{\partial x_2^2} + \frac{1}{4E^2G^2}\left(G\frac{\partial E}{\partial x_2} + E\frac{\partial G}{\partial x_2}\right)\frac{\partial E}{\partial x_2}$$

$$\begin{aligned}&= \frac{-1}{2\sqrt{EG}}\Bigg[\frac{1}{\sqrt{EG}}\frac{\partial^2 G}{\partial x_1^2} - \frac{1}{2(EG)^{3/2}}\left(G\frac{\partial E}{\partial x_1} + E\frac{\partial G}{\partial x_1}\right)\frac{\partial G}{\partial x_1} \\ &\qquad + \frac{1}{\sqrt{EG}}\frac{\partial^2 E}{\partial x_2^2} - \frac{1}{2(EG)^{3/2}}\left(G\frac{\partial E}{\partial x_2} + E\frac{\partial G}{\partial x_2}\right)\frac{\partial E}{\partial x_2}\Bigg]\end{aligned} \tag{4.1.9}$$

Now Consider the following:

$$\frac{\partial}{\partial x_1}\left(\frac{1}{\sqrt{EG}}\frac{\partial G}{\partial x_1}\right) = \frac{1}{\sqrt{EG}}\frac{\partial^2 G}{\partial x_1^2} - \frac{1}{2(EG)^{3/2}}\frac{\partial G}{\partial x_1}\left[E\frac{\partial G}{\partial x_1} + G\frac{\partial E}{\partial x_1}\right]$$

and,

$$\frac{\partial}{\partial x_2}\left(\frac{1}{\sqrt{EG}}\frac{\partial E}{\partial x_2}\right) = \frac{1}{\sqrt{EG}}\frac{\partial^2 E}{\partial x_2^2} - \frac{1}{2(EG)^{3/2}}\frac{\partial E}{\partial x_2}\left[E\frac{\partial G}{\partial x_2} + G\frac{\partial E}{\partial x_2}\right]$$

Putting these value in 4.1.9

$$\kappa = \frac{-1}{2\sqrt{EG}}\left[\frac{\partial}{\partial x_1}\left(\frac{1}{\sqrt{EG}}\frac{\partial G}{\partial x_1}\right) + \frac{\partial}{\partial x_2}\left(\frac{1}{\sqrt{EG}}\frac{\partial E}{\partial x_2}\right)\right].$$

which is the required result. □

Let us do an example to understand the application of the theorem.

Example 17.

Consider a regular surface S such that for a parametrization ψ the metric coefficients satisfy the following: $E(x_1,x_2) = G(x_1,x_2) = b$ and $F \equiv 0$. Using Theorem Egregium we will show that S is not locally isometric to sphere.
Since $F \equiv 0$. Therefore:

$$\Gamma_{11}^{1} = \frac{1}{2E}\frac{\partial E}{\partial x_1} = 0 \qquad \Gamma_{11}^{2} = \frac{-1}{2G}\frac{\partial E}{\partial x_2} = \frac{-1}{2x_2} \qquad \Gamma_{21}^{1} = \frac{1}{2E}\frac{\partial E}{\partial x_2} = \frac{1}{2x_2}$$

$$\Gamma_{21}^{2} = \frac{1}{2G}\frac{\partial G}{\partial x_1} = 0 \qquad \Gamma_{22}^{1} = \frac{-1}{2E}\frac{\partial G}{\partial x_1} = 0 \qquad \Gamma_{22}^{2} = \frac{1}{2G}\frac{\partial G}{\partial x_2} = \frac{1}{2b}$$

Let us apply Gauss's Theorem Egregium:

$$\kappa = \frac{1}{G}\left[\frac{\partial \Gamma_{22}^{1}}{\partial x_1} - \frac{\partial \Gamma_{12}^{1}}{\partial x_2} + \Sigma_{s=1}^{2}(\Gamma_{22}^{s}\Gamma_{1s}^{1} - \Gamma_{12}^{s}\Gamma_{2s}^{1})\right]$$

$$\frac{\partial \Gamma_{22}^{1}}{\partial x_1} = 0 \qquad ; \qquad \frac{\Gamma_{12}^{1}}{\partial x_2} = \frac{-1}{2x_2^2}$$

$$So, \qquad \kappa = \frac{1}{x_2}\left[0 + \frac{1}{2x_2^2} + (0 - \frac{1}{4x_2^2}) + \frac{1}{4x_2^2}\right]$$

$$\kappa = \frac{1}{2x_2^3}$$

which means that there is no open set in S which has constant curvature. But sphere has a constant curvature so it means that there is no subset of S which is isometric to a subset of sphere.

Chapter 5

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