Cycle Labelings and Eccentric Base Invariants of Graphs

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CIIT/FA14-BSM-014/LHR

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A Report Presented to

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In partial fulfillment

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By

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A Final Year Report is submitted to the Department of Mathematics as partial fulfillment of the requirement for the award of Degree of BS Mathematics.

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It is certified that Muhammad Azeem (**CIIT/FA14-BSM-014/LHR)** has carried out all the work related to this report under my supervision at the Department of Mathematics, COMSATS Institute of Information Technology, Lahore, and the work fulfills the requirement for award of BS degree.

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DEDICATION

To

My Family

ACKNOWLEDGEMENTS

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Abstract

A simple graph G admits an H-covering if each edge in $E(G)$ is a member of a subgraph of G isomorphic to H. An *H-magic labeling* λ of a graph G possessing an H-covering is a bijective function from the vertex-set and the edge-set of the graph G onto the set $\{1,2,\ldots,|V(G)|+|E(G)|\}$ with a property that for every subgraph H' of G isomorphic to H, the sum $wt_\lambda(H') = \sum\limits_{v\in V(H')} \lambda(v) + \sum\limits_{e\in E(H')} \lambda(e)$ is equal to μ. The sum $wt_\lambda(H')$ is termed as, the *H-weight*. A graph is called H-supermagic if it possesses an H-supermagic labeling. Let G be a graph and v be a vertex of G. The eccentricity of the vertex v is the maximum distance from v to any vertex. That is, e$(v) = max\{d(v,w) : w \in V(G)\}$. Firstly, we find the C_4-supermagic labelings for the disjoint union of non-isomorphic copies of ladder graph, which are the generalization of the result proved in [21].

Moreover, we find the new classes of 2-self centered graph namely uniform subdivided complete bipartite $K^k_{n,m}$ graph and line graph of uniform subdivided complete bipartite graph $L(K^k_{n,m})$. We also find their distance base topological indices of $K^k_{n,m}$ and $L(K^k_{n,m})$. Then we also find eccentricity of uniform subdivided wheel graph, line graph of uniform subdivided wheel graph and their distance based topological indices.

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Chapter 1

Introduction

1.1 Historical View

The graph theory started its journey from Königsberg city of Russia around 1736. The people were interested to find the solution of the famous bridge problem of Pregel river which was stated that whether there is a possible way to cross the seven bridges over the river exactly for once and finally reach at the starting point. Swiss mathematician **Leonard Euler** solved this mystery after a long time and gave birth to graph theory. The Königsberg bridge problem was an old puzzle concerning the possibility of finding a path over every one of seven bridges that span a forked river flowing past an island but without crossing any bridge twice. Euler argued that no such path exists. His proof involved only references to the physical arrangement of the bridges, but essentially he proved the first theorem in graph theory.

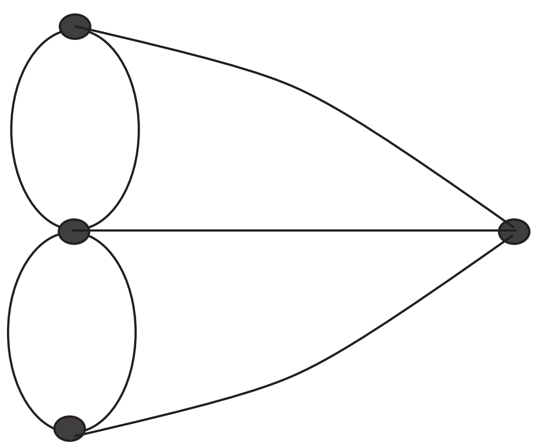

Figure 1.1: Graphical Representation of Königsberg

In the following section, we explain some notations, terminologies and basic definitions that will be helpful to understand our studies.

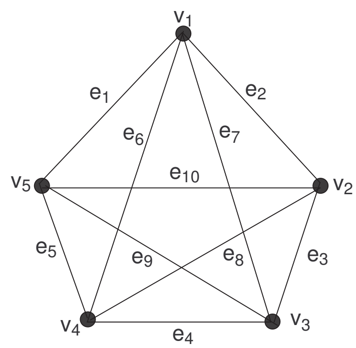

Figure 1.2: Graph with Vertex and Edge Set

1.2 Elementary Definitions

1.2.1 Graph

A graph $G(V,E)$ consists of a set of vertices or nodes $V(G)$ and set of edges or arcs $E(G)$. Here in fig: 1.2 the vertex set is $\{v_1, v_2, v_3, v_4, v_5\}$ the edge set is $\{e_1, e_2, e_3, \ldots, e_{10}\}$.

1.2.2 Order

For a graph G order of G is defined as total number of vertices of G. The graphs with order 1 is a *trivial* graphs. Here in fig: 1.2 the order is 5.

1.2.3 Size

The size of a graph G is the total number of edges in G. Here in fig: 1.2 size of graph is 10.

1.2.4 Simple Graph

A graph free of loops and multi-edge is a simple graph. Two vertices incident on an edge are called *adjacent vertices (neighbors)*.

1.2.5 Degree

The number of edges around some vertex is called degree of that vertex. The vertex is called even or odd if number of edges around that vertex are even or odd respectively. Here in fig: 1.2 the degree of each vertex is 4.

1.2.6 Subgraph

A graph H whose nodes and edges both are subsets of nodes and edges respectively, of parent graph G, is a subgraph.

1.2.7 Isomorphism

Two graphs having same number of elements (edges and nodes) and structural similarities are called isomorphic otherwise non-isomorphic. *Isomorphism* between graphs G and H is a bijection $f : V(G) \longrightarrow V(H)$ in such a way that x and y share an edge in G so is $f(x)$ and $f(y)$ in H.

1.2.8 Walk

A finite sequence of edges $\{v_1v_2, v_2v_3, \ldots, v_{m-1}v_m\}$, in a graph G is called a walk between v_0 and v_m in graph G. If $v_0 = v_m$, then walk is a *closed walk*. If v_1 and v_m are different vertices, then walk is called an *open walk*. Here in fig: 1.3 we denote walk $v_1v_2v_3v_4v_5v_6$ between v_1 and v_6 . A walk with no repeated edges is a *Trail*. A closed trail is a *circuit*. A cycle is a circuit in which no vertex except the first (which is also the last) appears more than once. Alternatively, a cycle can be defined as a closed path. An n-cycle is a cycle with n vertices. It is denoted as C_n. The number of edges in a walk is its *length*. In fig: 1.3 length is 5. A trail with all the vertices distinct is called a *path*.

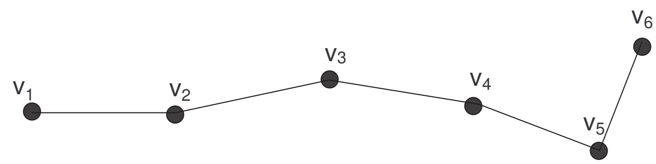

Figure 1.3: Walk, Length of Graph

1.2.9 Planar

If no two edges of graph cross each other when the graph is drawn on a plane, it is a Planar graph. For example, all trees, paths, $K_{2,n}$ and cycles are planar graphs. K_n for $n > 4$ are not planar graph. An example of planar graph is given below.

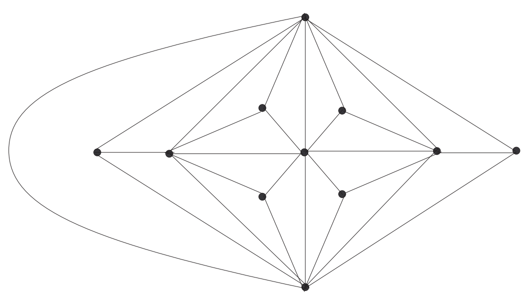

Figure 1.4: The Planar Goldner-Harary graph

1.2.10 Distance

The distance between two vertices in a graph is the number of edges in a shortest path (also called a graph geodesic) connecting them. This is also known as the geodesic distance.

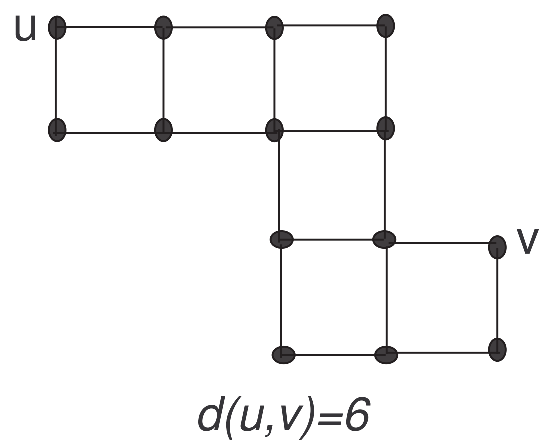

Figure 1.5: Distance of a Graph

1.2.11 Eccentricity of a Vertex

Let G be a graph and v be a vertex of G. The eccentricity of the vertex v is the maximum distance from v to any vertex. That is, e$(v) = max\{d(v,w) : w \in V(G)\}$.

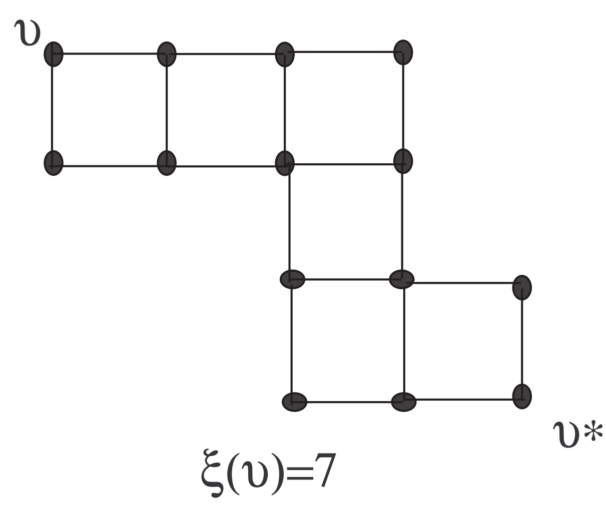

Figure 1.6: Eccentricity of Vertex

1.2.12 Radius of a Vertex

The radius of G is the minimum eccentricity among the vertices of G. Therefore, $radius(G) = min\{e(v) : v \in V(G)\}$.

1.2.13 Diameter of a Vertex

The diameter of G is the maximum eccentricity among the vertices of G. Thus, $diameter(G) = max\{e(v) : v \in V(G)\}$.

1.2.14 Center of a Graph

The center of G is the set of vertices of eccentricity equal to the radius. Hence, $center(G) = \{v \in V(G) : e(v) = radius(G)\}$.

1.3 Operations on Graphs

1.3.1 Cartesian Product

The Cartesian product $G \times H$ of graphs G and H is a graph such that the vertex set of is $G \times H$ the Cartesian product $V(G) times V(H)$; and any two vertices (u, u') and (v, v') are adjacent in $G \times H$ if and only if either $u = v$ and u' is adjacent to v' in H, or $u' = v'$ and u is adjacent to v in G. For example ladder graph L_m is a cartesian product of path graphs P_2 and P_m having order $2m$ and size $3m - 2$.

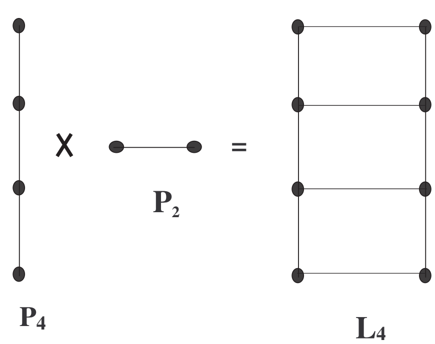

Figure 1.7: Cartesian Product Resulting Ladder Graph

1.3.2 Subdivision or Expansion

A subdivision is a graph operation resulting from the subdivision of edge or edges in G. For example, consider path graph P_2 with endpoints namely v_1, v_2 and single edge namely e_1. Now adding a vertex say v_3, splits e_1 into two edges e_1 and e_2, so P_2 become P_3, is an operation called subdivision.

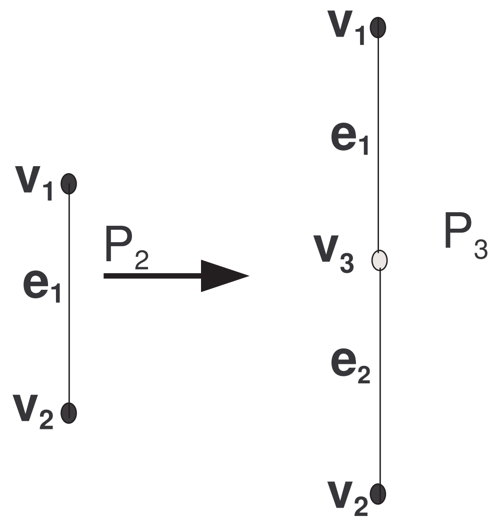

Figure 1.8: Subdivision of Path P_2

1.3.3 Line Graph

Starting with a graph G, we can associate a new graph with it, graph H, which we can also note as $L(G)$ and which we call the line graph of G. This kind of graph is obtained by creating a vertex per edge in G and linking two vertices in $H = L(G)$ if, and only if, the corresponding edges in G have an end in common.

1.3.4 Sum of Graphs

Sum of graph is an operation on graphs. Formally, let G_1 and G_2 be two disjoint graphs having order, n_1, n_2 and size e_1 and e_2 respectively. Then the sum of G_1 and G_2 is a graph $G = G_1 + G_2$ having order $n = n_1 + n_2$ and size $e = e_1 + e_2 + q$, where $q = |E(K_{n_1,n_2})| = n_1 n_2$. Here K_{n_1,n_2} is complete bipartite graph on n_1 and n_2 number of vertices. Generally, $G = G_1 + G_2 + G_3 + ... + G_k$ with order $n = n_1 + n_2 + n_3 + ... + n_k$ and size $e = e_1 + e_2 + e_3 + ... + e_k + |E(K_{n_1,n_2,n_3,...,n_k})|$.

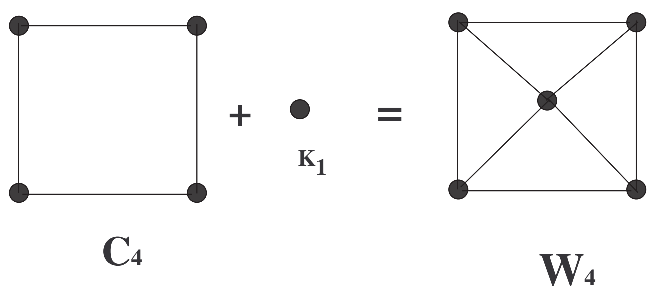

Figure 1.9: Sum of Two Graphs Resulting in Wheel Graph

1.4 Graph Families

1.4.1 Complete Graph

A complete graph is a simple graph in which each pair of distinct vertices are adjacent. A complete graph on n vertices is denoted by K_n, for $n \geq 2$.

1.4.2 Complete Bipartite Graph

A complete bipartite graph is a graph whose vertex-set can be split into set U and V in a way that for every edge of graph G has one end point in U and another end point in V. The number of vertices of U and V are m and n respectively. This graph is denoted as $B_{m,n}$. A complete bipartite graph has an additional property that is, every vertex of U is joined to every vertex of V.

1.4.3 Regular Graph

A regular graph is a graph in which all vertices have same degree. A k-regular graph is a regular graph whose common degree is k.

1.4.4 Connected and Disconnected

A graph G in which there exists at least a path between any two distinct vertices otherwise disconnected graph.

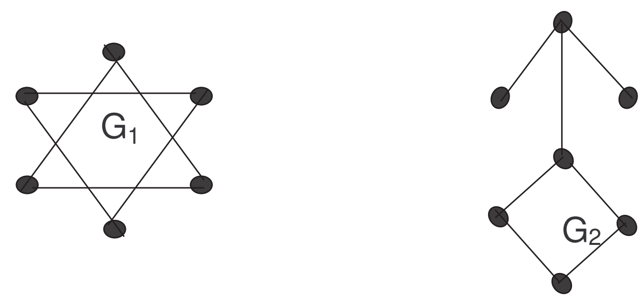

Figure 1.10: G_1 is Disconnected and G_2 is Connected

1.4.5 Ladder Graph

The ladder graph L_n is a planar undirected graph with $2n$ vertices and $n+2(n-1)$ edges. The ladder graph can be obtained as the Cartesian product of two path graphs, one of which has only one edge: $L_n = P_n \times P_2$. For example in fig: 1.7 Cartesian Product of P_4 and P_2 resulting in ladder graph L_4.

1.4.6 Wheel Graph

A wheel graph W_n is a graph of order $n+1$, containing a cycle of length n and a center vertex connected to all the vertices of the cycle. A wheel graph W_n is also called a sum of two graphs K_1 and C_n. For example in fig: 1.9 Sum of of C_4 and K_1 resulting in Wheel graph W_4.

1.5 Applications

Graph theory has proven useful in the design of integrated circuits (ICs) for computers and other electronic devices. These components, more often called chips, contain complex, layered microcircuits that can be represented as sets of points interconnected by lines or arcs. Using graph theory, engineers develop chips with maximum component density and minimum total interconnecting conductor length. This is important for optimizing processing speed and electrical efficiency. Graph theory has application in other fields of mathematics, science and technology. Graphs are used as graph data bases in data base designing [38]. Graph representation with nodes, edges and properties are used in graph database to depict and stock data.

Image Analysis [38] is the derivation of information from images. Image analysis is largely executed on digital image processing methods. Graph theoretic approach improves the image processing techniques. For image processing graph search algorithms in segmentation help to find edge boundaries. It helps calculate the picture alignment. Also the shortest path algorithms are applied to set the distance transforms of pixels and to find distance between the interior pixels. The concept of edge coloring and matching in graph theory are useful to plan the time table problems. The concept of vertex coloring is applied to color map coloring and allocation of frequencies in GSM mobile phone networks. Minimum vertex covers are useful to DNA sequencing and computer network security [40]. To determine the synchronization of some network, subgraphs and their complementary subgraphs can be employed. Such graphs especially are chains, cycles, bipartite and product graphs. These can be used to to represent network and then synchronization of the network can be characterized by few clear bounds that can be attained for the eigen ratio of the structural matrix of the underlying graph. BIM software is a fantastic tool that can help builders and designers to go through a process of decision making effectively as it is accumulation of the knowledge of mathematical values from the graph theory. With its help the conceptual graph replaces the actual layout of the building and then can be analyzed mathematically. Alternative solution for the designs can be easily assessed.

Chapter 2

Graph Labelings

Graph Labeling has its roots in late 1960s. Labeling is a function that allocates the elements of a graph to real numbers, usually positive integers. The elements of domain are vertices or edges or both. Vertices and edges are called graph elements. The labeling is called *total labeling* if it has all vertices and edges as its domain.

2.1 Edge Magic Total Labeling

An edge magic total labeling [6] is defined as a bijective mapping $\digamma$ from $V \cup E$ to $\{1,2,\ldots,|V \cup E|\}$ such that for any edge ab of graph G, $\digamma(a)+\digamma(b)+\digamma(ab)$ is a constant say ς is called weight of edge ab. A graph possessing edge magic total labeling is called edge magic total.

An edge magic total labeling [13] $\digamma : V \cup E \longrightarrow \{1,2,\ldots,v+e\}$ with an additional property that the vertex labels are 1 to $|v|$, is called super edge magic, where v represents the number of vertices and e represents the number of edges.
Gray [19] has also shown $C_4 \cup C_n$ to be super edge magic with condition that $n \geq 5$. Using the computer work he has shown that $C_5 \cup 2C_2$ are not supermagic. Super edge magicness of $St(1,m,n)$; $St(3,m,m+1)$; and $St(n,n+1,n+2)$ have been proven by Gao et al. [11].
Javed et al. [31] have contributed super edge magic labeling of subdivisions of $K_{1,4}$ and W-trees. Swaminath et al. [43], Hussain et al. [26] have contributed super edge magic total labeling of various classes of banana trees and theit forest. Ahmad et al. [3] have shown super edge magic total labeling of subdivisions and disjoint union of banana trees. Ahmad et al. [2] studied super edge magicness of graphs. Imran et al. [28] discussed labeling of uniform subdivision of wheel, politehn. Afzal et al. [17] have shown super edge magic labeling of volvox and pancycle graph.
Singgish [20] gave super edge magic total labelings for union of books $mB(n)$ for odd m; $m(P_2 \times P_n)$ for m and n odd; $r(P_m \times P_n)$ for odd r and $(m,n) \neq (2,2)$; $r(P_3 \times mP_n)$ for odd r; mP_n for $m \equiv 2(mod 4)$, $n \neq 2,3$; and mP_{4n} for $m \equiv 2(mod 4)$, $n \geq 1$ Fukuchi

et al. [20] describe a construction of super edge magic labelings of some families of trees with diameter 4. López et al. [20] introduced a new construction for edge magic labelings 2-regular graphs which allows loops and is related to the knight jump in the game of chess. They also studied the super edge magic properties of cycles with chords. Slamin [20] proved that the friendship graph consisting of n triangles is super edge magic iff n is 3, 4, 5, 7.
For further details, the reader is referred to the dynamic survey of graph labeling [20].

2.2 H-Magic Labeling

Inspired by edge magic labeling, Gutierréz and Lladó [5] in 2005 introduced the concept of H-magic labeling. They defined H-magic labeling as follows:
Let H and $G = (V,E)$ be finite simple graphs with the property that every edge of G belongs to at least one subgraph isomorphic to H. A bijection $f : V \cup E \to \{1,2,3,...,|V|+|E|\}$ is an H-magic labeling of G if $\exists$ a positive integer $m(f)$, called the magic sum, such that for any subgraph $H' = (V',E')$ of G isomorphic to H the sum $\sum_{v\in V'} f(v) + \sum_{e\in E} f(e)$ is equal to the magic sum, $m(f)$. A graph admitting H-magic labeling is called H-magic. A graph satisfying the property of assignment of first $|V|$ integers to set of vertices is called H-supermagic.
Every edge magic graph is H-magic as it is K_2-magic as well. Gutierréz and Lladó after defining H-magic labeling, formulated such labeling for complete graph K_n and complete bipartite graph $K_{m,n}$. In these cases H was considered star or path [37].
Lladó et al. [7] have concentrated on $cycle$-magicness of different connected graph. Their results include that all wheel graphs with n-vertices are C_3-magic for n at least 5 and odd; windmill graphs C_r^k are C_r-supermagic where $r \geq$ and $k \geq 2$. They have also shown C_{2r+k}-magicness of subdivision of wheel graphs $W_n(r,k)$ for any odd $n \neq \frac{2r}{k+1}$ and C_{2r+1}-supermagicness of $W_n(r,k)$ with $k = 1$. They have proven that the graph which is obtained by joining the endpoints of any number of internally disjoint path

of length $P \geq 2$ is C_{2p}-supermagic. Ngurah et al. [1] have constructed all possible C_h-magic labelings of fans and ladder graphs. Ngurah et al. in [1] discussed the C_3-supermagic labeling of fans, $K_{1,n}+K_1$; C_4-supermagic labeling of book graph, C_n-supermagic labeling of chain graphs in which identical blocks are isomorphic to C_n. Their results also include that cycles are P_t-supermagic. Roswitha et al. [35] have proven results on generalize Jahangir graph $J_{k,s}$ to be C_{s+2}-supermagic and $K_{2,n}$ to be C_4-supermagic and W_n to be C_3-supermagic for $n \geq 4$ and even. Roswitha et al. [34] proved that caterpillar, double stars, banana trees and firecrackers are star-supermagic. Rizvi et al. [42] studied cycle-supermagic labelings of disjoint union of isomorphic copies of odd wheel, fan and ladder graphs. They provided the C_3-supermagic labeling of disjoint union of non-isomorphic copies of fans and ladder graphs. Ali et al. [21] also discussed C_4-supermagic labelings of disjoint union of prism graphs. Also Rizvi et al. [42] found C_n-supermagic labeling for disjoint union of non-isomorphic copies of some families of graphs.

2.3 Applications of Graph Labelings

A particular type of graph labeling called radio labeling is used to assign radio channels to radio transmitters. Channels are assigned in a way to avoid disruption of close transmitters. In such labeling locations of the transmitters are being represented by vertices and the labels on the vertices corresponds to the channel or frequencies assigned to stations [9].

Some applications of such graphs in different fields are studied in [27]. In field of electronics ladder networks of resistors are economical to do conversions from digital to analog . One of such devices is weighted ladder and the other one is R/2R ladder. Better choice is R/2R. The reason is its superiority in accuracy and that it is easily manufactured. The function of both devices is conversions from digital voltage information in to analog. In telecommunication systems, a network can be defined by some structure specifically called graph topology. It can be a cycle, star, wheel, fan and moreover a friendship graph. The underlying communication network that is described by some graph consists of nodes and links. In such systems, magic labeling can help detecting route. A magic constant will be defined which is public to network, then router can move along a path that fits the magic constant. This type of routing is usually automatic. For security purposes, reduction of the size of packets and application of labeling scheme help transferring the small information packets quickly and securely as information will be in incomplete chunks. To find a solution of a problem where a system analyst must link its distinct communication ends with unique addresses, edge magic labeling helps creating unique addresses by distinct edge weights. The graph type complete, star, mesh cycle and bus are depicted to be used in computer network system.

Chapter 3

Topological Indices

3.1 Topological Indices

In the fields of chemical graph theory, molecular topology, and mathematical chemistry, the topological index also known as a connectivity index, is a type of a molecular descriptor that is calculated based on the molecular graph of a chemical compound [16]. Topological descriptors are derived from hydrogen-suppressed molecular graphs, in which the atoms are represented by vertices and the bonds by edges. The connections between the atoms can be described by various types of topological matrices (e.g., distance or adjacency matrices), which can be mathematically manipulated to derive a single number, usually known as graph invariant, graph-theoretical index or topological index [15], [14].

1. Degree based topological indices.
2. Distance based topological indices.
3. Eccentricity based topological indices.

3.1.1 Degree based topological indices

- Randic index (R_γ)
- Zagreb index (M)
- Atom-bond Connectivity index (ABC)
- Augmented Zegreb index (AZI)
- Geometric-arithmetic index (GA)
- Harmonic index (H)
- Sum-connectivity index (SCI)

Randic index

In 1975, Milan Randic introduced Randic index. It is also called connectivity index. It is denoted by $R_{-\frac{1}{2}}$(G). The Randic index defined as:

$$R_{-\frac{1}{2}}(G) = \sum_{\alpha\beta\in E(G)} \frac{1}{\sqrt{deg(\alpha)deg(\beta)}}$$

The general Randic index $R_\gamma(G)$ is defined as:

$$R_\gamma(G) = \sum_{\alpha\beta\in E(G)} (deg(\alpha)deg(\beta))^\gamma$$

Zagreb index

I. Gutman and Trinajstic introduced Zargeb index. Balaban et al. included M_1 and M_2 among topological indices and named them Zagreb group indices. The name Zagreb group index was soon abbrev-aviated to Zagreb index, and M_1 is called the first Zagreb index and M_2 is called the second Zagreb index.

First and second Zagreb index can be defined as:

$$M_1(G) = \sum_{\alpha\beta\in E(G)} [deg(\alpha) + deg(\beta)]$$

$$M_2(G) = \sum_{\alpha\beta\in E(G)} [deg(\alpha)deg(\beta)]$$

Atom-bond connectivity index

Estrada et al. introduced a well known connectivity index ABC, which is called Atomic-bound connectivity index. The ABC index can be defined as:

$$ABC(G) = \sum_{\alpha\beta\in E(G)} \sqrt{\frac{d_\alpha + d_\beta - 2}{d_\alpha d_\beta}}$$

In 2010, Ghorbani et al. introduced the fourth version of ABC index, which is defined as:

$$ABC_4(G) = \sum_{\alpha\beta \in E(G)} \sqrt{\frac{S_\alpha + S_\beta - 2}{S_\alpha S_\beta}}$$

Augmented Zagreb index

By the success of the ABC index, Furtula et al. put forward its modified version, that they named augmented Zagreb index. It is defined as:

$$AZI(G) = \sum_{\alpha\beta \in E(G)} (\frac{d_\alpha(G) d_\beta(G)}{d_\alpha(G) d_\beta(G) - 2})^3$$

Geometric-arithmetic index

Geometric-arithmetic GA index which is introduced by Vukicevic et al. and defined as:

$$GA(G) = \sum_{\alpha\beta \in E(G)} \frac{2(\sqrt{d_\alpha(G) d_\beta(G)})}{(d_\alpha(G) + d_\beta(G))}$$

Fifth version of GA index GA_5 is proposed by Graovac et al. in 2011 and it defined as:

$$GA_5(G) = \sum_{\alpha\beta \in E(G)} \frac{2(\sqrt{S_\alpha(G) S_\beta(G)})}{(S_\alpha(G) + S_\beta(G))}$$

Harmonic index

Zhang introduced H(G) quantity, and called it harmonic index. It is defined as follows:

$$H(G) = \sum_{\alpha\beta \in E(G)} \frac{2}{d_\alpha(G) + d_\beta(G)}$$

Sum-connectivity Index

B. Zhou and N. Trinajstic introduced the sum-connectivity index. It is defined as follows:

$$SCI(G) = \sum_{\alpha\beta \in E(G)} \frac{1}{d_\alpha(G) + d_\beta(G)}$$

3.1.2 Distance based topological indices

The distance between two vertices α and β of G is equal to the length of the shortest path connecting α and β vertices. It is defined by $d(\alpha, \beta)$.
The history of distance based topological indices goes back to 1947, when Harold Wiener calculated the boiling points of alkanes, he used the distances in the molecular graphs of alkanes. This research of topological index is called wiener index that became most popular molecular structure descriptors.
We describe some distance based topological indices as follows:

- Wiener index (W)
- Wiener polarity index (W_P)
- Hyper Wiener index (WW)
- Terminal Wiener index (TW)
- Electric Connectivity index (ξ)
- Schultz index (MTI)
- Gutman index (Gut)
- Harary index (H)
- Reciprocal Complementary Wiener index (RCW)
- Balaban index (J)
- Branching (B)
- First Order Connectivity (χ)

- Szeged index (Sz)
- Log RB index
- PI index

3.1.3 Eccentricity based topological indices

Let $deg_G(\alpha)$ be the degree of a vertex α in graph G. The length of the shortest path between α and β is the distance between two vertices α and β $d_G(\alpha,\beta)$. The eccentricity of a vertex α in a connected graph is defined as:

$$ecc_G(\alpha) = max\{d_G(\alpha,\beta) : \alpha \in V(G)\}$$

In 2013 One of the most famous topological index based on eccentricity is New Atomoc bound connectivity index ABC_5 is introduced by M. R. Farhani [33] is defined as:

$$ABC_5(G) = \sum_{uv \in E(G)} \sqrt{\frac{ecu(u)+ecu(v)-2}{ecu(u)ecu(v)}}$$

In 1997 Sharma et al. [44] introduced distance based topological index, the eccentric connectivity index, of the graph G, defined as :

$$\xi^c(G) = \sum_{\alpha \in V} ecc_G(\alpha) deg_G(\alpha)$$

which has been employed successfully for the development of numerous mathematical models for the prediction of biological activities of diverse nature.

Suppose $D_G(\alpha)$ be the sum of distance of all vertices in G from α, which can be written as;

$$D_G(\alpha) = \sum_{\alpha \in V} d_G(\alpha,\beta)$$

The eccentric distance is defined as:

$$\xi^d(G) = \sum_{\alpha \in V} ecc_G(\alpha) D_G(\alpha)$$

3.2 Literature Review

The topological index in chemistry was firstly studied by Wiener and he used its application in the study of paraffin boiling points [18]. The electron energy of a chemical graph depends on the Zagreb indices, defined by Gutman [23]. Topological indices found broad applications in the correlation and prediction of several molecular properties. Recent advances in this area are due to Diudea. Substantial work on the eccentricity distribution dates back to 1975, when the term *eccentric sequence* was introduced to denote the sequence that counts the frequency of each eccentricity value.
Baca et al. studied two degree based topological indices, the atom-bond connectivity ABC and the geometric arithmetic GA indices of fullerene networks and carbon nanotube networks [24]. Li and Ye [10] worked on second order Randic index of fluoranthene-type benzenoid systems, they gave the expression of R in terms of their inlet features. And they found the minimal and maximal value of the second order Randic index over the set of cata- catacondensed fluoranthene-type benzenoid systems, and characterized their corresponding graphs in [10]. Deng et al. worked on molecular graph and they proved that the first Zagreb index M_1 , which is equal to the sum of squares of the degrees of vertices, and the second Zagreb index M_2 is equal to the sum of the products of the degrees of pairs of adjacent vertices. They also worked on first and second Zagreb indices of four operations on graphs in [12]. Nadeem et al. utilized the generalized Randic index, Zagreb index, general sum-connectivity index, atom-bond connectivity ABC index and geometric-arithmetic GA index to guess the bioactivity of chemical compounds. They computed ABC_4 and GA indices of the line graph of tadpole, wheel and ladder graphs using the notion of subdivision in [25]. Baig et al. worked on different interconnection networks and derive analytical closed

results of the general Randic index R_γ(G) for $\gamma = 1$, $\frac{1}{2}$, -1 and $-\frac{1}{2}$ for dominating oxide network (DOX), dominating silicate network (DSL), and regular triangulene oxide network (RTOX). All of the studied interconnection networks were motivated by the molecular structure of a chemical compound SiO_4. They also computed the general first Zagreb, ABC, GA, ABC, ABC_4 and GA_5 indices and give closed formulae of these indices for these interconnection networks [8].
Vaidya et al. determined the bounds of Estrade index, First Zagreb index and some other indices that based on eccentricities of vertices and they find the formulas of these indices for cycloalkanes. Vaidya et al. determined the bounds of Estrade index, first zagreb index and some other indices that based on eccentricities of vertices and they find the formulas of these indices for cycloalkanes.
Ranjini and Lokesha [36] studied the eccentric connectivity index of the subdivision graph of the wheel graphs, tadpole graphs, and complete graphs. Morgan et al. [30] deduced the exact lower bound on by means of order and presented the sharpness of this bound.

3.3 Applications

The facility location problem is an example of the usefulness of eccentricity as a measure of centrality. When considering the placement of emergency facilities such as a hospital or fire station, assuming the map is modeled as a graph, the node with the lowest eccentricity value might be a good location for such a facility. In other situations, for example, for the placement of a shopping center, a related measure called closeness centrality, defined as a node's average distance to every other node $c(v) = \frac{1}{n}\sum_{w\in V} d(v,w)$, is more suitable. Generally speaking, the eccentricity is a relevant measure when some strict criterion (the firetruck has to be able to reach every location within 10 minutes) has to be met. An application of eccentricity as a measure on a larger scale is the network routing graph, where the eccentricity of a node says something about the worst-case response time between one machine and all other machines. The most well-known eccentricity-based measure is the diameter, which has been extensively investigated. Several measures related to the eccentricity and diameter have also been considered. Kang et al. study the effective radius, which they define as the 90th-percentile of all the shortest distances from a node. In a similar way, the effective diameter can be defined, which is shown to be decreasing over time for many large real-world graphs. Each of these measures is computed by using an approximation algorithm to determine the neighborhood of a node. To the best of our knowledge, there are no efficient techniques that have been specifically designed to determine the exact eccentricity distribution of a graph. Obviously, the naive approach, for example as suggested, is too time-consuming. Efficient approaches for solving the APSP problem (which then makes determining the eccentricity distribution a trivial task) have been developed, for example using matrix multiplication. Unfortunately, such approaches are still too complex in terms of time and memory requirements.

Chapter 4

Supermagic Labeling of Ladder Graph

4.1 C_4-Supermagic Labeling of Ladder Graph

In this paper we formulate C_4-supermagic labelings for the disjoint union of non-isomorphic copies of ladder are studied which are the generalization of the result proved in [21]. Rizvi et al. [21] found the C_4-supermagic labeling for graph $\alpha L_n \cong \beta L_{n-1}$.Hence we generalize the results by finding C_4-supermagic for the graph $\alpha L_n \cong \beta L_{n-i}$, $1 \leq i \leq n-2$

Theorem 4.1.1. *For any positive integer α, β and $n \geq 3$, the graph $G \cong \alpha L_n \cup \beta L_{n-i}$ is C_4-supermagic, where $1 \leq i \leq n-2$ and $n-i=m$.*

Proof.

Consider $\tau = |V(G)|$ and $\Omega = |E(G)|$, then $\tau = 2(\alpha n + \beta m)$, $\Omega = \alpha(3n-2) + \beta(3m-2)$. The vertex and edge sets of graph G are as follows:

$V(G) = \left\{a_\imath^\ell : 1 \leq \ell \leq \alpha, 1 \leq \imath \leq n\right\} \cup \left\{b_\imath^\ell : 1 \leq \ell \leq \alpha, 1 \leq \imath \leq n\right\}$
$\cup \{c_\imath^w : 1 \leq w \leq \beta, 1 \leq \imath \leq m\} \cup \{d_\imath^w : 1 \leq w \leq \beta, 1 \leq \imath \leq m\}$,
$E(G) = \left\{a_\imath^\ell\, b_\imath^\ell : 1 \leq \ell \leq \alpha,\ 1 \leq \imath \leq m\right\} \cup \left\{a_\imath^\ell\, b_\imath^\ell : 1 \leq \ell \leq \alpha,\ m+1 \leq \imath \leq n\right\}$
$\cup \{c_\imath^w\, d_\imath^w : 1 \leq w \leq \beta,\ 1 \leq \imath \leq m-1\} \cup \{c_\imath^w\, d_\imath^w : 1 \leq w \leq \beta, \imath = m\}$
$\cup \left\{b_\imath^\ell\, b_{\imath+1}^\ell : 1 \leq \ell \leq \alpha,\ 1 \leq \imath \leq m\right\} \cup \left\{b_\imath^\ell\, b_{\imath+1}^\ell : 1 \leq \ell \leq \alpha,\ m+1 \leq \imath \leq n-1\right\}$
$\cup \left\{d_\imath^w\, d_{\imath+1}^w : 1 \leq w \leq \beta,\ 1 \leq \imath \leq m-1\right\} \cup \left\{a_\imath^\ell\, a_{\imath+1}^\ell : 1 \leq \ell \leq \alpha,\ 1 \leq \imath \leq m-1\right\}$
$\cup \left\{a_\imath^\ell\, a_{\imath+1}^\ell : 1 \leq \ell \leq \alpha, m \leq \imath \leq n-1\right\} \cup \left\{c_\imath^w\, c_{\imath+1}^w : 1 \leq w \leq \beta,\ 1 \leq \imath \leq m-2\right\}$
$\cup \left\{c_\imath^w\, c_{\imath+1}^w : 1 \leq w \leq \beta,\ \imath = m-1\right\}$.

Define a total labeling
$\Psi : V(G) \cup E(G) \to \{1, 2, \ldots, 5\alpha n + 5\beta m) - 2\alpha - 2\beta)\}$, as follows:

We will consider $1 \leq \ell \leq \alpha,\ 1 \leq w \leq \beta$

For $1 \leq \imath \leq n$

$$\Psi(a_\imath^\ell) = \imath + n(\ell - 1)$$

For $1 \leq \imath \leq m-1$

$$\Psi(a_\imath^\ell\, a_{\imath+1}^\ell) = \tau + \alpha n + \beta m + \alpha(n-1) + \beta(m-1) + (\alpha+\beta)(\imath-1) + \ell$$

For $m \leq \imath \leq n-1$

$$\Psi(a_\imath^\ell\, a_{\imath+1}^\ell) = \tau + \alpha n + \beta m + \alpha n + \beta(m-1) + (\alpha+\beta)(m-2) + \alpha(\imath-m) + \ell$$

For $1 \leq \imath \leq n$

$$\Psi(b_\imath^\ell) = 2\alpha n + 2\beta m - n(\ell-1) - \imath + 1$$

For $1 \leq \imath \leq m$

$$\Psi(b_\imath^\ell\, b_{\imath+1}^\ell) = \tau + \alpha n + \beta m + 1 + (\imath-1)(\alpha+\beta) + (\ell-1)$$

For $m+1 \leq \imath \leq n-1$

$$\Psi(b_\imath^\ell\, b_{\imath+1}^\ell) = \tau + 2\beta m - \beta + \alpha(m+n) + 1 + \alpha(\imath-(m+1)) + (\ell-1)$$

For $1 \leq \imath \leq m$

$$\Psi(a_\imath^\ell\, b_\imath^\ell) = \tau + \alpha n + \beta m - (\imath-1)(\alpha+\beta) - (\ell-1)$$

For $m+1 \leq \imath \leq n$

$$\Psi(a_\imath^\ell\, b_\imath^\ell) = \tau + \alpha(n-m) + \beta - \alpha[\imath-(m+1)] - (\ell-1)$$

For $1 \leq \imath \leq m$

$$\Psi(c_\imath^w) = \alpha n + \imath + m(w-1)$$

For $\mathbf{1 \leq \iota \leq m-2}$

$$\Psi(c_\iota^w \, c_{\iota+1}^w) = \tau + \alpha n + \beta m + \alpha n + \beta(m-1) + (\alpha+\beta)(\iota-1) + w$$

For $\iota = \mathbf{m-1}$

$$\Psi(c_\iota^w \, c_{\iota+1}^w) = \tau + \Omega - (\beta-1) + (w-1)$$

For $\mathbf{1 \leq \iota \leq m}$

$$\Psi(d_\iota^w) = \alpha n + 2\beta m - (\iota-1) - m(w-1)$$

For $\mathbf{1 \leq \iota \leq m-1}$

$$\Psi(d_\iota^w \, d_{\iota+1}^w) = \tau + \alpha n + \beta m + (\alpha+1) + (\alpha+\beta)(\iota-1) + (w-1)$$

$$\Psi(c_\iota^w \, d_\iota^w) = \tau + \alpha n + \beta m - \alpha(\iota-1)(\alpha+\beta) - (w-1) - \alpha$$

For $\iota = \mathbf{m}$

$$\Psi(c_\iota^w \, d_\iota^w) = \tau + \beta - (w-1) - (\iota-m)$$

To show that Ψ is a C_4-supermagic labeling of G, let C_4^n, $1 \leq n \leq \alpha(n-1) + \beta(m-1)$, be the subcycles of $G \cong \alpha L_n \cup \beta L_{n-i}$. Since the weight of every cycle C_4^n is $4 + 17(\alpha n + \beta m) - 2(\alpha+\beta)$, therefore, G is a C_4-supermagic.

□

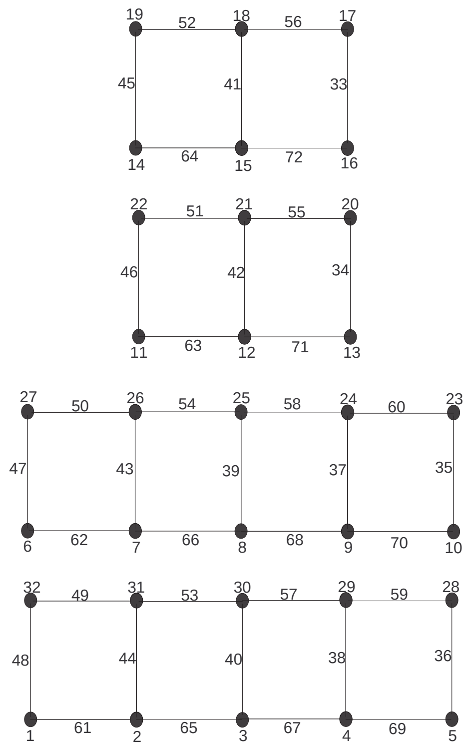

Figure 4.1: C_4-Supermagic Labeling of $2L_3 \cup 2L_5$

Chapter 5

Topological Indices of Some Graphs

5.1 Topological Indices of Wheel Related Graphs

Theorem 5.1.1. *Let W_n be a wheel graph having order $n+1$, then the*

$$ABC_5(W_n) = \begin{cases} 0 & if \quad n = 3 \\ \sqrt{2}n & if \quad n \geq 4 \end{cases}$$

Proof.

Tahe wheel graph W_n has order $n+1$ and size $2n$. Define vertex and edge set of W_n as follows:

$V(W_n) = \{\alpha_i : 1 \leq i \leq n\} \cup \{c\}$ and $E(W_n)\{\alpha_i\alpha_{i+1} : 1 \leq i \leq n-1\} \cup \{\alpha_i c : 1 \leq i \leq n\}$.

Now we define eccentricities of the vertex set of graph W_n.

$$\xi(\alpha_i) = 2$$

and

$$\xi(c) = 1$$

For $\mathbf{n = 3}$

In wheel graph having order $n+1$ and size become $2n$ so for $n = 3$ the size is 6.

The edge type in W_3 is only $(1,1)$ with count $2(3) = 6$ so by formula of new atomic bond connectivity index

$$ABC_5(G) = \sum_{uv \in E(G)} \sqrt{\frac{ecc(u) + ecc(v) - 2}{ecc(u)ecc(v)}}$$

$$ABC_5(W_n) = 2(3)\sqrt{\frac{1+1-2}{(1)(1)}} = 6(0) = 0$$

Now for $\mathbf{n \geq 4}$

The wheel graph has two types of edges $(2,2)$ and $(2,1)$ with respect to eccentricity with both have count n. Now, we find new atomic bond connectivity index

$$ABC_5(G) = \sum_{uv \in E(G)} \sqrt{\frac{ecc(u)+ecc(v)-2}{ecc(u)ecc(v)}}$$

$$ABC_5(W_n) = n\sqrt{\frac{2+2-2}{(2)(2)}} + n\sqrt{\frac{2+1-2}{(2)(1)}} = n\sqrt{\frac{2}{4}} + n\sqrt{\frac{1}{2}} = n\sqrt{\frac{1}{2}} + n\sqrt{\frac{1}{2}} = n\sqrt{2}$$

.

□

Theorem 5.1.2. *Let W_n^k be k-subdivided wheel graph, then for $n = 3$,*

$$ABC_5(W_n^k) = \Big(9+3(k-1)\Big)\sqrt{\frac{3k+1+\left\lceil\frac{k+1}{2}\right\rceil - \left\lceil\frac{k-1}{2}\right\rceil}{\Big(k+1+\left\lceil\frac{k+1}{2}\right\rceil\Big)\Big(2(k+1)-\left\lceil\frac{k-1}{2}\right\rceil\Big)}}$$

$$+3\sqrt{\frac{4k+1-\left\lceil\frac{k-1}{2}\right\rceil + \left\lceil\frac{k+1}{2}\right\rceil}{\Big(k+1+\left\lceil\frac{k+1}{2}\right\rceil\Big)\Big(2(k+1)-\left\lceil\frac{k-1}{2}\right\rceil\Big)}} + 3(k-1)$$

$$\times\sqrt{\frac{4k+2-2\left\lceil\frac{k-1}{2}\right\rceil + \sum_{q=1}^{\left\lceil\frac{k}{2}\right\rceil}(q-1) - \sum_{q=\left\lceil\frac{k}{2}\right\rceil+1}^{k}(q-k)}{\Big(2(k+1)-\left\lceil\frac{k-1}{2}\right\rceil + \sum_{q=1}^{\left\lceil\frac{k}{2}\right\rceil}(q-1)\Big)\Big(2(k+1)-\left\lceil\frac{k-1}{2}\right\rceil - \sum_{q=\left\lceil\frac{k}{2}\right\rceil+1}^{k}(q-k)\Big)}}$$

For $\mathbf{n = 4}$,

$$ABC_5(W_n) = \Big(12+4(k-1)\Big)\frac{\sqrt{4k+2}}{2(k+1)} + 4(k-1)\frac{\sqrt{4k+2-\sum_{q=\left\lceil\frac{k}{2}\right\rceil+1}^{k}(q-\left\lceil\frac{k}{2}\right\rceil)}}{2(k+1)-\sum_{q=\left\lceil\frac{k}{2}\right\rceil+1}^{k}(q-\left\lceil\frac{k}{2}\right\rceil)}$$

$$+4\sqrt{\frac{2k+1+\left\lceil\frac{k+1}{2}\right\rceil+\left\lceil\frac{k}{2}\right\rceil}{\left(2(k+1)-(k-\left\lceil\frac{k}{2}\right\rceil)\right)\left(k+1+\left\lceil\frac{k+1}{2}\right\rceil\right)}}$$

***For* $\mathbf{n=5}$,**

$$ABC_5(W_n)=10\sqrt{\frac{4k+2+\left\lceil\frac{k+1}{2}\right\rceil+\left\lceil\frac{k}{2}\right\rceil-\left\lfloor\frac{k}{2}\right\rfloor}{\left(2(k+1)+\left\lceil\frac{k+1}{2}\right\rceil\right)\left(2(k+1)+\left\lceil\frac{k}{2}\right\rceil-\left\lfloor\frac{k}{2}\right\rfloor\right)}}$$

$$+5(k-1)\sqrt{\frac{4k+2+2\left\lceil\frac{k}{2}\right\rceil-2\left\lfloor\frac{k}{2}\right\rfloor+\sum_{q=1}^{\left\lceil\frac{k}{2}\right\rceil}(q-1)-\sum_{q=\left\lceil\frac{k}{2}\right\rceil+1}^{k}(q-k)}{2(k+1)+\left\lceil\frac{k}{2}\right\rceil-\left\lfloor\frac{k}{2}\right\rfloor+\sum_{q=1}^{\left\lceil\frac{k}{2}\right\rceil}(q-1)}}$$

$$\times\frac{1}{\sqrt{2(k+1)+\left\lceil\frac{k}{2}\right\rceil-\left\lfloor\frac{k}{2}\right\rfloor-\sum_{q=\left\lceil\frac{k}{2}\right\rceil+1}^{k}(q-k)}}$$

$$+5\sqrt{\frac{4k+2+\left\lceil\frac{k+1}{2}\right\rceil+\left\lfloor\frac{2k-1}{4}\right\rfloor}{\left(2(k+1)+\left\lceil\frac{k+1}{2}\right\rceil\right)\left(2(k+1)+\left\lfloor\frac{2k-1}{4}\right\rfloor\right)}}$$

$$+5(k-1)\frac{\sqrt{4k+2+2\left\lfloor\frac{2k-1}{4}\right\rfloor}}{2(k+1)+\left\lfloor\frac{2k-1}{4}\right\rfloor}$$

$$+5\sqrt{\frac{3k+1+\left\lceil\frac{k+1}{2}\right\rceil+\left\lfloor\frac{2k-1}{4}\right\rfloor}{\left(2(k+1)+\left\lfloor\frac{2k-1}{4}\right\rfloor\right)\left(k+1+\left\lceil\frac{k+1}{2}\right\rceil\right)}}$$

***For* $\mathbf{n\geq 6}$,**

$$ABC_5(W_n)=2n\sqrt{\frac{4k+2+\left\lceil\frac{k+1}{2}\right\rceil+2\left\lceil\frac{k}{2}\right\rceil-\left\lfloor\frac{k-1}{2}\right\rfloor}{\left(2(k+1)+\left\lceil\frac{k+1}{2}\right\rceil\right)\left(2(k+1)+\left\lceil\frac{k}{2}\right\rceil-\left\lfloor\frac{k-1}{2}\right\rfloor\right)}}$$

$$+n(k-1)\sqrt{\frac{4k+2+4\left\lceil\frac{k}{2}\right\rceil-2\left\lfloor\frac{k-1}{2}\right\rfloor+\sum_{q=1}^{\left\lceil\frac{k}{2}\right\rceil}(q-1)-\sum_{q=\left\lceil\frac{k}{2}\right\rceil+1}^{k}(q-k)}{2(k+1)+2\left\lceil\frac{k}{2}\right\rceil-\left\lfloor\frac{k-1}{2}\right\rfloor+\sum_{q=1}^{\left\lceil\frac{k}{2}\right\rceil}(q-1)}}$$

$$\times\frac{1}{\sqrt{2(k+1)+2\left\lceil\frac{k}{2}\right\rceil-\left\lfloor\frac{k-1}{2}\right\rfloor-\sum_{q=\left\lceil\frac{k}{2}\right\rceil+1}^{k}(q-k)}}$$

$$+n\sqrt{\frac{4k+2+\left\lceil\frac{k+1}{2}\right\rceil+\left\lfloor\frac{2k-1}{4}\right\rfloor}{\left(2(k+1)+\left\lceil\frac{k+1}{2}\right\rceil\right)\left(2(k+1)+\left\lfloor\frac{2k-1}{4}\right\rfloor\right)}}$$

$$+n(k-1)\frac{\sqrt{4k+2+2\left\lfloor\frac{2k-1}{4}\right\rfloor-2\sum_{q=1}^{\left\lceil\frac{k}{2}\right\rceil}(q-1)}}{2(k+1)+\left\lfloor\frac{2k-1}{4}\right\rfloor-\sum_{q=1}^{\left\lceil\frac{k}{2}\right\rceil}(q-1)}$$

$$+n\sqrt{\frac{3k+1+\left\lceil\frac{k+1}{2}\right\rceil+\left\lfloor\frac{2k-1}{4}\right\rfloor}{\left(2(k+1)+\left\lceil\frac{k+1}{2}\right\rceil\right)\left(2(k+1)+\left\lfloor\frac{2k-1}{4}\right\rfloor-(k-1)\right)}}$$

Proof.

Let W_n^k be k-subdivided wheel graph with order $(2k+1)n+1$ and size $2n(k+1)$.
Define vertex set as follow:
$\{\alpha_i : 1 \leq i \leq n\} \cup \{c\} \cup \{\beta_p^q : 1 \leq p \leq n,\ \ 1 \leq q \leq k\} \cup \{\gamma_p^q : 1 \leq p \leq n,\ \ 1 \leq q \leq k\}$,
and edge set as $\{\alpha_i\beta_p^q : 1 \leq i \leq n,\ 1 \leq p \leq n,\ q=1\} \cup \{\beta_p^q\beta_p^{q+1} : 1 \leq p \leq n,\ 1 \leq q \leq k-1,\ k \neq 1\} \cup \{\alpha_i\gamma_p^q : 1 \leq i \leq n,\ 1 \leq p \leq n,\ q=1\} \cup \{\gamma_p^q\gamma_p^{q+1} : 1 \leq p \leq n,\ 1 \leq q \leq k-1,\ k \neq 1\} \cup \{\beta_p^q\alpha_i : 1 \leq i \leq n,\ 1 \leq p \leq n,\ q=k\} \cup \{\gamma_p^q c : 1 \leq p \leq n,\ q=k\}$.
Now we define eccentricities of vertex set of W_n^k as below:

$$\xi(c) = k+1+\left\lceil\frac{k+1}{2}\right\rceil$$

Throughout $1 \leq i \leq n$, $1 \leq p \leq n$

For $\mathbf{n=3}$

$$\xi(\alpha_i) = k+1+\left\lceil\frac{k+1}{2}\right\rceil$$

$$\xi(\beta_p^q) = \xi(\gamma_p^q) = \begin{cases} 2(k+1)-\left\lceil\frac{k-1}{2}\right\rceil+(q-1) & \text{if } 1 \leq q \leq \left\lceil\frac{k}{2}\right\rceil \\ 2(k+1)-\left\lceil\frac{k-1}{2}\right\rceil-(q-k) & \text{if } q = \left\lceil\frac{k}{2}\right\rceil+1,\ldots,k \end{cases}$$

For $\mathbf{n=4}$, $\mathbf{1 \leq q \leq k}$

$$\xi(\alpha_i) = \xi(\beta_p^q) = 2(k+1)$$

$$\xi(\gamma_p^q) = \begin{cases} 2(k+1) & \text{if } 1 \leq q \leq \left\lceil\frac{k}{2}\right\rceil \\ 2(k+1)-\left(q-\left\lceil\frac{k}{2}\right\rceil\right) & \text{if } q = \left\lceil\frac{k}{2}\right\rceil+1,...,k \end{cases}$$

For $\mathbf{n=5}$

$$\xi(\beta_p^q) = \begin{cases} 2(k+1)+\left\lceil\frac{k}{2}\right\rceil-\left\lfloor\frac{k-1}{2}\right\rfloor+(q-1) & \text{if } 1 \leq q \leq \left\lceil\frac{k}{2}\right\rceil \\ 2(k+1)+\left\lceil\frac{k}{2}\right\rceil-\left\lfloor\frac{k-1}{2}\right\rfloor-(q-k) & \text{if } q = \left\lceil\frac{k}{2}\right\rceil+1,...,k \end{cases}$$

For $\mathbf{n \geq 5}$

$$\xi(\alpha_i) = 2(k+1)+\left\lceil\frac{k+1}{2}\right\rceil$$

$\mathbf{1 \leq q \leq k}$

$$\xi(\gamma_p^q) = 2(k+1)+\left\lfloor\frac{2k-1}{4}\right\rfloor-(q-1)$$

For $\mathbf{n \geq 6}$

$$\xi(\beta_p^q) = \begin{cases} 2(k+1)+2\left\lceil\frac{k}{2}\right\rceil-\left\lfloor\frac{k-1}{2}\right\rfloor+(q-1) & \text{if } 1 \leq q \leq \left\lceil\frac{k}{2}\right\rceil \\ 2(k+1)+2\left\lceil\frac{k}{2}\right\rceil-\left\lfloor\frac{k-1}{2}\right\rfloor-(q-k) & \text{if } q = \left\lceil\frac{k}{2}\right\rceil+1,...,k \end{cases}$$

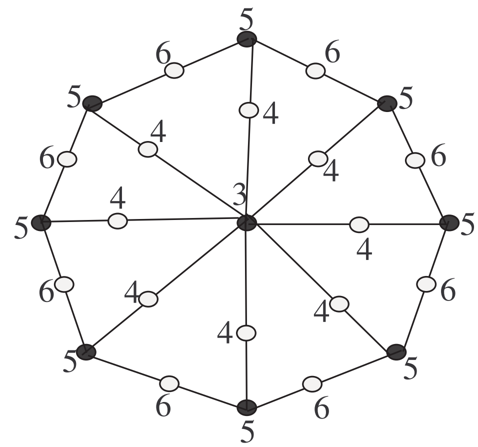

Figure 5.1: Uniformly Subdivided Wheel Graph W_8^1

Here six types of edges in k-subdivided wheel graph are $\Big(\xi(\alpha_i),\xi(\beta_p^q)\Big)$, $q=1$; $\Big(\xi(\beta_p^q),\xi(\beta_p^q)\Big)$, $1\leq q\leq k$; $\Big(\xi(\beta_p^q),\xi(\alpha_i)\Big)$, $q=k$; $\Big(\xi(\alpha_i),\xi(\gamma_p^q)\Big)$, $q=1$; $\Big(\xi(\gamma_p^q),\xi(\gamma_p^q)\Big)$, $1\leq q\leq k$; $\Big(\xi(\gamma_p^q),c\Big)$, $q=k$ having count n; $n(k-1)$; n; n; $n(k-1)$; n. Then according to formula of new atomic bond connectivity index

$$ABC_5(G)=\sum_{uv\in E(G)}\sqrt{\frac{ecc(u)+ecc(v)-2}{ecc(u)ecc(v)}}$$

Firstly **for** $\mathbf{n=3}$,

$$ABC_5(W_n^k)=3\sqrt{\frac{k+1+\left\lceil\frac{k+1}{2}\right\rceil+2(k+1)-\left\lceil\frac{k-1}{2}\right\rceil-2}{\left(k+1+\left\lceil\frac{k+1}{2}\right\rceil\right)\left(2(k+1)-\left\lceil\frac{k-1}{2}\right\rceil\right)}}+3(k-1)$$

$$\sqrt{\frac{2(k+1)-\left\lceil\frac{k-1}{2}\right\rceil+2(k+1)-\left\lceil\frac{k}{2}\right\rceil+\sum_{q=1}^{\left\lceil\frac{k}{2}\right\rceil}(q-1)-\sum_{q=\left\lceil\frac{k}{2}\right\rceil+1}^{k}(q-k)-2}{\left(2(k+1)-\left\lceil\frac{k-1}{2}\right\rceil+\sum_{q=1}^{\left\lceil\frac{k}{2}\right\rceil}(q-1)\right)\left(2(k+1)-\left\lceil\frac{k-1}{2}\right\rceil-\sum_{q=\left\lceil\frac{k}{2}\right\rceil+1}^{k}(q-k)\right)}}$$

$$+3\sqrt{\frac{2(k+1)-\left\lceil\frac{k-1}{2}\right\rceil+k+1+\left\lceil\frac{k+1}{2}\right\rceil-2}{\left(2(k+1)-\left\lceil\frac{k-1}{2}\right\rceil\right)\left(k+1+\left\lceil\frac{k+1}{2}\right\rceil\right)}}$$

$$+3\sqrt{\frac{k+1+\left\lceil\frac{k+1}{2}\right\rceil+2(k+1)-\left\lceil\frac{k-1}{2}\right\rceil-2}{\left(k+1+\left\lceil\frac{k+1}{2}\right\rceil\right)\left(2(k+1)-\left\lceil\frac{k-1}{2}\right\rceil\right)}}$$

$$+3(k-1)\sqrt{\frac{2\times 2(k+1)-2\left\lceil\frac{k-1}{2}\right\rceil-2}{\left(2(k+1)-\left\lceil\frac{k-1}{2}\right\rceil\right)^2}}$$

$$+3\sqrt{\frac{2(k+1)-\left\lceil\frac{k-1}{2}\right\rceil+k+1+\left\lceil\frac{k+1}{2}\right\rceil-2}{\left(k+1+\left\lceil\frac{k+1}{2}\right\rceil\right)\left(2(k+1)-\left\lceil\frac{k-1}{2}\right\rceil\right)}}$$

$$ABC_5(W_n^k)=\left(9+3(k-1)\right)\sqrt{\frac{3k+1+\left\lceil\frac{k+1}{2}\right\rceil-\left\lceil\frac{k-1}{2}\right\rceil}{\left(k+1+\left\lceil\frac{k+1}{2}\right\rceil\right)\left(2(k+1)-\left\lceil\frac{k-1}{2}\right\rceil\right)}}$$

$$+3\sqrt{\frac{4k+1-\left\lceil\frac{k-1}{2}\right\rceil+\left\lceil\frac{k+1}{2}\right\rceil}{\left(k+1+\left\lceil\frac{k+1}{2}\right\rceil\right)\left(2(k+1)-\left\lceil\frac{k-1}{2}\right\rceil\right)}}+3(k-1)$$

$$\sqrt{\frac{4k+2-2\left\lceil\frac{k-1}{2}\right\rceil+\sum_{q=1}^{\left\lceil\frac{k}{2}\right\rceil}(q-1)-\sum_{q=\left\lceil\frac{k}{2}\right\rceil+1}^{k}(q-k)}{\left(2(k+1)-\left\lceil\frac{k-1}{2}\right\rceil+\sum_{q=1}^{\left\lceil\frac{k}{2}\right\rceil}(q-1)\right)\left(2(k+1)-\left\lceil\frac{k-1}{2}\right\rceil-\sum_{q=\left\lceil\frac{k}{2}\right\rceil+1}^{k}(q-k)\right)}}$$

for $\mathbf{n=4}$

$$ABC_5(W_n^k)=4\sqrt{\frac{2(k+1)+2(k+1)-2}{\left(2(k+1)\right)^2}}+4(k-1)\sqrt{\frac{2(k+1)+2(k+1)-2}{\left(2(k+1)\right)^2}}$$

$$+4\sqrt{\frac{2(k+1)+2(k+1)-2}{\Big(2(k+1)\Big)^2}}+4\sqrt{\frac{2(k+1)+2(k+1)-2}{\Big(2(k+1)\Big)^2}}$$

$$+4(k-1)\sqrt{\frac{2\times 2(k+1)-2\sum_{q=\left\lceil\frac{k}{2}\right\rceil+1}^{k}(q-\left\lceil\frac{k}{2}\right\rceil)-2}{\Big(2(k+1)-2\sum_{q=\left\lceil\frac{k}{2}\right\rceil+1}^{k}(q-\left\lceil\frac{k}{2}\right\rceil)\Big)^2}}$$

$$+4\sqrt{\frac{2(k+1)-(k-\left\lceil\frac{k}{2}\right\rceil)+k+1+\left\lceil\frac{k+1}{2}\right\rceil-2}{\Big(2(k+1)-(k-\left\lceil\frac{k}{2}\right\rceil)\Big)\Big(k+1+\left\lceil\frac{k+1}{2}\right\rceil\Big)}}$$

$$ABC_5(W_n^k)=\Big(12+4(k-1)\Big)\frac{\sqrt{4k+2}}{2(k+1)}+4(k-1)\frac{\sqrt{4k+2-\sum_{q=\left\lceil\frac{k}{2}\right\rceil+1}^{k}(q-\left\lceil\frac{k}{2}\right\rceil)}}{2(k+1)-\sum_{q=\left\lceil\frac{k}{2}\right\rceil+1}^{k}(q-\left\lceil\frac{k}{2}\right\rceil)}$$

$$+4\sqrt{\frac{2k+1+\left\lceil\frac{k+1}{2}\right\rceil+\left\lceil\frac{k}{2}\right\rceil}{\Big(2(k+1)-(k-\left\lceil\frac{k}{2}\right\rceil)\Big)\Big(k+1+\left\lceil\frac{k+1}{2}\right\rceil\Big)}}$$

for $\mathbf{n=5}$,

$$ABC_5(W_n^k)=5\sqrt{\frac{2(k+1)+\left\lceil\frac{k+1}{2}\right\rceil+2(k+1)+\left\lceil\frac{k}{2}\right\rceil-\left\lfloor\frac{k-1}{2}\right\rfloor-2}{\Big(2(k+1)+\left\lceil\frac{k+1}{2}\right\rceil\Big)\Big(2(k+1)+\left\lceil\frac{k}{2}\right\rceil-\left\lfloor\frac{k-1}{2}\right\rfloor\Big)}}+5(k-1)$$

$$\times\sqrt{\frac{2\times 2(k+1)+2\left\lceil\frac{k}{2}\right\rceil-2\left\lfloor\frac{k-1}{2}\right\rfloor+\sum_{q=1}^{\left\lceil\frac{k}{2}\right\rceil}(q-1)-\sum_{q=\left\lceil\frac{k}{2}\right\rceil+1}^{k}(q-k)-2}{2(k+1)+\left\lceil\frac{k}{2}\right\rceil-\left\lfloor\frac{k-1}{2}\right\rfloor+\sum_{q=1}^{\left\lceil\frac{k}{2}\right\rceil}(q-1)}}$$

$$\times \frac{1}{\sqrt{2(k+1)+\left\lceil \frac{k}{2}\right\rceil - \left\lfloor \frac{k-1}{2}\right\rfloor - \sum_{q=\left\lceil \frac{k}{2}\right\rceil+1}^{k}(q-k)}}$$

$$+5\sqrt{\frac{2(k+1)+\left\lceil \frac{k}{2}\right\rceil - \left\lfloor \frac{k-1}{2}\right\rfloor + 2(k+1)+\left\lceil \frac{k+1}{2}\right\rceil - 2}{\left(2(k+1)+\left\lceil \frac{k}{2}\right\rceil - \left\lfloor \frac{k-1}{2}\right\rfloor\right)\left(2(k+1)+\left\lceil \frac{k+1}{2}\right\rceil\right)}}$$

$$+5\sqrt{\frac{2(k+1)+\left\lceil \frac{k+1}{2}\right\rceil + 2(k+1)+\left\lfloor \frac{2k-1}{4}\right\rfloor - 2}{\left(2(k+1)+\left\lceil \frac{k+1}{2}\right\rceil\right)\left(2(k+1)+\left\lfloor \frac{2k-1}{4}\right\rfloor\right)}}$$

$$+5(k-1)\sqrt{\frac{2\times 2(k+1)+2\left\lfloor \frac{2k-1}{4}\right\rfloor - 2}{\left(2(k+1)+2\left\lfloor \frac{2k-1}{4}\right\rfloor\right)^2}}$$

$$+5\sqrt{\frac{2(k+1)+2\left\lfloor \frac{2k-1}{4}\right\rfloor + k+1+\left\lceil \frac{k+1}{2}\right\rceil - 2}{\left(2(k+1)+2\left\lfloor \frac{2k-1}{4}\right\rfloor\right)\left(k+1+\left\lceil \frac{k+1}{2}\right\rceil\right)}}$$

$$ABC_5(W_n^k) = 10\sqrt{\frac{4k+2+\left\lceil \frac{k+1}{2}\right\rceil + \left\lceil \frac{k}{2}\right\rceil - \left\lfloor \frac{k}{2}\right\rfloor}{\left(2(k+1)+\left\lceil \frac{k+1}{2}\right\rceil\right)\left(2(k+1)+\left\lceil \frac{k}{2}\right\rceil - \left\lfloor \frac{k}{2}\right\rfloor\right)}} + 5(k-1)$$

$$\times \sqrt{\frac{4k+2+2\left\lceil \frac{k}{2}\right\rceil - 2\left\lfloor \frac{k-1}{2}\right\rfloor + \sum_{q=1}^{\left\lceil \frac{k}{2}\right\rceil}(q-1) - \sum_{q=\left\lceil \frac{k}{2}\right\rceil+1}^{k}(q-k)}{\left(2(k+1)+\left\lceil \frac{k}{2}\right\rceil - \left\lfloor \frac{k-1}{2}\right\rfloor + \sum_{q=1}^{\left\lceil \frac{k}{2}\right\rceil}(q-1)\right)}}$$

$$\times \frac{1}{\sqrt{\left(2(k+1)+\left\lceil \frac{k}{2}\right\rceil - \left\lfloor \frac{k-1}{2}\right\rfloor - \sum_{q=\left\lceil \frac{k}{2}\right\rceil+1}^{k}(q-k)\right)}}$$

$$+5\sqrt{\frac{4k+2+\left\lceil\frac{k+1}{2}\right\rceil+\left\lfloor\frac{2k-1}{4}\right\rfloor}{\left(2(k+1)+\left\lceil\frac{k+1}{2}\right\rceil\right)\left(2(k+1)+\left\lfloor\frac{2k-1}{4}\right\rfloor\right)}}$$

$$+5(k-1)\frac{\sqrt{4k+2+2\left\lfloor\frac{2k-1}{4}\right\rfloor}}{2(k+1)+\left\lfloor\frac{2k-1}{4}\right\rfloor}$$

$$+5\sqrt{\frac{3k+1+\left\lceil\frac{k+1}{2}\right\rceil+\left\lfloor\frac{2k-1}{4}\right\rfloor}{\left(2(k+1)+\left\lfloor\frac{2k-1}{4}\right\rfloor\right)\left(k+1+\left\lceil\frac{k+1}{2}\right\rceil\right)}}$$

for $\mathbf{n \geq 6}$

$$ABC_5(W_n^k)=n\sqrt{\frac{2(k+1)+\left\lceil\frac{k+1}{2}\right\rceil\Big)\Big(2(k+1)+\left\lceil\frac{k}{2}\right\rceil-\left\lfloor\frac{k-1}{2}\right\rfloor-2}{\left(2(k+1)+\left\lceil\frac{k+1}{2}\right\rceil\right)\left(2(k+1)+\left\lceil\frac{k}{2}\right\rceil-\left\lfloor\frac{k-1}{2}\right\rfloor\right)}}$$

$$+n(k-1)\sqrt{\frac{4k+2+4\left\lceil\frac{k}{2}\right\rceil-2\left\lfloor\frac{k-1}{2}\right\rfloor+\sum_{q=1}^{\left\lceil\frac{k}{2}\right\rceil}(q-1)-\sum_{q=\left\lceil\frac{k}{2}\right\rceil+1}^{k}(q-k)}{2(k+1)+\left\lceil\frac{k}{2}\right\rceil-\left\lfloor\frac{k-1}{2}\right\rfloor+\sum_{q=1}^{\left\lceil\frac{k}{2}\right\rceil}(q-1)}}$$

$$\times\frac{1}{\sqrt{2(k+1)+2\left\lceil\frac{k}{2}\right\rceil-\left\lfloor\frac{k-1}{2}\right\rfloor-\sum_{q=\left\lceil\frac{k}{2}\right\rceil+1}^{k}(q-k)}}$$

$$+n\sqrt{\frac{2(k+1)+\left\lceil\frac{k}{2}\right\rceil-\left\lfloor\frac{k-1}{2}\right\rfloor+2(k+1)+\left\lceil\frac{k+1}{2}\right\rceil-2}{\left(2(k+1)+\left\lceil\frac{k}{2}\right\rceil-\left\lfloor\frac{k-1}{2}\right\rfloor\right)\left(2(k+1)+\left\lceil\frac{k+1}{2}\right\rceil\right)}}$$

$$+n\sqrt{\frac{2(k+1)+\left\lceil\frac{k+1}{2}\right\rceil+2(k+1)+\left\lfloor\frac{2k-1}{4}\right\rfloor-2}{\left(2(k+1)+\left\lceil\frac{k+1}{2}\right\rceil\right)\left(2(k+1)+\left\lfloor\frac{2k-1}{4}\right\rfloor\right)}}$$

$$+n(k-1)\frac{\sqrt{4k+2+2\left\lfloor\frac{2k-1}{4}\right\rfloor-2\sum_{q=1}^{k}(q-1)}}{2(k+1)+\left\lfloor\frac{2k-1}{4}\right\rfloor-\sum_{q=1}^{k}(q-1)}$$

$$+n\sqrt{\frac{2(k+1)+\left\lceil\frac{k+1}{2}\right\rceil+2(k+1)+\left\lfloor\frac{2k-1}{4}\right\rfloor-(k-1)-2}{\left(2(k+1)+\left\lceil\frac{k+1}{2}\right\rceil\right)\left(2(k+1)+\left\lfloor\frac{2k-1}{4}\right\rfloor-(k-1)\right)}}$$

$$ABC_5(W_n^k)=2n\sqrt{\frac{4k+2+\left\lceil\frac{k+1}{2}\right\rceil+2\left\lceil\frac{k}{2}\right\rceil-\left\lfloor\frac{k-1}{2}\right\rfloor}{\left(2(k+1)+\left\lceil\frac{k+1}{2}\right\rceil\right)\left(2(k+1)+\left\lceil\frac{k}{2}\right\rceil-\left\lfloor\frac{k-1}{2}\right\rfloor\right)}}$$

$$+n(k-1)\sqrt{\frac{4k+2+4\left\lceil\frac{k}{2}\right\rceil-2\left\lfloor\frac{k-1}{2}\right\rfloor+\sum_{q=1}^{\left\lceil\frac{k}{2}\right\rceil}(q-1)-\sum_{q=\left\lceil\frac{k}{2}\right\rceil+1}^{k}(q-k)}{2(k+1)+2\left\lceil\frac{k}{2}\right\rceil-\left\lfloor\frac{k-1}{2}\right\rfloor+\sum_{q=1}^{\left\lceil\frac{k}{2}\right\rceil}(q-1)}}$$

$$\times\frac{1}{\sqrt{2(k+1)+2\left\lceil\frac{k}{2}\right\rceil-\left\lfloor\frac{k-1}{2}\right\rfloor-\sum_{q=\left\lceil\frac{k}{2}\right\rceil+1}^{k}(q-k)}}$$

$$+n\sqrt{\frac{4k+2+\left\lceil\frac{k+1}{2}\right\rceil+\left\lfloor\frac{2k-1}{4}\right\rfloor}{\left(2(k+1)+\left\lceil\frac{k+1}{2}\right\rceil\right)\left(2(k+1)+\left\lfloor\frac{2k-1}{4}\right\rfloor\right)}}$$

$$+n(k-1)\frac{\sqrt{4k+2+2\left\lfloor\frac{2k-1}{4}\right\rfloor-2\sum_{q=1}^{\left\lceil\frac{k}{2}\right\rceil}(q-1)}}{2(k+1)+\left\lfloor\frac{2k-1}{4}\right\rfloor-\sum_{q=1}^{\left\lceil\frac{k}{2}\right\rceil}(q-1)}$$

$$+n\sqrt{\frac{3k+1+\left\lceil\frac{k+1}{2}\right\rceil+\left\lfloor\frac{2k-1}{4}\right\rfloor}{\left(2(k+1)+\left\lceil\frac{k+1}{2}\right\rceil\right)\left(2(k+1)+\left\lfloor\frac{2k-1}{4}\right\rfloor-(k-1)\right)}}$$

$\square$

Theorem 5.1.3. *Let $L(W_n^k)$ be a line graph of k-subdivided wheel graph having order*

$2n(k+1)$ *and size* $\frac{n^2+5n+4nk}{2}$*, eccentricities of graph* $L(W_n^k)$ *are below.*

Proof.

Now let a graph $L(W_n^k)$ be a line graph of k-subdivided wheel graph having order $2n(k+1)$ and size $\frac{n^2+5n+4nk}{2}$.

The vertex set of $L(W_n^k)$ as follow:

$\{\alpha_i^l : 1 \leq l \leq n,\ 1 \leq i \leq 3n\} \cup \{\beta_l : 1 \leq l \leq n\} \cup \{\gamma_p^q : 1 \leq p \leq n,\ 1 \leq q \leq k-1,\ k \neq 1\} \cup \{\omega_p^q : 1 \leq p \leq n,\ 1 \leq q \leq k-1,\ k \neq 1\}$, and edge set for $k = 1$ as $\{\alpha_i^l \beta_l : 1 \leq l \leq n,\ i = 1\} \cup \{\alpha_3^l \alpha_2^{l+1} : 1 \leq l \leq n-1\} \cup \{\alpha_3^n \alpha_2^1\} \cup \{\alpha_i^l \alpha_j^l : i \neq j,\ i,\ j = 1,\ 2,\ 3\} \cup \{\beta_l \beta_m : l \neq m,\ l,\ m = 1,\ 2,\ 3,\ ...,\ n\}$ and edge set for $k \geq 2$ as $\{\beta_l \beta_m : l \neq m,\ l,\ m = 1,\ 2,\ 3,\ ...,\ n\} \cup \{\alpha_i^l \alpha_j^l : i \neq j,\ i,\ j = 1,\ 2,\ 3\} \cup \{\alpha_3^l \gamma_p^q : q = 1,\ 1 \leq l,\ p \leq n\} \cup \{\gamma_p^q \alpha_2^l : q = k,\ 1 \leq l,\ p \leq n\} \cup \{\gamma_p^q \gamma_p^{q+1} : 1 \leq p \leq n,\ 1 \leq q \leq k-2,\ k \neq 1,\ 2\} \cup \{\alpha_1^l \omega_p^q : 1 \leq l,\ p \leq n,\ q = 1\} \cup \{\omega_p^q \beta_l : 1 \leq l,\ p \leq n,\ q = k\} \cup \{\omega_p^q \omega_p^{q+1} : 1 \leq p \leq n,\ 1 \leq q \leq k-2,\ k \neq 1,\ 2\}$ Now we define eccentricities of vertex set of $L(W_n^k)$ as below:

Throughout $1 \leq l \leq n$

for $\mathbf{n = 3}$

$\mathbf{1 \leq i \leq 3n}$

$$\xi(\alpha_i^l) = \begin{cases} n & \text{when} \quad k = 1 \\ \\ n+k & \text{when} \quad k \geq 2 \end{cases}$$

$\mathbf{1 \leq i \leq n}$

$$\xi(\beta_l = \begin{cases} n & \text{when} \quad k = 1 \\ \\ n+k & \text{when} \quad k \geq 2 \end{cases}$$

$\mathbf{1 \leq p \leq n}$

$$\xi(\gamma_p^q) = \xi(\omega_p^q) = \begin{cases} k+3+\left\lceil \frac{k}{2} \right\rceil + (q-1) & \text{if } \ 1 \leq q \leq \left\lfloor \frac{k}{2} \right\rfloor \\ k+3+\left\lceil \frac{k}{2} \right\rceil - (q-k+1) & \text{if } \ q = \left\lceil \frac{k}{2} \right\rceil + 1, \ ..., \ k-1 \end{cases}$$

for $\mathbf{n = 4}$

$\mathbf{1 \leq i \leq 3n}, \ \mathbf{1 \leq q \leq k-1}$

$$\xi(\alpha_i^l) = \xi(\gamma_p^q) = 2(k+1)$$

$\mathbf{1 \leq i \leq n}$

$$\xi(\beta_l = \begin{cases} n & \text{when} \quad k = 1 \\ 2+k+\left\lfloor \frac{k}{2} \right\rfloor & \text{when} \quad k \geq 2 \end{cases}$$

$$\xi(\omega_p^q) = \begin{cases} 2(k+1) & \text{if } \ 1 \leq q \leq \left\lfloor \frac{k}{2} \right\rfloor \\ 2(k+1) - \left(q - \left\lfloor \frac{k}{2} \right\rfloor\right) & \text{if } \ q = \left\lceil \frac{k}{2} \right\rceil + 1, ..., k-1 \end{cases}$$

For $\mathbf{n = 5}$

$\mathbf{1 \leq i \leq 3n}$

$$\xi(\alpha_i^l) = \begin{cases} 2k+3 & \text{when} \quad k = 1, 2 \\ 2(k+2) & \text{when} \quad k \geq 3 \end{cases}$$

$$\xi(\gamma_p^q) = \begin{cases} 5+2(k-1)+(q-1) & \text{if } \ 1 \leq q \leq \left\lfloor \frac{k}{2} \right\rfloor \\ 5+2(k-1)-(q-k+1) & \text{if } \ q = \left\lceil \frac{k}{2} \right\rceil + 1, ..., k-1 \end{cases}$$

for $\mathbf{n \geq 5}$

$\mathbf{1 \leq i \leq n}, \ \mathbf{1 \leq q \leq k-1}$

$$\xi(\beta_l = 2+k+\left\lfloor \frac{k}{2} \right\rfloor$$

$$\xi(\omega_p^q) = 2k+1+\left\lfloor \frac{k}{2} \right\rfloor - (q-1)$$

for $\mathbf{n \geq 6}$

$\mathbf{1 \leq i \leq 3n}$, $\mathbf{1 \leq q \leq k-1}$

$$\xi(\alpha_i^l) = 2(k+1) + \left\lceil \frac{k+1}{2} \right\rceil$$

$$\xi(\gamma_p^q) = \begin{cases} 2k+4+\left\lfloor \frac{k}{2} \right\rfloor + (q-1) & \text{if} \quad 1 \leq q \leq \left\lfloor \frac{k}{2} \right\rfloor \\ \\ 2k+4+\left\lfloor \frac{k}{2} \right\rfloor - (q-k+1) & \text{if} \quad q = \left\lceil \frac{k}{2} \right\rceil + 1, \ldots, k-1 \end{cases}$$

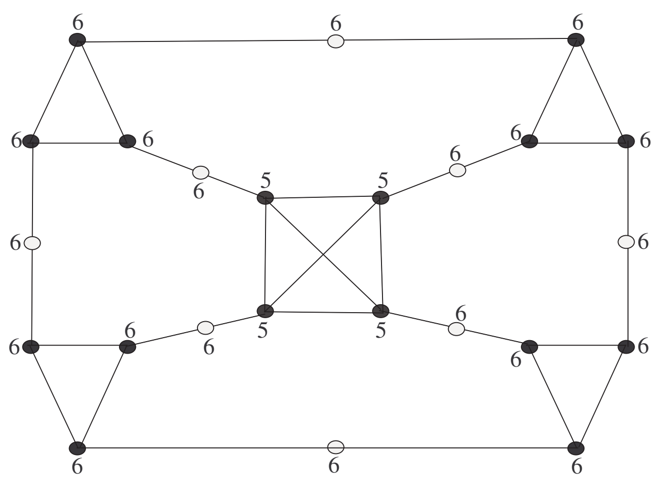

Figure 5.2: $L(W_4^2)$

□

Theorem 5.1.4. *Let W_n is a wheel graph having order $n+1$, then the eccentric connectivity index is*

$$\xi^c(W_n) = \begin{cases} 12 & \text{if} \quad n = 3 \\ \\ 7n & \text{if} \quad n \geq 4 \end{cases}$$

Proof.

Let W_n is a wheel graph having order $n+1$ and size $2n$. In wheel graph W_n, n-vertices have degree 3 and only center vertex have degree n. Then by the formula of eccentric connectivity index i.e.

$$\xi^c(G) = \sum_{\alpha \in V} ecc(\alpha) deg(\alpha)$$

Now putting values like degree and eccentricities from previous theorems we get

Firstly **for $\mathbf{n = 3}$**

$$\xi^c(W_n) = 3 \times 1 \times 3 + 1 \times 1 \times 3 = 9 + 3 = 12$$

For $\mathbf{n \geq 4}$

$$\xi^c(W_n) = n \times 2 \times 3 + 1 \times 1 \times n = 6n + n = 7n$$

□

Theorem 5.1.5. *Let W_n^k is a wheel graph having order $(2k+1)+1$, then the eccentric connectivity index below*

For $\mathbf{n = 3}$

$$\xi^c(W_n^k) = 12(k+1+\left\lceil \frac{k+1}{2} \right\rceil) + 6\Big[\left\lceil \frac{k}{2} \right\rceil \sum_{q=1}^{\left\lceil \frac{k}{2} \right\rceil} \Big(2(k+1) - \left\lceil \frac{k-1}{2} \right\rceil - (q-1)\Big)$$

$$+ \left(k - \left\lceil \frac{k}{2} \right\rceil\right) \sum_{q=\left\lceil \frac{k}{2} \right\rceil + 1}^{k} \Big(2(k+1) - \left\lceil \frac{k-1}{2} \right\rceil - (q-k)\Big)\Big]$$

For $\mathbf{n = 4}$

$$\xi^c(W_n^k) = 24(k+1) + 4\left(k+1+\left\lceil \frac{k+1}{2} \right\rceil\right) + 4k(k+1) + 16(k+1)\left\lceil \frac{k}{2} \right\rceil$$

$$8(k - \left\lceil \frac{k}{2} \right\rceil)\Big[2(k+1) - \sum_{q=\left\lceil \frac{k}{2} \right\rceil + 1}^{k} \Big(q - \left\lceil \frac{k}{2} \right\rceil\Big)\Big]$$

For $\mathbf{n=5}$

$$\xi^c(W_n^k) = 15\left(2(k+1)+\left\lceil\frac{k}{2}\right\rceil\right)+5\left(k+1+\left\lceil\frac{k}{2}\right\rceil\right)+10\left\lceil\frac{k}{2}\right\rceil$$

$$\times\sum_{q=1}^{\left\lceil\frac{k}{2}\right\rceil}\left(2(k+1)+\left\lceil\frac{k}{2}\right\rceil-\left\lfloor\frac{k-1}{2}\right\rfloor+(q-1)\right)$$

$$+10\left(k-\left\lceil\frac{k}{2}\right\rceil\right)\sum_{q=\left\lceil\frac{k}{2}\right\rceil+1}^{k}\left[2(k+1)+\left\lceil\frac{k}{2}\right\rceil-\left\lfloor\frac{k-1}{2}\right\rfloor+(q-k)\right]$$

$$+10k\sum_{q=1}^{k}\left[2(k+1)+\left\lfloor\frac{2k-1}{4}\right\rfloor+(q-1)\right]$$

For $\mathbf{n\geq 6}$

$$\xi^c(W_n^k) = 3n\left(2(k+1)+\left\lceil\frac{k}{2}\right\rceil\right)+n\left(k+1+\left\lceil\frac{k}{2}\right\rceil\right)+2n\left\lceil\frac{k}{2}\right\rceil$$

$$\times\sum_{q=1}^{\left\lceil\frac{k}{2}\right\rceil}\left(2(k+1)+2\left\lceil\frac{k}{2}\right\rceil-\left\lfloor\frac{k-1}{2}\right\rfloor+(q-1)\right)$$

$$+2n\left(k-\left\lceil\frac{k}{2}\right\rceil\right)\sum_{q=\left\lceil\frac{k}{2}\right\rceil+1}^{k}\left[2(k+1)+2\left\lceil\frac{k}{2}\right\rceil-\left\lfloor\frac{k-1}{2}\right\rfloor-(q-k)\right]$$

$$+2nk\sum_{q=1}^{k}\left[2(k+1)+\left\lfloor\frac{2k-1}{4}\right\rfloor-(q-1)\right]$$

Proof.

Let W_n^k be k-subdivided wheel graph with order $(2k+1)n+1$ and size $2n(k+1)$.

In W_n^k graph n-vertices have degree 3 and only center vertex have degree n. Moreover,

$2nk$-vertices have degree 2. Then by the formula of eccentric connectivity index i.e.

$$\xi^c(W_n^k) = \sum_{\alpha \in V} ecc(\alpha)deg(\alpha)$$

Now putting values like degree and eccentricities from previous theorems we get

For $\mathbf{n = 3}$

$$\xi^c(W_n^k) = 3 \times 3\Big(k+1+\Big\lceil \frac{k+1}{2} \Big\rceil\Big) + 3\Big(k+1+\Big\lceil \frac{k+1}{2} \Big\rceil\Big) + 6\Big\lceil \frac{k}{2} \Big\rceil$$

$$\sum_{q=1}^{\lceil \frac{k}{2} \rceil} \Big(2(k+1) - \Big\lceil \frac{k-1}{2} \Big\rceil - (q-1)\Big) + 6\Big(k - \Big\lceil \frac{k}{2} \Big\rceil\Big)$$

$$\sum_{q=\lceil \frac{k}{2} \rceil + 1}^{k} \Big(2(k+1) - \Big\lceil \frac{k-1}{2} \Big\rceil - (q-k)\Big)\Big]$$

$$\xi^c(W_n^k) = 12(k+1+\Big\lceil \frac{k+1}{2} \Big\rceil) + 6\Big[\Big\lceil \frac{k}{2} \Big\rceil \sum_{q=1}^{\lceil \frac{k}{2} \rceil} \Big(2(k+1) - \Big\lceil \frac{k-1}{2} \Big\rceil - (q-1)\Big) +$$

$$\Big(k - \Big\lceil \frac{k}{2} \Big\rceil\Big) \sum_{q=\lceil \frac{k}{2} \rceil + 1}^{k} \Big(2(k+1) - \Big\lceil \frac{k-1}{2} \Big\rceil - (q-k)\Big)\Big]$$

For $\mathbf{n = 4}$

$$\xi^c(W_n^k) = 4 \times 3 \times 2(k+1) + 1 \times 4 \times 4\Big(k+1+\Big\lceil \frac{k+1}{2} \Big\rceil\Big) + 2 \times 2k(k+1) +$$

$$2 \times 2(k+1) \times 4\Big\lceil \frac{k}{2} \Big\rceil + 2 \times 4\Big(k - \Big\lceil \frac{k}{2} \Big\rceil\Big)\Big[2(k+1) - \sum_{q=\lceil \frac{k}{2} \rceil + 1}^{k} \Big(q - \Big\lceil \frac{k}{2} \Big\rceil\Big)\Big]$$

$$\xi^c(W_n^k) = 24(k+1) + 4\left(k+1+\left\lceil\frac{k+1}{2}\right\rceil\right) + 4k(k+1) + 16(k+1)\left\lceil\frac{k}{2}\right\rceil$$

$$8(k-\left\lceil\frac{k}{2}\right\rceil)\left[2(k+1) - \sum_{q=\left\lceil\frac{k}{2}\right\rceil+1}^{k}\left(q-\left\lceil\frac{k}{2}\right\rceil\right)\right]$$

For $\mathbf{n = 5}$

$$\xi^c(W_n^k) = 5\times 3\times\left(2(k+1)+\left\lceil\frac{k}{2}\right\rceil\right) + 1\times 5\times\left(k+1+\left\lceil\frac{k}{2}\right\rceil\right) + 2\times 5\times\left\lceil\frac{k}{2}\right\rceil$$

$$\sum_{q=1}^{\left\lceil\frac{k}{2}\right\rceil}\left(2(k+1)+\left\lceil\frac{k}{2}\right\rceil-\left\lfloor\frac{k-1}{2}\right\rfloor+(q-1)\right) + 2\times 5\left(k-\left\lceil\frac{k}{2}\right\rceil\right)$$

$$\sum_{q=\left\lceil\frac{k}{2}\right\rceil+1}^{k}\left[2(k+1)+\left\lceil\frac{k}{2}\right\rceil-\left\lfloor\frac{k-1}{2}\right\rfloor+(q-k)\right] + 2\times 5k\sum_{q=1}^{k}\left[2(k+1)+\left\lfloor\frac{2k-1}{4}\right\rfloor+(q-1)\right]$$

$$\xi^c(W_n^k) = 15\left(2(k+1)+\left\lceil\frac{k}{2}\right\rceil\right) + 5\left(k+1+\left\lceil\frac{k}{2}\right\rceil\right) + 10\left\lceil\frac{k}{2}\right\rceil$$

$$\times\sum_{q=1}^{\left\lceil\frac{k}{2}\right\rceil}\left(2(k+1)+\left\lceil\frac{k}{2}\right\rceil-\left\lfloor\frac{k-1}{2}\right\rfloor+(q-1)\right)$$

$$+10\left(k-\left\lceil\frac{k}{2}\right\rceil\right)\sum_{q=\left\lceil\frac{k}{2}\right\rceil+1}^{k}\left[2(k+1)+\left\lceil\frac{k}{2}\right\rceil-\left\lfloor\frac{k-1}{2}\right\rfloor+(q-k)\right]+$$

$$10k\sum_{q=1}^{k}\left[2(k+1)+\left\lfloor\frac{2k-1}{4}\right\rfloor+(q-1)\right]$$

For $\mathbf{n \geq 6}$

$$\xi^c(W_n^k) = 3n\left(2(k+1) + \left\lceil \frac{k}{2} \right\rceil\right) + n\left(k+1+\left\lceil \frac{k}{2} \right\rceil\right) + 2n\left\lceil \frac{k}{2} \right\rceil$$

$$\times \sum_{q=1}^{\left\lceil \frac{k}{2} \right\rceil} \left(2(k+1) + 2\left\lceil \frac{k}{2} \right\rceil - \left\lfloor \frac{k-1}{2} \right\rfloor + (q-1)\right)$$

$$+2n\left(k - \left\lceil \frac{k}{2} \right\rceil\right) \sum_{q=\left\lceil \frac{k}{2} \right\rceil+1}^{k} \left[2(k+1) + 2\left\lceil \frac{k}{2} \right\rceil - \left\lfloor \frac{k-1}{2} \right\rfloor - (q-k)\right]$$

$$+2nk \sum_{q=1}^{k} \left[2(k+1) + \left\lfloor \frac{2k-1}{4} \right\rfloor - (q-1)\right]$$

□

Theorem 5.1.6. *Let $L(W_n^k)$ be a line graph of k-subdivided wheel graph having order $2n(k+1)$ and size $\frac{n^2+5n+4nk}{2}$, then the eccentric connectivity index below*

For $\mathbf{n = 3}$, $\mathbf{k = 1}$

$$\xi^c(L(W_n^k)) = 252$$

For $\mathbf{n = 3}$, $\mathbf{k \geq 2}$

$$\xi^c(L(W_n^k)) = 189 + 36k + 12\left\lfloor \frac{k}{2} \right\rfloor \sum_{q=1}^{\left\lfloor \frac{k}{2} \right\rfloor} \left(k+2+\left\lceil \frac{k}{2} \right\rceil + q\right) + 12\left(k - \left\lfloor \frac{k}{2} \right\rfloor\right)$$

$$\sum_{q=\left\lfloor \frac{k}{2} \right\rfloor+1}^{k-1} \left(2k+2+\left\lceil \frac{k}{2} \right\rceil - q\right)$$

For $\mathbf{n = 4}$, $\mathbf{k = 1}$

$$\xi^c(L(W_n^k)) = 288$$

For $\mathbf{n=4},\ \mathbf{k\geq 2}$

$$\xi^c(L(W_n^k)) = 104+104k+16k^2+32\left\lfloor\frac{k}{2}\right\rfloor+16k\left\lfloor\frac{k}{2}\right\rfloor$$

$$+8\sum_{q=\left\lfloor\frac{k}{2}\right\rfloor+1}^{k-1}\left(2k+2-q+\left\lfloor\frac{k}{2}\right\rfloor\right)\left(k-\left\lfloor\frac{k}{2}\right\rfloor\right)$$

For $\mathbf{n=5},\ \mathbf{k=1}$

$$\xi^c(L(W_n^k)) = 300$$

For $\mathbf{n=5},\ \mathbf{k=2}$

$$=\xi^c(L(W_n^k)) = 480$$

For $\mathbf{n=5},\ \mathbf{k\geq 3}$

$$\xi^c(L(W_n^k)) = 230+115k+25\left\lfloor\frac{k}{2}\right\rfloor+10\left\lfloor\frac{k}{2}\right\rfloor\sum_{q=1}^{\left\lfloor\frac{k}{2}\right\rfloor}\left(2+2k+q\right)+10\left(k-\left\lfloor\frac{k}{2}\right\rfloor\right)$$

$$\sum_{q=\left\lfloor\frac{k}{2}\right\rfloor+1}^{k-1}(2+3k-q)+10(k-1)\sum_{q=1}^{k-1}(2k+2+\left\lfloor\frac{k}{2}\right\rfloor-q)$$

For $\mathbf{n\geq 6}$

$$\xi^c(L(W_n^k)) = 9n\left(2(k+1)+\left\lceil\frac{k+1}{2}\right\rceil\right)+n^2\left(2+k+\left\lfloor\frac{k}{2}\right\rfloor\right)+2n\left\lfloor\frac{k}{2}\right\rfloor$$

$$\sum_{q=1}^{\left\lfloor\frac{k}{2}\right\rfloor}\left(2k+3+q+\left\lfloor\frac{k}{2}\right\rfloor\right)+2n\left(k-\left\lfloor\frac{k}{2}\right\rfloor\right)\sum_{q=\left\lfloor\frac{k}{2}\right\rfloor+1}^{k-1}\left(3k+3-q+\left\lfloor\frac{k}{2}\right\rfloor\right)$$

$$+2n(k-1)\sum_{q=1}^{k-1}\left(2k+2-q+\left\lfloor\frac{k}{2}\right\rfloor\right)$$

Proof.

For $\mathbf{n=3}$, $\mathbf{k=1}$

$$\xi^{c}(L(W_n^k)) = 3\times 3\times 9+(3+1)\times 3\times 9+3\times 3\times 3+(3+1)\times 3\times 3 = 252$$

For $\mathbf{n=3}$, $\mathbf{k\geq 1}$

$$\xi^{c}(L(W_n^k)) = 3\times 3\times 9+(3+1)\times 3\times 9+(3+1)\times 3\times 3+2\times \sum_{q=1}^{\left\lfloor \frac{k}{2} \right\rfloor}\left(k+3+\left\lceil \frac{k}{2}\right\rceil+(q-1)\right)$$

$$\times 3\left\lfloor \frac{k}{2}\right\rfloor+2\sum_{q=\left\lfloor \frac{k}{2}\right\rfloor+1}^{k-1}\left(k+3+\left\lceil \frac{k}{2}\right\rceil-(q-k+1)\right)\times 3\times\left(k-\left\lfloor \frac{k}{2}\right\rfloor\right)+2\times 3\left\lfloor \frac{k}{2}\right\rfloor$$

$$\times\sum_{q=1}^{\left\lfloor \frac{k}{2} \right\rfloor}\left(k+3+\left\lceil \frac{k}{2}\right\rceil+(q-1)\right)$$

$$+2\times 3\left(k-\left\lfloor \frac{k}{2}\right\rfloor\right)\sum_{q=\left\lfloor \frac{k}{2}\right\rfloor+1}^{k-1}\left(k+3+\left\lceil \frac{k}{2}\right\rceil-(q-k+1)\right)$$

$$\xi^{c}(L(W_n^k)) = 189+36k+12\left\lfloor \frac{k}{2}\right\rfloor\sum_{q=1}^{\left\lfloor \frac{k}{2} \right\rfloor}(k+2+\left\lceil \frac{k}{2}\right\rceil+q)+12\left(k-\left\lfloor \frac{k}{2}\right\rfloor\right)$$

$$\sum_{q=\left\lfloor \frac{k}{2}\right\rfloor+1}^{k-1}\left(2k+2+\left\lceil \frac{k}{2}\right\rceil-q\right)$$

For $\mathbf{n=4}$, $\mathbf{k=1}$

$$\xi^{c}(L(W_n^k)) = 3\times 2(1+1)\times 12+4\times 4\times 4+4\times 4\times (2+1)+2\times 2(1+1)\times 4\times 1 = 288$$

For $\mathbf{n=4,\ \ k\geq 2}$

$$\xi^c(L(W_n^k)) = 3\times 2(1+1)\times 12+4\times 4\times(2+1)+2\times 2(1+1)\times 4\times 1$$

$$+2\times 4\times \sum_{q=\left\lfloor \frac{k}{2}\right\rfloor+1}^{k-1}\left(2(k+1)-q+\left\lfloor\frac{k}{2}\right\rfloor\right)\times\left(k-\left\lfloor\frac{k}{2}\right\rfloor\right)+2(k+1)\times 2\times 4\times\left\lfloor\frac{k}{2}\right\rfloor$$

$$\xi^c(L(W_n^k)) = 104+104k+16k^2+32\left\lfloor\frac{k}{2}\right\rfloor+16k\left\lfloor\frac{k}{2}\right\rfloor$$

$$+8\sum_{q=\left\lfloor \frac{k}{2}\right\rfloor+1}^{k-1}\left(2k+2-q+\left\lfloor\frac{k}{2}\right\rfloor\right)\left(k-\left\lfloor\frac{k}{2}\right\rfloor\right)$$

For $\mathbf{n=5,\ \ k=1}$

$$\xi^c(L(W_n^k)) = 3\times(2\times 2+1)\times 15+5\times(2+1)\times 5 = 300$$

For $\mathbf{n=5,\ \ k=2}$

$$\xi^c(L(W_n^k)) = 3(2\times 2+1)\times 15+5\times 5(2+2+1)+2\times 5\times(5+2\times 1)+2\times 5\times(5+1)$$

$$=480$$

For $\mathbf{n=5,\ \ k\geq 3}$

$$\xi^c(L(W_n^k)) = 3\times 15\times 2(k+2)+5\times 5\times\left(2+k+\left\lfloor\frac{k}{2}\right\rfloor\right)+2\times 5\times\left\lfloor\frac{k}{2}\right\rfloor$$

$$\sum_{q=1}^{\left\lfloor \frac{k}{2}\right\rfloor}\left(5+2(k-1)+(q-1)\right)+2\times 5\left(k-\left\lfloor\frac{k}{2}\right\rfloor\right)\sum_{q=\left\lfloor \frac{k}{2}\right\rfloor+1}^{k-1}(5+2(k-1)-(q-k+1)$$

$$+2\times 5(k-1)\sum_{q=1}^{k-1}\left(2k+1+\left\lfloor\frac{k}{2}\right\rfloor-(q-1)\right)$$

$$\xi^{c}(L(W_n^k))=230+115k+25\left\lfloor\frac{k}{2}\right\rfloor+10\left\lfloor\frac{k}{2}\right\rfloor\sum_{q=1}^{\left\lfloor\frac{k}{2}\right\rfloor}\left(2+2k+q\right)+10\left(k-\left\lfloor\frac{k}{2}\right\rfloor\right)$$

$$\sum_{q=\left\lfloor\frac{k}{2}\right\rfloor+1}^{k-1}(2+3k-q)+10(k-1)\sum_{q=1}^{k-1}(2k+2+\left\lfloor\frac{k}{2}\right\rfloor-q)$$

For vn $\geq$ 6

$$\xi^{c}(L(W_n^k))=3\times n\times 3\left(2(k+1)+\left\lceil\frac{k+1}{2}\right\rceil\right)+n\times n\left(2+k+\left\lfloor\frac{k}{2}\right\rfloor\right)+2\times n\left\lfloor\frac{k}{2}\right\rfloor$$

$$\sum_{q=1}^{\left\lfloor\frac{k}{2}\right\rfloor}(2k+4+\left\lfloor\frac{k}{2}\right\rfloor+(q-1))+2\times n\left(k-\left\lfloor\frac{k}{2}\right\rfloor\right)\sum_{q=\left\lfloor\frac{k}{2}\right\rfloor+1}^{k-1}(2k+4+\left\lfloor\frac{k}{2}\right\rfloor$$

$$-(q-k+1))+2\times n(k-1)\sum_{q=1}^{k-1}(2k+1+\left\lfloor\frac{k}{2}\right\rfloor+(q-1))$$

$$\xi^{c}(L(W_n^k))=9n\left(2(k+1)+\left\lceil\frac{k+1}{2}\right\rceil\right)+n^2\left(2+k+\left\lfloor\frac{k}{2}\right\rfloor\right)+2n\left\lfloor\frac{k}{2}\right\rfloor$$

$$\sum_{q=1}^{\left\lfloor\frac{k}{2}\right\rfloor}\left(2k+3+q+\left\lfloor\frac{k}{2}\right\rfloor\right)+2n\left(k-\left\lfloor\frac{k}{2}\right\rfloor\right)\sum_{q=\left\lfloor\frac{k}{2}\right\rfloor+1}^{k-1}\left(3k+3-q+\left\lfloor\frac{k}{2}\right\rfloor\right)$$

$$+2n(k-1)\sum_{q=1}^{k-1}\left(2k+2-q+\left\lfloor\frac{k}{2}\right\rfloor\right)$$

□

5.2 Topological Indices of Complete Bipartite Related Graphs

Theorem 5.2.1. *Let a complete bipartite graph $K_{n,m}$ with order $n+m$ and size nm. Then the $ABC_5(K_{n,m}) = \frac{mn}{\sqrt{2}}$.*

Proof.

Let a complete bipartite graph $K_{n,m}$ with order $n+m$ and size nm. Define vertex set as follows:

$\{\alpha_i : 1 \leq i \leq n\} \cup \{\beta_j : 1 \leq j \leq m\}$ and edge set as $\{\alpha_i\beta_j : 1 \leq i \leq n,\ 1 \leq j \leq m\}$.

Now we define eccentricities of the vertex set of graph $K_{n,m}$.

$1 \leq i \leq n,\ 1 \leq j \leq m$

$$\xi(\alpha_i) = \xi(\beta_j) = 2$$

Only one type of edges in $K_{n,m}$ that is $(2,2)$ with count mn so by formula of new atomic bond connectivity index

$$ABC_5(G) = \sum_{uv \in E(G)} \sqrt{\frac{ecc(u)+ecc(v)-2}{ecc(u)ecc(v)}}$$

$$ABC_5(K_{n,m}) = mn\sqrt{\frac{2+2-2}{(2)(2)}} = mn\sqrt{\frac{2}{(2)(2)}} = \frac{mn}{\sqrt{2}}$$

.

□

Theorem 5.2.2. *Let $K^k_{n,m}$ be a k-subdivided complete bipartite graph. Then the*

$$ABC_5(K^k_{n,m}) = mn\sqrt{\frac{2k+1}{2}}$$

.

Proof.

Now let $K_{n,m}^{k}$ be k-subdivided complete bipartite graph with order $m+n+mnk$ and size $nm(k+1)$.

Define vertex set as follow:

$\{\alpha_i : 1 \leq i \leq n\} \cup \{\beta_j : 1 \leq j \leq m\} \cup \{\gamma_p^q : 1 \leq p \leq nm, \ \ 1 \leq q \leq k\}$, and edge set as $\{\alpha_i \gamma_p^q : 1 \leq i \leq n, \ 1 \leq p \leq m, \ q = 1\} \cup \{\gamma_p^q \beta_j : 1 \leq p \leq nm, \ 1 \leq j \leq m, \ q = k\} \cup \{\gamma_p^q \gamma_p^{q+1} : 1 \leq p \leq nm, \ 1 \leq q \leq k-1, \ k \neq 1\}$.

Now we define eccentricities of vertex set of $K_{n,m}^{k}$ as below:

$1 \leq p \leq nm$, $1 \leq q \leq k$, $1 \leq i \leq n$, $1 \leq j \leq m$

$$\xi(\alpha_i) = \xi(\beta_j) = \xi(\gamma_p^q) = 2(k+1)$$

Only one type of edges in $K_{n,m}^{k}$ with respect to eccentricity is $\bigg(2(k+1), 2(k+1)\bigg)$ having count $mn(k+1)$ so according to formula of new atomic bond connectivity index

$$ABC_5(G) = \sum_{uv \in E(G)} \sqrt{\frac{ecc(u)+ecc(v)-2}{ecc(u)ecc(v)}}$$

$$\begin{aligned} ABC_5(K_{n,m}^{k}) &= mn(k+1)\sqrt{\tfrac{2(k+1)+2(k+1)-2}{\big(2(k+1)\big)\big(2(k+1)\big)}} \\ &= mn(k+1)\sqrt{\tfrac{2(k+1+k+1-1)}{4(k+1)^2}} \\ &= mn(k+1)\tfrac{\sqrt{2(2k+1)}}{2(k+1)} \\ &= mn\sqrt{\tfrac{2k+1}{2}} \end{aligned}$$

$\square$

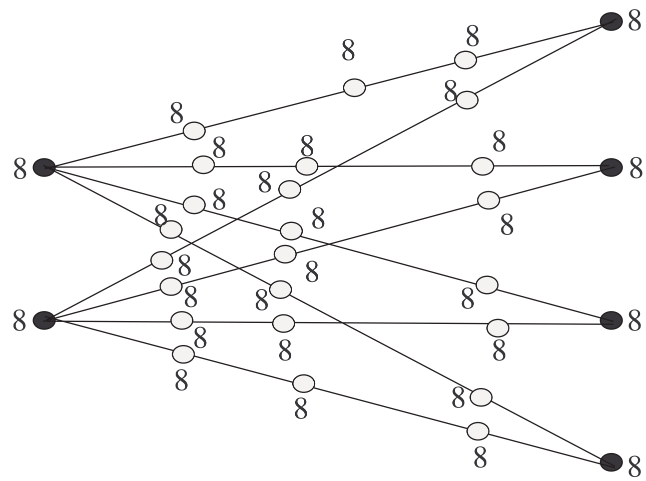

Figure 5.3: Uniform Subdivided Bipartite Graph $K^3_{2,4}$

Theorem 5.2.3. *Let $L(K^k_{n,m})$ be a line graph of k-subdivided complete bipartite graph. Then the* $ABC_5(L(K^k_{n,m})) = \left(\frac{m^2n+n^2m}{2} + mn(k-1)\right)\frac{\sqrt{2(2k+1)}}{2(k+1)}$.

Proof.

Let a graph $L(K^k_{n,m})$ be a line graph of k-subdivided complete bipartite graph with order $mn(k+1)$ and size $\frac{nm^2+mn^2}{2} + nm(k-1)$.

The vertex set of $L(K^k_{n,m})$ as follows:

$\{\alpha_i : 1 \leq i \leq nm\} \cup \{\beta_j : 1 \leq j \leq nm\} \cup \{\gamma^q_p : 1 \leq p \leq nm,\ \ 1 \leq q \leq k-1,\ \ k \neq 1\}$, and edge set as $\{\alpha_i \gamma^q_p : 1 \leq i \leq nm,\ 1 \leq p \leq nm,\ q = 1\} \cup \{\gamma^q_p \beta_j : 1 \leq p \leq nm,\ 1 \leq j \leq nm,\ q = k\} \cup \{\gamma^q_p \gamma^{q+1}_p : 1 \leq p \leq nm,\ 1 \leq q \leq k-2,\ k \neq 1,2\}$.

Now we define eccentricities of vertex set of $L(K^k_{n,m})$ as below:

$1 \leq p \leq nm$, $1 \leq q \leq k-1$, $1 \leq i \leq nm,\ 1 \leq j \leq nm$

$$\xi(\alpha_i) = \xi(\beta_j) = \xi(\gamma^q_p) = 2(k+1)$$

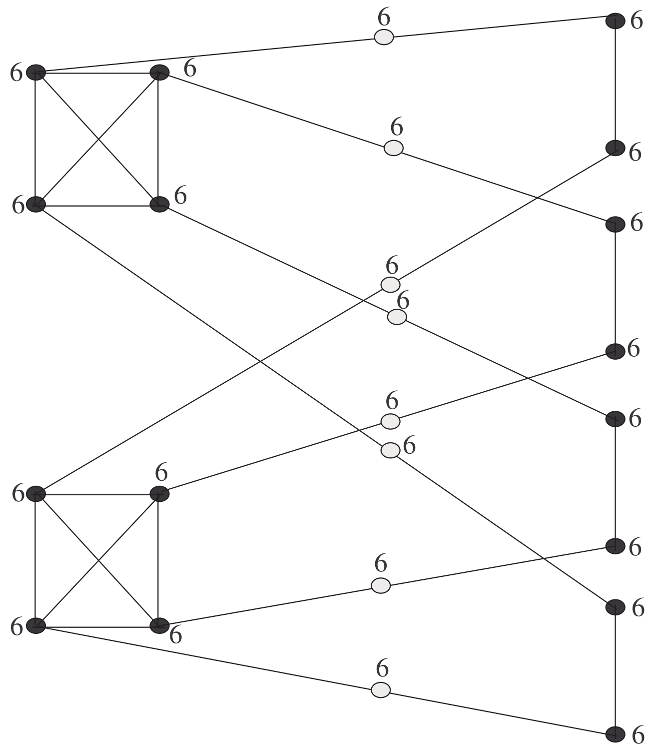

Figure 5.4: Line Graph of Uniformly Subdivided Bipartite Graph $L(K_{2,4}^{2})$

Only single type of edges in $L(K_{n,m}^{k})$ is $\Big(2(k+1), 2(k+1)\Big)$ having count $\frac{nm^2+mn^2}{2} + nm(k-1)$ so according to formula of new atomic bond connectivity index

$$ABC_5(G) = \sum_{uv \in E(G)} \sqrt{\frac{ecc(u)+ecc(v)-2}{ecc(u)ecc(v)}}$$

$$ABC_5(L(K_{n,m}^{k})) = \left[\frac{nm^2+mn^2}{2} + nm(k-1)\right] \sqrt{\frac{2(k+1)+2(k+1)-2}{\Big(2(k+1)\Big)\Big(2(k+1)\Big)}}$$

$$= \left[\frac{nm^2+mn^2}{2} + nm(k-1)\right] \sqrt{\frac{2k+2+2k+2-2)}{4(k+1)^2}}$$

$$= \left[\frac{nm^2+mn^2}{2} + nm(k-1)\right] \frac{\sqrt{4k+2}}{2(k+1)}$$

$$= \left[\frac{nm^2+mn^2}{2} + nm(k-1)\right] \frac{\sqrt{2(2k+1)}}{2(k+1)}$$

□

Theorem 5.2.4. *Let $K_{n,m}$ be a complete bipartite graph, then the eccentric connectivity index is $\xi^c(K_{n,m}) = 2(m+n)^2$.*

Proof.

Let $K_{n,m}$ be a complete bipartite graph with order $m+n$ and size mn. In $K_{n,m}$ graph n vertices have degree m and m vertices have degree n. Only one type of edges according to eccentricity in $K_{n,m}$ that is $(2,2)$ with count mn so by formula of eccentric connectivity index

$$\xi^c(G) = \sum_{\alpha \in V} ecc(\alpha)deg(\alpha)$$

putting values in formula we get

$$\xi^c(K_{n,m}) = (m+n)(2n+2m) = 2(m+n)(m+n) = 2(m+n)^2$$

□

Theorem 5.2.5. *Let $K^k_{n,m}$ be a k-subdivided complete bipartite graph, then the eccentric connectivity index is $\xi^c(K^k_{n,m}) = 2(k+1)\Big((m+n)^2 + 2mnk\Big)$.*

Proof.

Let $K^k_{n,m}$ be a k-subdivided complete bipartite graph with order $m+n+mnk$ and size $nm(k+1)$.

In $K^k_{n,m}$ graph n vertices have degree m and m vertices have degree n. Moreover, new vertices mnk have degree 2. Only one type of edges according to eccentricity in $K^k_{n,m}$ $\Big(2(k+1), 2(k+1)\Big)$ having count $mn(k+1)$ so by formula of eccentric connectivity index

$$\xi^c(G) = \sum_{\alpha \in V} ecc(\alpha)deg(\alpha)$$

putting values in formula we get

$$
\begin{aligned}
\xi^c(K^k_{n,m}) &= (m+n)\Big(2(k+1)(m+n)\Big) + \Big(2\times 2(k+1)\times mnk\Big)\\
&= 2(k+1)(m+n)^2 + 4mnk(k+1)\\
&= (k+1)\Big(2(m+n)^2 + 4mnk\Big)\\
&= 2(k+1)\Big((m+n)^2 + 2mnk\Big)
\end{aligned}
$$

□

Theorem 5.2.6. *Let $L(K^k_{n,m})$ be a line graph of k-subdivided complete bipartite graph. Then the eccentric connectivity index*

$$
\xi^c(L(K^k_{n,m})) = 4mn(k+1)(m+n+k-1)
$$

Proof.

Let $L(K^k_{n,m})$ be a graph $L(K^k_{n,m})$ be a line graph of k-subdivided complete bipartite graph with order $mn(k+1)$ and size $\frac{nm^2+mn^2}{2}+nm(k-1)$. In $L(K^k_{n,m})$ graph $n,\ \ m$ vertices have degree $m,\ \ n$ respectively and new vertices $mn(k-1)$ have degree 2. Here is formula of eccentric connectivity index

$$
\xi^c(G) = \sum_{\alpha\in V} ecc(\alpha)deg(\alpha)
$$

putting values

$$
\begin{aligned}
\xi^c(L(K^k_{n,m})) &= 2(mn)\times\Big(2(k+1)(m+n)\Big) + 2(k+1)\times 2\times mn(k-1)\\
&= 4mn(k+1)(m+n) + 4mn(k+1)(k-1)\\
&= 4mn(k+1)(m+n+k-1)
\end{aligned}
$$

□

Chapter 6

Conclusions and Open Problem

6.1 Conclusions

In this thesis we formulate the C_4-supermagic labelings of ladder graph more generalize results proved in [21] by Kashif Ali and Syed Tahir Raza Rizvi. Moreover, we find the eccentricity of wheel graph and complete bipartite graph, also for uniform subdivision of both graphs then line graph of uniformly subdivided wheel and complete bipartite graphs. In addition we find some topological indices like eccentric connectivity index and new atomic bond connectivity index of these above graphs.

6.2 Open Problems

Open Problem 1: Find L_n-supermagic labelings of non-isomorphic ladder graph.

Open Problem 2: Find new atomic bond connectivity index for line graph of uniformly subdivided wheel graph.

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