

Cosmological Bouncing Solutions with Different Forms of Hubble Parameter in $f(R,T)$ Gravity



By

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CIIT/FA17-RMT -020/LHR

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Declaration

I **Muhammad Ehsan Khalid** with registration number **CIIT/FA14-RMT-020/LHR** hereby declare that I have produced the work presented in this thesis, during the scheduled period of study. I also declare that I have not taken any material from any source except referred to wherever due, that amount of plagiarism is within acceptable range. If a violation of HEC rules on research has occurred in this thesis, I shall be liable to punishable action under the plagiarism rules of the HEC.

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Certificate

It is certified that **Muhammad Ehsan Khalid (CIIT/FA17-RMT-020/LHR)** has carried out all the work related to this thesis under my supervision at the Department of Mathematics, COMSATS University Islamabad, Lahore Campus and the work fulfills the requirement for award of MS degree.

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DEDICATION

To

My Parents

ACKNOWLEDGEMENTS

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Abstract

According to the recent research, our universe is going through the cosmic accelerated expansion. All observational data is in favor to this theory. It is considered that DE is the main reason behind this cosmic accelerated expansion. In present time, the key interest of cosmologists is to find out the source of this energy. The source of this energy is still under consideration. Cosmologists have proposed various modifications to gravitational field equations to find the exact solution to overcome this problem. We are interested to discuss the bouncing scenario in cosmic evolution for an homogeneous isotropic model by using $f(R, T)$ theory in the frame work of FLRW metric. We considered different forms of HP to construct cosmological models by choosing perfect fluid and viscous fluid. By choosing $f(R, T) = R + 2f(T)$ with $f(T) = \lambda T$ (for perfect fluid case), while $f(T) = \alpha T$ (for viscous fluid case), where λ and α are arbitrary constants. We have derived expressions for EoS (ω), energy density (ρ) and isotropic pressure (p). We have also analyzed the graphical behavior for some specific values of parameters H , q and ω to have successful bounce.

Abbreviations

In this thesis, the signature convention $(+, -, -, -)$ is used. We will use the following list of notations and abbreviations.

SNeIa:	Supernovae Type Ia
CMBR:	Cosmic Microwave Background Radiation
DE:	Dark Energy
DM:	Dark Matter
MTG:	Modified Theories of Gravity
GR:	General Relativity
RDE:	Ricci Dark Energy
FLRW:	Friedmann-Lemaitre-Robertson-Walker
EoS:	Equation of State
HP:	Hubble Parameter
DP:	Deceleration Parameter
EMT:	Energy Momentum Tensor
EFEs:	Einstein Field Equations

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Chapter 1

Introduction

The recent observations indicate that the universe is undergoing accelerated expansion. In current era, supernovae type Ia (SNeIa) [1] has provided a first clue about the transition (deceleration to accelerating) of the cosmos. The recent observations from cosmic microwave background radiation (CMBR), baryonic acoustic oscillations (BAO), observational Hubble parameter data, abundance of galaxy clusters, strong and weak gravitational lensing and the large scale structure confirmed cosmic accelerated expansion of the universe [2]. The reason behind the accelerated expansion is DE, the suitable explanation that it is still a dominant factor causing this. This energy is hypothesized to permeate all over the space and its sharing is 68.3% . The DE creates negative pressure which causes the expansion. The source for this energy is still under consideration, in this regard many theories have been proposed but none of them is able to answer satisfactorily. In order to address these issues different models were developed by doing some amendments in Einstein-Hilbert action, and known as modified gravity. These theories are an important source to understand the issues regarding the DE. Einstein proposed cosmological constant Λ that associates to vacuum energy or DE. The Λ helps to explain the temporarily expansion against attractive effects of gravity, but this also have some issues that need to be resolved. So the doors for modified theories are still open.

Alternative gravitational frameworks (large-distance modifications of gravity) are important to cosmologists as they have brought a fresh insight in the fields of high-energy physics, astrophysics and modern cosmology. To investigate the consequences of cosmic acceleration, MTG gained significant consideration [3, 4]. To address the issues of cosmic expansion, we have two ways, firstly to introduce the scalar field models in Einstein gravity like phantom [5], quintessence [6], secondly by doing some amendments in Einstein Hilbert action. Among the MTG which have been studied so far are $f(R)$ (R is the Ricci scalar) [7], $f(\tau)$ (τ is the torsion) [8], $f(G)$ (G is Gauss-Bonnet invariant) [9]. The

$f(\tau)$ gravity is an important theory and supports the idea that our universe is expanding at an increasing rate. However, many alternative theories were considered for enlightening the expansion of the universe and a couple of them are mentioned here as Brans-Dicke theory [10] and $f(T, T_G)$ [11] to address the issues of cosmology. The $f(R)$ gravity is the simplest and most important theory among others. The $f(R)$ MTG firstly used by Starobinsky to explain the inflation (theory regarding the early time expansion of universe) [12], this theory has key importance because of its models and also helpful to check the behavior of DM. Harko et al., [13] introduced $f(R, T)$ MTG using two parameters namely R , T , he formulated three models by considering three functions in $f(R, T)$ MTG and studied the solutions accordingly.

Houndjo et al., [14] considered $f(R, T)$ MTG and remodeled this theory to show the historical background of cosmic expansion of our universe like in GR through DM and holographic DE. Reddy et al. [15] used coupling like $f(R, T) = R + 2f(T)$ with perfect fluid $L_m = -p$ for constructing a cosmological model by using negative DP q . Alvarenga et al., [16] studied scalar perturbation in $f(R, T)$ MTG and considered coupling like $f_1(R) + f_2(T)$ to develop a model which satisfy the energy conservation equation. To have a suitable cosmological model, Alvarenga et al., [17] tested the stability of both de sitter and power law models in $f(R, T)$ MTG for some parametric values. Samanta and Dhal [18] worked in $f(R, T)$ MTG with perfect fluid as matter source to get such models having high dimensions, also discussed the phenomena like luminosity distance (d_L), distance modulus (μ_z) and $H(z)$ through red-shift. Singh and Vijay [19] studied two scale factors namely de Sitter and power law by choosing two fluids as perfect fluid and exotic fluid. They analyzed the behavior of $f(R, T)$ fluid in the presence and absence of usual matter. The $f(R, T)$ MTG shows behavior of DE and cosmological constant with matter and without matter respectively. They also explored the behavior of universe in power law case by using EoS ω

as well as density ρ . Sharif and Zubair [20] used Ricci DE model besides modified RDE model in $f(R, T)$ theory and concluded with the same result as in the present observational data, also interrogated the continuity equation by choosing particular model in $f(R, T)$ MTG [21]. Azizi studied the possibilities for wormhole geometry through $f(R, T)$ MTG [22]. Katore and Shaikh [23] considered Kantowski-Sachs space that consisting of perfect fluid as a source to get solution in $f(R, T)$ MTG.

The cosmological models say that our universe is the resultant of Big Bang. It was started from a singularity. In recent times the term big bang replaced by big bounce that has gained significant attention, these cosmological models are known as oscillatory or simply cyclic universe. By cyclic universe, it means firstly our universe contracts to singularity then expands to a new one and this process continues. According to cosmology, our universe was born in a massive explosion and an idea to support this theory is that it is still expanding. This theory is still hard to believe.

Alan Guth developed a theory called inflationary theory (early time expansion of the universe) to overcome cosmological problems which is helpful to describe properties of universe. To answer the initial singularity problem, an alternative theory called bouncing model of the universe was developed during mid of 1980's. Kuiroukidis [24] analyzed $f(R)$ MTG in the context of flat FLRW metric to construct a power law model, he also get solutions for some parametric values to explain the bouncing model. Bamba et al., [25] constructed $f(R)$ besides $f(R)$ MTG models considering exponential and power law forms to observe bounce. They also analyzed scale factors in $f(R)$ MTG to observe the expansion history of the universe and the stability of the solutions of models. Makarenko and Myagky [26] constructed a cosmic bouncing model in $f(G)$ MTG with exponential scale factor. They analyzed this model to observe bounce for early universe and studied the asymptotic behavior for late-time cosmic ac-

celeration. T. Singh et al., [27] considered Brans-Dicke theory with non-minimal coupling to have bouncing universe. Shabani and Ziaie [28] analyzed the results of bouncing models in $f(R, T)$ MTG with flat FLRW metric by choosing $f(R, T) = R + h(T)$ and perfect fluid, and also studied $f(R, T)$ MTG with different models to get nonsingular bounce. Shabani and Ziaie also concluded that if certain form of function $f(R, T)$ is considered, then models solutions will be free of future singularities. C.P Singh and V. Singh [29] examined particle creation through $f(R, T)$ MTG by considering FLRW space time. They used a functional form as $f(R, T) = R + 2f(T)$ as well as "Gamma law" EoS to study the role of particle creation in early as well as late time growth of the universe. Moraes and Sahoo [30] constructed the cosmological model by using $f(R, T) = R + \alpha RT$ in $f(R, T)$ MTG, also determined that by assigning values to parameters and using energy conditions, we can predict the phase from decelerated to accelerated. Myrzakulov and Sebastiani [31] analyzed cosmological model in context of FRW metric by considering viscous fluid as matter content filled in the universe and also determined the role of the fluid to built a relation among bounce and finite-time singularities. P. S. Debnath [32] studied the cosmic models in $f(R, T)$ MTG in context of FLRW metric by choosing $f(R, T) = R + 2\lambda T$ to get solution and ended with the result same as present observational data. The behavior of the model is described through Eckart, Truncated Israel Stewart (TIS) and Full Israel Stewart (FIS) theories. These theories are also helpful to study the deviation in the bulk viscous pressure, energy density (ρ), scale factor, H and q by cosmic evolution. S. Ram and P. Kumari [33] explored the Bianchi I (BI) and Bianchi V (BV) to construct cosmological models by using viscous fluid as matter source and $f(R, T) = R + 2f(T)$ with some values to scale factor to get numerical solution, the dynamical behavior was checked as well of the model in $f(R, T)$ MTG, and examined that at first model accelerates then shows deceleration. Kiran and Reddy [34] explored the Bianchi III (BIII) by considering bulk viscous

fluid and analyzed that this model is not viable in $f(R, T)$ MTG. Reddy et al. [35] used Bianchi *II* to construct a model with viscous fluid as a matter source in $f(R, T)$ MTG. Chakraborty [36] considered conserved EMT with same field equation and explored the energy conditions with perfect fluid in $f(R, T)$ MTG, the properties of homogeneous isotropic cosmic model were studied as well. We aimed to construct a suitable cosmological model to study the bouncing solution besides evolution of universe through MTG namely $f(R, T)$. Hence, we get solution to EFEs in $f(R, T)$ MTG with reference to homogeneous and isotropic flat FLRW space-time. We consider both perfect fluid and viscous fluid separately as a matter source filled in the universe. In particular, we considered five different forms of scale factor $a(t)$ [37], which is time dependent for both cases as viscous and perfect fluids separately. We used the specific functional form $f(R, T) = R + 2f(T)$ with $f(T) = \lambda T$ (for perfect fluid case) while $f(T) = \alpha T$ (for viscous fluid case), where α and λ are constants. The matter lagrangian L_m for perfect and viscous fluid is considered as $L_m = -p$ and $L_m = -\bar{p}$ respectively. We obtained the expressions for DP q , energy density ρ as well as EoS parameter ω for each form of HP. We are interested to analyze the restrictions on model parameters for observation of cosmic accelerated expansion of universe. The dynamical behavior of model parameters is also analyzed. The outline for thesis has following sequence: Chapter 2 consists of brief review about important preliminaries regarding our work. In chapter 3, we analyzed the restrictions to model parameters in order to get a suitable model for successful bounce of the universe. We also studied the behavior of HP (H), DP (q) and EoS (ω) for constraints to get bouncing model. In chapter 4, we checked the results for viscous fluid case as well for having successful bounce. In chapter 5, we summarized the results.

Chapter 2

Preliminaries

This chapter has brief review of important terminologies which are necessary to understand this thesis.

2.1 Dark Energy

In our universe, DE is among those factors that are the cause of accelerated expansion of the universe, the suitable explanation to this expansion is that the DE is still a dominant factor in our universe. This energy covered the whole space, according to Planck data its sharing is 68.3 % , whereas the DM contributes 26.8% . The DE creates negative pressure which overcomes the positive pressure (attractive just like gravitational force). It counteracts to gravity and as a result our universe observe accelerated expansion. The cosmologist tried to figure out the source of DE, but it is still under consideration. In this regard, many theories have been proposed to identify the source.

2.2 FLRW Space Time

In GR, FLRW space time helps to explain that the universe is homogeneous, isotropic and observing the expansion as well as contraction. The scale factor which is time dependent derived by using only EFE's. The flat FLRW space time is

$$ds^2 = dt^2 - a^2(t)dx^2 - a^2(t)dy^2 - a^2(t)dz^2,$$

here $a(t)$ is scale factor and it is a time dependent quantity. By observational and experimental data like WMAP and Planck, cosmologists concluded that our universe is homogeneous as well as isotropic and confirmed FLRW metric.

2.3 Hubble Parameter

HP H is an important parameter in cosmology. It is a time dependent quantity. It lays emphasis on that the universe is expanding and it is defined through a scale factor or FLRW factor $a(t)$. Indeed this parameter can also be defined by the ratio of $\dot{a}(t)$ and $a(t)$, simply rate of expansion. It is given as

$$H = \frac{\dot{a}(t)}{a(t)},$$

where the dot in $\dot{a}(t)$ represents time derivative. The sign of the HP is very crucial in this regard that either the universe is observing expansion, contraction or having a successful bounce. The HP is also helpful to get bouncing solutions to Einstein field equations.

2.4 Deceleration Parameter

The deceleration parameter q is among those important parameters used in cosmology and it is a time dependent quantity. This parameter helps in description that how universe expands and gradually slows down just because of self gravitation. It is given as

$$q = -\frac{\ddot{a}a}{\dot{a}^2} = -1 - \frac{\dot{H}}{H^2},$$

here $a(t)$ is a scaling factor and overhead dot in $\dot{a}(t)$ represents time derivative. The recent observations reveal that the universe is undergoing accelerated expansion. We are interested in accelerating case only. It is possible only when $\ddot{a} > 0$, and for this case the deceleration parameter has negative sign. The negative sign of DP is because of DE. Before the era dominated by DE, it was considered that the DP was positive.

2.5 Fluids

Any substance that has the property to change its shape or deform when the external force or shear stress is applied. It contains three states of the matter named by liquid, gas and plasma. Fluid has some physical properties like pressure, density and viscosity etc.

2.5.1 Perfect Fluid

A perfect fluid or ideal fluid is one that needs energy density ρ besides isotropic pressure p to define it or for its complete description. Perfect fluid does not exist but sometimes considered only to solve problems. Perfect fluid has not acted by external force not having properties like viscosity as well as heat conduction. The EMT for an isotropic perfect fluid is

$$T_{\mu\nu} = (\rho_m + P_m)u_\mu u_\nu - P_m g_{\mu\nu},$$

here u_μ represents four velocity.

2.5.2 Imperfect Fluid

A fluid which is not perfect means having properties like heat conduction and viscosity. The properties of the isotropic fluid are measured regardless of the direction. The EMT for an isotropic fluid is given as

$$T_{\mu\nu} = (\rho_m + P_m - \xi\theta)u_\mu u_\nu + (p + \xi\theta)g_{\mu\nu} + q_\mu u_\nu + q_\nu u_\mu - 2\eta\sigma_{\mu\nu},$$

in above expression the parameters q_μ , θ , ξ , η , $\sigma_{\mu\nu}$ represent heat flux, expansion, bulk viscosity, shear viscosity and shear tensor respectively.

2.5.3 Viscous Fluid

The ability of the fluid to show resistance against the external force is known as viscosity. The viscous fluid has large amount of viscosity unlike perfect fluid. Honey and ketchup are examples of viscous fluid. The EMT for viscous fluid is given as

$$T_{\mu\nu} = \rho u_\mu u_\nu - \bar{p} h_{\mu\nu},$$

here $\bar{p} = p - 3\xi H$, and $\xi = \xi_0 + \xi_0 H$.

2.6 EoS Parameter

The most valuable parameter in cosmology is EoS parameter and may be a constant or vary with time. This parameter is not only helpful for description of how universe expands, but plays a vital role as well to determine the properties besides evolution of DE. It is given by ω that is a dimensionless quantity defined by the ratio of matter pressure $P(t)$ and energy density $\rho(t)$. It is given mathematically as

$$\omega = \frac{P}{\rho}.$$

The EoS is considered in order to classify universe into deceleration and acceleration phases. For deceleration and acceleration phases, we have the following regions:

- Radiation: ($0 < \omega < \frac{1}{3}$)
- Cold DM: ($\omega = 0$)
- Quintessence region: ($-1 < \omega < \frac{-1}{3}$)
- Phantom region: ($\omega < -1$)
- Cosmological constant: ($\omega = -1$)

2.7 Properties of Fluid

There are some important properties of the fluid like density, viscosity, pressure etc. These properties are explained as:

2.7.1 Energy Density

Density ρ is the mass per unit volume. It is given by $\rho = \frac{m}{v}$. The difference between mass and density is on the basis of measurement. Mass measures by the whole of the object whereas density is measured by per unit volume. A fluid with large density will flow slowly than the fluid with less density. Compressibility change with pressure for compressible fluid and does not change for incompressible fluid.

2.7.2 Viscosity

The ability of the fluid to show resistance against the external force is known as viscosity. It is determined by the ratio of shearing stress and velocity gradient. It is represented by η and given as:

$$\eta = \frac{\frac{F}{A}}{\frac{dv_x}{dz}}.$$

There are two kinds of viscosity, absolute viscosity η and kinematic viscosity ν . The kinematic viscosity is the measurement of resistance of fluid due to gravity, depends on the temperature and nature of the fluid. It does not depend on pressure unless extremely high pressure is applied. The viscous fluid has large amount of viscosity unlike perfect fluid. Honey and ketchup are examples of viscous fluid.

2.7.3 Pressure

The pressure is a force that is exerted per unit area perpendicularly to the surface of the object. A material with more pressure will have large gravitational effects and the universe will go through decelerated expansion phase. In GR, pressure is an important source for gravitational effect. The DE has negative pressure which causes the universe to observe accelerated expansion confirmed by the recent time observational data. It means that the pressure in present universe will be negative. The EoS parameter is important factor in cosmology to explain expansion history and phases of expansion and pressure is needed to define ω .

2.8 Bouncing Cosmology

Cosmology is the branch of metaphysics that studies the properties, origin, deceleration and acceleration of universe. It studies the nature besides structure of the universe. It is concerned with the future events as well. Cosmologists are still confused over the theory that either there is only one universe or many exist. The big bang theory says that our universe was born from singularity then started expansion and it still continues. An important term or theory that gained attention is big bounce which is a replacement of big bang. The big bounce is a cosmological model used to know about the origin and the transition from contraction to expansion of the universe. The cosmological models also known as oscillatory or cyclic universe. By cyclic universe, it means our universe contracts to singularity then expands to a new one and this process continues.

The bounce is a single event or for an oscillatory universe it repeats in systematic way and can be separated through the phases of contraction and expansion. GR says that the universe is either facing expansion or contraction. The cosmological constant Λ was used by Einstein and it was intended to support his idea that the universe is static. In 1929 Edwin Hubble discovered that we are living in a universe which is expanding. The observational data like supernova shows that the universe is accelerating. An "inflation" theory explains expansion of universe in early times. This is completely understood through the space stretching, means distance among the galaxies increasing with time. This phenomena is verified by redshift.

The Hubble law shows that billion years ago the matter in universe was closer and more dense than today, then universe starts expanding and still continues. This phenomena is known as big bang theory and not only shows how universe was begin, it tells us about the conversion of all forms of energy into matter as well as radiations. Thus a new phase of expansion and contraction starts. In this regard many MTG are proposed to overcome the cosmological issues. The $f(R, T)$ MTG is the most fruitful theory in recent times. Harko et al. first used this theory and then Houndjo remodeled it by considering particular functional form.

2.9 $f(R, T)$ Theory of Gravity

To address the issues regarding sources and behavior of DE, many MTG were proposed. The $F(R)$, $f(\tau)$, $f(G)$ and $f(R, T)$ are few among these theories. The $f(R)$ MTG although the simplest but has great impact and it was firstly used by Starobinsky to explain the inflation [12]. This theory has key importance because of its models. The $f(\tau)$ gravity is an important theory and supports the idea that our universe is expanding at an increasing rate. Harko et al., [13] used

$f(R, T)$ MTG through considering two parameters R and T , also formulated three models like

- $f(R, T) = R + 2f(R)$,
- $f(R, T) = f_1(R) + f_2(T)$,
- $f(R, T) = f_1(R) + f_2(R)f_3(T)$.

Houndjo et al., [14] remodeled $f(R, T)$ MTG to show the historical background of cosmic expansion of our universe like in GR through DM and holographic DE. Reddy et al., [15] used coupling like $f(R, T) = R + 2f(T)$ with perfect fluid $L_m = -p$ for constructing a cosmological model by using negative DP. Alvarenga et al., [16] studied scalar perturbation in $f(R, T)$ MTG and chosen $f_1(R) + f_2(T)$ to develop a model which satisfies the energy conservation equation. The action for $f(R, T)$ MTG is

$$S = \frac{1}{16\pi G} \int f(R, T) \sqrt{-g} dx^4 + \int L_m \sqrt{-g} dx^4, \quad (2.1)$$

L_m denotes Lagrangian matter and G , g represents the gravitational constant and determinant of $g_{\mu\nu}$ respectively. By varying the Eq.(2.1) action for metric components, we get generalized field equation as:

$$\begin{aligned} 8\pi T_{\mu\nu} - f_T(R, T)T_{\mu\nu} - f_T(R, T)\Theta_{\mu\nu} &= f_R(R, T)R_{\mu\nu} - \frac{1}{2}f(R, T)g_{\mu\nu} \\ &+ (g_{\mu\nu}\square - \nabla_\mu \nabla_\nu)f_R(R, T), \end{aligned} \quad (2.2)$$

where $\Theta_{\mu\nu}$ is defined by

$$\Theta_{\mu\nu} = \frac{g^{\alpha\beta}\delta T_{\mu\nu}}{\delta g^{\mu\nu}} = -2T_{\mu\nu} + g_{\mu\nu}L_m - 2g^{\alpha\beta} \frac{\partial^2 L_m}{\partial g^{\mu\nu} \partial g^{\alpha\beta}}.$$

By contracting the Eq.(2.2), we acquire an expression for Ricci scalar R and T (Trace of EMT) as:

$$8\pi T - f_T(R, T)T - f_T(R, T)\Theta = f_R(R, T)R + 3\square f_R(R, T) - 2f(R, T). \quad (2.3)$$

2.9.1 Case-I

We considered here perfect fluid as a matter source filled in our universe and the matter lagrangian is given by $\mathcal{L}_m = -P_m$, which leads to the following expression for $\Theta_{\mu\nu}$

$$\Theta_{\mu\nu} = -2T_{\mu\nu} - P_m g_{\mu\nu}. \quad (2.4)$$

With the help of Eq.(2.4) and the EMT for perfect fluid, the Eq.(2.2) has the following form

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = k_{eff}^2 T_{\mu\nu}^{eff}, \quad (2.5)$$

where $k_{eff}^2 = \frac{k^2 + f_T}{f_R}$ is the effective gravitational constant and

$$T_{\mu\nu}^{eff} = T_{\mu\nu} + \frac{1}{k^2 + f_T} \left[\frac{1}{2}g_{\mu\nu}(f - Rf_R) + f_T P_m g_{\mu\nu} - (g_{\mu\nu} - \nabla_\mu \nabla_\nu) f_R \right],$$

is the effective energy momentum tensor, where f_T , f_R represent the derivative of $f(R, T)$ w.r.t T and R respectively. Applying the covariant divergence to the field Eq.(2.2), one can find [?]

$$\nabla^\alpha T_{\alpha\beta} = \frac{f_T}{\kappa^2 - f_T} \left[(T_{\alpha\beta} + \Theta_{\alpha\beta}) \nabla^\alpha \ln f_T + \nabla^\alpha \Theta_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} \nabla^\alpha T \right]. \quad (2.6)$$

We use $f(R, T) = R + 2f(T)$ with $f(T) = \alpha T$ and $f(T) = \lambda T$ in Eq.(2.6), where α and λ are arbitrary constants.

$$G_{\mu\nu} + (g_{\mu\nu} \square - \nabla_\mu \nabla_\nu) = [8p + 2f_T] T_{\mu\nu} + 2[f_T P + \frac{1}{2}f(T)] g_{\mu\nu}. \quad (2.7)$$

The field equations corresponding to Eq.(2.7) are

$$3H^2 = (8\pi + 3\lambda_1)\rho - \lambda_1 p, \quad (2.8)$$

$$2\dot{H} + 3H^2 = -(8\pi + 3\lambda_1)p + \lambda_1 \rho, \quad (2.9)$$

2.9.2 Case-II

We also considered viscous fluid as a matter source filled in our universe and the matter lagrangian is given by $\mathcal{L}_m = -\bar{p}$, which leads to the following expression for $\Theta_{\mu\nu}$

$$\Theta_{\mu\nu} = -2T_{\mu\nu} - \bar{p}g_{\mu\nu}. \quad (2.10)$$

Similarly, we can obtain the results for this case as obtained for case-I.

Chapter 3

Bouncing Solutions in $f(R, T)$ Gravity with Perfect Fluid

3.0.3 Exponential Evolution Model

In this section, we examine the first case by considering scale factor as

$$a(t) = A \exp\left(\alpha \frac{t^2}{t_*^2}\right), \quad (3.1)$$

with t_* arbitrary time, while $A > 0$, $\alpha > 0$ are constants. For model defined in Eq.(3.1), the HP is

$$H = \frac{2\alpha t}{t_*^2}, \quad (3.2)$$

and the expression for DP is

$$q = -1 - \frac{t_*^2}{2\alpha t^2}. \quad (3.3)$$

By solving Eq.(2.8) and Eq.(2.9) simultaneously and using assumption of HP, we get the expression for energy density ρ and pressure P as:

$$\rho = \frac{-\alpha}{2t_*^2(4\pi + \lambda_1)(2\pi + \lambda_1)} \left[\lambda_1 + \frac{3\lambda_1\alpha t^2}{t_*^2} - \frac{3\alpha t^2(8\pi + 3\lambda_1)}{t_*^2} \right], \quad (3.4)$$

$$P = \frac{-\alpha}{2(2\pi + \lambda_1)} \left[\frac{(8\pi + 3\lambda_1)}{(4\pi + \lambda_1)} + \frac{6\alpha t^2(8\pi + 3\lambda_1)}{t_*^2} \right]. \quad (3.5)$$

The EoS parameter is

$$\omega = \frac{t_*^2(4\pi + \lambda_1) \left[\frac{(8\pi + 3\lambda_1)}{(4\pi + 3\lambda_1)} + \frac{6\alpha t^2(8\pi + 3\lambda_1)}{t_*^2} \right]}{\left[\lambda_1 + \frac{3\lambda_1\alpha t^2}{t_*^2} - \frac{3\alpha t^2(8\pi + 3\lambda_1)}{t_*^2} \right]}. \quad (3.6)$$

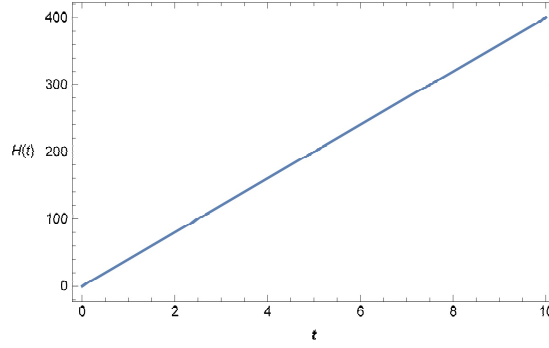


Figure 3.1: The graph of H vs t , while $\alpha = 5$ and $t_* = 0.3$ are fixed here. We can easily observe from the figure that at $t = 0$, HP is $H = 0$, so the universe will have bounce. For $t > 0$, HP is $H > 0$, which is the case of expansion of the universe.

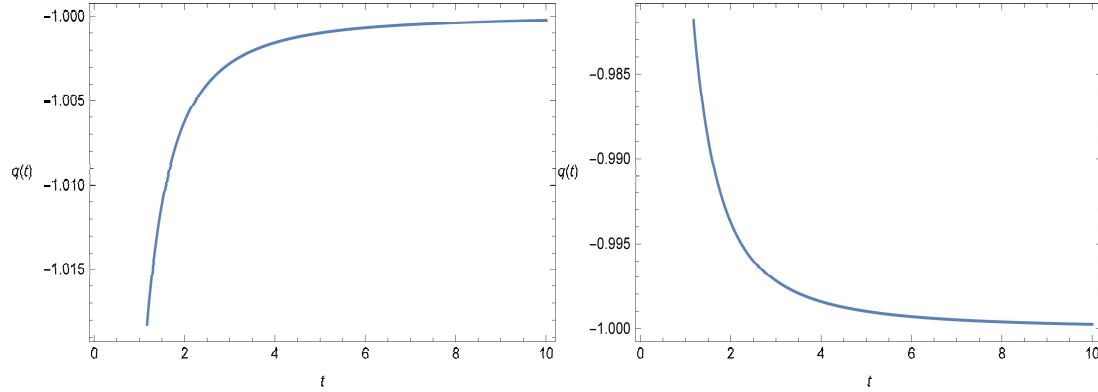


Figure 3.2: The graph on the left side is of q vs t , while $\alpha = 5$ and $t_* = 0.3$ are fixed here. The graph on the right side also of q vs t , here $\alpha = -5$ and $t_* = 0.3$ are fixed parameters. q depends on t and its signature depends on the model parameters. We observed that in both cases with $\alpha = 5$ and $\alpha = -5$, q is approaching to -1 , and represents the accelerated expansion of the universe.

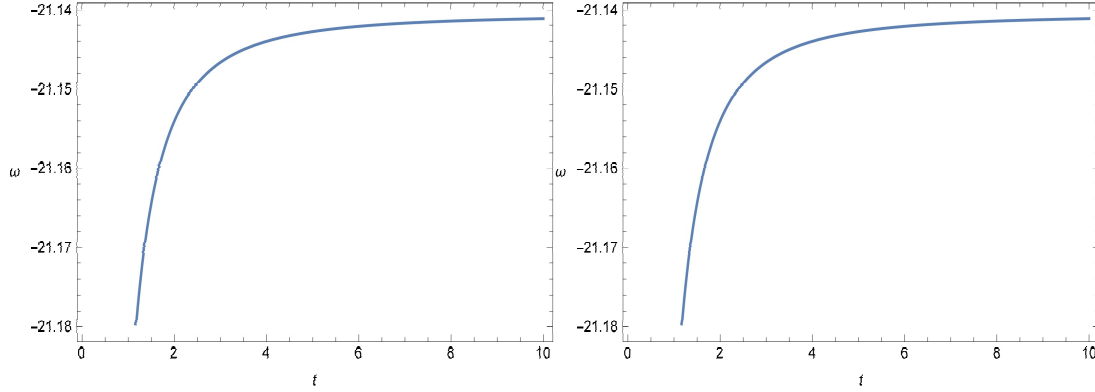


Figure 3.3: The graph on the left side is of ω vs t , while $\alpha = 5$, $\lambda_1 = 3$ and $t_* = 0.3$ are fixed. The graph on the right side also of ω vs t , while $\alpha = 5$, $\lambda_1 = -10$ and $t_* = 0.3$ are fixed. In both case with $\lambda_1 = 3$ and $\lambda_1 = -10$, EoS parameter is $\omega < -1$ which shows the phantom region and confirmed the accelerated expansion of the universe.

3.0.4 Oscillatory Model

We considered oscillatory scale factor for second bouncing model as

$$a(t) = A \sin^2 \left(B \frac{t}{t_*} \right), \quad (3.7)$$

here t_* represents arbitrary time, while $A > 0$, $B > 0$. Here t_* and $B > 0$ are simply nonzero. For model defined in Eq.(3.7), we have the following form of HP

$$H = \frac{2B}{t_*} \cot \left(B \frac{t}{t_*} \right). \quad (3.8)$$

For Hubble parameter defined in Eq.(3.8), the DP is

$$q = -1 + \frac{\csc^2 \left(\frac{Bt}{t_*} \right)}{\cot^2 \left(\frac{Bt}{t_*} \right)}. \quad (3.9)$$

With the help of Eq.(2.8) and Eq.(2.9) and using assumption of HP, we have expression for energy density ρ and pressure P of the form

$$\begin{aligned} \rho &= \frac{B}{2t_*(4\pi + \lambda_1)(2\pi + \lambda_1)} \left[2\lambda_1 \cot\left(B\frac{t}{t_*}\right) \csc^2\left(B\frac{t}{t_*}\right) - 3\lambda_1 \frac{B}{t_*} \cot^2\left(B\frac{t}{t_*}\right) \right. \\ &\quad \left. + 3\frac{B}{t_*}(8\pi + \lambda_1) \cot^2\left(B\frac{t}{t_*}\right) \right], \end{aligned} \quad (3.10)$$

$$\begin{aligned} P &= \frac{B(8\pi + 3\lambda_1)}{2t_*(4\pi + 3\lambda_1)(2\pi + \lambda_1)} \left[2 \cot\left(B\frac{t}{t_*}\right) \csc^2\left(B\frac{t}{t_*}\right) - 3\frac{B}{t_*} \cot^2\left(B\frac{t}{t_*}\right) \right. \\ &\quad \left. + 3\lambda_1 \frac{B}{t_*(8\pi + 3\lambda_1)} \cot^2\left(B\frac{t}{t_*}\right) \right]. \end{aligned} \quad (3.11)$$

The EoS parameter is

$$\omega = \frac{(8\pi + 3\lambda_1) \left[2 \csc^2\left(B\frac{t}{t_*}\right) - 3\frac{B}{t_*} \cot\left(B\frac{t}{t_*}\right) + 3\lambda_1 \frac{B}{t_*(8\pi + 3\lambda_1)} \cot\left(B\frac{t}{t_*}\right) \right]}{2\lambda_1 \csc^2\left(B\frac{t}{t_*}\right) - 3\lambda_1 \frac{B}{t_*} \cot\left(B\frac{t}{t_*}\right) + 3\frac{B}{t_*}(8\pi + \lambda_1) \cot\left(B\frac{t}{t_*}\right)}. \quad (3.12)$$

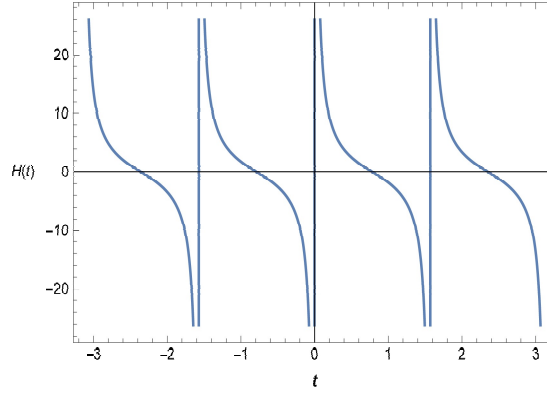


Figure 3.4: The graph of H vs t , while $B = 3$ and $t_* = 1.5$ are fixed.

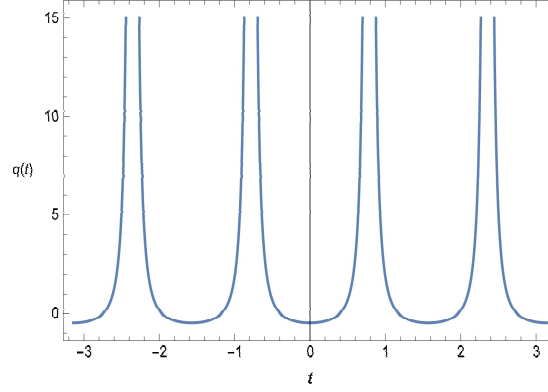


Figure 3.5: The graph of q vs t , while $B = 0.2$, $\lambda_1 = 1.3$ and $t_* = 1.5$ are fixed.

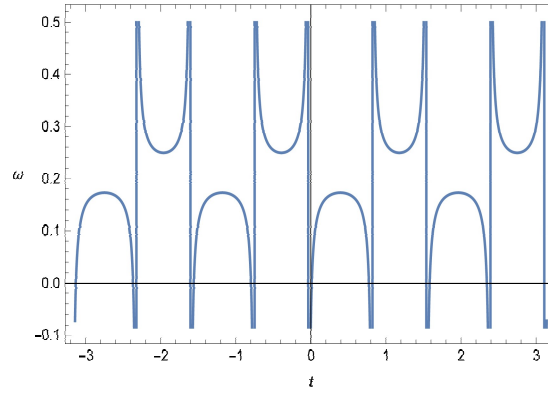


Figure 3.6: The graph of ω vs t , while $B = 0.2$, $\lambda_1 = 1.3$ and $t_* = 1.5$ are fixed.

3.0.5 Power Law Model

The scale factor for power law model is considered as

$$a(t) = \left(\frac{t_s - t}{t_o} \right)^{\frac{2}{c^2}}, \quad (3.13)$$

with t_s time when bounce occurs, t_o is arbitrary time parameter which defines scale factor to be 1 when $t = t_s + t_o$ and c is any constant. The HP for Eq.(3.13) is

$$H = -\frac{2}{c^2(t_s - t)}. \quad (3.14)$$

For Eq.(3.14), the expression for DP is

$$q = -1 + \frac{c^2}{2}. \quad (3.15)$$

By using Eq.(2.8) and Eq.(2.9), we have

$$\rho = \frac{1}{c^2(t_s - t)^2(4\pi + \lambda_1)(2\pi + \lambda_1)} \left[\lambda_1 - \frac{3\lambda_1}{c^2} + \frac{3(8\pi + 3\lambda_1)}{c^2} \right], \quad (3.16)$$

$$P = \frac{(8\pi + 3\lambda_1)}{2c^2(t_s - t)^2(4\pi + \lambda_1)(2\pi + \lambda_1)} \left[1 - \frac{3}{c^2} + \frac{3\lambda_1}{c^2} \right]. \quad (3.17)$$

The EoS parameter is

$$\omega = \frac{(8\pi + 3\lambda_1) \left[1 - \frac{3}{c^2} + \frac{3\lambda_1}{c^2} \right]}{2 \left[\lambda_1 - \frac{3\lambda_1}{c^2} + \frac{3(8\pi + 3\lambda_1)}{c^2} \right]}. \quad (3.18)$$

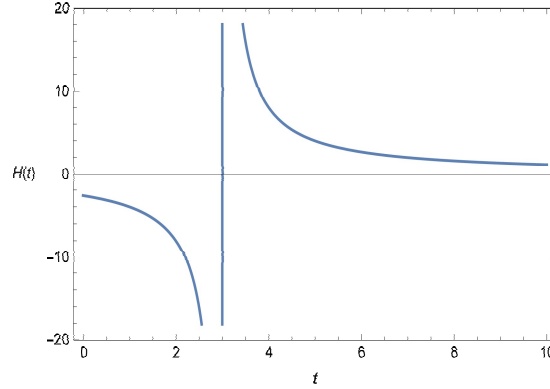


Figure 3.7: The graph of H vs t , while $c = 2$ and $t_s = 0.5$ are fixed here. We observed that for $t < 3$, HP is $H < 0$ and shows the contraction of the universe. For $t > 3$, HP is $H > 0$ is the expansion case, while at $t = 3$, HP is $H = 0$ and universe observes bounce.

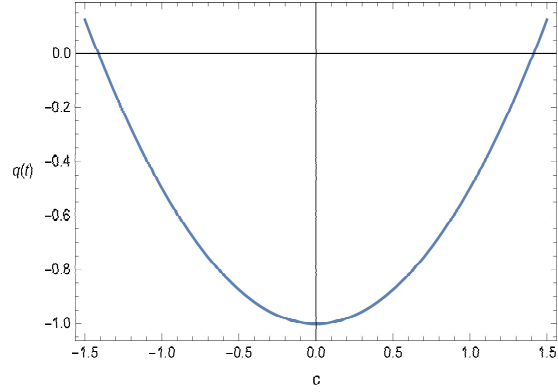


Figure 3.8: The graph of q vs c , while $c = 2$, $\lambda_1 = 3$ and $t_* = 0.5$ are fixed here. We can see that q approaches to -1 , while for $q < 0$ and $q > 0$, universe observes expansion and contraction respectively.

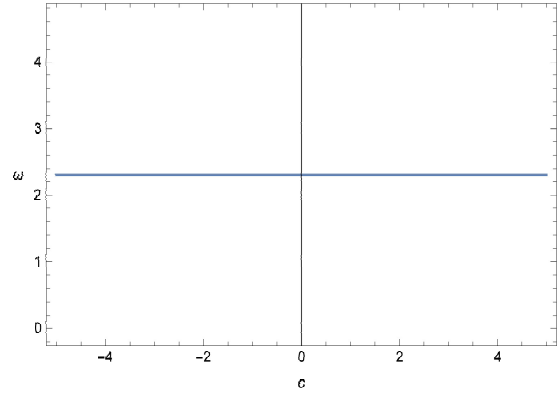


Figure 3.9: The graph of ω vs t , while $c = 2$, $\lambda_1 = 3$ and $t_* = 0.5$ are fixed here. We observed that $\omega > 0$ and represents the radiation region.

3.0.6 Critical Density Model

The scale factor for critical density model is defined by

$$a(t) = A \left(\frac{3}{2} \rho_{cr} t^2 + 1 \right)^{\frac{1}{3}}, \quad (3.19)$$

with ρ_{cr} as critical density, while $A > 0$ simply a non zero constant. The corresponding HP for Eq.(3.19) is

$$H = \frac{2t\rho_{cr}}{2 + 3\rho_{cr}t^2}. \quad (3.20)$$

For Eq.(3.20), the expression for DP is

$$q = \frac{1}{2} - \frac{1}{t\rho_{cr}}. \quad (3.21)$$

We used Eq.(2.8) and Eq.(2.9) to have expression for energy density ρ and pressure P as

$$\rho = -\frac{\rho_{cr}}{2(4\pi + \lambda_1)(2\pi + \lambda_1)(2 + 3\rho_{cr}t^2)^2} [2\lambda_1 - 3\rho_{cr}t^2(8\pi + 3\lambda_1)], \quad (3.22)$$

$$P = -\frac{\rho_{cr}}{2(4\pi + \lambda_1)(2\pi + \lambda_1)(2 + 3\rho_{cr}t^2)^2} [16\pi(1 - \rho_{cr}t^2) + 3\lambda_1(2 - 3\rho_{cr}t^2)]. \quad (3.23)$$

And the EoS parameter is

$$\omega = \frac{16\pi(1 - \rho_{cr}t^2) + 3\lambda_1(2 - 3\rho_{cr}t^2)}{2\lambda_1 - 3\rho_{cr}t^2(8\pi + 3\lambda_1)}. \quad (3.24)$$

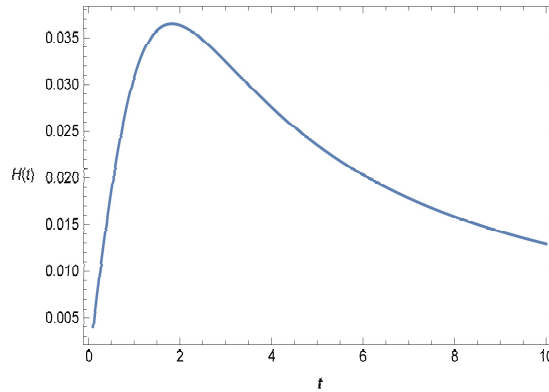


Figure 3.10: The graph of H vs t , while $\lambda_1 = 3$ and $\rho_{cr} = 0.2$ are fixed here. We observed that the $H > 0$ and thus represents the expansion of the universe.

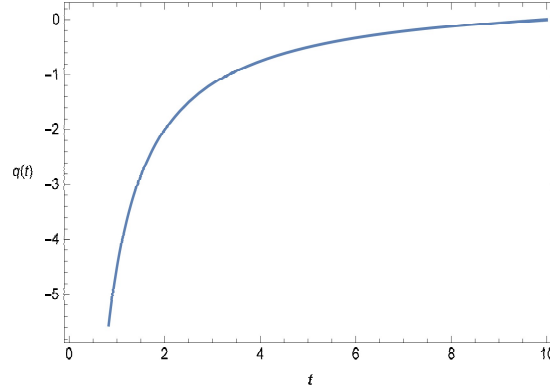


Figure 3.11: The graph of q vs t , while $\lambda_1 = 3$ and $\rho_{cr} = 0.2$ are fixed parameters here. The DP is approaching to 0 but less than zero ($q < 0$) and thus universe observes accelerated expansion.

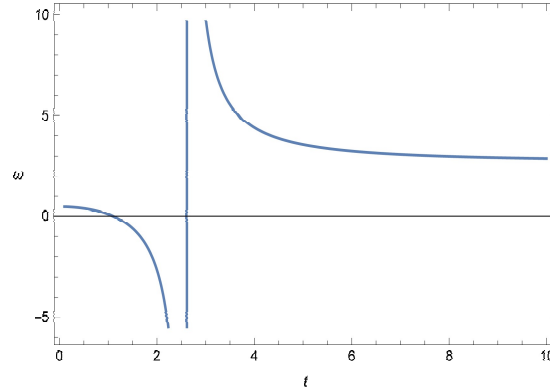


Figure 3.12: The graph of ω vs t , while $\lambda_1 = 3$ and $\rho_{cr} = 0.2$ are fixed. We observed that for $t < 2.5$ and $t > 2.5$, ω represents radiation region, while cold DM for $t = 2.5$.

3.0.7 Exponential Model-II

We considered exponential scale factor as

$$a(t) = A \exp \left[\frac{f_0}{\alpha + 1} (t - t_s)^{\alpha+1} \right], \quad (3.25)$$

with $A > 0$ dimensionless constant, f_0 is arbitrary constant, t_s is time when bounce occurs and α is another constant. The corresponding HP is obtained as

$$H = f_0(t - t_s)^\alpha. \quad (3.26)$$

The expression for DP is

$$q = -1 - \frac{\alpha}{f_0(t - t_s)^{\alpha+1}}. \quad (3.27)$$

The Eq.(2.8) and Eq.(2.9) yields the expression for energy density ρ and P as

$$\begin{aligned} \rho = & -\frac{f_0(t - t_s)^\alpha(8\pi + 3\lambda_1)}{8(4\pi + \lambda_1)(2\pi + \lambda_1)} \left[\frac{2\lambda_1 f_0 + 3\lambda_1 f_0(t - t_s)^{2\alpha}}{(8\pi + 3\lambda_1)(t - t_s)^\alpha} \right. \\ & \left. - 3f_0(t - t_s)^\alpha \right], \end{aligned} \quad (3.28)$$

$$\begin{aligned} P = & -\frac{f_0(t - t_s)^\alpha(8\pi + 3\lambda_1)}{8(4\pi + \lambda_1)(2\pi + \lambda_1)} \left[\frac{2\alpha f_0 + 3f_0(t - t_s)^2}{(t - t_s)} \right. \\ & \left. - \frac{3\lambda_1 f_0(t - t_s)^\alpha}{(8\pi + 3\lambda_1)} \right]. \end{aligned} \quad (3.29)$$

The EoS parameter is

$$\omega = \frac{\frac{2\alpha f_0 + 3f_0(t - t_s)^2}{(t - t_s)} - \frac{3\lambda_1 f_0(t - t_s)^\alpha}{(8\pi + 3\lambda_1)}}{\frac{2\lambda_1 f_0 + 3\lambda_1 f_0(t - t_s)^{2\alpha}}{(8\pi + 3\lambda_1)(t - t_s)^\alpha} - 3f_0(t - t_s)^\alpha}. \quad (3.30)$$

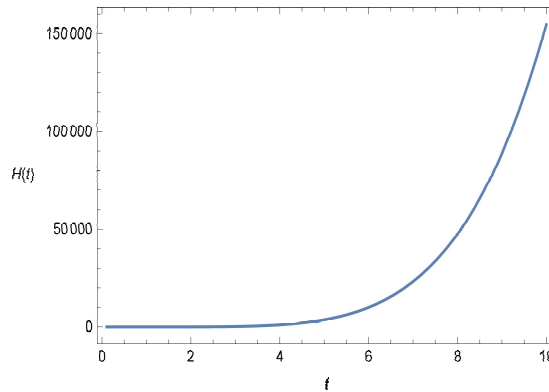


Figure 3.13: The graph of H vs t , while $\alpha = 5$, $t_s = 0.5$ and $f_0 = 2$ are fixed here.

We observed that at $t = 0$, HP is $H = 0$ and universe observes bounce, while for $t > 0$, HP is $H > 0$, universe shows expansion.

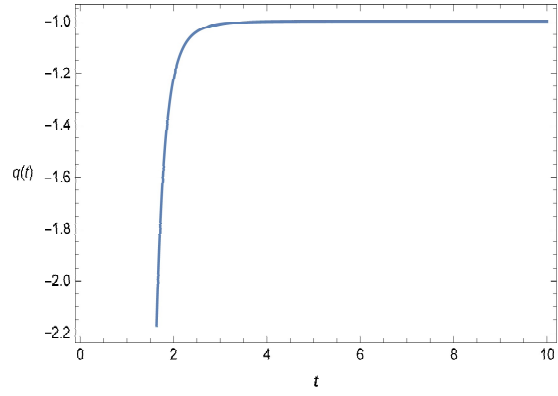


Figure 3.14: The graph of q vs t , while $\alpha = 5$, $t_s = 0.5$ and $f_0 = 2$ are some fixed parameters here. We observed that q is approaching to -1 , for which universe observes expansion.

Chapter 4

Bouncing Solutions in $f(R, T)$ Gravity with Viscous Fluid

4.0.8 Model 1

The HP considered for this model as

$$H(t) = \alpha t \ln \left(\frac{c - \lambda_2 \arctan t}{t} \right), \quad (4.1)$$

here $\alpha, c > 0$ are simply non-zero constants, while λ_2 is a constant. These constants are used to describe the dynamical behavior of universe.

By solving Eq.(2.8) and Eq.(2.9) simultaneously and using assumption of HP, we get the expression for energy density ρ and pressure P as

$$\begin{aligned} \rho &= \frac{\alpha^2}{(1 + 6\alpha + 8\alpha^2)} \left[(8\alpha + 3)t^2 \ln \left(\frac{c - \lambda_2 \arctan t}{t} \right)^2 \right. \\ &\quad \left. - 2 - \frac{2\lambda_2 t}{(1 + t^2)(c - \lambda_2 \arctan t)} + 2 \ln \left(\frac{c - \lambda_2 \arctan t}{t} \right) \right], \end{aligned} \quad (4.2)$$

$$\begin{aligned} P &= \frac{1}{(1 + 6\alpha + 8\alpha^2)} \left[2\alpha \left(1 + \frac{\lambda_2 t}{(1 + t^2)(c - \lambda_2 \arctan t)} - \ln \left(\frac{c - \lambda_2 \arctan t}{t} \right) \right) \right. \\ &\quad - 3\alpha t^2 \ln \left(\frac{c - \lambda_2 \arctan t}{t} \right)^2 (1 + \xi_1 + 6\alpha\xi_1 + \alpha + 8\alpha^2\xi_1) \\ &\quad \left. + 3\alpha t \ln \left(\frac{c - \lambda_2 \arctan t}{t} \right) (\xi_0 + 6\alpha\xi_0 + 8\alpha^2\xi_0) \right]. \end{aligned} \quad (4.3)$$

The EoS parameter is

$$\begin{aligned} \omega &= \frac{\left[2 + \frac{2\lambda_2 t}{(1+t^2)(c-\lambda_2 \arctan t)} - 2 \ln \left(\frac{c-\lambda_2 \arctan t}{t} \right) \right]}{\alpha \left[(8\alpha t^2 + 3t^2 + 1) \ln \left(\frac{c-\lambda_2 \arctan t}{t} \right)^2 - 2 - \frac{2\lambda_2 t}{(1+t^2)(c-\lambda_2 \arctan t)} \right]} \\ &\quad + \frac{3t \ln \left(\frac{c-\lambda_2 \arctan t}{t} \right) (\xi_0 + 6\alpha\xi_0 + 8\alpha^2\xi_0)}{\alpha \left[(8\alpha t^2 + 3t^2 + 1) \ln \left(\frac{c-\lambda_2 \arctan t}{t} \right)^2 - 2 - \frac{2\lambda_2 t}{(1+t^2)(c-\lambda_2 \arctan t)} \right]} \\ &\quad - \frac{3\alpha t^2 \ln \left(\frac{c-\lambda_2 \arctan t}{t} \right)^2 (1 + \xi_1 + 6\alpha\xi_1 + \alpha + 8\alpha^2\xi_1)}{\alpha \left[(8\alpha t^2 + 3t^2 + 1) \ln \left(\frac{c-\lambda_2 \arctan t}{t} \right)^2 - 2 - \frac{2\lambda_2 t}{(1+t^2)(c-\lambda_2 \arctan t)} \right]}. \end{aligned} \quad (4.4)$$

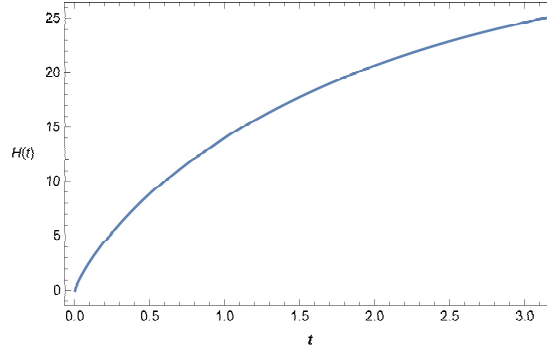


Figure 4.1: The graph of H vs t , while $\alpha = 5$, $\lambda_2 = 2$, and $t_* = 0.3$ are some fixed parameters here. HP is observed as $H > 0$ for $t > 0$, thus it is the case of expansion of the universe, while at $t = 0$, HP is $H = 0$ and universe observes bounce.

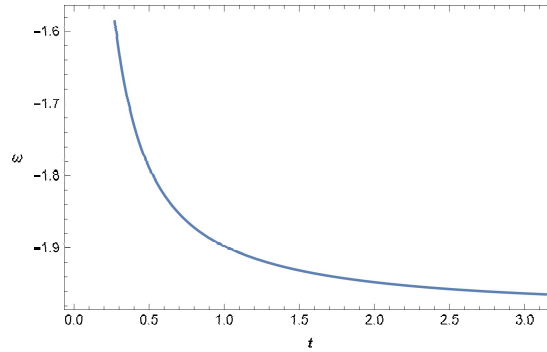


Figure 4.2: The graph of ω vs t , while $\alpha = 5$, $c = 18$, $\lambda_2 = 2$, $\xi_0 = 0.1$ and $\xi_1 = 0.1$ are fixed here. We observed that $\omega < -1$, represents phantom region and thus universe observes accelerated expansion.

4.0.9 Exponential Evolution Model

In this section, we examine the results by considering scale factor as

$$a(t) = A \exp\left(\alpha \frac{t^2}{t_*^2}\right), \quad (4.5)$$

with t_* arbitrary time and $A > 0, \alpha > 0$ are constants. For model Eq.(4.5), the HP is

$$H = \frac{2\alpha t}{t_*^2}. \quad (4.6)$$

For Eq.(4.6), the expression for DP is

$$q = -1 - \left(\frac{t_*}{t}\right)^2. \quad (4.7)$$

By solving Eq.(2.8) and Eq.(2.9) simultaneously and using assumption of HP, we get the expression for energy density ρ and pressure P as:

$$\rho = \frac{4\alpha^2}{t_*^2(1 + 6\alpha + 8\alpha^2)} \left[\frac{(8\alpha + 3)t^2}{t_*^2} + 1 \right], \quad (4.8)$$

$$P = \frac{2\alpha}{t_*^2(1 + 6\alpha + 8\alpha^2)} \left[-2 - \left(\frac{6\alpha t^2}{t_*^2}\right) (1 + \xi_1 + \alpha + 6\alpha\xi_1 + 8\alpha^2\xi_1) \right. \\ \left. + 3t(\xi_0 + 6\alpha\xi_0 + 8\alpha^2\xi_0) \right]. \quad (4.9)$$

The EoS parameter is

$$\omega = \frac{\left[-2 - \left(\frac{6\alpha t^2}{t_*^2}\right) (1 + \xi_1 + \alpha + 6\alpha\xi_1 + 8\alpha^2\xi_1) + 3t(\xi_0 + 6\alpha\xi_0 + 8\alpha^2\xi_0) \right]}{2\alpha \left[\frac{(8\alpha + 3)t^2}{t_*^2} + 1 \right]}. \quad (4.10)$$

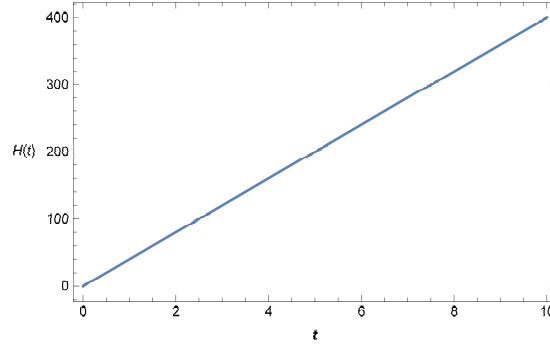


Figure 4.3: The graph of H vs t , while $\alpha = 5$ and $t_* = 0.3$ are fixed here. We can easily observe from the figure that $H > 0$ for $t > 0$, which is the case of expansion of the universe, while at $t = 0$, HP is $H = 0$ and universe observes bounce.

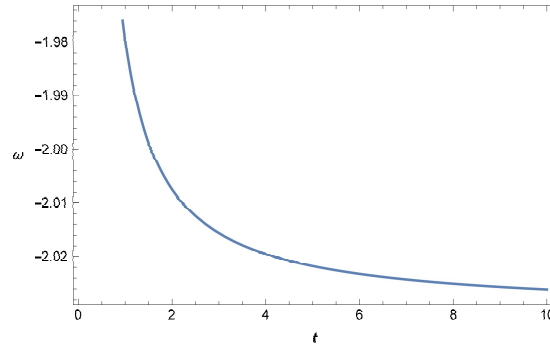


Figure 4.4: The graph of ω vs t , while $\alpha = 5$, $t_* = 0.5$, $\xi_0 = 0.1$ and $\xi_1 = 0.1$ are some fixed parameters here. In this case, we observed $\omega < -1$ shows the phantom region. Thus confirmed the accelerated expansion of the universe.

4.0.10 Oscillatory Model

We considered oscillatory scale factor as

$$a(t) = A \sin^2 \left(B \frac{t}{t_*} \right), \quad (4.11)$$

here t_* , $A > 0$ and $B > 0$ are simply nonzero, while t_* is for arbitrary time. For model Eq.(4.11), the HP is

$$H = \frac{2B}{t_*} \cot \left(B \frac{t}{t_*} \right). \quad (4.12)$$

For Hubble parameter Eq.(4.12), the DP is

$$q = -1 + \frac{\frac{t_*}{B} \csc^2 \left(B \frac{t}{t_*} \right)}{\cot \left(B \frac{t}{t_*} \right)}. \quad (4.13)$$

With the help of Eq.(2.8) and Eq.(2.9) and using assumption of HP, we have expression for energy density ρ and pressure P of the form

$$\rho = \frac{4B^2}{t_*^2(1 + 6\alpha + 8\alpha^2)} \left[(8\alpha + 3) \cot^2 \left(\frac{Bt}{t_*} \right) - \alpha \csc^2 \left(\frac{Bt}{t_*} \right) \right], \quad (4.14)$$

$$\begin{aligned} P = & \frac{2B}{t_*(1 + 6\alpha + 8\alpha^2)} \left[2 \left(\frac{B}{t_*} \csc^2 \left(\frac{Bt}{t_*} \right) \right) - 6 \left(\left(\frac{B}{t_*} \cot^2 \left(\frac{Bt}{t_*} \right) \right) \right. \right. \\ & \left. \left. (1 + \xi_1 + \alpha + 6\alpha\xi_1 + 8\alpha^2\xi_1) + 3 \cot \left(\frac{Bt}{t_*} \right) \right. \right. \\ & \left. \left. (\xi_0 + 6\alpha\xi_0 + 8\alpha^2\xi_0) \right] \right]. \end{aligned} \quad (4.15)$$

The EoS parameter is

$$\omega = \frac{t_* \left[\frac{2B}{t_*} \csc^2 \left(\frac{Bt}{t_*} \right) - 6 \frac{B}{t_*} \cot^2 \left(\frac{Bt}{t_*} \right) (1 + \xi_1 + \alpha + 6\alpha\xi_1 + 8\alpha^2\xi_1) + 3 \cot \left(\frac{Bt}{t_*} \right) (\xi_0 + 6\alpha\xi_0 + 8\alpha^2\xi_0) \right]}{2B \left[(8\alpha + 3) \cot^2 \left(\frac{Bt}{t_*} \right) - \alpha \csc^2 \left(\frac{Bt}{t_*} \right) \right]}. \quad (4.16)$$

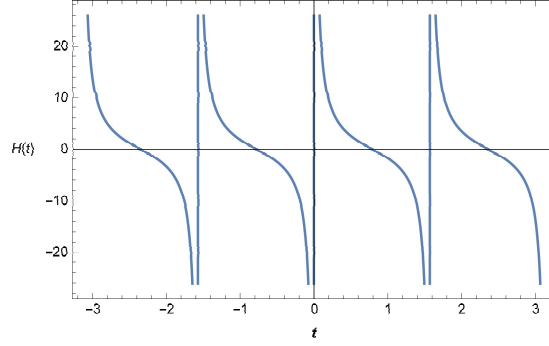


Figure 4.5: The graph of H vs t , while $B = 3$ and $t_* = 1.5$ are fixed here.

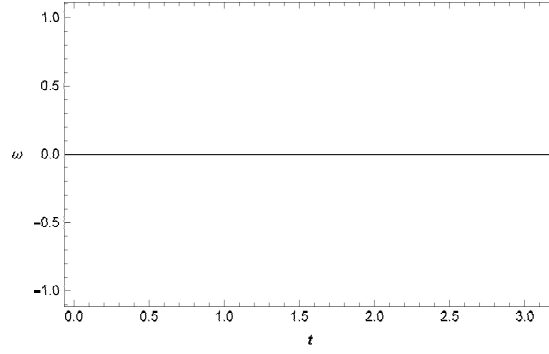


Figure 4.6: The graph of ω vs t , while $\alpha = 5$, $B = 3$, $t_* = 3$, $\xi_0 = 0.1$ and $\xi_1 = 0.1$ are some fixed parameters here.

4.0.11 Power Law Model

The power law model has scale factor as

$$a(t) = \left(\frac{t_s - t}{t_o} \right)^{\frac{2}{c^2}}, \quad (4.17)$$

with t_s time when bounce occurs, t_o is arbitrary time and c is any constant. The HP for Eq.(4.17) is

$$H = -\frac{2}{c^2(t_s - t)}. \quad (4.18)$$

For Hubble parameter in Eq.(4.18), the DP is

$$q = -1 + \frac{c^2}{2}. \quad (4.19)$$

By using Eq.(2.8) and Eq.(2.9), we have

$$\rho = \frac{4}{c^2(t_s - t)^2(1 + 6\alpha + 8\alpha^2)} \left[\frac{(8\alpha + 3)}{c^2} - \alpha \right], \quad (4.20)$$

$$\begin{aligned} P &= \frac{2}{(1 + 6\alpha + 8\alpha^2)c^2(t_s - t)^2} \left[2 - \frac{6}{c^2} (1 + \xi_1 + \alpha + 6\alpha\xi_1 + 8\alpha^2\xi_1) \right. \\ &\quad \left. - 3(t_s - t)(\xi_0 + 6\alpha\xi_0 + 8\alpha^2\xi_0) \right]. \end{aligned} \quad (4.21)$$

The EoS parameter is

$$\omega = \frac{\left[2 - \frac{6}{c^2} (1 + \xi_1 + \alpha + 6\alpha\xi_1 + 8\alpha^2\xi_1) - 3(t_s - t)(\xi_0 + 6\alpha\xi_0 + 8\alpha^2\xi_0) \right]}{2 \left[\frac{(8\alpha + 3)}{c^2} - \alpha \right]}. \quad (4.22)$$

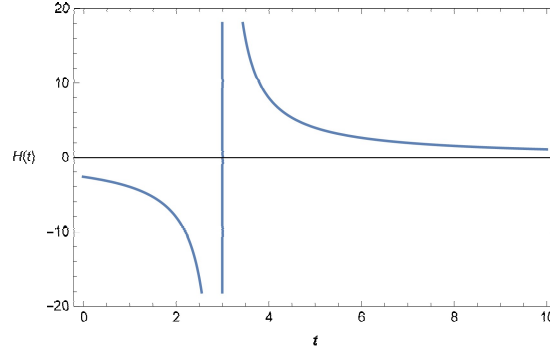


Figure 4.7: The graph of H vs t , while $c = 0.5$, $t_* = 3$ are fixed here. We can observe that at $t = 3$, HP is $H = 0$ represents the bounce. While for $t > 3$, HP is $H > 0$, universe observes expansion and $t < 3$, $H < 0$ is the contraction case.

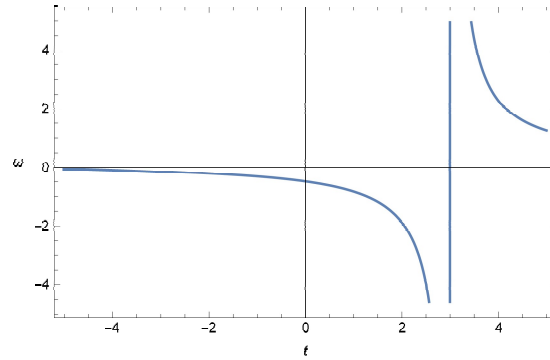


Figure 4.8: The graph of ω vs t , while $\alpha = 5$, $c = 0.5$, $t_* = 3$, $\xi_0 = 0.1$ and $\xi_1 = 0.1$ are some fixed parameters here. We observed that $\omega > -1$ represents quintessence region for $t < 3$, and cold DM ($\omega = 0$) at $t = 3$.

4.0.12 Critical Density Model

The scale factor for critical density model is defined by

$$a(t) = A \left(\frac{3}{2} \rho_{cr} t^2 + 1 \right)^{\frac{1}{3}}, \quad (4.23)$$

with ρ_{cr} as critical density, while $A > 0$ a non-zero constant. The corresponding HP for Eq.(4.23) is

$$H = \frac{2t\rho_{cr}}{2 + 3\rho_{cr}t^2}. \quad (4.24)$$

For Hubble parameter H in Eq.(4.24), the DP is

$$q = \frac{1}{2} - \frac{1}{t\rho_{cr}}. \quad (4.25)$$

We used Eq.(2.8) and Eq.(2.9) to have expression for energy density ρ and pressure P as

$$\rho = \frac{4t\rho_{cr}}{(1 + 6\alpha + 8\alpha^2)(2 + 3\rho_{cr}t^2)^2} \left[t\rho_{cr}(8\alpha + 3) - 3t\alpha\rho_{cr} + \frac{2\alpha}{t} \right], \quad (4.26)$$

$$P = \frac{2\rho_{cr}}{(1 + 6\alpha + 8\alpha^2)(2 + 3\rho_{cr}t^2)^2} \left[\frac{(6\rho_{cr}t^2 - 4)}{(2 + 3\rho_{cr}t^2)} - \frac{(6\rho_{cr}t^2)}{(2 + 3\rho_{cr}t^2)} \right. \\ \left. (1 + \xi_1 + \alpha + 6\alpha\xi_1 + 8\alpha^2\xi_1) + 3t(\xi_0 + 6\alpha\xi_0 + 8\alpha^2\xi_0) \right]. \quad (4.27)$$

And the EoS parameter is

$$\omega = \frac{\left[\frac{6\rho_{cr}t^2 - 4}{2 + 3\rho_{cr}t^2} - \frac{6\rho_{cr}t^2}{2 + 3\rho_{cr}t^2} (1 + \xi_1 + \alpha + 6\alpha\xi_1 + 8\alpha^2\xi_1) + 3t(\xi_0 + 6\alpha\xi_0 + 8\alpha^2\xi_0) \right]}{2t \left[t\rho_{cr}(8\alpha + 3) - 3t\alpha\rho_{cr} + \frac{2\alpha}{t} \right]}. \quad (4.28)$$

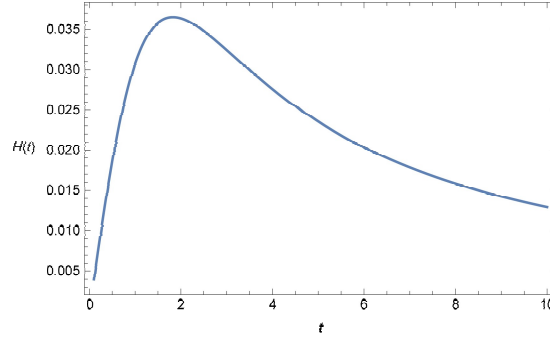


Figure 4.9: The graph of H vs t , while $\rho_{cr} = 0.2$ is fixed here. We observed that the $H > 0$, and thus represents the expansion of the universe.

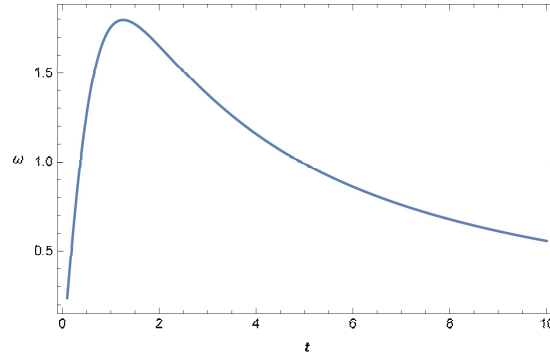


Figure 4.10: The graph of ω vs t , while $\alpha = 5$, $\rho_{cr} = 0.2$, $\xi_0 = 0.1$ and $\xi_1 = 0.1$ are some fixed parameters here. We observed that $\omega > 0$, thus represents the radiation region.

4.0.13 Exponential Model-II

The final model has scale factor as following

$$a(t) = A \exp \left[\frac{f_0}{\alpha + 1} (t - t_s)^{\alpha+1} \right], \quad (4.29)$$

with $A > 0$ dimensionless constant, f_0 is arbitrary constant, t_s is time when bounce occurs and α is another constant. The corresponding HP to Eq.(4.29) is

obtained as

$$H = f_0(t - t_s)^\alpha. \quad (4.30)$$

For Hubble parameter Eq.(4.30), the DP is

$$q = -1 - \frac{\alpha}{f_0(t - t_s)^{\alpha+1}}. \quad (4.31)$$

The solution of Eq.(2.8) and Eq.(2.9) yield the expression for energy density ρ and pressure P as

$$\rho = \frac{1}{(1 + 6\alpha + 8\alpha^2)} [(8\alpha + 3)f_0^2(t - t_s)^{2\alpha} + 2\alpha^2 f_0(t - t_s)^{\alpha-1}], \quad (4.32)$$

$$\begin{aligned} P = & \frac{1}{(1 + 6\alpha + 8\alpha^2)} [-2\alpha f_0(t - t_s)^{\alpha-1} - 3f_0^2(t - t_s)^{2\alpha} \\ & (1 + \xi_1 + \alpha + 6\alpha\xi_1 + 8\alpha^2\xi_1) \\ & + 3f_0(t - t_s)^\alpha (\xi_0 + 6\alpha\xi_0 + 8\alpha^2\xi_0)]. \end{aligned} \quad (4.33)$$

The EoS parameter is

$$\begin{aligned} \omega = & \frac{[-2\alpha f_0(t - t_s)^{\alpha-1} - 3f_0^2(t - t_s)^{2\alpha} (1 + \xi_1 + \alpha + 6\alpha\xi_1 + 8\alpha^2\xi_1)]}{[(8\alpha + 3)f_0^2(t - t_s)^{2\alpha} + 2\alpha^2 f_0(t - t_s)^{\alpha-1}]} \\ & + \frac{[3f_0(t - t_s)^\alpha (\xi_0 + 6\alpha\xi_0 + 8\alpha^2\xi_0)]}{[(8\alpha + 3)f_0^2(t - t_s)^{2\alpha} + 2\alpha^2 f_0(t - t_s)^{\alpha-1}]}. \end{aligned} \quad (4.34)$$

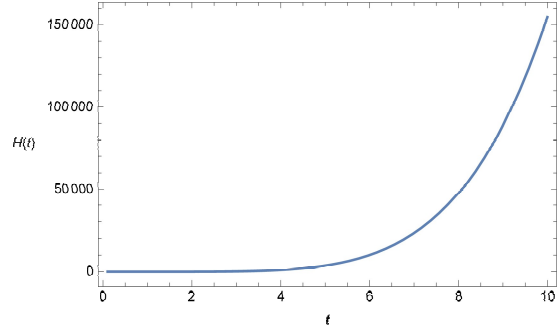


Figure 4.11: The graph of H vs t , while $\alpha = 5$, $t_s = 0.5$ and $f_0 = 2$ are fixed here. We observed that at $t = 0$, $H = 0$, thus universe observes bounce, while for $t > 0$, $H > 0$ universe shows expansion.

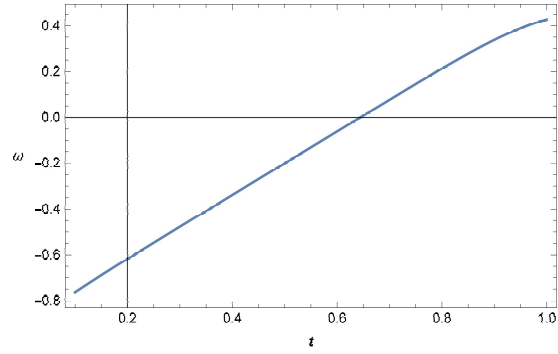


Figure 4.12: The graph of ω vs t , while $\alpha = 5$, $t_s = 0.5$, $f_0 = 2$, $\xi_0 = 0.1$ and $\xi_1 = 0.1$ are some fixed parameters here. We observed that at $t = 0.6$, EoS parameter ω represents the cold DM, while for $t > 0.6$ and $t < 0.6$, it shows the radiation and quintessence region respectively.

Chapter 5

Discussion and Conclusion

In recent times, the bouncing cosmology has become the viable theory. It is just because of that the models do not need the initial singularity and conditions. Alternative gravitational frameworks (large-distance modifications of gravity) are important as they have brought a fresh insight in the fields of high-energy physics, astrophysics and modern cosmology. To investigate the consequences of cosmic acceleration, MTG gained significant consideration. The cosmological models are known as oscillatory or simply cyclic universe. By cyclic universe, it means our universe contracts to singularity, then expands to a new one and this process continues.

In this thesis, we studied the FLRW space time with perfect and viscous fluids as matter source. We used different forms of HP and studied the behavior of DP and EoS parameter. We also got the numerical solution to EFEs for each form of HP by assuming $\lambda_1 = \lambda_2 = \lambda$ and also assumed $f(t) = \lambda_1 T$. we discussed the necessary conditions to models parameter in order to get bouncing solutions or to have a successful bounce.

The scale factor $a(t)$ and deceleration parameter are crucial to know about the expansion, contraction and bouncing point of the universe. Universe shows contraction for $(\dot{a}(t) < 0)$ and $(q > 0)$, while universe expands for $(\dot{a}(t) > 0)$ and $(q < 0)$, they both approach to zero for bouncing point.

The sign of HP is very important in order to check the behavior of universe either the universe shows expansion or contraction. The Hubble parameter observes as $H < 0$, $H > 0$ and $H = 0$ when universe contracts, expands and having a bounce respectively. For a model to observe a successful bounce there must satisfy the condition $\dot{H} = 4\pi G\rho(1 + \omega) > 0$, that is the violation of null energy condition. The EoS parameter ω passes through the phantom region $\omega < -1$ to $\omega > -1$ and thus helpful to study DE with EoS. For perfect and viscous fluid both, we have analyzed that the exponential and power law models show the acceleration expansion of the universe.

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