ABSTRACT

M-Polynomials of New Families of Graphs

In this thesis, we calculated the M-polynomials for uniformly k-subdivided graphs and line graphs of k-subdivided graphs of connected families of the tetracanic nanotubes and nanotori. Then we calculated degree based topological indices for the k-subdivided graphs and the line graphs of the k-subdivided graphs of nanotubes by using the means of M-polynomials of the graphs. The precise and definite formulas of these indices for different families of nanotubes are presented in this section. We also compute M-polynomials for the sum of two graphs $G+H$, k-subdivided graph of G denoted by $G(k)$ and line graph of $G(k)$, for $k \geq 1$.

Contents

List of Figures

Chapter 1

Introduction and Basic Definitions

1.1 Elementary Definitions

1.1.1 Graph

A graph is an ordered pair $G = (V, E)$ where, V is the vertex set of the graph, represented by $V(G)$ and $E(G)$ is the edge set of the graph, represented by $E(G)$. The

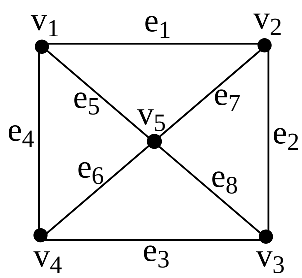

Figure 1.1

number of vertices in a graph is called order of graph and count of edges in a graph is named as size of a graph. In the following figure 1.1 the order and size of graph is 5 and 8 respectively.

1.1.2 Degree of a Vertex

The total number of edges attached to a vertex is called the degree of that vertex, with loops counted twice. Degree of each vertex is 4 in the graph below.

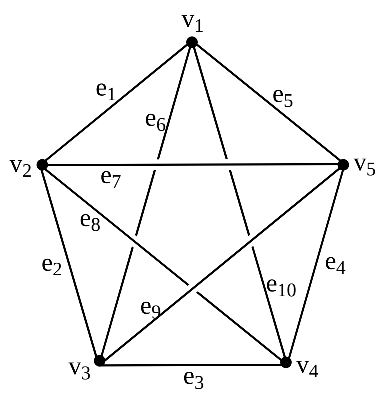

Figure 1.2

1.1.3 Path

A path is alternate sequence of vertices and edges in which no edge and vertex are repeated.

1.2 Graph Operations

1.2.1 k-subdivided Graph G_k

For $k \geq 1$, the graph G_k is a k-subdivided graph of graph G, obtained by replacing every edge of G by a path P_{k+2}.

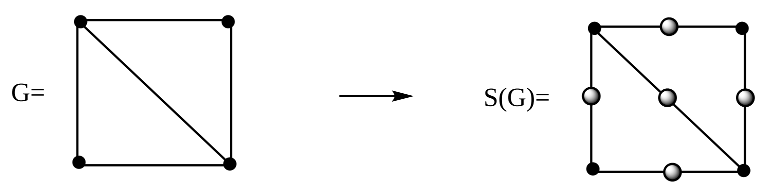

Figure 1.3: 1-subdivided Graph

1.2.2 Line Graph

The line graph $L(G)$ of graph G is the graph in which vertex set is $V[L(G)] = E(G)$ and there will be an edge between vertices of $L(G)$ if they are joined by a edge if the associated edges in G have common vertex in G.

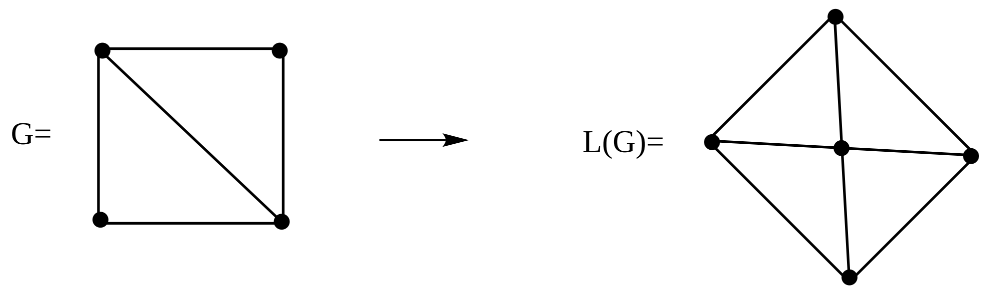

Figure 1.4: Line Graph of Graph G

1.2.3 Sum of Two Graphs

Let $G_1(p_1,q_1)$ and $G_2(p_2,q_2)$ be two graphs then
$G = G_1 + G_2$ is a graph with

$$\begin{aligned} V(G) &= V(G_1) \cup V_(G_2) \\ E(G) &= E(G_1) \cup E_(G_2) \cup E_3 \; with \\ E_3 &= \{xy : x \in G_1, and \; y \in G_2\} \end{aligned}$$

1.3 Topological Indices

1.3.1 Topological Indices

In the field of chemical graph theory a topological index is a quantity that describes the molecular structure and the pattern of branching of the molecule.

1.3.2 Degree Based Topological Indices

Topological indices were introduced by wiener in 1947 which are also familiar as connectivity indices. First and mostly studied index was the wiener index [7]. Degree based indices were studied in 1970. Randic in 1976 found the randic index [11]. After that Bollobos and Erdos generalized the randic index by replacing with some real number α and then it was called generalized randic index [5]. First zagreb index which is calculated by adding the squares of degrees of vertices of that particular graph was found in 1972 by Trijanstic and Gutman [11]. Second zagreb index which is calculated by adding the products of degrees of vertices of that particular graph was also found in 1972 by Trijanstic and Gutman [11]. Zagreb indices were redeveloped into Modified zagreb indices by Milicevi [20]. Harmonic index is one of the alternative of randic index was found by Fajtlowicz [7]. The augmented Zagreb index for a graph G was found by Furtula et al. [9]. Symmetric Division index and Inverse sum index were proposed by Vukicevic and Gasperov [29] [30]. Some familiar Degree based topological indices are written bellow:

- First Zagreb Index
- Second Zagreb Index
- Second Modified Zagreb Index
- General Randic (GR)
- General Randic (GR)
- Symmetric Division Index (SDI)
- Harmonic Index (HI)
- Inverse Sum Index (ISI)
- Augmented Zagreb index (AZI)

1.4 Introduction to M-Polynomials

M-polynomials were invented in 2015 and presented by Deutsch and Klavzar [6]. There are many polynomials which plays an important role in chemical graph theory. M-polynomials helps to determine the structures of the various chemical compounds. M-polynomials are important because it contains too much information about the various degree-based topological indices which are described below. M-polynomial of a graph G is described in the following way [6];
$M(G,\alpha,\beta)=\sum\limits_{\delta\leq b\leq c\leq\Delta} m_{bc}(G)\alpha^b\beta^c$ where,
$\Delta=max[d_v:v\in V(G)]$, $\delta=min[d_v:v\in V(G)]$
and m_{bc} (G) the quantity of edges $uv\in E(G)$ such that $(d_v,d_u)=(b,c)$.

1.5 Literature Review

M-polynomial was firstly studied in 2015 and proposed by Deutsch and Klavzar [6] and degree based indices were studied in 1970. Kwun et al. formulated M-polynomials of V-Phenylenic Nanotubes and Nanotori [17]. Munir et al. established M-polynomials of Nanostar Dendrimers [22]. Gao et al. proposed the M-polynomials and topological indices of crystallographic structures of the molecules [10]. Kwun et al. worked out the M-polynomials of Bismuth Tri-Iodide [18]. Liu et al. calculated M-polynomials of VC5C7[p,q] and HC5C7[p,q] Nanotubes [19]. Ajmal et al. established M-polynomials of toroidal polyhex networks [3]. Rezaei et al computed M-polynomials for Titania nanotubes $TiO_2[m,n]$ [26]. Nizami et al. [24] discussed the general closed forms of the M-polynomial of the generalized Mobius ladder and its line graph. They also computed Zagreb Indices, generalized Randic indices, and symmetric division index of these graphs via the M-polynomial.In the field of chemical graph theory a topological index is a quantity that describes the molecular structure and the pattern of branching of the molecule. Topological indices were introduced by wiener in 1947 which are also familiar as connectivity indices. First and mostly studied index was the wiener index [7]. Later on many indices were studied and found by different authors. Computation of Second Modified Zagreb, General Randic, Symmetric Division Index, Harmonic Index, Inverse Sum Index, Augmented Zagreb Index etc., were computed by M-polynomials by Ali et al. [4], increase its importance. In the area of mathematical chemistry some valuable basic properties of a substance can be found by using simple mathematical methods of polynomials rather than using tools of quantum mechanics. Hex-derived networks has various helpful uses in the medicine and mechanics. In this proposal Kang et al. discuss the precise type of the M-polynomial of the hex-derived systems HDN1[n] and HDN2[n], which left n-dimensional hexagonal work [15]. And moreover, Kang et al. also found the close types of a few degree based topological index related to these systems. Ajmal et al. studied the close types of M-polynomial for generalized prism network [2].The different indices for n-dimensional dominating

David derived and David derived networks were calculated by Imran et al. [12]. In the report [14], Kang et al. found M-polynomials for the dominating Derived networks of 1-3 kinds of dimension n. They also computed topological indices by utilizing M-polynomials. A line graph has various uses in physical chemistry. Ahmad et al. [1] calculated the close types of the M-polynomials for the line graph of firecracker graph. They also computed topological index by using M-polynomials. Ali et al. [4] studied M-polynomials for rhombic benzenoid and zigzag systems. Riaz et al. established the close types of M-polynomials for three main categories of convex polytopes [27]. By using them they also calculated topological indices for Zagreb index of first and second kind, modified Zagreb indices, symmetric division indices and so forth. Javaid et al. discussed the M-polynomial of the oxide, chain silicate and silicate network [13]. Munir et al. found M-polynomials of poly hex-nanotubes [23] and Munir et al. also computed M-polynomials for Titania nanotubes [21]. Soleimani et al. computed the topological indices for nanostructures [28]. Kwun et al also find M-polynomials and indices that are based on degree for Triangular, Hourglass, and Jagged-Rectangle Benzenoid Systems [16].

Chapter 2

M-Polynomials of New Families of Graphs

2.1 M-Polynomials of k-subdivided Graph of $2D$ Graph Lattice of Tetracenic Nanotubes

In this section, we find the M-Polynomials of k-subdivided graph of $2D$ graph lattice of α-Tetracenic Nanotubes, where $\alpha = F, H, W, Z$.

In the following lemma, we find the type of edges of k-subdivided graph of $2D$ graph lattice of $F[b,c]$.

Lemma 1: Let $G_F(k)$ be the k-subdivided graph of $2D$ graph lattice of $F[b,c]$, where

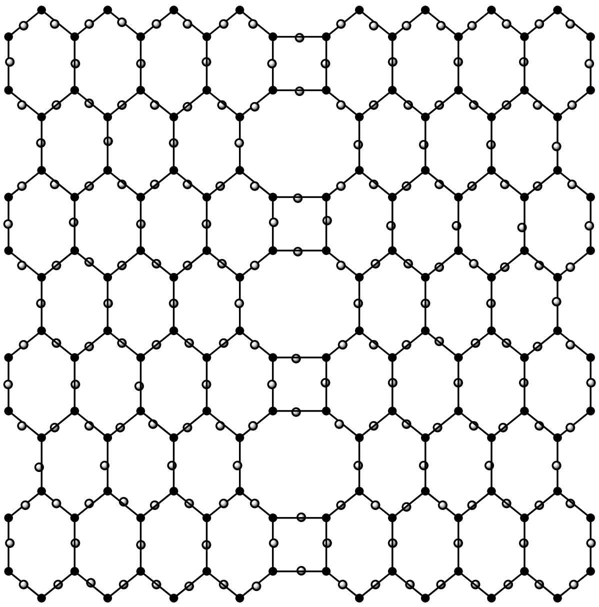

Figure 2.1: 1-subdivided Graph of F

$k \geq 1$. The number of vertices, edges and type of edges of $G_F(k)$ with respect to degree are as follows:

$\lvert V(F)\rvert$	$\lvert E(F)\rvert$	$\lvert E_1(F)\rvert = \lvert(2,2)\rvert$	$\lvert E_2(F)\rvert = \lvert(2,3)\rvert$
$18bc + ke$	$27bc - 2c - 4b + ke$	$16b + 8c + (k-1)e$	$54bc - 24b - 12c$

where e represents the size of $F = F[b,c]$, and $e = 27bc - 2c - 4b$.

Proof:

For $k \geq 1$, the graph F has $|V(F)| = 18bc + ke$ and $|E(F)| = 27bc - 2c - 4b + ke$. There are two types of the edges, named as $[E_1(F)] = mn$ and $[E_2(F)] = rs$. Here $d_m = d_n = 2$ and $d_r = 2, d_s = 3$. The edges $(2,2)$ and $(2,3)$ have count $16b+8c+(k-1)e$ and $54bc-24b-12c$ respectively. $\square$

In the following theorem, we find the M-polynomial of k-subdivided graph of $2D$ graph lattice of $F[b,c]$.

Theorem 1: The M-polynomial of $G_F(k)$ is as follows:

$M(F = F[b,c];x,y) = \big(16b+8c+(k-1)e\big)x^2y^2 + \big(54bc-24b-12c\big)x^2y^3.$

Proof:

Suppose the k-subdivided $2D$ graph lattice of $F[b,c]$ then by using Lemma 1, we have two edge types as : E_1(F)=[e=mn∈E(F): d_m=d_n=2], E_2(F)=[e=rs∈E(F): d_r=2, d_s=3] with $|E_1(F)| = 16b+8c+(k-1)e$, $|E_2(F)| = 54bc-24b-12c$.

Now by using definition of M-polynomial, we have:

$$\begin{aligned}
M(F[b,c];x,y) &= \sum_{i\le j} m_{ij}x^iy^j \\
&= \sum_{2\le 2} m_{22}x^2y^2 + \sum_{2\le 3} m_{23}x^2y^3 \\
&= |E_1(F)|x^2y^2 + |E_2(F)|x^2y^3 \\
&= \big(16b+8c+(k-1)e\big)x^2y^2 + (54bc-24b-12c)x^2y^3.
\end{aligned}$$

$\square$

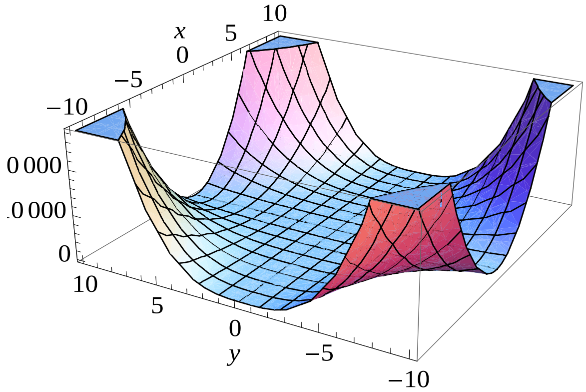

Figure 2.2: 3D Graph of 1-subdivided M-Polynomial of F

Topological Index	Derivation from M-polynomial
"First zagreb	$(D_x{+}D_y)[M(G_\alpha^\beta,x,y)]_{y=x=1}$
Second zagreb	$(D_xD_y)[M(G_\alpha^\beta,x,y)]_{y=x=1}$
Second Modified zagreb	$(S_xS_y)[M(G_\alpha^\beta,x,y)]_{y=x=1}$
General Randic (GR)	$(D_x^mD_y^m)[M(G_\alpha^\beta,x,y)]_{y=x=1}$
General Randic (GR)	$(S_x^mS_y^m)[M(G_\alpha^\beta,x,y)]_{y=x=1}$
Symmetric Division Index	$(D_xS_y+D_yS_x)[M(G_\alpha^\beta,x,y)]_{y=x=1}$
Harmonic Index	$2S_xJ[M(G_\alpha^\beta,x,y)]_{y=x=1}$"
Inverse Sum Index	$S_xJD_xD_y[M(G_\alpha^\beta,x,y)]_{y=x=1}$
Augmented Zagreb Index	$S_x^3Q_{-2}JD_x^3D_y^3[M(G_\alpha^\beta,x,y)]_{y=x=1}$

Table 2.1

2.2 Degree Based Topological Indices of k-subdivided Graph of F

Now we calculate some important degree based topological indices of the k-subdivided graph of $F[b,c]$ by using the M-polynomial. The following table defined in [4] shows the derivation of different degree based topological indices by using M-polynomial. Where;

$D_xg = x\frac{\delta(g(x,y))}{\delta x}$, $D_yg = y\frac{\delta(g(x,y))}{\delta y}$, $S_x{=}\int_0^x \frac{g(y,t)}{t}dt$, $S_y{=}\int_0^y \frac{g(x,t)}{t}dt$

$J(g(x,y)) = g(x,x)$

By using the Table 2.1 we have following:

Let $M(F[b,c];x,y)$=f(x, y)=$\big(16b+8c+(k-1)e\big)x^2y^2$+$(54bc-24b-12c)x^2y^3$

$D_xf(x,y) = 2\big(16b+8c+(k-1)e\big)x^2y^2+2(54bc-24b-12c)x^2y^3$

$D_xf(x,y) = \big(32b+16c+2(k-1)e\big)x^2y^2+(108bc-48b-24c)x^2y^3$

$D_yf(x,y) = 2\big(16b+8c+(k-1)e\big)x^2y^2+3(54bc-24b-12c)x^2y^3$

$D_yf(x,y) = \big(32b+16c+2(k-1)e\big)x^2y^2+(162bc-72b-36c)x^2y^3$

$D_yD_xf(x,y) = 4\big(16b+8c+(k-1)e\big)x^2y^2+6(54bc-24b-12c)x^2y^3$

$D_yD_xf(x,y) = \big(64b+32c+4(k-1)e\big)x^2y^2+(324bc-144b-72c)x^2y^3$

$S_y(f(x,y)) = \frac{1}{2}\big(16b+8c+(k-1)e\big)x^2y^2+\frac{1}{3}(54bc-24b-12c)x^2y^3$

$S_xS_y(f(x,y)) = \frac{1}{4}\big(16b+8c+(k-1)e\big)x^2y^2+\frac{1}{6}(54bc-24b-12c)x^2y^3$

$S_xS_y(f(x,y)) = \big(4b+2c+\frac{(k-1)e}{4}\big)x^2y^2+(9bc-4b-2c)x^2y^3$

$$D_y^m(f(x,y)) = 2^m\big(16b+8c+(k-1)e\big)x^2y^2 + 3^m(54bc-24b-12c)x^2y^3$$
$$D_xD_y^m(f(x,y)) = 2.2^m\big(16b+8c+(k-1)e\big)x^2y^2 + 2.3^m(54bc-24b-12c)x^2y^3$$
$$D_x^mD_y^m(f(x,y)) = 2^m.2^m\big(16b+8c+(k-1)e\big)x^2y^2 + 2^m.3^m(54bc-24b-12c)x^2y^3$$
$$D_x^mD_y^m(f(x,y)) = 2^{2m}\big(16b+8c+(k-1)e\big)x^2y^2 + 6^m(54bc-24b-12c)x^2y^3$$
$$S_y^m(f(x,y)) = \tfrac{1}{2^m}\big(16b+8c+(k-1)e\big)x^2y^2 + \tfrac{1}{3^m}(54bc-24b-12c)x^2y^3$$
$$S_xS_y^m(f(x,y)) = \tfrac{1}{2.2^m}\big(16b+8c+(k-1)e\big)x^2y^2 + \tfrac{1}{2.3^m}(54bc-24b-12c)x^2y^3$$
$$S_x^mS_y^m(f(x,y)) = \tfrac{1}{2^m.2^m}\big(16b+8c+(k-1)e\big)x^2y^2 + \tfrac{1}{2^m.3^m}(54bc-24b-12c)x^2y^3$$
$$S_x^mS_y^m(f(x,y)) = \tfrac{1}{2^{2m}}\big(16b+8c+(k-1)e\big)x^2y^2 + \tfrac{1}{6^m}(54bc-24b-12c)x^2y^3$$
$$S_x^mS_y^m(f(x,y)) = \tfrac{1}{4^m}\big(16b+8c+(k-1)e\big)x^2y^2 + \tfrac{1}{6^m}(54bc-24b-12c)x^2y^3$$
$$S_yD_x(f(x,y)) = \big(16b+8c+(k-1)e\big)x^2y^2 + \tfrac{2}{3}(54bc-24b-12c)x^2y^3$$
$$S_xD_y(f(x,y)) = \big(16b+8c+(k-1)e\big)x^2y^2 + \tfrac{3}{2}(54bc-24b-12c)x^2y^3$$
$$Jf(x,y) = \big(16b+8c+(k-1)e\big)x^4 + (54bc-24b-12c)x^5$$
$$S_xJf(x,y) = \tfrac{1}{4}\big(16b+8c+(k-1)e\big)x^2y^2 + \tfrac{1}{5}(54bc-24b-12c)x^2y^3$$
$$JD_xD_yf(x,y) = 4\big(16b+8c+(k-1)e\big)x^4 + 6(54bc-24b-12c)x^5$$
$$S_xJD_xD_yf(x,y) = \big(16b+8c+(k-1)e\big)x^4 + \tfrac{6}{5}(54bc-24b-12c)x^5$$
$$D_y^3f(x,y) = (2^3)\big(16b+8c+(k-1)e\big)x^2y^2 + (3^3)\big(54bc-24b-12c\big)x^2y^3$$
$$D_y^3f(x,y) = 8\big(16b+8c+(k-1)e\big)x^2y^2 + 27(54bc-24b-12c)x^2y^3$$
$$D_x^3D_y^3f(x,y) = 2^{2.3}\big(16b+8c+(k-1)e\big)x^2y^2 + (6^3)(54bc-24b-12c)x^2y^3$$
$$D_x^3D_y^3f(x,y) = 2^6\big(16b+8c+(k-1)e\big)x^2y^2 + (6^3)(54bc-24b-12c)x^2y^3$$
$$D_x^3D_y^3f(x,y) = 64\big(16b+8c+(k-1)e\big)x^2y^2 + 216\big(54bc-24b-12c\big)x^2y^3$$
$$JD_x^3D_y^3f(x,y) = 64\big(16b+8c+(k-1)e\big)x^4 + 216\big(54bc-24b-12c\big)x^5$$
$$Q_{-}2JD_x^3D_y^3f(x,y) = 64\big(16b+8c+(k-1)e\big)x^{4-2} + 216\big(54bc-24b-12c\big)x^{5-2}$$
$$Q_{-}2JD_x^3D_y^3f(x,y) = 64\big(16b+8c+(k-1)e\big)x^2 + 216\big(54bc-24b-12c\big)x^3$$
$$S_xQ_{-}2JD_x^3D_y^3f(x,y) = \tfrac{64}{2}\big(16b+8c+(k-1)e\big)x^2 + \tfrac{216}{3}(54bc-24b-12c)x^3$$
$$S_x^mQ_{-}2JD_x^3D_y^3f(x,y) = \tfrac{64}{2^m}\big(16b+8c+(k-1)e\big)x^2 + \tfrac{216}{3^m}(54bc-24b-12c)x^3$$
$$S_x^3Q_{-}2JD_x^3D_y^3f(x,y) = \tfrac{64}{2^3}\big(16b+8c+(k-1)e\big)x^2 + \tfrac{216}{3^3}(54bc-24b-12c)x^3$$
$$S_x^3Q_{-}2JD_x^3D_y^3f(x,y) = \tfrac{64}{8}\big(16b+8c+(k-1)e\big)x^2 + \tfrac{216}{27}(54bc-24b-12c)x^3$$
$$S_x^3Q_{-}2JD_x^3D_y^3f(x,y) = 8\big(16b+8c+(k-1)e\big)x^2 + 8(54bc-24b-12c)x^3$$

First Zagreb:

$$(D_x + D_y)[M(F_\alpha^\beta, x, y)]_{y=x=1} = 4(k-1)e + 270bc - 56b - 28c$$

Second Zagreb:

$$(D_x D_y)[M(F_\alpha^\beta, x, y)]_{y=x=1} = 4(k-1)e + 324bc - 80b - 40c$$

Second Modified Zagreb:

$$(S_x S_y)[M(F_\alpha^\beta, x, y)]_{y=x=1} = \frac{k-1}{4}e + 9bc$$

General Randic (GR):

$$\begin{aligned}(D_x^m D_y^m)[M(F_\alpha^\beta, x, y)]_{y=x=1} &= 4^m(16b) + 4^m(8c) + 4^m(k-1)e + 6^m(54bc) \\ &- 6^m(24b) - 6^m(12c)\end{aligned}$$

General Randic (GR):

$$(S_x^m S_y^m)[M(F_\alpha^\beta, x, y)]_{y=x=1} = \frac{1}{4^m} \times \left(16b + 8c + (k-1)e\right) + \frac{1}{6^m} \times (54bc - 24b - 12c)$$

"Symmetric Division Index (SDI):

$$(D_x S_y + D_y S_x)[M(F_\alpha^\beta, x, y)]_{y=x=1}" = 2(k-1)e + 117bc - 20b - 10c$$

Harmonic Index (HI):

$$2S_x J[M(F_\alpha^\beta, x, y)]_{y=x=1} = \frac{(k-1)e}{2} + \frac{108bc}{5} - \frac{8b}{5} - \frac{4c}{5}$$

”Inverse Sum Index (ISI):

$$S_xJD_xD_y[M(F_{\alpha}^{\beta},x,y)]_{y=x=1}\text{”} \quad = \quad (k-1)e+\frac{324bc}{5}-\frac{64b}{5}-\frac{32c}{5}$$

Augmented Zagreb Index (AZI):

$$S_x^3Q_{-}2JD_x^3D_y^3[M(F_{\alpha}^{\beta},x,y)]_{y=x=1} \quad = \quad 8(k-1)e+432bc-64b-32c$$

$\square$

In the following lemma, we find the type of edges of k-subdivided graph of $2D$ graph lattice of $H[b,c]$.

Lemma 2: Let $G_H(k)$ be the k-subdivided graph of $2D$ graph lattice of $H[b,c]$, where

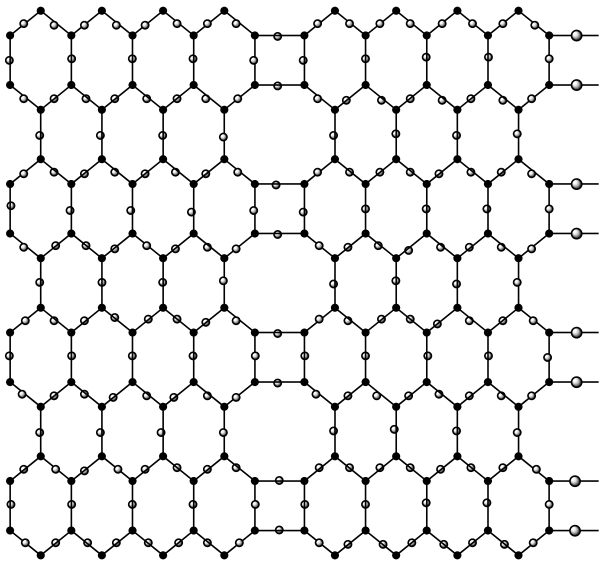

Figure 2.3: 1-subdivided Graph of H

$k \geq 1$. The number of vertices, edges and type of edges of $G_H(k)$ with respect to degree are as follows:

$\lvert V(H)\rvert$	$\lvert E(H)\rvert$	$\lvert E_1(H)\rvert = \lvert(2,2)\rvert$	$\lvert E_2(H)\rvert = \lvert(2,3)\rvert$
$18bc+ke$	$27bc-4b+ke$	$16b+(k-1)e$	$54bc-24b$

where e represents the size of $H[b,c]$, and $e = 27bc - 4b$.

Proof: For $k \geq 1$, the graph H has $|V(H)| = 18bc + ke$ and $|E(H)| = 27bc - 4b + ke$. There are two types of the edges in H. We name $[E_1(H) = (2,2)]$ as type 1 and $[E_2(H) = (2,3)]$ as type 2. By calculating these edge types, we get that $16b + (k-1)e$ and $54bc - 24b$ are counts of the edges of type 1 and type 2 respectively. □

In the following theorem, we find the M-polynomial of k-subdivided graph of $2D$ graph lattice of $H[b,c]$.

Theorem 2: "The M-polynomial of $G_H(k)$ is as follows":

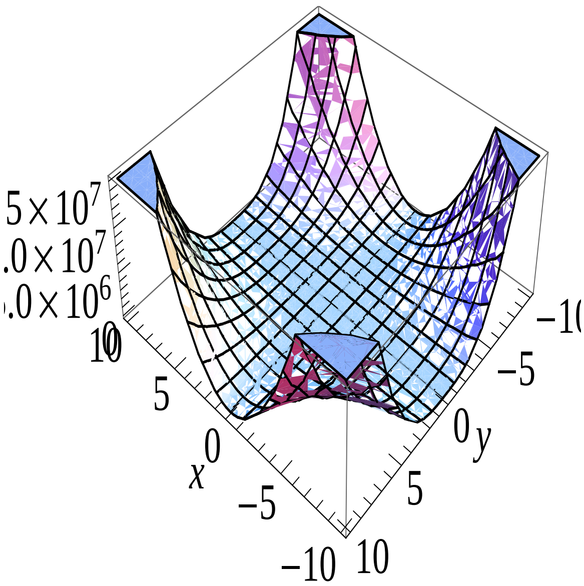

Figure 2.4: 3D Graph of 1-subdivided M-Polynomial of H

$M(H[b,c];x,y) = \big(16b + (k-1)e\big)x^2y^2 + \big(54bc - 24b\big)x^2y^3$.

Proof:

Suppose the k-subdivided $2D$ graph lattice of $H[b,c]$ then by using Lemma 2, we have two edge types as : E_1(H)=[e=mn∈E(H): d_m=d_n=2], E_2(H)=[e=rs∈E(H): d_r=2, d_s=3] with $|E_1(H)| = 16b + (k-1)e$, $|E_2(H)| = 54bc - 24b$.

Now by using definition of M-polynomial, we have:

$$\begin{aligned} M(H[b,c];x,y) &= \sum_{i\leq j} m_{ij}x^iy^j \\ &= \sum_{2\leq 2} m_{22}x^2y^2 + \sum_{2\leq 3} m_{23}x^2y^3 \\ &= |E_1(H)|x^2y^2 + |E_2(H)|x^2y^3 \\ &= \big(16b+(k-1)e\big)x^2y^2 + (54bc-24b)x^2y^3. \end{aligned}$$

$\square$

By using the Table 2.1 we have following:

Let $M(H[b,c];x,y)$=h(x, y)=$\big(16b+(k-1)e\big)x^2y^2$+$(54bc-24b)x^2y^3$

$$\begin{aligned} D_xh(x,y) &= 2\big(16b+(k-1)e\big)x^2y^2 + 2(54bc-24b)x^2y^3 \\ D_xh(x,y) &= \big(32b+2(k-1)e\big)x^2y^2 + (108bc-48b)x^2y^3 \\ D_yh(x,y) &= 2\big(16b+(k-1)e\big)x^2y^2 + 3(54bc-24b)x^2y^3 \\ D_yh(x,y) &= \big(32b+2(k-1)e\big)x^2y^2 + (162bc-72b)x^2y^3 \\ D_yD_xh(x,y) &= 4\big(16b+(k-1)e\big)x^2y^2 + 6(54bc-24b)x^2y^3 \\ D_yD_xh(x,y) &= \big(64b+4(k-1)e\big)x^2y^2 + (324bc-144b)x^2y^3 \\ S_y(h(x,y)) &= \frac{1}{2}\big(16b+(k-1)e\big)x^2y^2 + \frac{1}{3}(54bc-24b)x^2y^3 \\ S_xS_y(h(x,y)) &= \frac{1}{4}\big(16b+(k-1)e\big)x^2y^2 + \frac{1}{6}(54bc-24b)x^2y^3 \\ S_xS_y(h(x,y)) &= \big(4b+\frac{(k-1)e}{4}\big)x^2y^2 + (9bc-4b)x^2y^3 \\ D_y^m(h(x,y)) &= 2^m\big(16b+(k-1)e\big)x^2y^2 + 3^m(54bc-24b)x^2y^3 \\ D_xD_y^m(h(x,y)) &= 2.2^m\big(16b+(k-1)e\big)x^2y^2 + 2.3^m(54bc-24b)x^2y^3 \\ D_x^mD_y^m(h(x,y)) &= 2^m.2^m\big(16b+(k-1)e\big)x^2y^2 + 2^m.3^m(54bc-24b)x^2y^3 \\ D_x^mD_y^m(h(x,y)) &= 2^{2m}\big(16b+(k-1)e\big)x^2y^2 + 6^m(54bc-24b)x^2y^3 \\ S_y^m(h(x,y)) &= \frac{1}{2^m}\big(16b+(k-1)e\big)x^2y^2 + \frac{1}{3^m}(54bc-24b)x^2y^3 \end{aligned}$$

$$\begin{aligned}
S_xS_y^m(h(x,y)) &= \frac{1}{2.2^m}\big(16b+(k-1)e\big)x^2y^2+\frac{1}{2.3^m}(54bc-24b)x^2y^3\\
S_x^mS_y^m(h(x,y)) &= \frac{1}{2^m.2^m}\big(16b+(k-1)e\big)x^2y^2+\frac{1}{2^m.3^m}(54bc-24b)x^2y^3\\
S_x^mS_y^m(h(x,y)) &= \frac{1}{2^{2m}}\big(16b+(k-1)e\big)x^2y^2+\frac{1}{6^m}(54bc-24b)x^2y^3\\
S_x^mS_y^m(h(x,y)) &= \frac{1}{4^m}\big(16b+(k-1)e\big)x^2y^2+\frac{1}{6^m}(54bc-24b)x^2y^3\\
S_yD_x(h(x,y)) &= \big(16b+(k-1)e\big)x^2y^2+\frac{2}{3}(54bc-24b)x^2y^3\\
S_xD_y(h(x,y)) &= \big(16b+(k-1)e\big)x^2y^2+\frac{3}{2}(54bc-24b)x^2y^3\\
Jh(x,y) &= \big(16b+(k-1)e\big)x^4+(54bc-24b)x^5\\
S_xJh(x,y) &= \frac{1}{4}\big(16b+(k-1)e\big)x^2y^2+\frac{1}{5}(54bc-24b)x^2y^3\\
JD_xD_yh(x,y) &= 4\big(16b+(k-1)e\big)x^4+6(54bc-24b)x^5\\
S_xJD_xD_yh(x,y) &= \big(16b+(k-1)e\big)x^4+\frac{6}{5}(54bc-24b)x^5\\
D_y^3h(x,y) &= (2^3)\big(16b+(k-1)e\big)x^2y^2+(3^3)\big(54bc-24b\big)x^2y^3\\
D_y^3h(x,y) &= 8\big(16b+(k-1)e\big)x^2y^2+27(54bc-24b)x^2y^3\\
D_x^3D_y^3h(x,y) &= 2^{2.3}\big(16b+(k-1)e\big)x^2y^2+(6^3)(54bc-24b)x^2y^3\\
D_x^3D_y^3h(x,y) &= 2^6\big(16b+(k-1)e\big)x^2y^2+(6^3)(54bc-24b)x^2y^3\\
D_x^3D_y^3h(x,y) &= 64\big(16b+(k-1)e\big)x^2y^2+216\big(54bc-24b\big)x^2y^3\\
JD_x^3D_y^3h(x,y) &= 64\big(16b+(k-1)e\big)x^4+216\big(54bc-24b\big)x^5\\
Q_-2JD_x^3D_y^3h(x,y) &= 64\big(16b+(k-1)e\big)x^{4-2}+216\big(54bc-24b\big)x^{5-2}\\
Q_-2JD_x^3D_y^3h(x,y) &= 64\big(16b+(k-1)e\big)x^2+216\big(54bc-24b\big)x^3\\
S_xQ_-2JD_x^3D_y^3h(x,y) &= \frac{64}{2}\big(16b+(k-1)e\big)x^2+\frac{216}{3}(54bc-24b)x^3\\
S_x^mQ_-2JD_x^3D_y^3h(x,y) &= \frac{64}{2^m}\big(16b+(k-1)e\big)x^2+\frac{216}{3^m}(54bc-24b)x^3\\
S_x^3Q_-2JD_x^3D_y^3h(x,y) &= \frac{64}{2^3}\big(16b+(k-1)e\big)x^2+\frac{216}{3^3}(54bc-24b)x^3\\
S_x^3Q_-2JD_x^3D_y^3h(x,y) &= \frac{64}{8}\big(16b+(k-1)e\big)x^2+\frac{216}{27}(54bc-24b)x^3\\
S_x^3Q_-2JD_x^3D_y^3h(x,y) &= 8\big(16b+(k-1)e\big)x^2+8(54bc-24b)x^3
\end{aligned}$$

Degree Based Topological Indices of k-subdivided graph of H

First Zagreb:

$$(D_x + D_y)[M(H_{\alpha}^{\beta}, x, y)]_{y=x=1} = 4(k-1)e + 270bc - 56b$$

Second Zagreb:

$$(D_x D_y)[M(H_{\alpha}^{\beta}, x, y)]_{y=x=1} = 4(k-1)e + 324bc - 80b$$

Second Modified Zagreb:

$$(S_x S_y)[M(H_{\alpha}^{\beta}, x, y)]_{y=x=1} = \frac{k-1}{4}e + 9bc$$

General Randic (GR):

$$(D_x^m D_y^m)[M(H_{\alpha}^{\beta}, x, y)]_{y=x=1} = 4^m(16b) + 4^m(k-1)e + 6^m(54bc) - 6^m(24b)$$

General Randic (GR):

$$(S_x^m S_y^m)[M(H_{\alpha}^{\beta}, x, y)]_{y=x=1} = \frac{1}{4^m} \times \big(16b + (K-1)e\big) + \frac{1}{6^m} \times (54bc - 24b)$$

"Symmetric Division Index (SDI):

$$(D_x S_y + D_y S_x)[M(H_{\alpha}^{\beta}, x, y)]_{y=x=1}\text{''} = 2(k-1)e + 117bc - 20b$$

Harmonic Index (HI):

$$2S_x J[M(H_{\alpha}^{\beta}, x, y)]_{y=x=1} = \frac{(k-1)e}{2} + \frac{108bc}{5} - \frac{8b}{5}$$

”Inverse Sum Index (ISI):

$$S_xJD_xD_y[M(H_\alpha^\beta,x,y)]_{y=x=1}\text{”} \quad = \quad (k-1)e+\frac{324bc}{5}-\frac{64b}{5}$$

Augmented Zagreb Index (AZI):

$$S_x^3Q_{-2}JD_x^3D_y^3[M(H_\alpha^\beta,x,y)]_{y=x=1} \quad = \quad 8(k-1)e+432bc-64b$$

□

In the following lemma, we find the type of edges of k-subdivided graph of $2D$ graph lattice of $W[b,c]$.

Lemma 3: Let $G_W(k)$ be the k-subdivided graph of $2D$ graph lattice of $W[b,c]$, where

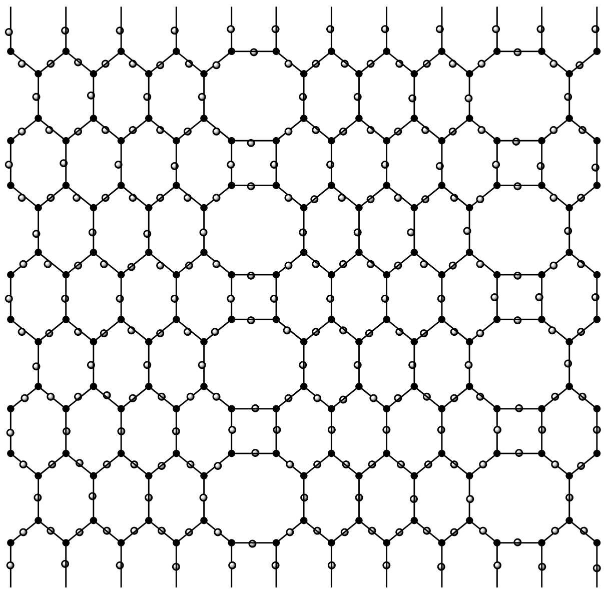

Figure 2.5: 1-subdivided Graph of W

$k \geq 1$. The number of vertices, edges and type of edges of $G_W(k)$ with respect to degree are as follows:

$\|V(W)\|$	$\|E(W)\|$	$\|E_1(W)\|=\|(2,2)\|$	$\|E_2(W)\|=\|(2,3)\|$
$18bc+ke$	$27bc-2c+ke$	$8c+(k-1)e$	$54bc-12c$

where e represents the size of $W[b,c]$, and $e = 27bc - 2c$.

Proof:

For $k \geq 1$, the graph W has $|V(W)| = 18bc + ke$ and $|E(W)| = 27bc - 2c + ke$. There are two types of the edges in W. We name $[E_1(W) = (2,2)]$ as type 1 and $[E_2(W) = (2,3)]$ as type 2. By calculating these edge types, we get that $8c + (k-1)e$ and $54bc - 12c$ are counts of the edges of type 1 and type 2 respectively. $\square$

In the following theorem, we find the M-polynomial of k-subdivided graph of $2D$ graph lattice of $W[b,c]$.

Theorem 3:

The M-polynomial of $G_W(k)$ is as follows:

$M(W[b,c];x,y) = \big(8c + (k-1)e\big)x^2y^2 + \big(54bc - 12c\big)x^2y^3.$

Proof:

Suppose the k-subdivided $2D$ graph lattice of $W[b,c]$ then by using Lemma 3, we have two edge types as : E_1(W)=[e=mn∈E(W): d_m=d_n=2], E_2(W)=[e=rs∈E(W): d_r=2, d_s=3] with $|E_1(W)| = 8c + (k-1)e$, $|E_2(W)| = 54bc - 12c$.

Now by using definition of M-polynomial, we have:

$$\begin{aligned}
M(W[b,c];x,y) &= \sum_{i\leq j} m_{ij}x^iy^j \\
&= \sum_{2\leq 2} m_{22}x^2y^2 + \sum_{2\leq 3} m_{23}x^2y^3 \\
&= |E_1(W)|x^2y^2 + |E_2(W)|x^2y^3 \\
&= \big(8c + (k-1)e\big)x^2y^2 + (54bc - 12c)x^2y^3.
\end{aligned}$$

$\square$

$M(W[b,c];x,y)$=w(x, y)=$\big(8c + (k-1)e\big)x^2y^2$+$(54bc - 12c)x^2y^3$

By using the Table 2.1 we have following:

$$
\begin{aligned}
D_x w(x,y) &= 2\big(8c+(k-1)e\big)x^2y^2+2(54bc-12c)x^2y^3\\
D_x w(x,y) &= \big(16c+2(k-1)e\big)x^2y^2+(108bc-24c)x^2y^3\\
D_y w(x,y) &= 2\big(8c+(k-1)e\big)x^2y^2+3(54bc-12c)x^2y^3\\
D_y w(x,y) &= \big(16c+2(k-1)e\big)x^2y^2+(162bc-36c)x^2y^3\\
D_yD_x w(x,y) &= 4\big(8c+(k-1)e\big)x^2y^2+6(54bc-12c)x^2y^3\\
D_yD_x w(x,y) &= \big(32c+4(k-1)e\big)x^2y^2+(324bc-72c)x^2y^3\\
S_y(w(x,y)) &= \frac{1}{2}\big(8c+(k-1)e\big)x^2y^2+\frac{1}{3}(54bc-12c)x^2y^3\\
S_xS_y(w(x,y)) &= \frac{1}{4}\big(8c+(k-1)e\big)x^2y^2+\frac{1}{6}(54bc-12c)x^2y^3\\
S_xS_y(w(x,y)) &= \big(2c+\frac{(k-1)e}{4}\big)x^2y^2+(9bc-2c)x^2y^3\\
D_y^m(w(x,y)) &= 2^m\big(8c+(k-1)e\big)x^2y^2+3^m(54bc-12c)x^2y^3\\
D_xD_y^m(w(x,y)) &= 2.2^m\big(8c+(k-1)e\big)x^2y^2+2.3^m(54bc-12c)x^2y^3\\
D_x^mD_y^m w(x,y)) &= 2^m.2^m\big(8c+(k-1)e\big)x^2y^2+2^m.3^m(54bc-12c)x^2y^3\\
D_x^mD_y^m(w(x,y)) &= 2^{2m}\big(8c+(k-1)e\big)x^2y^2+6^m(54bc-12c)x^2y^3\\
S_y^m(w(x,y)) &= \frac{1}{2^m}\big(8c+(k-1)e\big)x^2y^2+\frac{1}{3^m}(54bc-12c)x^2y^3\\
S_xS_y^m w(x,y)) &= \frac{1}{2.2^m}\big(8c+(k-1)e\big)x^2y^2+\frac{1}{2.3^m}(54bc-12c)x^2y^3\\
S_x^mS_y^m(w(x,y)) &= \frac{1}{2^m.2^m}\big(8c+(k-1)e\big)x^2y^2+\frac{1}{2^m.3^m}(54bc-12c)x^2y^3\\
S_x^mS_y^m(w(x,y)) &= \frac{1}{2^{2m}}\big(8c+(k-1)e\big)x^2y^2+\frac{1}{6^m}(54bc-12c)x^2y^3\\
S_x^mS_y^m(w(x,y)) &= \frac{1}{4^m}\big(8c+(k-1)e\big)x^2y^2+\frac{1}{6^m}(54bc-12c)x^2y^3\\
S_yD_x(w(x,y)) &= \big(8c+(k-1)e\big)x^2y^2+\frac{2}{3}(54bc-12c)x^2y^3\\
S_xD_y(w(x,y)) &= \big(8c+(k-1)e\big)x^2y^2+\frac{3}{2}(54bc-12c)x^2y^3\\
Jw(x,y) &= \big(8c+(k-1)e\big)x^4+(54bc-12c)x^5\\
S_xJw(x,y) &= \frac{1}{4}\big(8c+(k-1)e\big)x^2y^2+\frac{1}{5}(54bc-12c)x^2y^3
\end{aligned}
$$

$$
\begin{aligned}
JD_xD_yw(x,y) &= 4\big(8c+(k-1)e\big)x^4+6(54bc-12c)x^5\\
S_xJD_xD_yw(x,y) &= \big(8c+(k-1)e\big)x^4+\frac{6}{5}(54bc-12c)x^5\\
D_y^3w(x,y) &= (2^3)\big(8c+(k-1)e\big)x^2y^2+(3^3)\big(54bc-12c\big)x^2y^3\\
D_y^w(x,y) &= 8\big(8c+(k-1)e\big)x^2y^2+27(54bc-12c)x^2y^3\\
D_x^3D_y^3w(x,y) &= 2^{2.3}\big(8c+(k-1)e\big)x^2y^2+(6^3)(54bc-12c)x^2y^3\\
D_x^3D_y^3w(x,y) &= 2^6\big(8c+(k-1)e\big)x^2y^2+(6^3)(54bc-12c)x^2y^3\\
D_x^3D_y^3w(x,y) &= 64\big(8c+(k-1)e\big)x^2y^2+216\big(54bc-12c\big)x^2y^3\\
JD_x^3D_y^3w(x,y) &= 64\big(8c+(k-1)e\big)x^4+216\big(54bc-12c\big)x^5\\
Q_-2JD_x^3D_y^3w(x,y) &= 64\big(8c+(k-1)e\big)x^{4-2}+216\big(54bc-12c\big)x^{5-2}\\
Q_-2JD_x^3D_y^3w(x,y) &= 64\big(8c+(k-1)e\big)x^2+216\big(54bc-12c\big)x^3\\
S_xQ_-2JD_x^3D_y^3w(x,y) &= \frac{64}{2}\big(8c+(k-1)e\big)x^2+\frac{216}{3}(54bc-12c)x^3\\
S_x^mQ_-2JD_x^3D_y^3w(x,y) &= \frac{64}{2^m}\big(8c+(k-1)e\big)x^2+\frac{216}{3^m}(54bc-12c)x^3\\
S_x^3Q_-2JD_x^3D_y^3w(x,y) &= \frac{64}{2^3}\big(8c+(k-1)e\big)x^2+\frac{216}{3^3}(54bc-12c)x^3\\
S_x^3Q_-2JD_x^3D_y^3w(x,y) &= \frac{64}{8}\big(8c+(k-1)e\big)x^2+\frac{216}{27}(54bc-12c)x^3\\
S_x^3Q_-2JD_x^3D_y^3w(x,y) &= 8\big(8c+(k-1)e\big)x^2+8(54bc-12c)x^3
\end{aligned}
$$

Degree Based Topological Indices of k-subdivided graph of W

First Zagreb:

$$(D_x+D_y)[M(W_\alpha^\beta,x,y)]_{y=x=1} = 4(k-1)e+270bc-28c$$

Second Zagreb:

$$(D_xD_y)[M(W_\alpha^\beta,x,y)]_{y=x=1} = 4(k-1)e+324bc-40c$$

Second Modified Zagreb:

$$(S_xS_y)[M(W_\alpha^\beta,x,y)]_{y=x=1} = \frac{k-1}{4}e+9bc$$

General Randic (GR):

$$(D_x^mD_y^m)[M(W_\alpha^\beta,x,y)]_{y=x=1} = 4^m(8c)+4^m(k-1)e+6^m(54bc)-6^m(12c)$$

General Randic (GR):

$$(S_x^mS_y^m)[M(W_\alpha^\beta,x,y)]_{y=x=1} = \frac{1}{4^m}\times\big(8c+(K-1)e\big)+\frac{1}{6^m}\times(54bc-12c)$$

"Symmetric Division Index (SDI):

$$(D_xS_y+D_yS_x)[M(W_\alpha^\beta,x,y)]_{y=x=1}" = 2(k-1)e+117bc-10c$$

Harmonic Index (HI):

$$2S_xJ[M(W_\alpha^\beta,x,y)]_{y=x=1} = \frac{(k-1)e}{2}+\frac{108bc}{5}-\frac{4c}{5}$$

"Inverse Sum Index (ISI):

$$S_xJD_xD_y[M(W_\alpha^\beta,x,y)]_{y=x=1}" = (k-1)e+\frac{324bc}{5}-\frac{32c}{5}$$

Augmented Zagreb Index (AZI):

$$S_x^3Q_{-2}JD_x^3D_y^3[M(W_\alpha^\beta,x,y)]_{y=x=1} = 8(k-1)e+432bc-32c$$

□

In the following lemma, we find the type of edges of k-subdivided graph of $2D$ graph lattice of $Z[b,c]$.

Lemma 4: Let $G_Z(k)$ be the k-subdivided graph of $2D$ graph lattice of $Z[b,c]$, where

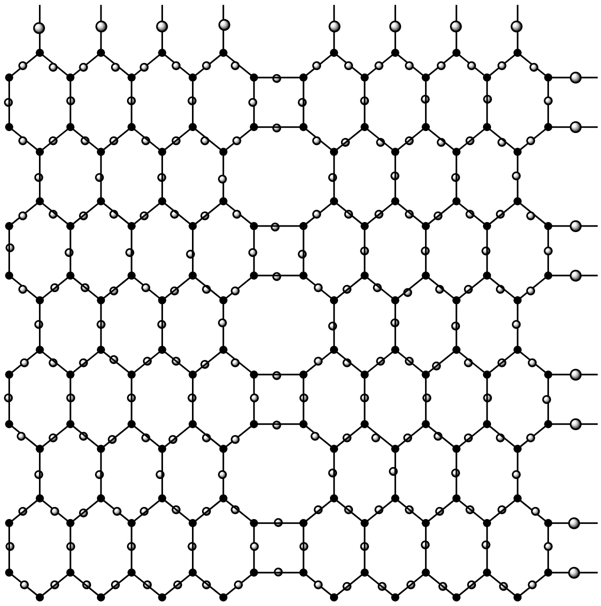

Figure 2.6: 1-subdivided Graph of Z

$k \geq 1$. The number of vertices, edges and type of edges of $G_Z(k)$ with respect to degree are as follows:

$\lvert V(Z)\rvert$	$\lvert E(Z)\rvert$	$\lvert E_1(Z)\rvert = \lvert(2,2)\rvert$	$\lvert E_2(Z)\rvert = \lvert(2,3)\rvert$
$18bc+ke$	$27bc+ke$	$(k-1)e$	$54bc$

where e represents the size of $Z[b,c]$, and $e = 27bc$.

Proof:

For $k \geq 1$, the graph Z has $|V(Z)| = 18bc+ke$ and $|E(Z)| = 27bc+ke$. There are two types of the edges in Z. We name $[E_1(Z) = (2,2)]$ as type 1 and $[E_2(Z) = (2,3)]$ as type 2. By calculating these edge types, we get that $(k-1)e$ and $54bc$ are counts of the edges of type 1 and type 2 respectively. □

In the following theorem, we find the M-polynomial of k-subdivided graph of $2D$ graph lattice of $Z[b,c]$.

Theorem 4:

The M-polynomial of $G_Z(k)$ is as follows:

$M(Z[b,c];x,y) = \big((k-1)e\big)x^2y^2 + 54bcx^2y^3$.

Proof:

Suppose the k-subdivided $2D$ graph lattice of $Z[b,c]$ then by using Lemma 4, we have two edge types as : E_1(Z)=[e=mn∈E(Z): d_m=d_n=2], E_2(Z)=[e=rs∈E(Z): d_r=2, d_s=3] with $|E_1(Z)| = (k-1)e$, $|E_2(Z)| = 54bc$.

Now by using definition of M-polynomial, we have:

$$\begin{aligned}
M(Z[b,c];x,y) &= \sum_{i\leq j} m_{ij}x^iy^j \\
&= \sum_{2\leq 2} m_{22}x^2y^2 + \sum_{2\leq 3} m_{23}x^2y^3 \\
&= |E_1(Z)|x^2y^2 + |E_2(Z)|x^2y^3 \\
&= \big((k-1)e\big)x^2y^2 + 54bcx^2y^3.
\end{aligned}$$

□

By using the Table 2.1 we have following:

Let $M(Z[b,c];x,y)$=z(x, y)=$\big((k-1)e\big)x^2y^2$+$(54bc)x^2y^3$

$$\begin{aligned}
D_xz(x,y) &= 2\big((k-1)e\big)x^2y^2 + 2(54bc)x^2y^3 \\
D_xz(x,y) &= \big(2(k-1)e\big)x^2y^2 + (108bc)x^2y^3 \\
D_yz(x,y) &= 2\big((k-1)e\big)x^2y^2 + 3(54bc)x^2y^3 \\
D_yz(x,y) &= \big(2(k-1)e\big)x^2y^2 + (162bc)x^2y^3 \\
D_yD_xz(x,y) &= 4\big((k-1)e\big)x^2y^2 + 6(54bc)x^2y^3 \\
D_yD_xz(x,y) &= \big(4(k-1)e\big)x^2y^2 + (324bc)x^2y^3 \\
S_y(z(x,y)) &= \frac{1}{2}\big((k-1)e\big)x^2y^2 + \frac{1}{3}(54bc)x^2y^3 \\
S_xS_y(z(x,y)) &= \frac{1}{4}\big((k-1)e\big)x^2y^2 + \frac{1}{6}(54bc)x^2y^3 \\
S_xS_y(z(x,y)) &= (\frac{(k-1)e}{4})x^2y^2 + (9bc)x^2y^3
\end{aligned}$$

$$
\begin{aligned}
D_y^m(z(x,y)) &= 2^m\big((k-1)e\big)x^2y^2 + 3^m(54bc)x^2y^3 \\
D_xD_y^m(z(x,y)) &= 2.2^m\big((k-1)e\big)x^2y^2 + 2.3^m(54bc)x^2y^3 \\
D_x^mD_y^m z(x,y)) &= 2^m.2^m\big((k-1)e\big)x^2y^2 + 2^m.3^m(54bc)x^2y^3 \\
D_x^mD_y^m(z(x,y)) &= 2^{2m}\big((k-1)e\big)x^2y^2 + 6^m(54bc)x^2y^3 \\
S_y^m(z(x,y)) &= \frac{1}{2^m}\big((k-1)e\big)x^2y^2 + \frac{1}{3^m}(54bc)x^2y^3 \\
S_xS_y^m z(x,y)) &= \frac{1}{2.2^m}\big((k-1)e\big)x^2y^2 + \frac{1}{2.3^m}(54bc)x^2y^3 \\
S_x^mS_y^m(z(x,y)) &= \frac{1}{2^m.2^m}\big((k-1)e\big)x^2y^2 + \frac{1}{2^m.3^m}(54bc)x^2y^3 \\
S_x^mS_y^m(z(x,y)) &= \frac{1}{2^{2m}}\big((k-1)e\big)x^2y^2 + \frac{1}{6^m}(54bc)x^2y^3 \\
S_x^mS_y^m z(x,y)) &= \frac{1}{4^m}\big((k-1)e\big)x^2y^2 + \frac{1}{6^m}(54bc)x^2y^3 \\
S_yD_x(z(x,y)) &= \big((k-1)e\big)x^2y^2 + \frac{2}{3}(54bc)x^2y^3 \\
S_xD_y(z(x,y)) &= \big((k-1)e\big)x^2y^2 + \frac{3}{2}(54bc)x^2y^3 \\
Jz(x,y) &= \big((k-1)e\big)x^4 + (54bc)x^5 \\
S_xJz(x,y) &= \frac{1}{4}\big((k-1)e\big)x^2y^2 + \frac{1}{5}(54bc)x^2y^3 \\
JD_xD_yz(x,y) &= 4\big((k-1)e\big)x^4 + 6(54bc)x^5 \\
S_xJD_xD_yz(x,y) &= \big((k-1)e\big)x^4 + \frac{6}{5}(54bc)x^5 \\
D_y^3z(x,y) &= (2^3)\big((k-1)e\big)x^2y^2 + (3^3)\big(54bc\big)x^2y^3 \\
D_y^z(x,y) &= 8\big((k-1)e\big)x^2y^2 + 27(54bc)x^2y^3 \\
D_x^3D_y^3z(x,y) &= 2^{2.3}\big((k-1)e\big)x^2y^2 + (6^3)(54bc)x^2y^3 \\
D_x^3D_y^3z(x,y) &= 2^6\big((k-1)e\big)x^2y^2 + (6^3)(54bc)x^2y^3 \\
D_x^3D_y^3z(x,y) &= 64\big((k-1)e\big)x^2y^2 + 216\big(54bc\big)x^2y^3 \\
JD_x^3D_y^3z(x,y) &= 64\big((k-1)e\big)x^4 + 216\big(54bc\big)x^5 \\
Q_{-}2JD_x^3D_y^3z(x,y) &= 64\big((k-1)e\big)x^{4-2} + 216\big(54bc\big)x^{5-2} \\
Q_{-}2JD_x^3D_y^3z(x,y) &= 64\big((k-1)e\big)x^2 + 216\big(54bc\big)x^3 \\
S_xQ_{-}2JD_x^3D_y^3z(x,y) &= \frac{64}{2}\big((k-1)e\big)x^2 + \frac{216}{3}(54bc)x^3
\end{aligned}
$$

$$
\begin{aligned}
S_x^m Q_{-2} J D_x^3 D_y^3 z(x,y) &= \frac{64}{2^m}\big((k-1)e\big)x^2 + \frac{216}{3^m}(54b)x^3 \\
S_x^3 Q_{-2} J D_x^3 D_y^3 z(x,y) &= \frac{64}{2^3}\big(8c+(k-1)e\big)x^2 + \frac{216}{3^3}(54bc-12c)x^3 \\
S_x^3 Q_{-2} J D_x^3 D_y^3 z(x,y) &= \frac{64}{8}\big(8c+(k-1)e\big)x^2 + \frac{216}{27}(54bc-12c)x^3 \\
S_x^3 Q_{-2} J D_x^3 D_y^3 z(x,y) &= 8\big(8c+(k-1)e\big)x^2 + 8(54bc-12c)x^3
\end{aligned}
$$

Degree Based Topological Indices of k-subdivided graph of Z

First Zagreb:

$$(D_x+D_y)[M(Z_\alpha^\beta,x,y)]_{y=x=1} = 4(k-1)e+270bc$$

Second Zagreb:

$$(D_xD_y)[M(Z_\alpha^\beta,x,y)]_{y=x=1} = 4(k-1)e+324bc$$

Second Modified Zagreb:

$$(S_xS_y)[M(Z_\alpha^\beta,x,y)]_{y=x=1} = \frac{k-1}{4}e+9bc$$

General Randic (GR):

$$(D_x^mD_y^m)[M(Z_\alpha^\beta,x,y)]_{y=x=1} = 4^m(k-1)e+6^m(54bc)$$

General Randic (GR):

$$(S_x^mS_y^m)[M(Z_\alpha^\beta,x,y)]_{y=x=1} = \frac{1}{4^m}\times\big((K-1)e\big)+\frac{1}{6^m}\times(54bc)$$

"Symmetric Division Index (SDI):

$$(D_xS_y+D_yS_x)[M(Z_\alpha^\beta,x,y)]_{y=x=1}\text{''} = 2(k-1)e+117bc$$

"Harmonic Index (HI):

$$2S_xJ[M(Z_{\alpha}^{\beta},x,y)]_{y=x=1}" \;=\; \frac{(k-1)e}{2}+\frac{108bc}{5}$$

"Inverse Sum Index (ISI):

$$S_xJD_xD_y[M(Z_{\alpha}^{\beta},x,y)]_{y=x=1}" \;=\; (k-1)e+\frac{324bc}{5}$$

Augmented Zagreb Index (AZI):

$$S_x^3Q_{-2}JD_x^3D_y^3[M(Z_{\alpha}^{\beta},x,y)]_{y=x=1} \;=\; 8(k-1)e+432bc$$

$\square$

2.3 M-Polynomials of Line Graphs of k-subdivided Graph of $2D$ Graph Lattice of Tetracenic Nanotubes: $L\big(G_{[\alpha]}(k)\big)$, $\alpha = F, H, W, Z$.

In the following lemma, we find the type of edges of lien graphs of k-subdivded grahps of $2D$ graph lattice of $F[b,c]$.

Lemma 5: Let $G_{L[F]}(k)$ be the lien graphs of k-subdivded grahps of $2D$ graph lattice of $F[b,c]$, where $k \geq 2$. The number of vertices, edges and type of edges of $G_{L[F]}(k)$ with respect to degree are as follows:

$\lvert V(L[F])\rvert$	$\lvert E(L[F])\rvert$	$\lvert E_1(L[F])\rvert = \lvert(2,2)\rvert$	$\lvert E_2(L[F])\rvert = \lvert(2,3)\rvert$	$\lvert E_3(L[F])\rvert = \lvert(3,3)\rvert$
27bc-2c-4b+ke	108bc-24b-12c	24b+12c+(k-2)e	54bc-24b-12c	54bc-24b-12c

where e represents the size of $F[b,c]$, and $e = 27bc-2c-4b$.

Proof:

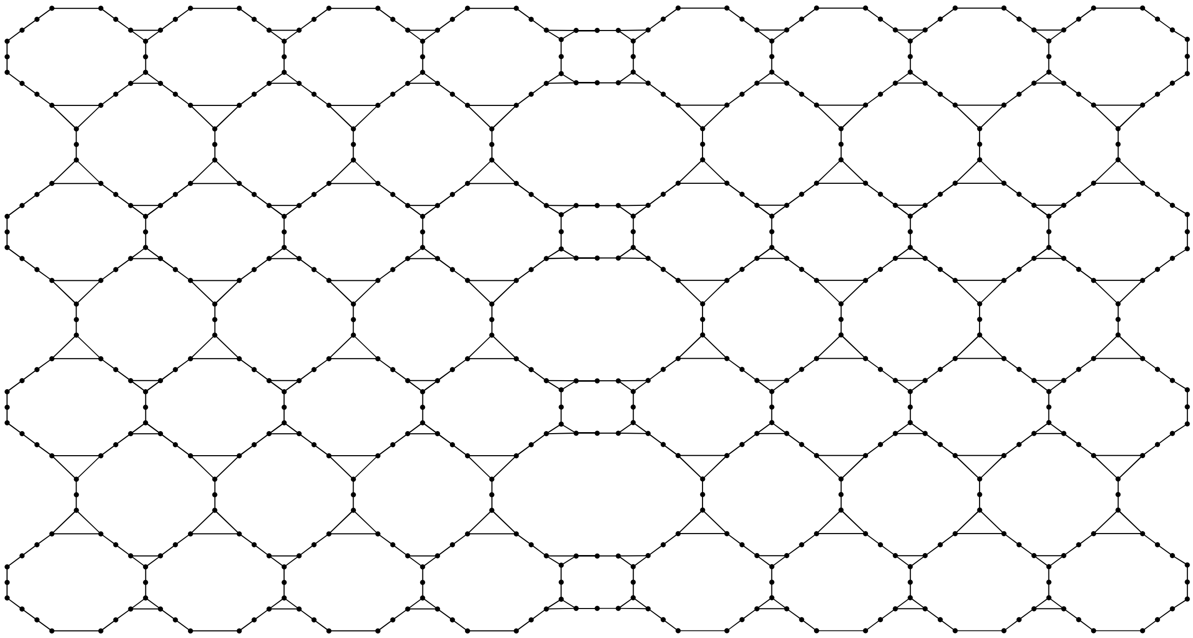

Figure 2.7: Line Graph of 2-subdivided Graph of F

For $k \geq 2$, the graph $L[F]$ has $|V(L[F])| = 27bc - 2c - 4b + ke$ and $|E(L[F])| = 108bc - 24b - 12c + ke$. There are three types of the edges, named as $[E_1(L[F])] = mn$, $|E_2(L[F])| = pq$ and $[E_3(L[F])] = rs$. Here $d_m = d_n = 2$, d_p= 2, d_q= 3 and $d_r = d_s = 3$. The edges $(2,2)$, $(2,3)$ and (3,3) have count $16b + 8c + (k-1)e$, $54bc - 24b - 12c$ and $54bc - 24b - 12c$ respectively. $\square$

In the following theorem, we find the M-polynomial of line graph of k-subdivided graph of $2D$ graph lattice of $F[b,c]$."

Theorem 5:

The M-polynomial of $G_{L[F]}(k)$ is as follows:

$M(L[F] = L\big[F[b,c];x,y\big] = \big(24b + 12c + (k-2)e\big)x^2y^2 + \big(54bc - 24b - 12c\big)x^2y^3 + \big(54bc - 24b - 12c\big)x^3y^3$

Proof:

Suppose the lien graphs of k-subdivded $2D$ graph lattice of $F[b,c]$ then by using Lemma 5, we have three edge types as : E_1(L[F])=[e=mn∈E(L[F]): d_m=d_n=2], E_2(L[F])=[e=pq∈E(L[F]): d_p=2, d_q=3], E_3(L[F])=[e=rs∈E(L[F]): d_r=3, d_s=3] with

$|E_1(L[F])|$=$24b+12c+(k-2)e$, $|E_2(L[F])|$=$54bc-24b-12c$, $|E_3(L[F])|$=$54bc-24b-12c$.

Now by using definition of M-polynomial, we have:

$$\begin{aligned}
M(L[F]=L\big[F[b,c]\big];x,y) &= \sum_{i\leq j} m_{ij}x^i y^j \\
&= \sum_{2\leq 2} m_{22}x^2y^2+\sum_{2\leq 3} m_{23}x^2y^3+\sum_{3\leq 3} m_{33}x^3y^3 \\
&= |E_1(L[F])|x^2y^2+|E_2(L[F])|x^2y^3+|E_3(L[F])|x^3y^3 \\
&= \big(24b+12c+(k-2)e\big)x^2y^2+\big(54bc-24b-12c\big)x^2y^3 \\
&+ \big(54bc-24b-12c\big)x^3y^3
\end{aligned}$$

$\square$

By using the Table 2.1 we have following:

$$\begin{aligned}
M(L[F]=(\big[F[b,c]\big];x,y)=f(x,y) &= \big(24b+12c+(k-2)e\big)x^2y^2+(54bc-24b \\
&- 12c)x^2y^3+(54bc-24b-12c)x^3y^3
\end{aligned}$$

$$\begin{aligned}
D_xf(x,y) &= 2\big(24b+12c+(k-2)e\big)x^2y^2+2(54bc-24b-12c)x^2y^3 \\
&+ 3(54bc-24b-12c)x^3y^3 \\
D_xf(x,y) &= \big(48b+24c+2(k-2)e\big)x^2y^2+(108bc-48b-24c)x^2y^3 \\
&+ (162bc-72b-36c)x^3y^3 \\
D_yf(x,y) &= 2\big(24b+12c+(k-2)e\big)x^2y^2+3(54bc-24b-12c)x^2y^3 \\
&+ 3(54bc-24b-12c)x^3y^3 \\
D_yf(x,y) &= \big(48b+24c+2(k-2)e\big)x^2y^2+(162bc-72b-36c)x^2y^3 \\
&+ (162bc-72b-36c)x^3y^3 \\
D_yD_xf(x,y) &= 4\big(24b+12c+(k-2)e\big)x^2y^2+6(54bc-24b-12c)x^2y^3 \\
&+ 9(54bc-24b-12c)x^3y^3
\end{aligned}$$

$$
\begin{aligned}
D_yD_xf(x,y) &= \big(96b+48c+4(k-2)e\big)x^2y^2+(324bc-144b-72c)x^2y^3\\
&+ (486bc-216b-108c)x^3y^3\\
S_y(f(x,y)) &= \frac{1}{2}\big(24b+12c+(k-2)e\big)x^2y^2+\frac{1}{3}(54bc-24b-12c)x^2y^3\\
&+ \frac{1}{3}(54bc-24b-12c)x^3y^3\\
S_xS_y(f(x,y)) &= \frac{1}{4}\big(24b+12c+(k-2)e\big)x^2y^2+\frac{1}{6}(54bc-24b-12c)x^2y^3\\
&+ \frac{1}{9}(54bc-24b-12c)x^3y^3\\
S_xS_y(f(x,y)) &= \big(6b+3c+\frac{(k-2)e}{4}\big)x^2y^2+(9bc-4b-2c)x^2y^3\\
&+ (6bc-\frac{8b}{3}-\frac{4c}{3})x^3y^3\\
D_y^m(f(x,y)) &= 2^m\big(24b+12c+(k-2)e\big)x^2y^2+3^m(54bc-24b-12c)x^2y^3\\
&+ 3^m(54bc-24b-12c)x^3y^3\\
D_xD_y^m(f(x,y)) &= 2.2^m\big(24b+12c+(k-2)e\big)x^2y^2+2.3^m(54bc-24b-12c)x^2y^3\\
&+ 3.3^m(54bc-24b-12c)x^3y^3\\
D_x^mD_y^m(f(x,y)) &= 2^m.2^m\big(24b+12c+(k-2)e\big)x^2y^2+2^m.3^m(54bc-24b\\
&- 12c)x^2y^3+3^m.3^m(54bc-24b-12c)x^3y^3\\
D_x^mD_y^m(f(x,y)) &= 2^{2m}\big(24b+12c+(k-2)e\big)x^2y^2+6^m(54bc-24b-12c)x^2y^3\\
&+ 9^m(54bc-24b-12c)x^3y^3\\
S_y^m(f(x,y)) &= \frac{1}{2^m}\big(24b+12c+(k-2)e\big)x^2y^2+\frac{1}{3^m}(54bc-24b-12c)x^2y^3\\
&+ \frac{1}{3^m}(54bc-24b-12c)x^3y^3\\
S_xS_y^m(f(x,y)) &= \frac{1}{2.2^m}\big(24b+12c+(k-2)e\big)x^2y^2+\frac{1}{2.3^m}(54bc-24b\\
&- 12c)x^2y^3+\frac{1}{3.3^m}(54bc-24b-12c)x^3y^3\\
S_x^mS_y^m(f(x,y)) &= \frac{1}{2^m.2^m}\big(24b+12c+(k-2)e\big)x^2y^2+\frac{1}{2^m.3^m}(54bc-24b\\
&- 12c)x^2y^3+\frac{1}{3^m.3^m}(54bc-24b-12c)x^3y^3
\end{aligned}
$$

$$
\begin{aligned}
S_x^m S_y^m(f(x,y)) &= \frac{1}{2^{2m}}\big(24b+12c+(k-2)e\big)x^2y^2+\frac{1}{6^m}(54bc-24b-12c)x^2y^3\\
&+ \frac{1}{9^m}(54bc-24b-12c)x^3y^3\\
S_x^m S_y^m(f(x,y)) &= \frac{1}{4^m}\big(24b+12c+(k-2)e\big)x^2y^2+\frac{1}{6^m}(54bc-24b-12c)x^2y^3\\
&+ \frac{1}{9^m}(54bc-24b-12c)x^3y^3\\
S_yD_x(f(x,y)) &= \big(24b+12c+(k-2)e\big)x^2y^+\frac{2}{3}(54bc-24b-12c)x^2y^3\\
&+ (54bc-24b-12c)x^3y^3\\
S_xD_y(f(x,y)) &= \big(24b+12c+(k-2)e\big)x^2y^2+\frac{3}{2}(54bc-24b-12c)x^2y^3\\
&+ (54bc-24b-12c)x^3y^3\\
Jf(x,y) &= \big(24b+12c+(k-2)e\big)x^4+(54bc-24b-12c)x^5\\
&+ (54bc-24b-12c)x^6\\
S_xJf(x,y) &= \frac{1}{4}\big(24b+12c+(k-2)e\big)x^4+\frac{1}{5}(54bc-24b-12c)x^5\\
&+ \frac{1}{6}(54bc-24b-12c)x^6\\
JD_xD_yf(x,y) &= 4\big(24b+12c+(k-2)e\big)x^4+6(54bc-24b-12c)x^5\\
&+ 9(54bc-24b-12c)x^6\\
S_xJD_xD_yf(x,y) &= \big(24b+12c+(k-2)e\big)x^4+\frac{6}{5}(54bc-24b-12c)x^5\\
&+ \frac{3}{2}(54bc-24b-12c)x^6\\
D_y^3f(x,y) &= (2^3)\big(24b+12c+(k-2)e\big)x^2y^2+(3^3)\big(54bc-24b\\
&- 12c\big)x^2y^3+(3^3)\big(54bc-24b-12c\big)x^3y^3\\
D_y^3f(x,y) &= 8\big(24b+12c+(k-2)e\big)x^2y^2+27(54bc-24b-12c)x^2y^3\\
&+ 27(54bc-24b-12c)x^3y^3\\
D_x^3D_y^3f(x,y) &= 2^{2.3}\big(24b+12c+(k-2)e\big)x^2y^2+(6^3)(54bc-24b\\
&- 12c)x^2y^3+(9^3)(54bc-24b-12c)x^3y^3\\
D_x^3D_y^3f(x,y) &= 2^6\big(24b+12c+(k-2)e\big)x^2y^2+(6^3)(54bc-24b\\
&- 12c)x^2y^3+(9^3)(54bc-24b-12c)x^3y^3
\end{aligned}
$$

$$
\begin{aligned}
D_x^3D_y^3f(x,y) &= 64\big(24b+12c+(k-2)e\big)x^4+216\big(54bc-24b\\
&- 12c\big)x^5+729\big(54bc-24b-12c\big)x^6\\
JD_x^3D_y^3f(x,y) &= 64\big(24b+12c+(k-2)e\big)x^{4-2}+216\big(54bc-24b\\
&- 12c\big)x^{5-2}+729\big(54bc-24b-12c\big)x^{6-2}\\
Q_{-2}JD_x^3D_y^3f(x,y) &= 64\big(24b+12c+(k-2)e\big)x^{4-2}+216\big(54bc-24b\\
&- 12c\big)x^{5-2}+729\big(54bc-24b-12c\big)x^{6-2}\\
Q_{-2}JD_x^3D_y^3f(x,y) &= 64\big(24b+12c+(k-2)e\big)x^2+216\big(54bc-24b\\
&- 12c\big)x^3+729\big(54bc-24b-12c\big)x^4\\
S_xQ_{-2}JD_x^3D_y^3f(x,y) &= \frac{64}{2}\big(24b+12c+(k-2)e\big)x^2+\frac{216}{3}(54bc-24b\\
&- 12c)x^3+\frac{729}{4}\big(54bc-24b-12c\big)x^4\\
S_x^mQ_{-2}JD_x^3D_y^3f(x,y) &= \frac{64}{2^m}\big(24b+12c+(k-2)e\big)x^2+\frac{216}{3^m}(54bc-24b\\
&- 12c)x^3+\frac{729}{4^m}\big(54bc-24b-12c\big)x^4\\
S_x^3Q_{-2}JD_x^3D_y^3f(x,y) &= \frac{64}{2^3}\big(24b+12c+(k-2)e\big)x^2+\frac{216}{3^3}(54bc-24b\\
&- 12c)x^3+\frac{729}{4^3}\big(54bc-24b-12c\big)x^4\\
S_x^3Q_{-2}JD_x^3D_y^3f(x,y) &= \frac{64}{8}\big(24b+12c+(k-2)e\big)x^2+\frac{216}{27}(54bc-24b\\
&- 12c)x^3+\frac{729}{64}\big(54bc-24b-12c\big)x^4\\
S_x^3Q_{-2}JD_x^3D_y^3f(x,y) &= 8\big(24b+12c+(k-2)e\big)x^2+8(54bc-24b-12c)x^3\\
&+ \frac{729}{64}\big(54bc-24b-12c\big)x^4
\end{aligned}
$$

2.4 Degree Based Topological Indices of Line Graph of k-subdivided Graph of F

"First Zagreb:

$$(D_x+D_y)[M(L[F]_{\alpha}^{\beta},x,y)]_{y=x=1}\text{”} \;=\; 4(k-2)e+594bc-168b-84c$$

Second Zagreb:

$$(D_xD_y)[M(L[F]_{\alpha}^{\beta},x,y)]_{y=x=1} \quad = \quad 4(k-2)e+810bc-264b-132c$$

”Second Modified Zagreb:

$$(S_xS_y)[M(L[F]_{\alpha}^{\beta},x,y)]_{y=x=1}\text{”} \quad = \quad \frac{k-2}{4}e+15bc-\frac{2b}{3}-\frac{c}{3}$$

General Randic (GR):

$$\begin{aligned}(D_x^mD_y^m)[M(L[F]_{\alpha}^{\beta},x,y)]_{y=x=1} &= 4^m(24b)+4^m(12c)+4^m(k-2)e+6^m(54bc)-6^m(24b)\\ &- 6^m(12c)+9^m(54bc)-9^m(24b)-9^m(12c)\end{aligned}$$

General Randic (GR):

$$\begin{aligned}(S_x^mS_y^m)[M(L[F]_{\alpha}^{\beta},x,y)]_{y=x=1} &= \frac{1}{4^m}\times\big(24b+12c+(k-2)e\big)+\frac{1}{6^m}\times(54bc-24b-12c)\\ &+ \frac{1}{9^m}\times(54bc-24b-12c)\end{aligned}$$

Symmetric Division Index (SDI):

$$(D_xS_y+D_yS_x)[M(L[F]_{\alpha}^{\beta},x,y)]_{y=x=1} \quad = \quad 2(k-2)e+225bc-52b-26c$$

Harmonic Index (HI):

$$2S_xJ[M(L[[F]_{\alpha}^{\beta},x,y)]_{y=x=1} \quad = \quad \frac{(k-2)e}{2}+\frac{198bc}{5}-\frac{28b}{5}-\frac{14c}{5}$$

Inverse Sum Index (ISI):

$$S_xJD_xD_y[M(L[F]_{\alpha}^{\beta},x,y)]_{y=x=1} \quad = \quad (k-2)e+\frac{729bc}{5}-\frac{204b}{5}-\frac{102c}{5}$$

Augmented Zagreb Index (AZI):

$$S_x^3 Q_{-2} J D_x^3 D_y^3 [M(L[F]_{\alpha}^{\beta}, x, y)]_{y=x=1} \quad = \quad 8(k-2)e + \frac{67014bc}{64} - \frac{17496b}{64} - \frac{8748c}{64}$$

□

In the following lemma, we find the type of edges of lien graphs of k-subdivded grahp of $2D$ graph lattice of $H[b,c]$.

Lemma 6: Let $G_{L[H]}(k)$ be the lien graphs of k-subdivded grahp of $2D$ graph lattice of $H[b,c]$, where $k \geq 2$. The number of vertices, edges and type of edges of $G_{L[H]}(k)$ with respect to degree are as follows:

$\lvert V(L[H])\rvert$	$\lvert E(L[H])\rvert$	$\lvert E_1(L[H])\rvert = \lvert(2,2)\rvert$	$\lvert E_2(L[H])\rvert = \lvert(2,3)\rvert$	$\lvert E_3(L[H])\rvert = \lvert(3,3)\rvert$
$27bc-4b+ke$	$108bc-24b$	$24b+(k-2)e$	$54bc-24b$	$54bc-24b$

where e represents the size of line graph of $H[b,c]$, and $e = 27bc-4b$.

Proof:

For $k \geq 2$, the graph $L[H]$ has $|V(L[H])| = 27bc-4b+ke$ and $|E(L[H])| = 108bc-24b$. There are three types of the edges in $L[H]$. We name $[E_1(L[H]) = (2,2)]$ as type 1, $[E_2(H) = (2,3)]$ as type 2 and $[E_3(L[H]) = (3,3)]$ as type 3. By calculating these edge types, we get that $24b+(k-2)e$, $54bc-24b$ and $54bc-24b$ are counts of the edges of type 1, type 2 and type 3 respectively. □

In the following theorem, we find the M-polynomial of lien graphs of k-subdivded grahp of $2D$ graph lattice of $H[b,c]$.

Theorem 6:

"The M-polynomial of $G_{L[H]}(k)$ is as follows":

$M(L[H] = L\big[H[b,c];x,y\big] = \big(24b+(k-2)e\big)x^2y^2 + \big(54bc-24b\big)x^2y^3 + \big(54bc-24b\big)x^3y^3$

Proof:

Suppose thelien graphs of k-subdivded $2D$ graph lattice of $H[b,c]$ then by using Lemma 6, we have three edge types as : E_1(L[H])=[e=mn∈E(L[H]): d_m=d_n=2],

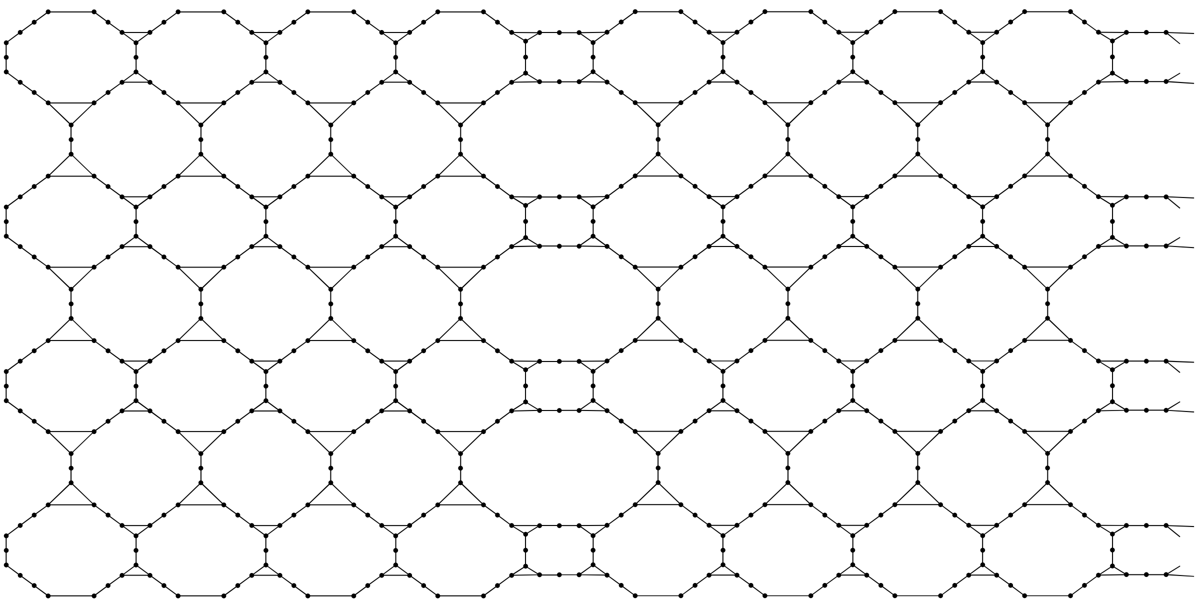

Figure 2.8: Line Graph of 2-subdivided Graph of H

E_2(L[H])=[e=pq∈E(L[H]): d_p=2, d_q=3], E_3(L[H])=[e=rs∈E(L[H]): d_r=3, d_s=3] with $|E_1(L[H])|$=$24b+(k-2)e$, $|E_2(L[H])|$=$54bc-24b$, $|E_3(L[H])|$=$54bc-24b$.

Now by using definition of M-polynomial, we have:

$$\begin{aligned}
M(L[H]=L\big[H[b,c]\big];x,y) &= \sum_{i\leq j} m_{ij}x^iy^j \\
&= \sum_{2\leq 2} m_{22}x^2y^2+\sum_{2\leq 3} m_{23}x^2y^3+\sum_{3\leq 3} m_{33}x^3y^3 \\
&= |E_1(L[H])|x^2y^2+|E_2(L[H])|x^2y^3+|E_3(L[H])|x^3y^3 \\
&= \big(24b+(k-2)e\big)x^2y^2+\big(54bc-24b\big)x^2y^3 \\
&+ \big(54bc-24b\big)x^3y^3
\end{aligned}$$

□

By using the Table 2.1 we have following:

$$\begin{aligned}
M(L[H]=(\big[H[b,c]\big];x,y) &= \big(24b+(k-2)e\big)x^2y^2+(54bc-24b)x^2y^3 \\
&+ (54bc-24b)x^3y^3
\end{aligned}$$

$$
\begin{aligned}
D_x h(x,y) &= 2\big(24b+(k-2)e\big)x^2y^2+2(54bc-24b)x^2y^3 \\
&+ 3(54bc-24b)x^3y^3 \\
D_x h(x,y) &= \big(48b+2(k-2)e\big)x^2y^2+(108bc-48b)x^2y^3 \\
&+ (162bc-72b)x^3y^3 \\
D_y h(x,y) &= 2\big(24b+(k-2)e\big)x^2y^2+3(54bc-24b)x^2y^3 \\
&+ 3(54bc-24b)x^3y^3 \\
D_y h(x,y) &= \big(48b+2(k-2)e\big)x^2y^2+(162bc-72b)x^2y^3 \\
&+ (162bc-72b)x^3y^3 \\
D_yD_x h(x,y) &= 4\big(24b+(k-2)e\big)x^2y^2+6(54bc-24b)x^2y^3 \\
&+ 9(54bc-24b)x^3y^3 \\
D_yD_x h(x,y) &= \big(96b+4(k-2)e\big)x^2y^2+(324bc-144b)x^2y^3 \\
&+ (486bc-216b)x^3y^3 \\
S_y(h(x,y)) &= \frac{1}{2}\big(24b+(k-2)e\big)x^2y^2+\frac{1}{3}(54bc-24b)x^2y^3 \\
&+ \frac{1}{3}(54bc-24b)x^3y^3 \\
S_xS_y(h(x,y)) &= \frac{1}{4}\big(24b+(k-2)e\big)x^2y^2+\frac{1}{6}(54bc-24b)x^2y^3 \\
&+ \frac{1}{9}(54bc-24b)x^3y^3 \\
S_xS_y(h(x,y)) &= \big(6b+\frac{(k-2)e}{4}\big)x^2y^2+(9bc-4b)x^2y^3 \\
&+ (6bc-\frac{8b}{3})x^3y^3 \\
D_y^m(h(x,y)) &= 2^m\big(24b+(k-2)e\big)x^2y^2+3^m(54bc-24b)x^2y^3 \\
&+ 3^m(54bc-24b)x^3y^3 \\
D_xD_y^m(h(x,y)) &= 2.2^m\big(24b+(k-2)e\big)x^2y^2+2.3^m(54bc-24b)x^2y^3 \\
&+ 3.3^m(54bc-24b)x^3y^3 \\
D_x^mD_y^m(h(x,y)) &= 2^m.2^m\big(24b+(k-2)e\big)x^2y^2+2^m.3^m(54bc \\
&- 24b)x^2y^3+3^m.3^m(54bc-24b)x^3y^3
\end{aligned}
$$

$$
\begin{aligned}
D_x^m D_y^m(h(x,y)) &= 2^{2m}\big(24b+(k-2)e\big)x^2y^2+6^m(54bc-24b)x^2y^3\\
&+ 9^m(54bc-24b)x^3y^3\\
S_y^m(h(x,y)) &= \frac{1}{2^m}\big(24b+(k-2)e\big)x^2y^2+\frac{1}{3^m}(54bc-24b)x^2y^3\\
&+ \frac{1}{3^m}(54bc-24b)x^3y^3\\
S_xS_y^m(h(x,y)) &= \frac{1}{2.2^m}\big(24b+(k-2)e\big)x^2y^2+\frac{1}{2.3^m}(54bc\\
&- 24b)x^2y^3+\frac{1}{3.3^m}(54bc-24b)x^3y^3\\
S_x^mS_y^m(h(x,y)) &= \frac{1}{2^m.2^m}\big(24b+(k-2)e\big)x^2y^2+\frac{1}{2^m.3^m}(54bc\\
&- 24b)x^2y^3+\frac{1}{3^m.3^m}(54bc-24b)x^3y^3\\
S_x^mS_y^m(h(x,y)) &= \frac{1}{2^{2m}}\big(24b+(k-2)e\big)x^2y^2+\frac{1}{6^m}(54bc-24b)x^2y^3\\
&+ \frac{1}{9^m}(54bc-24b)x^3y^3\\
S_x^mS_y^m(h(x,y)) &= \frac{1}{4^m}\big(24b+(k-2)e\big)x^2y^2+\frac{1}{6^m}(54bc-24b)x^2y^3\\
&+ \frac{1}{9^m}(54bc-24b)x^3y^3\\
S_yD_x(h(x,y)) &= \big(24b+(k-2)e\big)x^2y^+\frac{2}{3}(54bc-24b)x^2y^3\\
&+ (54bc-24b)x^3y^3\\
S_xD_y(h(x,y)) &= \big(24b+(k-2)e\big)x^2y^2+\frac{3}{2}(54bc-24b)x^2y^3\\
&+ (54bc-24b)x^3y^3\\
Jh(x,y) &= \big(24b+(k-2)e\big)x^4+(54bc-24b)x^5\\
&+ (54bc-24b)x^6\\
S_xJh(x,y) &= \frac{1}{4}\big(24b+(k-2)e\big)x^4+\frac{1}{5}(54bc-24b)x^5\\
&+ \frac{1}{6}(54bc-24b)x^6\\
JD_xD_yh(x,y) &= 4\big(24b+(k-2)e\big)x^4+6(54bc-24b)x^5\\
&+ 9(54bc-24b)x^6
\end{aligned}
$$

$$
\begin{aligned}
S_xJD_xD_yh(x,y) &= \big(24b+(k-2)e\big)x^4+\frac{6}{5}(54bc-24b)x^5\\
&+ \frac{3}{2}(54bc-24b)x^6\\
D_y^3h(x,y) &= (2^3)\big(24b+(k-2)e\big)x^2y^2+(3^3)\big(54bc\\
&- 24b\big)x^2y^3+(3^3)\big(54bc-24b\big)x^3y^3\\
D_y^3h(x,y) &= 8\big(24b+(k-2)e\big)x^2y^2+27(54bc-24b)x^2y^3\\
&+ 27(54bc-24b)x^3y^3\\
D_x^3D_y^3h(x,y) &= 2^{2.3}\big(24b+(k-2)e\big)x^2y^2+(6^3)(54bc\\
&- 24b)x^2y^3+(9^3)(54bc-24b)x^3y^3\\
D_x^3D_y^3h(x,y) &= 2^6\big(24b+(k-2)e\big)x^2y^2+(6^3)(54bc\\
&- 24b)x^2y^3+(9^3)(54bc-24b)x^3y^3\\
D_x^3D_y^3h(x,y) &= 64\big(24b+(k-2)e\big)x^4+216\big(54bc\\
&- 24b\big)x^5+729\big(54bc-24b\big)x^6\\
JD_x^3D_y^3h(x,y) &= 64\big(24b+(k-2)e\big)x^{4-2}+216\big(54bc\\
&- 24b\big)x^{5-2}+729\big(54bc-24b\big)x^{6-2}\\
Q_{-2}JD_x^3D_y^3h(x,y) &= 64\big(24b+(k-2)e\big)x^{4-2}+216\big(54bc\\
&- 24b\big)x^{5-2}+729\big(54bc-24b\big)x^{6-2}\\
Q_{-2}JD_x^3D_y^3h(x,y) &= 64\big(24b+(k-2)e\big)x^2+216\big(54bc\\
&- 24b\big)x^3+729\big(54bc-24b\big)x^4\\
S_xQ_{-2}JD_x^3D_y^3h(x,y) &= \frac{64}{2}\big(24b+(k-2)e\big)x^2+\frac{216}{3}(54bc\\
&- 24b)x^3+\frac{729}{4}\big(54bc-24b\big)x^4\\
S_x^mQ_{-2}JD_x^3D_y^3h(x,y) &= \frac{64}{2^m}\big(24b+(k-2)e\big)x^2+\frac{216}{3^m}(54bc\\
&- 24b)x^3+\frac{729}{4^m}\big(54bc-24b\big)x^4\\
S_x^3Q_{-2}JD_x^3D_y^3h(x,y) &= \frac{64}{2^3}\big(24b+(k-2)e\big)x^2+\frac{216}{3^3}(54bc\\
&- 24b)x^3+\frac{729}{4^3}\big(54bc-24b\big)x^4
\end{aligned}
$$

$$
\begin{aligned}
S_x^3 Q_{-2}JD_x^3 D_y^3 h(x,y) &= \frac{64}{8}\big(24b+(k-2)e\big)x^2+\frac{216}{27}(54bc \\
&- 24b)x^3+\frac{729}{64}\big(54bc-24b\big)x^4 \\
S_x^3 Q_{-2}JD_x^3 D_y^3 h(x,y) &= 8\big(24b+(k-2)e\big)x^2+8(54bc-24b)x^3 \\
&+ \frac{729}{64}\big(54bc-24b\big)x^4
\end{aligned}
$$

Degree Based Topological Indices of Line Graph of *k*-subdivided graph of *H*

"First Zagreb:

$$
(D_x+D_y)[M(L[H]_{\alpha}^{\beta},x,y)]_{y=x=1}\text{"} = 4(k-2)e+594bc-168b
$$

Second Zagreb:

$$
(D_xD_y)[M(L[H]_{\alpha}^{\beta},x,y)]_{y=x=1} = 4(k-2)e+810bc-264b
$$

"Second Modified Zagreb:

$$
(S_xS_y)[M(L[H]_{\alpha}^{\beta},x,y)]_{y=x=1}\text{"} = \frac{k-2}{4}e+15bc-\frac{2b}{3}
$$

General Randic (GR):

$$
\begin{aligned}
(D_x^m D_y^m)[M(L[H]_{\alpha}^{\beta},x,y)]_{y=x=1} &= 4^m(24b)+4^m(k-2)e+6^m(54bc)-6^m(24b) \\
&+ 9^m(54bc)-9^m(24b)
\end{aligned}
$$

General Randic (GR):

$$
\begin{aligned}
(S_x^m S_y^m)[M(L[H]_{\alpha}^{\beta},x,y)]_{y=x=1} &= \frac{1}{4^m}\times\big(24b+(K-2)e\big)+\frac{1}{6^m}\times(54bc-24b) \\
&+ \frac{1}{9^m}\times(54bc-24b)
\end{aligned}
$$

”Symmetric Division Index (SDI):

$$(D_xS_y + D_yS_x)[M(L[H]^{\beta}_{\alpha}, x, y)]_{y=x=1}\text{”} \quad = \quad 2(k-2)e + 225bc - 52b$$

Harmonic Index (HI):

$$\text{”}2S_xJ[M(L[[H]^{\beta}_{\alpha}, x, y)]_{y=x=1}\text{”} \quad = \quad \frac{(k-2)e}{2} + \frac{198bc}{5} - \frac{28b}{5}$$

Inverse Sum Index (ISI):

$$S_xJD_xD_y[M(L[H]^{\beta}_{\alpha}, x, y)]_{y=x=1} \quad = \quad (k-2)e + \frac{729bc}{5} - \frac{204b}{5}$$

”Augmented Zagreb Index (AZI):

$$S_x^3Q_{-}2JD_x^3D_y^3[M(L[H]^{\beta}_{\alpha}, x, y)]_{y=x=1}\text{”} \quad = \quad 8(k-2)e + \frac{67014bc}{64} - \frac{17496b}{64}$$

□

In the following lemma, we find the type of edges of lien graphs of k-subdivded grahp of $2D$ graph lattice of $W = W[b, c]$.

Lemma 7: Let $G_{L[W]}(k)$ be the line graph of k-subdivided graph of $2D$ graph lattice of $W[b, c]$, where $k \geq 2$. The number of vertices, edges and type of edges of $G_{L[W]}(k)$ with respect to degree are as follows:

$\lvert V(L[W])\rvert$	$\lvert E(L[W])\rvert$	$\lvert E_1(L[W])\rvert = \lvert(2,2)\rvert$	$\lvert E_2(L[W])\rvert = \lvert(2,3)\rvert$	$\lvert E_3(L[W])\rvert = \lvert(3,3)\rvert$
$27bc - 2c + ke$	$108bc - 12c$	$12c + (k-2)e$	$54bc - 12c$	$54bc - 12c$

where e represents the size of line graph of $W[b, c]$, and $e = 27bc - 2c$.

Proof:

For $k \geq 2$, the graph $L[W]$ has $|V(L[W])| = 27bc - 2c + ke$ and $|E(L[W])| = 108bc - 12c$. There are three types of the edges in $L[W]$. We name $[E_1(L[W]) = (2,2)]$ as type 1, $[E_2(W) = (2,3)]$ as type 2 and $[E_3(L[W]) = (3,3)]$ as type 3. By calculating these

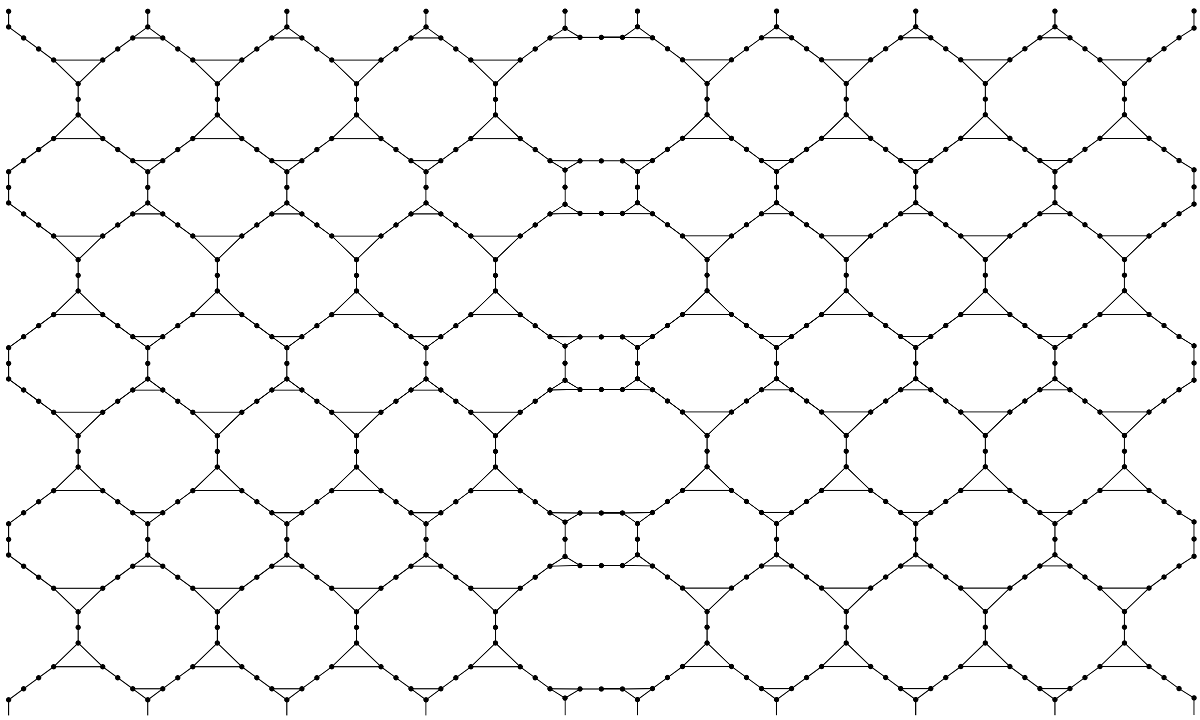

Figure 2.9: Line Graph of 2-subdivided Graph of W

edge types, we get that $12c+(k-2)e$, $54bc-12c$ and $54bc-12c$ are counts of the edges of type 1, type 2 and type 3 respectively. □

In the following theorem, we find the M-polynomial of line graph of k-subdivided graph of $2D$ graph lattice of $W[b,c]$.

Theorem 7:

The M-polynomial of $G_{L[W]}(k)$ is as follows:

$$M(L[W]=L\big[W[b,c];x,y\big]=\big(12c+(k-2)e\big)x^2y^2+\big(54bc-12c\big)x^2y^3+\big(54bc-12c\big)x^3y^3$$

Proof:

Suppose the lien graphs of k-subdivded grahp lattice of 2D $W[b,c]$ then by using Lemma 7, we have three edge types as : E_1(L[W])=[e=mn∈E(L[W]): d_m=d_n=2], E_2(L[W])=[e=pq∈E(L[W]): d_p=2, d_q=3], E_3(L[W])=[e=rs∈E(L[W]): d_r=3, d_s=3] with $|E_1(L[W])|$=$12c+(k-2)e$, $|E_2(L[W])|$=$54bc-12c$, $|E_3(L[W])|$=$54bc-12c$.

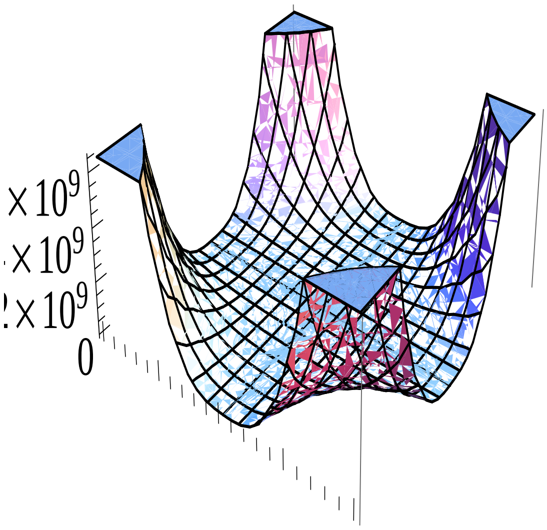

Figure 2.10: 3D Graph of 2-subdivided M-Polynomial of W

Now by using definition of M-polynomial, we have:

$$\begin{aligned}
M(L[W] = L\big[W[b,c]\big];x,y) &= \sum_{i\leq j} m_{ij}x^i y^j \\
&= \sum_{2\leq 2} m_{22}x^2y^2 + \sum_{2\leq 3} m_{23}x^2y^3 + \sum_{3\leq 3} m_{33}x^3y^3 \\
&= |E_1(L[W])|x^2y^2 + |E_2(L[W])|x^2y^3 + |E_3(L[W])|x^3y^3 \\
&= \big(12c+(k-2)e\big)x^2y^2 + \big(54bc-12c\big)x^2y^3 \\
&+ \big(54bc-12c\big)x^3y^3
\end{aligned}$$

□

By using the Table 2.1 we have following:

Let

$$\begin{aligned}
M(L[W] = (\big[W[b,c]\big];x,y) &= \big(12c+(k-2)e\big)x^2y^2 + (54bc-12c)x^2y^3 \\
&+ (54bc-12c)x^3y^3
\end{aligned}$$

$$
\begin{aligned}
D_x w(x,y) &= 2\big(12c+(k-2)e\big)x^2y^2+2(54bc-12c)x^2y^3 \\
&+ 3(54bc-12c)x^3y^3 \\
D_x w(x,y) &= \big(24c+2(k-2)e\big)x^2y^2+(108bc-24c)x^2y^3 \\
&+ (162bc-36c)x^3y^3 \\
D_y w(x,y) &= 2\big(12c+(k-2)e\big)x^2y^2+3(54bc-12c)x^2y^3 \\
&+ 3(54bc-12c)x^3y^3 \\
D_y w(x,y) &= \big(24c+2(k-2)e\big)x^2y^2+(162bc-36c)x^2y^3 \\
&+ (162bc-36c)x^3y^3 \\
D_yD_x w(x,y) &= 4\big(12c+(k-2)e\big)x^2y^2+6(54bc-12c)x^2y^3 \\
&+ 9(54bc-12c)x^3y^3 \\
D_yD_x w(x,y) &= \big(48c+4(k-2)e\big)x^2y^2+(324bc-72c)x^2y^3 \\
&+ (486bc-108c)x^3y^3 \\
S_y(w(x,y)) &= \frac{1}{2}\big(12c+(k-2)e\big)x^2y^2+\frac{1}{3}(54bc-12c)x^2y^3 \\
&+ \frac{1}{3}(54bc-12c)x^3y^3 \\
S_xS_y(w(x,y)) &= \frac{1}{4}\big(12c+(k-2)e\big)x^2y^2+\frac{1}{6}(54bc-12c)x^2y^3 \\
&+ \frac{1}{9}(54bc-12c)x^3y^3 \\
S_xS_y(w(x,y)) &= \big(3c+\frac{(k-2)e}{4}\big)x^2y^2+(9bc-2c)x^2y^3 \\
&+ (6bc-\frac{4c}{3})x^3y^3 \\
D_y^m(w(x,y)) &= 2^m\big(12c+(k-2)e\big)x^2y^2+3^m(54bc-12c)x^2y^3 \\
&+ 3^m(54bc-12c)x^3y^3 \\
D_xD_y^m(w(x,y)) &= 2.2^m\big(12c+(k-2)e\big)x^2y^2+2.3^m(54bc-12c)x^2y^3 \\
&+ 3.3^m(54bc-12c)x^3y^3 \\
D_x^mD_y^m(w(x,y)) &= 2^m.2^m\big(12c+(k-2)e\big)x^2y^2+2^m.3^m(54bc \\
&- 12c)x^2y^3+3^m.3^m(54bc-12c)x^3y^3
\end{aligned}
$$

$$
\begin{aligned}
D_x^m D_y^m(w(x,y)) &= 2^{2m}\big(12c+(k-2)e\big)x^2y^2 + 6^m(54bc-12c)x^2y^3 \\
&+ 9^m(54bc-12c)x^3y^3 \\
S_y^m(w(x,y)) &= \frac{1}{2^m}\big(12c+(k-2)e\big)x^2y^2 + \frac{1}{3^m}(54bc-12c)x^2y^3 \\
&+ \frac{1}{3^m}(54bc-12c)x^3y^3 \\
S_xS_y^m(w(x,y)) &= \frac{1}{2.2^m}\big(12c+(k-2)e\big)x^2y^2 + \frac{1}{2.3^m}(54bc \\
&- 12c)x^2y^3 + \frac{1}{3.3^m}(54bc-12c)x^3y^3 \\
S_x^mS_y^m(w(x,y)) &= \frac{1}{2^m.2^m}\big(12c+(k-2)e\big)x^2y^2 + \frac{1}{2^m.3^m}(54bc \\
&- 12c)x^2y^3 + \frac{1}{3^m.3^m}(54bc-12c)x^3y^3 \\
S_x^mS_y^m(w(x,y)) &= \frac{1}{2^{2m}}\big(12c+(k-2)e\big)x^2y^2 + \frac{1}{6^m}(54bc-12c)x^2y^3 \\
&+ \frac{1}{9^m}(54bc-12c)x^3y^3 \\
S_x^mS_y^m(w(x,y)) &= \frac{1}{4^m}\big(12c+(k-2)e\big)x^2y^2 + \frac{1}{6^m}(54bc-12c)x^2y^3 \\
&+ \frac{1}{9^m}(54bc-12c)x^3y^3 \\
S_yD_x(w(x,y)) &= \big(12c+(k-2)e\big)x^2y^3 + \frac{2}{3}(54bc-12c)x^2y^3 \\
&+ (54bc-12c)x^3y^3 \\
S_xD_y(w(x,y)) &= \big(12c+(k-2)e\big)x^2y^2 + \frac{3}{2}(54bc-12c)x^2y^3 \\
&+ (54bc-12c)x^3y^3 \\
Jw(x,y) &= \big(12c+(k-2)e\big)x^4 + (54bc-12c)x^5 \\
&+ (54bc-12c)x^6 \\
S_xJw(x,y) &= \frac{1}{4}\big(12c+(k-2)e\big)x^4 + \frac{1}{5}(54bc-12c)x^5 \\
&+ \frac{1}{6}(54bc-12c)x^6 \\
JD_xD_yw(x,y) &= 4\big(12c+(k-2)e\big)x^4 + 6(54bc-12c)x^5 \\
&+ 9(54bc-12c)x^6 \\
S_xJD_xD_yw(x,y) &= \big(12c+(k-2)e\big)x^4 + \frac{6}{5}(54bc-12c)x^5 \\
&+ \frac{3}{2}(54bc-12c)x^6
\end{aligned}
$$

$$
\begin{aligned}
D_y^3 w(x,y) &= (2^3)\big(12c+(k-2)e\big)x^2y^2+(3^3)\big(54bc \\
&- 12c\big)x^2y^3+(3^3)\big(54bc-12c\big)x^3y^3 \\
D_y^3 w(x,y) &= 8\big(12c+(k-2)e\big)x^2y^2+27(54bc-12c)x^2y^3 \\
&+ 27(54bc-12c)x^3y^3 \\
D_x^3D_y^3 w(x,y) &= 2^{2.3}\big(12c+(k-2)e\big)x^2y^2+(6^3)(54bc \\
&- 12c)x^2y^3+(9^3)(54bc-12c)x^3y^3 \\
D_x^3D_y^3 w(x,y) &= 2^6\big(12c+(k-2)e\big)x^2y^2+(6^3)(54bc \\
&- 12c)x^2y^3+(9^3)(54bc-12c)x^3y^3 \\
D_x^3D_y^3 w(x,y) &= 64\big(12c+(k-2)e\big)x^4+216\big(54bc \\
&- 12c\big)x^5+729\big(54bc-12c\big)x^6 \\
JD_x^3D_y^3 w(x,y) &= 64\big(12c+(k-2)e\big)x^{4-2}+216\big(54bc \\
&- 12c\big)x^{5-2}+729\big(54bc-12c\big)x^{6-2} \\
Q_-2JD_x^3D_y^3 w(x,y) &= 64\big(12c+(k-2)e\big)x^{4-2}+216\big(54bc \\
&- 12c\big)x^{5-2}+729\big(54bc-12c\big)x^{6-2} \\
Q_-2JD_x^3D_y^3 w(x,y) &= 64\big(12c+(k-2)e\big)x^2+216\big(54bc \\
&- 12c\big)x^3+729\big(54bc-12c\big)x^4 \\
S_xQ_-2JD_x^3D_y^3 w(x,y) &= \frac{64}{2}\big(12c+(k-2)e\big)x^2+\frac{216}{3}(54bc \\
&- 12c)x^3+\frac{729}{4}\big(54bc-12c\big)x^4 \\
S_x^mQ_-2JD_x^3D_y^3 w(x,y) &= \frac{64}{2^m}\big(12c+(k-2)e\big)x^2+\frac{216}{3^m}(54bc \\
&- 12c)x^3+\frac{729}{4^m}\big(54bc-12c\big)x^4 \\
S_x^3Q_-2JD_x^3D_y^3 w(x,y) &= \frac{64}{2^3}\big(12c+(k-2)e\big)x^2+\frac{216}{3^3}(54bc \\
&- 12c)x^3+\frac{729}{4^3}\big(54bc-12c\big)x^4 \\
S_x^3Q_-2JD_x^3D_y^3 w(x,y) &= \frac{64}{8}\big(12c+(k-2)e\big)x^2+\frac{216}{27}(54bc \\
&- 12c)x^3+\frac{729}{64}\big(54bc-12c\big)x^4
\end{aligned}
$$

$$
\begin{aligned}
S_x^3 Q_{-2} J D_x^3 D_y^3 w(x,y) &= 8\big(12c+(k-2)e\big)x^2 + 8(54bc-12c)x^3 \\
&+ \frac{729}{64}\big(54bc-12c\big)x^4
\end{aligned}
$$

Degree Based Topological Indices of Line Graph of k-subdivided graph of W

"First Zagreb:

$$
(D_x+D_y)[M(L[W]_{\alpha}^{\beta},x,y)]_{y=x=1}\text{"} = 4(k-2)e+594bc-84c
$$

Second Zagreb:

$$
(D_xD_y)[M(L[W]_{\alpha}^{\beta},x,y)]_{y=x=1} = 4(k-2)e+810bc-132c
$$

"Second Modified Zagreb:

$$
(S_xS_y)[M(L[W]_{\alpha}^{\beta},x,y)]_{y=x=1}\text{"} = \frac{k-2}{4}e+15bc-\frac{c}{3}
$$

General Randic (GR):

$$
\begin{aligned}
(D_x^mD_y^m)[M(L[W]_{\alpha}^{\beta},x,y)]_{y=x=1} &= 4^m(12c)+4^m(k-2)e+6^m(54bc) \\
&- 6^m(12c)+9^m(54bc)-9^m(12c)
\end{aligned}
$$

General Randic (GR):

$$
\begin{aligned}
(S_x^mS_y^m)[M(L[W]_{\alpha}^{\beta},x,y)]_{y=x=1} &= \frac{1}{4^m}\times\big(12c+(K-2)e\big)+\frac{1}{6^m}\times(54bc-12c) \\
&+ \frac{1}{9^m}\times(54bc-12c)
\end{aligned}
$$

"Symmetric Division Index (SDI):

$$
(D_xS_y+D_yS_x)[M(L[W]_{\alpha}^{\beta},x,y)]_{y=x=1}\text{"} = 2(k-2)e+225bc-26c
$$

Harmonic Index (HI):

$$2S_xJ[M(L[[W]_{\alpha}^{\beta},x,y)]_{y=x=1} = \frac{(k-2)e}{2}+\frac{198bc}{5}-\frac{14c}{5}$$

Inverse Sum Index (ISI):

$$S_xJD_xD_y[M(L[W]_{\alpha}^{\beta},x,y)]_{y=x=1} = (k-2)e+\frac{729bc}{5}-\frac{102c}{5}$$

Augmented Zagreb Index (AZI):

$$S_x^3Q_{-2}JD_x^3D_y^3[M(L[W]_{\alpha}^{\beta},x,y)]_{y=x=1} = 8(k-2)e+\frac{67014bc}{64}-\frac{8748c}{64}$$

□ In the following lemma, we find the type of edges of lien graphs of k-subdivded grahp of $2D$ graph lattice of $Z[b,c]$.

Lemma 8: Let $G_{L[Z]}(k)$ be the line graph of k-subdivided graph of $2D$ graph lattice of

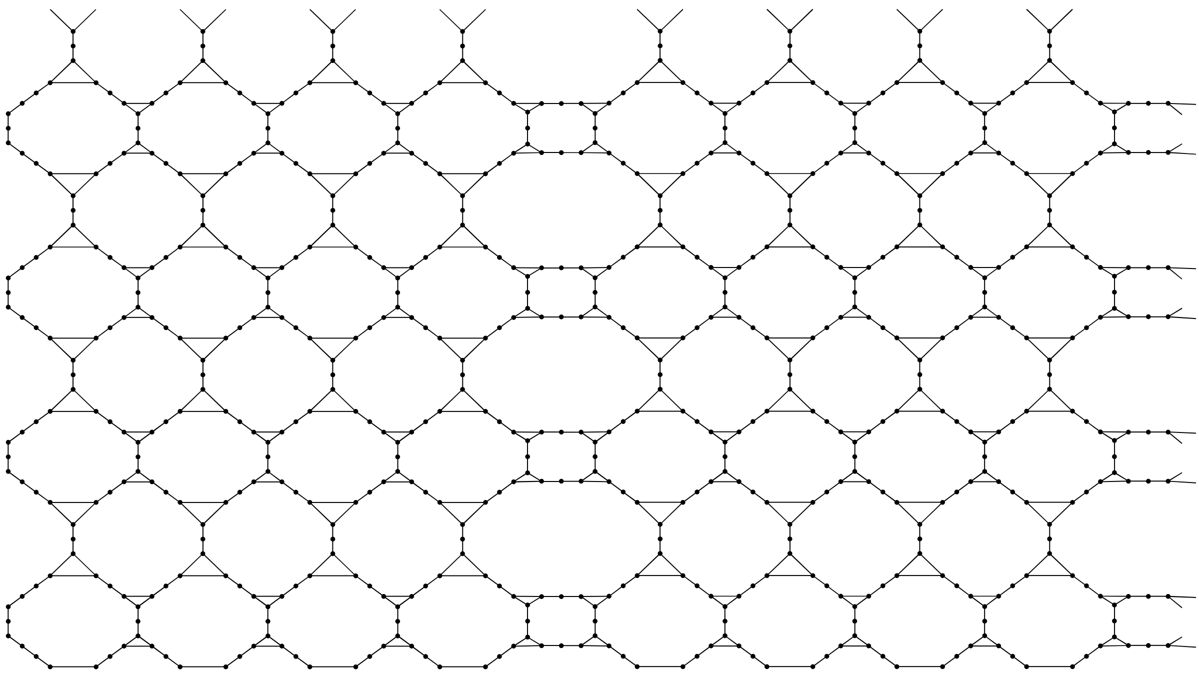

Figure 2.11: Line Graph of 2-subdivided Graph of Z

$Z[b,c]$, where $k \geq 2$. The number of vertices, edges and type of edges of $G_{L[Z]}(k)$ with respect to degree are as follows:

$\lvert V(L[Z])\rvert$	$\lvert E(L[Z])\rvert$	$\lvert E_1(L[Z])\rvert = \lvert(2,2)\rvert$	$\lvert E_2(L[Z])\rvert = \lvert(2,3)\rvert$	$\lvert E_3(L[Z])\rvert = \lvert(3,3)\rvert$
$27bc+ke$	$108bc$	$(k-2)e$	$54bc$	$54bc$

where e represents the size of line graph of $Z[b,c]$, and $e = 27bc$.

Proof:

For $k \geq 2$, the graph $L[Z]$ has $|V(L[Z])| = 27bc+ke$ and $|E(L[Z])| = 108bcb$. There are three types of the edges in $L[Z]$. We name $[E_1(L[Z]) = (2,2)]$ as type 1, $[E_2(Z) = (2,3)]$ as type 2 and $[E_3(L[Z]) = (3,3)]$ as type 3. By calculating these edge types, we get that $(k-2)e$, $54bc$ and $54bc$ are counts of the edges of type 1, type 2 and type 3 respectively. □

In the following theorem, we find the M-polynomial of lien graphs of k-subdivded grahp of $2D$ graph lattice of $Z[b,c]$. **Theorem 8:**

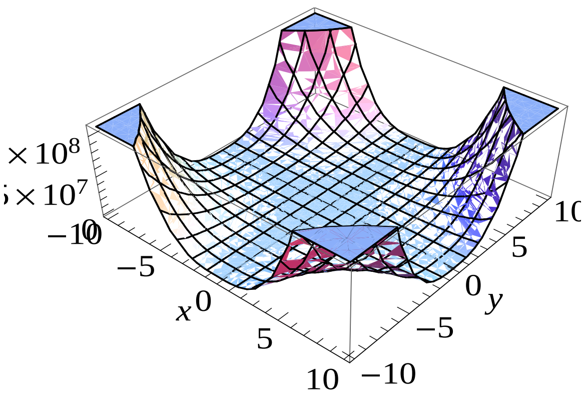

Figure 2.12: 3D Graph of 2-subdivided M-Polynomial of Z

The M-polynomial of $G_{L[Z]}(k)$ is as follows:

$M(L[Z] = L\left[Z[b,c];x,y\right] = \left((k-2)e\right)x^2y^2 + 54bcx^2y^3 + 54bcx^3y^3$

Proof:

Suppose the lien graphs of k-subdivded grahps lattice of $Z[b,c]$ then by using Lemma 8, we have three edge types as : E_1(L[Z])=[e=mn∈E(L[Z]): d_m=d_n=2],

E_2(L[Z])=[e=pq∈E(L[Z]): d_p=2, d_q=3], E_3(L[Z])=[e=rs∈E(L[Z]): $d_r = 3$, $d_s = 3$] with $|E_1(L[Z])|$=$(k-2)e$, $|E_2(L[Z])|$=54bc, $|E_3(L[Z])|$=54bc.

Now by using definition of M-polynomial, we have:

$$\begin{aligned}
M(L[Z] = L\big[Z[b,c]\big];x,y) &= \sum_{i\leq j} m_{ij}x^iy^j \\
&= \sum_{2\leq 2} m_{22}x^2y^2 + \sum_{2\leq 3} m_{23}x^2y^3 + \sum_{3\leq 3} m_{33}x^3y^3 \\
&= |E_1(L[Z])|x^2y^2 + |E_2(L[Z])|x^2y^3 + |E_3(L[Z])|x^3y^3 \\
&= \big((k-2)e\big)x^2y^2 + 54bcx^2y^3 + 54bcx^3y^3
\end{aligned}$$

□

By using the Table 2.1 we have following:

Let $M(L[Z] = (\big[Z[b,c]\big];x,y)$=z(x, y)=$\big((k-2)e\big)x^2y^2$+$(54bc)x^2y^3$+$(54bc)x^3y^3$

$$\begin{aligned}
D_xz(x,y) &= 2\big((k-2)e\big)x^2y^2 + 2(54bc)x^2y^3 + 3(54bc)x^3y^3 \\
D_xz(x,y) &= \big(2(k-2)e\big)x^2y^2 + (108bc)x^2y^3 + (162bc)x^3y^3 \\
D_yz(x,y) &= 2\big((k-2)e\big)x^2y^2 + 3(54bc)x^2y^3 + 3(54bc)x^3y^3 \\
D_yz(x,y) &= \big(2(k-2)e\big)x^2y^2 + (162bc)x^2y^3 + (162bc)x^3y^3 \\
D_yD_xz(x,y) &= 4\big((k-2)e\big)x^2y^2 + 6(54bc)x^2y^3 + 9(54bc)x^3y^3 \\
D_yD_xz(x,y) &= \big(4(k-2)e\big)x^2y^2 + (324bc)x^2y^3 + (486bc)x^3y^3 \\
S_y(z(x,y)) &= \frac{1}{2}\big((k-2)e\big)x^2y^2 + \frac{1}{3}(54bc)x^2y^3 + \frac{1}{3}(54bc)x^3y^3 \\
S_xS_y(z(x,y)) &= \frac{1}{4}\big((k-2)e\big)x^2y^2 + \frac{1}{6}(54bc)x^2y^3 + \frac{1}{9}(54bc)x^3y^3 \\
S_xS_y(z(x,y)) &= (\frac{(k-2)e}{4})x^2y^2 + (9bc)x^2y^3 + (6bc)x^3y^3 \\
D_y^m(z(x,y)) &= 2^m\big((k-2)e\big)x^2y^2 + 3^m(54bc)x^2y^3 + 3^m(54bc)x^3y^3 \\
D_xD_y^m(z(x,y)) &= 2.2^m\big((k-2)e\big)x^2y^2 + 2.3^m(54bc)x^2y^3 + 3.3^m(54bc)x^3y^3
\end{aligned}$$

$$
\begin{aligned}
D_x^m D_y^m(z(x,y)) &= 2^m.2^m\big((k-2)e\big)x^2y^2+2^m.3^m(54bc)x^2y^3+3^m.3^m(54bc)x^3y^3\\
D_x^m D_y^m(z(x,y)) &= 2^{2m}\big((k-2)e\big)x^2y^2+6^m(54bc)x^2y^3+9^m(54bc)x^3y^3\\
S_y^m(z(x,y)) &= \frac{1}{2^m}\big((k-2)e\big)x^2y^2+\frac{1}{3^m}(54bc)x^2y^3+\frac{1}{3^m}(54bc)x^3y^3\\
S_xS_y^m(z(x,y)) &= \frac{1}{2.2^m}\big((k-2)e\big)x^2y^2+\frac{1}{2.3^m}(54bc)x^2y^3+\frac{1}{3.3^m}(54bc)x^3y^3\\
S_x^mS_y^m(z(x,y)) &= \frac{1}{2^m.2^m}\big((k-2)e\big)x^2y^2+\frac{1}{2^m.3^m}(54bc)x^2y^3+\frac{1}{3^m.3^m}(54bc)x^3y^3\\
S_x^mS_y^m(z(x,y)) &= \frac{1}{2^{2m}}\big((k-2)e\big)x^2y^2+\frac{1}{6^m}(54bc)x^2y^3+\frac{1}{9^m}(54bc)x^3y^3\\
S_x^mS_y^m(z(x,y)) &= \frac{1}{4^m}\big((k-2)e\big)x^2y^2+\frac{1}{6^m}(54bc)x^2y^3+\frac{1}{9^m}(54bc)x^3y^3\\
S_yD_x(z(x,y)) &= \big((k-2)e\big)x^2y^3+\frac{2}{3}(54bc)x^2y^3+(54bc)x^3y^3\\
S_xD_y(z(x,y)) &= \big((k-2)e\big)x^2y^2+\frac{3}{2}(54bc)x^2y^3+(54bc)x^3y^3\\
Jz(x,y) &= \big((k-2)e\big)x^4+(54bc)x^5+(54bc)x^6\\
S_xJz(x,y) &= \frac{1}{4}\big((k-2)e\big)x^4+\frac{1}{5}(54bc)x^5+\frac{1}{6}(54bc)x^6\\
JD_xD_yz(x,y) &= 4\big((k-2)e\big)x^4+6(54bc)x^5+9(54bc)x^6\\
S_xJD_xD_yz(x,y) &= \big((k-2)e\big)x^4+\frac{6}{5}(54bc)x^5+\frac{3}{2}(54bc)x^6\\
D_y^3z(x,y) &= (2^3)\big((k-2)e\big)x^2y^2+(3^3)\big(54bc\big)x^2y^3+(3^3)\big(54bc\big)x^3y^3\\
D_y^3z(x,y) &= 8\big((k-2)e\big)x^2y^2+27(54bc)x^2y^3\\
&+ 27(54bc)x^3y^3\\
D_x^3D_y^3z(x,y) &= 2^{2.3}\big((k-2)e\big)x^2y^2+(6^3)(54bc)x^2y^3+(9^3)(54bc)x^3y^3\\
D_x^3D_y^3z(x,y) &= 2^6\big((k-2)e\big)x^2y^2+(6^3)(54bc)x^2y^3+(9^3)(54bc)x^3y^3\\
D_x^3D_y^3z(x,y) &= 64\big((k-2)e\big)x^4+216\big(54bc\big)x^5+729\big(54bc\big)x^6\\
JD_x^3D_y^3z(x,y) &= 64\big((k-2)e\big)x^{4-2}+216\big(54bc\big)x^{5-2}+729\big(54bc\big)x^{6-2}\\
Q_{-}2JD_x^3D_y^3z(x,y) &= 64\big((k-2)e\big)x^{4-2}+216\big(54bc\big)x^{5-2}+729\big(54bc\big)x^{6-2}\\
Q_{-}2JD_x^3D_y^3z(x,y) &= 64\big((k-2)e\big)x^2+216\big(54bc\big)x^3+729\big(54bc\big)x^4\\
S_xQ_{-}2JD_x^3D_y^3z(x,y) &= \frac{64}{2}\big((k-2)e\big)x^2+\frac{216}{3}(54bc)x^3+\frac{729}{4}\big(54bc\big)x^4\\
S_x^mQ_{-}2JD_x^3D_y^3z(x,y) &= \frac{64}{2^m}\big((k-2)e\big)x^2+\frac{216}{3^m}(54bc)x^3+\frac{729}{4^m}\big(54bc\big)x^4
\end{aligned}
$$

$$S_x^3 Q_{-2} J D_x^3 D_y^3 z(x,y) = \frac{64}{2^3}\big((k-2)e\big)x^2 + \frac{216}{3^3}(54bc)x^3 + \frac{729}{4^3}\big(54bc\big)x^4$$
$$S_x^3 Q_{-2} J D_x^3 D_y^3 z(x,y) = \frac{64}{8}\big((k-2)e\big)x^2 + \frac{216}{27}(54bc)x^3 + \frac{729}{64}\big(54bc\big)x^4$$
$$S_x^3 Q_{-2} J D_x^3 D_y^3 z(x,y) = 8\big((k-2)e\big)x^2 + 8(54bc)x^3 + \frac{729}{64}\big(54bc\big)x^4$$

Degree Based Topological Indices of Line Graph of k-subdivided graph of Z

"First Zagreb:

$$(D_x + D_y)[M(L[Z]^{\beta}_{\alpha}, x, y)]_{y=x=1}\text{''} = 2(k-2)e + 270bc + 2(K-2)e + 324bc$$

Second Zagreb:

$$(D_x D_y)[M(L[Z]^{\beta}_{\alpha}, x, y)]_{y=x=1} = 4(k-2)e + 810bc$$

"Second Modified Zagreb:

$$(S_x S_y)[M(L[Z]^{\beta}_{\alpha}, x, y)]_{y=x=1}\text{''} = \frac{k-2}{4}e + 15bc$$

General Randic (GR):

$$(D_x^m D_y^m)[M(L[Z]^{\beta}_{\alpha}, x, y)]_{y=x=1} = 4^m(k-2)e + 6^m(54bc) + 9^m(54bc)$$

General Randic (GR):

$$(S_x^m S_y^m)[M(L[Z]^{\beta}_{\alpha}, x, y)]_{y=x=1} = \frac{1}{4^m} \times \big((K-2)e\big) + \frac{1}{6^m} \times (54bc) + \frac{1}{9^m} \times (54bc)$$

"Symmetric Division Index (SDI):

$$(D_x S_y + D_y S_x)[M(L[Z]^{\beta}_{\alpha}, x, y)]_{y=x=1}\text{''} = 2(k-2)e + 225bc$$

Harmonic Index (HI):

$$2S_xJ[M(L[[Z]_{\alpha}^{\beta},x,y)]_{y=x=1} \;=\; \frac{(k-2)e}{2}+\frac{198bc}{5}$$

Inverse Sum Index (ISI):

$$S_xJD_xD_y[M(L[Z]_{\alpha}^{\beta},x,y)]_{y=x=1} \;=\; (k-2)e+\frac{729bc}{5}$$

Augmented Zagreb Index (AZI):

$$S_x^3Q_{-2}JD_x^3D_y^3[M(L[Z]_{\alpha}^{\beta},x,y)]_{y=x=1} \;=\; 8(k-2)e+\frac{67014bc}{64}$$

□

2.5 M-Polynomial For Graph $(G+H)$

Theorem 2.5.1. *Let m_1, m_2 be the orders and w_1, w_2 be the sizes of finite graphs G and H respectively. "Then M-polynomial of $G+H$" is*

$$\begin{aligned} M(G+H;\alpha,\beta) &= \sum_{a=1}^{w_1}\alpha_a\alpha^{dx_a+m_2}\beta^{dy_a+m_2}+\sum_{b=1}^{w_2}\beta_b\alpha^{dx_b+m_1}\beta^{dy_b+m_1} \\ &+ \sum_{a^*=1}^{m_1}\sum_{b^*=1}^{m_2}\alpha^{dx_{a^*}+m_2}\beta^{dy_{b^*}+m_1}. \end{aligned}$$

Proof.

Let graph G has w_1 type of edges $g_a=(dx_a,dy_a)$, with respect to degrees having count α_a and graph H has w_2 type of edges $h_a=(dx_b,dy_b)$, with respect to degrees having count β_b. "Then in $G+H$, the edges g_a becomes" (dx_a+m_2,dy_a+m_2), $1\le a\le w_1$ and edges h_a becomes (dx_b+m_1,dy_b+m_1) where $1\le b\le w_2$. Further, $(dx_{a^*}+m_2,dy_{b^*}+m_1)$, is a third degree based edge in $G+H$, where $1\le a^*\le m_1$ and $1\le b^*\le m_2$.

”The M-polynomial of $G+H$, is calculated as follows”:

$$M(G+H;\alpha,\beta)=\sum_{xy\in E(G)} m_{ij}\alpha^{a}\beta^{b}, \tag{2.5.1}$$

now,

$$\begin{aligned} M(G+H;\alpha,\beta) &= \sum_{a=1}^{w_1}\alpha_a\alpha^{dx_a+m_2}\beta^{dy_a+m_2}+\sum_{b=1}^{w_2}\beta_b\alpha^{dx_b+m_1}\beta^{dy_b+m_1} \\ &+ \sum_{a^*=1}^{m_1}\sum_{b^*=1}^{m_2}\alpha^{dx_{a^*}+m_2}\beta^{dy_{b^*}+m_1}. \end{aligned}$$

$$\begin{aligned} M(G+H;\alpha,\beta) &= \alpha^{m_2}\beta^{m_2}\sum_{a=1}^{w_1}\alpha_a\alpha^{dx_a}\beta^{dy_a}+\alpha^{m_1}\beta^{m_1}\sum_{b=1}^{w_2}\beta_b\alpha^{dx_b}\beta^{dy_b} \\ &+ \alpha^{m_2}\beta^{m_1}\sum_{a^*=1}^{m_1}\sum_{b^*=1}^{m_2}\alpha^{dx_{a^*}}\beta^{dy_{b^*}}. \end{aligned}$$

2.6 M-Polynomials For $G(k)$

Lemma 2.6.1. *Let $G(p,q)$ be a graph with degree sequence $d_1\leq d_2\leq d_3\leq .....\leq d_p$ and (d_{ui},d_{vi}) type of edge according to its degree with count N_i then graph $G(k)$ has following type of edges according to degree,*

$(2,d_{ui})$	$(2,2)$	$(2,d_{vi})$
N_i	$(k-1)q$	N_i

where; $1\leq i\leq z$

Proof:

Since the edge (d_{ui},d_{vi}) becomes path having degree sequence $d_{ui},2,2,2,....,2,2,2,d_{vi}$. Therefore we will have three types of edges $(d_{ui},2),(2,2)$ and $(2,d_{vi})$ in $G(k)$ with count N_i, $(k-1)q$ and N_i respectively.

$$\begin{aligned}
M\big(G(k);x,y\big) &= \sum_{i\leq j} m_{ij}x^i y^j \\
&= \sum_{d_{u_i}\leq 2} m_{d_{u_i}2}x^{d_{u_i}}y^2 + \sum_{2\leq 2} m_{22}x^2y^2 + \sum_{2\leq d_{v_i}} m_{2d_{v_i}}x^2y^{d_{v_i}} \\
&= N_1x^{d_{u_1}}y^2 + (k-1)qx^2y^2 + N_1x^2y^{d_{v_1}} + N_2x^{d_{u_2}}y^2 + N_2x^2y^{d_{v_2}} \\
&+ N_3x^{d_{u_3}}y^2 + N_3x^2y^{d_{v_3}} + \ldots.. + N_zx^{d_{u_z}}y^2 + N_zx^2y^{d_{v_z}} \\
&= y^2(N_1x^{d_{u_1}} + N_2x^{d_{u_2}} + N_3x^{d_{u_3}} + \ldots + N_zx^{d_{u_z}}) + (k-1)qx^2y^2 \\
&+ x^2(N_1y^{d_{v_1}} + N_2y^{d_{v_2}} + N_3y^{d_{v_3}} + \ldots + N_zy^{d_{v_z}}) \\
&= y^2\sum_{i=1}^{z} N_ix^{d_{u_i}} + (k-1)qx^2y^2 + x^2\sum_{i=1}^{z} N_iy^{d_{v_i}}
\end{aligned}$$

$\square$

2.7 M-Polynomials For Line Graph of $G(k)$

Lemma 2.7.1. *Let $G(p,q)$ be a graph with degree sequence $d_1 \leq d_2 \leq d_3 \leq \ldots.. \leq d_p$ and (d_{ui}, d_{vi}) type of edge according to its degree with count N_i then the line graph of graph $G(k)$ has following type of edges according to degree,*

$(2, d_{ui})$	$(2,2)$	$(2, d_{vi})$	(d_{ui}, d_{ui})
N_i	$(k-2)q$	N_i	$\frac{d_{ui}(d_{ui}-1)}{2}$

where; $1 \leq i \leq z$

Proof:

Since the edge (d_{ui}, d_{vi}) becomes path having degree sequence $d_{ui}, 2, 2, 2, \ldots., 2, 2, 2, d_{vi}$. Therefore we will have these types of edges $(d_{ui}, 2), (2,2), (2, d_{vi})$ and (d_{ui}, d_{ui}) in $G(k)$

with count N_i, $(k-2)q$, N_i and $\frac{d_{ui}(d_{ui}-1)}{2}$ respectively.

$$
\begin{aligned}
M\big(G(k);x,y\big) &= \sum_{i\leq j} m_{ij}x^i y^j \\
&= \sum_{d_{u_i}\leq 2} m_{d_{u_i}2}x^{d_{u_i}}y^2 + \sum_{2\leq 2} m_{22}x^2y^2 + \sum_{2\leq d_{v_i}} m_{2d_{v_i}}x^2y^{d_{v_i}} \\
&+ \sum_{d_{u_i}\leq d_{u_i}} m_{d_{u_i}d_{u_i}}x^{d_{u_i}}y^{d_{u_i}} \\
&= N_1x^{d_{u_1}}y^2 + (k-2)qx^2y^2 + N_1x^2y^{d_{v_1}} + N_2x^{d_{u_2}}y^2 + N_2x^2y^{d_{v_2}} \\
&+ N_3x^{d_{u_3}}y^2 + N_3x^2y^{d_{v_3}} + \ldots\ldots + N_zx^{d_{u_z}}y^2 + N_zx^2y^{d_{v_z}} \\
&+ \frac{d_{ui}(d_{ui}-1)}{2}x^{d_{ui}}y^{d_{ui}} \\
&= y^2(N_1x^{d_{u_1}} + N_2x^{d_{u_2}} + N_3x^{d_{u_3}} + \ldots + N_zx^{d_{u_z}}) + (k-2)qx^2y^2 \\
&+ x^2(N_1y^{d_{v_1}} + N_2y^{d_{v_2}} + N_3y^{d_{v_3}} + \ldots + N_zy^{d_{v_z}}) + \frac{d_{ui}(d_{ui}-1)}{2}x^{d_{ui}}y^{d_{ui}} \\
&= y^2\sum_{i=1}^{z} N_ix^{d_{u_i}} + (k-2)qx^2y^2 + x^2\sum_{i=1}^{z} N_iy^{d_{v_i}} + \frac{d_{ui}(d_{ui}-1)}{2}x^{d_{ui}}y^{d_{ui}}
\end{aligned}
$$

$\square$

Chapter 3

Conclusion

3.1 Conclusion

In this proposal we have found M-Polynomials for different families of chemical graphs.We found M-polynomials for k-subdivided and line graphs of k-subdivided of nanotubes. We have also found the M-Polynomials for the new families of graphs obtained by using some graph operations. We focussed on M-Polynomials of $G+H, G(k), L[G(k)]$, where G,H are simple connected graphs and $L[G(k)]$ is line graph of uniformly k-subdivided graph of G, for k 1. We have found M-Polynomials for different families of chemical graphs.

Chapter 4

References

Bibliography

[1] M. S. Ahmad, W. Nazeer, S.M. Kang, C. Y. Jung, M-Polynomial and Degree Based Topological Indices for the Line Graph of Firecracker Graph, *Global journal of pure and applied mathematics.,* 13 (2017), 0973-1768.

[2] M. Ajmal, W. Nazeer, S. M. Kang, C. Y. Jung, The M-Polynomial and Topological Indices of Generalized Prism Network, *International journal of mathematical analysis.,* (March 2017).

[3] M. Ajmal, W. Nazeer, M. Munir, S. M. Kang, C. Y. Jung, The M-Polynomials and Topological Indices of Toroidal Polyhex Network, *International journal of mathematical analysis.,* 11 (2017) 305-315.

[4] A. Ali, W. Nazeer, et al, M-Polynomials and Topological Indices of Zigzag and Rhombic Benzenoid Systems, 16 (2017), 73-78.

[5] B. Bollobas and P. Erdos, Graphs of Extremal Weights, *Ars Combin.,* 50 (1998), 225233.

[6] E. Deutsch and S. Klavzar, M-Polynomial and Degree-Based Topological Indices, *Iran. J. Math. Chem.,* 6 (2015), 93-102.

[7] A. A. Dobrynin, R. Entringer, I. Gutman, Wiener Index of Trees: Theory and Applications, *Acta Appl. Math.,* 66 (2001), 211-249.

[8] S. Fajtlowicz, On Conjectures of Graffiti-II, *Congr. Numer.,* 60 (1987), 187-197.

[9] B. Furtula, A. Graovac, D. Vukicevic, Augmented Zagreb index, *J. Math. Chem.,* 48 (2010) 370380.

[10] W. Gao, M. Younas, A. Farooq, A. Mahboob, W. Nazeer, M-Polynomials and Degree-Based Topological Indices of the Crystallographic Structure of Molecules, *Biomolecules.,* 107 (October 3, 2018) 1-17.

[11] I. Gutman, N. Trinajstic, Graph Theory and Molecular orbitalsm. Total electron energy of alternant hydrocarbons, *Chem. Phys. Lett.,* 17 (1972) 535-538.

[12] M. Imran, A. Q. Baig, H. Ali, On topological properties of dominating David derived networks, *Canadian Journal of Chemistry.,* (2015) vol.94, 137-148.

[13] M. Javaid, C. Y. Jung, M-polynomials and topological indices of silicate and oxide networks, *Internal journal of pure and applied mathematics.,* 115 (2017), 129-152.

[14] S. M. Kang, W. Nazeer, W. Gao, D. Afzal, M-polynomials and topological indices of dominating David derived networks, *open chemistry.,* 16 (March 2018), 201-213.

[15] S. M. Kang, W. Nazeer, M. A. Zahid, A. R. Nizami, M-polynomials and the topological indices of hex-derived networks, 16 (July 2018), 394-403.

[16] Y. C. Kwun, A. Ali, W. Nazeer, M. A. Chaudhary, S. M. Kang, M-Polynomials and Degree-Based Topological Indices of Triangular, Hourglass, and Jagged-Rectangle Benzenoid Systems, *Journal of Chemistry.,* (September 12,2018), 1-8.

[17] Y. C. Kwun, M. Munir, W. Nazeer, S. Rafique, S. M. Kang, M-Polynomials and topological indices of V-Phenylenic Nanotubes and Nanotori, *Scientific Reports.,* 8756 (August 18,2017) 1-9.

[18] Y. C. Kwun, A. u. R. Virk, W. Nazeer, S. M. Kang, M-Polynomials and Degree-Based Topological Indices of Bismuth Tri-Iodide, *Preprints.,* (2018).

[19] J. B. Liu, M. Younas, M. Habib, M. Yousas, W. Nazeer, M-Polynomials and Degree-Based Topological Indices of VC5C7[p,q] and HC5C7[p,q] Nanotubes, *IEEE Xplore Digital Library.,* 7 (March 27, 2019) 41125 - 41132.

[20] A. Milicevic, S. Nikolic, N. Trinajstic, On reformulated Zagreb indices, *Mol. Divers.,* 8.

[21] M. Munir, W. Nazeer, A. R. Nizami, S. Rafique, S. M. Kang, M-Polynomials and Topological Indices of Titania Nanotubes, *Symmetry.,* 117 (October 2016) 1-9.

[22] M. Munir, W. Nazeer, S. Rafique, S. M. Kang, M-Polynomial and Related Topological Indices of Nanostar Dendrimers, *Symmetry.,* 97 (September 21,2016) 1-12.

[23] M. Munir, W. Nazeer, S. Rafique, S. M. Kang, M-polynomial and Degree-Based Topological Indices of Polyhex Nanotubes, *Symmetry.,* 149 (December 6,2016).

[24] A. R. Nizami, M. Idrees, The M-Polynomial and Topological Indices of Generalized Mobius Ladder and its Line Graph, *Math.co.,* (2017).

[25] M. Randic, On Characterization of Molecular Branching, *J. Am. Chem. Soc.,* 97 (1975) 6609-6615.

[26] M. Rezaei, W. Gao, M. K. Siddiqui, M. R. Farahani, Computing Hyper Zagreb Index and M-Polynomials of Titania Nanotubes $TiO_2[m,n]$, *Sigma J Eng And Nat Sci.,* 35 (2017), 707-714.

[27] M. Riaz, W. Gao, A. Q. Baig, M-Polynomials and Degree Based Topological Indices of some Families of Convex Polytopes, *Open Journal of Math. Sci.,* 2 (2018), 18-28.

[28] N. Soleimani, M. J. Nikmehr, H. A. Tavallaee, Computation of the Different Topological Indices of Nanostructures, *J. Natn. Sci. Foundation.,* 43 (2015), 127-133.

[29] D. Vukicevic, Bond Additive Modeling 2 Mathematical Properties of max-min Rodeg Index, *Croatica Chemica. acta.,* 54 (2010) 261-273.

[30] D. Vukicevic, M. Gasperov, Bond Additive Modeling 1. Adriatic Indices, *Croat.Chem. Acta.,* 83 (2010) 243-260.