# Hybrid Projection Algorithm for Generalized Split Equilibrium Problem in Hibert Spaces

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A Post Graduate Thesis submitted to the Department of Mathematics as partial fulfillment of the requirement for the award of Degree of M.S Mathematics.


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# Declaration

I Hammad Sarwar **CIIT/SP17-RMT-008/LHR** hereby declare that I have produced the work presented in this thesis, during the scheduled period of study. I also declare that I have not taken any material from any source except referred to wherever due, that amount of plagiarism is within acceptable range. If a violation of HEC rules on research has occurred in this thesis, I shall be liable to punishable action under the plagiarism rules of the HEC.

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It is certified that Hammad Sarwar, CIIT/SP17-RMT-008/LHR has carried out all the work related to this thesis under my supervision at the Department of Mathematics, COMSATS University Islamabad, Lahore campus and the work fulfills the requirement for award of MS degree.

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# DEDICATION

*To*

My Parents

# ACKNOWLEDGEMENTS

In the name of Allah, the most gracious and the most merciful.
The Prophet (S.A.W) said: “Seeking knowledge is obligatory for every Muslim”.
First, I recognize the creator and finisher of my faith and in memorial of my life, I say thank ALLAH Almighty and Prophet Muhammad (S.A.W) for everything. I am grateful to Almighty ALLAH, without Whose endowments it was difficult to finish this proposal.
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# ABSTRACT

In this thesis we suggest and examine Extra Gradient-Proximal Point algorithm to find the solution of Split Equilibrium Problem and Fixed Point Problem for nonexpansive semigroup in real Hilbert spaces. Our purpose is to address fixed point problems(FPP), variational inequality problems (VIP) and generalized Split Equilibrium Problems(GSEP) in the setting of Hilbert spaces. We propose a hybrid shrinking projection algorithm, which exhibits strong convergence, for the construction of a common element in the set of common fixed points of an infinite family of k-strict pseudo-contractions and the set of common solutions of a finite family of Generalized Split Equilibrium Problems(GSEP) in Hilbert spaces. Our results generalize and improve various existing results in the current literature.

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# Chapter 1

# Introduction and Basic Definitions

In this part we focused on introduction of fixed point theory (FPT), split equilibrium theory (SET) and shrinking projection method in area of Hilbert spaces. Class of Hilbert spaces has an important geometrical structures which is important for the analysis of problems related to metric fixed point theory (MFPT) and equilibrium problem theory (EPT). In the continuation, distinctive central thoughts of FPT and EPT are described in such settings.

## 1.1 Fixed Point Theory

Most of the phenomena happening in daily life are commonly nonlinear in nature. Subsequently, mathematicians and researchers are always in mission for finding new techniques to deal with a nonlinear problems. The most effective techniques to manage nonlinear problem is to frame it as an equivalent FPP. The FPT is one of the most amazing system of latest mathematical sciences. It join topology ,analysis and geometry together. Specifically, it has been associated in such areas as economics, arithmetic, biology, game theory, physics and chemistry etc.

"Fixed point theory(FPT) is a branch of nonlinear analysis through which the fixed points of any function $\mathfrak{f}$ can approximated. For example,

$$\text{if for any point } \mathfrak{g} \in \mathsf{C} \text{ such that } \mathfrak{f}(\mathfrak{g}) = \mathfrak{g}$$

where $\mathsf{C} \neq \emptyset$ is the domain set and $\mathfrak{f}$ is a self nonlinear map . "

Fixed Point Theory(FPT) has proved itself a helpful apparatus for the examination of various nonlinear real world problems. FPT has been extended broadly with the progression of scientific procedures to build the fixed points. The notion $\mathfrak{f}(\mathfrak{g}) = \mathfrak{g}$ can be simplified into $(\mathbb{I} - \mathfrak{f})\mathfrak{g} = 0$ which relates the FPP to corresponding problems in theories of integral equation, nonlinear differential equations and rising in various fields of study, for example, accounting, game theory, biology, physics, designing, chemistry, software engineering and arithmetic related fields.

Generally, FPT is designed for the investigation of the presence or understanding of related fixed points for the nonlinear maps under consideration. The presence and/or interpretation of related fixed points for the nonlinear maps can be used to tackle issues in different parts of mathematical sciences. We remark that the iterative calculation of fixed points has inalienable significance in different branches of applied mathematics and appropriate for investigation of fixed points in such cases where the derivation of constrains is important and it is hard to extract those preconditions for existence.

Fixed Point Theory (FPT) is classified in following parts
$(i)$Discrete FPT
$(ii)$ Topological FPT
$(iii)$ Metric FPT

One of the fundamental inspiration for the study of MFPT is to analyze the existence as well as uniqueness of the solution related with the differential equations. Famous mathematicians of nineteenth century to be specific, Liouville, Picard, Lipschitz, Peano and especially, Cauchy contributed fundamentally to the theory of differential equations with the help of MFPT. MFPT accepts to be started from the well-known Banach Constraction Principle (BCP) basically due to Stephan Banach [1].

**Theorem 1.1.1** (BCP). *Consider a metric space $\mathcal{C}$ which is complete and a selfmap $\mathfrak{f} : \mathcal{C} \longrightarrow \mathcal{C}$ satisfying*

$$d(\mathfrak{f}\mathfrak{g}, \mathfrak{f}q) \leq e[d(\mathfrak{g}, q)], \quad for\ all \quad \mathfrak{g}, q \in \mathcal{C},$$

*while $1 > e > 0$. Then $\exists$ a unique fixed point $'\ p\ '$ $for$ $'\mathfrak{f}'$ s.t, sequence $\mathfrak{g}^n = \mathfrak{f}(\mathfrak{g}^{n-1})$ converges to that unique fixed point $'p'$.*

BCP has critical function in estimation of nonlinear equations. BCP is a bond for the presence of fixed point and in addition the uniqueness which can be evaluated by using progressive estimation.

In 1965,the nonexpansive maps theory was started self-rulingly by F. E. Browder [2] , W. A. Kirk [35] and D. Göhde [28]. In nonlinear functional analysis the MFPT is considerational and an operational field of research. The theory of nonexpansive maps has vital applications. Similarly, we can find the solutions of convex minimization problem, monotone operators, and variational inequality problem (VIP) of a functions using nonexpansive maps. For more discussion of the FPT of nonexpansive maps, see [27, 36, 37].

## 1.2 Some Definitions

This part of thesis is concentrated to describe some central thoughts and ideas related to MFPT required in the continuation:

**Definition 1.2.1.** *"Suppose an arbitrary space $H$ and an arbitrary sequence $\{\mathfrak{g}^n\} \in H$. if $\{\mathfrak{g}^n\} \in H$ is cauchy then the space $H$ is known as a complete space. The cauchy criterion is defined as, given $\epsilon_{m,n} > 0$ we have $M$ such that if $m, n > M$ then:*

$$|\mathfrak{g}^m - \mathfrak{g}^n| < \epsilon_{m,n} \,; \mathfrak{g}^m, \mathfrak{g}^n \in H. \tag{1.1}$$

*Any arbitrary sequence is said to be Cauchy if the terms of the sequence in the long run, all turn out to be relatively near. "*

Suppose 'H' be a Hilbert space (real) having $\emptyset \neq \mathcal{C} \subset \mathsf{H}$ a closed as well as convex subset. The set containing fixed points is demonstrated as $\mathbb{F}(\mathfrak{f})$ : $\{\mathfrak{g} \in \mathcal{C} : \mathfrak{f}\mathfrak{g} = \mathfrak{g}\}$ under the action of a map $\mathfrak{f}$.

We now characterize a few maps defined below:

Suppose $\emptyset \neq \mathcal{C} \subset \mathsf{H}$ be a convex as well as closed subset. Suppose $\mathfrak{g}, q \in \mathcal{C}$. and $\mathfrak{f} : \mathcal{C} \to \mathcal{C}$ be a map:

**Definition 1.2.2** ([1]). *"A map* $\mathfrak{f} : \mathcal{C} \to \mathcal{C}$ *such that:*

$$\|\mathfrak{f}\mathfrak{g} - \mathfrak{f}q\| \leq \beta \|\mathfrak{g} - q\| \ ; \ \beta \in (0,1) \tag{1.2}$$

*is called contraction ."*

**Definition 1.2.3** ([2, 28, 35]). *"A map* $\mathfrak{f} : \mathcal{C} \to \mathcal{C}$ *such that:*

$$\|\mathfrak{f}\mathfrak{g} - \mathfrak{f}q\| \leq \beta \|\mathfrak{g} - q\| \ ; \ \beta \in (0,1)$$

*if* $\beta = 1$.

$$\|\mathfrak{f}\mathfrak{g} - \mathfrak{f}q\| \leq \|\mathfrak{g} - q\| \tag{1.3}$$

*is called nonexpansive map ."*

Nonexpansive map gets a fixed point for a subset which is also convex of a Banach space (uniformly convex) which is closed as well as bounded. This result was unveild by F.E.Browder [2] and D.Göhde [28] in 1965, autonomously. As a consequence this conclusion was showed by W.A.Kirk [35] with marginally weaker conditions. In 1969 K.Goebel [26] gave another verification which is more elementary geometric in nature.

Let for $\mathfrak{g}, q \in \mathcal{C}$ we have $\mathfrak{f} : \mathcal{C} \to \mathcal{C}$ a map then:

**Definition 1.2.4.** *"A map* $\mathfrak{f} : \mathcal{C} \to \mathcal{C}$ *for which we have*

$$\|\mathfrak{f}\mathfrak{g} - \mathfrak{f}q\| \leq \beta \|\mathfrak{g} - q\| \textit{for some } \beta > 0 \tag{1.4}$$

*is known as Lipschitzian. "*

**Definition 1.2.5.** *"A map* $\mathfrak{f} : \mathcal{C} \to \mathcal{C}$ *is known as firmly nonexpansive if*

$$\|\mathfrak{f}\mathfrak{g} - \mathfrak{f}q\|^2 + \|(\mathbb{I} - \mathfrak{f})\mathfrak{g} - (\mathbb{I} - \mathfrak{f})q\|^2 \leq \|\mathfrak{g} - q\|^2 \tag{1.5}$$

*where* $\mathbb{I}$ *is an identity map."*

**Definition 1.2.6** ([3]). *"A map $\mathfrak{f} : \mathcal{C} \to \mathcal{C}$ for which we have*

$$\|\mathfrak{f}\mathfrak{g} - \mathfrak{f}q\|^2 \leq \|\mathfrak{g} - q\|^2 + \|(\mathbb{I} - \mathfrak{f})\mathfrak{g} - (\mathbb{I} - \mathfrak{f})q\|^2 \tag{1.6}$$

*is known as pseudo-contraction. "*

**Definition 1.2.7** ([3]). *"A map $\mathfrak{f} : \mathcal{C} \to \mathcal{C}$ if for any $m \in [0, 1)$ we have*

$$\|\mathfrak{f}\mathfrak{g} - \mathfrak{f}q\|^2 \leq \|\mathfrak{g} - q\|^2 + m\|(\mathbb{I} - \mathfrak{f})\mathfrak{g} - (\mathbb{I} - \mathfrak{f})q\|^2 \tag{1.7}$$

*is called strict pseudo contraction."*

## 1.3 Equilibrium Problem Theory

In this section, we summerize some important ideas and thoughts of Equilibrium Problem Theory which is required in the continuation:

**Definition 1.3.1.** *"If in any vector space the domain is $D(\mathfrak{f})$ and $R(\mathfrak{f})$ is the range and they are subset of a vector space over the same field, provided that for a scalar $\lambda$ and $\forall$ $\mathfrak{g}, q \in D(\mathfrak{f})$, we have an operator $\mathfrak{f}$*

$$\begin{aligned} \mathfrak{f}(\mathfrak{g} + q) &= \mathfrak{f}\mathfrak{g} + \mathfrak{f}q \\ \mathfrak{f}(\lambda\mathfrak{g}) &= \lambda\mathfrak{f}\mathfrak{g} \end{aligned}$$

*then an operator $\mathfrak{f}$ is called linear operator."*

**Definition 1.3.2.** *"Let any arbitrary space $\mathcal{C}$ and a linear operator $\mathfrak{f}$ defined as*

$$\mathfrak{f} : D(\mathfrak{f}) \longrightarrow \mathcal{C}$$

*here $D(\mathfrak{f}) \subset \mathcal{C}$. If $\mathcal{L}$ is a real number such that $\forall$ $\mathfrak{g} \in D(\mathfrak{f})$.*

$$\|(\mathfrak{f})\mathfrak{g}\| \leq \mathcal{L}\,\|\mathfrak{g}\| \tag{1.8}$$

*Then the operator $\mathfrak{f}$ is known as BLO."*

**Definition 1.3.3.** *"A Hilbert space $H$ and a closed as well as convex subset $\mathcal{C} \neq \emptyset$. Let the real numbers set $\mathcal{R}$ and a functional F defined as $F : \mathcal{C} \times \mathcal{C} \rightarrow \mathcal{R}$. The equilibrium problem (EP) is the evaluation of $\mathfrak{g}^* \in \mathcal{C}$ s.t:*

$$F(\mathfrak{g}^*, \mathfrak{g}) \geq 0, \ \forall \ \mathfrak{g} \in \mathcal{C}. \tag{1.9}$$

*The solutions set of EP is expressed as $Sol(\mathcal{C}, F)$). "*

**Definition 1.3.4** ([43]). *"Let the arbitrary sets $\mathcal{C} \neq \emptyset$ $\mathcal{Q} \neq \emptyset$ which is a subset of $H_1$, and $H_2$ respectively. Suppose a BLO $A : H_1 \rightarrow H_2$ be given and two bifunctions F and G defined as*

$$F : \mathcal{C} \times \mathcal{C} \rightarrow \mathcal{R}$$

*and*

$$G : \mathcal{Q} \times \mathcal{Q} \rightarrow \mathcal{R}.$$

*Then the SEP is to find:*

$$\mathfrak{g}^* \in \mathcal{C} \quad \textit{such that } F(\mathfrak{g}^*, \mathfrak{g}) \geq 0, \ \forall \ \mathfrak{g} \in \mathcal{C}. \tag{1.10}$$

*and*

$$q^* = A\mathfrak{g}^* \in \mathcal{Q} \quad \textit{such that } G\,(q^*, q) \geq 0. \tag{1.11}$$

*for all $q \in \mathcal{Q}$. while the solution set of SEP is defined as:*

$$\Omega = \{\mathfrak{g}^* \in Sol(\mathcal{C}, F); A\mathfrak{g}^* \in Sol(\mathcal{Q}, G)\}$$

*Moudafi [43] , in 2011 initiated the SEP ."*

**Remark 1.3.5.** *It can be easily seen that inequality (1.9) represents the classical EP [18] and the solution set is described as $Sol(\mathcal{C}, F)$. Moreover, the inequalities (1.10) and (1.11) consisted of a couple of EPs and are focused to find the solution of (1.10) i.e $\mathfrak{g}^*$ such that its image $q^* = A\mathfrak{g}^*$ also solves further (1.11) under action of a bounded linear operator (BLO).*

EPT characterize an extensive area research in mathematical sciences as various problems in game theory, operational research, financial mathematics, optimization, economics, physics and mechanics can be shown as an equilibrium problem .EP includes variational inequalities problem (VIP), FPP and optimization problems . The evolution of benificial and executable iterative schemes are significant and important in the theory of VIP and EPs. In recent years, some iterative schemes have been investigated to approximate the EP and VIP in Banach spaces and Hilbert spaces. See, for example [4, 9, 22, 25].

## 1.4 Iterative Algorithms for Nonlinear Maps

As the idea of FPT is arithmetic. So one can without much of a stretch characterize the iterative algorithms to find the fixed points of different nonlinear problems. Anybody can evaluate a broad variety of iterative algorithms in the current writing. We currently characterize some iterative designs in the accompanying continuation:

**Definition 1.4.1.** *"Let $\mathfrak{f} : \mathcal{C} \to \mathcal{C}$ be a map and a sequence $\{\mathfrak{g}^n\} \in \mathcal{C}$. Suppose an initial guess is $\mathfrak{g}^0 \in \mathcal{C}$ then:*

$$\mathfrak{g}^{n+1} = \mathfrak{f}^n \mathfrak{g} \tag{1.12}$$

*is called Picard's Iterative scheme."*

This iterative plan is very direct and much of the time actualized for the approximation of desired solution in many problems from various zones of mathematics especially in fixed point theory. In 1955 , Krasnoselskii [38] showed that if $\mathfrak{f}$ is a nonexpansive map and has a unique fixed point then the Picard's iterative scheme is not convergent to the unique fixed point of $\mathfrak{f}$.

**Definition 1.4.2** ([40])**.** *"A map $\mathfrak{f} : \mathcal{C} \to \mathcal{C}$. Then a sequence $\{\mathfrak{g}^{n+1}\}$ defined by :*

$$\mathfrak{g}^{n+1} = \beta^n \mathfrak{g}^n + (1 - \beta^n) \mathfrak{f}\mathfrak{g}^n. \tag{1.13}$$

*is said to be Mann's Iterative scheme."*

Mann's iterative scheme [40] has been broadly used and investigated in documentation for instance observe [44, 7, 16]. In [5], D. Borwein and J. M. Borwein showed that Mann iteration is not convergent for a $\mathcal{L}$-Lipschitz map (not pseudo contractive), and posses a unique fixed point. While Hicks and Kubicek [29], examined another case for the Mann iteration is not convegent under the activity of a discontinuous pseudo constraction having a unique fixed point.

**Definition 1.4.3** ([30]). *"In 1974 Ishikawa introduced this scheme. A map $\mathfrak{f} : \mathcal{C} \to \mathcal{C}$. A sequence $\{z^n\}$ defined by*

$$\begin{aligned} \mathfrak{g}^{n+1} &= (1-\gamma^n)\,\mathfrak{f}\mathfrak{g}^n + \gamma^n z^n \\ z^n &= (1-\gamma^n)\,\mathfrak{f}\mathfrak{g}^n + \gamma^n \mathfrak{g}^n. \end{aligned} \tag{1.14}$$

*is said to be Ishikawa's Iterative scheme."*

**Definition 1.4.4** ([46]). *"In 1991 Schu studied the Mann iteration and starts the modified Mann iterative scheme. A map $\mathfrak{f} : \mathcal{C} \to \mathcal{C}$. A sequence $\{\mathfrak{g}^{n+1}\}$ defined by*

$$\mathfrak{g}^{n+1} = (1-\beta^n)\mathfrak{g}^n + \beta^n \mathfrak{f}^n \mathfrak{g}^n. \tag{1.15}$$

*is said to be Ishikawa's Iterative scheme."*

**Definition 1.4.5.** *"Let $\mathcal{C} \subset H_1$, where $\mathcal{C}$ is convex and closed . A map $P_{\mathcal{C}} : H_1 \to \mathcal{C}$. If for each $\mathfrak{g} \in H_1$, there is a unique fixed point of $\mathcal{C}$, denoted by $P_{\mathcal{C}}\mathfrak{g}$, s.t,*

$$\|\mathfrak{g} - P_{\mathcal{C}}\mathfrak{g}\| \leq \|\mathfrak{g} - q\| \textit{ for all } q \in \mathcal{C}.$$

*is known as a metric projection of $H_1$ onto $\mathcal{C}$. "*

Moreover, $P_{\mathcal{C}}$ satisfies the characteristics of nonexpansive maps in a Hilbert space and

$$\langle \mathfrak{g} - P_{\mathcal{C}}\mathfrak{g}, P_{\mathcal{C}}\mathfrak{g} - q\rangle \geq 0 \text{ for all } \mathfrak{g}, q \in \mathcal{C}.$$

**Remark 1.4.6.** *It is remarked that $P_{\mathcal{C}}$ is firmly nonexpansive map from $H_1$ onto $\mathcal{C}$, i.e,*

$$\|P_{\mathcal{C}}\mathfrak{g} - P_{\mathcal{C}}q\|^2 \leq \langle \mathfrak{g} - q, P_{\mathcal{C}}\mathfrak{g} - P_{\mathcal{C}}q \rangle \text{, for all } \mathfrak{g}, q \in \mathcal{C}.$$

**Definition 1.4.7** ([20]). *"Let $\{\mathcal{C}_{n+1}\}$ be a closed and convex subset of $H_1$, then the following iterative scheme:*

$$\begin{aligned}
\mathfrak{g}^0 &\in \mathcal{C} \text{ chosen arbitrarily where } \mathcal{C}^0 = \mathcal{C}, \\
q^n &= \beta^n \mathfrak{g}^n + (1-\beta^n)\mathfrak{f}\mathfrak{g}^n, \\
\mathcal{C}_{n+1} &= \{z \in \mathcal{C}_n : \|z - q^n\| \leq \|z - \mathfrak{g}^n\|\}, \\
\mathfrak{g}^{n+1} &= P_{\mathcal{C}_{n+1}}\mathfrak{g}^0.
\end{aligned} \tag{1.16}$$

*is often called Shrinking Projection Method ."*

Later on , Takahashi et al. [47] started the hybrid scheme to establish a strong convergence result using the shrinking effect in 2008, and in 2011 Dong et al.[20], introduced a Shrinking Projection process like $(1.16)$ under the action of nonexpansive maps in Hilbert Space.

## 1.5 Motivation and Statement of Problem

In this section, we will talk about the Motivation of our work and at last we will eleborate the statement of our Problem.

### Motivation:

Motivated by the the recently referenced results and the continuing research towards this path, we intend to use Extragradient-Proximal point algorithm for finding the common solution of SEFPP for nonexpansive Semigroup in real Hilbert Space and a Hybrid Shrinking Projection algorithm to locate a common component in solution set of an infinite family of k-strict pseudo-contraction mapping with common fixed points set of a GSEP and GVIP in Hilbert Spaces .These outcomes can be observe as a speculation of various existing results in the present composition.

**Problem Statement:**

In this thesis we suggest and examine Extra Gradient-Proximal point algorithm to find the solution of SEP and FPP for nonexpansive semigroup in real Hilbert spaces.Our purpose is to address FPP, VIP and GSEP in Hilbert spaces. We proposed a Hybrid Shrinking Projection algorithm for the construction of a common element in common fixed points sets of an infinite family of k-strict pseudo-contractions and the common solution set of family of GSEP in Hilbert Spaces.Our outcomes sum up and enhance different existing outcomes in the present literature.

## 1.6 Arrangements of the Thesis

Remaining thesis is manage as following:

- In 2nd Chapter, we will show the strong convergence and weak convergence results using iterative algorithms formulated by the Extragradient-Proximal point algorithm for the SEP and FPP in Hilbert Spaces.

- In 3rd Chapter, we will show the strong convergence result using iterative algorithms formulated by the Hybrid Shrinking algorithm for Solving GSEP and system of VIP for k-Strict Pseudo-Contraction Maps in Hilbert Spaces.

# Chapter 2

# Convergence Analysis of Extragradient-Proximal Point Algorithm

This chapter is focused to initiate the weak and strong convergence results generated by the Hybrid projection strategy in Hilbert spaces.The proposed approximant sequence is used to find solution of SEP and FPP for nonexpansive semigroup in real Hilbert spaces by examining Extragradient Proximal point algorithm.

In the sixties, the interest in the study of nonexpansive operators increased enormously. This was motivated by following two facts: Browder's works which develops an association linking two broader classes of nonlinear operators such as monotone operators and nonexpansive operators, and the seminal paper by the kirk where the significance of the geometric properties of the norm are studied to established the existence result for fixed points of nonexpansive operators. The theory for the presence of fixed-point for the class of nonexpansive operators was initiated, independently by Göhde and Browder in real Banach spaces. This theory develops the fixed-point property of such operator which is defined on non-empty subset of a banach space having uniform convex structure. The subset additionally satisfies the condition of closeness, boundedness and convexity in such setting. On the other hand, this outcome was also unveiled by Kirk with slightly weaker conditions. In 1969, Goebel gave another proof of the same result established which is more elementary and geometric in nature. The Class of nonexpansive operators is exceptionally appropriate to different world problems. As a consequence, this class of operators has been contemplated widely with regards to issues related with real-world phenomenon. Additionally, this class of operators made ready for the improvement of different new ideas in fixed-point theory.

"A family $\mathcal{T} = \{\mathrm{T}(s) : 0 \leq s < \infty\}$ of map on $\mathcal{C}$ is called nonexpansive semigroup if it fulfills the accompanying condition;

(1) $\mathrm{T}(0)\mathfrak{g} = \lim_{s\to 0} \mathrm{T}(s)\mathfrak{g} = \mathfrak{g}$ ,$\forall \mathfrak{g} \in \mathcal{C}$

(2) $\mathrm{T}(\mathfrak{g} + q) = \mathrm{T}(\mathfrak{g})\mathrm{T}(q)$ $\forall \mathfrak{g}, q > 0$

(3) $\|\mathrm{T}(s)\mathfrak{g} - \mathrm{T}(s)q\| \leq \|\mathfrak{g} - q\| \; \forall \mathfrak{g}, q \in \mathcal{C}$ and $0 \leq s$

(4)$s \mapsto \mathrm{T}(s)\mathfrak{g}$ is continuous $\forall \mathfrak{g} \in \mathcal{C}$ ."

Using $\mathbb{F}(\mathcal{T})$ to show the common fixed-point set of the semigroup T means $\mathbb{F}(\mathcal{T}) = \{\mathfrak{g} \in \mathcal{C} : \mathfrak{g} = \mathrm{T}(s)\mathfrak{g}\} \; where \; s \in [0, \infty)$. It is notable that semigroup result has implication in PDE theories and also in FPT. Nonexpansive semigroup is directly attached to the DE and it has been examined by numerous researchers. Takahashi introduced an iterartive method for computing the common component of the set of fixed-points of nonexpansive map and set of solution to GEP in a (real) Hilbert space.

In [39] ,Li et al considerd the accompanying two iterative techniques for finding common fixed-point of a One parameter nonexpansive semigroup $\mathcal{T} = \{\mathrm{T}(s); 0 \leq s < \infty\}$ on a (real) Hilbert space $\mathsf{H}$.

$$\mathfrak{g}^n = (\mathbb{I} - \gamma_n \mathsf{A})\frac{1}{t_n}\int_0^{t_n} \mathrm{T}(s)\mathfrak{g}^n ds + \gamma_n \beta \mathrm{F}(\mathfrak{g}^n)$$

$$q^{n+1} = (\mathbb{I} - \gamma_n \mathsf{A})\frac{1}{t_n}\int_0^{t_n} \mathrm{T}(s)q^n ds + \gamma_n \beta \mathrm{F}(q^n)$$

They demonstrated that the sequences $\{\mathfrak{g}^n\}$, $\{q^n\}$ converge strongly to a common fixed-point $\mathfrak{g}^* \in \mathbb{F}(\mathcal{T})$, which gives solution of the following Variational Inequality

$$\langle (\mathsf{A} - \beta \mathrm{F})\mathfrak{g}^*, \mathfrak{g}^* - w \rangle \leq 0, w \in \mathbb{F}(\mathcal{T})$$

Cianciaruso et al.[24] presented the accompanying implicit and explicit algorithm for approximating a common element of the common fixed-points set of a one parameter nonexpansive semigroup $\mathcal{T} = \{\mathrm{T}(s); 0 \leq s < \infty\}$ which is solution to an EP in a (real) Hilbert space.

$$\forall q \in \mathsf{H}; \mathrm{F}(u^n, q) + \frac{1}{r_n}\langle q - u^n, u^n - \mathfrak{g}^n \rangle \geq 0$$

$$\mathfrak{g}^n = (\mathbb{I} - \gamma_n \mathsf{A})\frac{1}{t_n}\int_0^{t_n} \mathrm{T}(s)u^n ds = \gamma_n \beta \mathrm{F}(\mathfrak{g}^n)$$

$$\forall q \in \mathsf{H}; \mathrm{F}(u^n, q) + \frac{1}{r_n}\langle q - u^n, u^n - \mathfrak{g}^n \rangle \geq 0$$

$$\mathfrak{g}^{n+1} = (\mathbb{I} - \gamma_n \mathsf{A})\frac{1}{t_n}\int_0^{t_n} \mathrm{T}(s)u^n ds = \gamma_n \beta \mathrm{F}(\mathfrak{g}^n)$$

They demonstrate that the sequences $\{u^n\}$ , $\{\mathfrak{g}^n\}$ converge strongly to a common fixed-point $\mathfrak{g}^* \in \mathbb{F}(\mathcal{T})$ ,which is also a solution to an EP in a (real) Hilbert space. By inspiring from on going examination in this field depicted above, we aim to study SEP and FPP for nonexpansive semigroup by using extragradient and proximal point algorithm in Hilbert spaces.

## 2.1 Preliminaries

In this segment, we review a few lemmas and assumptions required in the continuation.

**Lemma 2.1.1.** *"Let $\mathcal{C} \subset \mathsf{H}$ where $\mathcal{C}$ is closed and convex. Then $P_{\mathcal{C}}$ has accompanying properties:*

*(1) For every ,$\mathfrak{g} \in \mathsf{H}$, we will get $P_{\mathcal{C}}(\mathfrak{g})$ singleton and well defined .*

*(2) For each $\mathfrak{g} \in \mathsf{H}$ ,$w = P_{\mathcal{C}}(\mathfrak{g})$ iff $\langle \mathfrak{g} - w, q - w\rangle \leq 0$ for all $q \in \mathcal{C}$*

*(3) $\|P_{\mathcal{C}}(\mathfrak{g}) - P_{\mathcal{C}}(q)\|^2 \leq \langle P_{\mathcal{C}}(\mathfrak{g}) - P_{\mathcal{C}}(q), \mathfrak{g} - q\rangle \ \forall \mathfrak{g}, q \in \mathsf{H}$*

*(4) $\|P_{\mathcal{C}}(\mathfrak{g}) - P_{\mathcal{C}}(q)\|^2 \leq \|\mathfrak{g} - q\|^2 - \|\mathfrak{g} - P_{\mathcal{C}}(\mathfrak{g}) - q + P_{\mathcal{C}}(q)\|^2 \ \forall \mathfrak{g}, q \in \mathsf{H}$."*

**Lemma 2.1.2.** *"Let a real Hilbert space $\mathsf{H}$ , then for all $\mathfrak{g}, q \in \mathsf{H}$ and $\gamma \in [0,1]$ we have*

$\|\gamma(\mathfrak{g}) + (1-\gamma)q\|^2 = \gamma\|\mathfrak{g}\|^2 + (1-\gamma)\|q\|^2 - \gamma(1-\gamma)\|\mathfrak{g} - q\|^2$"

**Lemma 2.1.3.** *"A sequence $\{\mathfrak{g}^k\} \subset \mathsf{H}$ with $\mathfrak{g}^k$ conveges weakly to $\mathfrak{g}$, the inequality $\lim_{k\to+\infty} inf\|\mathfrak{g}^k - \mathfrak{g}\| < \lim_{k\to\infty} inf\|\mathfrak{g}^k - q\|$ holds for each $q \in \mathsf{H}$ with $q \neq \mathfrak{g}$ ; $(Opial'sCondition)$ "*

*"Now, let $G : \mathcal{Q} \times \mathcal{Q} \to R$ and $F : \mathcal{C} \times \mathcal{C} \to R$ be bifunctions.Let the following widely used assumptions in monotone and pseudumonotone equilibrium problems.*

**Assumptions$\mathfrak{A}$;**

$(A_1)$ $\mathrm{G}(q,q)=0$, $\forall\, q\in\mathcal{Q}$

$(A_2)$ G is monotone on $\mathcal{Q}$;

$(A_3)$ for each $\mathfrak{g},q,w\in\mathcal{Q}$

$$\lim_{\lambda\to 0}\sup \mathrm{G}(\lambda w+(1-\lambda)\mathfrak{g},q)\leq \mathrm{G}(\mathfrak{g},q)$$

$(A_4)$ $\mathrm{G}(q,.)$ is convex and lower semicontinous on $\mathcal{Q}$ for each $q\in\mathcal{Q}$.

**Assumptions$\mathfrak{B}$;**

$(B_1)$ $\mathrm{F}(\mathfrak{g},\mathfrak{g})=0$, $\forall\, \mathfrak{g}\in\mathcal{C}$

$(B_2)$ F is pseudomonotone on $\mathcal{C}$ with respect to $Sol(\mathcal{C},\mathrm{F})$.

$(B_3)$ F is jointly weakly continues on $\mathcal{C}\times\mathcal{C}$ in the sense that, if $\mathfrak{g},q\in\mathcal{C}$ and $\{\mathfrak{g}^k\},\{q^k\}\subset\mathcal{C}$ converges weakly to $\mathfrak{g}$ and $q$ ,respectively,then $\mathrm{F}(\mathfrak{g}^k,q^k)\to\mathrm{F}(\mathfrak{g},q)$ as $k\to\infty$.

$(B_4)$ F($\mathfrak{g}$,.) is convex, subdifferentiable on $\mathcal{C}$, $\forall\mathfrak{g}\in\mathcal{C}$

$(B_5)$ F is Lipschitz-type continous on $\mathcal{C}$ with constants $c_1>0$ and $c_2>0$. "

**Lemma 2.1.4.** *"Let G satisfy assumption $\mathfrak{A}$ Then, for each $\gamma>0$ and $\mathfrak{g}\in$ H, there exists $w\in\mathcal{Q}$ such that*

$$G(w,q)+\frac{1}{\gamma}\langle q-w,w-\mathfrak{g}\rangle\geq 0$$

*for all $q\in\mathcal{Q}$."*

**Lemma 2.1.5.** *"Under assumptions of Lemma 2.1.4, the map $T_\gamma^G$ defined on H by*

$$T_\gamma^G(\mathfrak{g})=\{w\in\mathcal{Q};G(w,q)+\frac{1}{\gamma}\langle q-w,w-\mathfrak{g}\rangle\geq 0\}.$$

*for all $q\in\mathcal{Q}$ has the following properties ;*

*(1) $T_\gamma^G$ is single valued.*

*(2) $T^G_\gamma$ is firmly nonexpansive ,i.e for any $\mathfrak{g}, q \in$ H*

$$\|T^G_\gamma(\mathfrak{g}) - T^G_\gamma(q)\|^2 \le \langle T^G_\gamma(\mathfrak{g}) - T^G_\gamma(q), \mathfrak{g} - q\rangle.$$

*(3) $Fix(T^G_\gamma) = Sol(\mathcal{Q}, G)$.*

*(4) $Sol(\mathcal{Q}, G)$ is closed and convex."*

**Lemma 2.1.6.** *"By assumptions of lemma 2.1.5, for $\gamma, \beta > 0$ and $\mathfrak{g}, q \in$ H, one has*

$$\|T^G_\gamma(\mathfrak{g}) - T^G_\gamma(q)\| \le \|q - \mathfrak{g}\| + \frac{|\beta - \gamma|}{\beta}\|T^g_\beta(q) - q\|.$$

*"*

**Lemma 2.1.7.** *"Suppose that $\mathfrak{g}^* \in Sol(\mathcal{C}, F), F(\mathfrak{g}, .)$ is convex and subdifferentiable on $\mathcal{C}$ for all $\mathfrak{g} \in \mathcal{C}$, and F is pseudomonotone on $\mathcal{C}$. Then, we have*

*(1) $\lambda_n[F(\mathfrak{g}^n, q) - F(\mathfrak{g}^n, q^n)] \ge \langle q^n - \mathfrak{g}^n, q^n - q\rangle\ \forall q \in \mathcal{C}$*

*(2) $\|w^n - \mathfrak{g}^*\|^2 \le \|\mathfrak{g}^n - \mathfrak{g}^*\|^2 - (1 - 2\lambda_n c_1)\|\mathfrak{g}^n - q^n\|^2 - (1 - 2\lambda_n c_2)\|q^n - w^n\|^2$ for all n. "*

We have some properties of Innerproduct

**Remark 2.1.8.** $\langle \mathfrak{g} - q, w - q\rangle = \frac{1}{2}[\|\mathfrak{g} - q\|^2 + \|w - q\|^2 - \|\mathfrak{g} - w\|^2]$.

**Remark 2.1.9.**

$$\langle \mathfrak{g} - q, q\rangle = \langle \mathfrak{g}, q\rangle - \|q\|^2.$$

**Remark 2.1.10.**

$$\langle \mathfrak{g} - q, \mathfrak{g} - w\rangle = \frac{1}{2}[\|\mathfrak{g} - q\|^2 - \|w - q\|^2 + \|\mathfrak{g} - w\|^2].$$

**Remark 2.1.11.**

$$\|\mathfrak{g} + q\|^2 = \|\mathfrak{g}\|^2 + \|q\|^2 + 2\langle \mathfrak{g}, q\rangle.$$

## 2.2 Main Results

We presently demonstrate our principle consequence of this area.

**Theorem 2.2.1.** *"Let $H_1$ and $H_2$ be two (real) Hilbert spaces and suppose $\emptyset \neq \mathcal{C} \subset H_1$ and $\emptyset \neq \mathrm{Q} \subset H_2$ where both the subsets are closed and convex. Let $T(s) : \mathcal{C} \to \mathcal{C}$,$T(m) : \mathcal{Q} \to \mathcal{Q}$ be nonexpansive semigroup. Let $A : H_1 \to H_2$ be a BLO with its adjoint $A^*$. Assume $F : \mathcal{C} \times \mathcal{C} \to R$ and $G : \mathcal{Q} \times \mathcal{Q} \to \mathcal{R}$ are bifunctions satisfying assumptions $\mathfrak{B}$ and $\mathfrak{A}$ respectively , Take $\mathfrak{g}^1 \in \mathcal{C} : \{\lambda_n\} \subset [a, b]$ for some $a, b \in (0, min\{\frac{1}{2c_1}, \frac{1}{2c_2}\}); 0 < \gamma < 1; 0 < \xi < \frac{1}{\|A\|^2}; \{\gamma_n\} \subset (0, \infty)$ with $\liminf_{n\to\infty} \gamma_n > 0$ and consider the sequences $\{\mathfrak{g}^n\}, \{q^n\}, \{w^n\}, \{t^n\}$ and $\{u^n\}$ defined by*

$$\begin{cases} q^n = \arg\min\{\lambda_n F(\mathfrak{g}^n, q) + \frac{1}{2}\|q - \mathfrak{g}^n\|^2 : q \in \mathcal{C}\}, \\ w^n = \arg\min\{\lambda_n F(q^n, q) + \frac{1}{2}\|q - \mathfrak{g}^n\|^2 : q \in \mathcal{C}\}, \\ t^n = (1-\gamma)w^n + \gamma \frac{1}{s_t}\int_0^{s_t} T(s) w^n ds, \\ u^n = T_{\gamma_n}^{G} A t^n, \\ \mathfrak{g}^{n+1} = P_{\mathcal{C}}(t^n + \xi A^*(\frac{1}{m_t}\int_0^{m_t} T(m) u^n dm - A t^n)), \end{cases} \tag{2.1}$$

*if $\omega = \{\mathfrak{g}^* \in Sol(\mathcal{C}, F) \cap Fix(T(s)); A\mathfrak{g}^* \in Sol(\mathcal{Q}, G) \cap Fix(T(m))\} \neq \emptyset$ so, $\{\mathfrak{g}^n\}$ and $\{w^n\} \rightharpoonup p \in \omega$ and $\{u^n\} \rightharpoonup Ap \in Sol(\mathcal{Q}, G) \cap Fix(T(m))$".*

**Proof**: We will partition the proof into four steps.

**Step 1.**For all $\mathfrak{g}^* \in \omega$ ,$\lim_{n\to\infty} \|\mathfrak{g}^n - \mathfrak{g}^*\|$ exist . Where $\omega = [\mathfrak{g}^* \in Sol(\mathcal{C}, \mathrm{F}) \cap$

$Fix(\mathsf{T(s)})\text{and}\mathsf{A}\mathfrak{g}^* \in Sol(\mathcal{Q}, \mathrm{G}) \cap Fix(\mathrm{T}(m))]$. By definition of $t^n$ ,we have

$$\begin{aligned}
\| t^n - \mathfrak{g}^*\| &= \|(1-\gamma)w^n + \frac{\gamma}{s_t}\int_0^{s_t} \mathsf{T(s)}w^n ds - \mathfrak{g}^*\| \\
&= \|(1-\gamma)w^n + \frac{\gamma}{s_t}\int_0^{s_t} \mathsf{T(s)}w^n ds - (\gamma+1-\gamma)\mathfrak{g}^*\| \\
&= \|(1-\gamma)w^n + \frac{\gamma}{s_t}\int_0^{s_t} \mathsf{T(s)}w^n ds - \gamma\mathfrak{g}^* - (1-\gamma)\mathfrak{g}^*\| \\
&= \|(1-\gamma)(w^n - \mathfrak{g}^*) + \frac{\gamma}{s_t}\int_0^{s_t} \mathsf{T(s)}w^n ds - \gamma\mathfrak{g}^*\| \\
&= \|(1-\gamma)(w^n - \mathfrak{g}^*) + \frac{\gamma}{s_t}\int_0^{s_t} \mathsf{T(s)}w^n ds - \gamma\mathsf{T(s)}\mathfrak{g}^*\| \\
&\le (1-\gamma)\|w^n - \mathfrak{g}^*\| + \frac{\gamma}{s_t}\int_0^{s_t} \|\mathsf{T(s)}w^n - \mathsf{T(s)}\mathfrak{g}^*\| ds \\
&\le (1-\gamma)\|w^n - \mathfrak{g}^*\| + \gamma\|w^n - \mathfrak{g}^*\| \\
&= \|w^n - \mathfrak{g}^*\|. \qquad (2.2)
\end{aligned}$$

From $\lambda_n \subset [a,b] \subset (0, \min(\frac{1}{2c_1}, \frac{1}{2c_2}))$ and Lemma 2.1.7 . we have

$$\begin{aligned}
\|w^n - \mathfrak{g}^*\|^2 &\le \|\mathfrak{g}^n - \mathfrak{g}^*\|^2 - (1-2\lambda_n c_1)\|\mathfrak{g}^n - q^n\|^2 - (1-2\lambda_n c_2)\|q^n - w^n\|^2 \\
&\le \|\mathfrak{g}^n - \mathfrak{g}^*\|^2
\end{aligned}$$

so we have

$$\|w^n - \mathfrak{g}^*\| \le \|\mathfrak{g}^n - \mathfrak{g}^*\|.$$

Combining this with (2.2)

$$\|t^n - \mathfrak{g}^*\| \le \|w^n - \mathfrak{g}^*\| \le \|\mathfrak{g}^n - \mathfrak{g}^*\|. \qquad (2.3)$$

Now using the Lemma 2.1.5

$$\begin{aligned}
\|\mathrm{T}^{\mathrm{G}}_{\gamma_n}\mathsf{A}t^n - \mathsf{A}\mathfrak{g}^*\|^2 &= \|\mathrm{T}^{\mathrm{G}}_{\gamma_n}\mathsf{A}t^n - \mathrm{T}^{g}_{\gamma_n}\mathsf{A}\mathfrak{g}^*\|^2 \\
&\le \langle \mathrm{T}^{\mathrm{G}}_{\gamma_n}\mathsf{A}t^n - \mathrm{T}^{\mathrm{G}}_{\gamma_n}\mathsf{A}\mathfrak{g}^*, \mathsf{A}t^n - \mathsf{A}\mathfrak{g}^*\rangle \\
&= \langle \mathrm{T}^{\mathrm{G}}_{\gamma_n}\mathsf{A}t^n - \mathsf{A}\mathfrak{g}^*, \mathsf{A}t^n - \mathsf{A}\mathfrak{g}^*\rangle \\
&= \frac{1}{2}[\|\mathrm{T}^{\mathrm{G}}_{\gamma_n}\mathsf{A}t^n - \mathsf{A}\mathfrak{g}^*\|^2 + \|\mathsf{A}t^n - \mathsf{A}\mathfrak{g}^*\|^2 \\
&- \|\mathrm{T}^{\mathrm{G}}_{\gamma_n}\mathsf{A}t^n - \mathsf{A}t^n\|^2].
\end{aligned}$$

By arranging the above inequality

$$\begin{aligned}
\|\mathrm{T}^{\mathrm{G}}_{\gamma_n}\mathsf{A}t^n-\mathsf{A}\mathfrak{g}^*\|^2 &\le \frac{1}{2}[\|\mathrm{T}^{\mathrm{G}}_{\gamma_n}\mathsf{A}t^n-\mathsf{A}\mathfrak{g}^*\|^2+\|\mathsf{A}t^n-\mathsf{A}\mathfrak{g}^*\|^2-\|\mathrm{T}^{\mathrm{G}}_{\gamma_n}\mathsf{A}t^n-\mathsf{A}t^n\|^2]\\
(1-\frac{1}{2})[\|\mathrm{T}^{\mathrm{G}}_{\gamma_n}\mathsf{A}t^n-\mathsf{A}\mathfrak{g}^*\|^2] &\le \frac{1}{2}[\|\mathsf{A}t^n-\mathsf{A}\mathfrak{g}^*\|^2-\|\mathrm{T}^{\mathrm{G}}_{\gamma_n}\mathsf{A}t^n-\mathsf{A}t^n\|^2]\\
\frac{1}{2}[\|\mathrm{T}^{\mathrm{G}}_{\gamma_n}\mathsf{A}t^n-\mathsf{A}\mathfrak{g}^*\|^2] &\le \frac{1}{2}[\|\mathsf{A}t^n-\mathsf{A}\mathfrak{g}^*\|^2-\|\mathrm{T}^{\mathrm{G}}_{\gamma_n}\mathsf{A}t^n-\mathsf{A}t^n\|^2].
\end{aligned}$$

Hence

$$[\|\mathrm{T}^{\mathrm{G}}_{\gamma_n}\mathsf{A}t^n-\mathsf{A}\mathfrak{g}^*\|^2]\le[\|\mathsf{A}t^n-\mathsf{A}\mathfrak{g}^*\|^2-\|\mathrm{T}^{\mathrm{G}}_{\gamma_n}\mathsf{A}t^n-\mathsf{A}t^n\|^2]. \tag{2.4}$$

Uniting this reality with the nonexpansiveness of the semigroup $\mathrm{T}(m)$

$$\begin{aligned}
\|\mathrm{T}(m)\mathrm{T}^{\mathrm{G}}_{\gamma_n}\mathsf{A}t^n-\mathsf{A}\mathfrak{g}^*\|^2 &= \|\mathrm{T}(m)\mathrm{T}^{\mathrm{G}}_{\gamma_n}\mathsf{A}t^n-\mathrm{T}(m)\mathsf{A}\mathfrak{g}^*\|^2\\
&\le \|\mathrm{T}^{\mathrm{G}}_{\gamma_n}\mathsf{A}t^n-\mathsf{A}\mathfrak{g}^*\|^2\\
&\le \|\mathsf{A}t^n-\mathsf{A}\mathfrak{g}^*\|^2-\|\mathrm{T}^{\mathrm{G}}_{\gamma_n}\mathsf{A}t^n-\mathsf{A}t^n\|^2.
\end{aligned}$$

By using $u^n=\mathrm{T}^{\mathrm{G}}_{\gamma_n}\mathsf{A}t^n$

$$\begin{aligned}
\|\mathrm{T}(m)u^n-\mathsf{A}\mathfrak{g}^*\|^2 &\le \|\mathsf{A}t^n-\mathsf{A}\mathfrak{g}^*\|^2-\|u^n-\mathsf{A}t^n\|^2\\
\|\mathrm{T}(m)u^n-\mathsf{A}\mathfrak{g}^*\|^2-\|\mathsf{A}t^n-\mathsf{A}\mathfrak{g}^*\|^2 &\le -\|u^n-\mathsf{A}t^n\|^2.
\end{aligned} \tag{2.5}$$

Using 2.5 we obtain

$$\begin{aligned}
&\langle\mathsf{A}(t^n-\mathfrak{g}^*),\tfrac{1}{m_t}\textstyle\int_0^{m_t}\mathrm{T}(m)u^n dm-\mathsf{A}t^n\rangle\\
=&\langle(\frac{1}{m_t}\int_0^{m_t}\mathrm{T}(m)u^n dm-\mathsf{A}\mathfrak{g}^*)-(\frac{1}{m_t}\int_0^{m_t}\mathrm{T}(m)u^n dm-\mathsf{A}t^n),\frac{1}{m_t}\int_0^{m_t}\mathrm{T}(m)u^n dm-\mathsf{A}t^n\rangle.
\end{aligned}$$

By using the Remark 2.1.9

$$= \langle \frac{1}{m_t}\int_0^{m_t} \mathrm{T}(m)u^n dm - \mathsf{A}\mathfrak{g}^*, \frac{1}{m_t}\int_0^{m_t} \mathrm{T}(m)u^n dm - \mathsf{A}t^n\rangle - \| \frac{1}{m_t}\int_0^{m_t} \mathrm{T}(m)u^n dm - \mathsf{A}t^n\rangle\|^2$$

Now using Remark 2.1.10

$\langle \frac{1}{m_t}\int_0^{m_t} \mathrm{T}(m)u^n dm - \mathsf{A}\mathfrak{g}^*, \frac{1}{m_t}\int_0^{m_t} \mathrm{T}(m)u^n dm - \mathsf{A}t^n\rangle - \| \frac{1}{m_t}\int_0^{m_t} \mathrm{T}(m)u^n dm - \mathsf{A}t^n\rangle\|^2$

$$\begin{aligned}
&= \frac{1}{2}[\|\frac{1}{m_t}\int_0^{m_t} \mathrm{T}(m)u^n dm - \mathsf{A}\mathfrak{g}^*\|^2 - \|\mathsf{A}t^n - \mathsf{A}\mathfrak{g}^*\|^2 \\
&+ \|\frac{1}{m_t}\int_0^{m_t} \mathrm{T}(m)u^n dm - \mathsf{A}t^n\|^2] - \|\frac{1}{m_t}\int_0^{m_t} \mathrm{T}(m)u^n dm - \mathsf{A}t^n\|^2 \\
&= \frac{1}{2}[\|\frac{1}{m_t}\int_0^{m_t} \mathrm{T}(m)u^n dm - \mathsf{A}\mathfrak{g}^*\|^2 - \|\mathsf{A}t^n - \mathsf{A}\mathfrak{g}^*\|^2] \\
&+ [\frac{1}{2}\|\frac{1}{m_t}\int_0^{m_t} \mathrm{T}(m)u^n dm - \mathsf{A}t^n\|^2 - \|\frac{1}{m_t}\int_0^{m_t} \mathrm{T}(m)u^n dm - \mathsf{A}t^n\|^2] \\
&= \frac{1}{2}[\|\frac{1}{m_t}\int_0^{m_t} \mathrm{T}(m)u^n dm - \mathsf{A}\mathfrak{g}^*\|^2 - \|\mathsf{A}t^n - \mathsf{A}\mathfrak{g}^*\|^2] \\
&+ (\frac{1}{2} - 1)[\|\frac{1}{m_t}\int_0^{m_t} \mathrm{T}(m)u^n dm - \mathsf{A}t^n\|^2 \\
&= \frac{1}{2}[\|\frac{1}{m_t}\int_0^{m_t} \mathrm{T}(m)u^n dm - \mathsf{A}\mathfrak{g}^*\|^2 - \|\mathsf{A}t^n - \mathsf{A}\mathfrak{g}^*\|^2] \\
&- \frac{1}{2}[\|\frac{1}{m_t}\int_0^{m_t} \mathrm{T}(m)u^n dm - \mathsf{A}t^n\|^2] \\
&\leq \frac{1}{2m_t}\int_0^{m_t} \|\mathrm{T}(m)u^n - \mathsf{A}\mathfrak{g}^*\|^2 dm - \frac{1}{2}\|\mathsf{A}t^n - \mathsf{A}\mathfrak{g}^*\|^2 \\
&- \frac{1}{2m_t}\int_0^{m_t} \|\mathrm{T}(m)u^n - \mathsf{A}t^n\|^2 dm \\
&= \frac{1}{2}\|\mathrm{T}(m)u^n - \mathsf{A}\mathfrak{g}^*\|^2 - \frac{1}{2}\|\mathsf{A}t^n - \mathsf{A}\mathfrak{g}^*\|^2 - \frac{1}{2}\|\mathrm{T}(m)u^n - \mathsf{A}t^n\|^2 \\
&= \frac{1}{2}[\|\mathrm{T}(m)u^n - \mathsf{A}\mathfrak{g}^*\|^2 - \|\mathsf{A}t^n - \mathsf{A}\mathfrak{g}^*\|^2] - \frac{1}{2}\|\mathrm{T}(m)u^n - \mathsf{A}t^n\|^2 \\
&\leq \frac{1}{2}[-\|u^n - \mathsf{A}t^n\|^2] - \frac{1}{2}\|\mathrm{T}(m)u^n - \mathsf{A}t^n\|^2.
\end{aligned}$$

So we have

$$\langle \mathsf{A}(t^n - \mathfrak{g}^*), \frac{1}{m_t}\int_0^{m_t} \mathrm{T}(m)u^n dm - \mathsf{A}t^n\rangle \leq \frac{1}{2}[-\|u^n - \mathsf{A}t^n\|^2] - \frac{1}{2}\|\mathrm{T}(m)u^n - \mathsf{A}t^n\|^2. \tag{2.6}$$

It shows from (2.6) that

$$\begin{aligned}\|\mathfrak{g}^{n+1}-\mathfrak{g}^*\|^2 &= \|P_{\mathcal{C}}(t^n+\xi\mathsf{A}^*(\frac{1}{m_t}\int_0^{m_t}\mathrm{T}(m)u^n dm-\mathsf{A}t^n)-P_{\mathcal{C}}(\mathfrak{g}^*)\|^2\\ &\leq \|(t^n-\mathfrak{g}^*)+\xi\mathsf{A}^*(\frac{1}{m_t}\int_0^{m_t}\mathrm{T}(m)u^n dm-\mathsf{A}t^n)\|^2.\end{aligned}$$

Using Remark 2.1.11 and (2.6), we get

$\|(t^n-\mathfrak{g}^*)+\xi\mathsf{A}^*(\frac{1}{m_t}\int_0^{m_t}\mathrm{T}(m)u^n dm-\mathsf{A}t^n)\|$

$$\begin{aligned}&=\|t^n-\mathfrak{g}^*\|^2+\|\xi\mathsf{A}^*(\frac{1}{m_t}\int_0^{m_t}\mathrm{T}(m)u^n dm-\mathsf{A}t^n)\|^2\\
&+2\langle(t^n-\mathfrak{g}^*),\xi\mathsf{A}^*(\frac{1}{m_t}\int_0^{m_t}\mathrm{T}(m)u^n dm-\mathsf{A}t^n)\rangle\\
&\leq\|t^n-\mathfrak{g}^*\|^2+\xi^2\|\mathsf{A}^*\|^2\|\frac{1}{m_t}\int_0^{m_t}\mathrm{T}(m)u^n dm-\mathsf{A}t^n\|^2\\
&+2\xi\langle\mathsf{A}(t^n-\mathfrak{g}^*),\frac{1}{m_t}\int_0^{m_t}\mathrm{T}(m)u^n dm-\mathsf{A}t^n\rangle\\
&\leq\|t^n-\mathfrak{g}^*\|^2+\xi^2\|\mathsf{A}^*\|^2\|\frac{1}{m_t}\int_0^{m_t}\mathrm{T}(m)u^n dm-\mathsf{A}t^n\|^2\\
&+2\xi[\frac{1}{2}[-\|u^n-\mathsf{A}t^n\|^2]-\frac{1}{2}\|\mathrm{T}(m)u^n-\mathsf{A}t^n\|^2]\\
&=\|t^n-\mathfrak{g}^*\|^2+\xi^2\|\mathsf{A}^*\|^2\|\frac{1}{m_t}\int_0^{m_t}\mathrm{T}(m)u^n dm-\mathsf{A}t^n\|^2\\
&+2\xi\frac{1}{2}[-\|u^n-\mathsf{A}t^n\|^2]-\|\mathrm{T}(m)u^n-\mathsf{A}t^n\|^2]\\
&=\|t^n-\mathfrak{g}^*\|^2+\xi^2\|\mathsf{A}^*\|^2\|\frac{1}{m_t}\int_0^{m_t}\mathrm{T}(m)u^n dm-\mathsf{A}t^n\|^2\\
&+\xi[-\|u^n-\mathsf{A}t^n\|^2]-\|\mathrm{T}(m)u^n-\mathsf{A}t^n\|^2]\\
&=\|t^n-\mathfrak{g}^*\|^2+\xi^2\|\mathsf{A}^*\|^2\|\frac{1}{m_t}\int_0^{m_t}\mathrm{T}(m)u^n dm-\mathsf{A}t^n\|^2\\
&-\xi\|u^n-\mathsf{A}t^n\|^2-\xi\|\mathrm{T}(m)u^n-\mathsf{A}t^n\|^2\\
&=\|t^n-\mathfrak{g}^*\|^2-\xi\|\frac{1}{m_t}\int_0^{m_t}\mathrm{T}(m)u^n dm-\mathsf{A}t^n\|[1-\xi\|\mathsf{A}^*\|^2]\\
&-\xi\|u^n-\mathsf{A}t^n\|^2.\end{aligned}$$

So we have

$$\|\mathfrak{g}^{n+1}-\mathfrak{g}^*\|^2\leq\|t^n-\mathfrak{g}^*\|^2-\xi\|\frac{1}{m_t}\int_0^{m_t}\mathrm{T}(m)u^n dm-\mathsf{A}t^n\|[1-\xi\|\mathsf{A}^*\|^2]-\xi\|u^n-\mathsf{A}t^n\|^2. \tag{2.7}$$

As $0 < \xi < \frac{1}{\|\mathsf{A}\|^2}$ so we get

$$\|\mathfrak{g}^{n+1} - \mathfrak{g}^*\|^2 \leq \|t^n - \mathfrak{g}^*\|^2.$$

so we have

$$\|\mathfrak{g}^{n+1} - \mathfrak{g}^*\| \leq \|t^n - \mathfrak{g}^*\|. \tag{2.8}$$

combining (2.3) and (2.8) we get

$$\|\mathfrak{g}^{n+1} - \mathfrak{g}^*\| \leq \|t^n - \mathfrak{g}^*\| \leq \|w^n - \mathfrak{g}^*\| \leq \|\mathfrak{g}^n - \mathfrak{g}^*\|. \tag{2.9}$$

from (2.9) we conclude, $\lim_{n\to\infty} \|\mathfrak{g}^n - \mathfrak{g}^*\|$ exist.
From (2.7) we have

$$\xi\|\frac{1}{m_t}\int_0^{m_t} \mathsf{T}(m)u^n dm - \mathsf{A}t^n\|[1-\xi\|\mathsf{A}^*\|^2] + \xi\|u^n - \mathsf{A}t^n\|^2 \leq \|t^n - \mathfrak{g}^*\|^2 - \|\mathfrak{g}^{n+1} - \mathfrak{g}^*\|^2. \tag{2.10}$$

So we obtain from (2.10)

$$\lim_{n\to\infty} \|\mathfrak{g}^n - \mathfrak{g}^*\| = \lim_{n\to\infty} \|t^n - \mathfrak{g}^*\| = \lim_{n\to\infty} \|w^n - \mathfrak{g}^*\|.$$

and

$$\lim_{n\to\infty} \|\frac{1}{m_t}\int_0^{m_t} \mathsf{T}(m)u^n dm - \mathsf{A}t^n\| = \lim_{n\to\infty} \|u^n - \mathsf{A}t^n\| = 0. \tag{2.11}$$

**Step2**:

$$\lim_{n\to\infty} \|w^n - \mathfrak{g}^n\| = \lim_{n\to\infty} \|\frac{1}{s_t}\int_0^{s_t} \mathsf{T(s)}w^n ds - w^n\| = \lim_{n\to\infty} \|t^n - \mathfrak{g}^n\| = 0.$$

By using the inequality

$$\|\frac{1}{m_t}\int_0^{m_t} \mathsf{T}(m)u^n dm - u^n\| \leq \|\frac{1}{m_t}\int_0^{m_t} \mathsf{T}(m)u^n dm - \mathsf{A}t^n\| + \|u^n - \mathsf{A}t^n\|.$$

By using (2.11) it is clear that

$$\lim_{n\to\infty} \|\frac{1}{m_t}\int_0^{m_t} \mathsf{T}(m)u^n dm - u^n\| = 0. \tag{2.12}$$

From $\lambda_n \subset [a, b] \subset (0, \min(\frac{1}{2c_1}, \frac{1}{2c_2}))$ and Lemma 2.1.7.

$$\|w^n - \mathfrak{g}^*\|^2 \leq \|\mathfrak{g}^n - \mathfrak{g}^*\|^2 - (1 - 2\lambda_n c_1)\|\mathfrak{g}^n - q^n\|^2 - (1 - 2\lambda_n c_2)\|q^n - w^n\|^2$$

Arranging this we get

$$(1-2\lambda_n c_1)\|\mathfrak{g}^n - q^n\|^2 + (1-2\lambda_n c_2)\|q^n - w^n\|^2 \leq \|\mathfrak{g}^n - \mathfrak{g}^*\|^2 - \|w^n - \mathfrak{g}^*\|^2 \quad (2.13)$$

From this we have

$$(1-2bc_1)\|\mathfrak{g}^n - q^n\|^2 \leq \|\mathfrak{g}^n - \mathfrak{g}^*\|^2 - \|w^n - \mathfrak{g}^*\|^2,$$

$$(1-2bc_2)\|q^n - w^n\|^2 \leq \|\mathfrak{g}^n - \mathfrak{g}^*\|^2 - \|w^n - \mathfrak{g}^*\|^2. \quad (2.14)$$

From (2.11) we get

$$\lim_{n\to\infty}(\|\mathfrak{g}^n - \mathfrak{g}^*\|^2 - \|w^n - \mathfrak{g}^*\|^2) = 0.$$

Combining this fact with (2.14), we get

$$\lim_{n\to\infty}\|\mathfrak{g}^n - q^n\| = 0, \lim_{n\to\infty}\|q^n - w^n\| = 0 \quad (2.15)$$

By using triangle property

$$\|w^n - \mathfrak{g}^n\| \leq \|\mathfrak{g}^n - q^n\| + \|q^n - w^n\|,$$

By using (2.15) we compute

$$\lim_{n\to\infty}\|w^n - \mathfrak{g}^n\| = 0. \quad (2.16)$$

Using
$t^n = (1-\gamma)w^n + \gamma\frac{1}{s_t}\int_0^{s_t}$ T(s)$w^n ds$ in $\|t^n - \mathfrak{g}^*\|^2$

$$\begin{aligned}\|t^n - \mathfrak{g}^*\|^2 =& \|(1-\gamma)w^n + \gamma\frac{1}{s_t}\int_0^{s_t} \text{T(s)}w^n ds - \mathfrak{g}^*\|^2 \\ =& \|(1-\gamma)w^n + \gamma\frac{1}{s_t}\int_0^{s_t} \text{T(s)}w^n ds - (1-\gamma+\gamma)\mathfrak{g}^*\|^2 \\ =& \|(1-\gamma)(w^n - \mathfrak{g}^*) + \gamma(\frac{1}{s_t}\int_0^{s_t} \text{T(s)}w^n ds - \mathfrak{g}^*)\|^2.\end{aligned}$$

from Lemma 2.1.2, we have

$$\begin{aligned}
&\|\gamma(\tfrac{1}{s_t}\textstyle\int_0^{s_t}\text{T(s)}w^n ds-\mathfrak{g}^*)+(1-\gamma)(w^n-\mathfrak{g}^*)\|^2\\
&=\gamma\|\frac{1}{s_t}\int_0^{s_t}\text{T(s)}w^n ds-\mathfrak{g}^*\|^2+(1-\gamma)\|w^n-\mathfrak{g}^*\|^2\\
&-\gamma(1-\gamma)\|\frac{1}{s_t}\int_0^{s_t}\text{T(s)}w^n ds-\mathfrak{g}^*-(w^n-\mathfrak{g}^*)\|^2\\
&=\gamma\|\frac{1}{s_t}\int_0^{s_t}\text{T(s)}w^n ds-\mathfrak{g}^*\|^2+(1-\gamma)\|w^n-\mathfrak{g}^*\|^2\\
&-\gamma(1-\gamma)\|\frac{1}{s_t}\int_0^{s_t}\text{T(s)}w^n ds-\mathfrak{g}^*-w^n+\mathfrak{g}^*\|^2\\
&=\gamma\|\frac{1}{s_t}\int_0^{s_t}\text{T(s)}w^n ds-\mathfrak{g}^*\|^2+(1-\gamma)\|w^n-\mathfrak{g}^*\|^2\\
&-\gamma(1-\gamma)\|\frac{1}{s_t}\int_0^{s_t}\text{T(s)}w^n ds-w^n\|^2\\
&\le\gamma\frac{1}{s_t}\int_0^{s_t}\|\text{T(s)}w^n-\text{T(s)}\mathfrak{g}^*\|^2 ds+(1-\gamma)\|w^n-\mathfrak{g}^*\|^2\\
&-\gamma(1-\gamma)\|\frac{1}{s_t}\int_0^{s_t}\text{T(s)}w^n ds-w^n\|^2\\
&\le\gamma\|w^n-\mathfrak{g}^*\|^2+(1-\gamma)\|w^n-\mathfrak{g}^*\|^2\\
&-\gamma(1-\gamma)\|\frac{1}{s_t}\int_0^{s_t}\text{T(s)}w^n ds-w^n\|^2\\
&=\gamma\|w^n-\mathfrak{g}^*\|^2+\|w^n-\mathfrak{g}^*\|^2-\gamma\|w^n-\mathfrak{g}^*\|^2\\
&-\gamma(1-\gamma)\|\frac{1}{s_t}\int_0^{s_t}\text{T(s)}w^n ds-w^n\|^2\\
&=\|w^n-\mathfrak{g}^*\|^2-\gamma(1-\gamma)\|\frac{1}{s_t}\int_0^{s_t}\text{T(s)}w^n ds-w^n\|^2.
\end{aligned}$$

Finally we have

$$\|t^n-\mathfrak{g}^*\|^2\le\|w^n-\mathfrak{g}^*\|^2-\gamma(1-\gamma)\|\frac{1}{s_t}\int_0^{s_t}\text{T(s)}w^n ds-w^n\|^2. \tag{2.17}$$

By arranging

$$\gamma(1-\gamma)\|\frac{1}{s_t}\int_0^{s_t}\text{T(s)}w^n ds-w^n\|^2\le\|w^n-\mathfrak{g}^*\|^2-\|t^n-\mathfrak{g}^*\|^2.$$

By using (2.11)

$$\gamma(1-\gamma)\lim_{n\to\infty}\|\frac{1}{s_t}\int_0^{s_t}\text{T(s)}w^n ds-w^n\|^2\le\lim_{n\to\infty}[\|w^n-\mathfrak{g}^*\|^2-\|t^n-\mathfrak{g}^*\|^2]=0.$$

So we get

$$\lim_{n\to\infty} \|\frac{1}{s_t}\int_0^{s_t} \text{T(s)}w^n ds - w^n\| = 0. \tag{2.18}$$

We have

$$\begin{aligned}
\|t^n - \mathfrak{g}^n\| &\leq \|t^n - w^n\| + \|w^n - \mathfrak{g}^n\| \\
&= \|(1-\gamma)w^n + \gamma\frac{1}{s_t}\int_0^{s_t} \text{T(s)}w^n ds - w^n\| + \|w^n - \mathfrak{g}^n\| \\
&= \|w^n - \gamma w^n + \gamma\frac{1}{s_t}\int_0^{s_t} \text{T(s)}w^n ds - w^n\| + \|w^n - \mathfrak{g}^n\| \\
&= \| - \gamma w^n + \gamma\frac{1}{s_t}\int_0^{s_t} \text{T(s)}w^n ds\| + \|w^n - \mathfrak{g}^n\| \\
&= \gamma\|\frac{1}{s_t}\int_0^{s_t} \text{T(s)}w^n ds - w^n\| + \|w^n - \mathfrak{g}^n\| \\
&\leq \gamma\|\frac{1}{s_t}\int_0^{s_t} \text{T(s)}w^n ds - w^n\| + \|w^n - \mathfrak{g}^n\|.
\end{aligned}$$

So,we get

$$\lim_{n\to\infty} \|t^n - \mathfrak{g}^n\| \leq \gamma \lim_{n\to\infty} \|\frac{1}{s_t}\int_0^{s_t} \text{T(s)}w^n ds - w^n\| + \lim_{n\to\infty} \|w^n - \mathfrak{g}^n\|.$$

by using (2.16) and (2.18), we get

$$\lim_{n\to\infty} \|t^n - \mathfrak{g}^n\| = 0. \tag{2.19}$$

**Step3**:

The sequences $\{\mathfrak{g}^n\}, \{w^n\} \rightharpoonup p \in Sol(\mathcal{C}, \text{F}) \cap Fix(\text{T(s)})$ and $\{u^n\} \rightharpoonup \mathsf{A}p \in Sol(\mathcal{Q}, \text{G}) \cap Fix(\text{T}(m))$ .

By Step1, $\lim_{n\to\infty} \|\mathfrak{g}^n - \mathfrak{g}^*\|$ exist and the sequence $\{\mathfrak{g}^n\}$ is bounded. In this way ,there exist a subsequence $\{\mathfrak{g}^{n_j}\}$ of $\{\mathfrak{g}^n\}$ such that $\{\mathfrak{g}^{n_j}\} \rightharpoonup p \in \mathcal{C}$ as $j \to \infty$. Then ,it showed from $(2.19)$ that $\{t^{n_j}\} \rightharpoonup p$ and $\mathsf{A}t^{n_j} \rightharpoonup \mathsf{A}p$. As $\lim_{n\to\infty}\|u^n - \mathsf{A}t^n\| = 0$ ,from which we conclude that $u^{n_j} \rightharpoonup \mathsf{A}p$. Remembering that $\{u^n\} \subset \mathcal{Q}$ so $\mathsf{A}p \in \mathcal{Q}$. Also $\lim_{n\to\infty}\|w^n - \mathfrak{g}^n\| = 0$ and $\mathfrak{g}^{n_j} \rightharpoonup p$ . Hence $w^{n_j} \rightharpoonup p$.

if T(s)$p \neq p$ then by Lemma 2.1.3 and (2.18) .we have

$$
\begin{aligned}
\lim_{n\to\infty}\inf\|w^{n_j}-p\| &< \lim_{j\to\infty}\inf\|w^{n_j}-\frac{1}{s_t}\int_0^{s_t}\text{T(s)}pds\| \\
&= \lim_{j\to\infty}\inf\|w^{n_j}-\frac{1}{s_t}\int_0^{s_t}\text{T(s)}w^{n_j}ds+\frac{1}{s_t}\int_0^{s_t}\text{T(s)}w^{n_j}ds-\frac{1}{s_t}\int_0^{s_t}\text{T(s)}pds\| \\
&\le \lim_{j\to\infty}\inf(\|w^{n_j}-\frac{1}{s_t}\int_0^{s_t}\text{T(s)}w^{n_j}ds)\| \\
&+\|\frac{1}{s_t}\int_0^{s_t}\text{T(s)}w^{n_j}ds-\frac{1}{s_t}\int_0^{s_t}\text{T(s)}pds)\|)
\end{aligned}
$$

by using (2.18) we have

$$
\begin{aligned}
&= \lim_{j\to\infty}\inf\|\frac{1}{s_t}\int_0^{s_t}\text{T(s)}w^{n_j}ds-\frac{1}{s_t}\int_0^{s_t}\text{T(s)}pds)\|) \\
&= \lim_{j\to\infty}\inf\|\text{T(s)}w^{n_j}-\text{T(s)}p)\|) \\
&\le \lim_{j\to\infty}\inf\|w^{n_j}-p\|.
\end{aligned}
$$

This is a contradiction.So , T(s)$p=p$, that is, $p\in Fix(\text{T(s)})$.

From Lemma 2.1.7, we have

$$\lambda_{n_j}[\mathrm{F}(\mathfrak{g}^{n_j},q)-\mathrm{F}(\mathfrak{g}^{n_j},q^{n_j})]\ge\langle q^{n_j}-\mathfrak{g}^{n_j},q^{n_j}-q\rangle\forall q\in\mathcal{C}.$$

Let $j\to\infty$,we have $\mathrm{F}(p,q)\ge 0$ for all $q\in\mathcal{C}$ and so $p\in Sol(\mathcal{C},\mathrm{F})$. Therefore,

$$p\in Sol(\mathcal{C},\mathrm{F})\cap Fix(\text{T(s)}). \tag{2.20}$$

Next we need to demonstrate that

$$\mathsf{A}p\in Sol(\mathcal{Q},\mathrm{G})\cap Fix(\mathrm{T}(m)).$$

First ,if $\mathrm{T}(m)\mathsf{A}p\ne\mathsf{A}p$

for all $m\in[0,\infty)$,then by the Lemma 2.1.3

$\lim_{j\to\infty} \inf \|u^{n_j} - \mathsf{A}p\|$

$$
\begin{aligned}
&< \lim_{j\to\infty} \inf \|u^{n_j} - \frac{1}{m_t}\int_0^{m_t} \mathrm{T}(m)\mathsf{A}p\| \\
&= \lim_{j\to\infty} \inf \|u^{n_j} - \frac{1}{m_t}\int_0^{m_t} \mathrm{T}(m)u^{n_j}dm + \frac{1}{m_t}\int_0^{m_t} \mathrm{T}(m)u^{n_j}dm - \frac{1}{m_t}\int_0^{m_t} \mathrm{T}(m)\mathsf{A}pdm\| \\
&\leq \lim_{j\to\infty} \inf(\|u^{n_j} - \frac{1}{m_t}\int_0^{m_t} \mathrm{T}(m)u^{n_j}dm\| + \|\frac{1}{m_t}\int_0^{m_t} \mathrm{T}(m)u^{n_j}dm - \frac{1}{m_t}\int_0^{m_t} \mathrm{T}(m)\mathsf{A}pdm\|) \\
&= \lim_{j\to\infty} \inf \|\frac{1}{m_t}\int_0^{m_t} \mathrm{T}(m)u^{n_j}dm - \frac{1}{m_t}\int_0^{m_t} \mathrm{T}(m)\mathsf{A}pdm\| \\
&= \lim_{j\to\infty} \inf \|\mathrm{T}(m)u^{n_j} - \mathrm{T}(m)\mathsf{A}p\| \\
&\leq \lim_{j\to\infty} \inf \|u^{n_j} - \mathsf{A}p\|.
\end{aligned}
$$

which is a contradiction.

So , $\mathsf{A}p \in Fix(\mathrm{T}(m))$.

On the other hand ,$Sol(\mathcal{Q}, \mathrm{G}) = Fix(T_\gamma^{\mathrm{G}})$.

so if $T_\gamma^{\mathrm{G}}\mathsf{A}p \neq \mathsf{A}p$, Then consolidating $(2.11)$, the Lemma 2.1.3 and we get

$$
\begin{aligned}
\lim_{j\to\infty} \inf \|\mathsf{A}t^{n_j} - \mathsf{A}p\| &< \lim_{j\to\infty} \inf \|\mathsf{A}t^{n_j} - T_\gamma^{\mathrm{G}}\mathsf{A}p\| \\
&= \lim_{j\to\infty} \inf \|\mathsf{A}t^{n_j} - u^{n_j} + u^{n_j} - T_\gamma^{\mathrm{G}}\mathsf{A}p\| \\
&\leq \lim_{j\to\infty} \inf(\|\mathsf{A}t^{n_j} - u^{n_j}\| + \|T_\gamma^{\mathrm{G}}\mathsf{A}p - u^{n_j}\|) \\
&= \lim_{j\to\infty} \inf \|T_\gamma^{\mathrm{G}}\mathsf{A}p - u^{n_j}\| \\
&= \lim_{j\to\infty} \inf \|T_\gamma^{\mathrm{G}}\mathsf{A}p - T_{\gamma_{n_j}}^{\mathrm{G}}\mathsf{A}t^{n_j}\|.
\end{aligned}
$$

By lemma 2.1.6

$$
\begin{aligned}
\|T_\gamma^{\mathrm{G}}(\mathsf{A}p) - T_{\gamma_{n_j}}^{\mathrm{G}}(\mathsf{A}t^{n_j})\| &\leq \lim_{j\to\infty} \inf\{\|\mathsf{A}t^{n_j} - \mathsf{A}p\| + \frac{|\gamma_{n_j} - \gamma|}{\gamma_{n_j}}\|T_{\gamma_{n_j}}^{\mathrm{G}}\mathsf{A}t^{n_j} - \mathsf{A}t^{n_j}\|\} \\
&= \lim_{j\to\infty} \inf\{\|\mathsf{A}t^{n_j} - \mathsf{A}p\| + \frac{|\gamma_{n_j} - \gamma|}{\gamma_{n_j}}\|u^{n_j} - \mathsf{A}t^{n_j}\|\}.
\end{aligned}
$$

By using (2.11) we have

$$\lim \|u^{n_j} - \mathsf{A}t^{n_j}\| = 0.$$

So we get

$$\|T_{\gamma}^{\mathrm{G}}(\mathsf{A}p) - T_{\gamma_{n_j}}^{\mathrm{G}}(\mathsf{A}t^{n_j})\| = \lim_{j\to\infty}\inf \|\mathsf{A}t^{n_j} - \mathsf{A}p\|.$$

So finally we get

$$\lim_{j\to\infty}\inf \|\mathsf{A}t^{n_j} - \mathsf{A}p\| < \lim_{j\to\infty}\inf \|\mathsf{A}t^{n_j} - \mathsf{A}p\|.$$

which is a contradiction,thus $\mathsf{A}p \in Fix(T_{\gamma}^{\mathrm{G}}) = Sol(\mathcal{Q}, \mathrm{G})$.Therefore

$$\mathsf{A}p \in Sol(\mathcal{Q}, \mathrm{G}) \cap Fix(\mathrm{T}(m)). \tag{2.21}$$

we obtain from (2.20) and (2.21) that $p \in \omega$.

**Step4**:

The sequence $\{\mathfrak{g}^n\} \rightharpoonup p$ and $\{u^n\} \rightharpoonup$ to $\mathsf{A}p$. On the other hand , $\exists$ a subsequence $\{\mathfrak{g}^{m_i}\}$ of $\{\mathfrak{g}^n\}$ such that $\mathfrak{g}^{m_i} \rightharpoonup q$ where $q \in \omega$ with $q \neq p$,and once again by opial's Condition ,we get

$$\begin{aligned}
\lim_{i\to\infty}\inf \|\mathfrak{g}^{m_i} - q\| &< \lim_{i\to\infty}\inf \|\mathfrak{g}^{m_i} - p\| \\
&= \lim_{j\to\infty}\inf \|\mathfrak{g}^{n_j} - p\| \\
&< \lim_{j\to\infty}\inf \|\mathfrak{g}^{n_j} - q\| \\
&= \lim_{i\to\infty}\inf \|\mathfrak{g}^{m_i} - q\|.
\end{aligned}$$

This is a contradiction . Hence, $\{\mathfrak{g}^n\} \rightharpoonup p$.

From both (2.16) and (2.19) we additionally get $w^n \rightharpoonup p$ and $t^n \rightharpoonup p$, so $\mathsf{A}t^n \rightharpoonup \mathsf{A}p$.Connecting with (2.11),we get that $u^n \rightharpoonup \mathsf{A}p$ where $\mathsf{A}p \in Sol(\mathcal{Q}, \mathrm{G}) \cap Fix(\mathrm{T}(m))$ .Hence theorem is proved.□

**Corollary 2.2.2.** *"Let $H_1$ and $H_2$ be two (real) Hilbert spaces and suppose $\emptyset \neq \mathcal{C} \subset H_1$ and $\emptyset \neq$ Q $\subset H_2$ where both the subsets are closed and convex. Let T(s) : $\mathcal{C} \to \mathcal{C}$,$T(m)$ : $\mathcal{Q} \to \mathcal{Q}$ be nonexpansive semigroup. Let $A : H_1 \to H_2$ be a BLO with its adjoint $A^*$. Suppose $\mathrm{F} : \mathcal{C} \times \mathcal{C} \to R$ and $\mathrm{G} : \mathcal{Q} \times \mathcal{Q} \to R$ are bifunctions satisfying assumptions$\mathfrak{B}$ and $\mathfrak{A}$ respectively , Take $\mathfrak{g}^1 \in \mathcal{C}$ : $\{\lambda_n\} \subset [a, b]$, for some $a, b \in (0, min\{\frac{1}{2c_1}, \frac{1}{2c_2}\}); 0 < \gamma < 1; 0 < \xi < \frac{1}{\|A\|^2}; \{\gamma_n\} \subset (0, \infty)$ with $\liminf_{n\to\infty} \gamma_n > 0$ and consider the sequences $\{\mathfrak{g}^n\}, \{q^n\}, \{w^n\}$ , and $\{u^n\}$ defined by*

$$\begin{cases} q^n = \arg\min\{\lambda_n \mathrm{F}(\mathfrak{g}^n, q) + \frac{1}{2}\|q - \mathfrak{g}^n\|^2 : q \in \mathcal{C}\}, \\ w^n = \arg\min\{\lambda_n \mathrm{F}(q^n, q) + \frac{1}{2}\|q - \mathfrak{g}^n\|^2 : q \in \mathcal{C}\}, \\ u^n = T^{\mathrm{G}}_{\gamma_n} A z^n, \\ \mathfrak{g}^{n+1} = P_{\mathcal{C}}(w^n + \xi A^*(u^n - A w^n)), \end{cases} \tag{2.22}$$

*if $\omega = \{\mathfrak{g}^* \in Sol(\mathcal{C}, \mathrm{F}); A\mathfrak{g}^* \in Sol(\mathcal{Q}, G)\} \neq \emptyset$ then ,the $\{\mathfrak{g}^n\}$ and $\{w^n\} \rightharpoonup p \in \omega$ and $\{u^n\} \rightharpoonup Ap \in Sol(\mathcal{Q}, G)$."*

**Proof;**

When T(s) $= I_{H_1}$ and $\mathrm{T}(m) = I_{H_2}$, then the SEFPP reduces to SEP.

**Corollary 2.2.3.** *"Let $\mathrm{F}$ be a bifunction which satisfy the assumption $\mathfrak{B}$. Take $\mathfrak{g}^1 \in \mathcal{C}$ : $\{\lambda_n\} \subset [a, b]$, for $a, b \in (0, min\{\frac{1}{2c_1}, \frac{1}{2c_2}\})$,and consider the sequences $\{\mathfrak{g}^n\}$ and $\{q^n\}$ generated by*

$$\begin{cases} q^n = \arg\min\{\lambda_n \mathrm{F}(\mathfrak{g}^n, q) + \frac{1}{2}\|q - \mathfrak{g}^n\|^2 : q \in \mathcal{C}\}, \\ \mathfrak{g}^{n+1} = \arg\min\{\lambda_n \mathrm{F}(q^n, q) + \frac{1}{2}\|q - \mathfrak{g}^n\|^2 : q \in \mathcal{C}\}, \end{cases} \tag{2.23}$$

*If $Sol(\mathcal{C}, \mathrm{F})$ is non empty, then $\{\mathfrak{g}^n\}$ and $\{q^n\} \rightharpoonup Sol(\mathcal{C}, \mathrm{F})$ "*

**Proof;**

The accompanying corollary is quick from the theorem 2.2.1 when we take $H_1 = H_2 = H$ , $\mathrm{G} = 0$ and T(s) $= \mathrm{T}(m) = I_H$.So we get the required outcome.

**Corollary 2.2.4.** *"Suppose G be bifunction which satisfy Assummption $\mathfrak{A}$ .Take $\mathfrak{g}^1 \in \mathcal{C}$ : $0 < \xi < 1$;$\{\gamma_n\} \subset (0, \infty)$ with $\liminf_{n\to\infty} \gamma_n > 0$ and consider the sequences*

$\{\mathfrak{g}^n\}, \{q^n\}, \{w^n\}$ *and* $\{u^n\}$ *defined by*

$$\begin{cases} u^n = T^{G}_{\gamma_n} x^n, \\ \mathfrak{g}^{n+1} = (1-\xi)\mathfrak{g}^n + \xi T(m) u^n, \end{cases} \tag{2.24}$$

*if* $\omega = \{\mathfrak{g}^* \in Sol(\mathcal{C}, \mathrm{F}) \cap Fix(T(m))\} \neq \emptyset$ *then* $\{\mathfrak{g}^n\}$ *and* $\{u^n\} \rightharpoonup p \in \omega$ *"*

**Proof**; When we put $H_1 = H_2 = H$, $\mathrm{F} = 0$ and $\mathrm{T}(s) = \mathsf{A} = I_H$ in theorem 2.2.1 so we get the required outcome.

**Theorem 2.2.5.** *"Let* $\mathfrak{g}^1 \in \mathcal{C}_1 = \mathcal{C}$*, consider the sequences* $\{t^n\}, \{u^n\}, \{\mathfrak{g}^n\}, \{q^n\} and \{w^n\}$ *generated as following*

$$\begin{cases} q^n = \arg\min\{\lambda_n \mathrm{F}(\mathfrak{g}^n, q) + \frac{1}{2}\|q - \mathfrak{g}^n\|^2 : q \in \mathcal{C}\}, \\ w^n = \arg\min\{\lambda_n \mathrm{F}(q^n, q) + \frac{1}{2}\|q - \mathfrak{g}^n\|^2 : q \in \mathcal{C}\}, \\ t^n = (1-\gamma) w^n + \gamma \frac{1}{s_t} \int_0^{s_t} T(s) w^n ds, \\ u^n = T^{G}_{\gamma_n} \mathsf{A} t^n, \\ s^n = P_{\mathcal{C}}(t^n + \xi \mathsf{A}^* (\frac{1}{m_t} \int_0^{m_t} T(m) u^n dm - \mathsf{A} t^n)), \\ \mathcal{C}_{n+1} = \{r \in \mathcal{C}_n : \|s^n - r\| \leq \|t^n - r\| \leq \|\mathfrak{g}^n - r\|\}, \\ \mathfrak{g}^{n+1} = P_{\mathcal{C}_{n+1}}(\mathfrak{g}^1), n \in \mathbb{N} \end{cases} \tag{2.25}$$

*where* $0 < \gamma < 1, 0 < \xi < \frac{1}{\|\mathsf{A}\|^2}, \{\gamma_n\} \subset (0, \infty)$ *with* $\liminf_{n\to\infty} \gamma_n > 0$*. Then under the assumptions of Theorem 2.2.1 and* $\omega = \{\mathfrak{g}^* \in Sol(\mathcal{C}, \mathrm{F}) \cap Fix(T(s)); \mathsf{A}\mathfrak{g}^* \in Sol(\mathcal{Q}, G) \cap Fix(T(m))\} \neq \emptyset$ *,the sequences* $\{w^n\}, \{\mathfrak{g}^n\}$ *converges strongly to an element* $p \in \omega$ *and* $\{u^n\}$ *converges strongly to* $\mathsf{A}p \in Sol(\mathcal{Q}, G) \cap Fix(T(m))$*."*

**Proof**; We separate the proof in several claims.

**Claim 1**: $\forall\, n \in \mathbb{N}$, $\mathcal{C}_n \neq \emptyset$,is closed convex set.

Let $\mathfrak{g}^* \in \omega$. Then it pursues from (2.3), (2.7) and (2.17) that

$$
\begin{aligned}
\|s^n - \mathfrak{g}^*\|^2 &\le \|t^n - \mathfrak{g}^*\|^2 - \xi(1-\xi\|\mathsf{A}\|^2)\|\frac{1}{m_t}\int_0^{m_t} \mathrm{T}(m)u^n dm - \mathsf{A}t^n\|^2 - \xi\|u^n - \mathsf{A}t^n\|^2 \\
&\le \|w^n - \mathfrak{g}^*\|^2 - \gamma(1-\gamma)\|\frac{1}{s_t}\int_0^{s_t} \mathsf{T(s)}dsw^n - w^n\|^2 \\
&- \xi(1-\xi\|\mathsf{A}\|^2)\|\frac{1}{m_t}\int_0^{m_t} \mathrm{T}(m)u^n dm - \mathsf{A}t^n\|^2 - \xi\|u^n - \mathsf{A}t^n\|^2. \qquad (2.26) \\
&\le \|\mathfrak{g}^n - \mathfrak{g}^*\|^2 - \gamma(1-\gamma)\|\frac{1}{s_t}\int_0^{s_t} \mathsf{T(s)}w^n ds - w^n\|^2 \\
&- \xi(1-\xi\|\mathsf{A}\|^2)\|\frac{1}{m_t}\int_0^{m_t} \mathrm{T}(m)u^n dm - \mathsf{A}t^n\|^2 - \xi\|u^n - \mathsf{A}t^n\|^2.
\end{aligned}
$$

since $0 < \gamma < 1, 0 < \xi < \frac{1}{\|\mathsf{A}\|^2}$ and by(2.26),we get

$$\|s^n - \mathfrak{g}^*\| \le \|t^n - \mathfrak{g}^*\| \le \|w^n - \mathfrak{g}^*\| \le \|\mathfrak{g}^n - \mathfrak{g}^*\|; \forall n. \qquad (2.27)$$

from (2.27) and $\mathfrak{g}^* \in \mathcal{C}_1$,we get by induction that $\mathfrak{g}^* \in \mathcal{C}_n$ for all $n \in \mathbb{N}$ i.e $\omega \subset \mathcal{C}_n$ for all n. Define

$$D_n = \{r \in \mathsf{H}_1; \|s^n - r\| \le \|t^n - r\| \le \|\mathfrak{g}^n - r\|\ ; n \in \mathbb{N}\}.$$

Then $\mathcal{C}_{n+1} = \mathcal{C}_n \cap D_n$. Since for all n, $\mathcal{C}_1\ and\ \mathcal{D}_n$ are closed so $\mathcal{C}_n$ is closed for all n.

Now to verify for all n, $D_n$ is convex. So we have to show that for $r^1, r^2 \in D_n$ and $\lambda \in [0,1]$ ; $\lambda r^1 + (1-\lambda)r^2 \in D_n$, so using Lemma 2.1.2,we have

$$
\begin{aligned}
\|s^n - (\lambda r^1 + (1-\lambda)r^2)\|^2 &= \|\lambda s^n - \lambda s^n + s^n - (\lambda r^1 + (1-\lambda)r^2)\|^2 \\
&= \|\lambda(s^n - r^1) + (1-\lambda)(s^n - r^2)\|^2 \\
&= \lambda\|s^n - r^1\|^2 + (1-\lambda)\|s^n - r^2\|^2 - \lambda(1-\lambda)\|r^1 - r^2\|^2 \\
&\le \lambda\|t^n - r^1\|^2 + (1-\lambda)\|t^n - r^2\|^2 - \lambda(1-\lambda)\|r^1 - r^2\|^2 \\
&= \|\lambda(t^n - r^1) + (1-\lambda)(t^n - r^2)\|^2 \\
&= \|\lambda t^n - \lambda t^n + t^n - (\lambda r^1 + (1-\lambda)r^2)\|^2 \\
&= \|t^n - (\lambda r^1 + (1-\lambda)r^2)\|^2.
\end{aligned}
$$

so

$$\|s^n - (\lambda r^1 + (1-\lambda)r^2\|^2 \le \|t^n - (\lambda r^1 + (1-\lambda)r^2)\|^2.$$

$$\Rightarrow \|s^n - (\lambda r^1 + (1-\lambda)r^2\| \le \|t^n - (\lambda r^1 + (1-\lambda)r^2)\|.$$

similarly

$$\|t^n - (\lambda r^1 + (1-\lambda)r^2) \le \|\mathfrak{g}^n - (\lambda r^1 + (1-\lambda)r^2\|.$$

Thus

$$\|s^n - (\lambda r^1 + (1-\lambda)r^2\| \le \|t^n - (\lambda r^1 + (1-\lambda)r^2)\| \le \|\mathfrak{g}^n - (\lambda r^1 + (1-\lambda)r^2\|.$$

Therefore

$$\lambda r^1 + (1-\lambda)r^2 \in D_n$$

since $\mathcal{C}_1$ and $D_n$ are convex $\forall\, n$, so $\mathcal{C}_n$ is convex $\forall\, n$.

**Claim 2**: Sequences $\{\mathfrak{g}^n\}, \{s^n\}$ are bounded and $\mathfrak{g}^n$ converges to $p$. By the definition of $\mathfrak{g}^{n+1}$,we have $\mathfrak{g}^{n+1} \in \mathcal{C}_{n+1} \subset \mathcal{C}_n$ and $\mathfrak{g}^n = P_{\mathcal{C}_n}(\mathfrak{g}^1)$,so we have by definition of metric projection

$$\|\mathfrak{g}^n - \mathfrak{g}^1\| \le \|\mathfrak{g}^{n+1} - \mathfrak{g}^1\|.$$

In addition ,$\mathfrak{g}^{n+1} = P_{\mathcal{C}_{n+1}}(\mathfrak{g}^1)$ and $\mathfrak{g}^* \in \mathcal{C}_{n+1}$.Hence,

$$\|\mathfrak{g}^n - \mathfrak{g}^1\| \le \|\mathfrak{g}^{n+1} - \mathfrak{g}^1\| \le \|\mathfrak{g}^* - \mathfrak{g}^1\|.$$

for all n. so $\lim_{n\to\infty} \|\mathfrak{g}^n - \mathfrak{g}^1\|$ and the sequence $\{\mathfrak{g}^n\}$ is bounded.Hence by (2.27),$\{s^n\}$ is also bounded. Furthermore, for all $d > n$,we have $\mathfrak{g}^d \in \mathcal{C}_d \subset \mathcal{C}_n, \mathfrak{g}^n = P_{\mathcal{C}_n}(\mathfrak{g}^1)$. Connecting this fact with Lemma 2.1.1, we get

$$\|\mathfrak{g}^d - \mathfrak{g}^n\|^2 \le \|\mathfrak{g}^d - \mathfrak{g}^1\|^2 - \|\mathfrak{g}^n - \mathfrak{g}^1\|^2.$$

Since $\lim_{n\to\infty} \|\mathfrak{g}^n - \mathfrak{g}^1\|$ exists,the sequence $\{\mathfrak{g}^n\}$ is a cauchy sequence. so for some $p \in \mathcal{C}$

$$\lim_{n\to\infty} \mathfrak{g}^n = p \tag{2.28}$$

**Claim 3**:For $p \in \omega$ ,we have $\lim_{n\to\infty} w^n = p$ and $lim_{n\to\infty}u^n = \mathsf{A}p$. By the definitions of $\mathcal{C}_{n+1}$ and $\mathfrak{g}^{n+1}$, we have

$$\|s^n - \mathfrak{g}^{n+1}\| \leq \|t^n - \mathfrak{g}^{n+1}\| \leq \|\mathfrak{g}^n - \mathfrak{g}^{n+1}\|.$$

so

$$\begin{aligned} \|s^n - \mathfrak{g}^n\| &\leq \|s^n - \mathfrak{g}^{n+1}\| + \|\mathfrak{g}^{n+1} - \mathfrak{g}^n\| \\ &\leq \|\mathfrak{g}^n - \mathfrak{g}^{n+1}\| + \|\mathfrak{g}^n - \mathfrak{g}^{n+1}\| \\ &= 2\|\mathfrak{g}^n - \mathfrak{g}^{n+1}\|. \end{aligned} \tag{2.29}$$

and

$$\begin{aligned} \|t^n - \mathfrak{g}^n\| &\leq \|t^n - \mathfrak{g}^{n+1}\| + \|\mathfrak{g}^{n+1} - \mathfrak{g}^n\| \\ &\leq \|\mathfrak{g}^n - \mathfrak{g}^{n+1}\| + \|\mathfrak{g}^n - \mathfrak{g}^{n+1}\| \\ &= 2\|\mathfrak{g}^n - \mathfrak{g}^{n+1}\|. \end{aligned} \tag{2.30}$$

Combining with (2.28), we have from (2.29) and (2.30) that

$$\lim_{n\to\infty} \|s^n - \mathfrak{g}^n\| = \lim_{n\to\infty} \|t^n - \mathfrak{g}^n\| = 0. \tag{2.31}$$

From(2.26),Claim.2 and (2.31),we can write

$$\gamma(1-\gamma)\|\frac{1}{s_t}\int_0^{s_t} \mathrm{T(s)}w^n ds - w^n\|^2 + \xi(1-\xi\|\mathsf{A}\|^2)\|\frac{1}{m_t}\int_0^{m_t} \mathrm{T}(m)u^n dm - \mathsf{A}t^n\|^2 + \xi\|u^n - \mathsf{A}t^n\|^2.$$

$$\begin{aligned} &\leq \|\mathfrak{g}^n - \mathfrak{g}^*\|^2 - \|s^n - \mathfrak{g}^*\|^2 \\ &= (\|\mathfrak{g}^n - \mathfrak{g}^*\| + \|s^n - \mathfrak{g}^*\|)(\|\mathfrak{g}^n - \mathfrak{g}^*\| - \|s^n - \mathfrak{g}^*\|) \\ &= (\|\mathfrak{g}^n - \mathfrak{g}^*\| + \|s^n - \mathfrak{g}^*\|)(\|\mathfrak{g}^n - \mathfrak{g}^* - s^n + \mathfrak{g}^*\|) \\ &\leq (\|\mathfrak{g}^n - \mathfrak{g}^*\| + \|s^n - \mathfrak{g}^*\|)(\|\mathfrak{g}^n - s^n\|) \to 0. \end{aligned} \tag{2.32}$$

as $n$ approaches to $\infty$

From $0 < \gamma < 1$ and $0 < \xi < \frac{1}{\|\mathsf{A}\|^2}$, we get from (2.32) that

$$\lim_{n\to\infty}\|\frac{1}{m_t}\int_0^{m_t}\mathrm{T}(m)u^n dm-\mathsf{A}t^n\|=\lim_{n\to\infty}\|u^n-\mathsf{A}t^n\|=0,\ \lim_{n\to\infty}\|\frac{1}{s_t}\int_0^{s_t}\text{T(s)}w^n ds-w^n\|=0. \tag{2.33}$$

so (2.33) and the inequality

$$\|\frac{1}{m_t}\int_0^{m_t}\mathrm{T}(m)u^n dm-u^n\|\leq\|\frac{1}{m_t}\int_0^{m_t}\mathrm{T}(m)u^n dm-\mathsf{A}t^n\|+\|u^n-\mathsf{A}t^n\|.$$

which imply that

$$\lim_{n\to\infty}\|\frac{1}{m_t}\int_0^{m_t}\mathrm{T}(m)u^n dm-u^n\|=0. \tag{2.34}$$

Then again, from (2.15 ),(2.16 ),(2.19 ),and $\lim_{n\to\infty}\mathfrak{g}^n=p$, we have

$$\lim_{n\to\infty}q^n=p,\ \lim_{n\to\infty}w^n=p,\ \lim_{n\to\infty}t^n=p. \tag{2.35}$$

Finally, it follows from (2.33) and (2.35) that

$$\begin{aligned}\|\frac{1}{s_t}\int_0^{s_t}\text{T(s)}p ds-p\| &\leq\|\frac{1}{s_t}\int_0^{s_t}\text{T(s)}p ds-\frac{1}{s_t}\int_0^{s_t}\text{T(s)}w^n ds\|\\ &+\|\frac{1}{s_t}\int_0^{s_t}\text{T(s)}w^n ds-w^n\|+\|w^n-p\|\\ &\leq\|p-w^n\|+\|\frac{1}{s_t}\int_0^{s_t}\text{T(s)}w^n ds-w^n\|+\|w^n-p\|\\ &=2\|w^n-p\|+\|\frac{1}{s_t}\int_0^{s_t}\text{T(s)}w^n ds-w^n\|\to 0.\end{aligned}$$

as $n\to\infty$

so, T(s)$p=p$ i.e., $p\in Fix$(T(s))

On the other side , we obtain from Lemma 2.1.7 that.

$$\lambda_n[\mathrm{F}(\mathfrak{g}^n,q)-\mathrm{F}(\mathfrak{g}^n,q^n)]\geq\langle q^n-\mathfrak{g}^n,q^n-q\rangle\forall q\in\mathcal{C}.$$

letting $n\to\infty$,using the joint weak continuity of F, $\lim_{n\to\infty}\mathfrak{g}^n=p$ and $(2.35)$, we get in the limit that for all $q\in\mathcal{C}$

$$\mathrm{F}(p,q)\geq 0$$

It is prompt that $p\in Sol(\mathcal{C},\mathrm{F})$. Thusly,

$$p\in Sol(\mathcal{C},\mathrm{F})\cap Fix(\text{T(s)}). \tag{2.36}$$

by (2.35) it follows that $\lim_{n\to\infty} \mathsf{A}t^n = \mathsf{A}p$.

joining this reality with (2.33), one has

$$\lim_{n\to\infty} u^n = \mathsf{A}p. \tag{2.37}$$

from (2.34),(2.37), one can deduce

$$\begin{aligned}
\|\frac{1}{m_t}\int_0^{m_t} \mathrm{T}(m)\mathsf{A}p dm - \mathsf{A}p\| &\leq \|\frac{1}{m_t}\int_0^{m_t} \mathrm{T}(m)\mathsf{A}p dm - \frac{1}{m_t}\int_0^{m_t} \mathrm{T}(m)u^n dm\| \\
&+ \|\frac{1}{m_t}\int_0^{m_t} \mathrm{T}(m)u^n dm - u^n\| + \|u^n - \mathsf{A}p\| \\
&\leq \|\mathsf{A}p - u^n\| + \|\frac{1}{m_t}\int_0^{m_t} \mathrm{T}(m)u^n dm - u^n\| + \|u^n - \mathsf{A}p\| \\
&= 2\|u^n - \mathsf{A}p\| + \|\frac{1}{m_t}\int_0^{m_t} \mathrm{T}(m)u^n dm - u^n\| \to 0.
\end{aligned}$$

as $n \to \infty$. Hence T(m) Ap=Ap,i.e.,A$p \in Fix(\mathrm{T}(m))$.

Furthermore , we get from Lemma 2.1.6, $\lim_{n\to\infty} \mathsf{A}t^n = \mathsf{A}p$ and (2.33) that

$$\begin{aligned}
\|T_\gamma^{\mathrm{G}}\mathsf{A}p - \mathsf{A}p\| &\leq \|T_\gamma^{\mathrm{G}}\mathsf{A}p - T_{\gamma_n}^{\mathrm{G}}\mathsf{A}t^n\| + \|T_{\gamma_n}^{\mathrm{G}}\mathsf{A}t^n - \mathsf{A}t^n\| + \|\mathsf{A}t^n - \mathsf{A}p\| \\
&= \|T_\gamma^{\mathrm{G}}\mathsf{A}p - T_{\gamma_n}^{\mathrm{G}}\mathsf{A}t^n\| + \|u^n - \mathsf{A}t^n\| + \|\mathsf{A}t^n - \mathsf{A}p\| \\
&\leq \|\mathsf{A}t^n - \mathsf{A}p\| + \frac{|\gamma_n - \gamma|}{\gamma_n}\|T_{\gamma_n}^{\mathrm{G}}\mathsf{A}t^n - \mathsf{A}t^n\| + \|u^n - \mathsf{A}t^n\| + \|\mathsf{A}t^n - \mathsf{A}p\| \\
&= 2\|\mathsf{A}t^n - \mathsf{A}p\| + \frac{|\gamma_n - \gamma|}{\gamma_n}\|u^n - \mathsf{A}t^n\| + \|u^n - \mathsf{A}t^n\| \to 0.
\end{aligned}$$

as $n \to \infty$ which implies that $T_\gamma^{\mathrm{G}}\mathsf{A}p = \mathsf{A}p$,i.e.,$\mathsf{A}p \in Fix(T_\gamma^{\mathrm{G}}) = Sol(\mathcal{Q}, \mathrm{G})$.So,

$$\mathsf{A}p \in Sol(\mathcal{Q}, \mathrm{G}) \cap Fix(\mathrm{T}(m)). \tag{2.38}$$

From (2.36) and (2.37), we get $p \in \omega$.

Consequently the proof is finished.

## 2.3 Consequently Results

**Corollary 2.3.1.** *"let* F *and G are two bifunctions satisfying assumptions* $\mathfrak{B}$ *and* $\mathfrak{A}$*,respectively. Let* $\mathfrak{g}^1 \in \mathcal{C}_1 = \mathcal{C}$*,consider the sequences* $\{t^n\}, \{u^n\}, \{\mathfrak{g}^n\}, \{q^n\} and \{w^n\}$ *generated as* *following*

$$\begin{cases} q^n = \arg\min\{\lambda_n \mathrm{F}(\mathfrak{g}^n, q) + \frac{1}{2}\|q - \mathfrak{g}^n\|^2 : q \in \mathcal{C}\}, \\ w^n = \arg\min\{\lambda_n \mathrm{F}(q^n, q) + \frac{1}{2}\|q - \mathfrak{g}^n\|^2 : q \in \mathcal{C}\}, \\ u^n = T^{\mathrm{G}}_{\gamma_n} \mathsf{A} z^n, \\ s^n = P_{\mathcal{C}}(w^n + \xi \mathsf{A}^*(u^n - \mathsf{A} z^n)), \\ \mathcal{C}_{n+1} = \{r \in \mathcal{C}_n : \|s^n - r\| \le \|w^n - r\| \le \|\mathfrak{g}^n - r\|\}, \\ \mathfrak{g}^{n+1} = P_{\mathcal{C}_{n+1}}(\mathfrak{g}^1), n \in \mathbb{N}. \end{cases} \tag{2.39}$$

*where* $0 < \gamma < 1, 0 < \xi < \frac{1}{\|\mathsf{A}\|^2}, \{\gamma_n\} \subset (0, \infty)$ *with* $\liminf_{n\to\infty} \gamma_n > 0$. *Then under the assumptions of Theorem 2.2.5 and* $\omega = \{\mathfrak{g}^* \in Sol(\mathcal{C}, \mathrm{F}); \mathsf{A}\mathfrak{g}^* \in Sol(\mathcal{Q}, G)\} \neq \emptyset$ *,the sequences* $\{w^n\}, \{\mathfrak{g}^n\}$ *converges strongly to an element* $p \in \omega$ *and* $\{u^n\}$ *converges strongly to* $\mathsf{A}p \in Sol(\mathcal{Q}, G)$ *"*

**Proof**; If we take in theorem 2.2.5 T$(m)$ = $I_{H_2}$ and T(s) = $I_{H_1}$ we will get immediate result.

# Chapter 3

# Convergence Analysis of Hybrid Shrinking Algorithm

In this part, we contemplate the convergence characteristics of an iterative algorithm, with the help of the Hybrid Shrinking method, in Hilbert spaces. The proposed algorithm provides an optimal solution in intersection of solutions sets of the SEP, the VIP and the FPP associated with the k-strict pseudo-contraction mappings in such setting. We expect to set up strong convergence results of the proposed scheme by employing reasonable conditions on the control sequences of parameters. As a consequence, these results can be considered as an improvement and generalization of various results in this field of study.
Class of strict pseudo-constrictions is conspicuous between different classes of nonlinear maps and has many applications, in particular, to solve inverse problems [45].
However, the odds in the development of iterative algorithms for strict pseudo-contractions is the second term in its definition. Moreover, various existing iterative algorithms involving k -strict pseudo-contractions exhibit weak convergence towards the required solution. Some successful attempts have recently been made to introduce new iterative algorithms which possess strong convergence characteristics. One such iterative algorithm is due to Takahashi et al [47] , which ensures the strong convergence by the shrinking effect of a sequence of closed convex sets $\{\mathcal{C}_{n+1}\}$. so it is, normal to propose and examine iterative algorithms for the development of fixed points of k-strict pseudo-contractions and built up results of strong convergence under reasonable conditions..

Censor and Elfving [11],in 1994, proposed the theory of SFP where as in 2012, Censor et al. [15] , introduced the theory of SVIP. This hypothesis gives a unified framework to address an assortment of inverse problems which produced from phase retrievals, sensor networks, computerized tomography and data compression ([8] ,[17]). It is worth mentioning that this theory has been successfully employed as a model in power balanced radiation treatment arranging and medicinal picture remaking,, see, for example,([12],[13]) the references refered to in that.

In 2011, Moudafi [43] presented the idea of split monotone variational inclusions problems(SMVIP) which includes, as a special case, SVIP, SFP, SFPP, split zeroes problem and SEP. Enlivened and inspired by the work of Censor and Elfving [11] , Censor et al. [15] and Moudafi [43] , introduced the concept of split generalized equilibrium problem (SGEP).
Let $\mathcal{C} \subset \mathsf{H}_1$,and $\mathrm{Q} \subset \mathsf{H}_2$ are closed and convex subset and let $\mathsf{A} : \mathsf{H}_1 \to \mathsf{H}_2$ is (BLO). "Suppose $\mathrm{F}, h_1 : \mathcal{C} \times \mathcal{C} \to \mathbb{R}$ and $\mathrm{G}, h_2 : \mathrm{Q} \times \mathrm{Q} \to \mathbb{R}$ be four bifunctions. Recall that (SGEP) is to find:

$$l^* \in \mathcal{C} \quad \text{such that } \mathrm{F}\left(l^*, l\right) + h_1\left(l^*, l\right) \geq 0 \text{ for all } l \in \mathcal{C}, (1.1)$$

and

$$q^* = \mathsf{A}l^* \in \mathrm{Q} \quad \text{such that } \mathrm{G}\left(q^*, q\right) + h_2\left(l^*, l\right) \geq 0 \text{ for all } q \in \mathrm{Q}.(1.2)$$

Note that the inequality (1.1) represents the generalized equilibrium problem whose solution set is denoted as $GEP(\mathrm{F}, h_1)$, where as the solution set of SGEP (1.1) and (1.2) is denoted as $\Omega = \{l \in GEP(\mathrm{F}, h_1) : \mathsf{A}l^* \in GEP(\mathrm{G}, h_2)\}$. It is remarked that if:
(i): if $h_1 = 0 = h_2$, then (1.1) and (1.2) becomes the SEP studied by Moudafi [43];
(ii): if $h_2 = 0 = \mathrm{G}$, then (1.1) and (1.2) becomes the equilibrium problem studied by Cianciaruso et al. [24];
(iii): if $h_1 = h_2 = \mathrm{G} = 0$, then $\mathrm{F}\left(l^*, l\right) \geq 0$ represents the classical equilibrium problem. "
In 2014, Deepho et al.[31] , studied a Hybrid Shrinking projection algorithm for approximate common solution of SGEP and FPP in Hilbert spaces.Previously, Sitthithakerngkiet et al. [34] proposed and analyzed an iterative algorithm for SGEP and FPP in Hilbert spaces.

′ Roused by the works of Moudafi [43] , Kazmi [33], Deepho et al. [31] and Sitthithakerngkiet et al. [34] , we make an iterative algorithm for the development of a common element in common fixed-points set of an infinite family of k-strict pseudo-contractions and the set of common solutions of family of SGEP in Hilbert spaces. We build up strong convergence of the proposed algorithm under mellow conditions on control arrangements of parameters and subsequently refine and enhance different outcomes reported in the present writing.′

## 3.1 Preliminaries

In this segment, we review a few lemmas and remarks which are required in the continuation. Suppose we write $\mathfrak{g}^n \to \mathfrak{g}$ (resp. $\mathfrak{g}^n \rightharpoonup \mathfrak{g}$) to indicate the strong convergence and the weak convergence respectively of a sequence $\{\mathfrak{g}^n\}_{n=1}^{\infty}$.

**Lemma 3.1.** "Suppose $\mathsf{H}_1$ be a (real) Hilbert space, then:
(1): $\forall\ \mathfrak{g}, q \in \mathsf{H}_1$,we have $\|\mathfrak{g}-q\|^2 = \|\mathfrak{g}\|^2 - \|q\|^2 - 2\langle \mathfrak{g}-q, q\rangle$ ;
(2): $\|\mathfrak{g}+q\|^2 \leq \|\mathfrak{g}\|^2 + 2\langle q, \mathfrak{g}+q\rangle, \forall\ \mathfrak{g}, q \in \mathsf{H}_1$;
(3): $2\langle \mathfrak{g}-q, u-v\rangle = \|\mathfrak{g}-v\|^2 + \|q-u\|^2 - \|\mathfrak{g}-u\|^2 - \|q-v\|^2, \forall\ \mathfrak{g}, q, u, v \in \mathsf{H}_1$;
(4): $\|\alpha\mathfrak{g}+(1-\alpha)q\|^2 = \alpha\|\mathfrak{g}\|^2 + (1-\alpha)\|q\|^2 - \alpha(1-\alpha)\|\mathfrak{g}-q\|^2\ \forall\ \mathfrak{g}, q \in \mathsf{H}_1$ and $\alpha \in [0,1]$."

**Lemma 3.2** ([41]). "Suppose $\mathcal{C} \subset \mathsf{H}$. If $S : \mathcal{C} \to H$ is a $k$-strict pseudo-contraction, then we have (FP) set F$(S)$ is closed and convex so the projection $P_{F(S)}$ is well defined."

**Lemma 3.3**([32])."Let $\mathcal{C} \subset \mathsf{H}_1$.Let $\{S_n\}_{n=1}^{\infty}$ be a infinite family of nonexpansive mappings of $\mathcal{C}$ into itself with $\bigcap_{n=1}^{\infty} F(S_n) \neq \emptyset$ .Let $S$ be the S-mapping generated by $S_N, S_{N-1}$.... and $\lambda_N, \lambda_{N-1}$..... then $F(S) = \bigcap_{n=1}^{\infty} F(S_n)$ ."

**Lemma 3.4** ([41]) ."Let for any real Hilbert space $\mathsf{H}_1$ ,we have a closed convex subset $\mathcal{C}$, and a self-map of $\mathcal{C}$, $S : \mathcal{C} \to \mathcal{C}$ .If we have $S$ as k-strict pseudo-contraction mapping then $S$ satisfy the Lipchitz conditions $\forall \mathfrak{g}, q \in \mathcal{C}$

$$||S(\mathfrak{g}) - S(q)|| = \frac{1+k}{1-k}||\mathfrak{g} - q||$$

"

**Remark**: For each $n \in \mathbb{N}$, $S_n$ is nonexpansive and $\lim_{n\to\infty} sup_{\mathfrak{g}\in E}||S_n\mathfrak{g} - S|| = 0$ where $E \subset \mathcal{C}$,where $E$ is bounded.

**Lemma 3.5** ([41])."Let $S : \mathcal{C} \to \mathcal{C}$ be a k-strict contraction and $\mathcal{C} \subset H$ . Then $(\mathbb{I} - S)$ is demiclosed, that is, if $\{\mathfrak{g}^n\} \in \mathcal{C}$ with $\mathfrak{g}^n \rightharpoonup \mathfrak{g}$ and $\mathfrak{g}^n - S\mathfrak{g}^n \to 0$, then $\mathfrak{g} \in \mathbb{F}(S)$."

**Lemma 3.6** ."Let $\{\mathcal{C}_n\} \in \mathsf{H}_1$ where $\{\mathcal{C}_n\}$ is closed and convex. If $\mathcal{C}_0 = \bigcap_{n=1}^{\infty} \mathcal{C}_n \neq \emptyset$ , then $\{P_{\mathcal{C}_n}\mathfrak{g}\} \rightharpoonup P_{\mathcal{C}_0}\mathfrak{g}$ for each $\mathfrak{g} \in \mathcal{C}$.

To solve an EP, the bifunction $F$ must satisfy certain conditions as summarized in the following lemma (c.f. [4] and [19])"

**Lemma 3.7**."Let F is same as given in Assumption $\mathfrak{B}$

let $h_1 : \mathcal{C} \times \mathcal{C} \to \mathbb{R}$ is a bifunction satisfying the accompanying conditions:

(1). $h_1(\mathfrak{g}, \mathfrak{g}) = 0$ for all $\mathfrak{g} \in \mathcal{C}$;

(2). for each $q \in \mathcal{C}$ fixed , the function $l \to h_1(\mathfrak{g}, q)$ is upper semicontinuous;

(3). for each $\mathfrak{g} \in \mathcal{C}$ fixed , the function $q \to h_1(\mathfrak{g}, q)$ is convex and lower semi-continuous, and assume that for fixed $r > 0$ and $w \in \mathcal{C}$, there exists a non empty compact convex subset $K$ of $\mathsf{H}_1$ and $\mathfrak{g} \in \mathcal{C} \cap K$ such that

$$\mathrm{F}(q, \mathfrak{g}) + h_1(q, \mathfrak{g}) + \frac{1}{r}\langle q - \mathfrak{g}, \mathfrak{g} - w\rangle < 0, \text{ for all } q \in \mathcal{C}.$$

"

**Lemma 3.8** ([19])."Let $\mathcal{C} \subset \mathsf{H}_1$ and let $\mathrm{F} : \mathcal{C} \times \mathcal{C} \to \mathbb{R}$ be a bifunction satisfying Lemma 3.7. For $r > 0$ and $\mathfrak{g} \in \mathsf{H}_1$, there exists $w \in \mathcal{C}$ such that

$$\mathrm{F}(w, q) + \frac{1}{r}\langle q - w, w - \mathfrak{g}\rangle \geq 0, \text{ for all } q \in \mathcal{C}.$$

Moreover, define a mapping $\mathsf{T}_r^{\mathrm{F}} : \mathsf{H}_1 \to \mathcal{C}$ same as in Lemma 2.1.5."

**Remark** "It is remarked that if $G : \mathrm{Q} \times \mathrm{Q} \to \mathbb{R}$ be a bifunction satisfying Lemma 3.7, then for $s > 0$ and $w \in \mathsf{H}_2$ we can define a mapping:

$$\mathsf{T}_s^F(\mathfrak{g}) = \left\{ d \in \mathcal{C} : G(d, e) + \frac{1}{s}\langle e - d, d - w\rangle \geq 0, \text{ for all } e \in \mathrm{Q} \right\},$$

which is, nonempty, single-valued and firmly nonexpansive. Moreover, $Sol(\mathcal{Q}, G)$ is closed and convex, and $F(\mathsf{T}_s^G) = Sol(\mathcal{Q}, G)$."

**Lemma 3.9**([10])."For given $\mathfrak{g}^*, q^* \in \mathcal{C}$,$(\mathfrak{g}^*, q^*)$ is solution of general system of VIP if and only if $\mathfrak{g}^*$ is fixed-point of $G^{\sim} : \mathcal{C} \to \mathcal{C}$ defined by

$$G^{\sim}(\mathfrak{g}) = P_{\mathcal{C}}[P_{\mathcal{C}}(\mathfrak{g} - \lambda_2 B_2 \mathfrak{g}) - \lambda_1 B_1 P_{\mathcal{C}}(\mathfrak{g} - \lambda_2 B_2 \mathfrak{g})]$$

for all $\mathfrak{g} \in \mathcal{C}$ here $q^* = P_{\mathcal{C}}(\mathfrak{g} - \lambda_2 B_2 \mathfrak{g})$, also here $\lambda_1, \lambda_2$ are positive constants and $B_1, B_2 : \mathcal{C} \to \mathsf{H}_1$ are two maps."

**Lemma 3.10**. "Let $G^{\sim} : \mathcal{C} \to \mathcal{C}$ defined by

$$G^{\sim}(\mathfrak{g}) = P_{\mathcal{C}}[P_{\mathcal{C}}(\mathfrak{g} - \lambda_2 B_2 \mathfrak{g}) - \lambda_1 B_1 P_{\mathcal{C}}(\mathfrak{g} - \lambda_2 B_2 \mathfrak{g})]$$

for all $\mathfrak{g} \in \mathcal{C}$ here $q^* = P_{\mathcal{C}}(\mathfrak{g} - \lambda_2 B_2 \mathfrak{g})$, also here $\lambda_1, \lambda_2$ are positive constants and if $B_1, B_2 : \mathcal{C} \to \mathsf{H}_1$ be $\beta_1, \beta_2$ inverse-strongly monotones and $\lambda_1 \in (0, 2\beta_1)$,$\lambda_2 \in (0, 2\beta_2)$ respectively then $G^{\sim}$ is nonexpansive mapping. "

## 3.2 Main Results

Now we will prove our main result of this section.

**Theorem 3.2.1.** *"Let $H_1$ and $H_2$ be two (real) Hilbert spaces and suppose $\emptyset \neq \mathcal{C} \subset H_1$ and $\emptyset \neq \mathrm{Q} \subset H_2$ where both the subsets are closed and convex. Let $B_1 : \mathcal{C} \to H_1$ is $\beta_1$ inverse strongly monotone map ,similarly,$B_2 : \mathcal{C} \to H_2$ is $\beta_2$ strongly monotone map. Let $A : H_1 \to H_2$ be a BLO. Let $\mathrm{F}, h_1 : \mathcal{C} \times \mathcal{C} \to R$ and $\mathrm{G}, h_2 : \mathrm{Q} \times \mathrm{Q} \to R$ satisfying Lemma 3.7; $h_1$, $h_2$ are monotone and $\mathrm{G}$ is upper semicontinuous and $\{S_n\}_{n=1}^{\infty}$ be a sequence of infinite family of $k^n$-strict pseudo-contractions from $\mathcal{C}$ into itself with $k = \sup^n k^n$, For every $n \in N$ ,$S_n$ and $S$ are S-mappings produced by $S_n, ..., S_1$ and $\lambda_n, \lambda_{n-1}...., \lambda_1$ and $S_n, S_{n-1}...$ and $\lambda_n, \lambda_{n-1}....$ respectively.Likewise $S$ be a mapping of $\mathcal{C}$ into itself characterized by $S\mathfrak{g} = \lim_{n\to\infty} S_n\mathfrak{g}$ such that*

$$\Delta := \cap_{n=1}^{\infty} Fix(S_n) \cap \Omega \cap GVI(\mathcal{C}, B_1, B_2) \neq \emptyset \tag{3.1}$$

*For a given $\mathfrak{g}^0 \in H_1, \mathcal{C}^1 = \mathcal{C}, \mathfrak{g}^1 = P_{\mathcal{C}}\mathfrak{g}^0, u^n \in \mathcal{C}$, let the iterative sequences $w^n$,$\mathfrak{g}^n$,$u^n$, and $q^n$ be generated by*

$$\begin{cases} u^n = T_{r^n}^{(\mathrm{F},h_1)}(\mathfrak{g}^n + \eta A^*(T_{r^n}^{(\mathrm{G},h_2)} - I)A\mathfrak{g}^n), \\ w^n = P_{\mathcal{C}}(u^n - \lambda_2 B_2 u^n), \\ q^n = \alpha^n \mathfrak{g}^n + (1-\alpha^n) S_n P_{\mathcal{C}}(w^n - \lambda_1 B_1 w^n), \\ \mathcal{C}_{n+1} = w \in \mathcal{C}_n : \|q^n - w\| \leq \|\mathfrak{g}^n - w\|, \\ \mathfrak{g}^{n+1} = P_{\mathcal{C}_{n+1}}\mathfrak{g}^0, \forall n \in N, \end{cases} \tag{3.2}$$

*where $\alpha_n \in (0,1), \lambda^1 \in [a_1, b_1] \subset (0, 2\beta_1), \lambda^2 \in [a_2, b_2] \subset (0, 2\beta_2), r^n \subset (0, \infty)$ and $\eta \in (0, \frac{1}{L}), L$ is the spectral radius of the operator $A^*A$ and $A^*$ is the adjoint of $A$ fulfilling the accompanying conditions:*

**C1** $0 < a_1 \leq \lambda_1 \leq b_1 < 2\beta_1$;

**C2** $0 < a_2 \leq \lambda_2 \leq b_2 < 2\beta_2$;

**C3** $\liminf_{n\to\infty} r^n > 0.$

*Then, the sequence* $\mathfrak{g}^n \to P_\Delta \mathfrak{g}^0$ *and* $(\mathfrak{g}^*, q^*)$ *is a solution of a GVIP, where* $q^* = P_\mathcal{C}(\mathfrak{g}^* - \lambda_2 B_2 \mathfrak{g}^*)$ ”

**Proof:** Its proof can be devided into six steps.

**Step 1.** We start our proof to establish that $\mathfrak{g}^n$ is well-defined and solution space $\mathcal{C}_n$ for any $n \in N$ is closed and convex . From the supposition, we note that $\mathcal{C}_1 = \mathcal{C}$ is closed and convex. Assume ,for some $k \geq 1$, $\mathcal{C}_n$ is closed and convex . Presently, we will demonstrate that for some $k$, $C_{n+1}$ is closed and convex . For any $\mathfrak{g}^* \in \mathcal{C}_n$, we get

$$\begin{aligned}
&\|q^n - \mathfrak{g}^*\| \leq \|\mathfrak{g}^n - \mathfrak{g}^*\| \\
&\Leftrightarrow \|q^n - \mathfrak{g}^*\|^2 \leq \|\mathfrak{g}^n - \mathfrak{g}^*\|^2 \\
&\Leftrightarrow \|q^n - \mathfrak{g}^n + \mathfrak{g}^n - \mathfrak{g}^*\|^2 \leq \|\mathfrak{g}^n - \mathfrak{g}^*\|^2 \\
&\Leftrightarrow \|q^n - \mathfrak{g}^*\|^2 + 2(q^n - \mathfrak{g}^n, \mathfrak{g}^n - \mathfrak{g}^*) + \|\mathfrak{g}^n - \mathfrak{g}^*\|^2 \leq \|\mathfrak{g}^n - \mathfrak{g}^*\|^2
\end{aligned}$$

This implies that $\|q^n - \mathfrak{g}^*\| \leq \|\mathfrak{g}^n - \mathfrak{g}^*\|$ is equivalent to $\|q^n - \mathfrak{g}^*\|^2 + 2(q^n - \mathfrak{g}^n, \mathfrak{g}^n - \mathfrak{g}^*) \leq 0$. Thus, $C_{n+1}$ is closed and convex. Then,for any $n \in N$, $\mathcal{C}_n$ is closed and convex .This suggests $\mathfrak{g}^n$ is well defined.

**Step 2.** We will demonstrate that $\Delta \subset \mathcal{C}_n$, for all $n \in N$ by mathematical induction. Since $\mathfrak{g}^* \in \Delta$, we get $\mathfrak{g}^* = S_n \mathfrak{g}^* = P_\mathcal{C}[P_\mathcal{C}(\mathfrak{g}^* - \lambda_2 B_2 \mathfrak{g}^*) - \lambda_1 B_1 P_\mathcal{C}(\mathfrak{g}^* - \lambda_2 B_2 \mathfrak{g}^*)] = \mathrm{T}_{r^n}^{(\mathrm{F},h_1)} \mathfrak{g}^*$ and $\mathsf{A}\mathfrak{g}^* = \mathrm{T}_{r^n}^{(\mathrm{G},h_2)} \mathsf{A}\mathfrak{g}^*$. From the supposition, we see that $\Delta \subset \mathcal{C} = \mathcal{C}_1$. Assume that $\Delta \subset \mathcal{C}_n$, for some $k \geq 1$.

Put $q^* = P_\mathcal{C}(\mathfrak{g}^* - \lambda_2 B_2 \mathfrak{g}^*)$ and $\upsilon^n = P_\mathcal{C}(w^n - \lambda_1 B_1 w^n)$. Then $\mathfrak{g}^* = P_\mathcal{C}(q^* - \lambda_1 B_1 q^*)$ and

$$
\begin{aligned}
\|\upsilon^n - \mathfrak{g}^*\| &= \|P_{\mathcal{C}}(w^n - \lambda_1 B_1 w^n) - P_{\mathcal{C}}(q^* - \lambda_1 B_1 q^*)\| \\
&\leq \|(w^n - \lambda_1 B_1 w^n) - (q^* - \lambda_1 B_1 q^*)\| \\
&= \|(I - \lambda_1 B_1) w^n - (I - \lambda_1 B_1) q^*\| \\
&\leq \|w^n - q^*\| \\
&= \|P_{\mathcal{C}}(u^n - \lambda_2 B_2 u^n) - P_{\mathcal{C}}(\mathfrak{g}^* - \lambda_2 B_2 \mathfrak{g}^*)\| \\
&\leq \|(u^n - \lambda_2 B_2 u^n) - (\mathfrak{g}^* - \lambda_2 B_2 \mathfrak{g}^*)\| \\
&= \|(I - \lambda_2 B_2) u^n - (I - \lambda_2 B_2)\mathfrak{g}^*\|
\end{aligned}
$$

As $(I - \lambda_2 B_2)$ is non-expansive so

$$
\|\upsilon^n - \mathfrak{g}^*\| \leq \|u^n - \mathfrak{g}^*\| \tag{3.3}
$$

Now since $\mathfrak{g}^* \in \Delta$, i.e., $\mathfrak{g}^* \in \Omega$, and we have $\mathfrak{g}^* = \mathsf{T}_{r^n}^{(\mathrm{F},h_1)}\mathfrak{g}^*$ and $\mathsf{A}\mathfrak{g}^* = \mathsf{T}_{r^n}^{(\mathrm{G},h_2)}\mathsf{A}\mathfrak{g}^*$. We estimate

$$
\begin{aligned}
\|u^n - \mathfrak{g}^*\|^2 &= \|\mathsf{T}_{r^n}^{(\mathrm{F},h_1)}(\mathfrak{g}^n + \eta \mathsf{A}^*(\mathsf{T}_{r^n}^{(\mathrm{G},h_2)} - I)\mathsf{A}\mathfrak{g}^n) - \mathfrak{g}^*\|^2 \\
&= \|\mathsf{T}_{r^n}^{(\mathrm{F},h_1)}(\mathfrak{g}^n + \eta \mathsf{A}^*(\mathsf{T}_{r^n}^{(\mathrm{G},h_2)} - I)\mathsf{A}\mathfrak{g}^n) - \mathsf{T}_{r^n}^{(\mathrm{F},h_1)}\mathfrak{g}^*\|^2 \\
&\leq \|\mathfrak{g}^n + \eta \mathsf{A}^*(\mathsf{T}_{r^n}^{(\mathrm{G},h_2)} - I)\mathsf{A}\mathfrak{g}^n - \mathfrak{g}^*\|^2 \\
&\leq \|\mathfrak{g}^n - \mathfrak{g}^*\|^2 + \eta^2 \|\mathsf{A}^*(\mathsf{T}_{r^n}^{(\mathrm{G},h_2)} - I)\mathsf{A}\mathfrak{g}^n\|^2 \\
&+ 2\eta(\mathfrak{g}^n - \mathfrak{g}^*, \mathsf{A}^*(\mathsf{T}_{r^n}^{(\mathrm{G},h_2)} - I)\mathsf{A}\mathfrak{g}^n).
\end{aligned} \tag{3.4}
$$

Thus, we have

$$
\begin{aligned}
\|u^n - \mathfrak{g}^*\|^2 \leq \|\mathfrak{g}^n - \mathfrak{g}^*\|^2 &+ \eta^2 \langle (\mathsf{T}_{r^n}^{(\mathrm{G},h_2)} - I)\mathsf{A}\mathfrak{g}^n, \mathsf{A}\mathsf{A}^*(\mathsf{T}_{r^n}^{(\mathrm{G},h_2)} - I)\mathsf{A}\mathfrak{g}^n\rangle \\
&+ 2\eta\langle \mathfrak{g}^n - \mathfrak{g}^*, \mathsf{A}^*(\mathsf{T}_{r^n}^{(\mathrm{G},h_2)} - I)\mathsf{A}\mathfrak{g}^n\rangle.
\end{aligned} \tag{3.5}
$$

Now, we have

$$
\begin{aligned}
&\eta^2 \langle (\mathsf{T}_{r^n}^{(\mathrm{G},h_2)} - I)\mathsf{A}\mathfrak{g}^n, \mathsf{A}\mathsf{A}^*(\mathsf{T}_{r^n}^{(\mathrm{G},h_2)} - I)\mathsf{A}\mathfrak{g}^n\rangle \\
&\leq L\eta^2 \langle (\mathsf{T}_{r^n}^{(\mathrm{G},h_2)} - I)\mathsf{A}\mathfrak{g}^n, (\mathsf{T}_{r^n}^{(\mathrm{G},h_2)} - I)\mathsf{A}\mathfrak{g}^n\rangle \\
&= L\eta^2 \|(\mathsf{T}_{r^n}^{(\mathrm{G},h_2)} - I)\mathsf{A}\mathfrak{g}^n\|^2.
\end{aligned} \tag{3.6}
$$

Denoting $\Lambda := 2\eta\langle \mathfrak{g}^n - \mathfrak{g}^*, \mathsf{A}^*(\mathrm{T}_{r^n}^{(\mathrm{G},h_2)} - I)\mathsf{A}\mathfrak{g}^n\rangle$ , we have

$$\begin{aligned}
\Lambda &= 2\eta\langle \mathfrak{g}^n - \mathfrak{g}^*, \mathsf{A}^*(\mathrm{T}_{r^n}^{(\mathrm{G},h_2)} - I)\mathsf{A}\mathfrak{g}^n\rangle \\
&= 2\eta\langle \mathsf{A}(\mathfrak{g}^n - \mathfrak{g}^*), (\mathrm{T}_{r^n}^{(\mathrm{G},h_2)} - I)\mathsf{A}\mathfrak{g}^n\rangle \\
&= 2\eta\langle \mathsf{A}(\mathfrak{g}^n - \mathfrak{g}^*) + (\mathrm{T}_{r^n}^{(\mathrm{G},h_2)} - I)\mathsf{A}\mathfrak{g}^n - (\mathrm{T}_{r^n}^{(\mathrm{G},h_2)} - I)\mathsf{A}\mathfrak{g}^n, (\mathrm{T}_{r^n}^{(\mathrm{G},h_2)} - I)\mathsf{A}\mathfrak{g}^n\rangle \\
&= 2\eta\{\langle \mathrm{T}_{r^n}^{(\mathrm{G},h_2)}\mathsf{A}\mathfrak{g}^n - \mathsf{A}\mathfrak{g}^*, (\mathrm{T}_{r^n}^{(\mathrm{G},h_2)} - I)\mathsf{A}\mathfrak{g}^n\rangle - \|(\mathrm{T}_{r^n}^{(\mathrm{G},h_2)} - I)\mathsf{A}\mathfrak{g}^n\|^2\} \\
&\leq 2\eta\{\frac{1}{2}\|(\mathrm{T}_{r^n}^{(\mathrm{G},h_2)} - I)\mathsf{A}\mathfrak{g}^n\|^2 - \|(\mathrm{T}_{r^n}^{(\mathrm{G},h_2)} - I)\mathsf{A}\mathfrak{g}^n\|^2\} \\
&\leq -\eta\|(\mathrm{T}_{r^n}^{(\mathrm{G},h_2)} - I)\mathsf{A}\mathfrak{g}^n\|^2
\end{aligned} \tag{3.7}$$

Using Equations (3.5), (3.6) and (3.7), we obtain

$$\|u^n - \mathfrak{g}^*\|^2 \leq \|\mathfrak{g}^n - \mathfrak{g}^*\|^2 + \eta(L\eta - 1)\|(\mathrm{T}_{r^n}^{(\mathrm{G},h_2)} - I)\mathsf{A}\mathfrak{g}^n\|^2. \tag{3.8}$$

Since $\eta \in (0, \frac{1}{L})$, we obtain

$$\|u^n - \mathfrak{g}^*\|^2 \leq \|\mathfrak{g}^n - \mathfrak{g}^*\|^2 \tag{3.9}$$

So from Equations (3.3) and (3.9), we have

$$\|\upsilon^n - \mathfrak{g}^*\| \leq \|w^n - q^*\| \leq \|u^n - \mathfrak{g}^*\| \leq \|\mathfrak{g}^n - \mathfrak{g}^*\| \tag{3.10}$$

Hence, we have

$$\begin{aligned}
\|q^n - \mathfrak{g}^*\| \quad &= \quad \|\alpha_n\mathfrak{g}^n + (1-\alpha_n)S_n\upsilon^n - \mathfrak{g}^*\| \\
&\leq \quad \alpha_n\|\mathfrak{g}^n - \mathfrak{g}^*\| + (1-\alpha_n)\|S_n\upsilon^n - \mathfrak{g}^*\| \\
&\leq \quad \alpha_n\|\mathfrak{g}^n - \mathfrak{g}^*\| + (1-\alpha_n)\|\upsilon^n - \mathfrak{g}^*\| \\
&\leq \quad \alpha_n\|\mathfrak{g}^n - \mathfrak{g}^*\| + (1-\alpha_n)\|\mathfrak{g}^n - \mathfrak{g}^*\| \\
&= \quad \|\mathfrak{g}^n - \mathfrak{g}^*\|.
\end{aligned}$$

Thus, we get $\mathfrak{g}^* \in C_{n+1}$. Which infers that $\Delta \subset \mathcal{C}_n$ for all $n \in N$.

**Step 3.**We will demonstrate that $\lim_{n\to\infty}\|\mathfrak{g}^n - \mathfrak{g}^0\|$ exists. From $\mathfrak{g}^n = P_{\mathcal{C}_n}\mathfrak{g}^0$, we have

$$\langle \mathfrak{g}^0 - \mathfrak{g}^n, q - \mathfrak{g}^n\rangle \leq 0. \tag{3.11}$$

then

$$\langle \mathfrak{g}^0 - \mathfrak{g}^n, \mathfrak{g}^n - q\rangle \geq 0. \tag{3.12}$$

for each $q \in \mathcal{C}_n$. Using $\Delta \subset \mathcal{C}_n$, we also have

$\langle \mathfrak{g}^0 - \mathfrak{g}^n, \mathfrak{g}^n - \mathfrak{g}^*\rangle \geq 0$ for each $\mathfrak{g}^* \in \Delta$ and $n \in N$.

Then, for $\mathfrak{g}^* \in \Delta$, we obtain

$$\begin{aligned}
0 \quad &\leq \langle \mathfrak{g}^0 - \mathfrak{g}^n, \mathfrak{g}^n - \mathfrak{g}^*\rangle \\
&= \langle \mathfrak{g}^0 - \mathfrak{g}^n, \mathfrak{g}^n - \mathfrak{g}^0 + \mathfrak{g}^0 - \mathfrak{g}^*\rangle \\
&= \langle \mathfrak{g}^0 - \mathfrak{g}^n, \mathfrak{g}^n - \mathfrak{g}^0\rangle + \langle \mathfrak{g}^0 - \mathfrak{g}^n, \mathfrak{g}^0 - \mathfrak{g}^*\rangle \\
&\leq -\langle \mathfrak{g}^0 - \mathfrak{g}^n, \mathfrak{g}^0 - \mathfrak{g}^n\rangle + \langle \mathfrak{g}^0 - \mathfrak{g}^n, \mathfrak{g}^0 - \mathfrak{g}^*\rangle \\
&\leq -\|\mathfrak{g}^0 - \mathfrak{g}^n\|^2 + \langle \mathfrak{g}^0 - \mathfrak{g}^n, \mathfrak{g}^0 - \mathfrak{g}^*\rangle \\
&\leq -\|\mathfrak{g}^0 - \mathfrak{g}^n\|^2 + \|\mathfrak{g}^0 - \mathfrak{g}^n\|\|\mathfrak{g}^0 - \mathfrak{g}^*\|
\end{aligned}$$

This implies that

$$\|\mathfrak{g}^0 - \mathfrak{g}^n\| \leq \|\mathfrak{g}^0 - \mathfrak{g}^*\| \tag{3.13}$$

for all $\mathfrak{g}^* \in \Delta$ and $n \in N$.

From $\mathfrak{g}^n = P_{\mathcal{C}_n}\mathfrak{g}^0$ and $\mathfrak{g}^{n+1} = P_{\mathcal{C}_{n+1}}\mathfrak{g}^0 \in \mathcal{C}_{n+1} \subset \mathcal{C}_n$, we have

$$\langle \mathfrak{g}^0 - \mathfrak{g}^n, \mathfrak{g}^n - \mathfrak{g}^{n+1}\rangle \geq 0. \tag{3.14}$$

Hence

$$\begin{aligned}
0 \quad &\leq \langle \mathfrak{g}^0 - \mathfrak{g}^n, \mathfrak{g}^n - \mathfrak{g}^{n+1}\rangle \\
&= \langle \mathfrak{g}^0 - \mathfrak{g}^n, \mathfrak{g}^n - \mathfrak{g}^0 + \mathfrak{g}^0 - \mathfrak{g}^{n+1}\rangle \\
&= \langle \mathfrak{g}^0 - \mathfrak{g}^n, \mathfrak{g}^n - \mathfrak{g}^0\rangle + \langle \mathfrak{g}^0 - \mathfrak{g}^n, \mathfrak{g}^0 - \mathfrak{g}^{n+1}\rangle \\
&= -\langle \mathfrak{g}^0 - \mathfrak{g}^n, \mathfrak{g}^0 - \mathfrak{g}^n\rangle + \langle \mathfrak{g}^0 - \mathfrak{g}^n, \mathfrak{g}^0 - \mathfrak{g}^{n+1}\rangle \\
&\leq -\|\mathfrak{g}^0 - \mathfrak{g}^n\|^2 + \langle \mathfrak{g}^0 - \mathfrak{g}^n, \mathfrak{g}^0 - \mathfrak{g}^{n+1}\rangle \\
&\leq -\|\mathfrak{g}^0 - \mathfrak{g}^n\|^2 + \|\mathfrak{g}^0 - \mathfrak{g}^n\|\|\mathfrak{g}^0 - \mathfrak{g}^{n+1}\|
\end{aligned}$$

It follows that

$$\|\mathfrak{g}^0 - \mathfrak{g}^n\| \leq \|\mathfrak{g}^0 - \mathfrak{g}^{n+1}\| \tag{3.15}$$

for all $n \in N$.

In this way, the sequence $\{\|\mathfrak{g}^0 - \mathfrak{g}^n\|\}$ is a bounded and nondecreasing , so $\lim_{n\to\infty} \|\mathfrak{g}^n - \mathfrak{g}^0\|$ exists, and afterward there exists $m$ such that

$$\lim_{n\to\infty} \|\mathfrak{g}^n - \mathfrak{g}^0\| = m \tag{3.16}$$

**Step 4.** We will show the following :

**(1)** $\lim_{n\to\infty} \|\mathfrak{g}^{n+1} - \mathfrak{g}^n\| = 0$

**(2)** $\lim_{n\to\infty} \|\mathfrak{g}^n - q^n\| = 0$

**(3)** $\lim_{n\to\infty} \|\mathfrak{g}^n - u^n\| = 0$

**(4)** $\lim_{n\to\infty} \|\mathfrak{g}^n - \upsilon^n\| = 0$

From (3.14), we get

$$\begin{aligned}
&\|\mathfrak{g}^n - \mathfrak{g}^{n+1}\|^2 \\
&\quad = \|\mathfrak{g}^n - \mathfrak{g}^0 + \mathfrak{g}^0 - \mathfrak{g}^{n+1}\|^2 \\
&\quad = \|\mathfrak{g}^n - \mathfrak{g}^0\|^2 + 2\langle \mathfrak{g}^n - \mathfrak{g}^0, \mathfrak{g}^0 - \mathfrak{g}^{n+1}\rangle + \|\mathfrak{g}^0 - \mathfrak{g}^{n+1}\|^2 \\
&\quad = \|\mathfrak{g}^n - \mathfrak{g}^0\|^2 + 2\langle \mathfrak{g}^n - \mathfrak{g}^0, \mathfrak{g}^0 - \mathfrak{g}^n + \mathfrak{g}^n - \mathfrak{g}^{n+1}\rangle + \|\mathfrak{g}^0 - \mathfrak{g}^{n+1}\|^2 \\
&\quad = \|\mathfrak{g}^n - \mathfrak{g}^0\|^2 + 2\langle \mathfrak{g}^n - \mathfrak{g}^0, \mathfrak{g}^0 - \mathfrak{g}^n\rangle + 2\langle \mathfrak{g}^n - \mathfrak{g}^0, \mathfrak{g}^n - \mathfrak{g}^{n+1}\rangle + \|\mathfrak{g}^0 - \mathfrak{g}^{n+1}\|^2 \\
&\quad \leq \|\mathfrak{g}^n - \mathfrak{g}^0\|^2 + 2\langle \mathfrak{g}^n - \mathfrak{g}^0, \mathfrak{g}^0 - \mathfrak{g}^n\rangle + \|\mathfrak{g}^0 - \mathfrak{g}^{n+1}\|^2 \\
&\quad = \|\mathfrak{g}^n - \mathfrak{g}^0\|^2 - 2\langle \mathfrak{g}^n - \mathfrak{g}^0, \mathfrak{g}^n - \mathfrak{g}^0\rangle + \|\mathfrak{g}^0 - \mathfrak{g}^{n+1}\|^2 \\
&\quad \leq \|\mathfrak{g}^n - \mathfrak{g}^0\|^2 - 2\|\mathfrak{g}^n - \mathfrak{g}^0\|^2 + \|\mathfrak{g}^0 - \mathfrak{g}^{n+1}\|^2 \\
&\quad = -\|\mathfrak{g}^n - \mathfrak{g}^0\|^2 + \|\mathfrak{g}^0 - \mathfrak{g}^{n+1}\|^2
\end{aligned}$$

From (3.16), we get $\|\mathfrak{g}^{n+1} - \mathfrak{g}^0\|^2 - \|\mathfrak{g}^n - \mathfrak{g}^0\|^2 \to 0$, therefore

$$\lim_{n\to\infty} \|\mathfrak{g}^n - \mathfrak{g}^{n+1}\| = 0 \tag{3.17}$$

By $\mathfrak{g}^{n+1} = P_{\mathcal{C}_{n+1}}\mathfrak{g}^0 \in \mathcal{C}_{n+1} \subset \mathcal{C}_n$, we have

$$\|\mathfrak{g}^{n+1} - q^n\| \leq \|\mathfrak{g}^{n+1} - \mathfrak{g}^n\| \tag{3.18}$$

Further, we also get

$$\|\mathfrak{g}^n - q^n\| \leq \|\mathfrak{g}^n - \mathfrak{g}^{n+1}\| + \|\mathfrak{g}^{n+1} - q^n\| \leq 2\|\mathfrak{g}^n - \mathfrak{g}^{n+1}\| \tag{3.19}$$

From (3.17), we get

$$\lim_{n\to\infty} \|\mathfrak{g}^n - q^n\| = 0 \tag{3.20}$$

and we also have

$$\begin{aligned}
\|q^n - \mathfrak{g}^n\| \qquad & = \|\alpha_n\mathfrak{g}^n + (1-\alpha_n)S_n\upsilon^n - \mathfrak{g}^n\| \\
& = \|-(1-\alpha_n)\mathfrak{g}^n + (1-\alpha_n)S_n\upsilon^n\| \\
& = \|(1-\alpha_n)(S_n\upsilon^n - \mathfrak{g}^n)\| \\
& = (1-\alpha_n)\|S_n\upsilon^n - \mathfrak{g}^n)\|.
\end{aligned}$$

Since $\alpha_n \in (0,1)$ and (3.20), we get

$$\lim_{n\to\infty} \|S_n\upsilon^n - \mathfrak{g}^n)\| = 0 \tag{3.21}$$

Consider from Equations (3.2), (3.8) and (3.10), we obtain

$$\begin{aligned}
\|q^n - \mathfrak{g}^*\|^2 & \leq \alpha_n\|\mathfrak{g}^n - \mathfrak{g}^*\|^2 + (1-\alpha_n)\|S_n\upsilon^n - \mathfrak{g}^*\|^2 \\
& \leq \alpha_n\|\mathfrak{g}^n - \mathfrak{g}^*\|^2 + (1-\alpha_n)\|\upsilon^n - \mathfrak{g}^*\|^2 \\
& \leq \alpha_n\|\mathfrak{g}^n - \mathfrak{g}^*\|^2 + (1-\alpha_n)\|u^n - \mathfrak{g}^*\|^2 \\
& \leq \alpha_n\|\mathfrak{g}^n - \mathfrak{g}^*\|^2 + (1-\alpha_n)(\|\mathfrak{g}^n - \mathfrak{g}^*\|^2 + \eta(L\eta - 1)\|(\mathrm{T}_{r_k}^{(\mathrm{G},h_2)} - I)\mathsf{A}\mathfrak{g}^n\|^2) \\
& = \alpha_n\|\mathfrak{g}^n - \mathfrak{g}^*\|^2 + \|\mathfrak{g}^n - \mathfrak{g}^*\|^2 - \alpha_n\|\mathfrak{g}^n - \mathfrak{g}^*\|^2 \\
& + (1-\alpha_n)\eta(L\eta - 1)\|(\mathrm{T}_{r^n}^{(\mathrm{G},h_2)} - I)\mathsf{A}\mathfrak{g}^n\| \\
& = \|\mathfrak{g}^n - \mathfrak{g}^*\|^2 - (1-\alpha_n)\eta(1 - L\eta)\|(\mathrm{T}_{r^n}^{(\mathrm{G},h_2)} - I)\mathsf{A}\mathfrak{g}^n\|^2
\end{aligned}$$

This implies that

$$
\begin{aligned}
(1-\alpha_n)\eta(1-L\eta)\|(\mathrm{T}_{r^n}^{(\mathrm{G},h_2)}-I)\mathsf{A}\mathfrak{g}^n\|^2 &\le \|\mathfrak{g}^n-\mathfrak{g}^*\|^2-\|q^n-\mathfrak{g}^*\|^2\\
&=(\|\mathfrak{g}^n-\mathfrak{g}^*\|-\|q^n-\mathfrak{g}^*\|)(\|\mathfrak{g}^n-\mathfrak{g}^*\|+\|q^n-\mathfrak{g}^*\|)\\
&=(\|\mathfrak{g}^n-\mathfrak{g}^*-q^n+\mathfrak{g}^*\|)(\|\mathfrak{g}^n-\mathfrak{g}^*\|+\|q^n-\mathfrak{g}^*\|)\\
&=\|\mathfrak{g}^n-q^n\|(\|\mathfrak{g}^n-\mathfrak{g}^*\|+\|q^n-\mathfrak{g}^*\|)
\end{aligned}
$$

From (3.20), $\alpha_n \in (0,1), (1-\alpha_n)\eta(1-L\eta)>0$, we obtain

$$
\lim_{n\to\infty}\|(\mathrm{T}_{r^n}^{(\mathrm{G},h_2)}-I)\mathsf{A}\mathfrak{g}^n\|=0 \tag{3.22}
$$

.For $\mathfrak{g}^*\in\Delta$ and $\mathrm{T}_{r^n}^{(\mathrm{F},h_1)}$ is firmly nonexpansive, we get

$$
\begin{aligned}
\|u^n-\mathfrak{g}^*\|^2 &= \|\mathrm{T}_{r^n}^{(\mathrm{F},h_1)}(\mathfrak{g}^n+\eta\mathsf{A}^*(\mathrm{T}_{r^n}^{(\mathrm{G},h_2)}-I)\mathsf{A}\mathfrak{g}^n)-\mathfrak{g}^*\|^2\\
&=\|\mathrm{T}_{r^n}^{(\mathrm{F},h_1)}(\mathfrak{g}^n+\eta\mathsf{A}^*(\mathrm{T}_{r^n}^{(\mathrm{G},h_2)}-I)\mathsf{A}\mathfrak{g}^n)-\mathrm{T}_{r^n}^{(\mathrm{F},h_1)}\mathfrak{g}^*\|^2\\
&\le\langle u^n-\mathfrak{g}^*,\mathfrak{g}^n+\eta\mathsf{A}^*(\mathrm{T}_{r^n}^{(\mathrm{G},h_2)}-I)\mathsf{A}\mathfrak{g}^n)-\mathfrak{g}^*\rangle\\
&=\frac{1}{2}\{\|u^n-\mathfrak{g}^*\|+\|\mathfrak{g}^n+\eta\mathsf{A}^*(\mathrm{T}_{r^n}^{(\mathrm{G},h_2)}-I)\mathsf{A}\mathfrak{g}^n)-\mathfrak{g}^*\|^2\\
&-\|(u^n-\mathfrak{g}^*)-[\mathfrak{g}^n+\eta\mathsf{A}^*(\mathrm{T}_{r^n}^{(\mathrm{G},h_2)}-I)\mathsf{A}\mathfrak{g}^n)-\mathfrak{g}^*]\|^2\}\\
&=\frac{1}{2}\{\|u^n-\mathfrak{g}^*\|^2+\|\mathfrak{g}^n-\mathfrak{g}^*\|^2-\|u^n-\mathfrak{g}^n-\eta\mathsf{A}^*(\mathrm{T}_{r^n}^{(\mathrm{G},h_2)}-I)\mathsf{A}\mathfrak{g}^n\|^2\}\\
&=\frac{1}{2}\{\|u^n-\mathfrak{g}^*\|^2+\|\mathfrak{g}^n-\mathfrak{g}^*\|^2-\|u^n-\mathfrak{g}^n\|^2+\eta^2\|\mathsf{A}^*(\mathrm{T}_{r^n}^{(\mathrm{G},h_2)}-I)\mathsf{A}\mathfrak{g}^n\|^2\\
&-2\eta\langle u^n-\mathfrak{g}^n,\mathsf{A}^*(\mathrm{T}_{r^n}^{(\mathrm{G},h_2)}-I)\mathsf{A}\mathfrak{g}^n\rangle\}
\end{aligned}
$$

Hence, we obtain

$$
\|u^n-\mathfrak{g}^*\|^2\le\|\mathfrak{g}^n-\mathfrak{g}^*\|^2-\|u^n-\mathfrak{g}^n\|^2+2\eta\|\mathsf{A}(u^n-\mathfrak{g}^n)\|\|(\mathrm{T}_{r^n}^{(\mathrm{G},h_2)}-I)\mathsf{A}\mathfrak{g}^n\|. \tag{3.23}
$$

By Equations (3.2), (3.10) and (3.23), it follows that

$$\begin{aligned}
\|q^n - \mathfrak{g}^*\|^2 &\leq \alpha_n\|\mathfrak{g}^n - \mathfrak{g}^*\|^2 + (1-\alpha_n)\|S_n\upsilon^n - \mathfrak{g}^*\|^2 \\
&\leq \alpha_n\|\mathfrak{g}^n - \mathfrak{g}^*\|^2 + (1-\alpha_n)\|\upsilon^n - \mathfrak{g}^*\|^2 \\
&\leq \alpha_n\|\mathfrak{g}^n - \mathfrak{g}^*\|^2 + (1-\alpha_n)\|u^n - \mathfrak{g}^*\|^2 \\
&\leq \alpha_n\|\mathfrak{g}^n - \mathfrak{g}^*\|^2 + (1-\alpha_n)(\|\mathfrak{g}^n - \mathfrak{g}^*\|^2 - \|u^n - \mathfrak{g}^*\|^2 \\
&+ 2\eta\|\mathsf{A}(u^n - \mathfrak{g}^n)\|\|(\mathsf{T}_{r^n}^{(\mathrm{G},h_2)} - I)\mathsf{A}\mathfrak{g}^n\|) \\
&= \|\mathfrak{g}^n - \mathfrak{g}^*\|^2 - (1-\alpha_n)\|u^n - \mathfrak{g}^n\|^2 + 2(1-\alpha_n)\eta\|\mathsf{A}(u^n - \mathfrak{g}^n)\|\|(\mathsf{T}_{r^n}^{(\mathrm{G},h_2)} - I)\mathsf{A}\mathfrak{g}^n\|.
\end{aligned}$$

So,

$$\begin{aligned}
(1-\alpha_n)\|\mathfrak{g}^n - u^n\|^2 &\leq \|\mathfrak{g}^n - \mathfrak{g}^*\|^2 - \|q^n - \mathfrak{g}^*\|^2 + 2(1-\alpha_n)\eta\|\mathsf{A}(u^n - \mathfrak{g}^n)\|\|(\mathsf{T}_{r^n}^{(\mathrm{G},h_2)} - I)\mathsf{A}\mathfrak{g}^n\| \\
&= \|\mathfrak{g}^n - q^n\|(\|\mathfrak{g}^n - \mathfrak{g}^*\| + \|q^n - \mathfrak{g}^*\|) + \\
&2(1-\alpha_n)\eta\|\mathsf{A}(u^n - \mathfrak{g}^n)\|\|(\mathsf{T}_{r^n}^{(\mathrm{G},h_2)} - I)\mathsf{A}\mathfrak{g}^n\|.
\end{aligned}$$

From Equations (3.20), (3.22) and $\alpha_n \in (0,1)$, we get

$$\lim_{n\to\infty} \|\mathfrak{g}^n - u^n\| = 0 \tag{3.24}$$

Since $\mathfrak{g}^* \in \Delta$ from Equations (3.2) and (3.10), we obtain

$$\begin{aligned}
\|q^n - \mathfrak{g}^*\|^2 &\leq \alpha_n\|\mathfrak{g}^n - \mathfrak{g}^*\|^2 + (1-\alpha_n)\|S_n\upsilon^n - \mathfrak{g}^*\|^2 \\
&\leq \alpha_n\|\mathfrak{g}^n - \mathfrak{g}^*\| + (1-\alpha_n)\|\upsilon^n - \mathfrak{g}^*\|^2 \\
&= \alpha_n\|\mathfrak{g}^n - \mathfrak{g}^*\|^2 + \|\upsilon^n - \mathfrak{g}^*\|^2 - \alpha_n\|\upsilon^n - \mathfrak{g}^*\|^2 \\
&= \|P_{\mathcal{C}}(w^n - \lambda_1 B_1 w^n) - P_{\mathcal{C}}(q^* - \lambda_1 B_1 q^*)\|^2 \\
&\leq \|(I - \lambda_1 B_1)w^n - (I - \lambda_1 B_1)q^*\|^2 \\
&\leq \|w^n - q^*\|^2 + \lambda_1(\lambda_1 - 2\beta_1)\|B_1 w^n - B_1 q^*\|^2 \\
&\leq \|\mathfrak{g}^n - \mathfrak{g}^*\|^2 + a_1(b_1 - 2\beta_1)\|B_1 w^n - B_1 q^*\|^2.
\end{aligned}$$

and also

$$\begin{aligned}
\|q^n - \mathfrak{g}^*\|^2 &\leq \alpha_n \|\mathfrak{g}^n - \mathfrak{g}^*\|^2 + (1-\alpha_n)\|S_n \upsilon^n - \mathfrak{g}^*\|^2 \\
&\leq \alpha_n \|\mathfrak{g}^n - \mathfrak{g}^*\| + (1-\alpha_n)\|\upsilon_n - \mathfrak{g}^*\|^2 \\
&= \alpha_n \|\mathfrak{g}^n - \mathfrak{g}^*\|^2 + \|w^n - q^*\|^2 - \alpha_n \|\mathfrak{g}^n - \mathfrak{g}^*\|^2 \\
&= \|P_{\mathcal{C}}(u^n - \lambda_2 B_2 u^n) - P_{\mathcal{C}}(\mathfrak{g}^* - \lambda_2 B_2 \mathfrak{g}^*)\|^2 \\
&\leq \|(I - \lambda_2 B_2)u^n - (I - \lambda_2 B_2)\mathfrak{g}^*\|^2 \\
&\leq \|u^n - \mathfrak{g}^*\|^2 + \lambda_2(\lambda_2 - 2\beta_2)\|B_2 u^n - B_2 \mathfrak{g}^*\|^2 \\
&\leq \|\mathfrak{g}^n - \mathfrak{g}^*\|^2 + a_2(b_2 - 2\beta_2)\|B_2 u^n - B_2 \mathfrak{g}^*\|^2.
\end{aligned}$$

Therefore, we have

$$\begin{aligned}
a_1(2\beta_1 - b_1)\|B_1 w^n - B_1 q^*\|^2 &\leq \|\mathfrak{g}^n - \mathfrak{g}^*\|^2 - \|q^n - \mathfrak{g}^*\|^2 \\
&\leq \|\mathfrak{g}^n - q^n\|(\|\mathfrak{g}^n - \mathfrak{g}^*\| + \|q^n - \mathfrak{g}^*\|)
\end{aligned}$$

and

$$\begin{aligned}
a_2(2\beta_2 - b_2)\|B_2 u^n - B_2 \mathfrak{g}^*\|^2 &\leq \|\mathfrak{g}^n - \mathfrak{g}^*\|^2 - \|q^n - \mathfrak{g}^*\|^2 \\
&\leq \|\mathfrak{g}^n - q^n\|(\|\mathfrak{g}^n - \mathfrak{g}^*\| + \|q^n - \mathfrak{g}^*\|)
\end{aligned}$$

From conditions ( C1), ( C2) and (3.20), we get

$$\lim_{n\to\infty} \|B_1 w^n - B_1 q^*\| = \lim_{n\to\infty} \|B_2 u^n - B_2 \mathfrak{g}^*\| = 0 \tag{3.25}$$

On the other hand, we have

$$\begin{aligned}
\|w^n - \mathfrak{g}^*\|^2 &= \|P_{\mathcal{C}}(u^n - \lambda_2 B_2 u^n) - P_{\mathcal{C}}(\mathfrak{g}^* - \lambda_2 B_2 \mathfrak{g}^*)\|^2 \\
&\leq \langle (u^n - \lambda_2 B_2 u^n) - (\mathfrak{g}^* - \lambda_2 B_2 \mathfrak{g}^*), w^n - q^* \rangle \\
&= \frac{1}{2}\{\|(u^n - \lambda_2 B_2 u^n) - (\mathfrak{g}^* - \lambda_2 B_2 \mathfrak{g}^*)\|^2 + \|w^n - q^*\|^2 \\
&- \|[(u^n - \lambda_2 B_2 u^n) - (\mathfrak{g}^* - \lambda_2 B_2 \mathfrak{g}^*)]^2 - (w^n - q^*)\|^2\} \\
&\leq \frac{1}{2}\{\|u^n - \mathfrak{g}^*\|^2 + \|w^n - q^*\|^2 - \|(u^n - w^n) - (\mathfrak{g}^* - q^*) - \lambda_2(B_2 u^n - B_2 \mathfrak{g}^*)\|^2\} \\
&\leq \frac{1}{2}\{\|\mathfrak{g}^n - \mathfrak{g}^*\|^2 + \|w^n - q^*\|^2 - \|(u^n - w^n) - (\mathfrak{g}^* - q^*)\|^2 \\
&+ 2\lambda_2 \langle (u^n - w^n) - (\mathfrak{g}^* - q^*), B_2 u^n - B_2 \mathfrak{g}^* \rangle - \lambda_2^2 \|B_2 u^n - B_2 \mathfrak{g}^*\|^2\}
\end{aligned}$$

So, we get

$$\|w^n - q^*\|^2 \leq \|\mathfrak{g}^n - \mathfrak{g}^*\|^2 - \|(u^n - w^n) - (\mathfrak{g}^* - q^*)\|^2 \\ + 2\lambda_2\|(u^n - w^n) - (\mathfrak{g}^* - q^*)\|\|B_2u^n - B_2\mathfrak{g}^*\| - \lambda_2^2\|B_2u^n - B_2\mathfrak{g}^*\|^2 \tag{3.26}$$

From (3.10) and (3.26), it follows that

$$\begin{aligned}\|q^n - \mathfrak{g}^*\|^2 &= \|\alpha_n\mathfrak{g}^n + (1-\alpha_n)S_n\upsilon^n - \mathfrak{g}^*\|^2 \\ &\leq \alpha_n\|\mathfrak{g}^n - \mathfrak{g}^*\|^2 + (1-\alpha_n)\|\upsilon^n - \mathfrak{g}^*\|^2 \\ &\leq \alpha_n\|\mathfrak{g}^n - \mathfrak{g}^*\|^2 + (1-\alpha_n)\|w^n - q^*\|^2 \\ &\leq \alpha_n\|\mathfrak{g}^n - \mathfrak{g}^*\|^2 + (1-\alpha_n)\{\|\mathfrak{g}^n - \mathfrak{g}^*\|^2 - \|(u^n - w^n) - (\mathfrak{g}^* - q^*)\|^2 \\ &+ 2\lambda_2\|(u^n - w^n) - (\mathfrak{g}^* - q^*)\|\|B_2u^n - B_2\mathfrak{g}^*\| - \lambda_2^2\|B_2u^n - B_2\mathfrak{g}^*\|^2\} \\ &= \|\mathfrak{g}^n - \mathfrak{g}^*\|^2 - (1-\alpha_n)\|(u^n - w^n) - (\mathfrak{g}^* - q^*)\|^2 \\ &+ 2\lambda_2(1-\alpha_n)\|(u^n - w^n) - (\mathfrak{g}^* - q^*)\|\|B_2u^n - B_2\mathfrak{g}^*\| \\ &- \lambda_2^2(1-\alpha_n)\|B_2u^n - B_2\mathfrak{g}^*\|^2\}.\end{aligned}$$

Thus, we get

$$\begin{aligned}(1-\alpha_n)\|(u^n - w^n) - (\mathfrak{g}^* - q^*)\|^2 &\leq \|\mathfrak{g}^n - \mathfrak{g}^*\|^2 - \|q^n - \mathfrak{g}^*\|^2 - \lambda_2^2(1-\alpha_n)\|B_2u^n - B_2\mathfrak{g}^*\|^2 \\ &+ 2\lambda_2(1-\alpha_n)\|(u^n - w^n) - (\mathfrak{g}^* - q^*)\|\|B_2u^n - B_2\mathfrak{g}^*\| \\ &\leq \|\mathfrak{g}^n - q^n\|(\|\mathfrak{g}^n - \mathfrak{g}^*\| + \|q^n - \mathfrak{g}^*\|) \\ &- \lambda_2^2(1-\alpha_n)\|B_2u^n - B_2\mathfrak{g}^*\|^2 \\ &+ 2\lambda_2(1-\alpha_n)\|(u^n - w^n) - (\mathfrak{g}^* - q^*)\|\|B_2u^n - B_2\mathfrak{g}^*\|.\end{aligned}$$

Since $\alpha_n \in (0,1)$, Equations (3.20) and (3.25), we obtain

$$\lim_{n\to\infty} \|(u^n - w^n) - (\mathfrak{g}^* - q^*)\| = 0$$

Now, we observe that

$$\|(w^n - \upsilon^n) + (\mathfrak{g}^* - q^*)\|^2$$

$$= \|(w^n - q^*) - (\upsilon^n - \mathfrak{g}^*)\|^2$$

$$= \|(w^n - \lambda_1 B_1 w^n) - (q^* - \lambda_1 B_1 q^*) - [P_{\mathcal{C}}(w^n - \lambda_1 B_1 w^n) - P_{\mathcal{C}}(q^* - \lambda_1 B_1 q^*)]$$

$$+ \lambda_1 (B_1 w^n - B_1 q^*)\|^2$$

$$\leq \|(w^n - \lambda_1 B_1 w^n) - (q^* - \lambda_1 B_1 q^*) - [P_{\mathcal{C}}(w^n - \lambda_1 B_1 w^n) - P_{\mathcal{C}}(q^* - \lambda_1 B_1 q^*)]\|^2$$

$$+ 2\lambda_1 \langle B_1 w^n - B_1 q^*, (w^n - \upsilon_n) + (\mathfrak{g}^* - q^*)\rangle$$

$$\leq \|(w^n - \lambda_1 B_1 w^n) - (q^* - \lambda_1 B_1 q^*)\|^2 - \|P_{\mathcal{C}}(w^n - \lambda_1 B_1 w^n) - P_{\mathcal{C}}(q^* - \lambda_1 B_1 q^*)\|^2$$

$$+ 2\lambda_1 \|B_1 w^n - B_1 q^*\| \|(w^n - \upsilon_n) + (\mathfrak{g}^* - q^*)\|$$

$$\leq \|(w^n - \lambda_1 B_1 w^n) - (q^* - \lambda_1 B_1 q^*)\|^2 - \|S_n P_{\mathcal{C}}(w^n - \lambda_1 B_1 w^n) -$$

$$S_n P_{\mathcal{C}}(q^* - \lambda_1 B_1 q^*)\|^2 + 2\lambda_1 \|B_1 w^n - B_1 q^*\| \|(w^n - \upsilon_n) + (\mathfrak{g}^* - q^*)\|$$

$$\leq \|(w^n - \lambda_1 B_1 w^n) - (q^* - \lambda_1 B_1 q^*)\|^2 - \|S_n \upsilon^n - S_n \mathfrak{g}^*\|^2$$

$$+ 2\lambda_1 \|B_1 w^n - B_1 q^*\| \|(w^n - \upsilon_n) + (\mathfrak{g}^* - q^*)\|$$

$$\leq \{\|(w^n - \lambda_1 B_1 w^n) - (q^* - \lambda_1 B_1 q^*)\| - \|S_n \upsilon^n - \mathfrak{g}^*\|\}$$

$$\times \{\|(w^n - \lambda_1 B_1 w^n) - (q^* - \lambda_1 B_1 q^*)\| + \|S_n \upsilon^n - \mathfrak{g}^*\|\}$$

$$+ 2\lambda_1 \|B_1 w^n - B_1 q^*\| \|(w^n - \upsilon_n) + (\mathfrak{g}^* - q^*)\|$$

$$\leq \|(w^n - \lambda_1 B_1 w^n) - (q^* - \lambda_1 B_1 q^*)\| - \|S_n \upsilon^n - \mathfrak{g}^*\|$$

$$\times \{\|(w^n - \lambda_1 B_1 w^n) - (q^* - \lambda_1 B_1 q^*)\| + \|S_n \upsilon^n - \mathfrak{g}^*\|\}$$

$$+ 2\lambda_1 \|B_1 w^n - B_1 q^*\| \|(w^n - \upsilon_n) + (\mathfrak{g}^* - q^*)\|$$

$$= \|(w^n - u^n + u^n - S_n \upsilon^n) + (\mathfrak{g}^* - q^*) - \lambda_1 (B_1 w^n - B_1 q^*)\|$$

$$\times \{\|(w^n - \lambda_1 B_1 w^n) - (q^* - \lambda_1 B_1 q^*)\| + \|S_n \upsilon^n - \mathfrak{g}^*\|\}$$

$$+ 2\lambda_1 \|B_1 w^n - B_1 q^*\| \|(w^n - \upsilon_n) + (\mathfrak{g}^* - q^*)\|$$

$$
\begin{aligned}
&= \|(u^n - S_n\upsilon^n) + (\mathfrak{g}^* - q^*) - (u^n - w^n) - \lambda_1\|(B_1w^n - B_1q^*)\| \\
&\times \{\|(w^n - \lambda_1 B_1 w^n) - (q^* - \lambda_1 B_1 q^*)\| + \|S_n\upsilon^n - \mathfrak{g}^*\|\} \\
&+ 2\lambda_1\|B_1w^n - B_1q^*\|\|(w^n - \upsilon_n) + (\mathfrak{g}^* - q^*)\| \\
&\leq \{\|(u^n - S_n\upsilon^n) + (\mathfrak{g}^* - q^*) - (u^n - w^n)\| + \lambda_1\|(B_1w^n - B_1q^*)\|\} \\
&\times \{\|(w^n - \lambda_1 B_1 w^n) - (q^* - \lambda_1 B_1 q^*)\| + \|S_n\upsilon^n - \mathfrak{g}^*\|\} \\
&+ 2\lambda_1\|B_1w^n - B_1q^*\|\|(w^n - \upsilon_n) + (\mathfrak{g}^* - q^*)\|.
\end{aligned}
$$

Since

$$\|u^n - S_n\upsilon^n\| \leq \|u^n - \mathfrak{g}^n\| + \|\mathfrak{g}^n - S_n\upsilon^n\|. \tag{3.27}$$

From equations (3.21) and (3.24), implies that

$$\lim_{n\to\infty} \|u^n - S_n\upsilon^n\| = 0 \tag{3.28}$$

We have from Equations (3.25), (3.27) and (3.28), it follows that

$$\lim_{n\to\infty} \|(w^n - \upsilon^n) + (\mathfrak{g}^* - q^*)\| = 0 \tag{3.29}$$

Also, observe that

$$
\begin{aligned}
&\|S_n\upsilon^n - \upsilon^n\| \\
&= \|S_n\upsilon^n - u^n + u^n - w^n + w^n - \mathfrak{g}^* + \mathfrak{g}^* - q^* + q^* - \upsilon^n\| \\
&\leq \|S_n\upsilon^n - u^n\| + \|(u^n - w^n) - (x^* - q^*)\| + \|(w^n - \upsilon^n) + (\mathfrak{g}^* - q^*)\|.
\end{aligned}
$$

From equations (3.27), (3.29) and (3.30), we get

$$\lim_{n\to\infty} \|S_n\upsilon^n - \upsilon^n\| = 0 \tag{3.30}$$

Since

$$\|\mathfrak{g}^n - \upsilon^n\| \leq \|\mathfrak{g}^n - S_n\upsilon^n\| + \|S_n\upsilon^n - \upsilon^n\|. \tag{3.31}$$

By Equations (3.21) and (3.31), we get

$$\lim_{n\to\infty} \|\mathfrak{g}^n - \upsilon^n\| = 0. \tag{3.32}$$

**Step 5.** We will show that $w \in \Delta := \bigcap_{n=1}^{\infty} Fix(S_n) \cap \Omega \cap GVI(\mathcal{C}, B_1, B_2)$.

**Step 5.1**:First, we show that $w \in \bigcap_{n=1}^{\infty} Fix(S_n)$. From $\|S_n \upsilon^n - \upsilon^n\| \to 0$, we obtain that $S_n \upsilon_{n_i} \to w$ as $i \to \infty$. By definition of demicloseness, we conclude that $w \in \bigcap_{n=1}^{\infty} Fix(S_n)$.

**Step 5.2**:we will demonstrate that $w \in \Omega$.

First, we will demonstrate that$w \in GEP(\mathrm{F}, h_1)$.

Since $u^n = \mathrm{T}_{r^n}^{\mathrm{F},h_1} \mathfrak{g}^n$, we have

$$\mathrm{F}(u^n, q) + h_1(u^n, q) + \frac{1}{r^n}\langle q - u^n, u^n - \mathfrak{g}^n \rangle \geq 0, \forall q \in \mathcal{C}. \tag{3.33}$$

It pursues from the monotonicity of $\mathrm{F}$ that

$$h_1(u^n, q) + \frac{1}{r^n}\langle q - u^n, u^n - \mathfrak{g}^n \rangle \geq \mathrm{F}(q, u^n), \tag{3.34}$$

and hence replacing $n$ by $n_i$, we get

$$h_1(u^{n_i}, q) + \langle q - u^{n_i}, \frac{u^{n_i} - \mathfrak{g}^{n_i}}{r^{n_i}} \rangle \geq \mathrm{F}(q, u^{n_i}). \tag{3.35}$$

Since $\|u^n - \mathfrak{g}^n\| \to 0$ we get $u^{n_i} \rightharpoonup w$ and $\frac{u^{n_i} - \mathfrak{g}^{n_i}}{r^{n_i}} \to 0$. It follows by Lemma 3.7 (4) that $0 \geq \mathrm{F}(q, w), \forall w \in \mathcal{C}$. For any $t$ with $0 < t \leq 1$ and $w \in \mathcal{C}$, let $q^t = tq + (1-t)w$. Since $q \in \mathcal{C}, w \in \mathcal{C}$, we have $q^t \in \mathcal{C}$, and hence, $\mathrm{F}(q^t, w) \leq 0$. So, from Lemma 3.7 (1) and (4), we have

$$\begin{aligned}
0 &= \mathrm{F}(q^t, q^t) + h_1(q^t, q^t) \\
&\leq t[\mathrm{F}(q^t, q) + h_1(q^t, q)] + (1-t)[\mathrm{F}(q^t, w) + h_1(q^t, w)] \\
&\leq t[\mathrm{F}(q^t, q) + h_1(q^t, q)] + (1-t)[\mathrm{F}(w, q^t) + h_1(w, q^t)] \\
&\leq [\mathrm{F}(q^t, q) + h_1(q^t, q)]
\end{aligned}$$

Therefore, $0 \leq \mathrm{F}(q^t, q) + h_1(q^t, q)$. From Lemma 2.7 (3), we have $0 \leq \mathrm{F}(w, q) + h_1(w, q)$. This infers $w \in GEP(\mathrm{F}, h_1)$.

Next, We demonstrate that $\mathsf{A}w \in GEP(\mathrm{G}, h_2)$. Since $\|u^n - \mathfrak{g}^n\| \to 0, u^n \rightharpoonup w$ as $n \to \infty$ and $\mathfrak{g}^n$ is bounded, there exists a subsequence $\mathfrak{g}^{n_i}$ of $\mathfrak{g}^n$ such that $\mathfrak{g}^{n_i} \rightharpoonup w$, and since $\mathsf{A}$ is BLO, so $\mathsf{A}\mathfrak{g}^{n_i} \rightharpoonup \mathsf{A}w$. Presently,setting $k^{n_i} = \mathsf{A}\mathfrak{g}^{n_i} - \mathrm{T}_{r^{n_i}}^{(\mathrm{G},h_2)}\mathsf{A}\mathfrak{g}^{n_i}$. It follows from (3.22) that $\lim_{n\to\infty} k^{n_i} = 0$ and $\mathsf{A}\mathfrak{g}^{n_i} - k^{n_i} = \mathrm{T}_{r^{n_i}}^{(\mathrm{G},h_2)}\mathsf{A}\mathfrak{g}^{n_i}$. Therefore, from Lemma 2.8, we have

$$\mathrm{G}(\mathsf{A}\mathfrak{g}^{n_i}-k^{n_i}, w^{'})+h_2(\mathsf{A}\mathfrak{g}^{n_i}-k^{n_i}, w^{'})+\frac{1}{r^{n_i}}\langle w^{'}-(\mathsf{A}\mathfrak{g}^{n_i}-k^{n_i}), (\mathsf{A}\mathfrak{g}^{n_i}-k^{n_i})-\mathsf{A}\mathfrak{g}^{n_i}\rangle \geq 0, \tag{3.36}$$

$\forall w^{'} \in \mathrm{Q}$. Since G and $h_2$ are upper semicontinuous, taking $lim\ sup$ to above inequality as $i \to \infty$ and using condition ( C3), we obtain

$$\mathrm{G}(\mathsf{A}w, w^{'}) + h_2(\mathsf{A}w, w^{'}) \geq 0, \forall w^{'} \in \mathrm{Q}, \tag{3.37}$$

which means that $\mathsf{A}w \in GEP(\mathrm{G}, h_2)$ and hence $w \in \Omega$.

**Step 5.3**:Lastly, we demonstrate that $w \in GVI(\mathcal{C}, B_1, B_2)$. Since $\lim_{n\to\infty} \|S_n\upsilon^n - \upsilon^n\| = 0, \lim_{n\to\infty} \|S_n\upsilon^n - \mathfrak{g}^n\| = 0, \lim_{n\to\infty} \|\mathfrak{g}^n - u^n\| = 0$, then we get

$$\|\upsilon^n - u^n\| \leq \|\upsilon^n - S_n\upsilon^n\| + \|S_n\upsilon^n - \mathfrak{g}^n\| + \|\mathfrak{g}^n - u^n\| \tag{3.38}$$

we conclude that $\lim_{n\to\infty} \|\upsilon^n - u^n\| = 0$. Moreover, by the nonexpansivity of $G^{\sim}$ in Lemma 3.10, we have

$$\begin{aligned}\|\upsilon^n - G^{\sim}(\upsilon^n)\| &= \|P_{\mathcal{C}}[P_{\mathcal{C}}(u^n - \lambda_2 B_2 u^n) - \lambda_1 B_1 P_{\mathcal{C}}(u^n - \lambda_2 B_2 u^n)] - G^{\sim}(\upsilon^n)\| \\ &= \|G^{\sim}(u^n) - G^{\sim}(\upsilon^n)\| \\ &\leq \|u^n - \upsilon^n\|\end{aligned}$$

Since $\|u^n - \upsilon^n\| \leq \|u^n - \mathfrak{g}^n\| + \|\mathfrak{g}^n - \upsilon^n\|$ and $\|u^n - \mathfrak{g}^n\| \to 0, \|\mathfrak{g}^n - \upsilon^n\| \to 0$, we get $\|u^n - \upsilon^n\| \to 0$ as $n \to \infty$. Thus, $\lim_{n\to\infty} \|\upsilon^n - G^{\sim}(\upsilon^n)\| = 0$. According to Demiclosedness principle, we obtain that $w \in GVI(\mathcal{C}, B_1, B_2)$. Hence, $w \in \Delta$.

**Step 6.** We will demonstrate that $\mathfrak{g}^n \to w$, where $w = P_\Delta \mathfrak{g}^0$. Since $\mathfrak{g}^n = P_{\mathcal{C}_n}\mathfrak{g}^0$ and $w \in \Delta \subset \mathcal{C}_n$, we have

$$\|\mathfrak{g}^n - \mathfrak{g}^0\| \leq \|w - \mathfrak{g}^0\|. \tag{3.39}$$

It follows from $w^{\sim} = P_\Delta \mathfrak{g}^0$ and the lower semicontinuity of the norm that

$$\|w^{\sim} - \mathfrak{g}^0\| \leq \|w - \mathfrak{g}^0\| \leq \liminf_{i\to\infty} \|\mathfrak{g}^{n_i} - \mathfrak{g}^0\| \leq \limsup_{i\to\infty} \leq \|w^{\sim} - \mathfrak{g}^0\|. \tag{3.40}$$

consequently we get $\lim_{n\to\infty}\|\mathfrak{g}^{n_i}-\mathfrak{g}^0\| = \|w-\mathfrak{g}^0\| = \|w^{\sim}-\mathfrak{g}^0\|$. Using the Kadec-Klee property of $\mathsf{H}_1$,we acquire that

$$\lim_{i\to\infty}\mathfrak{g}^{n_i} = w = w^{\sim}. \tag{3.41}$$

Since $\mathfrak{g}^{n_i}$ is an arbitrary subsequence of $\mathfrak{g}^n$ which is weakly convergent,we can infer that $\mathfrak{g}^n \longrightarrow w$, where $w = P_\Delta\mathfrak{g}^0$. By this our theorem is proved.□

## 3.3 Consequently Results

In this area, we find result from Theorem (3.2.1).

**Corollary 3.3.1.** *"Let $H_1$ and $H_2$ be two (real) Hilbert spaces and suppose $\emptyset \neq \mathcal{C} \subset H_1$ and $\emptyset \neq \mathrm{Q} \subset H_2$ where both the subsets are closed and convex. Let $B:\mathcal{C}\to H$ is $\beta$ inverse strongly monotone map ,. Let $A: H_1 \to H_2$ be a bounded linear operator. Let $\mathrm{F}, h_1 : \mathcal{C}\times\mathcal{C}\to R$ and $\mathrm{G}, h_2 : \mathrm{Q}\times\mathrm{Q}\to R$ satisfying Lemma 3.7; $h_1$, $h_2$ are monotone and $\mathrm{G}$ is upper semicontinuous and $\{S_n\}_{n=1}^{\infty}$ be a sequence of infinite family of $k^n$-strict pseudo-contractions from $\mathcal{C}$ into itself with $k = \sup^n k^n$, For every $n \in N$ ,$S_n$ and $S$ are S-mappings produced by $S_n, ..., S_1$ and $\lambda_n, \lambda_{n-1}...., \lambda_1$ and $S_n, S_{n-1}...$ and $\lambda_n, \lambda_{n-1}....$ respectively.Likewise $S$ be a mapping of $\mathcal{C}$ into itself characterized by $S\mathfrak{g} = \lim_{n\to\infty} S_n\mathfrak{g}$ such that*

$$\Delta := \cap_{n=1}^{\infty} Fix(S) \cap \Omega \cap VI(\mathcal{C}, B) \neq \emptyset \tag{3.42}$$

*For a given $\mathfrak{g}^0 \in H_1, \mathcal{C}_1 = \mathcal{C}, \mathfrak{g}_1 = P_{\mathcal{C}}\mathfrak{g}^0, u^n \in \mathcal{C}$, let the iterative sequences $w^n$,$\mathfrak{g}^n$,$u^n$, and $q^n$ be generated by*

$$\begin{cases} u^n = T_{r^n}^{(\mathrm{F},h_1)}(\mathfrak{g}^n + \eta A^*(T_{r^n}^{(\mathrm{G},h_2)} - I)A\mathfrak{g}^n), \\ w^n = P_{\mathcal{C}}(u^n - \lambda B u^n), \\ q^n = \alpha_n\mathfrak{g}^n + (1-\alpha_n)SP_{\mathcal{C}}(w^n - \lambda B w^n), \\ \mathcal{C}_{n+1} = w \in \mathcal{C}_n : \|q^n - w\| \leq \|\mathfrak{g}^n - w\|, \\ \mathfrak{g}^{n+1} = P_{\mathcal{C}_{n+1}}\mathfrak{g}^0, \forall n \in N, \end{cases} \tag{3.43}$$

*where* $\alpha_n \in (0,d], d \in (0,1), \lambda \in [a,b] \subset (0,2\beta), r^n \subset (0,\infty)$ *and* $\eta \in (0,\frac{1}{L}), L$ *is the spectral radius of the operator* $\mathsf{A}^*\mathsf{A}$ *and* $\mathsf{A}^*$ *is the adjoint of* $\mathsf{A}$ *satisfying the* $\liminf_{n\to\infty} r^n > 0$. *Then, the sequence* $\mathfrak{g}^n \to P_\Delta \mathfrak{g}^0$ *and* $(\mathfrak{g}^*, q^*)$ *is a solution of a GVIP, where* $q^* = P_{\mathcal{C}}(\mathfrak{g}^* - \lambda_2 B_2 \mathfrak{g}^*)$*".*

***Proof****:By setting* $S_n$=$S$*,for all* $n \in N$*,* $B = B_1 = B_2$*,* $\lambda = \lambda_1 = \lambda_2$ *in Theorem 3.1,we get the wanted outcome.*□

# Chapter 4

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