Multilayer Flows of Non-Newtonian Fluids Through a Cylindrical Pipe

By

MUHAMMAD IMRAN

CIIT/FA16-RMT-030/LHR

MS

In

Mathematics

COMSATS University Islamabad

Lahore Campus

Spring, 2018

COMSATS University Islamabad, Lahore Campus

Multilayer Flows of Non-Newtonian Fluids Through a Cylindrical Pipe

A Thesis Presented to

COMSATS University Islamabad, Lahore Campus

In partial fulfillment

of the requirement for the degree of

MS (Mathematics)

By

MUHAMMAD IMRAN

CIIT/FA16-RMT-030/LHR

Spring, 2018

Multilayer Flows of Non-Newtonian Fluids Through a Cylindrical Pipe

A Post Graduate Thesis is submitted to the Department of Mathematics as partial fulfillment of the requirement for the award of Degree of MS Mathematics.

Name	Registration Number
MUHAMMAD IMRAN	CIIT/FA16-RMT-030/LHR

Supervisor

Dr. Qurat-ul-Ain Azim

Assistant Professor, Department of Mathematics

COMSATS University Islamabad, Lahore Campus

Spring, 2018

Final Approval

This report titled

Multilayer Flows of Non-Newtonian Fluids Through a Cylindrical Pipe

By

MUHAMMAD IMRAN

CIIT/FA16-RMT-030/LHR

Has been approved

For the COMSATS University Islamabad,Lahore Campus

ExternalExaminer:_____________________________

Dr.

Supervisor: ____________________________

Dr. Qurat-ul-Ain Azim

Department of Mathematics /CUI, Lahore Campus

HoD:__________________________________

Department of Mathematics, Lahore Campus

Declaration

I **Muhammad Imran (CIIT/FA16-RMT-030/LHR)** hereby declare that I have produced the work presented in this thesis, during the scheduled period of study. I also declare that I have not taken any material from any source except referred to wherever due that amount of plagiarism is within acceptable range. If a violation of HEC rules on research has occurred in this thesis, I shall be liable to punishable action under the plagiarism rules of the HEC.

Date:_______________

____

Muhammad Imran

CIIT/FA16-RMT-030/LHR

Certificate

It is certified that **Muhammad Imran (CIIT/FA16-RMT-030/LHR)** has carried out all the work related to this thesis under my supervision at the Department of Mathematics, COMSATS University Islamabad, Lahore Campus, and the work fulfills the requirement for award of MS degree.

Date:__________________

Supervisor:

____

Dr. Qurat-ul-Ain Azim
Assistant Professor
Department of Mathematics

Head of Department:

____

Dr. Sarfraz Ahmad
Associate Professor
Department of Mathematics

DEDICATION

My success is really the fruit of sincerest prayers of my father (Saad Khan) and my Mother who always courage me and be my inspiration in study. And support me morally and financially during the entire period of studies. There is a big contribution of my family who ever remained ready to lend me a helping hand during the period of research.

Muhammad Imran

ACKNOWLEDGEMENTS

Thanks to Allah Almighty, the Master of the entire universe because without HIS help I would have never been able to fulfill the requirements of this thesis. HIS help in the form of guidance and knowledge assisted me throughout this research till its completion.

For the best support and supervision, a very special thanks goes to Dr. Qurat-ul-Ain Azim. Her kind supervision and productive recommendations empowered me to finish this work.

Next, my special thanks to Head of Department of Mathematics for providing a good and healthy research environment required for the accomplishment of this dissertation.

And finally, I am indebted to my parents and brothers and sister for their prayers and continuing support and love.

Muhammad Imran
CIIT/FA16-RMT-030/LHR

Abstract

The concentric n-layer flow for PTT fluid model and two-layer flow for Oldroyed six constant model through cylindrical pipe are considered. Such flows have great significance both theoretical and applications in the chemical processing industries, polymer extraction, mining industry, oil transportation etc and are also useful for biologists for biomedical fluid flow problems. The exact solutions of PTT fluid flow for velocity fields and volume flow rates are derived. Approximate analytical solutions of Oldroyed six constant fluid flow for velocity and volume flow are also presented. In PTT fluid flow problem, the effects of appropriate parameters on the velocity profiles are shown graphically for two-layer and multi-layer flows. In Oldroyed six constant fluid flow problem the effect of model parameters on the velocity profiles are also shown graphically.

Contents

List of Figures

Chapter 1

Preliminaries and Basic Concepts

1.1 Fluid

A fluid is a substance that flow continuously under the action of shear stress, it does not matter how shear stress may be smaller or larger. Fluids are classified as liquid, gases. For example air, oxygen, water, oil , paint etc.

1.2 Properties of Fluids

These properties of fluids have much significance to the study of fluid mechanics.

1.2.1 Density

Density is define as the quantity which contained in unit volume of the substance, there are following four types of density, given that

Mass Density

Mass density ρ of a substance represent the mass per unit volume. Mass density at a given point is derived by choosing the mass δm with small volume δV surrounding the point. To maintain the concept of continuum δV can not be smaller then x^3.

$$\rho = \lim_{\delta V \to x^3} \frac{\delta m}{\delta V} \tag{1.2.1}$$

Specific Weight

The specific weight w of a particle is define as the weight per unit volume. Specific weight varies from one point to another point because weight depends on the gravitational attraction. Using the value of gravitational acceleration g , connection between w and ρ can be conclude from Newton second law.

Relative Density

Relative density σ of a particle is the ratio between the mass density of particle with standard mass density. For example for solids and liquids picked standard mass density is the maximum density of water at 4^oC atmospheric pressure.
$\sigma = \frac{\rho_{\text{substance}}}{\rho_{H_2O}}$ at 4^oC

Specific Volume

Specific volume of a substance is define as the ratio of mass density.

1.2.2 Viscosity

Viscosity of the fluid is basically internal resistance between the layer of fluid, which resist the fluid flow . For example viscosity of honey is so high.

Coefficient of Dynamic Viscosity

The coefficient of dynamic viscosity μ can be defined as shear force per unit area required to drag one layer of the fluid with unit velocity past another layer a unit distance away from it in the fluid.

$$\mu = \frac{\tau}{\frac{dv}{dy}} \tag{1.2.2}$$

where $\tau = \frac{F}{A}$ and $\frac{du}{dy}$ is the local shear velocity.

Kinematic Viscosity

The kinematic viscosity v is define as the ratio of dynamic viscosity to mass density.

$$\nu = \frac{\mu}{\rho} \tag{1.2.3}$$

where μ is dynamic viscosity and ρ is the density of fluid.

1.2.3 Pressure

Pressure is ratio of applied force F and area A . Mathematically the pressure p at a point s is written as ,

$$p = \lim_{\delta A \to 0} \frac{\delta F}{\delta A} \tag{1.2.4}$$

where δA is the area of an element around point s and δF is the normal force due to fluid on δA.

1.3 Types Of Fluids

There are following types of fluids classified as:

1.3.1 Newtonian Fluids

Newtonian fluids are those fluids in which shear stress is directly proportional to the rate of change or fluids which satisfy the Newton law of the viscosity Mathematically :

$$\tau \propto \mu \frac{du}{dy} \tag{1.3.1}$$

Examples of such fluids are air , water etc.

1.3.2 Non-Newtonian Fluids

Non-Newtonian fluids are those type of fluids which does not satisfy the Newton law of viscosity. .

For example toothpaste, blood, paint etc.

The stress might be given by,for example :

$$\tau = K(\frac{du}{dy})^n, n \neq 1 \tag{1.3.2}$$

where n is flow behavior index and K is the regularity index. Above equation can be changes into Newton's law of viscosity for $n = 1$. Most fluids in nature are non-

Newtonian.

1.3.3 Real Fluids (Viscous Fluids)

All fluids for which $\mu \neq 0$ are called real fluids.

1.3.4 Ideal Fluids (Inviscid Fluids)

All fluids for which $\mu = 0$ are called ideal fluids.

1.4 Continuum Concept of a Fluid

While the properties of a fluid emerge from its molecular structure, usually engineering problems are scrutinize with the immensity of fluids. Normally at a microscopic level, a fluid consists of molecules with a lot of space in between. Mostly involved number of molecules are extensive, and separation between these molecules is normally negligible in comparison to the distance involved in the practical situation being studied. Under these conditions, it is usual to consider a fluid as a continuum.

1.5 Dimensions and Units

The measurement of physical quantities has physical importance in science and engineering. The primary or basic quantities are length L, mass M, and time T.

1.5.1 Units

Basically arbitrary names are units which are assigned to the fundamentals dimensions adopted as a standard for measurement.e.g the basic dimensions for length would be measured in terms of standardize unit of length such as meter.

1.5.2 Dimensions

It is used to refer any measurable quantity .e.g we refer to physical quantity such as length , time , mass and temperature as dimension.

1.6 International System of Units

There are many system of units that consider different quantities as fundamentals e.g

1.6.1 SI System

Mass $\{M\}$, Length $\{L\}$, Time $\{t\}$, Temperature $\{T\}$

1.6.2 British Gravitational System

Force $\{F\}$, Length $\{L\}$, Time $\{t\}$, Temperature $\{T\}$

1.6.3 English Engineering System

Force $\{F\}$, $Mass\{M\}$, Length $\{L\}$, Temperature $\{T\}$.
In fluid mechanics only 4 of the fundamentals quantities length, mass, time and temperature are needed.
Some commonly used derived quantities along with their units and dimensions are given in the table below.

1.7 Types Of Fluid Flow

There are following types of fluid flows

1.7.1 Uniform Flow

Such type of flow in which the the velocity of liquid particle at all segment of channel or pipe are not same.

Table 1.1: Dimensions and Units

Quantity	Defining Equation	Dimensions	SI Unit
Length	Length	$[L]$	m
Velocity	Distanc/time	$[LT^{-1}]$	ms^{-1}
Acceleration	Velocity/Time	$[LT^{-2}]$	ms^{-2}
Force	Mass$\times$ Acceleration	$[MLT^{-2}]$	$N(kgms^{-2})$
Mass density	Mass/Volume	$[ML^{-3}]$	kgm^{-3}
Impulse	Force$\times$Time	$[MLT^{-1}]$	$kgms^{-1}$
Viscosity dynamic	Shear stress/Velocity gradient	$[ML^{-1}T^{-1}]$	Nsm^{-2}
Viscosity kinematic	Dynamic Viscosity/Mass density	$[L^2T^{-1}]$	m^2s^{-1}
Pressure	Force/Area	$[ML^{-1}T^{-2}]$	$Pa(=Nm^{-2})$
Work,Energy	Force$\times$ Distance	$[ML^2T^{-2}]$	$J(=Nm)$
Power	Work/Time	$[ML^2T^{-3}]$	$Watt(Js^{-1})$

1.7.2 Non-Uniform Flow

If the velocity changes instantaneously from point to point then flow is called non-uniform flow.

1.7.3 Steady Flow

A fluid flow is steady if different quantities like velocity, pressure and cross-section, vary from point to point but do not change with time.

1.7.4 Unsteady Flow

If at a given point velocity, pressure and cross-section changes with time is known as unsteady flow.

1.7.5 Steady Uniform Flow

A steady uniform flow is a flow in which Conditions do not changes with position or time.

1.7.6 Steady non-uniform Flow

A flow in which quantities like velocity, pressure, cross-section may changes form point to point but do not changes with time.

1.7.7 Unsteady Uniform Flow

A flow in which velocity, pressure and cross-section remain same at each point but changes with time.

1.7.8 Unsteady Non-uniform Flow

A flow in which velocity, pressure and cross-section does not remain same at every point and also changes with time.

1.7.9 Laminar Flow

Laminar flow is a flow in which each particle have a specific path, none of particle cross each other.

1.7.10 Turbulent Flow

Turbulent flow is the flow in which each none of the particle have specific path. Each particle cross the other particle.

1.7.11 Incompressible Flow

Incompressible flow is the flow in which density remain constant throughout, do not changes with time and position.

1.7.12 Compressible Flow

A compressible flow is the flow in which density does remain constant with time and position.

1.7.13 Irrotational Flow

If curl vector $\vec{v} = 0$ and $\vec{v} = 0$ is the fluid velocity vector then flow field is Irrotational.

Mathematically :

$$\nabla \times \vec{v} = 0 \tag{1.7.1}$$

1.7.14 Rotational Flow

If curl vector $\vec{v} \neq 0$ then flow field is rotational.

Mathematically :

$$\nabla \times \vec{v} \neq 0 \tag{1.7.2}$$

1.8 Equations Governing Fluid Flows:-

1.8.1 Continuity Equation

Continuity equation in vector form is,

$$\frac{\partial \rho}{\partial t} + \nabla.(\rho \mathbf{v}) = 0 \tag{1.8.1}$$

In which ρ is the density of fluid at time t and v is the velocity field of the fluid.

For a fluid of constant density or incompressible fluid above equation becomes ,

$$\nabla.\mathbf{v} = 0 \tag{1.8.2}$$

1.8.2 Two Dimensional Continuity Equation For Incompressible Flow

For incompressible flow $\rho = constant$ therefore , continuity equation becomes

$$\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} = 0 \tag{1.8.3}$$

1.8.3 Continuity Equation For steady Flow

For steady flow $\frac{\partial}{\partial t} = 0$

$$div(\rho \vec{v}) = 0 \tag{1.8.4}$$

1.8.4 Momentum Equation

Vector Form of Momentum equation is ,

$$\rho \frac{D\mathbf{V}}{Dt} = \nabla.\tau - \nabla p + \rho \mathbf{b} \tag{1.8.5}$$

1.8.5 Momentum Equation In Cartesian Coordinates

- x-component

$$\rho \frac{Du}{Dt} = \rho b_x - \frac{\partial p}{\partial x} + \frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} \tag{1.8.6}$$

- y-component

$$\rho \frac{Dv}{Dt} = \rho b_y - \frac{\partial p}{\partial y} + \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \tau_{yy}}{\partial y} + \frac{\partial \tau_{zy}}{\partial z} \tag{1.8.7}$$

- z-component

$$\rho \frac{Dw}{Dt} = \rho b_z - \frac{\partial p}{\partial z} + \frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \tau_{zz}}{\partial z} \tag{1.8.8}$$

1.8.6 Momentum Equation In Cylindrical Coordinates

The equations of momentum for incompressible Newtonian fluid in cylindrical coordinates (r, θ, z),

- r-component

$$\rho \left(\frac{Du_r}{Dt} - \frac{u^2 \theta}{r} \right) = \rho b_r - \frac{\partial p}{\partial r} + \frac{1}{r} \frac{\partial (r \tau_{rr})}{\partial r} + \frac{1}{r} \frac{\partial \tau_{\theta r}}{\partial \theta} - \frac{\tau_{\theta\theta}}{r} \tag{1.8.9}$$

- θ-component

$$\rho \left(\frac{Du_\theta}{Dt} - \frac{u_r u_\theta}{r} \right) = \rho b_\theta - \frac{\partial p}{\partial \theta} + \frac{1}{r^2} \frac{\partial (r^2 \tau_{r\theta})}{\partial r} + \frac{1}{r} \frac{\partial \tau_{\theta\theta}}{\partial \theta} - \frac{\partial \tau_{z\theta}}{z} \tag{1.8.10}$$

• z-component

$$\rho \frac{Du_z}{Dt} = \rho b_z - \frac{\partial p}{\partial z} + \frac{1}{r}\frac{\partial (r\tau_{rz})}{\partial r} + \frac{1}{r}\frac{\partial \tau_{\theta z}}{\partial \theta} - \frac{\tau_{zz}}{z} \tag{1.8.11}$$

• Differential Equations

Such type of equations in which derivative of one or more dependent variable w.r.t independent variable involved is known as differential equation. These types of equations play a important rule in many disciplines including physics,engineering,economics and biology. Differential equations can be divided into many types.

• **Ordinary Differential Equations**

Ordinary differential equations are those differential equations contains one or more functions of one independent variable and its derivative.

$$\frac{d^2y}{dx^2} - \frac{dy}{dx} + 4y = 0 \tag{1.8.12}$$

• **Partial Differential Equations**

Partial differential equations are those equations which consist on multi-variables and their partial derivatives. e.g

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \tag{1.8.13}$$

• **Order of Differential Equations**

Order of differential equations is the order of maximum derivative involve. e.g

$$\frac{d^2y}{dx^2} - \frac{dy}{dx} + 4y = 0 \tag{1.8.14}$$

so order of above equation is 2.

• **Linear Differential Equations**

Linear differential equation variables of an equation does not appear in the power of one or appear in the power of more then one e.g

$$\frac{dy}{dx} + \frac{1}{\frac{d^2y}{dx}} = 0 \tag{1.8.15}$$

• Non-Differential Equations

Linear differential equation means such type of equations in which variable of an equation occur only in the the power of one.

$$(1-y)\frac{dy}{dx} + 2y = e^{(x)} \tag{1.8.16}$$

1.9 Some Basic Flows in Fluid Mechanics

1.9.1 Couette Flow

The couette flow or shear flow is the such type of flow in which two parallel plates are placed as one of which $y = 0$ is at rest and other at $y = h$ movies uniformly with velocity U as shown in figure 1.1. Flow is along x-axis as shown in the figure. There is no pressure gradient in this case $\frac{dp}{dx} = 0$. Thus such types of flow just depends on the motion of the upper plate. Since flow is steady , laminar and unidirectional then momentum equation reduces to

$$\frac{\partial p}{\partial x} = \mu \frac{\partial^2 u}{\partial y^2} \tag{1.9.1}$$

Integrating above w.r.t 'y'

$$u = \frac{1}{\mu}\frac{dp}{dx}\frac{y^2}{2} + Ay + B \tag{1.9.2}$$

Conditions for this problem becomes,

$$u = 0 \text{ at } y = 0 \text{ and } u = U \text{ at } y = h \tag{1.9.3}$$

such that

$$u(0) = 0 \text{ and } u(h) = U \tag{1.9.4}$$

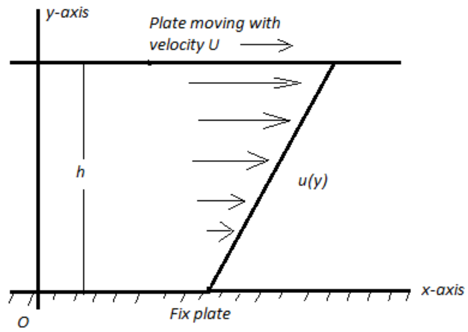

Figure 1.1: Couette Flow

As pressure gradient is equal to zero,then above equation for couette flow reduces to

$$u(y) = Ay + B \tag{1.9.5}$$

Using condition (1.9.4)

$$A = \frac{U}{h} \tag{1.9.6}$$

then (1.9.5) will become as,

$$u(y) = \frac{U}{h}y \tag{1.9.7}$$

In non-dimensional form,velocity distribution will become as,

$$\frac{u}{U} = \frac{y}{h} \tag{1.9.8}$$

Above velocity distribution induced in a fluid is due to the motion of upper plate at constant velocity. Couette flow is called plan couette flow.

1.9.2 Plane Poiseuille Flow

As we have already discussed that for simple Couette flow one plate is at rest but other plate is moving with velocity U but for plane poiseuille flow both upper and lower plates are fixed at a constant distance from the x-axis as shown in figure 1.2.

x-axis is along the center between these two fixed parallel plates. The fully developed

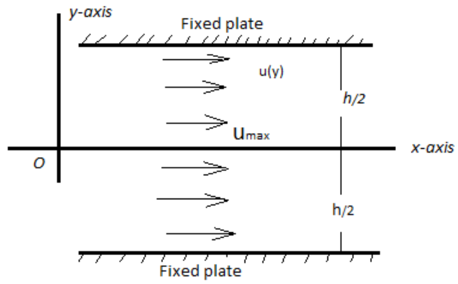

Figure 1.2: Plan Poiseuille Flow

flow between the plates referred to plan poiseuille flow. Also consider that both fixed plates be situated at $y = -\frac{h}{2}$ and $y = \frac{h}{2}$ while pressure gradient is $\frac{dp}{dx}$. So boundary conditions are $u = 0$ at $y = \pm\frac{h}{2}$.
Then velocity distribution is written as,

$$u = \frac{1}{\mu}\frac{dp}{dx}\frac{y^2}{2} + Ay + B \tag{1.9.9}$$

Using conditions $u = 0$ at $y = +\frac{h}{2}$

$$0 = \frac{h^2}{\mu}\frac{dp}{dx} + A\frac{h}{2} \tag{1.9.10}$$

Using conditions $u = 0$ at $y = -\frac{h}{2}$

$$0 = \frac{h^2}{\mu}\frac{dp}{dx} - A\frac{h}{2} \tag{1.9.11}$$

From (1.9.10) and (1.9.11)

$$A = 0, B = -\frac{h^2}{8\mu}\frac{dp}{dx} \tag{1.9.12}$$

By substituting values of A and B in (1.9.9) then,

$$u = -\frac{h^2}{8\mu}\frac{dp}{dx}\left[1 - 4\left(\frac{y}{h}\right)^2\right] \tag{1.9.13}$$

1.9.3 The Hagen Poiseuille Flow

Consider steady laminar flow of viscous incompressible fluids flow through an infinite long straight horizontal circular pipe of radius R as shown in figure 1.3. We will use cylindrical coordinates to describe the flow geometry. We also consider that $z-axis$ is along the axis of pipe. Such type of axially symmetric flow through a circular pipe is called Hagen poiseuille flow. As flow is along the z-axis then velocity along radial and tangential components will be zero.

$$\frac{1}{r}\frac{\partial}{\partial r}(rv_r)+\frac{1}{r}\frac{\partial v_\theta}{\partial \theta}+\frac{\partial v_z}{\partial z}=0 \tag{1.9.14}$$

velocity is the function of $'r'$.

$$v_z = v_z(r) \tag{1.9.15}$$

Momentum equation in cylindrical coordinates becomes ,

$$-\frac{1}{\rho}\frac{\partial p}{\partial r}=0 \tag{1.9.16}$$

$$-\frac{1}{\rho r}\frac{\partial p}{\partial \theta}=0 \tag{1.9.17}$$

$$-\frac{1}{\rho}\frac{\partial p}{\partial z}+\nu\left[\frac{\partial^2 v_z}{\partial r^2}+\frac{1}{r}\frac{\partial v_z}{\partial r}\right]=0 \tag{1.9.18}$$

$$\frac{\partial p}{\partial r}=\frac{\partial p}{\partial \theta}=0 \tag{1.9.19}$$

so,

$$\frac{dp}{dz}=\mu\left[\frac{d^2 v_z}{dr^2}+\frac{1}{r}\frac{dv_z}{dr}\right] \tag{1.9.20}$$

Integrating twice w.r.t $'r'$,

$$v_z=\frac{r^2}{4\mu}\frac{dp}{dz}+A\ln r+B \tag{1.9.21}$$

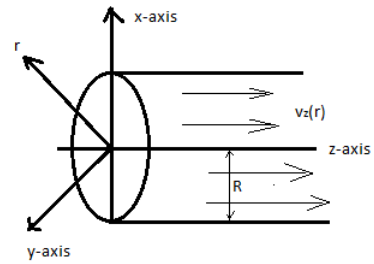

Figure 1.3: Hagen Poiseuille Flow

In which constants A and B are determined using boundary conditions. So $A = 0$ because v_z is infinite at $r = 0$, then

$$v_z = \frac{r^2}{4\mu}\frac{dp}{dz} + B \tag{1.9.22}$$

But second boundary condition is no-slip condition at the wall i.e ,

$$v_z = 0 \text{ at } r = R \tag{1.9.23}$$

Using no-slip boundary condition then

$$B = -\frac{1}{4\mu}\frac{dp}{dz}R^2 \tag{1.9.24}$$

substituting value of B then velocity will become as,

$$v_z = -\frac{R^2}{4\mu}\frac{dp}{dz}\left[1 - (\frac{r}{R})^2\right] \tag{1.9.25}$$

1.10 Approximate Analytical Techniques for Solving Non-linear Equations

Sometime exact solution of non-linear differential equation is not possible or difficult to find. In such situation we use some approximate analytical techniques to solve those equations. We will use one of the following approximate techniques

1.10.1 Homotopy Perturbation Method

To understand the basic idea of homotopy perturbation method first we consider a non-linear differential equation

$$L(w)+N(w)=f(r), r\in\Omega \tag{1.10.1}$$

With boundary conditions

$$C=(w,\frac{\partial w}{\partial n}), r\in\Gamma \tag{1.10.2}$$

Where L is linear operator and N is non-linear operator, C is the boundary operator, w is dependent variable and r is the independent variable. Γ is the boundary of domain Ω and $f(r)$ is analytical function.
we will construct a homotopy

$$v(r,s):\Omega\times[0,1]\rightarrow\Re \tag{1.10.3}$$

which satisfies

$$M(v,s)=(1-s)[L(v)-L(w_0)]+s[L(v)+N(v)-f(r)]=0 \tag{1.10.4}$$

or

$$M(v,s)=L(v)-L(w_0)+sL(w_0)+s[N(v)-f(r)]=0 \tag{1.10.5}$$

Where $s\in[0,1]$ is an embedding parameter, w_0 is initial approximation which satisfy the boundary conditions.
From (1.10.5) and (1.10.4) ,

$$M(v,0)=L(v)-L(w_0)=0 \tag{1.10.6}$$

$$M(v,1)=L(v)+N(v)-f(r) \tag{1.10.7}$$

By changing $'s'$ from 0 to 1 , $v(r,s)$ changes from $w_0(r)$ to $w(r)$. In topology this is called a deformation and $L(v)-L(w_0)$ and $L(v)+N(v)-f(r)$ are known as homotopic. (1.10.5) and (1.10.4) can be expresses as power series of $'s'$.

$$v = v_0 + sv_1 + s^2 v_2 + ... \tag{1.10.8}$$

Using (1.10.8) into (1.10.5) , we solve for different orders of the problem by comparing like coefficient of $'s'$ on both sides. Then approximate solution becomes ,

$$w = \lim_{s \longrightarrow 1} v = v_0 + v_1 + v_2 + ... \tag{1.10.9}$$

1.10.2 Modified Homotopy Perturbation

Consider a non-linear differential equation

$$L(w) + N(w) = f(r), r \in \Omega \tag{1.10.10}$$

With boundary conditions

$$C(w, \frac{\partial w}{\partial n}), r \in \Gamma \tag{1.10.11}$$

where L is linear operator and N is non-linear operator. C is the boundary operator, Γ is the boundary of domain Ω and $f(r)$ is analytical function.
Modified homotopy perturbation method depends on the idea of replacing $f(r)$ by a series of infinite components And in Tayler series form,

$$f(y) = \sum_{n=0}^{\infty} f_n(y) \tag{1.10.12}$$

According to supposition $f(r) = f_1(r) + f_2(r)$
we will construct homotopy,

$$v(y,q) : \Omega \times [0,1] \longrightarrow \Re \tag{1.10.13}$$

so,

$$N(v,q) = (1-q)[L(v) - L(w_0)] + q[L(v) + N(v) + N(v)] - \sum_0^\infty q^n f_n(y) = \quad (1.10.14)$$

equivalent to

$$N(v,q) = (L(v) - L(w_0) + qL(w_0) + qN(v) = \sum_{n=0}^\infty q^n f_n(y) \quad (1.10.15)$$

where $q \in [0,1]$ is an embedding parameter, w_0 is initial approximation we assume solution in the form of

$$v(y) = \sum_0^\infty q^n v_n(y) \quad (1.10.16)$$

if we choose $f_1 = f$ and $f_0 = f_1 = f_2 = ... = 0$ then (1.10.15) reduces to homotopy method. So Modified homotopy method depends on the selection of $f_0, f_1, f_2, ...$ then the solution of $w(r)$ of (1.10.10) can be obtained fast as HPM,

$$v(y,q) = \lim_{q \to 1} v(y,q) = v_0 + qv_1 + q^2 v_2 + ... \quad (1.10.17)$$

If we solve the problem using HPM the resulting expressions for velocity profile contain high power constant and solution of theses constants are not easy then use MHPM . Such problem arises for Poiseuille problems and for CouettePoiseuille flow problems.

1.10.3 Homotopy Analysis Method

Let us consider a non-linear differential equation such as,

$$A(w) + f(r) = 0 \quad (1.10.18)$$

where A is non-linear operator, $f(r)$ is known function, For homotopy analysis method we construct a family of equation which is

$$(1-q)\pounds[v(r,q)-w_0(r)] = q\hbar\{A[v(r,q)]-f(r)\} \tag{1.10.19}$$

where $\pounds$ is linear operator, $w_0(r)$ is initial guess, $\hbar$ is auxiliary parameter, $q \in [0,1]$ is an embedding parameter, Liao [21] [22] expended $v(r,q)$ in Tyler's series about embedding parameter

$$v(r,q) = w_0(r) + \sum_{m=1}^{\infty} w_m(r)p^m \tag{1.10.20}$$

where

$$w_m(r) = \frac{1}{m!}\frac{\partial^m v(r,q)}{\partial q^m}|_{q=0} \tag{1.10.21}$$

The convergence of (1.10.20) depends upon the parameter $\hbar$, if convergent at $q = 1$, then

$$w(r) = w_0(r) + \sum_{m=1}^{\infty} w_m(r) \tag{1.10.22}$$

Differentiating zeroth-order deformation of (1.10.19) m-times w.r.t q and the deviding by $m!$ and finally substitute $q = 0$ we obtain

$$\pounds[w_m(r) - \chi_m w_{m-1}(r)] = \hbar\Re_m(r) \tag{1.10.23}$$

In which

$$\chi_m = [^{0,m\leq 1}_{1,m>1} \tag{1.10.24}$$

$$\Re_m(r) = \frac{1}{(m-1)!}\{\frac{d^{k-1}}{dq^{k-1}}A[w_0(r)] + \sum_{m=1}^{\infty} w_m(r)q^m\}|_{q=0} \tag{1.10.25}$$

we can solve non-linear equations using this technique at different order of approximation.

1.11 Thesis Plan And Layout

Our proposed research work is based upon the analytical study of concentric flow of n-layers of PTT fluids and two-layer flow of Oldroyd six constant fluids. All the flows are considered to be immiscible and incompressible, through a cylindrical pipe. These types of flows are encountered in many industrial and manufacturing processes. In industry, the fluids that are mostly used are complex in nature and exhibit a variety of characteristics and rheological properties. Our aim is to study these complex non-Newtonian fluids in this thesis.

The thesis is planned as follows: In first chapter we have discussed some preliminaries and basic concepts. In second chapter we analyze the multilayer flow of PTT fluids through a cylindrical pipe. Using MATLAB for graphing purposes, we also discuss the effect of different parameters on the velocity profile. In third chapter we also analyze the two-layer flow of Oldroyd six constant fluids through a cylindrical pipe. Finally, the effect of different parameters on the velocity profile is presented graphically. In last chapter, we discuss conclusions and future work.

Chapter 2

The n-Layers Concentric Flow of (PTT) Fluids Through a Cylindrical Pipe

2.1 Introduction

Majority of scientists and engineers deal with different flow of different fluids which are Newtonian in nature. In most practical situations, Newtonian flow behavior is not valid and rather more complex non-Newtonian flow response must be modeled. Flow of multiple layers of fluids arises in chemical processing industries, polymer extraction, mining industry, oil transportation and also useful for biologists for biomedical fluid flow problems.

For example, in polymer processing, materials with desired characteristics can be produced with combination of various melt streams. Non-Newtonian fluids are also encountered in mining industry where slurries and muds are often handled. Use of double layer in non-Newtonian fluids is of much interest like double layer coating, double layer paints in different chemical processes.

In oil transportation when crude oil is transported through long pipelines, pumping costs can be reduced using low viscoelastic fluids which reduces the pressure gradient. The study of multiple layer flows is also important due to its logical connection to biomedical fluid flow applications.

Most of present work deals with two-layer fluid flows. Bird et al. [1] have analyzed the two-phase flow of incompressible, immiscible Newtonian fluids. Kapur and Shukla [2] investigated the flow of Newtonian fluids between two parallel plates. Lohrasbi et al. [3], Malashetty et al. [4], [5] have analyzed the Magnetohydrodynamics heat transfer in two-phase flow. Shail [6] have investigated the laminar two- phase flow in Magnetohydrodynamics.

In [7], [8], [9] instability analysis and stability analysis of two-phase flow for Newtonian fluids have been discussed. But [10], [11], [12], [13], [14] also investigated the stability analysis for non-Newtonian fluids. Peristaltic flow of two-layers of Power law fluids have been studied by Usha and Ramchandra Rao in [15]. Recently Taza Gul et al. [16] have investigated the exact solution of two thin film non-Newtonian immiscible fluids on a vertical belt.

Though most people worked on two- layers flow and these results are useful but their applicability is restricted because some industrial processes require addition of multi fluid layers. There is not much literature available on n-layer fluid flows in cylindrical pipe geometry because most of the above work deals with two-layer fluid flows. In present literature, Brauner et al. [17], [18] and Leib et al. [19] have worked on multilayer of Newtonian fluids. Siddiqui et al. [20] also worked on exact solutions of concentric n-layer flows of viscous fluids in pipe. Mostly fluids used in industry can only be modeled realistically using non-Newtonian fluid models.

So in this problem we have considered n-layer flows for non-Newtonian fluids through cylindrical pipe. Both linear PTT model and exponential PTT model are used. In both linear PTT model problem and exponential PTT model problem, we obtain non-linear differential equation for velocity which is accompanied by different boundary conditions for fluid velocity and shear stress. Exact solutions for some useful physical quantities like velocity profile and flow rates are derived. Some special cases of these problems for flow rate also discussed. Using MATLAB the effects of different parameters on the velocity profiles are presented graphically for two-layer and multi-layer flows.

2.2 Problem Formulation

2.2.1 Geometry of the Problem

We will consider the concentric n-layers flow for immiscible and incompressible PTT fluids through cylindrical geometry. Let $z-axis$ is along the axis of pipe and $'r'$ is radially outward from the axis. The pressure gradient $\frac{dP}{dz} = -P$ is along axis of pipe. We denote the radius of pipe by R_1.

Interfaces of two fluids starting from $R_2, R_3, ..., R_n$ from the axis of pipe as shown in figure.We also consider that fluid flow is steady, unidirectional and fully developed.

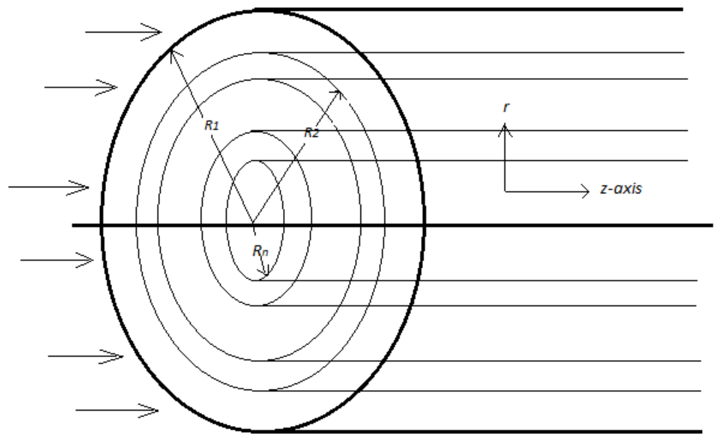

Figure 2.1: Geometry Of The Problem

2.2.2 Basic Equations

The basic equations governing the motion of an incompressible fluids flow with constant density are mass conservation

$$\nabla.\mathbf{V}^{(k)} = 0 \tag{2.2.1}$$

and momentum equation

$$\rho^{(k)} \frac{d\mathbf{V}^{(\mathbf{k})}}{dt} = \mathrm{div}\tau^{(k)} - \nabla p + \rho^{(k)}\mathbf{b} \tag{2.2.2}$$

where the superscripts $(k = 1, 2, ..., n)$ denote the number of different fluid layers. $k = 1$ indicates the first layer (outermost) along with solid boundary of pipe , and $k = 2$ is for the fluid layer which is adjacent to first layer (outermost) and so on. Innermost layer will be around the axis of cylindrical pipe. $V^{(k)}$ represents the velocity of kth layer of fluid, $\frac{d}{dt}$ expresses material derivative, p is the fluid pressure, $\tau^{(k)}$ is the extra stress tensor, $\rho^{(k)}$ is density of fluid and $\mathbf{b}$ is the body force vector. General form of constitutive equation defining Phan-Thien Tanner (PTT) fluid will be written as,

$$f\left(tr(\tau^{(k)})\right)\tau^{(k)} + \lambda^{(k)}\hat{\tau} = 2\eta^{(k)}\mathbf{A^{(k)}} \tag{2.2.3}$$

Where $f\left(tr(\tau^{(k)})\right)$ is the trace of stress tensor , $\lambda^{(k)}$ is relaxation time and $\mathbf{A^{(k)}}$ is deformation rate tensor and $\hat{\tau}$ represent upper convected derivative. And $\mathbf{A^{(k)}}$ is defined as,

$$\mathbf{A^{(k)}} = \frac{1}{2}\left[\mathbf{L^{(k)}} + (\mathbf{L^{(k)}})^T\right] \tag{2.2.4}$$

Where $\mathbf{L^{(k)}}$ is defined as,

$$\mathbf{L^{(k)}} = \nabla\mathbf{V^{(k)}} \tag{2.2.5}$$

And upper convected derivative is given by,

$$\hat{\tau} = \frac{d\tau^{(k)}}{dt} - \tau^{(k)}(\nabla\mathbf{V^{(k)}}) - (\nabla\mathbf{V^{(k)}})^T\tau^{(k)} \tag{2.2.6}$$

According to Tanner's classification, Linear form of (PTT) fluids model can be written as,

$$f(tr(\tau^{(k)}) = 1 + \frac{\varepsilon^{(k)}\lambda^{(k)}}{\eta^{(k)}}tr(\tau^{(k)}) \tag{2.2.7}$$

Exponential form of (PTT) fluid model is,

$$f(tr(\tau^{(k)}) = \exp\left(\frac{\varepsilon^{(k)}\lambda^{(k)}}{\eta^{(k)}}tr(\tau^{(k)})\right) \tag{2.2.8}$$

2.2.3 Boundary Conditions

The flow characteristics used for the boundary conditions of this problem are symmetry of flow about the axis of pipe. The continuity of stresses and velocities of fluids at the interfaces and there is no-slip conditions at the wall of the pipe.
Mathematically ,

$$\tau_{rz}^{(n)} = 0 \text{ at } r = 0 \tag{2.2.9}$$

and at axis of pipe,

$$\tau_{rz}^{(k)} = \tau_{rz}^{(k+1)} \text{ at } r = R_{k+1}, \quad k = 1, 2, \ldots, n-1. \tag{2.2.10}$$

$$u^{(k)} = u^{(k-1)} \text{ at } r = R_k, \quad k = 2, 3, \ldots, n. \tag{2.2.11}$$

at the interfaces.

No-slip conditions at the wall of pipe,

$$u^{(1)} = 0 \text{ at } r = R_1 \tag{2.2.12}$$

Velocities and extra stresses becomes,

$$\mathbf{V}^{(k)} = \left[0, 0, u^{(k)}(r)\right] \text{ and } \tau^{(k)} = \tau^{(k)}(r) \tag{2.2.13}$$

2.3 Exact Solution of the Linear (PTT) Fluid Model

2.3.1 Shear Stresses

By substituting (2.2.13) into (2.2.1) and (2.2.2) then (2.2.1) is identically satisfied z-component of (2.2.2) in the absence of body forces we get,

$$\frac{dp}{dz} - \frac{1}{r}\frac{d}{dr}(r\tau_{rz}^{(k)}) = 0, \quad k = 1, 2, \ldots, n. \tag{2.3.1}$$

integrating w.r.t $'r'$ we will get,

$$r\tau_{rz}^{(k)} = \frac{dp}{dz}\frac{r^2}{2} + C^{(k)} \tag{2.3.2}$$

where $C^{(k)}$ are constant of integration. In view of condition (2.2.9), $C^{(n)} = 0$ and above expression will be written as,

$$\tau_{rz}^{(n)} = \frac{dp}{dz}\frac{r}{2} \tag{2.3.3}$$

Equation (2.3.3) with conditions (2.2.10) when used in (2.3.2) then

$\Rightarrow C^{(k)}$ =0.

This result in the general expression,

$$\tau_{rz}^{(k)} = \frac{dp}{dz}\frac{r}{2}; \quad k = 1, 2, \ldots, n. \tag{2.3.4}$$

for shear stresses of the fluids.

2.3.2 Velocities

Now we need to find the velocities for the k fluids. First of all we incorporate geometry of our problem.From equation (2.2.4) we know that

$$\mathbf{A_1^{(k)}} = \frac{1}{2}\left[\mathbf{L^{(k)}} + (\mathbf{L^{(k)}})^T\right] \tag{2.3.5}$$

Where $\mathbf{L^{(k)}}$ is

$$\mathbf{L^{(k)}} = grad\mathbf{V^{(k)}} \tag{2.3.6}$$

$$L^{(k)} = \begin{bmatrix} \frac{\partial}{\partial r} \\ \frac{1}{r}\frac{\partial}{\partial \theta} \\ \frac{\partial}{\partial z} \end{bmatrix} \left[0, 0, u^{(k)}(r)\right] \tag{2.3.7}$$

$$L^{(k)} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \frac{\partial u^{(k)}}{\partial r} & 0 & 0 \end{bmatrix} \tag{2.3.8}$$

Replacing $\frac{\partial u^{(k)}}{\partial r} = \frac{du^{(k)}}{dr}$ then

$$L^{(k)} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \frac{du^{(k)}}{dr} & 0 & 0 \end{bmatrix} \tag{2.3.9}$$

$$(L^{(k)})^T = \begin{bmatrix} 0 & 0 & \frac{du^{(k)}}{dr} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{2.3.10}$$

Substituting values of $L^{(k)}$ and $(L^{(k)})^T$ into (2.2.4) we obtain

$$A_1^{(k)} = \frac{1}{2} \begin{bmatrix} 0 & 0 & \frac{du^{(k)}}{dr} \\ 0 & 0 & 0 \\ \frac{du^{(k)}}{dr} & 0 & 0 \end{bmatrix} \tag{2.3.11}$$

Now using equation (2.2.6)

$$\hat{\tau} = \frac{d\tau^{(k)}}{dt} - \tau^{(k)}(\nabla \mathbf{V}^{(\mathbf{k})}) - (\nabla \mathbf{V}^{(\mathbf{k})})^T \tau^{(k)} \tag{2.3.12}$$

$$\hat{\tau} = -\begin{bmatrix} \tau_{rr}^{(k)} & \tau_{r\theta}^{(k)} & \tau_{rz}^{(k)} \\ \tau_{\theta r}^{(k)} & \tau_{\theta\theta}^{(k)} & \tau_{\theta z}^{(k)} \\ \tau_{zr}^{(k)} & \tau_{zz}^{(k)} & \tau_{z\theta}^{(k)} \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \frac{du^{(k)}}{dr} & 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 0 & \frac{du^{(k)}}{dr} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \tau_{rr}^{(k)} & \tau_{r\theta}^{(k)} & \tau_{rz}^{(k)} \\ \tau_{\theta r}^{(k)} & \tau_{\theta\theta}^{(k)} & \tau_{\theta z}^{(k)} \\ \tau_{zr}^{(k)} & \tau_{z\theta}^{(k)} & \tau_{zz}^{(k)} \end{bmatrix} \tag{2.3.13}$$

$$\hat{\tau} = -\begin{bmatrix} \frac{du^{(k)}}{dr}\tau_{rz}^{(k)} & 0 & 0 \\ \frac{du^{(k)}}{dr}\tau_{\theta z}^{(k)} & 0 & 0 \\ \frac{du^{(k)}}{dr}\tau_{zz}^{(k)} & 0 & 0 \end{bmatrix} - \begin{bmatrix} \frac{du^{(k)}}{dr}\tau_{zr}^{(k)} & \frac{du^{(k)}}{dr}\tau_{z\theta}^{(k)} & \frac{du^{(k)}}{dr}\tau_{zz}^{(k)} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{2.3.14}$$

$$\hat{\tau} = \begin{bmatrix} -2\frac{du^{(k)}}{dr}\tau_{rz}^{(k)} & -\frac{du^{(k)}}{dr}\tau_{z\theta}^{(k)} & -\frac{du^{(k)}}{dr}\tau_{zz}^{(k)} \\ -\frac{du^{(k)}}{dr}\tau_{\theta z}^{(k)} & 0 & 0 \\ -\frac{du^{(k)}}{dr}\tau_{zz}^{(k)} & 0 & 0 \end{bmatrix} \tag{2.3.15}$$

Substituting (2.2.7) ,(2.3.15) , (2.3.11) in (2.2.3) then

$$\left[1 + \frac{\varepsilon^{(k)}\lambda^{(k)}}{\eta^{(k)}} tr(\tau^{(k)})\right]\tau^{(k)} + \lambda^{(k)} \begin{bmatrix} -2\frac{du^{(k)}}{dr}\tau_{rz}^{(k)} & -\frac{du^{(k)}}{dr}\tau_{z\theta}^{(k)} & -\frac{du^{(k)}}{dr}\tau_{zz}^{(k)} \\ -\frac{du^{(k)}}{dr}\tau_{\theta z}^{(k)} & 0 & 0 \\ -\frac{du^{(k)}}{dr}\tau_{zz}^{(k)} & 0 & 0 \end{bmatrix} = \eta^{(k)} \begin{bmatrix} 0 & 0 & \frac{du^{(k)}}{dr} \\ 0 & 0 & 0 \\ \frac{du^{(k)}}{dr} & 0 & 0 \end{bmatrix} \tag{2.3.16}$$

So

$$\left[1+\frac{\varepsilon^{(k)}\lambda^{(k)}}{\eta^{(k)}}tr(\tau^{(k)})\right]\begin{bmatrix}\tau_{rr}^{(k)} & \tau_{r\theta} & \tau_{rz}^{(k)}\\ \tau_{\theta r} & \tau_{\theta\theta} & \tau_{\theta z}\\ \tau_{zr}^{(k)} & \tau_{z\theta} & \tau_{zz}^{(k)}\end{bmatrix}+\lambda^{(k)}\begin{bmatrix}-2\frac{du^{(k)}}{dr}\tau_{rz}^{(k)} & -\frac{du^{(k)}}{dr}\tau_{z\theta}^{(k)} & -\frac{du^{(k)}}{dr}\tau_{zz}^{(k)}\\ -\frac{du^{(k)}}{dr}\tau_{\theta z}^{(k)} & 0 & 0\\ -\frac{du^{(k)}}{dr}\tau_{zz}^{(k)} & 0 & 0\end{bmatrix}$$
$$=\eta^{(k)}\begin{bmatrix}0 & 0 & \frac{du^{(k)}}{dr}\\ 0 & 0 & 0\\ \frac{du^{(k)}}{dr} & 0 & 0\end{bmatrix} \tag{2.3.17}$$

Now in component form we omit θ component altogether since there is no rotational flow and none of the fluid velocities or stresses depends on θ.

$$\left[1+\frac{\varepsilon^{(k)}\lambda^{(k)}}{\eta^{(k)}}tr(\tau^{(k)})\right]\tau_{rr}^{(k)}-2\lambda^{(k)}(\frac{du^{(k)}}{dr})\tau_{zr}^{(k)}=0 \tag{2.3.18}$$

$$\left[1+\frac{\varepsilon^{(k)}\lambda^{(k)}}{\eta^{(k)}}tr(\tau^{(k)})\right]\tau_{rz}^{(k)}-\lambda^{(k)}(\frac{du^{(k)}}{dr})\tau_{zz}^{(k)}=\eta^{(k)}\frac{du^{(k)}}{dr} \tag{2.3.19}$$

$$\left[1+\frac{\varepsilon^{(k)}\lambda^{(k)}}{\eta^{(k)}}tr(\tau^{(k)})\right]\tau_{zr}^{(k)}-\lambda^{(k)}(\frac{du^{(k)}}{dr})\tau_{zz}^{(k)}=\eta^{(k)}\frac{du^{(k)}}{dr} \tag{2.3.20}$$

$$\left[1+\frac{\varepsilon^{(k)}\lambda^{(k)}}{\eta^{(k)}}tr(\tau^{(k)})\right]\tau_{zz}^{(k)}=0 \tag{2.3.21}$$

From (2.3.21),

$$\tau_{zz}^{(k)}=0 \tag{2.3.22}$$

From (2.3.18),

$$\left[1+\frac{\varepsilon^{(k)}\lambda^{(k)}}{\eta^{(k)}}tr(\tau^{(k)})\right]\tau_{rr}^{(k)}=2\lambda^{(k)}(\frac{du^{(k)}}{dr})\tau_{zr}^{(k)} \tag{2.3.23}$$

From (2.3.19),

$$\left[1+\frac{\varepsilon^{(k)}\lambda^{(k)}}{\eta^{(k)}}tr(\tau^{(k)})\right]\tau_{rz}^{(k)}=\eta^{(k)}\frac{du^{(k)}}{dr} \tag{2.3.24}$$

Dividing by (2.3.23) and (2.3.24) we get,

$$\tau_{rr}^{(k)} = \frac{2\lambda^{(k)}}{\eta^{(k)}}(\tau_{rz}^{(k)})^2 \tag{2.3.25}$$

using equation (2.3.25) into equation (2.3.18) we obtain,

$$\left[1+\frac{\varepsilon^{(k)}\lambda^{(k)}}{\eta^{(k)}}(\tau_{rr}^{(k)}+\tau_{zz}^{(k)})\right]\tau_{rr}^{(k)} = 2\lambda^{(k)}(\frac{du^{(k)}}{dr})\tau_{zr}^{(k)} \tag{2.3.26}$$

As from (2.3.22) we know that $\tau_{zz}^{(k)} = 0$ and $\tau_{rr}^{(k)} = \frac{2\lambda^{(k)}}{\eta^{(k)}}(\tau_{rz}^{(k)})^2$ then above (2.3.26) equation will become as,

$$\left[1+\frac{\varepsilon^{(k)}\lambda^{(k)}}{\eta^{(k)}}(\frac{2\lambda^{(k)}}{\eta^{(k)}}(\tau_{rz}^{(k)})^2)\right]\frac{2\lambda^{(k)}}{\eta^{(k)}}(\tau_{rz}^{(k)})^2 = 2\lambda^{(k)}(\frac{du^{(k)}}{dr})\tau_{zr}^{(k)} \tag{2.3.27}$$

$$\frac{1}{\eta^{(k)}}(\tau_{rz}^{(k)})^2+\frac{2\varepsilon^{(k)}(\lambda^{(k)})^2}{(\eta^{(k)})^3}(\tau_{rz}^{(k)})^4 = \frac{du^{(k)}}{dr}\tau_{rz}^{(k)} \tag{2.3.28}$$

So,

$$\frac{du^{(k)}}{dr} = \frac{1}{\eta^{(k)}}\tau_{rz}^{(k)}+\frac{2\varepsilon^{(k)}(\lambda^{(k)})^2}{(\eta^{(k)})^3}(\tau_{rz}^{(k)})^3 \tag{2.3.29}$$

From equation (2.3.4) $\tau_{rz}^{(k)} = \frac{dp}{dz}\frac{r}{2}$ so equation(2.3.29) will becomes

$$\frac{du^{(k)}}{dr} = \frac{r}{2\eta^{(k)}}\frac{dp}{dz}+\frac{\varepsilon^{(k)}(\lambda^{(k)})^2}{4(\eta^{(k)})^3}(\frac{dp}{dz})^3 r^3 \tag{2.3.30}$$

using $\frac{dp}{dz} = -P$ in above equation

$$\frac{du^{(k)}}{dr} = -\frac{rP}{2\eta^{(k)}}-\frac{\varepsilon^{(k)}(\lambda^{(k)})^2P^3}{4(\eta^{(k)})^3}r^3 \tag{2.3.31}$$

For $k = 1$

$$\frac{du^{(1)}}{dr} = -\frac{rP}{2\eta^{(1)}}-\frac{\varepsilon^{(1)}(\lambda^{(1)})^2P^3}{4(\eta^{(1)})^3}r^3 \tag{2.3.32}$$

Integrating w.r.t $'r'$

$$u^{(1)}(r) = -\frac{Pr^2}{4\eta^{(1)}} - \frac{\varepsilon^{(1)}(\lambda^{(1)})^2 P^3 r^4}{16(\eta^{(1)})^3} + C^{(1)} \tag{2.3.33}$$

From boundary condition (2.2.12) , we get

$$C^{(1)} = \frac{PR_1^2}{4\eta^{(1)}} + \frac{\varepsilon^{(1)}(\lambda^{(1)})^2 P^3 R_1^4}{16(\eta^{(1)})^3} \tag{2.3.34}$$

substituting value of $C^{(1)}$ then $u^{(1)}(r)$ becomes

$$u^{(1)}(r) = \frac{P}{4\eta^{(1)}}(R_1^2 - r^2) + \frac{P^3\varepsilon^{(1)}(\lambda^{(1)})^2}{16(\eta^{(1)})^3}(R_1^4 - r^4) \tag{2.3.35}$$

Now for $k = 2$

$$\frac{du^{(2)}}{dr} = -\frac{rP}{2\eta^{(2)}} - \frac{\varepsilon^{(2)}(\lambda^{(2)})^2 P^3}{4(\eta^{(2)})^3} r^3 \tag{2.3.36}$$

After integrating

$$u^{(2)}(r) = -\frac{Pr^2}{4\eta^{(2)}} - \frac{\varepsilon^{(2)}(\lambda^{(2)})^2 P^3 r^4}{16(\eta^{(2)})^3} + C^{(2)} \tag{2.3.37}$$

By using B.C (2.2.11) then we will find value of $C^{(2)}$

$$\Big[u^{(1)}(r)\Big]_{r=R_2} = -\frac{PR_2^2}{4\eta^{(2)}} - \frac{\varepsilon^{(2)}(\lambda^{(2)})^2 P^3 R_2^4}{16(\eta^{(2)})^3} + C^{(2)} \tag{2.3.38}$$

$$C^{(2)} = \Big[u^{(1)}(r)\Big]_{r=R_2} + \frac{PR_2^2}{4\eta^{(2)}} + \frac{\varepsilon^{(2)}(\lambda^{(2)})^2 P^3 R_2^4}{16(\eta^{(2)})^3} \tag{2.3.39}$$

so,

$$C^{(2)} = \frac{PR_2^2}{4\eta^{(2)}} + \frac{\varepsilon^{(2)}(\lambda^{(2)})^2 P^3 R_2^4}{16(\eta^{(2)})^3} + \frac{P}{4\eta^{(1)}}(R_1^2 - R_2^2) + \frac{P^3\varepsilon^{(1)}(\lambda^{(1)})^2}{16(\eta^{(1)})^3}(R_1^4 - R_2^4) \tag{2.3.40}$$

by substituting values of $C^{(2)}$ in (2.3.37)

$$u^{(2)}(r)=\frac{P}{4\eta^{(2)}}(R_2^2-r^2)+\frac{P^3\varepsilon^{(2)}(\lambda^{(2)})^2}{16(\eta^{(2)})^3}(R_2^4-r^4)+\frac{P}{4\eta^{(1)}}(R_1^2-R_2^2)+\frac{P^3\varepsilon^{(1)}(\lambda^{(1)})^2}{16(\eta^{(1)})^3}(R_1^4-R_2^4) \tag{2.3.41}$$

$$u^{(2)}(r)=\frac{P}{4\eta^{(2)}}(R_2^2-r^2)+\frac{P^3\varepsilon^{(2)}(\lambda^{(2)})^2}{16(\eta^{(2)})^3}(R_2^4-r^4)+[u^{(1)}(r)]_{r=R_2} \tag{2.3.42}$$

The kth velocity can be written as,

$$u^{(k)}(r)=\frac{P}{4\eta^{(k)}}(R_k^2-r^2)+\frac{P^3\varepsilon^{(k)}(\lambda^{(k)})^2}{16(\eta^{(k)})^3}(R_k^4-r^4)+[u^{(k-1)}(r)]_{r=R_k} \quad k=2,3,\ldots,n. \tag{2.3.43}$$

The generalized form of velocity for $\quad k=1,2,3,\ldots,n.$

$$\begin{aligned}u^{(k)}(r)=&\frac{P}{4\eta^{(k)}}(R_k^2-r^2)+\frac{P^3\varepsilon^{(k)}(\lambda^{(k)})^2}{16(\eta^{(k)})^3}(R_k^4-r^4)\\&+\sum_{i=1}^{k-1}\left[\frac{P}{4\eta^{(k)}}(R_i^2-R_{i+1}^2)+\frac{P^3\varepsilon^{(k)}(\lambda^{(k)})^2}{16(\eta^{(k)})^3}(R_k^4-R_{k+1}^4)\right]\end{aligned} \tag{2.3.44}$$

2.3.3 Volume Flow Rates

Total volume flux or volume flow rate $'Q'$ for $n-layer's$ fluid flow is given by,

$$Q=\sum_{k=1}^{n}Q^{(k)} \tag{2.3.45}$$

where

$$Q^{(k)}=2\pi\int_{R_{k+1}}^{R_k}ru^{(k)}(r)dr \quad k=1,2,\ldots,n-1 \tag{2.3.46}$$

and

$$Q^{(n)}=2\pi\int_{0}^{R_n}ru^{(n)}(r)dr \tag{2.3.47}$$

For $k=1$

$$Q^{(1)}=2\pi\int_{R_2}^{R_1}ru^{(1)}(r)dr \tag{2.3.48}$$

$$Q^{(1)} = 2\pi \int_{R_2}^{R_1} r \left[\frac{P}{4\eta^{(1)}}(R_1^2 - r^2) + \frac{P^3 \varepsilon^{(1)} (\lambda^{(1)})^2}{16(\eta^{(1)})^3}(R_1^4 - r^4) \right] dr \tag{2.3.49}$$

$$\begin{aligned} Q^{(1)} = {} & \frac{\pi P R_1^2}{2\eta^{(1)}} \int_{R_2}^{R_1} r dr - \frac{\pi P}{2\eta^{(1)}} \int_{R_2}^{R_1} r^3 dr + \frac{\pi P^3 \varepsilon^{(1)} (\lambda^{(1)})^2 R_1^4}{8(\eta^{(1)})^3} \int_{R_2}^{R_1} r dr \\ & - \frac{\pi P^3 \varepsilon^{(1)} (\lambda^{(1)})^2}{8(\eta^{(1)})^3} \int_{R_2}^{R_1} r^5 dr \end{aligned} \tag{2.3.50}$$

So after integration w.r.t $'r'$ the above expression , we have

$$\begin{aligned} Q^{(1)} = {} & \frac{\pi P R_1^2}{4\eta^{(1)}}(R_1^2 - R_2^2) - \frac{\pi P}{8\eta^{(1)}}(R_1^4 - R_2^4) + \frac{\pi P^3 \varepsilon^{(1)} (\lambda^{(1)})^2 R_1^4}{16(\eta^{(1)})^3}(R_1^2 - R_2^2) \\ & - \frac{\pi P^3 \varepsilon^{(1)} (\lambda^{(1)})^2}{48(\eta^{(1)})^3}(R_1^6 - R_2^6) \end{aligned} \tag{2.3.51}$$

$$\begin{aligned} Q^{(1)} = {} & \frac{\pi P}{4\eta^{(1)}} R_1^4 - \frac{\pi P}{4\eta^{(1)}} R_1^2 R_2^2 - \frac{\pi P}{8\eta^{(1)}} R_1^4 + \frac{\pi P}{8\eta^{(1)}} R_2^4 + \frac{\pi P^3 \varepsilon^{(1)} (\lambda^{(1)})^2 R_1^4}{16(\eta^{(1)})^3} R_1^6 \\ & - \frac{\pi P^3 \varepsilon^{(1)} (\lambda^{(1)})^2 R_1^4}{16(\eta^{(1)})^3} R_1^4 R_2^2 - \frac{\pi P^3 \varepsilon^{(1)} (\lambda^{(1)})^2}{48(\eta^{(1)})^3} R_1^6 + \frac{\pi P^3 \varepsilon^{(1)} (\lambda^{(1)})^2}{48(\eta^{(1)})^3} R_2^6 \end{aligned} \tag{2.3.52}$$

$$Q^{(1)} = \frac{\pi P}{8\eta^{(1)}} R_1^4 + \frac{\pi P}{4\eta^{(1)}} R_2^2 (\frac{R_2^2}{2} - R_1^2) + \frac{\pi P^3 \varepsilon^{(1)} (\lambda^{(1)})^2}{24(\eta^{(1)})^3} R_1^6 + \frac{\pi P^3 \varepsilon^{(1)} (\lambda^{(1)})^2}{16(\eta^{(1)})^3} R_2^2 (\frac{R_2^4}{3} - R_1^4) \tag{2.3.53}$$

For $k = 2$

$$Q^{(2)} = 2\pi \int_{R_3}^{R_2} r u^{(2)}(r) dr \tag{2.3.54}$$

$$\begin{aligned} Q^{(2)} = {} & 2\pi \int_{R_3}^{R_2} r [\frac{P}{4\eta^{(2)}}(R_2^2 - r^2) + \frac{P^3 \varepsilon^{(2)} (\lambda^{(2)})^2}{16(\eta^{(2)})^3}(R_2^4 - r^4) \\ & + \frac{P}{4\eta^{(1)}}(R_1^2 - R_2^2) + \frac{P^3 \varepsilon^{(1)} (\lambda^{(1)})^2}{16(\eta^{(1)})^3}(R_1^4 - R_2^4)] dr \end{aligned} \tag{2.3.55}$$

$$\begin{aligned} Q^{(2)} = {} & \frac{\pi P R_2^2}{2\eta^{(2)}} \int_{R_3}^{R_2} r dr - \frac{\pi P}{2\eta^{(2)}} \int_{R_3}^{R_2} r^3 dr + \frac{\pi P^3 \varepsilon^{(2)} (\lambda^{(2)})^2 R_2^4}{8(\eta^{(2)})^3} \int_{R_3}^{R_2} r dr \\ & - \frac{\pi P^3 \varepsilon^{(2)} (\lambda^{(2)})^2}{8(\eta^{(2)})^3} \int_{R_3}^{R_2} r^5 dr + \frac{\pi P}{2\eta^{(1)}}(R_1^2 - R_2^2) \int_{R_3}^{R_2} r dr \\ & + \frac{\pi P^3 \varepsilon^{(1)} (\lambda^{(1)})^2}{8(\eta^{(1)})^3}(R_1^4 - R_2^4) \int_{R_3}^{R_2} r dr \end{aligned} \tag{2.3.56}$$

So integrating

$$Q^{(2)} = \frac{\pi P R_2^2}{4\eta^{(2)}}(R_2^2 - R_3^2) - \frac{\pi P}{8\eta^{(2)}}(R_2^4 - R_3^4) + \frac{\pi P^3 \varepsilon^{(2)}(\lambda^{(2)})^2 R_2^4}{16(\eta^{(2)})^3}(R_2^2 - R_3^2)$$
$$- \frac{\pi P^3 \varepsilon^{(2)}(\lambda^{(2)})^2}{48(\eta^{(2)})^3}(R_2^6 - R_3^6) + \frac{\pi P}{4\eta^{(1)}}(R_1^2 - R_2^2)(R_2^2 - R_3^2) \quad (2.3.57)$$
$$+ \frac{\pi P^3 \varepsilon^{(1)}(\lambda^{(1)})^2}{16(\eta^{(1)})^3}(R_1^4 - R_2^4)(R_2^2 - R_3^2)$$

Alternatively , we can write

$$Q^{(2)} = \frac{\pi P}{4\eta^{(2)}}R_2^4 - \frac{\pi P}{4\eta^{(2)}}R_2^2 R_3^2 - \frac{\pi P}{8\eta^{(2)}}R_2^4 + \frac{\pi P}{8\eta^{(2)}}R_3^4 + \frac{\pi P^3 \varepsilon^{(2)}(\lambda^{(2)})^2}{16(\eta^{(2)})^3}R_2^6$$
$$- \frac{\pi P^3 \varepsilon^{(2)}(\lambda^{(2)})^2}{16(\eta^{(2)})^3}R_2^4 R_3^2 - \frac{\pi P^3 \varepsilon^{(2)}(\lambda^{(2)})^2}{48(\eta^{(2)})^3}R_2^6 + \frac{\pi P^3 \varepsilon^{(2)}(\lambda^{(2)})^2}{48(\eta^{(2)})^3}R_3^6 \quad (2.3.58)$$
$$+ \frac{\pi P}{4\eta^{(1)}}(R_1^2 - R_2^2)(R_2^2 - R_3^2) + \frac{\pi P^3 \varepsilon^{(1)}(\lambda^{(1)})^2}{16(\eta^{(1)})^3}(R_1^4 - R_2^4)(R_2^2 - R_3^2)$$

So

$$Q^{(2)} = \frac{\pi P}{8\eta^{(2)}}R_2^4 + \frac{\pi P}{4\eta^{(2)}}R_3^2(\frac{R_3^2}{2} - R_2^2) + \frac{\pi P^3 \varepsilon^{(2)}(\lambda^{(2)})^2}{24(\eta^{(1)})^3}R_2^6 + \frac{\pi P^3 \varepsilon^{(2)}(\lambda^{(2)})^2}{16(\eta^{(2)})^3}R_3^2(\frac{R_3^4}{3} - R_2^4)$$
$$+ \frac{\pi P}{4\eta^{(1)}}(R_1^2 - R_2^2)(R_2^2 - R_3^2) + \frac{\pi P^3 \varepsilon^{(1)}(\lambda^{(1)})^2}{16(\eta^{(1)})^3}(R_1^4 - R_2^4)(R_2^2 - R_3^2)$$
$$(2.3.59)$$

For $k = 3$

$$Q^{(3)} = 2\pi \int_{R_4}^{R_3} r u^{(3)}(r) dr \quad (2.3.60)$$

$$Q^{(3)} = 2\pi \int_{R_4}^{R_3} r[\frac{P}{4\eta^{(3)}}(R_3^2 - r^2) + \frac{P^3 \varepsilon^{(3)}(\lambda^{(3)})^2}{16(\eta^{(3)})^3}(R_3^4 - r^4)$$
$$+ \frac{P}{4\eta^{(2)}}(R_2^2 - R_3^2) + \frac{P^3 \varepsilon^{(2)}(\lambda^{(2)})^2}{16(\eta^{(2)})^3}(R_2^4 - R_2^4) \quad (2.3.61)$$
$$+ \frac{P}{4\eta^{(1)}}(R_1^2 - R_2^2) + \frac{P^3 \varepsilon^{(1)}(\lambda^{(1)})^2}{16(\eta^{(1)})^3}(R_1^4 - R_2^4)] dr$$

$$
\begin{aligned}
Q^{(3)} = {} & \frac{\pi P R_3^2}{2\eta^{(3)}} \int_{R_4}^{R_3} r dr - \frac{\pi P}{2\eta^{(3)}} \int_{R_4}^{R_3} r^3 dr + \frac{\pi P^3 \varepsilon^{(3)} (\lambda^{(3)})^2 R_3^4}{8(\eta^{(3)})^3} \int_{R_4}^{R_3} r dr \\
& - \frac{\pi P^3 \varepsilon^{(3)} (\lambda^{(3)})^2}{8(\eta^{(3)})^3} \int_{R_4}^{R_3} r^5 dr + \frac{\pi P}{2\eta^{(2)}} (R_2^2 - R_3^2) \int_{R_4}^{R_3} r dr \\
& + \frac{\pi P^3 \varepsilon^{(2)} (\lambda^{(2)})^2}{8(\eta^{(2)})^3} (R_2^4 - R_3^4) \int_{R_4}^{R_3} r dr + \frac{\pi P}{2\eta^{(1)}} (R_1^2 - R_2^2) \int_{R_4}^{R_3} r dr \\
& + \frac{\pi P^3 \varepsilon^{(1)} (\lambda^{(1)})^2}{8(\eta^{(1)})^3} (R_1^4 - R_2^4) \int_{R_4}^{R_3} r dr
\end{aligned}
\tag{2.3.62}
$$

So after integration

$$
\begin{aligned}
Q^{(3)} = {} & \frac{\pi P R_3^2}{4\eta^{(3)}} (R_3^2 - R_4^2) - \frac{\pi P}{8\eta^{(3)}} (R_3^4 - R_4^4) + \frac{\pi P^3 \varepsilon^{(3)} (\lambda^{(3)})^2 R_3^4}{16(\eta^{(3)})^3} (R_3^2 - R_4^2) \\
& - \frac{\pi P^3 \varepsilon^{(3)} (\lambda^{(3)})^2}{48(\eta^{(3)})^3} (R_3^6 - R_4^6) + \frac{\pi P}{4\eta^{(2)}} (R_2^2 - R_3^2)(R_3^2 - R_4^2) \\
& + \frac{\pi P^3 \varepsilon^{(2)} (\lambda^{(2)})^2}{16(\eta^{(2)})^3} (R_2^4 - R_3^4)(R_3^2 - R_4^2) + \frac{\pi P}{4\eta^{(1)}} (R_1^2 - R_2^2)(R_3^2 - R_4^2) \\
& + \frac{\pi P^3 \varepsilon^{(1)} (\lambda^{(1)})^2}{16(\eta^{(1)})^3} (R_1^4 - R_2^4)(R_3^2 - R_4^2)
\end{aligned}
\tag{2.3.63}
$$

$$
\begin{aligned}
Q^{(3)} = {} & \frac{\pi P}{4\eta^{(3)}} R_3^4 - \frac{\pi P}{4\eta^{(3)}} R_3^2 R_4^2 - \frac{\pi P}{8\eta^{(3)}} R_3^4 + \frac{\pi P}{8\eta^{(3)}} R_4^4 + \frac{\pi P^3 \varepsilon^{(3)} (\lambda^{(3)})^2}{16(\eta^{(3)})^3} R_3^6 \\
& - \frac{\pi P^3 \varepsilon^{(3)} (\lambda^{(3)})^2}{16(\eta^{(3)})^3} R_3^4 R_4^2 - \frac{\pi P^3 \varepsilon^{(3)} (\lambda^{(3)})^2}{48(\eta^{(3)})^3} R_3^6 + \frac{\pi P^3 \varepsilon^{(3)} (\lambda^{(3)})^2}{48(\eta^{(3)})^3} R_4^6 \\
& + \frac{\pi P}{4\eta^{(2)}} (R_2^2 - R_3^2)(R_3^2 - R_4^2) + \frac{\pi P^3 \varepsilon^{(2)} (\lambda^{(2)})^2}{16(\eta^{(2)})^3} (R_2^4 - R_3^4)(R_3^2 - R_4^2) \\
& + \frac{\pi P}{4\eta^{(1)}} (R_1^2 - R_2^2)(R_3^2 - R_4^2) + \frac{\pi P^3 \varepsilon^{(1)} (\lambda^{(1)})^2}{16(\eta^{(1)})^3} (R_1^4 - R_2^4)(R_3^2 - R_4^2)
\end{aligned}
\tag{2.3.64}
$$

So

$$Q^{(3)} = \frac{\pi P}{8\eta^{(3)}}R_3^4 + \frac{\pi P}{4\eta^{(3)}}R_4^2(\frac{R_4^2}{2} - R_3^2) + \frac{\pi P^3 \varepsilon^{(3)}(\lambda^{(3)})^2}{24(\eta^{(3)})^3}R_3^6 + \frac{\pi P^3 \varepsilon^{(3)}(\lambda^{(3)})^2}{16(\eta^{(3)})^3}R_4^2(\frac{R_4^4}{3} - R_3^4)$$
$$+ \frac{\pi P}{4\eta^{(2)}}(R_2^2 - R_3^2)(R_3^2 - R_4^2) + \frac{\pi P^3 \varepsilon^{(2)}(\lambda^{(2)})^2}{16(\eta^{(2)})^3}(R_2^4 - R_3^4)(R_3^2 - R_4^2)$$
$$+ \frac{\pi P}{4\eta^{(1)}}(R_1^2 - R_2^2)(R_3^2 - R_4^2) + \frac{\pi P^3 \varepsilon^{(1)}(\lambda^{(1)})^2}{16(\eta^{(1)})^3}(R_1^4 - R_2^4)(R_3^2 - R_4^2) \tag{2.3.65}$$

Finally the generalized form of $Q^{(k)}$ for $\quad k = 1, 2, 3, \ldots, n-1.$

$$Q^{(k)} = \frac{\pi P}{8\eta^{(k)}}R_k^4 + \frac{\pi P}{4\eta^{(k)}}R_{k+1}^2(\frac{R_{k+1}^2}{2} - R_k^2)$$
$$+ \frac{\pi P^3 \varepsilon^{(k)}(\lambda^{(k)})^2}{24(\eta^{(k)})^3}R_k^6 + \frac{\pi P^3 \varepsilon^{(k)}(\lambda^{(k)})^2}{16(\eta^{(k)})^3}R_{k+1}^2(\frac{R_{k+1}^4}{3} - R_k^4)$$
$$+ \left[\sum_{i=1}^{k-1}(\frac{\pi P}{4\eta^{(i)}}(R_i^2 - R_{i+1}^2)(R_{i+1}^2 - R_{i+2}^2) + \frac{\pi \varepsilon^{(i)}(\lambda^{(i)})^2 P^3}{16(\eta^{(i)})^3}(R_i^4 - R_{i+1}^4)(R_{i+1}^2 - R_{i+2}^2)\right] \tag{2.3.66}$$

Now we will find $Q^{(n)}$ such as

$$Q^{(n)} = 2\pi \int_0^{R_n} r u^{(n)}(r) dr \tag{2.3.67}$$

After integrating

$$Q^{(n)} = \frac{\pi P}{8\eta^{(n)}}R_n^4 + \frac{\pi \varepsilon^{(n)}(\lambda^{(n)})^2 P^3 R_n^{(6)}}{24(\eta^{(n)})^3} + \pi R_n^{(2)}\left[u^{(n-1)}(r)\right]_{r=R_n} \tag{2.3.68}$$

2.3.4 Special Case For First-layer

We have a special case for or $n = 1$ just for the one layer flow

$$Q^{(1)} = 2\pi \int_0^{R_1} r u^{(1)}(r) dr \tag{2.3.69}$$

$$Q^{(1)} = 2\pi \int_0^{R_1} r\left[\frac{P}{4\eta^{(1)}}(R_1^2 - r^2) + \frac{\varepsilon^{(1)}\lambda^{(1)}P^3}{16(\eta^{(1)})^2}(R_1^4 - r^4)\right]dr \tag{2.3.70}$$

$$\begin{aligned} Q^{(1)} = &\frac{\pi P R_1^2}{2\eta^{(1)}}\int_0^{R_1} r dr - \frac{\pi P}{2\eta^{(1)}}\int_0^{R_1} r^3 dr + \frac{\pi P^3 \varepsilon^{(1)}(\lambda^{(1)})^2 R_1^4}{8(\eta^{(1)})^3}\int_0^{R_1} r dr \\ &- \frac{\pi P^3 \varepsilon^{(1)}(\lambda^{(1)})^2}{8(\eta^{(1)})^3}\int_0^{R_1} r^5 dr \end{aligned} \tag{2.3.71}$$

After integrating w.r.t $'r'$

$$Q^{(1)} = \frac{\pi P R_1^4}{4\eta^{(1)}} - \frac{\pi P R_1^4}{8\eta^{(1)}} + \frac{\pi P^3 \varepsilon^{(1)}(\lambda^{(1)})^2 R_1^6}{16(\eta^{(1)})^3} - \frac{\pi P^3 \varepsilon^{(1)}(\lambda^{(1)})^2 R_1^6}{48(\eta^{(1)})^3} \tag{2.3.72}$$

so,

$$Q^{(1)} = \frac{\pi P}{8\eta^{(1)}} R_1^4 + \frac{\pi P^3 \varepsilon^{(1)}(\lambda^{(1)})^2}{24(\eta^{(1)})^3} R_1^6 \tag{2.3.73}$$

2.3.5 Special Case For Second-Layers

Now we have a special case for two layers flow for this we substitute $n = 2$

$$Q^{(2)} = 2\pi \int_0^{R_2} r u^{(2)}(r) dr \tag{2.3.74}$$

$$\begin{aligned} Q^{(2)} = 2\pi \int_0^{R_2} r[&\frac{P}{4\eta^{(2)}}(R_2^2 - r^2) + \frac{\varepsilon^{(2)}(\lambda^{(2)})^2 P^3}{16(\eta^{(2)})^3}(R_2^4 - r^4) \\ &+ \frac{P}{4\eta(1)}(R_1^2 - R_2^2) + \frac{\varepsilon^{(1)}(\lambda^{(1)})^2 P^3}{16(\eta^{(1)})^3}(R_1^4 - R_2^4)]dr \end{aligned} \tag{2.3.75}$$

$$\begin{aligned} Q^{(2)} = &\frac{\pi P R_2^2}{2\eta^{(2)}}\int_0^{R_2} r dr - \frac{\pi P}{2\eta^{(2)}}\int_0^{R_2} r^3 dr + \frac{\pi P^3 \varepsilon^{(2)}(\lambda^{(2)})^2 R_2^4}{8(\eta^{(2)})^3}\int_0^{R_2} r dr \\ &- \frac{\pi P^3 \varepsilon^{(2)}(\lambda^{(2)})^2}{8(\eta^{(2)})^3}\int_0^{R_2} r^5 dr + \frac{\pi P}{2\eta^{(1)}}(R_1^2 - R_2^2)\int_0^{R_2} r dr \\ &+ \frac{\pi P^3 \varepsilon^{(1)}(\lambda^{(1)})^2}{8(\eta^{(1)})^3}(R_1^4 - R_2^4)\int_0^{R_2} r dr \end{aligned} \tag{2.3.76}$$

After integration,

$$Q^{(2)} = \frac{\pi P}{4\eta^{(2)}}R_2^4 - \frac{\pi P}{8\eta^{(2)}}R_2^4 + \frac{\pi P^3\varepsilon^{(2)}(\lambda^{(2)})^2}{16(\eta^{(2)})^3}R_2^6$$
$$- \frac{\pi P^3\varepsilon^{(2)}(\lambda^{(2)})^2}{48(\eta^{(2)})^3}R_2^6 + \frac{\pi P}{4\eta^{(1)}}(R_1^2 - R_2^2)R_2^2 + \frac{\pi P^3\varepsilon^{(1)}(\lambda^{(1)})^2}{16(\eta^{(1)})^3}(R_1^4 - R_2^4)R_2^2 \tag{2.3.77}$$

so,

$$Q^{(2)} = \frac{\pi P}{8\eta^{(2)}}R_2^4 + \frac{\pi P^3\varepsilon^{(2)}(\lambda^{(2)})^2}{24(\eta^{(2)})^3}R_2^6 + \frac{\pi P}{4\eta^{(1)}}R_2^2(R_1^2 - R_2^2) + \frac{\pi P^3\varepsilon^{(1)}(\lambda^{(1)})^2}{16(\eta^{(1)})^3}R_2^2(R_1^4 - R_2^4) \tag{2.3.78}$$

2.3.6 Special Case For Third-Layer

We have a special case of three layers flow for which we substitute $n = 3$

$$Q^{(3)} = 2\pi \int_0^{R_3} ru^{(3)}(r)dr \tag{2.3.79}$$

$$\begin{aligned} Q^{(3)} = 2\pi \int_0^{R_3} r[&\frac{P}{4\eta^{(3)}}(R_3^2 - r^2) + \frac{P^3\varepsilon^{(3)}(\lambda^{(3)})^2}{16(\eta^{(3)})^3}(R_3^4 - r^4) \\ &+ \frac{P}{4\eta^{(2)}}(R_2^2 - R_3^2) + \frac{P^3\varepsilon^{(2)}(\lambda^{(2)})^2}{16(\eta^{(2)})^3}(R_2^4 - R_2^4) \\ &+ \frac{P}{4\eta^{(1)}}(R_1^2 - R_2^2) + \frac{P^3\varepsilon^{(1)}(\lambda^{(1)})^2}{16(\eta^{(1)})^3}(R_1^4 - R_2^4)]dr \end{aligned} \tag{2.3.80}$$

$$\begin{aligned} Q^{(3)} = &\frac{\pi P R_3^2}{2\eta^{(3)}}\int_0^{R_3} rdr - \frac{\pi P}{2\eta^{(3)}}\int_0^{R_3} r^3dr + \frac{\pi P^3\varepsilon^{(3)}(\lambda^{(3)})^2 R_3^4}{8(\eta^{(3)})^3}\int_0^{R_3} rdr \\ &- \frac{\pi P^3\varepsilon^{(3)}(\lambda^{(3)})^2}{8(\eta^{(3)})^3}\int_0^{R_3} r^5dr + \frac{\pi P}{2\eta^{(2)}}(R_2^2 - R_3^2)\int_0^{R_3} rdr \\ &+ \frac{\pi P^3\varepsilon^{(2)}(\lambda^{(2)})^2}{8(\eta^{(2)})^3}(R_2^4 - R_3^4)\int_0^{R_3} rdr + \frac{\pi P}{2\eta^{(1)}}(R_1^2 - R_2^2)\int_0^{R_3} rdr \\ &+ \frac{\pi P^3\varepsilon^{(1)}(\lambda^{(1)})^2}{8(\eta^{(1)})^3}(R_1^4 - R_2^4)\int_0^{R_3} rdr \end{aligned} \tag{2.3.81}$$

After integrating

$$
\begin{aligned}
Q^{(3)} = &\frac{\pi P}{4\eta^{(3)}}R_3^4 - \frac{\pi P}{8\eta^{(3)}}R_2^4 + \frac{\pi P^3 \varepsilon^{(3)}(\lambda^{(3)})^2}{16(\eta^{(3)})^3}R_3^6 \\
&- \frac{\pi P^3 \varepsilon^{(3)}(\lambda^{(3)})^2}{48(\eta^{(3)})^3}R_3^6 \\
&+ \frac{\pi P}{4\eta^{(2)}}(R_2^2 - R_3^2)R_3^2 + \frac{\pi P^3 \varepsilon^{(2)}(\lambda^{(2)})^2}{16(\eta^{(2)})^3}(R_2^4 - R_3^4)R_3^2 \\
&+ \frac{\pi P}{4\eta^{(1)}}(R_1^2 - R_2^2)R_3^2 + \frac{\pi P^3 \varepsilon^{(1)}(\lambda^{(1)})^2}{16(\eta^{(1)})^3}(R_1^4 - R_2^4)R_3^2
\end{aligned}
\tag{2.3.82}
$$

So,

$$
\begin{aligned}
Q^{(3)} = &\frac{\pi P}{8\eta^{(3)}}R_3^4 + \frac{\pi P^3 \varepsilon^{(3)}(\lambda^{(3)})^2}{24(\eta^{(3)})^3}R_3^6 \\
&+ \frac{\pi P}{4\eta^{(2)}}(R_2^2 - R_3^2)R_3^2 + \frac{\pi P^3 \varepsilon^{(2)}(\lambda^{(2)})^2}{16(\eta^{(2)})^3}(R_2^4 - R_3^4)R_3^2 \\
&+ \frac{\pi P}{4\eta^{(1)}}(R_1^2 - R_2^2)R_3^2 + \frac{\pi P^3 \varepsilon^{(1)}(\lambda^{(1)})^2}{16(\eta^{(1)})^3}(R_1^4 - R_2^4)R_3^2
\end{aligned}
\tag{2.3.83}
$$

2.4 Exact Solution for the Exponential PTT Fluid Model

Now we solve for the exponential form of PTT fluid. Geometry of the problem remains same as used in linearized form.

2.4.1 Velocities:-

Substituting (2.2.8) and values of $\hat{\tau}$ and $A_1^{(k)}$ in (2.2.3) then

$$
\exp\left(\frac{\varepsilon^{(k)}\lambda^{(k)}}{\eta^{(k)}}tr(\tau^{(k)})\right)\tau^{(k)} + \lambda^{(k)}\begin{bmatrix} -2\frac{du^{(k)}}{dr}\tau_{rz}^{(k)} & -\frac{du^{(k)}}{dr}\tau_{zz}^{(k)} \\ -\frac{du^{(k)}}{dr}\tau_{zz}^{(k)} & 0 \end{bmatrix} = \eta^{(k)}\begin{bmatrix} 0 & \frac{du^{(k)}}{dr} \\ \frac{du^{(k)}}{dr} & 0 \end{bmatrix}
\tag{2.4.1}
$$

So

$$\left[1+\frac{\varepsilon^{(k)}\lambda^{(k)}}{\eta^{(k)}}tr(\tau^{(k)})\right]\begin{bmatrix}\tau_{rr}^{(k)} & \tau_{rz}^{(k)}\\ \tau_{zr}^{(k)} & \tau_{zz}^{(k)}\end{bmatrix}+\lambda^{(k)}\begin{bmatrix}-2\frac{du^{(k)}}{dr}\tau_{rz}^{(k)} & -\frac{du^{(k)}}{dr}\tau_{zz}^{(k)}\\ -\frac{du^{(k)}}{dr}\tau_{zz}^{(k)} & 0\end{bmatrix}=\eta^{(k)}\begin{bmatrix}0 & \frac{du^{(k)}}{dr}\\ \frac{du^{(k)}}{dr} & 0\end{bmatrix} \tag{2.4.2}$$

components form becomes,

$$\exp\left(\frac{\varepsilon^{(k)}\lambda^{(k)}}{\eta^{(k)}}tr(\tau^{(k)})\right)\tau_{rr}^{(k)}-2\lambda^{(k)}(\frac{du^{k}}{dr})\tau_{zr}^{(k)}=0 \tag{2.4.3}$$

$$\exp\left(\frac{\varepsilon^{(k)}\lambda^{(k)}}{\eta^{(k)}}tr(\tau^{(k)})\right)\tau_{rz}^{(k)}-\lambda^{(k)}(\frac{du^{k}}{dr})\tau_{zz}^{(k)}=\eta^{(k)}\frac{du^{(k)}}{dr} \tag{2.4.4}$$

$$\exp\left(\frac{\varepsilon^{(k)}\lambda^{(k)}}{\eta^{(k)}}tr(\tau^{(k)})\right)\tau_{zr}^{(k)}-\lambda^{(k)}(\frac{du^{k}}{dr})\tau_{zz}^{(k)}=\eta^{(k)}\frac{du^{(k)}}{dr} \tag{2.4.5}$$

$$\exp\left(\frac{\varepsilon^{(k)}\lambda^{(k)}}{\eta^{(k)}}tr(\tau^{(k)})\right)\tau_{zz}^{(k)}=0 \tag{2.4.6}$$

Using result of (2.4.6),

$$\tau_{zz}^{(k)}=0 \tag{2.4.7}$$

From (2.4.3) and (2.4.4)

$$\exp\left(\frac{\varepsilon^{(k)}\lambda^{(k)}}{\eta^{(k)}}tr(\tau^{(k)})\right)\tau_{rr}^{(k)}=2\lambda^{(k)}(\frac{du^{(k)}}{dr})\tau_{zr}^{(k)} \tag{2.4.8}$$

$$\exp\left(\frac{\varepsilon^{(k)}\lambda^{(k)}}{\eta^{(k)}}tr(\tau^{(k)})\right)\tau_{rz}^{(k)}=\eta^{(k)}\frac{du^{(k)}}{dr} \tag{2.4.9}$$

Dividing (2.4.8) by (2.4.9) , we obtain

$$\tau_{rr}^{(k)}=\frac{2\lambda^{(k)}}{\eta^{(k)}}(\tau_{rz}^{(k)})^{2} \tag{2.4.10}$$

using (2.4.10) into (2.4.8)

$$\exp\left(\frac{\varepsilon^{(k)}\lambda^{(k)}}{\eta^{(k)}}(\tau_{rr}^{(k)}+\tau_{zz}^{(k)})\right)\tau_{rr}^{(k)} = 2\lambda^{(k)}(\frac{du^{(k)}}{dr})\tau_{zr}^{(k)} \tag{2.4.11}$$

$$\exp\left(\frac{\varepsilon^{(k)}\lambda^{(k)}}{\eta^{(k)}}(\frac{2\lambda^{(k)}}{\eta^{(k)}}(\tau_{rz}^{(k)})^2)\right)\frac{2\lambda^{(k)}}{\eta^{(k)}}(\tau_{rz}^{(k)})^2 = 2\lambda^{(k)}(\frac{du^{(k)}}{dr})\tau_{zr}^{(k)} \tag{2.4.12}$$

So,

$$\frac{du^{(k)}}{dr} = \frac{1}{\eta^{(k)}}\exp\left(\frac{2\varepsilon^{(k)}(\lambda^{(k)})^2}{(\eta^{(k)})^2}(\tau_{rz}^{(k)})^2\right)(\tau_{rz}^{(k)}) \tag{2.4.13}$$

Substituting (2.3.4) in the above equation (2.4.13)

$$\frac{du^{(k)}}{dr} = \frac{1}{\eta^{(k)}}\exp\left(\frac{2\varepsilon^{(k)}(\lambda^{(k)})^2}{(\eta^{(k)})^2}\{\frac{dp}{dz}\frac{r}{2}\}^2\right)\left(\frac{dp}{dz}\frac{r}{2}\right) \tag{2.4.14}$$

Using $\frac{dp}{dz} = -P$

$$\frac{du^{(k)}}{dr} = -\frac{Pr}{2\eta^{(k)}}\exp\left(\frac{\varepsilon^{(k)}(\lambda^{(k)})^2P^2r^2}{2(\eta^{(k)})^2}\right) \tag{2.4.15}$$

For $k = 1$

$$\frac{du^{(1)}}{dr} = -\frac{Pr}{2\eta^{(1)}}\exp\left(\frac{\varepsilon^{(1)}(\lambda^{(1)})^2P^2r^2}{2(\eta^{(1)})^2}\right) \tag{2.4.16}$$

$$u^{(1)}(r) = -\frac{P}{2\eta^{(1)}}\int\exp\left(\frac{\varepsilon^{(1)}(\lambda^{(1)})^2P^2r^2}{2(\eta^{(1)})^2}\right)rdr \tag{2.4.17}$$

Substituting,

$$\frac{\varepsilon^{(1)}(\lambda^{(1)})^2P^2r^2}{2(\eta^{(1)})^2} = w \tag{2.4.18}$$

$$\frac{\varepsilon^{(1)}(\lambda^{(1)})^2P^2}{(\eta^{(1)})^2}rdr = dw \tag{2.4.19}$$

$$\frac{(\eta^{(1)})^2}{\varepsilon^{(1)}(\lambda^{(1)})^2P^2}dw = rdr \tag{2.4.20}$$

Using (2.4.18),(2.4.19),(2.4.20) in (2.4.17) , we have

$$u^{(1)}(r) = -\frac{\eta^{(1)}}{2\varepsilon^{(1)}(\lambda^{(1)})^2 P}\int \exp(w)\,dw \tag{2.4.21}$$

After integrating

$$u^{(1)}(r) = -\frac{\eta^{(1)}}{2\varepsilon^{(1)}(\lambda^{(1)})^2 P}\exp(w) + C^{(1)} \tag{2.4.22}$$

$$u^{(1)}(r) = -\frac{\eta^{(1)}}{2\varepsilon^{(1)}(\lambda^{(1)})^2 P}\exp\left(\frac{\varepsilon^{(1)}(\lambda^{(1)})^2 P^2 r^2}{2(\eta^{(1)})^2}\right) + C^{(1)} \tag{2.4.23}$$

Using boundary conditions in equation (2.2.12) we obtain

$$C^{(1)} = \frac{\eta^{(1)}}{2\varepsilon^{(1)}(\lambda^{(1)})^2 P}\exp\left(\frac{\varepsilon^{(1)}(\lambda^{(1)})^2 P^2 R_1^2}{2(\eta^{(1)})^2}\right) \tag{2.4.24}$$

So by substituting value of $C^{(1)}$ in (2.4.22)

$$u^{(1)}(r) = \frac{\eta^{(1)}}{2\varepsilon^{(1)}(\lambda^{(1)})^2 P}\left[\exp\left(\frac{\varepsilon^{(1)}(\lambda^{(1)})^2 P^2 R_1^2}{2(\eta^{(1)})^2}\right) - \exp\left(\frac{\varepsilon^{(1)}(\lambda^{(1)})^2 P^2 r^2}{2(\eta^{(1)})^2}\right)\right] \tag{2.4.25}$$

Now for $k = 2$

$$\frac{du^{(2)}}{dr} = -\frac{Pr}{2\eta^{(2)}}\exp\left(\frac{\varepsilon^{(2)}(\lambda^{(2)})^2 P^2 r^2}{2(\eta^{(2)})^2}\right) \tag{2.4.26}$$

$$u^{(2)}(r) = -\frac{P}{2\eta^{(2)}}\int \exp\left(\frac{\varepsilon^{(2)}(\lambda^{(2)})^2 P^2 r^2}{2(\eta^{(2)})^2}\right) rdr \tag{2.4.27}$$

After integrating , we obtain

$$u^{(2)}(r) = -\frac{\eta^{(2)}}{2\varepsilon^{(2)}(\lambda^{(2)})^2 P}\exp\left(\frac{\varepsilon^{(2)}(\lambda^{(2)})^2 P^2 r^2}{2(\eta^{(2)})^2}\right) + C^{(2)} \tag{2.4.28}$$

By using boundary conditions (2.2.11) then we will find value of $C^{(2)}$

$$\left[u^{(1)}(r)\right]_{r=R_2} = -\frac{\eta^{(2)}}{2\varepsilon^{(2)}(\lambda^{(2)})^2 P}\exp\left(\frac{\varepsilon^{(2)}(\lambda^{(2)})^2 P^2 R_2^2}{2(\eta^{(2)})^2}\right) + C_2 \tag{2.4.29}$$

$$C^{(2)} = \left[u^{(1)}(r)\right]_{r=R_2} + \frac{\eta^{(2)}}{2\varepsilon^{(2)}(\lambda^{(2)})^2 P} \exp\left(\frac{\varepsilon^{(2)}(\lambda^{(2)})^2 P^2 R_2^2}{2(\eta^{(2)})^2}\right) \tag{2.4.30}$$

$$C^{(2)} = \frac{\eta^{(2)}}{2\varepsilon^{(2)}(\lambda^{(2)})^2 P} \exp\left(\frac{\varepsilon^{(2)}(\lambda^{(2)})^2 P^2 R_2^2}{2(\eta^{(2)})^2}\right) + \frac{\eta^{(1)}}{2\varepsilon^{(1)}(\lambda^{(1)})^2 P}\left[\exp\left(\frac{\varepsilon^{(1)}(\lambda^{(1)})^2 P^2 R_1^2}{2(\eta^{(1)})^2}\right) - \exp\left(\frac{\varepsilon^{(1)}(\lambda^{(1)})^2 P^2 R_2^2}{2(\eta^{(1)})^2}\right)\right] \tag{2.4.31}$$

Using $C^{(2)}$ into the equation(2.4.28) then,

$$u^{(2)}(r) = \frac{\eta^{(2)}}{2\varepsilon^{(2)}(\lambda^{(2)})^2 P}\left[\exp\left(\frac{\varepsilon^{(2)}(\lambda^{(2)})^2 P^2 R_2^2}{2(\eta^{(2)})^2}\right) - \exp\left(\frac{\varepsilon^{(2)}(\lambda^{(2)})^2 P^2 r^2}{2(\eta^{(2)})^2}\right)\right] + \frac{\eta^{(1)}}{2\varepsilon^{(1)}(\lambda^{(1)})^2 P}\left[\exp\left(\frac{\varepsilon^{(1)}(\lambda^{(1)})^2 P^2 R_1^2}{2(\eta^{(1)})^2}\right) - \exp\left(\frac{\varepsilon^{(1)}(\lambda^{(1)})^2 P^2 R_2^2}{2(\eta^{(1)})^2}\right)\right] \tag{2.4.32}$$

So the generalized form of velocity profile for $k = 2, 3, \ldots, n.$ becomes ,

$$u^{(k)}(r) = \frac{\eta^{(k)}}{2\varepsilon^{(2)}(\lambda^{(k)})^2 P}\left[\exp\left(\frac{\varepsilon^{(k)}(\lambda^{(k)})^2 P^2 R_k^2}{2(\eta^{(k)})^2}\right) - \exp\left(\frac{\varepsilon^{(k)}(\lambda^{(k)})^2 P^2 r^2}{2(\eta^{(k)})^2}\right)\right] + \left[u^{(k-1)}(r)\right]_{r=R_k} \tag{2.4.33}$$

2.4.2 Volume Flow Rates:-

Total volume flux or volume flow rate 'Q' for n-layer's fluid flow is given by,

$$Q = \sum_{k=1}^{n} Q^{(k)} \tag{2.4.34}$$

where,

$$Q^{(k)} = 2\pi \int_{R_{k+1}}^{R_k} r u^{(k)}(r) dr, \quad k = 1, 2, \ldots, n-1 \tag{2.4.35}$$

and

$$Q^{(n)} = 2\pi \int_0^{R_n} r u^{(n)}(r) dr \tag{2.4.36}$$

for $k = 1$

$$Q^{(1)} = 2\pi \int_{R_2}^{R_1} r u^{(1)}(r) dr \tag{2.4.37}$$

$$Q^{(1)} = 2\pi \int_{R_2}^{R_1} \frac{\eta^{(1)}}{2\varepsilon^{(1)}(\lambda^{(1)})^2 P} \left[\exp\left(\frac{\varepsilon^{(1)}(\lambda^{(1)})^2 P^2 R_1^2}{2(\eta^{(1)})^2} \right) - \exp\left(\frac{\varepsilon^{(1)}(\lambda^{(1)})^2 P^2 r^2}{2(\eta^{(1)})^2} \right) \right] r dr \tag{2.4.38}$$

$$Q^{(1)} = \frac{\pi \eta^{(1)}}{\varepsilon^{(1)}(\lambda^{(1)})^2 P} \exp\left(\frac{\varepsilon^{(1)}(\lambda^{(1)})^2 P^2 R_1^2}{2(\eta^{(1)})^2} \right) \int_{R_2}^{R_1} r dr - \frac{\pi \eta^{(1)}}{\varepsilon^{(1)}(\lambda^{(1)})^2 P} \int_{R_2}^{R_1} \exp\left(\frac{\varepsilon^{(1)}(\lambda^{(1)})^2 P^2 r^2}{2(\eta^{(1)})^2} \right) r dr \tag{2.4.39}$$

Substituting,

$$\frac{\varepsilon^{(1)}(\lambda^{(1)})^2 P^2 r^2}{2(\eta^{(1)})^2} = w \tag{2.4.40}$$

$$\frac{\varepsilon^{(1)}(\lambda^{(1)})^2 P^2}{(\eta^{(1)})^2} r dr = dw \tag{2.4.41}$$

$$\frac{(\eta^{(1)})^2}{\varepsilon^{(1)}(\lambda^{(1)})^2 P^2} dw = r dr \tag{2.4.42}$$

Using (2.4.40),(2.4.41),(2.4.42) in equation (2.4.39) so above expression will become as,

$$Q^{(1)} = \frac{\pi \eta^{(1)}}{\varepsilon^{(1)}(\lambda^{(1)})^2 P} \exp\left(\frac{\varepsilon^{(1)}(\lambda^{(1)})^2 P^2 R_1^2}{2(\eta^{(1)})^2} \right) \int_{R_2}^{R_1} r dr - \frac{\pi (\eta^{(1)})^3}{(\varepsilon^{(1)})^2(\lambda^{(1)})^4 P^3} \int_{R_2}^{R_1} \exp(w) dw \tag{2.4.43}$$

After integrating,

$$Q^{(1)} = \frac{\pi \eta^{(1)}}{2\varepsilon^{(1)}(\lambda^{(1)})^2 P} \exp\left(\frac{\varepsilon^{(1)}(\lambda^{(1)})^2 P^2 R_1^2}{2(\eta^{(1)})^2} \right) \left[r^2 \right]_{R_2}^{R_1} - \frac{\pi (\eta^{(1)})^3}{(\varepsilon^{(1)})^2(\lambda^{(1)})^4 P^3} \left[\exp(w) \right]_{R_2}^{R_1} \tag{2.4.44}$$

$$Q^{(1)} = \frac{\pi \eta^{(1)}}{2\varepsilon^{(1)}(\lambda^{(1)})^2 P} \exp\left(\frac{\varepsilon^{(1)}(\lambda^{(1)})^2 P^2 R_1^2}{2(\eta^{(1)})^2} \right) \left[r^2 \right]_{R_2}^{R_1} - \frac{\pi (\eta^{(1)})^3}{(\varepsilon^{(1)})^2(\lambda^{(1)})^4 P^3} \left[\exp\left(\frac{\varepsilon^{(1)}(\lambda^{(1)})^2 P^2 r^2}{2(\eta^{(1)})^2} \right) \right]_{R_2}^{R_1} \tag{2.4.45}$$

$$Q^{(1)} = \frac{\pi\eta^{(1)}}{2\varepsilon^{(1)}(\lambda^{(1)})^2 P} \exp\left(\frac{\varepsilon^{(1)}(\lambda^{(1)})^2 P^2 R_1^2}{2(\eta^{(1)})^2}\right)\left[R_1^2 - R_2^2\right]$$
$$- \frac{\pi(\eta^{(1)})^3}{(\varepsilon^{(1)})^2(\lambda^{(1)})^4 P^3}\left[\exp\left(\frac{\varepsilon^{(1)}(\lambda^{(1)})^2 P^2 R_1^2}{2(\eta^{(1)})^2}\right) - \exp\left(\frac{\varepsilon^{(1)}(\lambda^{(1)})^2 P^2 R_2^2}{2(\eta^{(1)})^2}\right)\right] \tag{2.4.46}$$

So,

$$Q^{(1)} = \frac{\pi\eta^{(1)}}{2\varepsilon^{(1)}(\lambda^{(1)})^2 P} \exp\left(\frac{\varepsilon^{(1)}(\lambda^{(1)})^2 P^2 R_1^2}{2(\eta^{(1)})^2}\right)\left[R_1^2 - R_2^2\right]$$
$$+ \frac{\pi(\eta^{(1)})^3}{(\varepsilon^{(1)})^2(\lambda^{(1)})^4 P^3}\left[\exp\left(\frac{\varepsilon^{(1)}(\lambda^{(1)})^2 P^2 R_2^2}{2(\eta^{(1)})^2}\right) - \exp\left(\frac{\varepsilon^{(1)}(\lambda^{(1)})^2 P^2 R_1^2}{2(\eta^{(1)})^2}\right)\right] \tag{2.4.47}$$

for $k = 2$

$$Q^{(2)} = 2\pi \int_{R_3}^{R_2} r u^{(2)}(r) dr \tag{2.4.48}$$

$$Q^{(2)} = 2\pi \int_{R_3}^{R_2} \frac{\eta^{(2)}}{2\varepsilon^{(2)}(\lambda^{(2)})^2 P}\left[\exp\left(\frac{\varepsilon^{(2)}(\lambda^{(2)})^2 P^2 R_2^2}{2(\eta^{(2)})^2}\right) - \exp\left(\frac{\varepsilon^{(2)}(\lambda^{(2)})^2 P^2 r^2}{2(\eta^{(2)})^2}\right)\right] r dr$$
$$+ 2\pi \int_{R_3}^{R_2} \frac{\eta^{(1)}}{2\varepsilon^{(1)}(\lambda^{(1)})^2 P}\left[\exp\left(\frac{\varepsilon^{(1)}(\lambda^{(1)})^2 P^2 R_1^2}{2(\eta^{(1)})^2}\right) - \exp\left(\frac{\varepsilon^{(1)}(\lambda^{(1)})^2 P^2 R_2^2}{2(\eta^{(1)})^2}\right)\right] r dr \tag{2.4.49}$$

$$Q^{(2)} = \frac{\pi\eta^{(2)}}{\varepsilon^{(2)}(\lambda^{(2)})^2 P} \exp\left(\frac{\varepsilon^{(2)}(\lambda^{(2)})^2 P^2 R_2^2}{2(\eta^{(2)})^2}\right)\int_{R_3}^{R_2} r dr - \frac{\pi\eta^{(2)}}{\varepsilon^{(2)}(\lambda^{(2)})^2 P}\int_{R_3}^{R_2} \exp\left(\frac{\varepsilon^{(2)}(\lambda^{(2)})^2 P^2 r^2}{2(\eta^{(2)})^2}\right) r dr$$
$$+ \frac{\pi\eta^{(1)}}{\varepsilon^{(1)}(\lambda^{(1)})^2 P} \exp\left(\frac{\varepsilon^{(1)}(\lambda^{(1)})^2 P^2 R_1^2}{2(\eta^{(1)})^2}\right)\int_{R_3}^{R_2} r dr - \frac{\pi\eta^{(1)}}{\varepsilon^{(1)}(\lambda^{(1)})^2 P} \exp\left(\frac{\varepsilon^{(1)}(\lambda^{(1)})^2 P^2 R_2^2}{2(\eta^{(1)})^2}\right)\int_{R_3}^{R_2} r dr \tag{2.4.50}$$

After integrating,

$$Q^{(2)} = \frac{\pi\eta^{(2)}}{2\varepsilon^{(2)}(\lambda^{(2)})^2 P} \exp\left(\frac{\varepsilon^{(2)}(\lambda^{(2)})^2 P^2 R_2^2}{2(\eta^{(2)})^2}\right) \left[r^2\right]_{R_3}^{R_2} - \frac{\pi(\eta^{(2)})^3}{(\varepsilon^{(2)})^2(\lambda^{(2)})^4 P^3} \left[\exp\left(\frac{\varepsilon^{(2)}(\lambda^{(2)})^2 P^2 r^2}{2(\eta^{(2)})^2}\right)\right]_{R_3}^{R_2}$$
$$+ \frac{\pi\eta^{(1)}}{2\varepsilon^{(1)}(\lambda^{(1)})^2 P} \exp\left(\frac{\varepsilon^{(1)}(\lambda^{(1)})^2 P^2 R_1^2}{2(\eta^{(1)})^2}\right) \left[r^2\right]_{R_3}^{R_2} - \frac{\pi\eta^{(1)}}{2\varepsilon^{(1)}(\lambda^{(1)})^2 P} \exp\left(\frac{\varepsilon^{(1)}(\lambda^{(1)})^2 P^2 R_2^2}{2(\eta^{(1)})^2}\right) \left[r^2\right]_{R_3}^{R_2} \tag{2.4.51}$$

$$\begin{aligned} Q^{(2)} = & \frac{\pi\eta^{(2)}}{2\varepsilon^{(2)}(\lambda^{(2)})^2 P} \exp\left(\frac{\varepsilon^{(2)}(\lambda^{(2)})^2 P^2 R_2^2}{2(\eta^{(2)})^2}\right) \left[R_2^2 - R_3^2\right] \\ & - \frac{\pi(\eta^{(2)})^3}{(\varepsilon^{(2)})^2(\lambda^{(2)})^4 P^3} \left[\exp\left(\frac{\varepsilon^{(2)}(\lambda^{(2)})^2 P^2 R_2^2}{2(\eta^{(2)})^2}\right) - \exp\left(\frac{\varepsilon^{(2)}(\lambda^{(2)})^2 P^2 R_3^2}{2(\eta^{(2)})^2}\right)\right] \\ & + \frac{\pi\eta^{(1)}}{2\varepsilon^{(1)}(\lambda^{(1)})^2 P} \exp\left(\frac{\varepsilon^{(1)}(\lambda^{(1)})^2 P^2 R_1^2}{2(\eta^{(1)})^2}\right) \left[R_2^2 - R_3^2\right] \\ & - \frac{\pi\eta^{(1)}}{2\varepsilon^{(1)}(\lambda^{(1)})^2 P} \exp\left(\frac{\varepsilon^{(1)}(\lambda^{(1)})^2 P^2 R_2^2}{2(\eta^{(1)})^2}\right) \left[R_2^2 - R_3^2\right] \end{aligned} \tag{2.4.52}$$

So,

$$\begin{aligned} Q^{(2)} = & \frac{\pi\eta^{(2)}}{2\varepsilon^{(2)}(\lambda^{(2)})^2 P} \exp\left(\frac{\varepsilon^{(2)}(\lambda^{(2)})^2 P^2 R_2^2}{2(\eta^{(2)})^2}\right) \left[R_2^2 - R_3^2\right] \\ & + \frac{\pi(\eta^{(2)})^3}{(\varepsilon^{(2)})^2(\lambda^{(2)})^4 P^3} \left[\exp\left(\frac{\varepsilon^{(2)}(\lambda^{(2)})^2 P^2 R_3^2}{2(\eta^{(2)})^2}\right) - \exp\left(\frac{\varepsilon^{(2)}(\lambda^{(2)})^2 P^2 R_2^2}{2(\eta^{(2)})^2}\right)\right] \\ & + \frac{\pi\eta^{(1)}}{2\varepsilon^{(1)}(\lambda^{(1)})^2 P} \left[\exp\left(\frac{\varepsilon^{(1)}(\lambda^{(1)})^2 P^2 R_1^2}{2(\eta(1))^2}\right) - \exp\left(\frac{\varepsilon^{(1)}(\lambda^{(1)})^2 P^2 R_2^2}{2(\eta(1))^2}\right)\right] \left[R_2^2 - R_3^2\right] \end{aligned} \tag{2.4.53}$$

for $k = 3$

$$Q^{(3)} = 2\pi \int_{R_4}^{R_3} r u^{(3)}(r) dr \tag{2.4.54}$$

$$
\begin{aligned}
Q^{(3)} &= 2\pi\int_{R_4}^{R_3}\frac{\eta^{(3)}}{2\varepsilon^{(3)}(\lambda^{(3)})^2P}\left[\exp\left(\frac{\varepsilon^{(3)}(\lambda^{(3)})^2P^2R_3^2}{2(\eta^{(3)})^2}\right)-\exp\left(\frac{\varepsilon^{(3)}(\lambda^{(3)})^2P^2r^2}{2(\eta^{(3)})^2}\right)\right]rdr\\
&+2\pi\int_{R_4}^{R_3}\frac{\eta^{(2)}}{2\varepsilon^{(2)}(\lambda^{(2)})^2P}\left[\exp\left(\frac{\varepsilon^{(2)}(\lambda^{(2)})^2P^2R_2^2}{2(\eta^{(2)})^2}\right)-\exp\left(\frac{\varepsilon^{(2)}(\lambda^{(2)})^2P^2R_3^2}{2(\eta^{(2)})^2}\right)\right]rdr\\
&+2\pi\int_{R_4}^{R_3}\frac{\eta^{(1)}}{2\varepsilon^{(1)}(\lambda^{(1)})^2P}\left[\exp\left(\frac{\varepsilon^{(1)}(\lambda^{(1)})^2P^2R_1^2}{2(\eta^{(1)})^2}\right)-\exp\left(\frac{\varepsilon^{(1)}(\lambda^{(1)})^2P^2R_2^2}{2(\eta^{(1)})^2}\right)\right]rdr
\end{aligned}
\tag{2.4.55}
$$

$$
\begin{aligned}
Q^{(3)} &= \frac{\pi\eta^{(3)}}{\varepsilon^{(3)}(\lambda^{(3)})^2P}\exp\left(\frac{\varepsilon^{(3)}(\lambda^{(3)})^2P^2R_3^2}{2(\eta^{(3)})^2}\right)\int_{R_4}^{R_3}rdr-\frac{\pi\eta^{(3)}}{\varepsilon^{(3)}(\lambda^{(3)})^2P}\int_{R_4}^{R_3}\exp\left(\frac{\varepsilon^{(3)}(\lambda^{(3)})^2P^2r^2}{2(\eta^{(3)})^2}\right)rdr\\
&+\frac{\pi\eta^{(2)}}{\varepsilon^{(2)}(\lambda^{(2)})^2P}\exp\left(\frac{\varepsilon^{(2)}(\lambda^{(2)})^2P^2R_2^2}{2(\eta^{(1)})^2}\right)\int_{R_4}^{R_3}rdr-\frac{\pi\eta^{(2)}}{\varepsilon^{(2)}(\lambda^{(2)})^2P}\exp\left(\frac{\varepsilon^{(2)}(\lambda^{(2)})^2P^2R_3^2}{2(\eta^{(2)})^2}\right)\int_{R_4}^{R_3}rdr\\
&+\frac{\pi\eta^{(1)}}{\varepsilon^{(1)}(\lambda^{(1)})^2P}\exp\left(\frac{\varepsilon^{(1)}(\lambda^{(1)})^2P^2R_1^2}{2(\eta^{(1)})^2}\right)\int_{R_4}^{R_3}rdr-\frac{\pi\eta^{(1)}}{\varepsilon^{(1)}(\lambda^{(1)})^2P}\exp\left(\frac{\varepsilon^{(1)}(\lambda^{(1)})^2P^2R_2^2}{2(\eta^{(1)})^2}\right)\int_{R_4}^{R_3}rdr
\end{aligned}
\tag{2.4.56}
$$

After integrating,

$$
\begin{aligned}
Q^{(3)} &= \frac{\pi\eta^{(3)}}{2\varepsilon^{(3)}(\lambda^{(3)})^2P}\exp\left(\frac{\varepsilon^{(3)}(\lambda^{(3)})^2P^2R_3^2}{2(\eta^{(3)})^2}\right)\left[r^2\right]_{R_4}^{R_3}-\frac{\pi(\eta^{(3)})^3}{(\varepsilon^{(3)})^2(\lambda^{(3)})^4P^3}\left[\exp(\frac{\varepsilon^{(3)}(\lambda^{(3)})^2P^2r^2}{2(\eta^{(3)})^2})\right]_{R_4}^{R_3}\\
&+\frac{\pi\eta^{(2)}}{2\varepsilon^{(2)}(\lambda^{(2)})^2P}\exp\left(\frac{\varepsilon^{(2)}(\lambda^{(2)})^2P^2R_2^2}{2(\eta^{(2)})^2}\right)\left[r^2\right]_{R_4}^{R_3}-\frac{\pi\eta^{(2)}}{2\varepsilon^{(2)}(\lambda^{(2)})^2P}\exp\left(\frac{\varepsilon^{(2)}(\lambda^{(2)})^2P^2R_3^2}{2(\eta^{(3)})^2}\right)\left[r^2\right]_{R_4}^{R_3}\\
&+\frac{\pi\eta^{(1)}}{2\varepsilon^{(1)}(\lambda^{(1)})^2P}\exp\left(\frac{\varepsilon^{(1)}(\lambda^{(1)})^2P^2R_1^2}{2(\eta^{(1)})^2}\right)\left[r^2\right]_{R_4}^{R_3}-\frac{\pi\eta^{(1)}}{2\varepsilon^{(1)}(\lambda^{(1)})^2P}\exp\left(\frac{\varepsilon^{(1)}(\lambda^{(1)})^2P^2R_2^2}{2(\eta^{(1)})^2}\right)\left[r^2\right]_{R_4}^{R_3}
\end{aligned}
\tag{2.4.57}
$$

$$
\begin{aligned}
Q^{(3)} = {} & \frac{\pi\eta^{(3)}}{2\varepsilon^{(3)}(\lambda^{(3)})^2 P}\exp\left(\frac{\varepsilon^{(3)}(\lambda^{(3)})^2P^2R_3^2}{2(\eta^{(3)})^2}\right)\left[R_3^2 - R_4^2\right] \\
& - \frac{\pi(\eta^{(3)})^3}{(\varepsilon^{(3)})^2(\lambda^{(3)})^4P^3}\left[\exp\left(\frac{\varepsilon^{(3)}(\lambda^{(3)})^2P^2R_3^2}{2(\eta^{(3)})^2}\right) - \exp\left(\frac{\varepsilon^{(3)}(\lambda^{(3)})^2P^2R_4^2}{2(\eta^{(3)})^2}\right)\right] \\
& + \frac{\pi\eta^{(2)}}{2\varepsilon^{(2)}(\lambda^{(2)})^2 P}\exp\left(\frac{\varepsilon^{(2)}(\lambda^{(2)})^2P^2R_2^2}{2(\eta^{(2)})^2}\right)\left[R_3^2 - R_4^2\right] \\
& - \frac{\pi\eta^{(2)}}{2\varepsilon^{(2)}(\lambda^{(2)})^2 P}\exp\left(\frac{\varepsilon^{(2)}(\lambda^{(2)})^2P^2R_3^2}{2(\eta^{(2)})^2}\right)\left[R_3^2 - R_4^2\right] \\
& + \frac{\pi\eta^{(1)}}{2\varepsilon^{(1)}(\lambda^{(1)})^2 P}\exp\left(\frac{\varepsilon^{(1)}(\lambda^{(1)})^2P^2R_1^2}{2(\eta^{(1)})^2}\right)\left[R_3^2 - R_4^2\right] \\
& - \frac{\pi\eta^{(1)}}{2\varepsilon^{(1)}(\lambda^{(1)})^2 P}\exp\left(\frac{\varepsilon^{(1)}(\lambda^{(1)})^2P^2R_2^2}{2(\eta^{(1)})^2}\right)\left[R_3^2 - R_4^2\right]
\end{aligned}
\tag{2.4.58}
$$

So,

$$
\begin{aligned}
Q^{(3)} = {} & \frac{\pi\eta^{(3)}}{2\varepsilon^{(3)}(\lambda^{(3)})^2 P}\exp\left(\frac{\varepsilon^{(3)}(\lambda^{(3)})^2P^2R_3^2}{2(\eta^{(3)})^2}\right)\left[R_3^2 - R_4^2\right] \\
& + \frac{\pi(\eta^{(3)})^3}{(\varepsilon^{(3)})^2(\lambda^{(3)})^4P^3}\left[\exp\left(\frac{\varepsilon^{(3)}(\lambda^{(3)})^2P^2R_4^2}{2(\eta^{(3)})^2}\right) - \exp\left(\frac{\varepsilon^{(3)}(\lambda^{(3)})^2P^2R_3^2}{2(\eta^{(3)})^2}\right)\right] \\
& + \frac{\pi\eta^{(2)}}{2\varepsilon^{(2)}(\lambda^{(2)})^2 P}\left[\exp\left(\frac{\varepsilon^{(2)}(\lambda^{(2)})^2P^2R_2^2}{2(\eta^(2))^2}\right) - \exp\left(\frac{\varepsilon^{(2)}(\lambda^{(2)})^2P^2R_3^2}{2(\eta^(2))^2}\right)\right]\left[R_3^2 - R_4^2\right] \\
& + \frac{\pi\eta^{(1)}}{2\varepsilon^{(1)}(\lambda^{(1)})^2 P}\left[\exp\left(\frac{\varepsilon^{(1)}(\lambda^{(1)})^2P^2R_1^2}{2(\eta^(1))^2}\right) - \exp\left(\frac{\varepsilon^{(1)}(\lambda^{(1)})^2P^2R_2^2}{2(\eta^(1))^2}\right)\right]\left[R_3^2 - R_4^2\right]
\end{aligned}
\tag{2.4.59}
$$

So,it's k^{th} term will be written as,

$$Q^{(k)} = \frac{\pi\eta^{(k)}}{2\varepsilon^{(k)}(\lambda^{(k)})^2P}\exp\left(\frac{\varepsilon^{(k)}(\lambda^{(k)})^2P^2R_k^2}{2(\eta^{(k)})^2}\right)\left[R_k^2 - R_{k+1}^2\right]$$
$$+\frac{\pi(\eta^{(k)})^3}{(\varepsilon^{(k)})^2(\lambda^{(k)})^4P^3}\left[\exp\left(\frac{\varepsilon^{(k)}(\lambda^{(k)})^2P^2R_{k+1}^2}{2(\eta^{(k)})^2}\right)-\exp\left(\frac{\varepsilon^{(k)}(\lambda^{(k)})^2P^2R_k^2}{2(\eta^{(k)})^2}\right)\right]$$
$$+\sum_{i=1}^{k-1}\frac{\pi\eta^{(i)}}{2\varepsilon^{(i)}(\lambda^{(i)})^2P}\left[\exp\left(\frac{\varepsilon^{(i)}(\lambda^{(i)})^2P^2R_i^2}{2(\eta^{(i)})^2}\right)-\exp\left(\frac{\varepsilon^{(i)}(\lambda^{(i)})^2P^2R_{i+1}^2}{2(\eta^{(i)})^2}\right)\right]\left[R_k^2 - R_{k+1}^2\right]$$
$$\tag{2.4.60}$$

Now we will find $Q^{(n)}$ such that

$$Q^{(n)} = 2\pi\int_0^{R_n} ru^{(n)}(r)dr \tag{2.4.61}$$

After integration

$$\begin{aligned} Q^{(n)} &= \frac{\pi\eta^{(n)}R_n^2}{2\varepsilon^{(n)}(\lambda^{(n)})^2P}\exp\left(\frac{\varepsilon^{(n)}(\lambda^{(n)})^2P^2R_n^2}{2(\eta^{(n)})^2}\right) \\ &+\frac{\pi(\eta^{(n)})^3}{(\varepsilon^{(n)})^2(\lambda^{(n)})^4P^3}\left[1-\exp\frac{\varepsilon^{(n)}(\lambda^{(n)})^2P^2R_n^2}{2(\eta^{(n)})^2}\right] \\ &+\pi R_n^2\left[u^{(n-1)}(r)\right]_{r=R_n} \end{aligned} \tag{2.4.62}$$

2.4.3 Special Case For First-Layer

For n=1

$$Q^{(1)} = 2\pi\int_0^{R_1} ru^{(1)}(r)dr \tag{2.4.63}$$

$$Q^{(1)} = 2\pi\int_0^{R_1}\frac{\eta^{(1)}}{2\varepsilon^{(1)}(\lambda^{(1)})^2P}\left[\exp\left(\frac{\varepsilon^{(1)}(\lambda^{(1)})^2P^2R_1^2}{2(\eta^{(1)})^2}\right)-\exp\left(\frac{\varepsilon^{(1)}(\lambda^{(1)})^2P^2r^2}{2(\eta^{(1)})^2}\right)\right]rdr \tag{2.4.64}$$

$$Q^{(1)} = \frac{\pi\eta^{(1)}}{\varepsilon^{(1)}(\lambda^{(1)})^2P}\exp\left(\frac{\varepsilon^{(1)}(\lambda^{(1)})^2P^2R_1^2}{2(\eta^{(1)})^2}\right)\int_0^{R_1} rdr - \frac{\pi\eta^{(1)}}{\varepsilon^{(1)}(\lambda^{(1)})^2P}\int_0^{R_1}\exp\left(\frac{\varepsilon^{(1)}(\lambda^{(1)})^2P^2r^2}{2(\eta^{(1)})^2}\right)rdr \tag{2.4.65}$$

After integrating,

$$Q^{(1)} = \frac{\pi\eta^{(1)}}{2\varepsilon^{(1)}(\lambda^{(1)})^2 P} \exp\left(\frac{\varepsilon^{(1)}(\lambda^{(1)})^2 P^2 R_1^2}{2(\eta^{(1)})^2}\right)\left[r^2\right]_0^{R_1} - \frac{\pi(\eta^{(1)})^3}{(\varepsilon^{(1)})^2(\lambda^{(1)})^4 P^3}\left[\exp\left(\frac{\varepsilon^{(1)}(\lambda^{(1)})^2 P^2 r^2}{2(\eta^{(1)})^2}\right)\right]_0^{R_1} \tag{2.4.66}$$

So,

$$Q^{(1)} = \frac{\pi\eta^{(1)} R_1^2}{2\varepsilon^{(1)}(\lambda^{(1)})^2 P} \exp\left(\frac{\varepsilon^{(1)}(\lambda^{(1)})^2 P^2 R_1^2}{2(\eta^{(1)})^2}\right) - \frac{\pi(\eta^{(1)})^3}{(\varepsilon^{(1)})^2(\lambda^{(1)})^4 P^3}\left[1 - \exp\left(\frac{\varepsilon^{(1)}(\lambda^{(1)})^2 P^2 R_1^2}{2(\eta^{(1)})^2}\right)\right] \tag{2.4.67}$$

2.4.4 Special Case For Second-Layer

We have a special case for two layers so we will substitute n=2

$$Q^{(2)} = 2\pi \int_0^{R_2} r u^{(2)}(r) dr \tag{2.4.68}$$

$$\begin{aligned} Q^{(2)} = {} & 2\pi \int_0^{R_2} \frac{\eta^{(2)}}{2\varepsilon^{(2)}(\lambda^{(2)})^2 P}\left[\exp\left(\frac{\varepsilon^{(2)}(\lambda^{(2)})^2 P^2 R_2^2}{2(\eta^{(2)})^2}\right) - \exp\left(\frac{\varepsilon^{(2)}(\lambda^{(2)})^2 P^2 r^2}{2(\eta^{(2)})^2}\right)\right] r dr \\ & + 2\pi \int_0^{R_2} \frac{\eta^{(1)}}{2\varepsilon^{(1)}(\lambda^{(1)})^2 P}\left[\exp\left(\frac{\varepsilon^{(1)}(\lambda^{(1)})^2 P^2 R_1^2}{2(\eta^{(1)})^2}\right) - \exp\left(\frac{\varepsilon^{(1)}(\lambda^{(1)})^2 P^2 R_2^2}{2(\eta^{(1)})^2}\right)\right] r dr \end{aligned} \tag{2.4.69}$$

$$\begin{aligned} Q^{(2)} = {} & \frac{\pi\eta^{(2)}}{\varepsilon^{(2)}(\lambda^{(2)})^2 P} \exp\left(\frac{\varepsilon^{(2)}(\lambda^{(2)})^2 P^2 R_2^2}{2(\eta^{(2)})^2}\right)\int_0^{R_2} r dr - \frac{\pi\eta^{(2)}}{\varepsilon^{(2)}(\lambda^{(2)})^2 P}\int_0^{R_2} \exp\left(\frac{\varepsilon^{(2)}(\lambda^{(2)})^2 P^2 r^2}{2(\eta^{(2)})^2}\right) r dr \\ & + \frac{\pi\eta^{(1)}}{\varepsilon^{(1)}(\lambda^{(1)})^2 P} \exp\left(\frac{\varepsilon^{(1)}(\lambda^{(1)})^2 P^2 R_1^2}{2(\eta^{(1)})^2}\right)\int_0^{R_2} r dr - \frac{\pi\eta^{(1)}}{\varepsilon^{(1)}(\lambda^{(1)})^2 P} \exp\left(\frac{\varepsilon^{(1)}(\lambda^{(1)})^2 P^2 R_2^2}{2(\eta^{(1)})^2}\right)\int_0^{R_2} r dr \end{aligned} \tag{2.4.70}$$

After integrating,

$$Q^{(2)} = \frac{\pi\eta^{(2)}}{2\varepsilon^{(2)}(\lambda^{(2)})^2 P} \exp\left(\frac{\varepsilon^{(2)}(\lambda^{(2)})^2 P^2 R_2^2}{2(\eta^{(2)})^2}\right)\left[r^2\right]_0^{R_2} - \frac{\pi(\eta^{(2)})^3}{(\varepsilon^{(2)})^2(\lambda^{(2)})^4 P^3}\left[\exp\left(\frac{\varepsilon^{(2)}(\lambda^{(2)})^2 P^2 r^2}{2(\eta^{(2)})^2}\right)\right]_0^{R_2}$$
$$+ \frac{\pi\eta^{(1)}}{2\varepsilon^{(1)}(\lambda^{(1)})^2 P} \exp\left(\frac{\varepsilon^{(1)}(\lambda^{(1)})^2 P^2 R_1^2}{2(\eta^{(1)})^2}\right)\left[r^2\right]_0^{R_2} - \frac{\pi\eta^{(1)}}{2\varepsilon^{(1)}(\lambda^{(1)})^2 P} \exp\left(\frac{\varepsilon^{(1)}(\lambda^{(1)})^2 P^2 R_2^2}{2(\eta^{(1)})^2}\right)\left[r^2\right]_0^{R_2} \tag{2.4.71}$$

So,

$$Q^{(2)} = \frac{\pi\eta^{(2)} R_2^2}{2\varepsilon^{(2)}(\lambda^{(2)})^2 P} \exp\left(\frac{\varepsilon^{(2)}(\lambda^{(2)})^2 P^2 R_2^2}{2(\eta^{(2)})^2}\right) + \frac{\pi(\eta^{(2)})^3}{(\varepsilon^{(2)})^2(\lambda^{(2)})^4 P^3}\left[1 - \exp\left(\frac{\varepsilon^{(2)}(\lambda^{(2)})^2 P^2 r^2}{2(\eta^{(2)})^2}\right)\right]$$
$$+ \frac{\pi\eta^{(1)} R_2^2}{2\varepsilon^{(1)}(\lambda^{(1)})^2 P}\left[\exp\left(\frac{\varepsilon^{(1)}(\lambda^{(1)})^2 P^2 R_1^2}{2(\eta^{(1)})^2}\right) - \exp\left(\frac{\varepsilon^{(1)}(\lambda^{(1)})^2 P^2 R_2^2}{2(\eta^{(1)})^2}\right)\right] \tag{2.4.72}$$

2.4.5 Special Case For Third-layer

Now we have a special case for three layers then we use $n = 3$

$$Q^{(3)} = 2\pi \int_0^{R_3} r u^{(3)}(r) dr \tag{2.4.73}$$

$$Q^{(3)} = 2\pi \int_0^{R_3} \frac{\eta^{(3)}}{2\varepsilon^{(3)}(\lambda^{(3)})^2 P}\left[\exp\left(\frac{\varepsilon^{(3)}(\lambda^{(3)})^2 P^2 R_3^2}{2(\eta^{(3)})^2}\right) - \exp\left(\frac{\varepsilon^{(3)}(\lambda^{(3)})^2 P^2 r^2}{2(\eta^{(3)})^2}\right)\right] r dr$$
$$+ 2\pi \int_0^{R_3} \frac{\eta^{(2)}}{2\varepsilon^{(2)}(\lambda^{(2)})^2 P}\left[\exp\left(\frac{\varepsilon^{(2)}(\lambda^{(2)})^2 P^2 R_2^2}{2(\eta^{(2)})^2}\right) - \exp\left(\frac{\varepsilon^{(2)}(\lambda^{(2)})^2 P^2 R_3^2}{2(\eta^{(2)})^2}\right)\right] r dr$$
$$+ 2\pi \int_0^{R_3} \frac{\eta^{(1)}}{2\varepsilon^{(1)}(\lambda^{(1)})^2 P}\left[\exp\left(\frac{\varepsilon^{(1)}(\lambda^{(1)})^2 P^2 R_1^2}{2(\eta^{(1)})^2}\right) - \exp\left(\frac{\varepsilon^{(1)}(\lambda^{(1)})^2 P^2 R_2^2}{2(\eta^{(1)})^2}\right)\right] r dr \tag{2.4.74}$$

$$
\begin{aligned}
Q^{(3)} = {} & \frac{\pi\eta^{(3)}}{\varepsilon^{(3)}(\lambda^{(3)})^2P}\exp\left(\frac{\varepsilon^{(3)}(\lambda^{(3)})^2P^2R_3^2}{2(\eta^{(3)})^2}\right)\int_0^{R_3} rdr - \frac{\pi\eta^{(3)}}{\varepsilon^{(3)}(\lambda^{(3)})^2P}\int_0^{R_3}\exp\left(\frac{\varepsilon^{(3)}(\lambda^{(3)})^2P^2r^2}{2(\eta^{(3)})^2}\right) rdr \\
& + \frac{\pi\eta^{(2)}}{\varepsilon^{(2)}(\lambda^{(2)})^2P}\exp\left(\frac{\varepsilon^{(2)}(\lambda^{(2)})^2P^2R_2^2}{2(\eta^{(1)})^2}\right)\int_0^{R_3} rdr - \frac{\pi\eta^{(2)}}{\varepsilon^{(2)}(\lambda^{(2)})^2P}\exp\left(\frac{\varepsilon^{(2)}(\lambda^{(2)})^2P^2R_3^2}{2(\eta^{(2)})^2}\right)\int_0^{R_3} rdr \\
& + \frac{\pi\eta^{(1)}}{\varepsilon^{(1)}(\lambda^{(1)})^2P}\exp\left(\frac{\varepsilon^{(1)}(\lambda^{(1)})^2P^2R_1^2}{2(\eta^{(1)})^2}\right)\int_0^{R_3} rdr - \frac{\pi\eta^{(1)}}{\varepsilon^{(1)}(\lambda^{(1)})^2P}\exp\left(\frac{\varepsilon^{(1)}(\lambda^{(1)})^2P^2R_2^2}{2(\eta^{(1)})^2}\right)\int_0^{R_3} rdr
\end{aligned}
\tag{2.4.75}
$$

After integrating,

$$
\begin{aligned}
Q^{(3)} = {} & \frac{\pi\eta^{(3)}}{2\varepsilon^{(3)}(\lambda^{(3)})^2P}\exp\left(\frac{\varepsilon^{(3)}(\lambda^{(3)})^2P^2R_3^2}{2(\eta^{(3)})^2}\right)\left[r^2\right]_0^{R_3} - \frac{\pi(\eta^{(3)})^3}{(\varepsilon^{(3)})^2(\lambda^{(3)})^4P^3}\left[\exp(\frac{\varepsilon^{(3)}(\lambda^{(3)})^2P^2r^2}{2(\eta^{(3)})^2})\right]_0^{R_3} \\
& + \frac{\pi\eta^{(2)}}{2\varepsilon^{(2)}(\lambda^{(2)})^2P}\exp\left(\frac{\varepsilon^{(2)}(\lambda^{(2)})^2P^2R_2^2}{2(\eta^{(2)})^2}\right)\left[r^2\right]_0^{R_3} - \frac{\pi\eta^{(2)}}{2\varepsilon^{(2)}(\lambda^{(2)})^2P}\exp\left(\frac{\varepsilon^{(2)}(\lambda^{(2)})^2P^2R_3^2}{2(\eta^{(3)})^2}\right)\left[r^2\right]_0^{R_3} \\
& + \frac{\pi\eta^{(1)}}{2\varepsilon^{(1)}(\lambda^{(1)})^2P}\exp\left(\frac{\varepsilon^{(1)}(\lambda^{(1)})^2P^2R_1^2}{2(\eta^{(1)})^2}\right)\left[r^2\right]_0^{R_3} - \frac{\pi\eta^{(1)}}{2\varepsilon^{(1)}(\lambda^{(1)})^2P}\exp\left(\frac{\varepsilon^{(1)}(\lambda^{(1)})^2P^2R_2^2}{2(\eta^{(1)})^2}\right)\left[r^2\right]_0^{R_3}
\end{aligned}
\tag{2.4.76}
$$

so,

$$
\begin{aligned}
Q^{(3)} = {} & \frac{\pi\eta^{(3)}R_3^2}{2\varepsilon^{(3)}(\lambda^{(3)})^2P}\exp\left(\frac{\varepsilon^{(3)}(\lambda^{(3)})^2P^2R_3^2}{2(\eta^{(3)})^2}\right) + \frac{\pi(\eta^{(3)})^3}{(\varepsilon^{(3)})^2(\lambda^{(3)})^4P^3}\left[1-\exp\left(\frac{\varepsilon^{(3)}(\lambda^{(3)})^2P^2R_3^2}{2(\eta^{(3)})^2}\right)\right] \\
& + \frac{\pi\eta^{(2)}R_3^2}{2\varepsilon^{(2)}(\lambda^{(2)})^2P}\left[\exp\left(\frac{\varepsilon^{(2)}(\lambda^{(2)})^2P^2R_2^2}{2(\eta^{(2)})^2}\right)-\exp\left(\frac{\varepsilon^{(2)}(\lambda^{(2)})^2P^2R_3^2}{2(\eta^{(2)})^2}\right)\right] \\
& + \frac{\pi\eta^{(1)}R_3^2}{2\varepsilon^{(1)}(\lambda^{(1)})^2P}\left[\exp\left(\frac{\varepsilon^{(1)}(\lambda^{(1)})^2P^2R_1^2}{2(\eta^{(1)})^2}\right)-\exp\left(\frac{\varepsilon^{(1)}(\lambda^{(1)})^2P^2R_2^2}{2(\eta^{(1)})^2}\right)\right]
\end{aligned}
\tag{2.4.77}
$$

2.5 Graphical Results and Discussion

In this section we will discuss about the velocity distribution for linear and exponential PTT fluid models plotted graphically against radius of cylinder for two layers and for multi-layers flows for different values of different parameters emerging in this model.

Graphs of linear PTT model for two-layers flow for $(n = 2)$ are shown in figure 2.2, figure 2.3, and figure 2.4 and multi-layers flow for $(n = 5)$ are shown in the figure 2.5 and figure 2.6.

Graphs of exponential PTT model for two layers $(n = 2)$ are shown in figure 2.7, figure 2.8 and 2.9 and for multi-layers $(n = 5)$ are shown in figure 2.10 and figure 2.11 respectively.

• Graphical Discussion For Two-Layers

For two layers we fixed $R_1 = 1$, $R_2 = 0.8$

. One inference from figure 2.2 and figure 2.7 is the existence of an inverse proportionality relationship between the ratio of viscosities $\eta^{(1)}$ and $\eta^{(2)}$ with fluid field. But other parameters pressure p , $\lambda^{(k)}$, $\varepsilon^{(k)}$ for $k = 1,2$ remain constant as shown in figure 2.2 and figure 2.7. This shows that fluid velocity is greater when fluid in outermost layer is less viscous. And significance of this important application is for the pumping of non-conducting. It also show that flow rate can also be controlled by adjusting relative viscosities of the fluid. In figure 2.2 (a) and figure 2.7 (a) we fixed $\eta^{(2)}$ and change $\eta^{(1)}$ for different values of m but in figure 2.2 (b) and figure 2.7 (b) we fixed $\eta^{(1)}$ and change $\eta^{(2)}$ for different values of $m.$ when we increase the value of m then viscosity also increases because viscosity is the multiple of $m.$ But with increasing viscosity , velocity profile and flow rate decreases as shown in figure 2.2 and figure 2.7 .

For two-layer linear PTT model in figure 2.4 we change pressure gradient for $p = 1$, $p = 1.2$, $p = 1.5$ and for two-layer exponential PTT model in figure 2.9 we change pressure gradient for $p = 0.6$, $p = 0.8$, $p = 1$. For both models effect of pressure gradient remain same , by increasing the pressure gradient velocity and flow rates increase, as expected.

For linear model figure 2.3 (a) and for exponential model 2.8 (c) we fixed $\lambda^{(1)}$ and change $\lambda^{(2)}$ for different values of n. By increasing the value of n, $\lambda^{(2)}$ also increasing and by increasing $\lambda^{(2)}$, velocity also increases.

In figure 2.3 (b) and figure 2.8 (d) we fixed $\varepsilon^{(1)}$ and change $\varepsilon^{(2)}$ by using different values of s, its effect on velocity has shown in the figure 2.3 (b) and figure 2.8 (d).

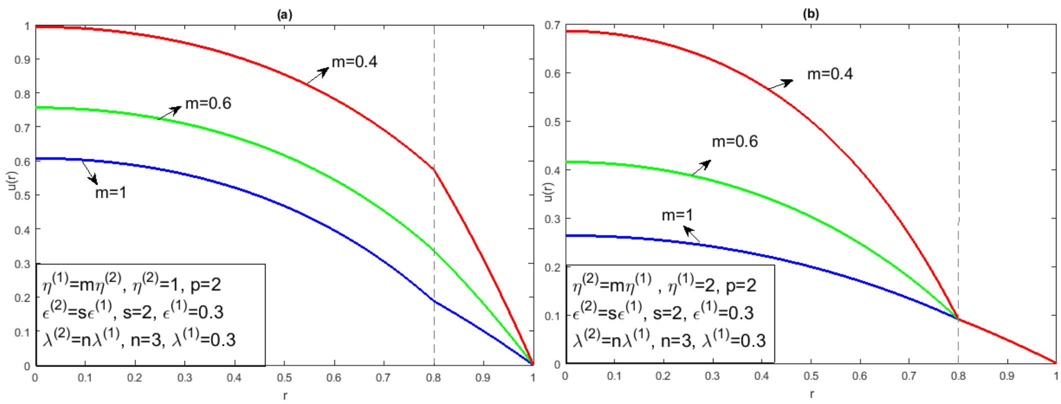

Figure 2.2: Effect of viscosities in two-layer flow (linear model)

• Graphical Discussion For Multi-layers

For multilayered flow we fixed $R_1 = 1$, $R_{k+1} = R_k - 0.2$ for $(k = 1,2,3,4)$. Figure 2.6 and figure 2.11 shows the effect of pressure gradient on the velocity profile for multilayered flows but as expected the effect of pressure gradient is same as it was for two-layers.

Figure 2.5 (a) and figure 2.10 (a) shows the behavior of flow rate for different values of $\lambda^{(k+1)}$ with fixed $\lambda^{(1)}$ for $(k = 1,2,3,4)$ by varying n.

Similarly figure 2.5 (b) and figure 2.10 (b) shows the effect of $\varepsilon^{(k+1)}$ with fixed $\lambda^{(1)}$ for $k = 1,2,3,4,5$ on velocity profile.

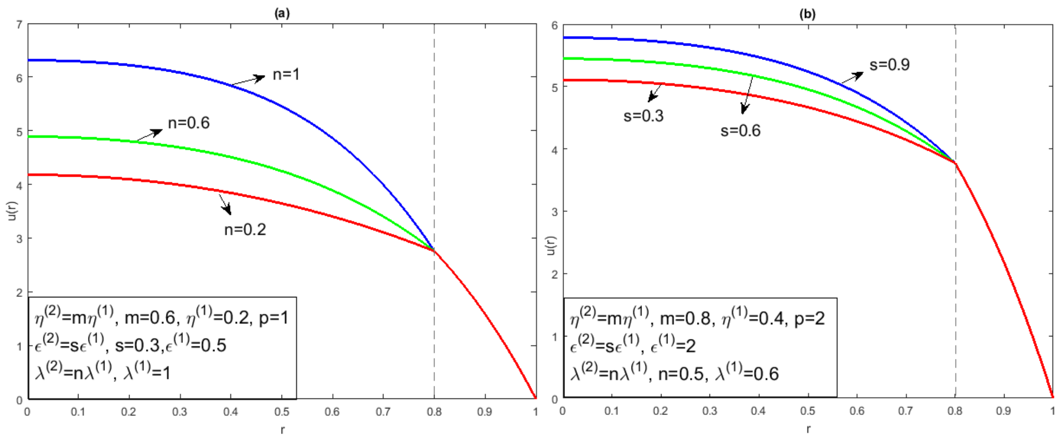

Figure 2.3: Effect of $\lambda^{(k)}$ and $\varepsilon^{(k)}$ in two-layer flow (linear model)

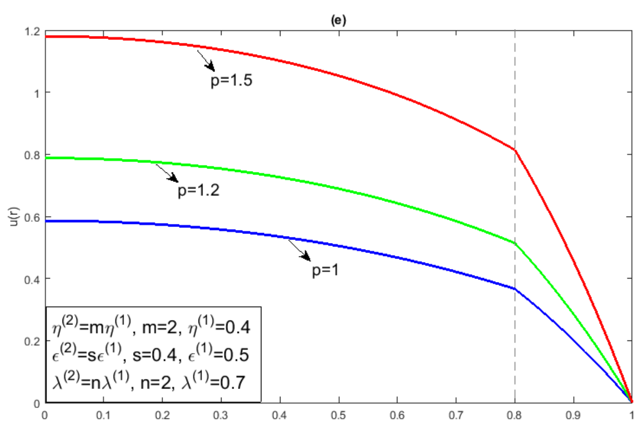

Figure 2.4: Effect of pressure gradient in two-layer flow (linear model)

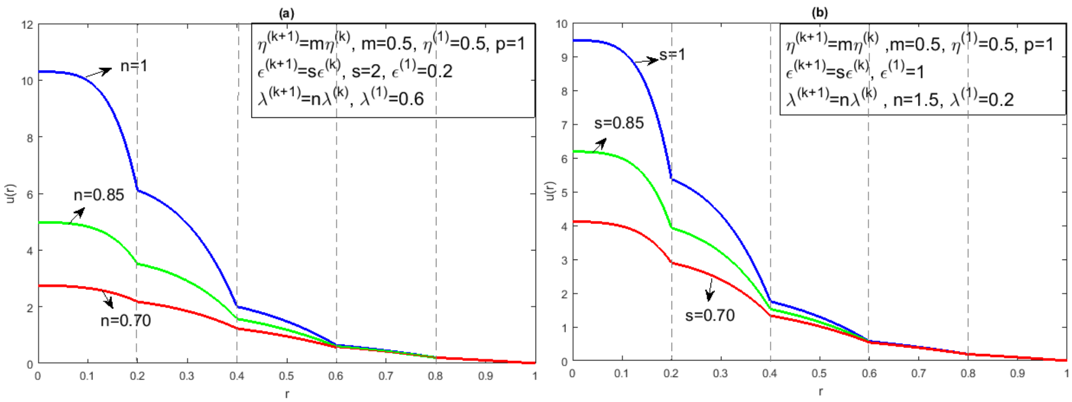

Figure 2.5: Effect of $\lambda^{(k)}$ and $\varepsilon^{(k)}$ in five-layers flow (linear model)

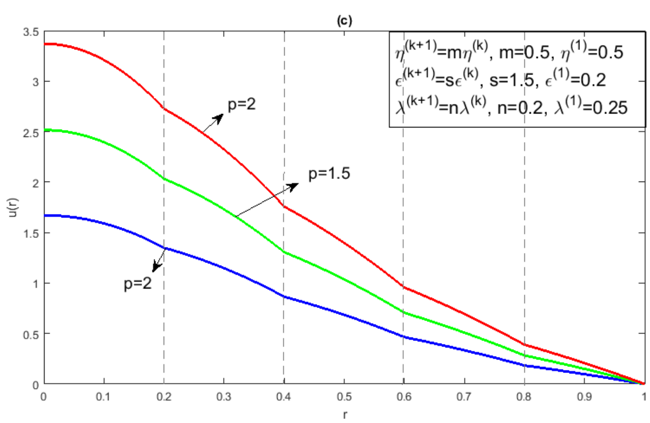

Figure 2.6: Effect of pressure in five-layers flow(linear model)

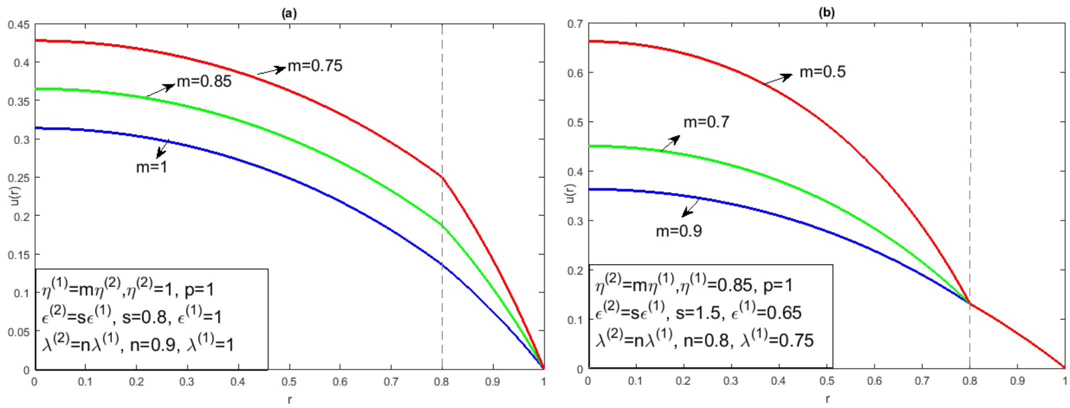

Figure 2.7: Effect of viscosities in two-layer flow (Exponential model)

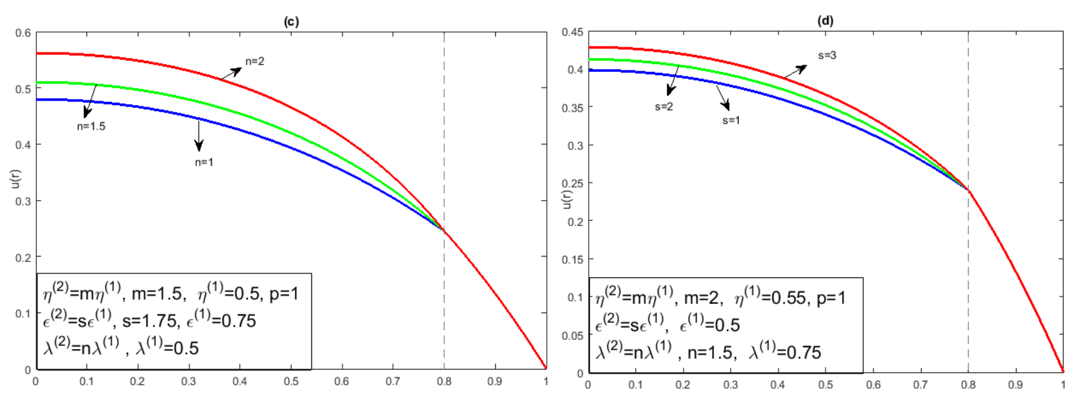

Figure 2.8: Effect of $\lambda^{(k)}$ and $\varepsilon^{(k)}$ in two-layer flow (Exponential model)

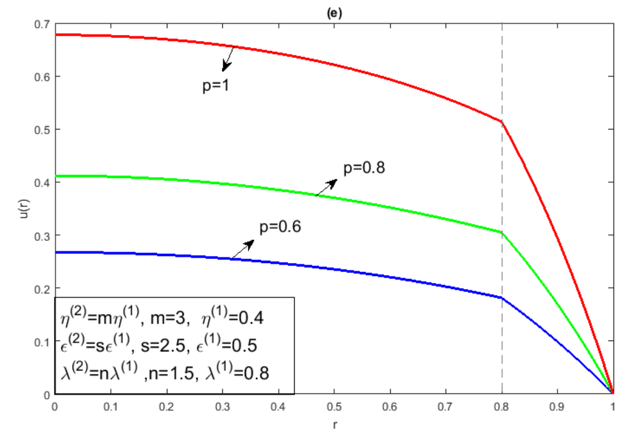

Figure 2.9: Effect of pressure gradient in two-layers flow (Exponential model)

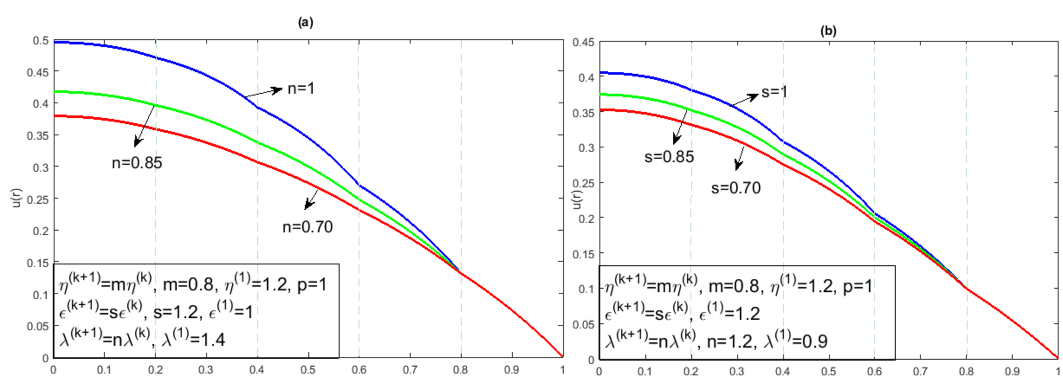

Figure 2.10: Effect of $\lambda^{(k)}$ and $\varepsilon^{(k)}$ in five-layers flow(Exponential model)

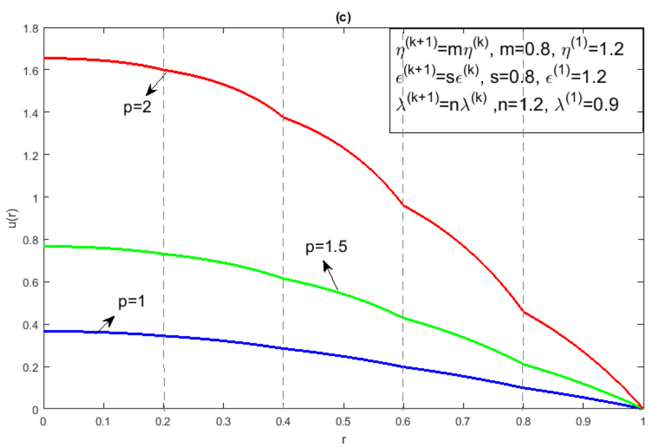

Figure 2.11: Effect of pressure gradient in five-layer flow(Exponential model)

Chapter 3

Two-layer Flows of Oldroyd six Constant Fluids through a Cylindrical Pipe

3.1 Introduction

It is known that mostly non-Newtonian fluid used in industry and technological applications. Non-Newtonian fluids include every day fluids like ketchup, certain oils and greases, shampoo, toothpaste and many other emulsions. Due to multipurpose behavior of fluids, there is no single model available in literature which exhibits all the properties. Therefore, the different constitutive equations of non-Newtonian fluids have been recommended. As we have already discussed that most of present work deals with two-layer fluid flows. Bird et al. [1] have analyzed the two-phase flow of incompressible, immiscible Newtonian fluids. Kapur and Shukla [2] investigated the flow of Newtonian fluids between two parallel plates. Lohrasbi et al. [3], Malashetty et al. [4] [5] have studied the Magnetohydrodynamics heat transfer in two-phase flow. Shail [6] have investigated the laminar two- phase flow in Magnetohydrodynamics. In [7] [8] [9] instability analysis and stability analysis of two-phase flow for Newtonian fluids have been discussed. But [10] [11] [12] [13] [14] also investigated the stability analysis for non-Newtonian fluids. Peristaltic flow for two-layer Power law fluids have been studied by Usha and Ramchandra Rao in [15]. Recently Taza Gul et al. [16] have investigated the exact solution for two-thin film non-Newtonian immiscible fluids on a vertical belt. The aim of this work is to obtain the solutions of the equations which behave the flow of two Oldroyd six constant fluids. Oldroyed six constant are fluids are mainly used to describe fluids that exhibit a variety of a rheological properties. Exact solution for Oldroyed six constant is not possible or difficult to find therefore we can use one of the recent methodologies [24], [26] like homotopy perturbation, modified homotopy perturbation techniques and homotopy analysis method. In present work, we choose to the study the problem using homotopy analysis method [21], [22], [23], [25].

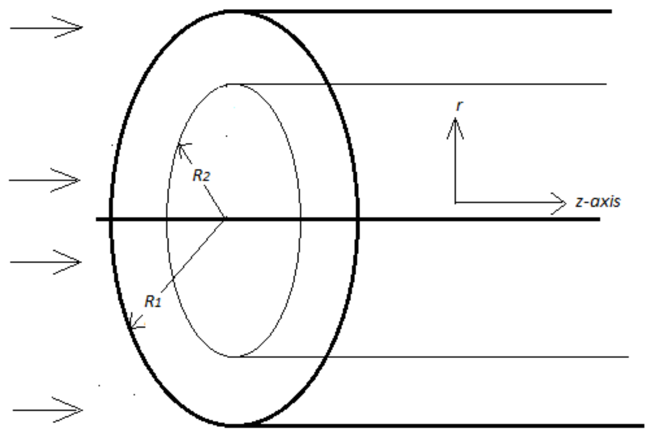

Figure 3.1: Geometry Of The Problem

3.2 Problem Formulation

3.2.1 Geometry of the Problem

We consider the two-layer Oldroyd six-constant fluid flow through cylindrical pipe geometry as shown in figure. We also consider that z-axis is along with the axis of pipe and r is radially in upward direction And pressure gradient $\frac{dP}{dz} = -P$ is along axis of pipe. Radius of the pipe is R_1.

Interface of two fluids is at R_2.We also assume that fluid flow is steady,unidirectional and fully developed.

3.2.2 Basic Equations:-

The basic equations governing the motion of an incompressible fluids flow with constant density are mass conservation

$$\nabla.\mathbf{V}^{(k)} = 0 \tag{3.2.1}$$

and momentum equation

$$\rho^{(k)}\frac{d\mathbf{V}^{(\mathbf{k})}}{dt} = \operatorname{div}\tau^{(k)} - \nabla p + \rho^{(k)}\mathbf{b} \tag{3.2.2}$$

where the superscripts $k = 1,2$ indicate the fluid layer number . $k = 1$ indicates the first layer (outermost) along with solid boundary of pipe so $k = 2$ is the inner layer along the axis of pipe . $V^{(k)}$ represents the velocity of the fluid, $\frac{d}{dt}$ expressing material derivative, p is the fluid pressure, $\tau^{(k)}$ is the stress rate tensor, $\rho^{(k)}$ is density of fluid and $\mathbf{b}$ is the body force vector. General form of the Oldroyd six-constant fluid is,

$$\begin{aligned}\tau^{(k)} + \lambda_1^{(k)}\frac{d\tau^{(k)}}{dt} + \frac{\lambda_3^{(k)}}{2}\left(\tau^{(k)}A_1^{(k)} + A_1^{(k)}\tau^{(k)}\right) + \frac{\lambda_5^{(k)}}{2}\left(tr\tau^{(k)}\right)A_1^{(k)} \\ = \mu^{(k)}\left(A_1^{(k)} + \lambda_2^{(k)}\frac{dA_1^{(k)}}{dt} + \lambda_4^{(k)}A_1^{(k)}\right)\end{aligned} \tag{3.2.3}$$

where

$$A_1^{(k)} = \mathbf{L}^{(\mathbf{k})} + (\mathbf{L}^{(\mathbf{k})})^T, \tag{3.2.4}$$

$$\mathbf{L}^{(\mathbf{k})} = gradV^{(k)} \tag{3.2.5}$$

Here $A_1^{(k)}$ is the first Rivilien-Ericksen tensor, $\mu^{(k)}, \lambda_1^{(k)}, \lambda_2^{(k)}, \lambda_3^{(k)}, \lambda_4^{(k)}$ and $\lambda_5^{(k)}$ are material constants.
The material time derivative $\frac{d}{dt}$ is define as,

$$\frac{d(.)}{dt} = \frac{d(.)}{dt} - \mathbf{L}^{(\mathbf{k})}(.) - (.)(\mathbf{L}^{(\mathbf{k})})^T \tag{3.2.6}$$

3.2.3 Boundary Conditions

Now we will discuss boundary conditions for two-layer flow of Oldroyed six constant .Continuity of stresses and velocities of fluids at the interfaces and no-slip conditions at the wall of the pipe. Mathematically written as,

$$u^{(1)}(r) = 0 \text{ at } r = R_1 \tag{3.2.7}$$

$$u^{(2)}(r) = u^{(1)}(r) \text{ at } r = R_2 \tag{3.2.8}$$

$$\tau_{rz}^{(2)} = 0 \text{ at } r = 0 \tag{3.2.9}$$

and

$$\tau_{rz}^{(1)}(r) = \tau_{rz}^{(2)}(r) \text{ at } r = R_2 \tag{3.2.10}$$

So velocities and extra stresses will be written as,

$$\mathbf{V}^{(\mathbf{k})} = \left[0, 0, u^{(k)}(r)\right] \text{ and } \tau^{(k)} = \tau^{(k)}(r). \tag{3.2.11}$$

3.3 Solution of the Problem

3.3.1 Shear Stresses

By substituting (3.2.11) into (3.2.1) and (3.2.2) then (3.2.1) is identically satisfied and z-component of (3.2.2) in the absence of body forces we get,

$$\frac{dp}{dz} - \frac{1}{r}\frac{d}{dr}(r\tau_{rz}^{(k)}) = 0, \quad k = 1, 2 \tag{3.3.1}$$

integrating w.r.t $'r'$ we will get,

$$r\tau_{rz}^{(k)} = \frac{dp}{dz}\frac{r^2}{2} + C^{(k)} \tag{3.3.2}$$

where $C^{(k)}$ are constant of integration. In view of condition (3.2.9), $C^{(n)} = 0$ and above expression will be written as,

$$\tau_{rz}^{(n)} = \frac{dp}{dz}\frac{r}{2} \tag{3.3.3}$$

Equation (3.3.3) with conditions (3.2.10) when used in (3.3.2) then

$\Rightarrow C^{(k)}$ =0.

This result in the general expression,

$$\tau_{rz}^{(k)} = \frac{dp}{dz}\frac{r}{2}; \quad k = 1,2 \tag{3.3.4}$$

for shear stresses of the fluids.

3.3.2 velocities

$$\tau^{(k)} = \begin{bmatrix} \tau_{rr}^{(k)} & \tau_{r\theta}^{(k)} & \tau_{rz}^{(k)} \\ \tau_{\theta r}^{(k)} & \tau_{\theta\theta}^{(k)} & \tau_{\theta z}^{(k)} \\ \tau_{zr}^{(k)} & \tau_{z\theta}^{(k)} & \tau_{zz}^{(k)} \end{bmatrix} \tag{3.3.5}$$

$$\mathbf{L^{(k)}} = \begin{bmatrix} \frac{\partial}{\partial r} \\ \frac{1}{r}\frac{\partial}{\partial \theta} \\ \frac{\partial}{\partial z} \end{bmatrix} \begin{bmatrix} 0 & 0 & u^{(k)} \end{bmatrix} \tag{3.3.6}$$

$$\mathbf{L^{(k)}} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \frac{\partial u^{(k)}}{\partial r} & 0 & 0 \end{bmatrix} \tag{3.3.7}$$

$$\mathbf{L^{(k)}} = \begin{bmatrix} 0 & 0 & \frac{\partial u^{(k)}}{\partial r} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{3.3.8}$$

So

$$A_1^{(k)} = \begin{bmatrix} 0 & 0 & \frac{\partial u^{(k)}}{\partial r} \\ 0 & 0 & 0 \\ \frac{\partial u^{(k)}}{\partial r} & 0 & 0 \end{bmatrix} \tag{3.3.9}$$

and

$$\frac{d\tau^{(k)}}{dt} = \frac{d\tau^{(k)}}{dt} - \mathbf{L^{(k)}}(\tau^{(k)}) - (\tau^{(k)})(\mathbf{L^{(k)}})^T \tag{3.3.10}$$

$$\mathbf{L}^{(\mathbf{k})}(\tau^{(k)}) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \frac{\partial u^{(k)}}{\partial r} & 0 & 0 \end{bmatrix} \begin{bmatrix} \tau_{rr}^{(k)} & \tau_{r\theta}^{(k)} & \tau_{rz}^{(k)} \\ \tau_{\theta r}^{(k)} & \tau_{\theta\theta}^{(k)} & \tau_{\theta z}^{(k)} \\ \tau_{zr}^{(k)} & \tau_{z\theta}^{(k)} & \tau_{zz}^{(k)} \end{bmatrix} \tag{3.3.11}$$

$$\mathbf{L}^{(\mathbf{k})}(\tau^{(k)}) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \frac{\partial u^{(k)}}{\partial r}\tau_{rr}^{(k)} & \frac{\partial u^{(k)}}{\partial r}\tau_{r\theta}^{(k)} & \frac{\partial u^{(k)}}{\partial r}\tau_{rz}^{(k)} \end{bmatrix} \tag{3.3.12}$$

$$(\tau^{(k)})(\mathbf{L}^{(\mathbf{k})})^T = \begin{bmatrix} 0 & 0 & \frac{\partial u^{(k)}}{\partial r}\tau_{rr}^{(k)} \\ 0 & 0 & \frac{\partial u^{(k)}}{\partial r}\tau_{\theta r}^{(k)} \\ 0 & 0 & \frac{\partial u^{(k)}}{\partial r}\tau_{zr} \end{bmatrix} \tag{3.3.13}$$

Using (3.3.12) and (3.3.13) in (3.3.10).

$$\frac{d\tau^{(k)}}{dt} = -\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \frac{\partial u^{(k)}}{\partial r}\tau_{rr}^{(k)} & \frac{\partial u^{(k)}}{\partial r}\tau_{r\theta}^{(k)} & \frac{\partial u^{(k)}}{\partial r}\tau_{rz}^{(k)} \end{bmatrix} - \begin{bmatrix} 0 & 0 & \frac{\partial u^{(k)}}{\partial r}\tau_{rr}^{(k)} \\ 0 & 0 & \frac{\partial u^{(k)}}{\partial r}\tau_{\theta r}^{(k)} \\ 0 & 0 & \frac{\partial u^{(k)}}{\partial r}\tau_{zr} \end{bmatrix} \tag{3.3.14}$$

So

$$\frac{d\tau^{(k)}}{dt} = \begin{bmatrix} 0 & 0 & -\frac{\partial u^{(k)}}{\partial r}\tau_{rr}^{(k)} \\ 0 & 0 & -\frac{\partial u^{(k)}}{\partial r}\tau_{\theta r}^{(k)} \\ \frac{\partial u^{(k)}}{\partial r}\tau_{rr}^{(k)} & \frac{\partial u^{(k)}}{\partial r}\tau_{r\theta}^{(k)} & -2\frac{\partial u^{(k)}}{\partial r}\tau_{rz} \end{bmatrix} \tag{3.3.15}$$

$$\tau^{(k)}A_1^{(k)} = \begin{bmatrix} \tau_{rr}^{(k)} & \tau_{r\theta}^{(k)} & \tau_{rz}^{(k)} \\ \tau_{\theta r}^{(k)} & \tau_{\theta\theta}^{(k)} & \tau_{\theta z}^{(k)} \\ \tau_{zr}^{(k)} & \tau_{z\theta}^{(k)} & \tau_{zz}^{(k)} \end{bmatrix} \begin{bmatrix} 0 & 0 & \frac{\partial u^{(k)}}{\partial r} \\ 0 & 0 & 0 \\ \frac{\partial u^{(k)}}{\partial r} & 0 & 0 \end{bmatrix} \tag{3.3.16}$$

$$\tau^{(k)}A_1^{(k)} = \begin{bmatrix} \frac{\partial u^{(k)}}{\partial r}\tau_{rz} & 0 & \frac{\partial u^{(k)}}{\partial r}\tau_{rr} \\ \frac{\partial u^{(k)}}{\partial r}\tau_{\theta z} & 0 & \frac{\partial u^{(k)}}{\partial r}\tau_{\theta r} \\ \frac{\partial u^{(k)}}{\partial r}\tau_{zz} & 0 & \frac{\partial u^{(k)}}{\partial r}\tau_{zr} \end{bmatrix} \tag{3.3.17}$$

$$A_1^{(k)}\tau^{(k)} = \begin{bmatrix} 0 & 0 & \frac{\partial u^{(k)}}{\partial r} \\ 0 & 0 & 0 \\ \frac{\partial u^{(k)}}{\partial r} & 0 & 0 \end{bmatrix} \begin{bmatrix} \tau_{rr}^{(k)} & \tau_{r\theta}^{(k)} & \tau_{rz}^{(k)} \\ \tau_{\theta r}^{(k)} & \tau_{\theta\theta}^{(k)} & \tau_{\theta z}^{(k)} \\ \tau_{zr}^{(k)} & \tau_{z\theta}^{(k)} & \tau_{zz}^{(k)} \end{bmatrix} \tag{3.3.18}$$

$$A_1^{(k)}\tau^{(k)} = \begin{bmatrix} \frac{\partial u^{(k)}}{\partial r}\tau_{zr}^{(k)} & \frac{\partial u^{(k)}}{\partial r}\tau_{z\theta}^{(k)} & \frac{\partial u^{(k)}}{\partial r}\tau_{zz}^{(k)} \\ 0 & 0 & 0 \\ \frac{\partial u^{(k)}}{\partial r}\tau_{rr}^{(k)} & \frac{\partial u^{(k)}}{\partial r}\tau_{r\theta}^{(k)} & \frac{\partial u^{(k)}}{\partial r}\tau_{rz}^{(k)} \end{bmatrix} \tag{3.3.19}$$

$$\left(A_1^{(k)}\right)^2 = \begin{bmatrix} 0 & 0 & \frac{\partial u^{(k)}}{\partial r} \\ 0 & 0 & 0 \\ \frac{\partial u^{(k)}}{\partial r} & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & \frac{\partial u^{(k)}}{\partial r} \\ 0 & 0 & 0 \\ \frac{\partial u^{(k)}}{\partial r} & 0 & 0 \end{bmatrix} \tag{3.3.20}$$

$$\left(A_1^{(k)}\right)^2 = \begin{bmatrix} (\frac{\partial u^{(k)}}{\partial r})^2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & (\frac{\partial u^{(k)}}{\partial r})^2 \end{bmatrix} \tag{3.3.21}$$

$$\frac{d(A_1^{(k)})}{dt} = \frac{dA_1^{(k)}}{dt} - \mathbf{L}^{(\mathbf{k})}(A_1^{(k)}) - (A_1^{(k)})(\mathbf{L}^{(\mathbf{k})})^T \tag{3.3.22}$$

$$\mathbf{L}^{(\mathbf{k})}(A_1^{(k)}) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \frac{\partial u^{(k)}}{\partial r} & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & \frac{\partial u^{(k)}}{\partial r} \\ 0 & 0 & 0 \\ \frac{\partial u^{(k)}}{\partial r} & 0 & 0 \end{bmatrix} \tag{3.3.23}$$

$$\mathbf{L}^{(\mathbf{k})}(A_1^{(k)}) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & (\frac{\partial u^{(k)}}{\partial r})^2 \end{bmatrix} \tag{3.3.24}$$

$$A_1^{(k)}(\mathbf{L}^{(\mathbf{k})})^T = \begin{bmatrix} 0 & 0 & \frac{\partial u^{(k)}}{\partial r} \\ 0 & 0 & 0 \\ \frac{\partial u^{(k)}}{\partial r} & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & \frac{\partial u^{(k)}}{\partial r} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{3.3.25}$$

$$A_1^{(k)}(\mathbf{L^{(k)}})^T = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & (\frac{\partial u^{(k)}}{\partial r})^2 \end{bmatrix} \tag{3.3.26}$$

Using (3.3.26) and (3.3.24) into (3.3.22),

$$\frac{dA_1^{(k)}}{dt} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -2(\frac{\partial u^{(k)}}{\partial r})^2 \end{bmatrix} \tag{3.3.27}$$

Substituting (3.3.5) , (3.3.15) , (3.3.17) , (3.3.19) , (3.3.27) in (3.2.3) we obtain

$$\begin{aligned}
&\begin{bmatrix} \tau_{rr}^{(k)} & \tau_{r\theta}^{(k)} & \tau_{rz}^{(k)} \\ \tau_{\theta r}^{(k)} & \tau_{\theta\theta}^{(k)} & \tau_{\theta z}^{(k)} \\ \tau_{zr}^{(k)} & \tau_{z\theta}^{(k)} & \tau_{zz}^{(k)} \end{bmatrix} + \lambda_1^{(k)} \begin{bmatrix} 0 & 0 & -\frac{\partial u^{(k)}}{\partial r}\tau_{rr}^{(k)} \\ 0 & 0 & -\frac{\partial u^{(k)}}{\partial r}\tau_{\theta r}^{(k)} \\ \frac{\partial u^{(k)}}{\partial r}\tau_{rr}^{(k)} & \frac{\partial u^{(k)}}{\partial r}\tau_{r\theta}^{(k)} & -2\frac{\partial u^{(k)}}{\partial r}\tau_{rz} \end{bmatrix} \\
&+ \frac{\lambda_3^{(k)}}{2}\left(\begin{bmatrix} \frac{\partial u^{(k)}}{\partial r}\tau_{rz} & 0 & \frac{\partial u^{(k)}}{\partial r}\tau_{rr} \\ \frac{\partial u^{(k)}}{\partial r}\tau_{\theta z} & 0 & \frac{\partial u^{(k)}}{\partial r}\tau_{\theta r} \\ \frac{\partial u^{(k)}}{\partial r}\tau_{zz} & 0 & \frac{\partial u^{(k)}}{\partial r}\tau_{zr} \end{bmatrix} + \begin{bmatrix} \frac{\partial u^{(k)}}{\partial r}\tau_{zr}^{(k)} & \frac{\partial u^{(k)}}{\partial r}\tau_{z\theta}^{(k)} & \frac{\partial u^{(k)}}{\partial r}\tau_{zz}^{(k)} \\ 0 & 0 & 0 \\ \frac{\partial u^{(k)}}{\partial r}\tau_{rr}^{(k)} & \frac{\partial u^{(k)}}{\partial r}\tau_{r\theta}^{(k)} & \frac{\partial u^{(k)}}{\partial r}\tau_{rz}^{(k)} \end{bmatrix} \right) \\
&+ \frac{\lambda_5^{(k)}}{2}\left((\tau_{rr}^{(k)} + \tau_{\theta\theta}^{(k)} + \tau_{zz}^{(k)}) \right) \begin{bmatrix} 0 & 0 & \frac{\partial u^{(k)}}{\partial r} \\ 0 & 0 & 0 \\ \frac{\partial u^{(k)}}{\partial r} & 0 & 0 \end{bmatrix} = \\
&\mu^{(k)}\left(\begin{bmatrix} 0 & 0 & \frac{\partial u^{(k)}}{\partial r} \\ 0 & 0 & 0 \\ \frac{\partial u^{(k)}}{\partial r} & 0 & 0 \end{bmatrix} + \lambda_2^{(k)} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -2(\frac{\partial u^{(k)}}{\partial r})^2 \end{bmatrix} + \lambda_4^{(k)} \begin{bmatrix} (\frac{\partial u^{(k)}}{\partial r})^2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & (\frac{\partial u^{(k)}}{\partial r})^2 \end{bmatrix} \right)
\end{aligned} \tag{3.3.28}$$

So in components form it becomes ,

$$\tau_{rr}^{(k)} + \lambda_3^{(k)} \frac{du^{(k)}}{dr}\tau_{rz}^{(k)} = \mu^{(k)}\lambda_4^{(k)} \left(\frac{du^{(k)}}{dr}\right)^2 \tag{3.3.29}$$

$$\tau_{r\theta}^{(k)} + \frac{\lambda_3^{(k)}}{2}\frac{du^{(k)}}{dr}\tau_{z\theta}^{(k)} = 0 \tag{3.3.30}$$

$$\tau_{rz}^{(k)} - \lambda_1^{(k)}\frac{du^{(k)}}{dr}\tau_{rr}^{(k)} + \frac{(\lambda_3^{(k)} + \lambda_5^{(k)})}{2}\left(\tau_{rr}^{(k)} + \tau_{zz}^{(k)}\right)\frac{du^{(k)}}{dr} = \mu^{(k)}\frac{du^{(k)}}{dr} \tag{3.3.31}$$

$$\tau_{z\theta}^{(k)} - \left[\lambda_1^{(k)} - \frac{\lambda_3^{(k)}}{2}\right]\frac{du^{(k)}}{dr}\tau_{r\theta}^{(k)} = 0 \tag{3.3.32}$$

$$\tau_{zz}^{(k)} - \left[2\lambda_1^{(k)} - \lambda_3^{(k)}\right]\frac{du^{(k)}}{dr}\tau_{rz}^{(k)} = \mu^{(k)}\left(\lambda_4^{(k)} - 2\lambda_2^{(k)}\right)\left(\frac{du^{(k)}}{dr}\right) \tag{3.3.33}$$

From (3.3.30) and (3.3.32) ,

$$\tau_{z\theta}^{(k)} = \tau_{r\theta}^{(k)} = 0 \tag{3.3.34}$$

Adding (3.3.29) and (3.3.33) , we get

$$\tau_{rr}^{(k)} + \tau_{zz}^{(k)} = 2\left(\lambda_1^{(k)} + \lambda_3^{(k)}\right)\frac{du^{(k)}}{dr}\tau_{rz}^{(k)} + 2\mu^{(k)}\left(\lambda_4^{(k)} - \lambda_2^{(k)}\right)\left(\frac{du^{(k)}}{dr}\right)^2 \tag{3.3.35}$$

Substituting (3.3.35) into (3.3.31) we obtain

$$\tau_{rz}^{(k)} = \lambda_1^{(k)}\frac{du^{(k)}}{dr}\tau_{rr}^{(k)} - \left(\frac{\lambda_3^{(k)} + \lambda_5^{(k)}}{2}\right)\left[2(\lambda_1^{(k)} - \lambda_3^{(k)})\frac{du^{(k)}}{dr}\tau_{rz}^{(k)} + 2\mu^{(k)}(\lambda_4^{(k)} - \lambda_2^{(k)})\left(\frac{du^{(k)}}{dr}\right)^2\right]\frac{du^{(k)}}{dr} + \mu^{(k)}\frac{du^{(k)}}{dr} \tag{3.3.36}$$

Using value of $\tau_{rr}^{(k)}$ from (3.3.29) into (3.3.36)

$$\tau_{rz}^{(k)} = \lambda_1^{(k)} \frac{du^{(k)}}{dr} \left[\mu^{(k)} \lambda_4^{(k)} \left(\frac{du^{(k)}}{dr} \right)^2 - \lambda_3^{(k)} \frac{du^{(k)}}{dr} \tau_{rz}^{(k)} \right] + \mu^{(k)} \frac{du^{(k)}}{dr} - \left(\frac{\lambda_3^{(k)} + \lambda_5^{(k)}}{2} \right) \left[2(\lambda_1^{(k)} - \lambda_3^{(k)}) \frac{du^{(k)}}{dr} \tau_{rz}^{(k)} + 2\mu^{(k)} (\lambda_4^{(k)} - \lambda_2^{(k)}) \left(\frac{du^{(k)}}{dr} \right)^2 \right] \frac{du^{(k)}}{dr} \tag{3.3.37}$$

$$\begin{aligned} \tau_{rz}^{(k)} = {} & \mu^{(k)} \lambda_1^{(k)} \lambda_4^{(k)} \left(\frac{du^{(k)}}{dr} \right)^3 - \lambda_1^{(k)} \lambda_3^{(k)} \left(\frac{du^{(k)}}{dr} \right)^2 \tau_{rz}^{(k)} - \lambda_3^{(k)} \lambda_1^{(k)} \left(\frac{du^{(k)}}{dr} \right)^2 \tau_{rz}^{(k)} \\ & + (\lambda_3^{(k)})^2 \left(\frac{du^{(k)}}{dr} \right)^2 \tau_{rz}^{(k)} - \lambda_1^{(k)} \lambda_5^{(k)} \left(\frac{du^{(k)}}{dr} \right)^2 \tau_{rz}^{(k)} + \lambda_5^{(k)} \lambda_3^{(k)} \left(\frac{du^{(k)}}{dr} \right)^2 \tau_{rz}^{(k)} \\ & - \lambda_3^{(k)} \lambda_4^{(k)} \mu^{(k)} \left(\frac{du^{(k)}}{dr} \right)^3 + \lambda_3^{(k)} \lambda_2^{(k)} \left(\frac{du^{(k)}}{dr} \right)^3 - \lambda_5^{(k)} \lambda_4^{(k)} \mu^{(k)} \left(\frac{du^{(k)}}{dr} \right)^3 \\ & + \lambda_5^{(k)} \lambda_2^{(k)} \mu^{(k)} \left(\frac{du^{(k)}}{dr} \right)^3 + \mu^{(k)} \frac{du^{(k)}}{dr} \end{aligned} \tag{3.3.38}$$

$$\begin{aligned} & \tau_{rz}^{(k)} \left[1 + \left(2\lambda_1^{(k)} \lambda_3^{(k)} - (\lambda_3^{(k)})^2 + \lambda_1^{(k)} \lambda_5^{(k)} - \lambda_5^{(k)} \lambda_3^{(k)} \right) \left(\frac{du^{(k)}}{dr} \right)^2 \right] = \\ & \mu^{(k)} \left[\frac{du^{(k)}}{dr} + \left(\lambda_1^{(k)} \lambda_4^{(k)} - \lambda_3^{(k)} \lambda_4^{(k)} + \lambda_3^{(k)} \lambda_2^{(k)} - \lambda_5^{(k)} \lambda_4^{(k)} + \lambda_5^{(k)} \lambda_2^{(k)} \right) \left(\frac{du^{(k)}}{dr} \right)^3 \right] \end{aligned} \tag{3.3.39}$$

We also take as,

$$\begin{aligned} \alpha_1^{(k)} &= \lambda_1^{(k)} \lambda_4^{(k)} - \lambda_3^{(k)} \lambda_4^{(k)} + \lambda_3^{(k)} \lambda_2^{(k)} - \lambda_5^{(k)} \lambda_4^{(k)} + \lambda_5^{(k)} \lambda_2^{(k)} \\ \alpha_2^{(k)} &= 2\lambda_1^{(k)} \lambda_3^{(k)} - (\lambda_3^{(k)})^2 + \lambda_1^{(k)} \lambda_5^{(k)} - \lambda_5^{(k)} \lambda_3^{(k)} \end{aligned} \tag{3.3.40}$$

$$\tau_{rz}^{(k)}\left[1+\alpha_2^{(k)}\left(\frac{du^{(k)}}{dr}\right)^2\right]=\mu^{(k)}\left[\frac{du^{(k)}}{dr}+\alpha_1^{(k)}\left(\frac{du^{(k)}}{dr}\right)^3\right] \tag{3.3.41}$$

$$M=\left[1+\alpha_2^{(k)}\left(\frac{du^{(k)}}{dr}\right)^2\right] \tag{3.3.42}$$

$$\tau_{rz}^{(k)}=\frac{\mu^{(k)}}{M}\left[\frac{du^{(k)}}{dr}+\alpha_1^{(k)}\left(\frac{du^{(k)}}{dr}\right)^3\right] \tag{3.3.43}$$

Using (3.3.4) ,

$$\frac{dP}{dz}\frac{r}{2}\left[1+\alpha_2^{(k)}\left(\frac{du^{(k)}}{dr}\right)^2\right]=\mu^{(k)}\left[\frac{du^{(k)}}{dr}+\alpha_1^{(k)}\left(\frac{du^{(k)}}{dr}\right)^3\right] \tag{3.3.44}$$

$$\mu^{(k)}\frac{du^{(k)}}{dr}-\alpha_2^{(k)}\left(\frac{du^{(k)}}{dr}\right)^2\frac{dP}{dz}\frac{r}{2}+\mu^{(k)}\alpha_1\left(\frac{du^{(k)}}{dr}\right)^3=\frac{dP}{dz}\frac{r}{2} \tag{3.3.45}$$

3.4 Velocity of First-layer

For first-layer we will use $k=1$.

3.4.1 Homotopy Analysis Method (HAM)

To apply homotopy analysis method for first-layer,for this first of all we select linear operator $L=\frac{\partial}{\partial r}$,$\frac{dP}{dz}=A$

Now we construct homotopy,

$$\begin{aligned}&(1-q)\left[L\left(u^{(1)}(r,q)\right)-u_0^{(1)}(r)\right]\\ &-qh\left[\frac{\partial u^{(1)}(r,q)}{\partial r}-\frac{\alpha_2^{(1)}A}{2\mu^{(1)}}\left(\frac{\partial u^{(1)}(r,q)}{\partial r}\right)^2 r+\mu^{(1)}\alpha_1^{(1)}\left(\frac{\partial u^{(1)}(r,q)}{\partial r}\right)^3-\frac{A}{2\mu^{(1)}}r\right]=0\end{aligned} \tag{3.4.1}$$

Together with boundary conditions,

$$u^{(k)}(r) = u^{(k-1)} \text{ at } r = R_k \tag{3.4.2}$$

Now

$$(1-q)\left(\frac{\partial u_1^{(1)}}{\partial r}q + \frac{\partial u_2^{(1)}}{\partial r}q^2 + \dots\right) - qh\left(\frac{\partial u_0^{(1)}}{\partial r} + \frac{\partial u_1^{(1)}}{\partial r}q + \frac{\partial u_2^{(1)}}{\partial r}q^2 + \dots\right)$$
$$+ \frac{\alpha_2^{(1)}A}{2\mu^{(1)}}qh\left(\frac{\partial u_0^{(1)}}{\partial r} + \frac{\partial u_1^{(1)}}{\partial r}q + \frac{\partial u_2^{(1)}}{\partial r}q^2 + \dots\right)^2 r + \alpha_1^{(1)}\left(\frac{\partial u_0^{(1)}}{\partial r} + \frac{\partial u_1^{(1)}}{\partial r}q + \frac{\partial u_2^{(1)}}{\partial r}q^2 + \dots\right)$$
$$+ \frac{A}{2\mu^{(1)}}qhr = 0 \tag{3.4.3}$$

$$q\frac{\partial u_1^{(1)}}{\partial r} + q^2\frac{\partial u_2^{(1)}}{\partial r} - q^2\frac{\partial u_1^{(1)}}{\partial r} - q^3\frac{\partial u_2^{(1)}}{\partial r} - qh\frac{\partial u_0^{(1)}}{\partial r} - q^2h\frac{\partial u_1^{(1)}}{\partial r} - q^3h\frac{\partial u_2^{(1)}}{\partial r}$$
$$+ \frac{\alpha_2^{(1)}A}{2\mu^{(1)}}qh\left((\frac{\partial u_0^{(1)}}{\partial r})^2 + 2q\frac{\partial u_0^{(1)}}{\partial r}\frac{\partial u_1^{(1)}}{\partial r} + q^2(\frac{\partial u_1^{(1)}}{\partial r})^2\right)r$$
$$-qh\alpha_1^{(1)}\left((\frac{\partial u_0^{(1)}}{\partial r})^3 + q^3(\frac{\partial u_1^{(1)}}{\partial r})^3 + 3q(\frac{\partial u_0^{(1)}}{\partial r})^2\frac{\partial u_1^{(1)}}{\partial r} + 3q^2(\frac{\partial u_1^{(1)}}{\partial r})^2\frac{\partial u_0^{(1)}}{\partial r}\right) + qh\frac{A}{2\mu^{(1)}}r = 0 \tag{3.4.4}$$

$$q\frac{\partial u_1^{(1)}}{\partial r} + q^2\frac{\partial u_2^{(1)}}{\partial r} - q^3\frac{\partial u_1^{(1)}}{\partial r} - q^3\frac{\partial u_2^{(1)}}{\partial r} - qh\frac{\partial u_0^{(1)}}{\partial r} - q^2h\frac{\partial u_1^{(1)}}{\partial r} - q^3h\frac{\partial u_2^{(1)}}{\partial r}q + qh\frac{\alpha_2^{(1)}A}{2\mu^{(1)}}(\frac{\partial u_0^{(1)}}{\partial r})^2 r$$
$$+q^2h\frac{\alpha_2^{(1)}A}{\mu^{(1)}}\frac{\partial u_0^{(1)}}{\partial r}\frac{\partial u_1^{(1)}}{\partial r}r + q^3h\frac{\alpha_2^{(1)}A}{2\mu^{(1)}}(\frac{\partial u_1^{(1)}}{\partial r})^2 r - qh\alpha_1^{(1)}(\frac{\partial u_0^{(1)}}{\partial r})^3 - 3q^2h\alpha_1^{(1)}(\frac{\partial u_0^{(1)}}{\partial r})^2\frac{\partial u_1^{(1)}}{\partial r}$$
$$-3q^3h\alpha_1^{(1)}\frac{\partial u_0^{(1)}}{\partial r}(\frac{\partial u_1^{(1)}}{\partial r})^2 - q^4h\alpha_1^{(1)}(\frac{\partial u_1^{(1)}}{\partial r})^3 + qhr\frac{A}{2\mu^{(1)}} = 0 \tag{3.4.5}$$

3.4.2 For Zeroth-Order Velocity

So our initial guess is,

$$\frac{\partial v_0^{(1)}}{\partial r} = \frac{A}{2\mu^{(1)}} r \qquad (3.4.6)$$

The zeroth-order velocity is

$$L(u_0^{(1)}) = L(v_0^{(1)}) \qquad (3.4.7)$$

$$u_0^{(1)} = v_0^{(1)} \qquad (3.4.8)$$

$$u_0^{(1)} = \frac{A}{4\mu^{(1)}}\left[r^2 - R_1^2\right] \qquad (3.4.9)$$

3.4.3 For First-Order Velocity

Differentiating (3.4.5) w.r.t $'q'$ (3.4.5)

$$\frac{\partial u_1^{(1)}}{\partial r} + 2q\frac{\partial u_2^{(1)}}{\partial r} - 2q\frac{\partial u_1^{(1)}}{\partial r} - 3q^2\frac{\partial u_2^{(1)}}{\partial r} - h\frac{\partial u_0^{(1)}}{\partial r} - 2qh\frac{\partial u_1^{(1)}}{\partial r} - 3q^2h\frac{\partial u_2^{(1)}}{\partial r} + h\frac{\alpha_2^{(1)}A}{2\mu^{(1)}}(\frac{\partial u_0^{(1)}}{\partial r})^2 r$$
$$+ 2qh\frac{\alpha_2^{(1)}A}{\mu^{(1)}}\frac{\partial u_0^{(1)}}{\partial r}\frac{\partial u_1^{(1)}}{\partial r}r + 3q^2h\frac{\alpha_2^{(1)}A}{2\mu^{(1)}}(\frac{\partial u_1^{(1)}}{\partial r})^2 r - h\alpha_1^{(1)}(\frac{\partial u_0^{(1)}}{\partial r})^3 - 6qh\alpha_1^{(1)}(\frac{\partial u_0^{(1)}}{\partial r})^2\frac{\partial u_1^{(1)}}{\partial r}$$
$$- 9q^2h\alpha_1^{(1)}\frac{\partial u_0^{(1)}}{\partial r}(\frac{\partial u_1^{(1)}}{\partial r})^2 - 4q^3h\alpha_1^{(1)}(\frac{\partial u_1^{(1)}}{\partial r})^3 + hr\frac{A}{2\mu^{(1)}} = 0 \quad (3.4.10)$$

Using $q = 0$ in (3.4.10) then ,

$$\frac{\partial u_1^{(1)}}{\partial r} - h\frac{\partial u_0^{(1)}}{\partial r} + h\frac{\alpha_2^{(1)}A}{2\mu^{(1)}}(\frac{\partial u_0^{(1)}}{\partial r})^2 r - h\alpha_1^{(1)}(\frac{\partial u_0^{(1)}}{\partial r})^3 + hr\frac{A}{2\mu^{(1)}} = 0 \qquad (3.4.11)$$

$$\frac{\partial u_1^{(1)}}{\partial r} = h\frac{\partial u_0^{(1)}}{\partial r} - h\frac{\alpha_2^{(1)}A}{2\mu^{(1)}}(\frac{\partial u_0^{(1)}}{\partial r})^2 r + h\alpha_1^{(1)}(\frac{\partial u_0^{(1)}}{\partial r})^3 - hr\frac{A}{2\mu^{(1)}} \qquad (3.4.12)$$

Substituting (3.4.7) we get ,

$$\frac{\partial u_1^{(1)}}{\partial r} = h(\frac{A}{2\mu^{(1)}}r) - h\frac{\alpha_2^{(1)}A}{2\mu^{(1)}}(\frac{A}{2\mu^{(1)}}r)^2 r + h\alpha_1^{(1)}(\frac{A}{2\mu^{(1)}}r)^3 - hr\frac{A}{2\mu^{(1)}} \qquad (3.4.13)$$

$$\frac{\partial u_1^{(1)}}{\partial r} = \frac{hA^3}{8(\mu^{(1)})^3}\left(\alpha_1^{(1)} - \alpha_2^{(1)}\right) r^3 \tag{3.4.14}$$

Now integrating ,

$$u_1^{(1)}(r) = \frac{hA^3}{32(\mu^{(1)})^3}\left(\alpha_1^{(1)} - \alpha_2^{(1)})\right) r^4 + C^{(1)} \tag{3.4.15}$$

using $u_1^{(1)}(r) = 0$ at $r = R_1$

$$C^{(1)} = -\frac{hA^3}{32(\mu^{(1)})^3}\left(\alpha_1^{(1)} - \alpha_2^{(1)}\right) R_1^4 \tag{3.4.16}$$

Substituting $C^{(1)}$ into above equation

$$u_1^{(1)}(r) = \frac{hA^3}{32(\mu^{(1)})^3}\left(\alpha_1^{(1)} - \alpha_2^{(1)}\right)\left(r^4 - R_1^4\right) \tag{3.4.17}$$

3.4.4 For Second-Order

Now for 2^{nd} order we will differentiate (3.4.10) w.r.t $'q'$ then

$$2\frac{\partial u_2^{(1)}}{\partial r} - 2\frac{\partial u_1^{(1)}}{\partial r} - 6q\frac{\partial u_2^{(1)}}{\partial r} - 2h\frac{\partial u_1^{(1)}}{\partial r} - 6qh\frac{\partial u_2^{(1)}}{\partial r} + \frac{2h\alpha_2^{(1)}A}{\mu^{(1)}}\frac{\partial u_0^{(1)}}{\partial r}\frac{\partial u_1^{(1)}}{\partial r}r$$
$$+\frac{6qh\alpha_2^{(1)}}{2\mu^{(1)}}(\frac{\partial u_1^{(1)}}{\partial r})^2 r - 6h\alpha_1^{(1)}(\frac{\partial u_0^{(1)}}{\partial r})^2\frac{\partial u_1^{(1)}}{\partial r} - 18qh\alpha_1^{(1)}\frac{\partial u_0^{(1)}}{\partial r}(\frac{\partial u_1^{(1)}}{\partial r})^2 - 12q^2h\alpha_1^{(1)})(\frac{\partial u_1^{(1)}}{\partial r})^3 = 0 \tag{3.4.18}$$

Using $q = 0$ into (3.4.18)

$$2\frac{\partial u_2^{(1)}}{\partial r} - 2\frac{\partial u_1^{(1)}}{\partial r} - 2h\frac{\partial u_1^{(1)}}{\partial r} + \frac{2h\alpha_2^{(1)}A}{\mu^{(1)}}\frac{\partial u_0^{(1)}}{\partial r}\frac{\partial u_1^{(1)}}{\partial r}r - 6h\alpha_1^{(1)}(\frac{\partial u_0^{(1)}}{\partial r})^2\frac{\partial u_1^{(1)}}{\partial r} = 0 \tag{3.4.19}$$

$$\frac{\partial u_2^{(1)}}{\partial r} = \frac{\partial u_1^{(1)}}{\partial r} + h\frac{\partial u_1^{(1)}}{\partial r} - \frac{h\alpha_2^{(1)}A}{\mu^{(1)}}\frac{\partial u_0^{(1)}}{\partial r}\frac{\partial u_1^{(1)}}{\partial r}r + 3h\alpha_1^{(1)}(\frac{\partial u_0^{(1)}}{\partial r})^2\frac{\partial u_1^{(1)}}{\partial r} \tag{3.4.20}$$

Substituting $\frac{\partial u_0^{(1)}}{\partial r} = \frac{Ar}{2\mu^{(1)}}$ and $\frac{\partial u_1^{(1)}}{\partial r} = \frac{hA^3 r^3}{8(\mu^{(1)})^3}\left(\alpha_1^{(1)} - \alpha_2^{(1)}\right)$ so,

$$\frac{\partial u_2^{(1)}}{\partial r} = (1+h)(\alpha_1^{(1)} - \alpha_2^{(1)})\frac{hA^3}{8(\mu)^3}r^3 - \frac{h^2A^5\alpha_2^{(1)}}{16(\mu^{(1)})^5}(\alpha_1^{(1)} - \alpha_2^{(1)})r^5 + \frac{3h^2A^5\alpha_1^{(1)}}{32(\mu^{(1)})^5}(\alpha_1^{(1)} - \alpha_2^{(1)})r^5 \tag{3.4.21}$$

Integrating w.r.t $'r'$ we obtain ,

$$u_2^{(1)}(r) = (1+h)\frac{hA^3}{32(\mu^{(1)})^3}(\alpha_1^{(1)} - \alpha_2^{(1)})r^4 - \frac{h^2A^5}{96(\mu^{(1)})^5}\alpha_2^{(1)}(\alpha_1^{(1)} - \alpha_2^{(1)})r^6 + \frac{h^2A^5}{64(\mu^{(1)})^5}\alpha_1^{(1)}(\alpha_1^{(1)} - \alpha_2^{(1)})r^6 + C^{(2)} \tag{3.4.22}$$

Using $u_2^{(1)}(r) = 0$ at $r = R_1$

$$C^{(2)} = -(1+h)\frac{hA^3}{32(\mu^{(1)})^3}(\alpha_1^{(1)} - \alpha_2^{(1)})R_1^4 + \frac{h^2A^5}{96(\mu^{(1)})^5}\alpha_2^{(1)}(\alpha_1^{(1)} - \alpha_2^{(1)})R_1^6 - \frac{h^2A^5}{64(\mu^{(1)})^5}\alpha_1^{(1)}(\alpha_1^{(1)} - \alpha_2^{(1)})R_1^6 \tag{3.4.23}$$

Substituting value of $C^{(2)}$ in c two eq

$$u_2^{(1)}(r) = (1+h)\frac{hA^3}{32(\mu^{(1)})^3}(\alpha_1^{(1)} - \alpha_2^{(1)})\left[r^4 - R_1\right] - \frac{h^2A^5}{96(\mu^{(1)})^5}\alpha_2^{(1)}(\alpha_1^{(1)} - \alpha_2^{(1)})\left[r^6 - R_1^6\right] + \frac{h^2A^5}{64(\mu^{(1)})^5}\alpha_1^{(1)}(\alpha_1^{(1)} - \alpha_2^{(1)})\left[r^6 - R_1^6\right] \tag{3.4.24}$$

$$u_2^{(1)}(r) = (1+h)\frac{hA^3}{32(\mu^{(1)})^3}(\alpha_1^{(1)} - \alpha_2^{(1)})\left[r^4 - R_1\right] + \frac{h^2A^5}{8(\mu^{(1)})^5}(\alpha_1^{(1)} - \alpha_2^{(1)})\left(\frac{\alpha_1^{(1)}}{8} - \frac{\alpha_2^{(1)}}{12}\right)\left[r^6 - R_1^6\right] \tag{3.4.25}$$

The velocity of first layer can be written as,

$$u^{(1)}(r) = u_0^{(1)}(r) + \frac{u_1^{(1)}(r)}{1!} + \frac{u_2^{(1)}(r)}{2!} + ... \tag{3.4.26}$$

Finally , we obtain

$$\begin{aligned}u^{(1)}(r) &= \frac{A}{4\mu^{(1)}}\left[r^2-R_1^2\right]+\frac{hA^3}{32(\mu^{(1)})^3}\left(\alpha_1^{(1)}-\alpha_2^{(1)}\right)\left[r^4-R_1^4\right]\\ &+(1+h)\frac{hA^3}{64(\mu^{(1)})^3}(\alpha_1^{(1)}-\alpha_2^{(1)})\left[r^4-R_1\right]\\ &+\frac{h^2A^5}{16(\mu^{(1)})^5}(\alpha_1^{(1)}-\alpha_2^{(1)})\left(\frac{\alpha_1^{(1)}}{8}-\frac{\alpha_2^{(1)}}{12}\right)\left[r^6-R_1^6\right]\end{aligned} \tag{3.4.27}$$

3.5 Velocity of Second-Layer

So velocity of second layer will be written as,

$$u^{(2)}(r)=u_0^{(2)}(r)+\frac{u_1^{(2)}(r)}{1!}+\frac{u_2^{(2)}(r)}{2!}+... \tag{3.5.1}$$

And boundary conditions will be written as,

$$u^{(2)}(r)=u^{(1)}(r) \text{ at } r=R_2 \tag{3.5.2}$$

3.5.1 For Zeroth-Order Velocity:-

Our initial guess is,

$$\frac{\partial v_0^{(2)}(r)}{\partial r}=\frac{A}{2\mu^{(2)}}r \tag{3.5.3}$$

The zeroth-order velocity is is

$$L(u_0^{(2)})=L(v_0^{(2)}) \tag{3.5.4}$$

$$u_0^{(2)}=v_0^{(2)} \tag{3.5.5}$$

Integrating and using Boundary conditions $u_0^{(2)}(r)=u_0^{(1)}(r)$ at $r=R_2$.

$$\left[u_0^{(1)}(r)\right]_{r=R_2}=\frac{A}{4\mu^{(2)}}R_2^2+C^{(1)} \tag{3.5.6}$$

$$C^{(1)} = \frac{A}{4\mu^{(1)}}\left[R_2^2 - R_1^2\right] - \frac{A}{4\mu^{(2)}}R_2^2 \tag{3.5.7}$$

Substituting value of $C^{(1)}$ then

$$u_0^{(2)}(r) = \frac{A}{4\mu^{(2)}}\left[r^2 - R_2^2\right] + \frac{A}{4\mu^{(1)}}\left[R_2^2 - R_1^2\right] \tag{3.5.8}$$

3.5.2 For First-Order Velocity:-

$$\frac{\partial u_1^{(2)}}{\partial r} = \frac{hA^3}{8(\mu^{(2)})^3}\left(\alpha_1^{(2)} - \alpha_2^{(2)}\right) r^3 \tag{3.5.9}$$

So integrating

$$u_1^{(2)}(r) = \frac{hA^3}{32(\mu^{(2)})^3}\left(\alpha_1^{(2)} - \alpha_2^{(2)})\right) R_2^4 + C^{(2)} \tag{3.5.10}$$

using $u_1^{(2)}(r) = u_1^{(1)}(r)$ at $r = R_2$

$$\left[u_1^{(1)}(r)\right]_{r=R_2} = \frac{hA^3}{32(\mu^{(2)})^3}\left(\alpha_1^{(2)} - \alpha_2^{(2)})\right) R_2^4 + C^{(2)} \tag{3.5.11}$$

$$\frac{hA^3}{32(\mu^{(1)})^3}\left(\alpha_1^{(1)} - \alpha_2^{(1)})\right)\left[R_2^4 - R_1^4\right] = \frac{hA^3}{32(\mu^{(2)})^3}\left(\alpha_1^{(2)} - \alpha_2^{(2)})\right) R_2^4 + C^{(2)} \tag{3.5.12}$$

So

$$C^{(2)} = -\frac{hA^3}{32(\mu^{(2)})^3}\left(\alpha_1^{(2)} - \alpha_2^{(2)})\right) R_2^4 + \frac{hA^3}{32(\mu^{(1)})^3}\left(\alpha_1^{(1)} - \alpha_2^{(1)})\right)\left[R_2^4 - R_1^4\right] \tag{3.5.13}$$

Substituting value of $C^{(2)}$

$$u_1^{(2)}(r) = \frac{hA^3}{32(\mu^{(2)})^3}\left(\alpha_1^{(2)} - \alpha_2^{(2)})\right)\left[r^4 - R_2^4\right] + \frac{hA^3}{32(\mu^{(1)})^3}\left(\alpha_1^{(1)} - \alpha_2^{(1)})\right)\left[R_2^4 - R_1^4\right] \tag{3.5.14}$$

3.5.3 For Second-Order velocity

Now we will find $u_2^{(2)}(r)$ and we also know that

$$\frac{\partial u_2^{(2)}}{\partial r} = (1+h)(\alpha_1^{(2)} - \alpha_2^{(2)})\frac{hA^3}{8(\mu^{(2)})^3}r^3 - \frac{h^2A^5\alpha_2^{(2)}}{16(\mu^{(2)})^5}(\alpha_1^{(2)} - \alpha_2^{(2)})r^5 + \frac{3h^2A^5\alpha_1^{(2)}}{32(\mu^{(2)})^5}(\alpha_1^{(2)} - \alpha_2^{(2)})r^5 \tag{3.5.15}$$

Integrating w.r.t $'r'$

$$u_2^{(2)}(r) = (1+h)\frac{hA^3}{32(\mu^{(2)})^3}(\alpha_1^{(2)} - \alpha_2^{(2)})r^4 - \frac{h^2A^5}{96(\mu^{(2)})^5}\alpha_2^{(2)}(\alpha_1^{(2)} - \alpha_2^{(2)})r^6 + \frac{h^2A^5}{64(\mu^{(2)})^5}\alpha_1^{(2)}(\alpha_1^{(2)} - \alpha_2^{(2)})r^6 + C^{(2)} \tag{3.5.16}$$

Using $u_2^{(2)}(r) = u_2^{(1)}(r)$ at $r = R_2$

$$\left[u_2^{(1)}(r)\right]_{r=R_2} = (1+h)\frac{hA^3}{32(\mu^{(2)})^3}(\alpha_1^{(2)} - \alpha_2^{(2)})R_2^4 - \frac{h^2A^5}{96(\mu^{(2)})^5}\alpha_2^{(2)}(\alpha_1^{(2)} - \alpha_2^{(2)})R_2^6 + \frac{h^2A^5}{64(\mu^{(2)})^5}\alpha_1^{(2)}(\alpha_1^{(2)} - \alpha_2^{(2)})R_2^6 + C^{(2)} \tag{3.5.17}$$

$$\begin{aligned} C^{(2)} = &-(1+h)\frac{hA^3}{32(\mu^{(2)})^3}(\alpha_1^{(2)} - \alpha_2^{(2)})R_2^4 + \frac{h^2A^5}{96(\mu^{(2)})^5}\alpha_2^{(2)}(\alpha_1^{(2)} - \alpha_2^{(2)})R_2^6 \\ &- \frac{h^2A^5}{64(\mu^{(2)})^5}\alpha_1^{(2)}(\alpha_1^{(2)} - \alpha_2^{(2)})R_2^6 + (1+h)\frac{hA^3}{32(\mu^{(1)})^3}(\alpha_1^{(1)} - \alpha_2^{(1)})\left[R_2^4 - R_1\right] \\ &+ \frac{h^2A^5}{8(\mu^{(1)})^5}(\alpha_1^{(1)} - \alpha_2^{(1)})\left(\frac{\alpha_1^{(1)}}{8} - \frac{\alpha_2^{(1)}}{12}\right)\left[R_2^6 - R_1^6\right] \end{aligned} \tag{3.5.18}$$

Substituting the value of$C^{(2)}$

$$u_2^{(2)}(r) = (1+h)\frac{hA^3}{32(\mu^{(2)})^3}(\alpha_1^{(2)}-\alpha_2^{(2)})\left[r^4-R_2\right] - \frac{h^2A^5}{96(\mu^{(2)})^5}\alpha_2^{(2)}(\alpha_1^{(2)}-\alpha_2^{(2)})\left[r^6-R_2^6\right]$$
$$+\frac{h^2A^5}{64(\mu^{(2)})^5}\alpha_1^{(2)}(\alpha_1^{(2)}-\alpha_2^{(2)})\left[r^6-R_2^6\right]+(1+h)\frac{hA^3}{32(\mu^{(1)})^3}(\alpha_1^{(1)}-\alpha_2^{(1)})\left[R_2^4-R_1\right]$$
$$+\frac{h^2A^5}{8(\mu^{(1)})^5}(\alpha_1^{(1)}-\alpha_2^{(1)})\left(\frac{\alpha_1^{(1)}}{8}-\frac{\alpha_2^{(1)}}{12}\right)\left[R_2^6-R_1^6\right] \tag{3.5.19}$$

$$u_2^{(2)}(r) = (1+h)\frac{hA^3}{32(\mu^{(2)})^3}(\alpha_1^{(2)}-\alpha_2^{(2)})\left[r^4-R_2^4\right]+\frac{h^2A^5}{8(\mu^{(2)})^5}(\alpha_1^{(2)}-\alpha_2^{(2)})\left(\frac{\alpha_1^{(2)}}{8}-\frac{\alpha_2^{(2)}}{12}\right)\left[r^6-R_2^6\right]$$
$$+(1+h)\frac{hA^3}{32(\mu^{(1)})^3}(\alpha_1^{(1)}-\alpha_2^{(1)})\left[R_2^4-R_1\right]+\frac{h^2A^5}{8(\mu^{(1)})^5}(\alpha_1^{(1)}-\alpha_2^{(1)})\left(\frac{\alpha_1^{(1)}}{8}-\frac{\alpha_2^{(1)}}{12}\right)\left[R_2^6-R_1^6\right] \tag{3.5.20}$$

$$u^{(2)}(r) = u_0^{(2)}(r)+\frac{u_1^{(2)}(r)}{1!}+\frac{u_2^{(2)}(r)}{2!}+\ldots \tag{3.5.21}$$

$$u^{(2)}(r) = \frac{A}{4\mu^{(2)}}\left[r^2-R_2^2\right]+\frac{A}{4\mu^{(1)}}\left[R_2^2-R_1^2\right]+\frac{hA^3}{32(\mu^{(2)})^3}\left(\alpha_1^{(2)}-\alpha_2^{(2)})\right)\left[r^4-R_2^4\right]$$
$$+\frac{hA^3}{32(\mu^{(1)})^3}\left(\alpha_1^{(1)}-\alpha_2^{(1)})\right)\left[R_2^4-R_1^4\right]+(1+h)\frac{hA^3}{64(\mu^{(2)})^3}(\alpha_1^{(2)}-\alpha_2^{(2)})\left[r^4-R_2^4\right]$$
$$+\frac{h^2A^5}{16(\mu^{(2)})^5}(\alpha_1^{(2)}-\alpha_2^{(2)})\left(\frac{\alpha_1^{(2)}}{8}-\frac{\alpha_2^{(2)}}{12}\right)\left[r^6-R_2^6\right]$$
$$+(1+h)\frac{hA^3}{64(\mu^{(1)})^3}(\alpha_1^{(1)}-\alpha_2^{(1)})\left[R_2^4-R_1\right]$$
$$+\frac{h^2A^5}{16(\mu^{(1)})^5}(\alpha_1^{(1)}-\alpha_2^{(1)})\left(\frac{\alpha_1^{(1)}}{8}-\frac{\alpha_2^{(1)}}{12}\right)\left[R_2^6-R_1^6\right] \tag{3.5.22}$$

So

$$
\begin{aligned}
u^{(2)}(r) = & \frac{A}{4\mu^{(2)}}\left[r^2 - R_2^2\right] + \frac{hA^3}{32(\mu^{(2)})^3}\left(\alpha_1^{(2)} - \alpha_2^{(2)})\right)\left[r^4 - R_2^4\right] \\
& + (1+h)\frac{hA^3}{64(\mu^{(2)})^3}(\alpha_1^{(2)} - \alpha_2^{(2)})\left[r^4 - R_2^4\right] \\
& + \frac{h^2A^5}{16(\mu^{(2)})^5}(\alpha_1^{(2)} - \alpha_2^{(2)})\left(\frac{\alpha_1^{(2)}}{8} - \frac{\alpha_2^{(2)}}{12}\right)\left[r^6 - R_2^6\right] + \left[u^{(1)}(r)\right]_{r=R_2}
\end{aligned}
\tag{3.5.23}
$$

3.6 Volume Flow Rate:-

We know that

$$
Q^{(n)} = 2\pi \int_0^{R_n} ru^{(n)}(r)dr \tag{3.6.1}
$$

3.6.1 Volume Flow Rate Of First-Layer

For $n = 1$

$$
Q^{(1)} = 2\pi \int_0^{R_1} ru^{(1)}(r)dr \tag{3.6.2}
$$

$$
\begin{aligned}
Q^{(1)} = 2\pi \int_0^{R_1} r[& \frac{A}{4\mu^{(1)}}(r^2 - R_1^2) + \frac{hA^3}{32(\mu^{(1)})^3}(\alpha_1^{(1)} - \alpha_2^{(1)})(r^4 - R_1^4) \\
& + (1+h)\frac{hA^3}{64(\mu^{(1)})^3}(\alpha_1^{(1)} - \alpha_2^{(1)})(r^4 - R_1) \\
& + \frac{h^2A^5}{16(\mu^{(1)})^5}(\alpha_1^{(1)} - \alpha_2^{(1)})(\frac{\alpha_1^{(1)}}{8} - \frac{\alpha_2^{(1)}}{12})(r^6 - R_1^6)]dr
\end{aligned}
\tag{3.6.3}
$$

$$
\begin{aligned}
Q^{(1)} = {} & \frac{\pi A}{2\mu^{(1)}}\int_0^{R_1} r^3 dr - \frac{\pi A R_1^2}{2\mu^{(1)}}\int_0^{R_1} r dr + \frac{\pi h A^3}{16(\mu^{(1)})^3}(\alpha_1^{(1)} - \alpha_2^{(1)})\int_0^{R_1} r^5 dr \\
& - \frac{\pi h A^3 R_1^4}{16(\mu^{(1)})^3}\int_0^{R_1} r dr + \frac{\pi h(1+h)A^3}{32(\mu^{(1)})^3}(\alpha_1^{(1)} - \alpha_2^{(1)})\int_0^{R_1} r^5 dr \\
& - \frac{\pi h(1+h)A^3 R_1^4}{32(\mu^{(1)})^3}(\alpha_1^{(1)} - \alpha_2^{(1)})\int_0^{R_1} r dr \\
& + \frac{\pi h^2 A^5}{8(\mu^{(1)})^5}(\alpha_1^{(1)} - \alpha_2^{(1)})(\frac{\alpha_1^{(1)}}{8} - \frac{\alpha_2^{(1)}}{12})\int_0^{R_1} r^7 dr \\
& - \frac{\pi h^2 A^5 R_1^6}{8(\mu^{(1)})^5}(\alpha_1^{(1)} - \alpha_2^{(1)})(\frac{\alpha_1^{(1)}}{8} - \frac{\alpha_2^{(1)}}{12})\int_0^{R_1} r dr
\end{aligned}
\tag{3.6.4}
$$

After integrating,

$$
\begin{aligned}
Q^{(1)} = {} & \frac{\pi A}{8\mu^{(1)}}\left[r^4\right]_0^{R_1} - \frac{\pi A R_1^2}{4\mu^{(1)}}\left[r^2\right]_0^{R_1} + \frac{\pi h A^3}{96(\mu^{(1)})^3}(\alpha_1^{(1)} - \alpha_2^{(1)})\left[r^6\right]_0^{R_1} \\
& - \frac{\pi h A^3 R_1^4}{32(\mu^{(1)})^3}\left[r^2\right]_0^{R_1} + \frac{\pi h(1+h)A^3}{192(\mu^{(1)})^3}(\alpha_1^{(1)} - \alpha_2^{(1)})\left[r^6\right]_0^{R_1} \\
& - \frac{\pi h(1+h)A^3 R_1^4}{64(\mu^{(1)})^3}(\alpha_1^{(1)} - \alpha_2^{(1)})\left[r^2\right]_0^{R_1} \\
& + \frac{\pi h^2 A^5}{64(\mu^{(1)})^5}(\alpha_1^{(1)} - \alpha_2^{(1)})(\frac{\alpha_1^{(1)}}{8} - \frac{\alpha_2^{(1)}}{12})\left[r^8\right]_0^{R_1} \\
& - \frac{\pi h^2 A^5 R_1^6}{16(\mu^{(1)})^5}(\alpha_1^{(1)} - \alpha_2^{(1)})(\frac{\alpha_1^{(1)}}{8} - \frac{\alpha_2^{(1)}}{12})\left[r^2\right]_0^{R_1}
\end{aligned}
\tag{3.6.5}
$$

$$
\begin{aligned}
Q^{(1)} = {} & \frac{\pi A R_1^4}{8\mu^{(1)}} - \frac{\pi A R_1^4}{4\mu^{(1)}} + \frac{\pi h A^3 R_1^6}{96(\mu^{(1)})^3}(\alpha_1^{(1)} - \alpha_2^{(1)}) - \frac{\pi h A^3 R_1^6}{32(\mu^{(1)})^3} \\
& + \frac{\pi h(1+h)A^3 R_1^6}{192(\mu^{(1)})^3}(\alpha_1^{(1)} - \alpha_2^{(1)}) - \frac{\pi h(1+h)A^3 R_1^6}{64(\mu^{(1)})^3}(\alpha_1^{(1)} - \alpha_2^{(1)}) \\
& + \frac{\pi h^2 A^5 R_1^8}{64(\mu^{(1)})^5}(\alpha_1^{(1)} - \alpha_2^{(1)})(\frac{\alpha_1^{(1)}}{8} - \frac{\alpha_2^{(1)}}{12}) - \frac{\pi h^2 A^5 R_1^8}{16(\mu^{(1)})^5}(\alpha_1^{(1)} - \alpha_2^{(1)})(\frac{\alpha_1^{(1)}}{8} - \frac{\alpha_2^{(1)}}{12})
\end{aligned}
\tag{3.6.6}
$$

So

$$Q^{(1)} = \frac{\pi h A^3 R_1^6}{48(\mu^{(1)})^3}(\alpha_2^{(1)} - \alpha_1^{(1)}) + \frac{\pi h(1+h)A^3 R_1^6}{96(\mu^{(1)})^3}(\alpha_2^{(1)} - \alpha_1^{(1)})$$
$$+ \frac{3\pi h^2 A^5 R_1^8}{64(\mu^{(1)})^5}(\alpha_2^{(1)} - \alpha_1^{(1)})(\frac{\alpha_1^{(1)}}{8} - \frac{\alpha_2^{(1)}}{12}) - \frac{\pi A R_1^4}{8\mu^{(1)}} \tag{3.6.7}$$

3.6.2 Volume Flow Rate of Second-Layer

Using $n = 2$

$$Q^{(2)} = 2\pi \int_0^{R_2} r u^{(2)}(r) dr \tag{3.6.8}$$

$$Q^{(2)} = 2\pi \int_0^{R_2} r[\frac{A}{4\mu^{(2)}}\left[r^2 - R_2^2\right] + \frac{hA^3}{32(\mu^{(2)})^3}\left(\alpha_1^{(2)} - \alpha_2^{(2)})\right)\left[r^4 - R_2^4\right]$$
$$+ (1+h)\frac{hA^3}{64(\mu^{(2)})^3}(\alpha_1^{(2)} - \alpha_2^{(2)})\left[r^4 - R_2^4\right] \tag{3.6.9}$$
$$+ \frac{h^2 A^5}{16(\mu^{(2)})^5}(\alpha_1^{(2)} - \alpha_2^{(2)})\left(\frac{\alpha_1^{(2)}}{8} - \frac{\alpha_2^{(2)}}{12}\right)\left[r^6 - R_2^6\right] + \left[u^{(1)}(r)\right]_{r=R_2}]dr$$

$$Q^{(2)} = 2\pi \int_0^{R_2} r[\frac{A}{4\mu^{(2)}}(r^2 - R_2^2) + \frac{hA^3}{32(\mu^{(2)})^3}(\alpha_1^{(2)} - \alpha_2^{(2)})(r^4 - R_2^4)$$
$$+ (1+h)\frac{hA^3}{64(\mu^{(2)})^3}(\alpha_1^{(2)} - \alpha_2^{(2)})(r^4 - R_2^4)$$
$$+ \frac{h^2 A^5}{16(\mu^{(2)})^5}(\alpha_1^{(2)} - \alpha_2^{(2)})(\frac{\alpha_1^{(2)}}{8} - \frac{\alpha_2^{(2)}}{12})(r^6 - R_2^6) + \frac{A}{4\mu^{(1)}}(R_2^2 - R_1^2)$$
$$+ \frac{hA^3}{32(\mu^{(1)})^3}(\alpha_1^{(1)} - \alpha_2^{(1)})(R_2^4 - R_1^4) + (1+h)\frac{hA^3}{64(\mu^{(1)})^3}(\alpha_1^{(1)} - \alpha_2^{(1)})(R_2^4 - R_1)$$
$$+ \frac{h^2 A^5}{16(\mu^{(1)})^5}(\alpha_1^{(1)} - \alpha_2^{(1)})(\frac{\alpha_1^{(1)}}{8} - \frac{\alpha_2^{(1)}}{12})(R_2^6 - R_1^6)]dr \tag{3.6.10}$$

$$
\begin{aligned}
Q^{(2)} = & \frac{\pi A}{2\mu^{(2)}}\int_0^{R_2} r^3 dr - \frac{\pi A R_2^2}{2\mu^{(2)}}\int_0^{R_2} r dr + \frac{\pi h A^3}{16(\mu^{(2)})^3}(\alpha_1^{(2)} - \alpha_2^{(2)})\int_0^{R_2} r^5 dr \\
& - \frac{\pi h A^3 R_2^4}{16(\mu^{(2)})^3}\int_0^{R_2} r dr + \frac{\pi h(1+h)A^3}{32(\mu^{(2)})^3}(\alpha_1^{(2)} - \alpha_2^{(2)})\int_0^{R_2} r^5 dr \\
& - \frac{\pi h(1+h)A^3 R_2^4}{32(\mu^{(2)})^3}(\alpha_1^{(2)} - \alpha_2^{(2)})\int_0^{R_2} r dr + \frac{\pi h^2 A^5}{8(\mu^{(2)})^5}(\alpha_1^{(2)} - \alpha_2^{(2)})(\frac{\alpha_1^{(2)}}{8} - \frac{\alpha_2^{(2)}}{12})\int_0^{R_2} r^7 dr \\
& - \frac{\pi h^2 A^5 R_2^6}{8(\mu^{(2)})^5}(\alpha_1^{(2)} - \alpha_2^{(2)})(\frac{\alpha_1^{(2)}}{8} - \frac{\alpha_2^{(2)}}{12})\int_0^{R_2} r dr + \frac{\pi A}{4\mu^{(1)}}(R_2^2 - R_1^2)\int_0^{R_2} r dr \\
& + \frac{\pi h A^3}{32(\mu^{(1)})^3}(\alpha_1^{(1)} - \alpha_2^{(1)})(R_2^4 - R_1^4)\int_0^{R_2} r dr \\
& + (1+h)\frac{\pi h A^3}{64(\mu^{(1)})^3}(\alpha_1^{(1)} - \alpha_2^{(1)})(R_2^4 - R_1)\int_0^{R_2} r dr \\
& + \frac{\pi h^2 A^5}{16(\mu^{(1)})^5}(\alpha_1^{(1)} - \alpha_2^{(1)})(\frac{\alpha_1^{(1)}}{8} - \frac{\alpha_2^{(1)}}{12})(R_2^6 - R_1^6)\int_0^{R_2} r dr
\end{aligned}
\tag{3.6.11}
$$

After Integrating finally $Q^{(2)}$ will become as,

$$
\begin{aligned}
Q^{(2)} = & \frac{\pi h A^3 R_2^6}{48(\mu^{(2)})^3}(\alpha_2^{(2)} - \alpha_1^{(2)}) + \frac{\pi h(1+h)A^3 R_2^6}{96(\mu^{(2)})^3}(\alpha_2^{(2)} - \alpha_1^{(2)}) \\
& + \frac{3\pi h^2 A^5 R_2^8}{64(\mu^{(2)})^5}(\alpha_2^{(2)} - \alpha_1^{(2)})(\frac{\alpha_1^{(2)}}{8} - \frac{\alpha_2^{(2)}}{12}) - \frac{\pi A R_1^4}{4\mu^{(2)}} + \frac{\pi A R_2^2}{4\mu^{(1)}}(R_2^2 - R_1^2) \\
& + \frac{\pi h A^3 R_2^2}{32(\mu^{(1)})^3}(\alpha_1^{(1)} - \alpha_2^{(1)})(R_2^4 - R_1^4) + (1+h)\frac{\pi h A^3 R_2^2}{64(\mu^{(1)})^3}(\alpha_1^{(1)} - \alpha_2^{(1)})(R_2^4 - R_1) \\
& + \frac{\pi h^2 A^5 R_2^2}{16(\mu^{(1)})^5}(\alpha_1^{(1)} - \alpha_2^{(1)})(\frac{\alpha_1^{(1)}}{8} - \frac{\alpha_2^{(1)}}{12})(R_2^6 - R_1^6)
\end{aligned}
\tag{3.6.12}
$$

3.7 Graphical Results And Discussion

In this section the velocity behavior of Oldroyed six constant is plotted graphically against different parameters .

Figure (3.2) show the behavior of velocity, against different values of viscosity. If

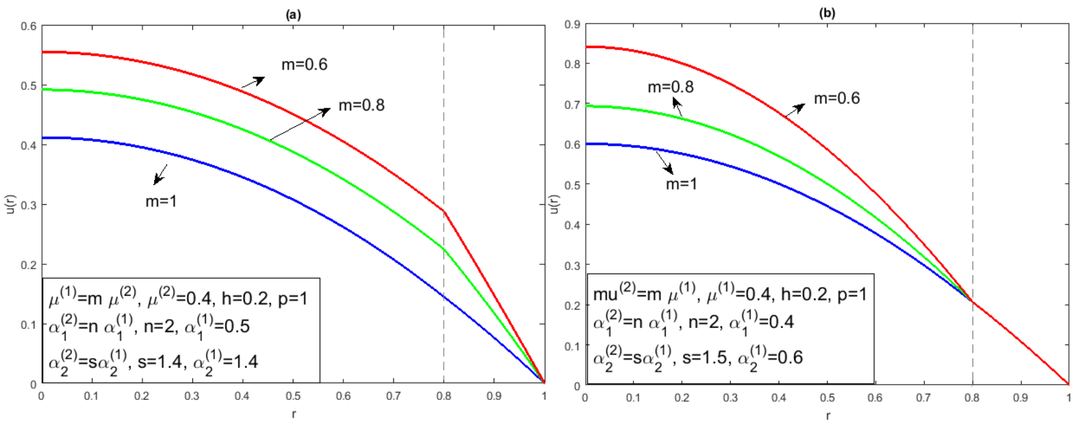

Figure 3.2: Effect of viscosity on two-layer flow of oldroyed six constant

viscosity is the ratio of $\mu^{(1)}$ and $\mu^{(2)}$. In figure (3.2) (a) we fixed $\mu^{(2)}$ and change $\mu^{(1)}$ using different values of m and in figure (3.2) (b) we fixed $\mu^{(1)}$ and change $\mu^{(2)}$ using different values of m other parameters remain same as shown in figure (3.2). By decreasing the viscosity , velocity profile increases as shown in figure (3.2) . We fixed $R_1 = 1$, $R_2 = 0.8$ and $h = -1$ for all figures.

Figure (3.4) shows the effect of pressure gradient on velocity, which shows pressure gradient is directly relate to the velocity. In figure (3.3) (c) we fixed the $\alpha_1^{(1)}$ and change $\alpha_2^{(1)}$ using different values of n, we observe that velocity increases by increasing the value of n. And figure (3.3) (d) shows the effect of $\alpha_2^{(2)}$ with fixed $\alpha_1^{(2)}$, for the different values of s.

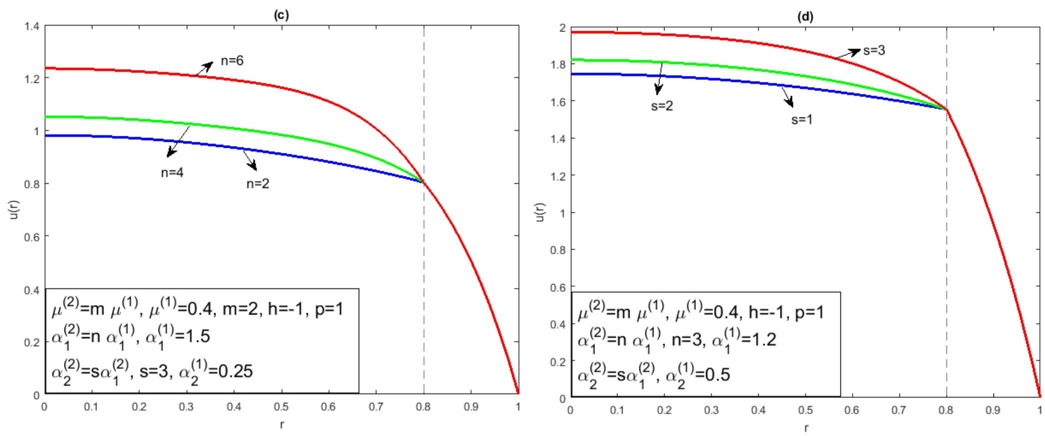

Figure 3.3: Effect of $\alpha_1^{(k)}$ and $\alpha_2^{(k)}$ on two-layer flow of oldroyed six constant

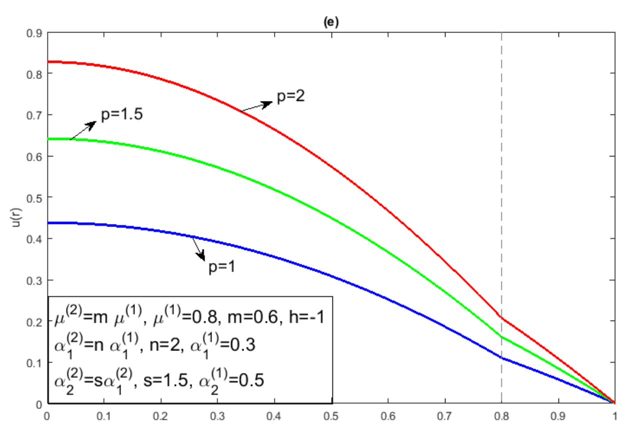

Figure 3.4: Effect of pressure gradient on Oldroyed six constant

Chapter 4

Conclusion and future work

In this thesis, we analyzed the n-layers flow of PTT fluids through cylindrical geometry. We also worked on the two-layer flow of Oldroyd six constant fluid with same geometry. Both the types of flows considered were under the influence of a constant pressure gradient alone. We concluded that the flows of these layers of fluids are affected greatly by varying material parameters and the level of pressure applied. This effect on the flow of all layers is also observed when the material parameters of even one of the layers are changed.

It is well-known that the fluids used in industry are mostly non-Newtonian in nature. This is because these fluids have a variety of characteristics like elasticity, shear-thinning and thickening etc that the Navier-Stokes equations are unable to capture. Our work in this thesis is based on cylindrical pipe geometry and addresses two different kinds of non-Newtonian fluids. There are many other future research openings in this area of multilayer flows.

Firstly, one can consider various types of stresses depending on different non-Newtonian fluids. Secondly, the flow geometry can be varied for these flows. For example multiple layer flows of fluids can studied in planar channel geometry, annulus of a pipe etc. Thirdly, the flows can also be generalized by considering the effects of heat transfer and MHD. Similarly, there are many more generalization of these types of layered flows in biological problems like peristaltic transport etc.

We aim to submit two papers from this thesis. The first one from chapter 2 about PTT fluids is in preparation. We intend to also apply heat transfer to the problem of chapter 3 and submit it as another paper for publication.

Chapter 5

References

Bibliography

[1] R. B . Bird, W. E Stewart. N. Lightfoot, Transport Phenomena, John Wiley, New York (1964).

[2] J. N. Kanpur. J. B. Shukla, The flow of incompressible, immiscible fluids between two plates, Applied Scientific Research, A13, 55 (1964).

[3] J. Lohrasbi, V. Sahai, Magneto hydrodynamics heat transfer in two-phase flow between parallel plates, Applied Scientific Research, 45 (1988) 53-66.

[4] M. S. Malashetty, V. Leela, Magneto hydrodynamics heat transfer in two-phase flow, International Journal of Engendering Science, 30 (1992) 371-377.

[5] M.S. Malashetty, J .C. Umavathi ,Two-phase Magneto hydrodynamics flow and heat transfer in an inclined channel , International Journal pf Multiphase Flow, 23 (1997) 545-560.

[6] R. Shail, On laminar two-phase flows in Magneto hydrodynamics International Journal of Engineering Science, 11 (1973) 1103-1108.

[7] H. Power, M. Vllegas, Weakly nonlinear instability of the flow of two immiscible liquids with different viscosities in a pipe, Fluid Dynamics Research, 7 (1991) 215-228.

[8] L. Dong, D. Johnson, Experimental and theoretical study of the inter facial instability between two shear fluids in a channel couette flow , International Journal of Heat and Fluid Flow, 26 (2005) 133-140.

[9] A. Pinarbasi , Interface stabilization in two-layer channel flow by heating or cooling, European Journal of Mechanics B/Fluids, 21 (2002) 225-236.

[10] V. Shankar, Stability of two layer viscoelastic plan couette flow past a deformable solid layer , Journal of Non-Newtonian Fluids Mechanics, 117 (2004) 163-182.

[11] W.T. Wong, C. C. Jeng, The Stability of Two concentric non-Newtonian Fluids in a circular pipe flow, Journal of Non-Newtonian Fluid Mechanics, 22 (1987) 359-380.

[12] C. T. Huang, B. Khomami, Role of dynamic modulation on stability of multi-layer Newtonian and viscoelastic flows down an inclined plan , Journal of Non-Newtonian Fluid Mechanics, 97 (2001) 67-86.

[13] N. D. Water, The stability of two stratified ”power-law liquids in couette flow, Journal of Non-Newtonian Fluid Mechanics, 12 (1983) 85-94.

[14] N. D. Water, A. M. Keely, The stability of two stratified Non-Newtonian liquids in Couette flow , Journal of Non-Newtonian Fluid Mechanics, 24 (1987) 161-181.

[15] S. Usha, A. R. Rao, Peristaltic transport of two-layered Power law fluids, Journal of Biomechanical Engineering, 119 (1997) 483-488.

[16] T. Gul, R. A. Shah, S. Islam, M. Ullah, M. A. Khan, A. Zaman, Z. Haq, Exact Solution of Two Thin Film Non-Newtonian Immiscible Fluids on a Vertical Belt, Journal of Basic and Applied Scientific Research, 4(6) (2014) 283-288.

[17] N. Brauner, D. M. Maron, Flow pattren transitions in two-phase liquid-liquid fiow in horizontal tube, International Journal of Multiphase Flow 18 (1992) 123-140.

[18] N. Brauner , Two phase liquid-liquid annular flow , International Journal of Multiphase Flow, 17 (1991) 59-76.

[19] T. M. Leib, M. Fink, D. Hasson, Heat tranfer in vertical annular laminar flow of two immiscible liquid, International Journal of Multiphase Flow, 3 (1977) 533-549.

[20] A.M. Siddiqui, Q. A. Azim , M. A. Rana, On Exact Solutions of Concentric N-layer Flows of Viscous Fluids in a Pipe , Nonl. Sci. Lett. A, Vol. 1, No.1, (2010) 67-76.

[21] Liao SJ. On the proposed homotopy analysis technique for nonlinear problems and its applications. PhD dissertation, Shanghai Jio Tong University; 1992.

[22] Liao SJ. Beyond perturbation: introduction to homotopy analysis method. Boca Raton: Chapman and Hall/CRC Press; 2003.

[23] T. Hayat , M. Sajid, The application of homotopy analysis method to thin film flows of a third order fluid, Chaos, Solitons and Fractals 38 (2008) 506-515.

[24] J. H. He Homotopy perturbation method a new non-linear analytical technique Appl Math Comput (2000) 135:73.

[25] A. M. Siddiqui, M. Ahmed, S. Islam, Q. K. Ghori, Homotopy analysis of Couette and Poiseuille flows for fourth-grade fluids, Acta Mechanica 180 (2005) 117-132.

[26] Zaid M. Odibat, A new modification of the homotopy perturbation method for linear and nonlinear operators, Appl. Math. Comput. 189 (2007) 746-753.