Hamiltonian Cycles in Certain Directed Toeplitz Graphs

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CIIT/FA14-BSMATH-002/LHR

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By

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To
Ammi and Abu

ACKNOWLEDGEMENT

All praise and glory to the mighty Lord, Allah, in whose hands lies the existence of this world. His blessings have guided me to complete my work and I can never be thankful enough for it.

It is a matter of great pleasure for me to thank my supervisor, Dr. M. Faisal Nadeem for his experienced and channelized guidance that helped me prosper in the last few years. I would also take this opportunity to acknowledge his patience and tolerance towards my occasional inability to focus and casual attitude.

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ABSTRACT

Hamiltonian Cycle in Certain Direted Toeplitz Graphs

Studies on the existence and properties of paths and cycles through a specified number of vertices in graph have been of considerable interest to both pure and applied mathematicians as well as researchers in other disciplines. Application of such studies can be found in elds related to electrical engineering, computer algorithm analysis, operations research and other areas of scientific research.

In chapter 1 we have discussed about graphs and some of its types, this chapter is based on basic definitions of graph theory.

In chapter 2 we have discussed known results about the Hamiltonicity of directed Toeplitz graph.

In chapter 3 we gave new results about the different classes of directed Toeplitz graph.

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Chapter 1

Preliminaries and Introduction

1 Preliminaries and Introduction

In this chapter we will discuss about graphs and some of its type, this chapter is based on basic definition of graph theory.

Graph

A *graph* G is an ordered pair (V_G, E_G), consisting of a non-empty set of vertices V_G, and a set E_G (possibly empty) of edges, which are unordered pairs of distinct vertices. If the edge e is the pair x, y, we write $e = xy$. Then e is said to join x and y, and the vertices x and y are called the ends of e. In this case, x and y are *adjacent vertices* and e and x are *incident*. If two edges have a common vertex, then these two edges are *adjacent*.

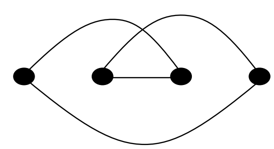

Figure 1: Graph

Cardinality

The cardinality of the set V_G is called the *order* of G, and is denoted by n, while the cardinality of the set E_G is the *size* of G, and is denoted by m. The order of a graph is at least 1 because every graph has a nonempty vertex set.

Trivial Graph

The graph of order 1 is called *trivial*.

Degree

The number of vertices adjacent to a vertex $x \in V_G$ is called the *degree* of x and is denoted by $d(x)$. The *minimum* degree of a graph G is defined as $\delta(G)$=min $\{d_G(x) \mid x \in G\}$, and the *maximum* degree of G is defined as $\Delta(G)$=max $\{d_G(x) \mid x \in G\}$. In a graph G of order n, for any vertex x of G, we have the following inequality

$$0 \leq \delta(G) \leq d(x) \leq \Delta(G) \leq n-1.$$

If all the vertices in a graph G have same degree, then we say that G is a *regular* graph.

Complete graph

A *complete graph* K_n is a graph with n vertices in which every pair of vertices is joined by an edge. In Fig. 2 K_4 is shown.

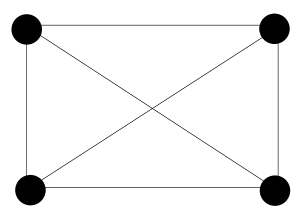

Figure 2: Complete Graph

1.1 Paths and Cycles in Graphs

Path and cycle

A *path graph* is a graph P with vertex set $V_P = \{x_1, x_2, ..., x_j\}$ of order j and edge set $E_P = \{x = x_1x_2, x_2x_3, ..., x_{j-1}x_j = y\}$. The vertices x and y are called endpoints of the path P. The number j is the *length* of P. A path of order n is denoted by P_n. Similarly, a *cycle* C is a graph with $V_C = \{x_1, x_2, ..., x_j\}$ and $E_C = \{x_1x_2, x_2x_3, ..., x_{j-1}x_j, x_jx_1\}$, for $j \geq 3$.

Figure 3: Cycle and Path

Fig. 3 shows a cycle C_8 and a path P_6.

Acyclic graph

A graph is called *acyclic* if it has no cycles. A *j-cycle* is a cycle of order j, denoted by C_j. It has length j. A cycle of even length is an *even cycle*; otherwise it is an *odd cycle*.

1.2 Subgraphs and Isomorphic graphs

Subgraphs

A graph H is said to be a *subgraph* of a graph G if $V_H \subseteq V_G$ and $E_H \subseteq E_G$. The graph G is called the *supergraph* of H.

Spanning subgraph

If a subgraph H of G has the same order as G, then H is called a *spanning subgraph* of G. We call a subgraph H of a graph G an *induced subgraph* of G if $x, y \in V_H$ and $xy \in E_G$ imply $xy \in E_H$.

A subgraph H of a graph G can be obtained by removing edges and vertices of the graph G. If x is a vertex of a graph G and $n \geq 2$, where n is the order of G, then $G - x$ denotes the subgraph with vertex set $V_G \setminus \{x\}$ and whose edges are all those edges of G which are not incident with x. Similarly, if $e \in E_G$, then $G - e$ is the subgraph having vertex set V_G and edge set $E_G \setminus \{e\}$.

Fig. 4 shows a graph G and its subgraph H.

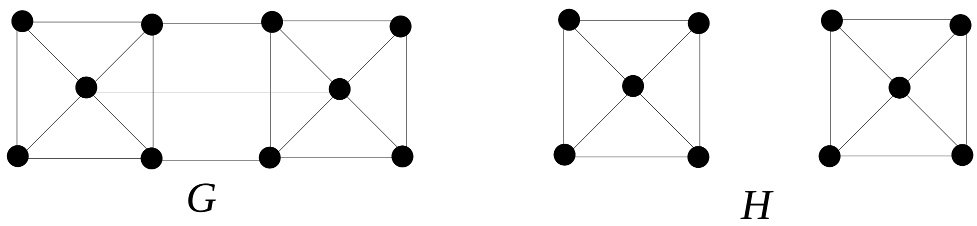

Figure 4: Graph G and subgraph H

1.3 Isomorphic graphs

Two graphs G and H are *isomorphic* if there exists a bijective function $g : V_G \to V_H$ such that $xy \in E_G$ if and only if $g(x)g(y) \in E_H$. The above function g is called an isomorphism. If G and H are isomorphic, we write $G \cong H$. If there is no such isomorphism exist as described above, then G and H are *non-isomorphic graphs*

The graphs G and H shown in Fig. 5 are isomorphic.

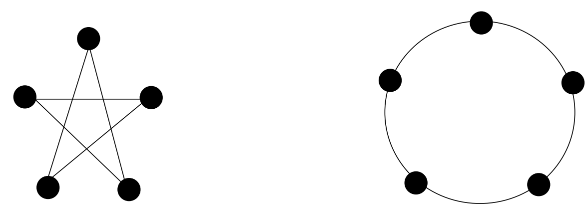

Figure 5: Isomorphic graphs

1.4 Connectivity

In this section we introduce the concepts of connected and disconnected graphs. A graph G is said to be *connected* if any two vertices $x, y \in V_G$ are joined by a path in G, otherwise we call it *disconnected*.

A connected graph G can be made disconnected by removing some of its vertices or edges. A vertex x in G is called *cut-vertex* if $G - x$ is disconnected. Analogously, an edge e of a connected graph G is a *bridge* of G if $G - e$ is disconnected. Graphs with bridges (except for K_2) contain cut-vertices as well, but the converse is not true.

In Fig. 6, x is a cut-vertex, e is a bridge.

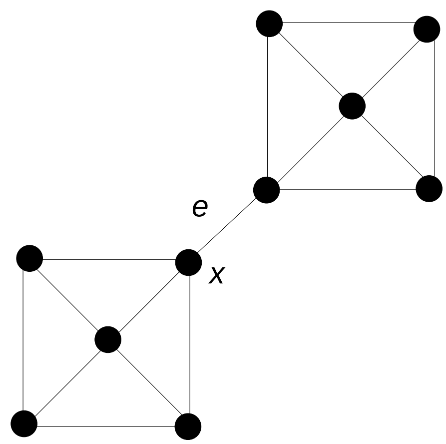

Figure 6: Connected Graph G

Vertex connectivity

The *vertex connectivity* $\kappa(G)$ of a graph G is the minimum number of vertices whose removal results in a disconnected or trivial graph. Analogously, we define the *edge-connectivity* $\lambda(G)$ of G.

Figure 7: Vertex disconnected Graph of G

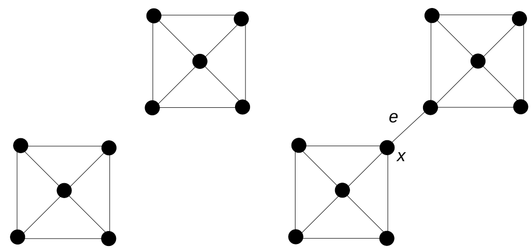

Figure 8: Edge disconnected Graph

k-connected graph

A finite graph G is said to be *k-connected* if $\kappa(G) \geq k$, where $k \geq 1$. So, the graph G is said to be k-connected if and only if the deletion of fewer than k vertices results in neither a disconnected nor a trivial graph.

The following inequalities show the relationship between the edge-connectivity, minimum degree and connectivity of a graph.

$$\kappa(G) \leq \lambda(G) \leq \delta(G).$$

1.5 Bipartite Graphs

A graph G is a *bipartite graph* if it is possible to partition the vertex set of G into two subsets X and Y, called *partite sets*, such that every edge $e \in E_G$ joins a vertex of X and a vertex of Y. Fig. 9 shows a bipartite graph G.

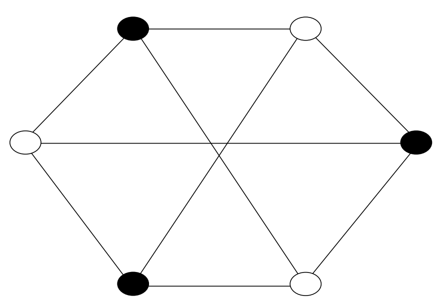

Figure 9: Bipartite graphs

The following theorem is the characterization of bipartite graphs due to König (1916).

Theorem 1.1. *A graph G is bipartite if and only if G has no odd cycle.*

Complete Bipartite Graph

A *complete bipartite graph* $K_{p,q}$ is the bipartite graph with bipartition (X,Y), where $|X| = p$ and $|Y| = q$ such that each vertex of X is joined with each vertex of Y. $K_{3,2}$ is shown in Fig. 10.

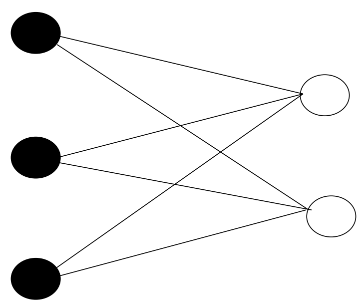

Figure 10: Complete Bipartite Graph $K_{3,2}$

1.6 Directed graph

A *directed graph* is graph, i.e., a set of objects (called vertices or nodes) that are connected together, where all the edges are directed from one vertex to another. A directed graph is sometimes called a digraph or a directed network.

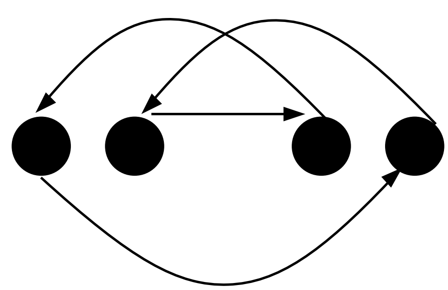

Figure 11: Directed Graph

In degree and Out degree

In case of directed graphs, number of edges going into a node is known as *in degree* of the corresponding node and number of edges coming out of a node is known as *out degree* of the corresponding node.

1.7 Matrices of graphs

A graph G is represented by a diagram of points (vertices) joined by lines (edges). Other useful representations involve matrices. Matrices of graphs enable us to store large graphs in a computer. There are several matrices such as adjacency matrix, incidence matrix, distance matrix, cycle matrix and co-cycle matrix which one can associate with any graph. We shall discuss here the adjacency matrix. We use these matrices to identify certain properties of a graph.

1.8 Adjacency matrices

Let G be a graph with vertex set $V_G = \{x_1, x_2, \ldots, x_n\}$ and edge set $E_G = \{e_1, e_2, \ldots, e_n\}$. The *adjacency matrix* $A = (a_{ij})$ of G is the $n \times n$ matrix in which $a_{ij} = 1$ if x_i is adjacent to y_j, and $a_{ij} = 0$ otherwise. Since edges have distinct ends, all entries of the main diagonal of the matrix A are zeros. A graph and its adjacency matrix are shown in Fig. 12.

$$\begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix}$$

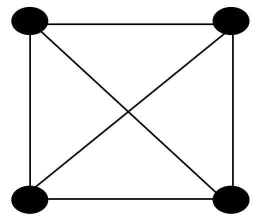

Figure 12: Adjacency Matrix

An adjacency matrix A is determined by a vertex ordering. Every adjacency matrix A is symmetric, i.e., $a_{ij} = a_{ji}$ for all $i,\ j$. The degree of x_i is equal to the sum of the entries in the row i of A. Presenting an adjacency matrix for a graph implicitly names the vertices by the order of the rows; the ith vertex corresponds to the ith row and column. The degree sequence can be obtained at once from the adjacency matrix. For any graph G of order n there are $n!$ different adjacency matrices, since there are $n!$ different orderings of the n vertices. However, any two such adjacency matrices are closely related in that one can be obtained from the other by simply interchanging rows and columns. The adjacency matrix of a graph G can also be used to check the connectedness of the graph G. For example, if G is a graph with n vertices, A the adjacency matrix of G and C the matrix

$$C = A + A^2 + A^3 + \cdots + A^{n-1},$$

then G is a connected graph if C has no zero entries on the main diagonal.

1.9 Toeplitz matrices and Toeplitz Graphs

An $(n \times n)$ matrix $A = (a_{ij})$ is called a *Toeplitz matrix* if $a_{ij} = a_{i+1,j+1}$ for each $i, j = 1, \ldots, n-1$. Toeplitz matrices were named after Otto Toeplitz (1881-1940). The following matrix is a Toeplitz matrix

$$\begin{pmatrix} a_0 & a_1 & a_2 & \dots & a_{n-2} & a_{n-1} \\ a_{-1} & a_0 & a_1 & \dots & a_{n-3} & a_{n-2} \\ a_{-2} & a_{-1} & a_0 & \dots & a_{n-4} & a_{n-3} \\ \dots & \dots & \dots & \dots & \dots & \dots \\ a_{-n+1} & a_{-n+2} & a_{-n+3} & \dots & a_{-1} & a_0 \end{pmatrix}.$$

A Toeplitz matrix has constant values along all diagonals parallel to the main diagonal, and thus a Toeplitz matrix is determined by its first row and first column. A directed or undirected Toeplitz graph is, by definition, a graph with a Toeplitz adjacency matrix.

Undirected Toeplitz Graphs

An $(n \times n)$ matrix $A = a_{ij}$ is called a Toeplitz matrix if $a_{ij} = a_{i+1,j+1}$ for each $i, j = 1 \dots n-1$. Toeplitz matrices are precisely those matrices that are constant along all diagonals parallel to the main diagonal, and thus a Toeplitz matrix is determined by its first row and column.

$$\begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

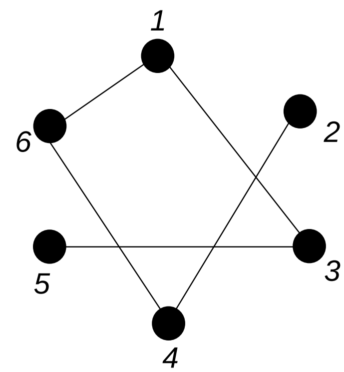

Figure 13: Undirected Toeplitz Graphs

Directed Toeplitz Graphs

The directed Toeplitz graph $T_n\langle s_1, s_2, \ldots, s_k; t_1, t_2, \ldots, t_l\rangle$ is the digraph with vertices $1, 2, \ldots, n$, in which the arc (i, j) occurs if and only if $j - i = s_p$ or $i - j = t_q$ for some integers p and q $(1 \leq p \leq k, 1 \leq q \leq l)$. Its adjacency matrix is a Toeplitz matrix, i.e., it has constant values along all diagonals parallel to the main diagonal. If the Toeplitz adjacency matrix is symmetric, the graph is said to be undirected. $T_n\langle t_1, t_2, \ldots, t_i\rangle$ denotes the undirected Toeplitz graph. Hamiltonicity of $T_n\langle t_1, t_2, \ldots, t_i\rangle$ means hamiltonicity of $T_n\langle t_1, \ldots, t_i, t_1, \ldots, t_i\rangle$.

Toeplitz matrices also arise in solutions to spline functions, differential and integral equations and problems and methods in mathematics, statistics, signal processing and physics. Toeplitz matrices also use in different areas in pure and applied mathematics and in computer science. For example, they are closely connected with Fourier series, they often appear when differential or integral equations are discretized, they arise in physical data-processing applications, in the theory of orthogonal polynomials, stationary processes, and moment problems; see Heinig and Rost [26]. For other references on Toeplitz matrices see [23], [25] and [28].

A special case of a Toeplitz matrix is a circulant matrix where each row is "rotated" one element to the right relative to the preceding row. Accordingly, an $(n \times n)$ matrix C is called a *circulant matrix* if it has the cyclic form

$$
\begin{pmatrix}
c_0 & c_1 & c_2 & \dots & c_{n-2} & c_{n-1} \\
c_{n-1} & c_0 & c_1 & \dots & c_{n-3} & c_{n-2} \\
c_{n-2} & c_{n-1} & c_0 & \dots & c_{n-4} & c_{n-3} \\
\dots & \dots & \dots & \dots & \dots & \dots \\
c_1 & c_2 & c_3 & \dots & c_{n-1} & c_0
\end{pmatrix}.
$$

For each $i, j = 1, \dots, n$ and $k = 0, 1, \dots, n-1$, all the elements (i, j) such that $j - i \equiv k(mod\, n)$ have the same value c_k these elements form the so-called kth stripe of C. It is easily verified that the addition and the product of two circulant matrices is a circulant matrix. Obviously, a circulant matrix is determined by its first column or row. It is clear that every circulant matrix is a Toeplitz matrix, but not conversely.

Circulant matrices and their properties have been studied in [13] and [25]. Circulants arise naturally in many applications, including signal processing and graph theory, have many interesting properties. Making a Fourier transformation is equivalent to diagonalizing a circulant matrix, and an inverse Fourier transformation takes a diagonal matrix into a circulant matrix.

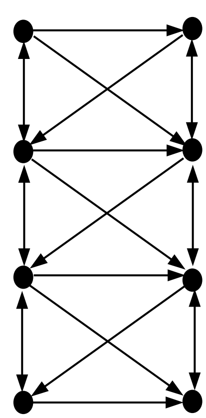

Figure 14: Directed Toeplitz Graph

$$\begin{pmatrix} 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \end{pmatrix}$$

Chapter 2

Historical Background and Literature Review

2 Historical Background and Literature Review

We begin with a historical note on Eulerian and hamiltonian graphs, and then define the concept of hamiltonicity in graphs. Finally, we discuss some known results on hamiltonicity of Toeplitz graphs.

2.1 Historical note

Few subjects in mathematics have as specified an origin as graph theory. Graph theory originated with the Königsberg bridge problem [14], which Leonhard Euler (1707-1783) solved in 1736. The problem is stated as follows:

The City of Königsberg (formerly in Germany, now part of Russia) consists of two islands and two banks of the Pregel River (the upper bank and the lower bank). Between these four land masses there were seven decorative bridges, as seen in the Fig. 15 below. As legend has it, the people of Königsberg wondered if a route could be found such that a person could leave his house, walk across all seven bridges in the town exactly once, and return home.

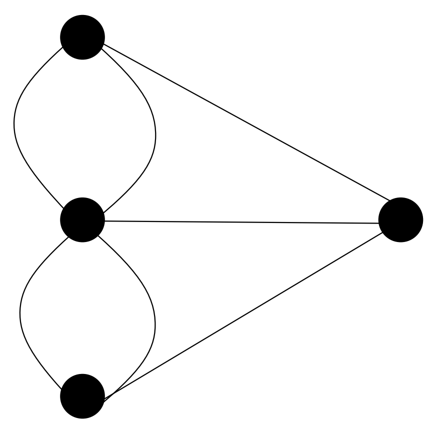

Figure 15: Königsberg

Euler's solution centered on the fact that for such a route to exist, all land masses must have an even number of bridges extending to them. This would allow for a person to both enter and leave a land mass each time they visit it. An odd number of bridges attaching to a land mass would lead to a person getting stuck eventually. In Königsberg, all land masses have an odd number of bridges. Thus, Euler proved that a route of the desired kind cannot exist. An important topic in graph theory, Eulerian circuits, gets its name from the solution of this problem. We are familiar with a popular puzzle where one is asked to trace all the lines of a given diagram without picking up the pencil and without retracing any already existing lines. This is precisely the problem of finding an Eulerian circuit.

It is unfortunate that Euler's work remained unknown for around 150 years. In [4] Biggs, Lloyd and Wilson wrote: ”Indeed, the problem was not well known until the end of the 19th century, when Lucas [31] and Rouse Ball [1] included it in their books on recreational mathematics although a French translation of Euler's paper had been published earlier by Coupy [11]”.

In 1758, over two decades after resolving the Königsberg bridge problem, Euler made his second major contribution to the subject of graph theory with his formula for planar graphs [15]. Euler's contribution to graph theory also includes another paper [16], on the Knight tour problem.

The first solution of this problem was discovered in 840 A.D. by the famous chess player al-Adli of Bagdad [36]. Many other solutions were discovered later but the first systematic analysis of the problem was carried out by Euler in 1759 in a long paper [16]. He first described a Knight route that begins at a corner and visits all squares without returning to the initial corner, and called this route *not re-entrant upon itself*. Subsequently, he gave an example of a closed tour, *a route that is re-entrant upon itself*, and remarked that this provides many routes starting in any square. Generalizing the problem in various ways, Euler discussed Knight tours with certain kind of symmetries, like visiting first one half (or quarter) of the board, then the rest. Finally he considered the problem on non-standard chess boards, such as rectangular or cross shaped.

Following Zamfirescu [43], ”Euler was the first mathematician who considered and investigated (in [16]) what later will be called hamiltonicity, traceability and homogeneous traceability of graphs”.

The same hamiltonicity problem, this time on the 1-skeleton of a polyhedron was first discussed by Kirkman in 1855. [4] describes his work as follows:

” In a paper [30] received by the Royal Society on 6 August 1855, Kirkman considered the following question. Given the graph of a polyhedron, can one always find a cycle which passes through each vertex once and only once? He gave a complicated condition which he claimed to be sufficient to ensure that such a cycle can be found, but unfortunately his claim is false.”

A few months later, as a consequence of his work on quaternions, William Rowan Hamilton (1805-1865) became interested in the existence of cycles passing through all the vertices of a regular dodecahedron and using only edges of it. He even built and sold such dodecahedra as a game, called the *Icosian Game* or *A Voyage round the world* [42]. It was this game of Hamilton that gave birth to the term *hamiltonian*.

2.2 Hamiltonian Graphs

Traceable Graphs

A traceable graph is a graph that possesses a Hamiltonian path. Hamiltonian graphs are therefore traceable, but the converse is not necessarily true. Fig. 16 shows a hamiltonian graph.

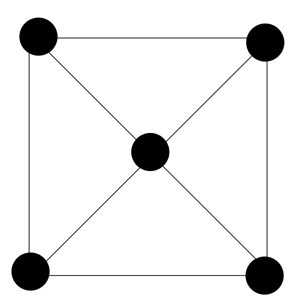

Figure 16: Hamiltonian Graph

Hamiltonian cycle

A hamiltonian cycle, also called a *hamiltonian circuit*, is a cycle (i.e., closed loop) through a graph that visits each node exactly once.

Hamiltonian connected

A graph is *Hamilton-connected*, if it has a $u-v$ hamiltonian path for all pairs of vertices u and v.

In 1963, Ore [37] defined a graph to be *hamiltonian-connected* if there is a hamiltonian path between every pair of distinct vertices. Since that time, a great deal of work has been done on this subject (see, for example, [8], [9], [38], [39]).

Hamiltonian cycles are useful in various operations research problems. For instance, in the routing of a delivery truck that must supply a chain of stores with goods, a hamiltonian cycle is desirable. Closely related to the topic of hamiltonian cycles is the Traveling Salesman problem, one of the most famous problems in graph theory. This problem seeks an optimal route for a salesman who is planning to visit multiple cities. The goal is to find a route that minimizes the cost of travel, based on the distance between the cities. Unlike the Eulerian circuit problem, however, the hamiltonian cycle problem and the Traveling Salesman problem do not have such concise solutions. Improving the efficiency of currently used algorithms is still a focus of intense study.

2.3 Literature Review

The hamiltonicity of undirected Toeplitz graphs was first studied by R. van Dal, G.Tijssen, Z. Tuza, J.A.A. van der Veen, Ch. Zamrescu and T. Zamrescu [12] in 1996 and then C. Heuberger [27] has extended this study in 2002. Moreover, hamiltonian connectedness of directed Toeplits graphs was studied in [35].

2.3.1 Known results of Directed Toeplitz Graphs

Hamiltonian properties of directed Toeplitz graphs have been investigated in [32], [33], and [34] as follows.

Theorem 2.1. *[32] $T_n\langle 1,3,4;2\rangle$ is hamiltonian for $n \in \{5,7\}$ all $n \cong 0,3$ or 4 modulo 6.*

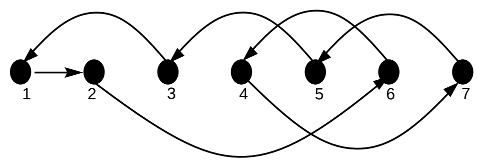

Figure 17: Hamiltonian circuit in $T_7\langle 1,3,4;2\rangle$

Lemma 2.2. *If $gcd\langle s,t\rangle = 1$ then $T_{s+1}\langle s;t\rangle$is a circuit.*

Theorem 2.3. *[32] $T_n\langle 1,3,4;3\rangle$ is hamiltonian for $n \in \{5,6,7,9\}$.*

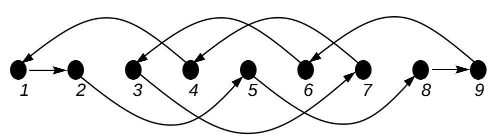

Figure 18: Hamiltonian circuit in $T_9\langle 1,3,4;3\rangle$

Theorem 2.4. *[32] $T_n\langle 1,3,4;4\rangle$ is hamiltonian for $n \in \{5,7,8,9,11,14,15,17,18,20,21\}$ all $n \geq 23$.*

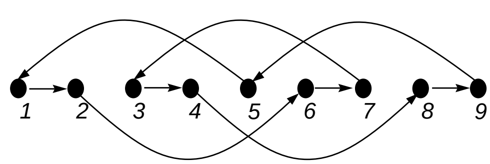

Figure 19: Hamiltonian circuit in $T_9\langle 1,3,4;4\rangle$

Theorem 2.5. *[32] $T_n\langle 1,3,4;5\rangle$ is hamiltonian for all n if and only if $n \neq 7$.*

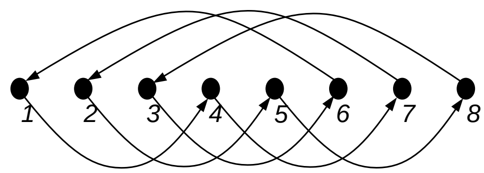

Figure 20: Hamiltonian circuit in $T_8\langle 1,3,4;5\rangle$

Theorem 2.6. *[32] $T_n\langle 1,3,4;6\rangle$ is hamiltonian for all n.*

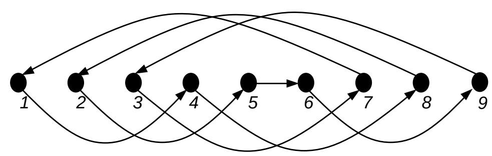

Figure 21: Hamiltonian circuit in $T_9\langle 1,3,4;6\rangle$

Theorem 2.7. *[32] $T_n\langle 1,3,4;7\rangle$ is hamiltonian for all n.*

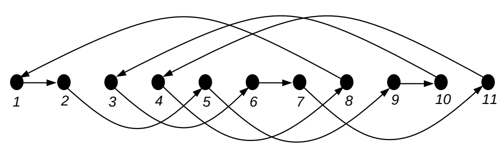

Figure 22: Hamiltonian circuit in $T_{11}\langle 1,3,4;7\rangle$

Theorem 2.8. *[32] $T_n\langle 1,3,4;8\rangle$ is hamiltonian for all n different from 14.*

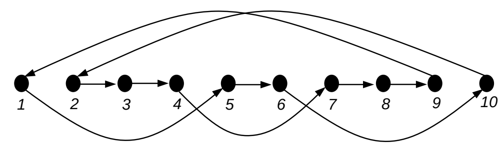

Figure 23: Hamiltonian circuit in $T_{10}\langle 1,3,4;8\rangle$

Theorem 2.9. *[32] $T_{14}\langle 1,3,4;8\rangle$ is non hamiltonian.*

Theorem 2.10. *[32] $T_n\langle 1,3,4;9\rangle$ is hamiltonian for all n different from 15.*

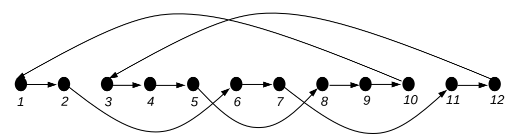

Figure 24: Hamiltonian circuit in $T_{12}\langle 1,3,4;9\rangle$

Theorem 2.11. *[32] $T_n\langle 1,3,4;t\rangle$, $t \geq 10$, is hamiltonian for all n.*

Theorem 2.12. *[33] $T_n\langle s,t\rangle$ is a cycle if and only if gcd $\langle s,t\rangle = 1$ and s +t=n.*

Theorem 2.13. *[33] $T_n\langle 1,3;1,2\rangle$, is hamiltonian for all n.*

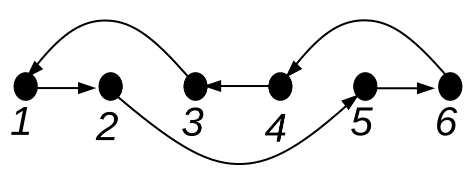

Figure 25: Hamiltonian circuit in $T_6\langle 1,3;1,2\rangle$

Theorem 2.14. *[33] $T_n\langle 1,3;1,4\rangle$, is hamiltonian for all n.*

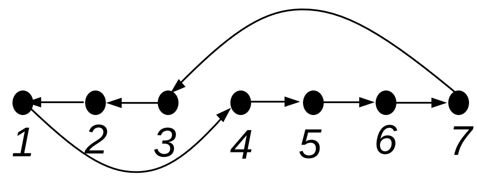

Figure 26: Hamiltonian circuit in $T_7\langle 1,3;1,4\rangle$

Theorem 2.15. *[33] $T_n\langle 1,3;1,6\rangle$, is hamiltonian for all n.*

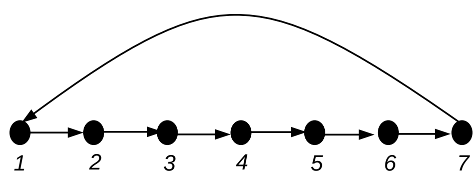

Figure 27: Hamiltonian circuit in $T_7\langle 1,3;1,6\rangle$

Theorem 2.16. *[33] $T_n\langle 1,3;1,t_2\rangle$, where $t_2(\geq 8)$ is even, is hamiltonian if $n \equiv 0,2,4,6,5,7,9,\ldots,t_2-$ 3 (mod(t_2-1)).*

Theorem 2.17. *[34] $T_n\langle 1,2;t_1\rangle$, is hamiltonian if $n \equiv 1(mod t_1)$ and $t_1 \geq 4$.*

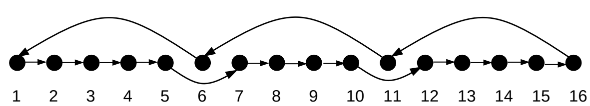

Figure 28: Hamiltonian circuit in $T_n\langle 1,2;t_1\rangle$

Theorem 2.18. *[34] For even $t_1, T_n\langle 1,2;t_1\rangle$, is hamiltonian if and only if n is odd.*

Theorem 2.19. *[34] $T_n\langle 1,2;3\rangle$ is hamiltonian if and only if $n = 5$ or $n \equiv 1(mod3)$. .*

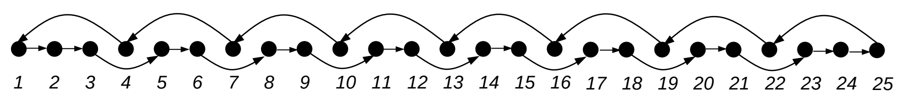

Figure 29: Hamiltonian circuit in $T_{25}\langle 1,2;3\rangle$

Theorem 2.20. *[34] $T_n\langle 1,2;5\rangle$ is hamiltonian for all $n \geq 29$. .*

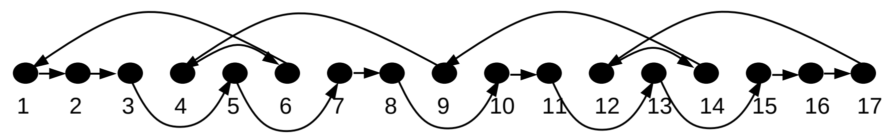

Figure 30: Hamiltonian circuit in $T_{17}\langle 1,2;5\rangle$

Theorem 2.21. *[34] Let $t_1 \geq 7$ be an odd integer. Then $T_n\langle 1,2;t_1\rangle$ is hamiltonian for all $n > 3t_1 + 5$.*

Theorem 2.22. *[34] $T_n\langle 1,2,3;2\rangle$ is hamiltonian if and only if $n = 4$ or $n \equiv 1 (mod 2)$. .*

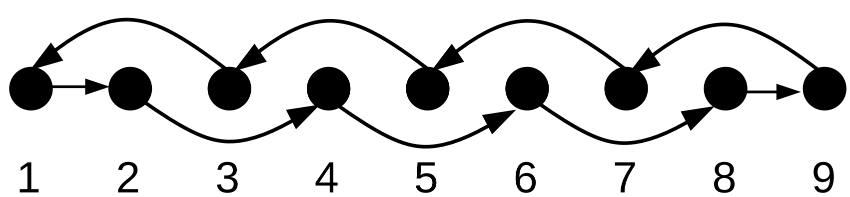

Figure 31: Hamiltonian circuit in $T_9\langle 1,2,3;2\rangle$

Chapter 3

Hamiltonicity in Directed Toeplitz Graph

3 Hamiltonian Circuits in Directed Toeplitz Graphs

Theorem 3.1. *$T_n\langle 1,4;2,3\rangle$ is hamiltonian for all $n > 4$.*

Proof

To show $T_n\langle 1,4;2,3\rangle$ contains a hamiltonian circuit we have the following four cases.

Case 1

If $n \equiv 0 (\mathrm{mod}\, 4)$ then we have the following hamiltonian circuit $(1,2,6,10,\ldots,n-10,n-6,n-2,n-1,n,n-3,n-5,n-4,n-7,\ldots,5,3,4,1)$.

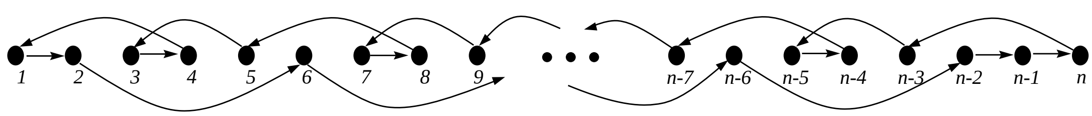

Figure 32: hamiltonian circuit in $T_n\langle 1,4;2,3\rangle$ when $n \equiv 0 (\mathrm{mod}\, 4)$

Case 2

If $n \equiv 1 (\mathrm{mod}\, 4)$ then we have the following hamiltonian circuit $(1,5,9,\ldots,n-8,n-4,n,n-2,n-1,n-3,n-6,n-5,n-7,\ldots,6,4,2,3,1)$.

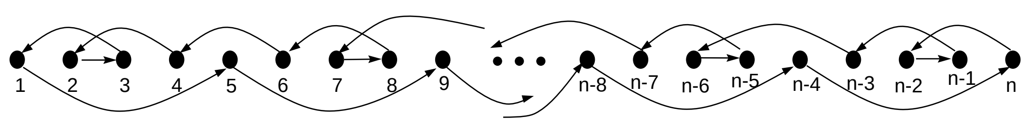

Figure 33: Hamiltonian circuit in $T_n\langle 1,4;2,3\rangle$ when $n \equiv 1 (\mathrm{mod}\, 4)$

Case 3

If $n \equiv 2 (\mathrm{mod}\, 4)$ then we have the following hamiltonian circuit $(1,2,6,10,\ldots,n-8,n-4,n,n-2,n-1,n-3,n-6,n-5,n-7,\ldots,8,9,7,4,5,3,1)$.

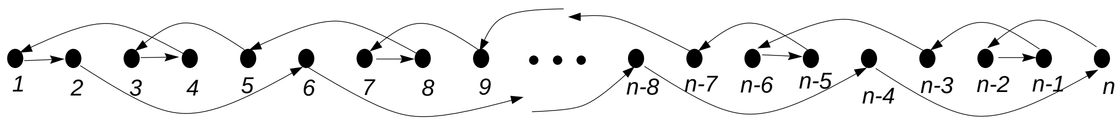

Figure 34: Hamiltonian circuit in $T_n\langle 1,4;2,3\rangle$ when $n \equiv 2(\mathrm{mod}\,4)$

Case 4

If $n \equiv 3(\mathrm{mod}\,4)$ then we have the following hamiltonian circuit $(1,2,6,10,\ldots,n-9,n-5,n-1,n,n-2,n-4,n-3,n-6,n-8,n-7,\ldots,7,8,5,3,4,1)$.

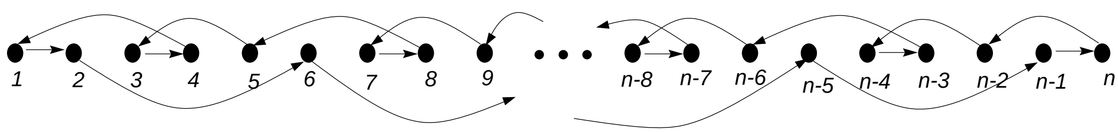

Figure 35: Hamiltonian circuit in $T_n\langle 1,4;2,3\rangle$ when $n \equiv 3(\mathrm{mod}\,4)$

Theorem 3.2. *$T_n\langle 1,5;2,3\rangle$ is hamiltonian for all n.*

Proof

To show that $T_n\langle 1,5;2,3\rangle$ possess a hamiltonian circuit for all $n > 5$ we have following five cases.

Case 1

If $n \equiv 0(\mathrm{mod}\,5)$ then following hamiltonian circuit $(1,2,3,8,13,\ldots,n-7,n-2,n-1,n,n-3,n-5,n-4,n-6,n-8,\ldots,14,12,10,11,9,7,5,6,4,1)$ will be formed.

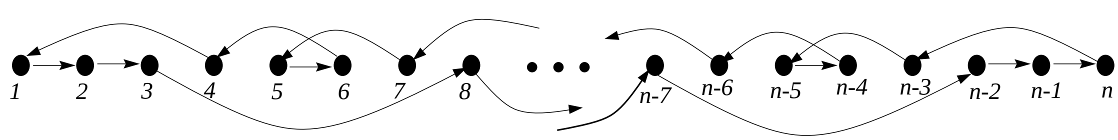

Figure 36: Hamiltonian circuit in $T_n\langle 1,5;2,3\rangle$ when $n \equiv 0(\mathrm{mod}\,5)$

Case 2

If $n \equiv 1 (\bmod 5)$ then following hamiltonian circuit $(1, 6, 11, \ldots, n-10, n-5, n, , n-2, n-1, n-4, n-3, n-6, \ldots, 7, 4, 5, 2, 3, 1)$.

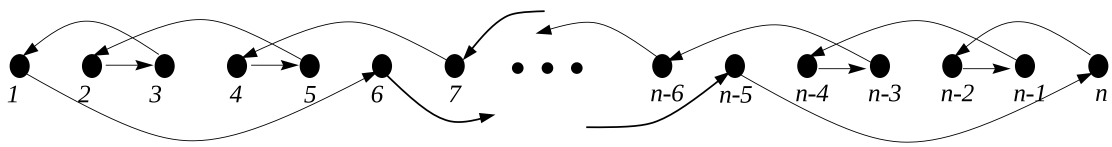

Figure 37: Hamiltonian circuit in $T_n\langle 1,5;2,3\rangle$ when $n \equiv 1 (\bmod 5)$

Case 3

If $n \equiv 2 (\bmod 5)$ then following hamiltonian circuit $(1, 2, 7, 12, \ldots, n-10, n-5, n, n-2, n-1, n-4, n-3, n-6, \ldots, 8, 5, 6, 3, 4, 1)$ will be formed.

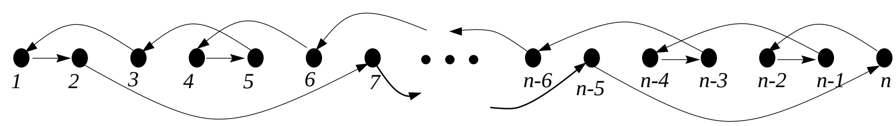

Figure 38: Hamiltonian circuit in $T_n\langle 1,5;2,3\rangle$ when $n \equiv 2 (\bmod 5)$

Case 4

If $n \equiv 3 (\bmod 5)$ then following hamiltonian circuit $(1, 2, 7, 12, \ldots, n-11, n-6, n-1, n, n-2, n-4, n-3, n-5, n-7, \ldots, 8, 6, 4, 5, 3, 1)$ will be formed.

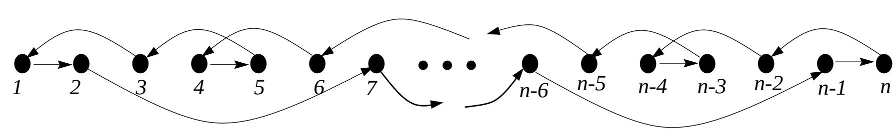

Figure 39: Hamiltonian circuit in $T_n\langle 1,5;2,3\rangle$when $n \equiv 3 (\bmod 5)$

Case 5

If $n \equiv 4 (\bmod 5)$ then following hamiltonian circuit $(1, 2, 7, 12, \ldots, n-7, n-2, n-1, n, n-3, n-5, n-4, n-6, n-8, \ldots, 8, 6, 4, 5, 3, 1)$ will be formed.

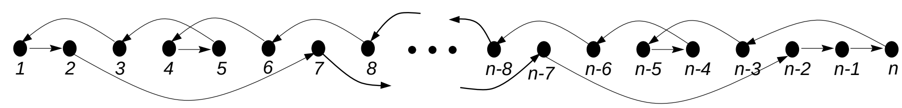

Figure 40: Hamiltonian circuit in $T_n\langle 1,5;2,3\rangle$ when $n \equiv 4 (\bmod 5)$

Theorem 3.3. *$T_n\langle 1,6;2,3\rangle$ is hamiltonian for all $n > 6$.*

Proof

We have following six cases to show this $T_n\langle 1,6;2,3\rangle$ Toeplitz graph is hamiltonian.

Case 1

If $n \equiv 0 (\bmod 6)$ then we have the following hamiltonian circuit $(1,2,8,6,12,10,16,14\ldots,n-8,n-10,n-4,n-6,n,n-2,n-1,n-3,n-5,n-7,n-9,\ldots,13,11,9,7,4,5,3,1)$.

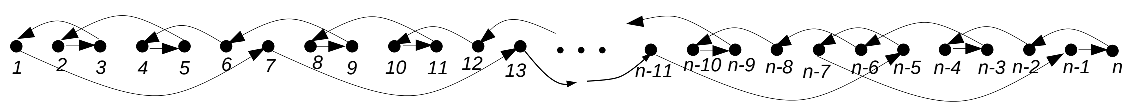

Figure 41: Hamiltonian circuit in $T_n\langle 1,6;2,3\rangle$ when $n \equiv 0 (\bmod 6)$

Case 2

If $n \equiv 1 (\bmod 6)$ then we have the following hamiltonian circuit $(1,7,13,19,\ldots,n-6,n,n-2,n-1,n-4,n-3,n-5,n-7,n-9,n-8,n-11,n-10,n-12,\ldots,6,4,5,2,3,1)$.

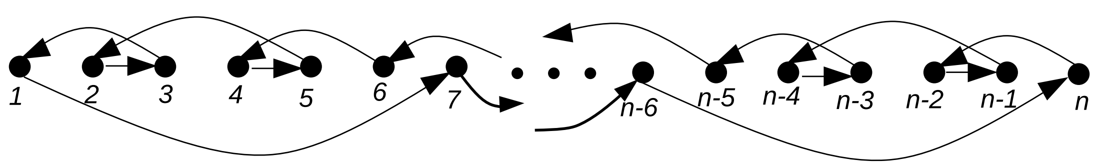

Figure 42: Hamiltonian circuit in $T_n\langle 1,6;2,3\rangle$ when $n \equiv 1 (\bmod 6)$

Case 3

If $n \equiv 2 (\bmod 6)$ then we have the following hamiltonian circuit $(1,2,8,14,\ldots,n-6,n,n-2,n-1,n-4,n-3,n-5,n-8,n-7,n-10,n-9,n-11,\ldots,6,7,4,5,3,1)$.

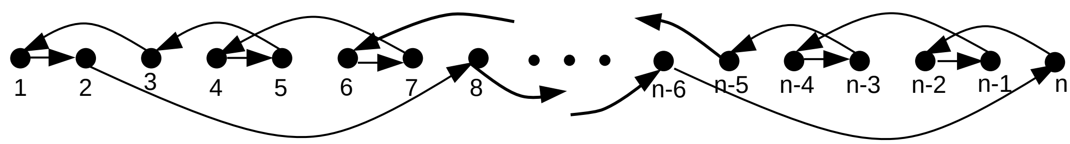

Figure 43: Hamiltonian circuit in $T_n\langle 1,6;2,3\rangle$ when $n \equiv 2 (\bmod 6)$

Case 4

If $n \equiv 3 (\bmod 6)$ then we have the following hamiltonian circuit $(1,2,8,14,\ldots,n-7,n-1,n,n-3,n-2,n-5,n-4,n-6,\ldots,6,7,4,5,3,1)$.

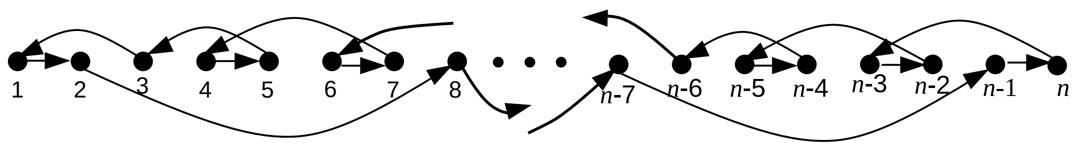

Figure 44: Hamiltonian circuit in $T_n\langle 1,6;2,3\rangle$ when $n \equiv 3 (\bmod 6)$

Case 5

If $n \equiv 4 (\bmod 6)$ then we have the following hamiltonian circuit $(1,2,8,14,20,\ldots,n-14,n-8,n-2,n-1,n,n-3,n-5,n-4,n-7,n-6,n-9,\ldots,7,5,6,3,4,1)$.

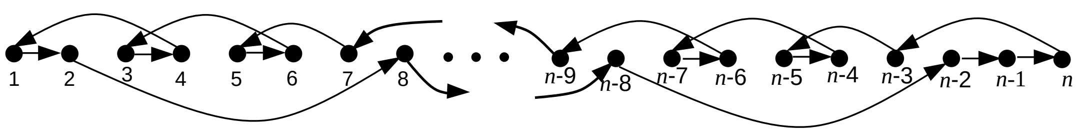

Figure 45: Hamiltonian circuit in $T_n\langle 1,6;2,3\rangle$ when $n \equiv 4 (\bmod 6)$

Case 6

If $n \equiv 5 (\bmod 6)$ then we have the following hamiltonian circuit $(1,2,3,9,15,21,\ldots,n-2,n-1,n,n-6,n,n-3,n-5,n-4,n-7,n-6,\ldots,10,11,8,5,6,7,4,1)$.

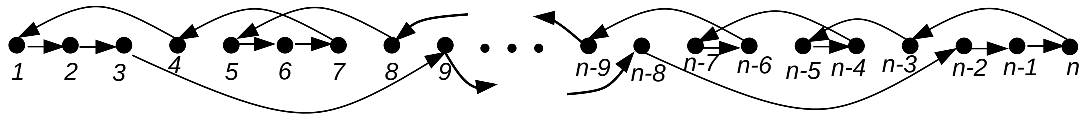

Figure 46: Hamiltonian circuit in $T_n\langle 1,6;2,3\rangle$ when $n \equiv 5(\bmod 6)$

Theorem 3.4. *$T_n\langle 1,7;2,3\rangle$ is hamiltonian for all $n > 7$.*

Proof

To show that $T_n\langle 1,7;2,3\rangle$ possess a hamiltonian circuit for all $n > 7$ we have following seven cases.

Case 1

If $n \equiv 0(\bmod 7)$ then following hamiltonian circuit $(1,8,6,13,11,18,16,\ldots,n-5,n-7,n,n-2,n-1,n-4,n-3,n-6,n-9,n-8,\ldots,9,10,7,4,5,2,3,1)$ will be formed.

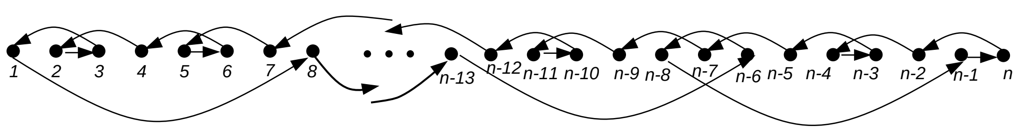

Figure 47: Hamiltonian circuit in $T_n\langle 1,7;2,3\rangle$ when $n \equiv 0(\bmod 7)$

Case 2

If $n \equiv 1(\bmod 7)$ then following hamiltonian circuit $(1,8,15,22,\ldots,n-21,n-14,n-7,n,n-2,n-1,n-4,n-3,n-6,n-5,n-8,\ldots,9,6,7,4,5,2,3,1)$ will be formed.

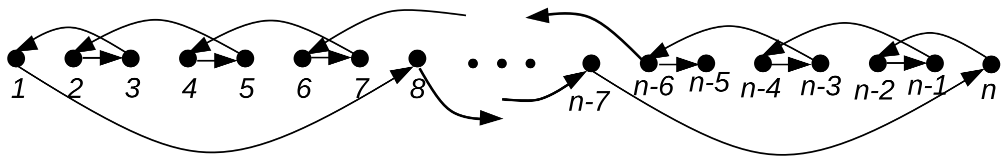

Figure 48: Hamiltonian circuit in $T_n\langle 1,7;2,3\rangle$ when $n \equiv 1(\bmod 7)$

Case 3

If $n \equiv 2(\bmod 7)$ then following hamiltonian circuit $(1,2,9,16,23,\ldots,n-21,n-14,n-7,n,n-2,n-1,n-4,n-3,n-5,n-6,n-5,n-8,\ldots,10,11,8,6,7,4,5,3,1)$ will be formed.

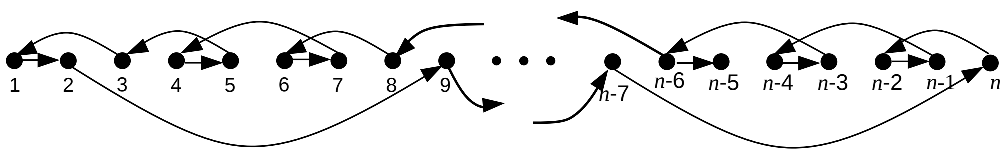

Figure 49: hamiltonian circuit in $T_n\langle 1,7;2,3\rangle$ when $n \equiv 2(\bmod 7)$

Case 4

If $n \equiv 3(\bmod 7)$ then following hamiltonian circuit $(1,2,9,16,23,30,\ldots,n-22,n-15,n-8,n-1,n,n-2,n-4,n-3,n-6,n-5,n-7,n-9,\ldots,10,8,6,7,4,5,3,1)$ will be formed.

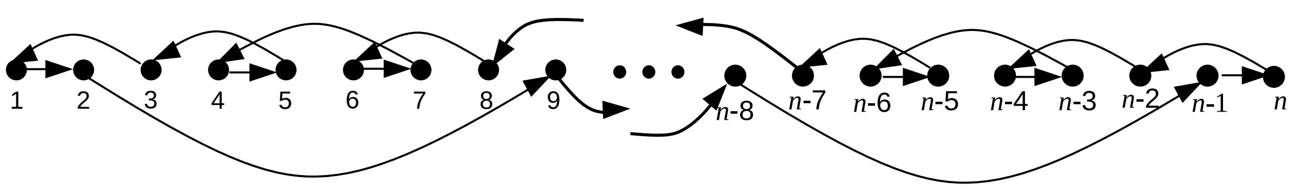

Figure 50: Hamiltonian circuit in $T_n\langle 1,7;2,3\rangle$ when $n \equiv 3(\bmod 7)$

Case 5

If $n \equiv 4 (\mathrm{mod}\, 7)$ then following hamiltonian circuit $(1, 2, 9, 16, 23, 30, \ldots, n-16, n-9, n-2, n-1, n, n-3, n-5, n-7, n-6, n-8, n-10, \ldots, 10, 8, 6, 7, 4, 5, 3, 1)$ will be formed.

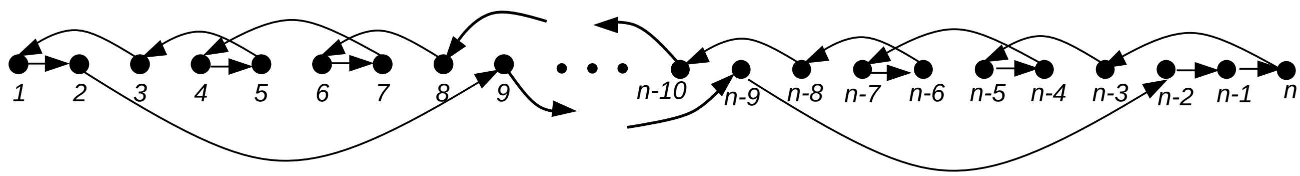

Figure 51: Hamiltonian circuit in $T_n\langle 1, 7; 2, 3\rangle$ when $n \equiv 4 (\mathrm{mod}\, 7)$

Case 6

If $n \equiv 5 (\mathrm{mod}\, 7)$ then following hamiltonian circuit $(1, 2, 3, 10, 17, 24, \ldots, n-16, n-9, n-2, n-1, n, n-3, n-5, n-4, n-7, n-6, n-8, n-10, \ldots, 11, 9, 7, 8, 5, 6, 4, 1)$ will be formed.

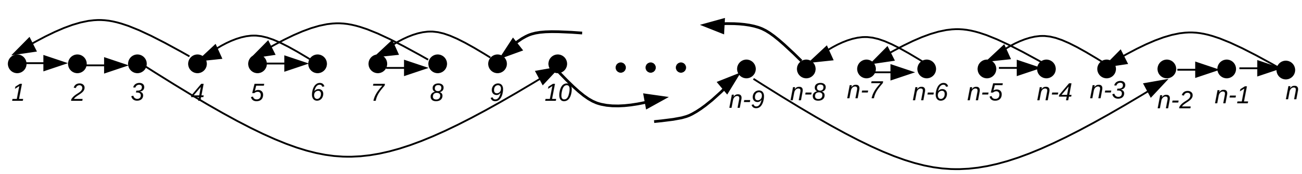

Figure 52: Hamiltonian circuit in $T_n\langle 1, 7; 2, 3\rangle$ when $n \equiv 5 (\mathrm{mod}\, 7)$

Case 7

If $n \equiv 6 (\mathrm{mod}\, 7)$ then following hamiltonian circuit $(1, 2, 9, \ldots, n-11, n-4, n-7, n, n-2, n-1, n-3, n-6, n-5, \ldots, 8, 6, 7, 4, 5, 3, 1)$ will be formed.

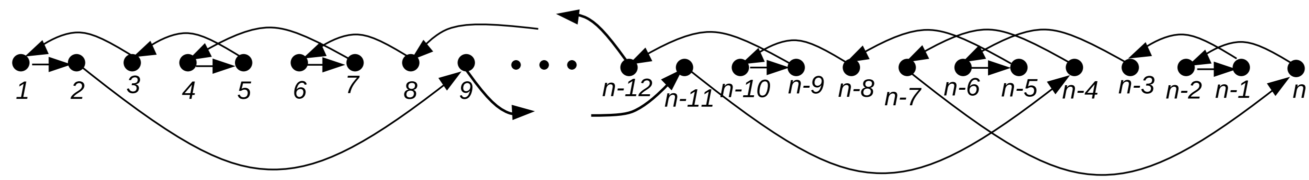

Figure 53: Hamiltonian circuit in $T_n\langle 1,7;2,3\rangle$ when $n \equiv 6 (\bmod 7)$

Theorem 3.5. *$T_n\langle 1,8;2,3\rangle$ is hamiltonian for all $n > 9$.*

Proof

We have following eight cases to show this $T_n\langle 1,8;2,3\rangle$ Toeplitz graph is hamiltonian.

Case 1

If $n \equiv 0 (\bmod 8)$ then we have the following hamiltonian circuit $(1,2,8,6,12,10,16,14\ldots,n-8,n-10,n-4,n-6,n,n-2,n-1,n-3,n-5,n-7,n-9,\ldots,13,11,9,7,4,5,3,1)$.

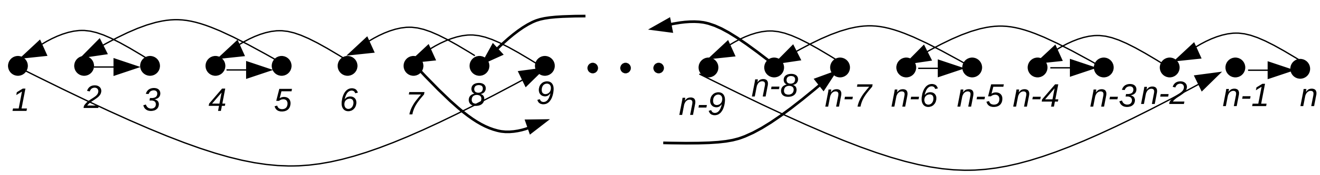

Figure 54: Hamiltonian circuit in $T_n\langle 1,8;2,3\rangle$ when $n \equiv 0 (\bmod 8)$

Case 2

If $n \equiv 1 (\bmod 8)$ then we have the following hamiltonian circuit $(1,9,17,25,\ldots,n-16,n-8,n,n-2,n-1,n-4,n-3,n-6,n-5,n-7,n-10,n-9,\ldots,10,8,6,7,4,5,2,3,1)$.

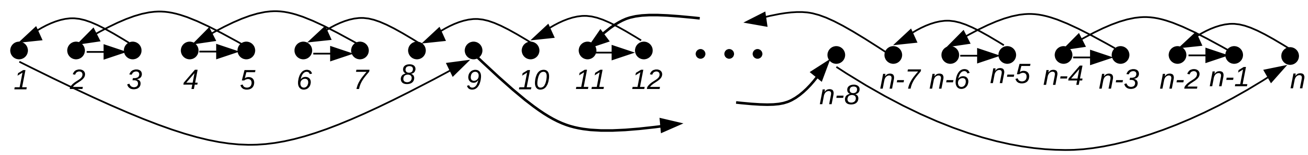

Figure 55: Hamiltonian circuit in $T_n\langle 1,8;2,3\rangle$ when $n \equiv 1 (\bmod 8)$

Case 3

If $n \equiv 2 (\bmod 8)$ then we have the following hamiltonian circuit $(1,2,10,18,26,\ldots,n-16,n-8,n,n-2,n-1,n-4,n-3,n-6,n-5,n-7,n-10,n-9,n-12,n-11,n-14,n-13,\ldots,11,8,9,6,7,4,5,3,1)$.

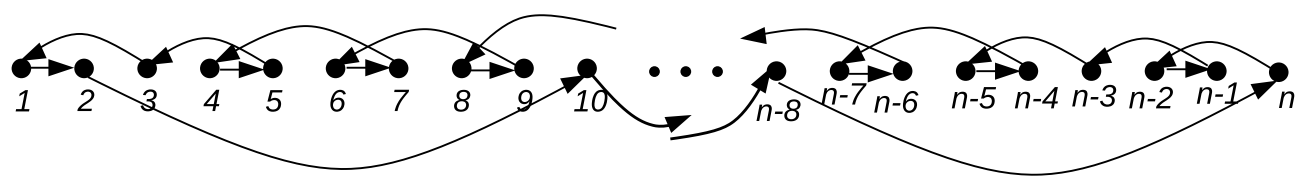

Figure 56: Hamiltonian circuit in $T_n\langle 1,8;2,3\rangle$ when $n \equiv 2 (\bmod 8)$

Case 4

If $n \equiv 3 (\bmod 8)$ then we have the following hamiltonian circuit $(1,2,10,18,26,\ldots,n-23,n-17,n-9,n-1,n,n-3,n-2,n-5,n-4,n-7,n-6,n-8,\ldots,11,8,9,6,7,4,5,3,1)$.

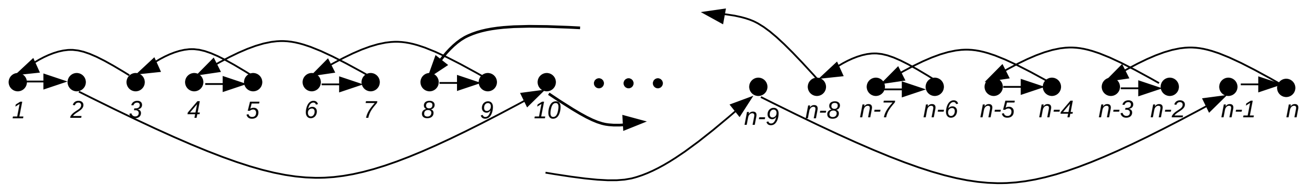

Figure 57: Hamiltonian circuit in $T_n\langle 1,8;2,3\rangle$ when $n \equiv 3(\bmod 8)$

Case 5

If $n \equiv 4(\bmod 8)$ then we have the following hamiltonian circuit $(1,2,10,18,26,\ldots,n-18,n-10,n-2,n-1,n,n-3,n-5,n-4,n-7,n-6,n-9,n-8,n-11,\ldots,11,12,9,7,8,5,6,3,1)$.

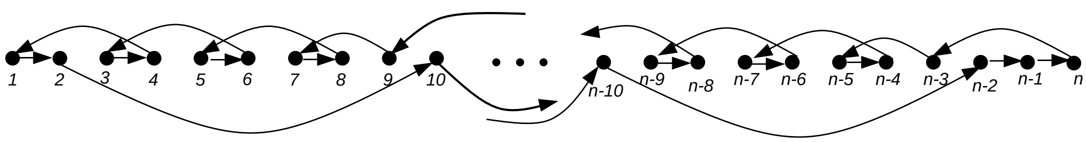

Figure 58: hamiltonian circuit in $T_n\langle 1,8;2,3\rangle$ when $n \equiv 4(\bmod 8)$

Case 6

If $n \equiv 5(\bmod 8)$ then we have the following hamiltonian circuit $(1,9,17,15,23,\ldots,n-14,n-6,n-8,n,n-2,n-1,n-4,n-3,n-5,n-7,\ldots,10,11,8,6,4,5,2,3,1)$.

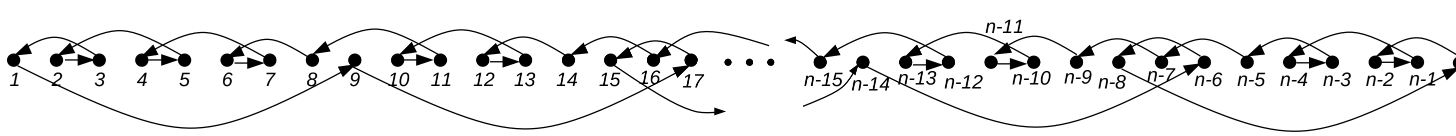

Figure 59: Hamiltonian circuit in $T_n\langle 1,8;2,3\rangle$ when $n \equiv 5(\bmod 8)$

Case 7

If $n \equiv 6 (\mathrm{mod}\, 8)$ then we have the following hamiltonian circuit $(1, 2, 10, 18, 16, \ldots, n-12, n-14, n-6, n-8, n, n-2, n-1, n-4, n-3, n-5, n-7, \ldots, 11, 12, 9, 7, 8, 5, 6, 3, 4, 1)$.

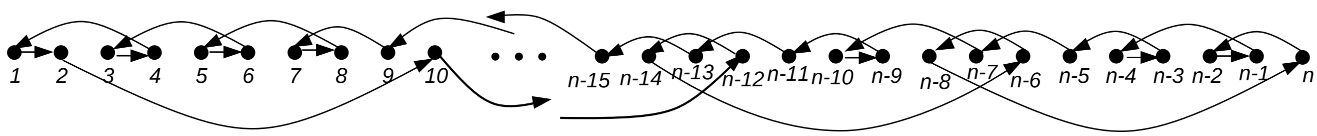

Figure 60: Hamiltonian circuit in $T_n\langle 1, 8; 2, 3\rangle$ when $n \equiv 6 (\mathrm{mod}\, 8)$

Case 8

If $n \equiv 7 (\mathrm{mod}\, 8)$ then we have the following hamiltonian circuit $(1, 2, 10, 18, \ldots, n-13, n-15, n-7, n-9, n-1, n, n-2, n-4, n-3, n-6, n-5, \ldots, 11, 12, 9, 7, 8, 5, 6, 3, 4, 1)$.

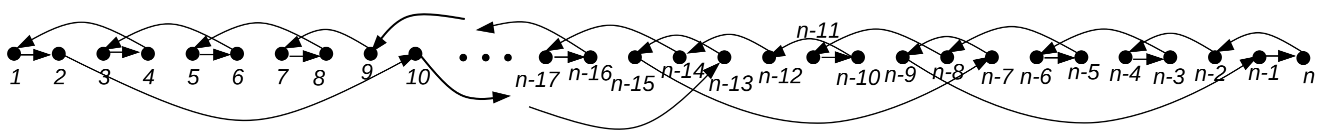

Figure 61: Hamiltonian circuit in $T_n\langle 1, 8; 2, 3\rangle$ when $n \equiv 7 (\mathrm{mod}\, 8)$

Theorem 3.6. *$T_n\langle 1, 9; 2, 3\rangle$ is hamiltonian for all n.*

Proof

To show $T_n\langle 1, 9; 2, 3\rangle$ contains a hamiltonian circuit we have the following nine cases.

Case 1

If $n \equiv 0 (\mathrm{mod}\, 9)$ then we have the following hamiltonian circuit $(1, 10, 19, \ldots, n-8, n-10, n-1, n, n-2, n-4, n-3, n-6, n-5, n-9, n-11, \ldots, 11, 8, 6, 7, 4, 5, 2, 3, 1)$.

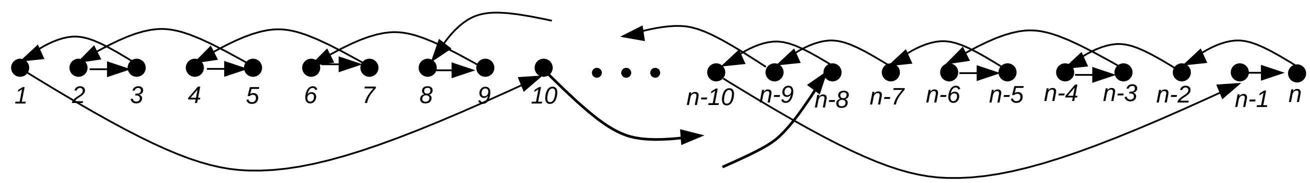

Figure 62: Hamiltonian circuit in $T_n\langle 1,9;2,3\rangle$ when $n \equiv 0 (\mathrm{mod}\, 9)$

Case 2

If $n \equiv 1 (\mathrm{mod}\, 9)$ then we have the following hamiltonian circuit $(1,10,19,\ldots,n-9,n,n-2,n-1,n-4,n-3,n-6,n-5,n-8,n-7,n-10,\ldots,11,8,9,6,7,4,5,2,3,1)$.

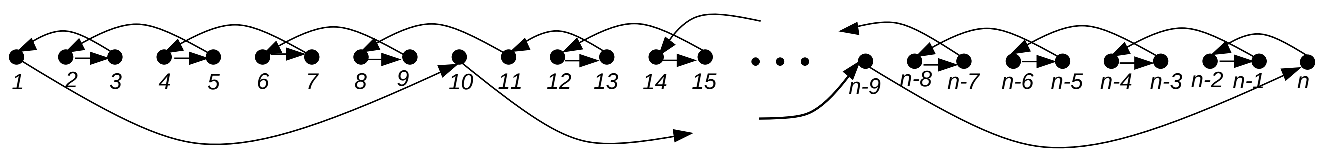

Figure 63: Hamiltonian circuit in $T_n\langle 1,9;2,3\rangle$ when $n \equiv 1 (\mathrm{mod}\, 9)$

Case 3

If $n \equiv 2 (\mathrm{mod}\, 9)$ then we have the following hamiltonian circuit $(1,2,11,9,\ldots,n-11,n-12,n-1,n,n-3,n-5,n-4,n-7,n-6,n-9,n-8,n-10,n-13,n-12,n-15,n-14,\ldots,10,7,8,5,6,3,4,1)$.

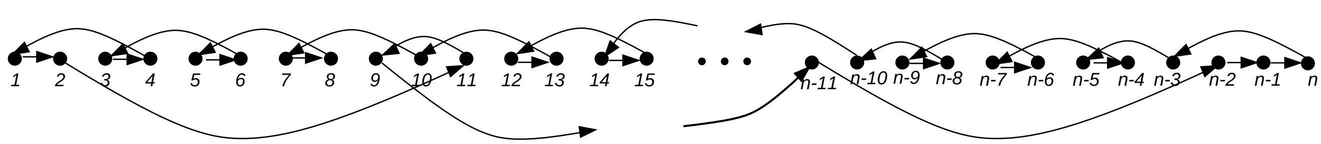

Figure 64: Hamiltonian circuit in $T_n\langle 1,9;2,3\rangle$ when $n \equiv 2 (\mathrm{mod}\, 9)$

Case 4

If $n \equiv 3 (\bmod 9)$ then we have the following hamiltonian circuit $(1,2,3,12,10,\ldots,n-11,n-2n-1,n,n-3,n-5,n-4,n-7,n-6,n-9,n-8,n-10,n-13,n-12,n-15,n-14,\ldots,13,11,9,7,8,5,6,4,1)$.

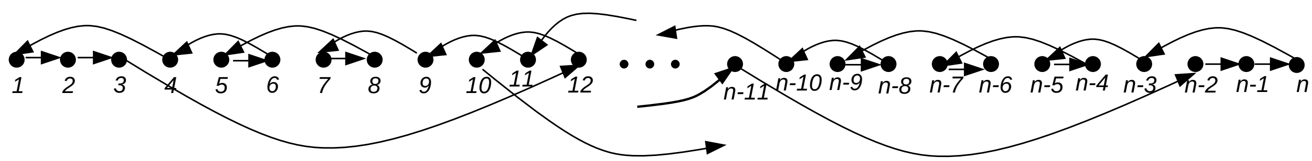

Figure 65: Hamiltonian circuit in $T_n\langle 1,9;2,3\rangle$ when $n \equiv 3 (\bmod 9)$

Case 5

If $n \equiv 4 (\bmod 9)$ then we have the following hamiltonian circuit $(1,10,8,17,15,\ldots,n-7,n-9,n,n-2,n-1,n-4,n-3,n-6,n-5,n-8,n-11,n-10,\ldots,11,12,9,7,8,5,6,3,1)$.

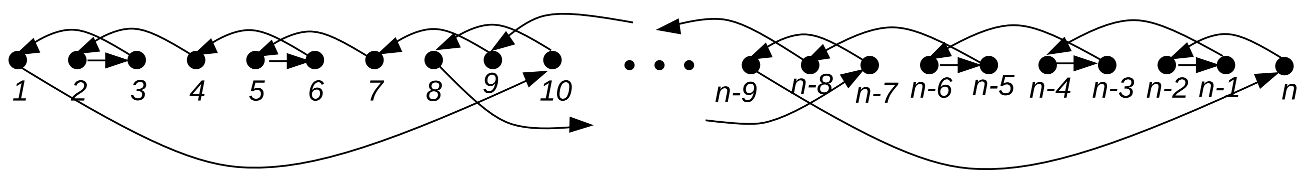

Figure 66: Hamiltonian circuit in $T_n\langle 1,9;2,3\rangle$ when $n \equiv 4 (\bmod 9)$

Case 6

If $n \equiv 5 (\bmod 9)$ then we have the following hamiltonian circuit $(1,2,11,9,\ldots,n-7,n-9,n,n-2,n-1,n-4,n-3,n-6,n-5,n-8,n-11,n-10,\ldots,10,7,8,5,6,3,4,1)$.

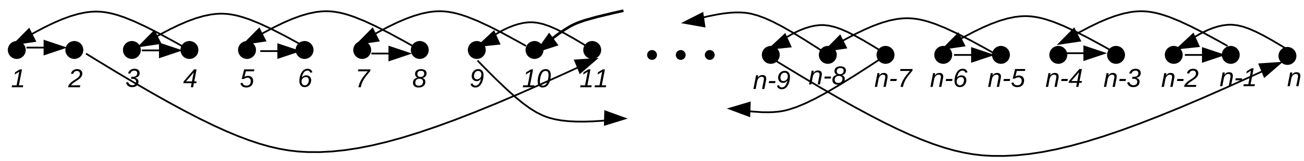

Figure 67: Hamiltonian circuit in $T_n\langle 1,9;2,3\rangle$ when $n \equiv 5 (\bmod 9)$

Case 7

If $n \equiv 6 (\bmod 9)$ then we have the following hamiltonian circuit $(1,2,11,9,\ldots,n-15,n-17,n-8,n-10,n-1,n,n-2,n-4,n-3,\ldots,6,4,2,3,1)$.

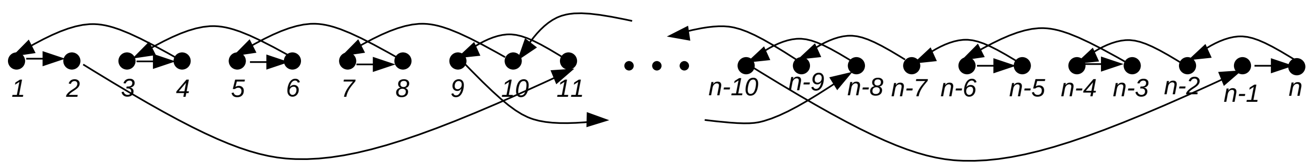

Figure 68: Hamiltonian circuit in $T_n\langle 1,9;2,3\rangle$ when $n \equiv 6 (\bmod 9)$

Case 8

If $n \equiv 7 (\bmod 9)$ then we have the following hamiltonian circuit $(1,7,5,11,9,15,13,\ldots,n-8,n-10,n-4,n-6,n,n-2,n-1,n-3,n-5,n-7,\ldots,6,4,2,3,1)$.

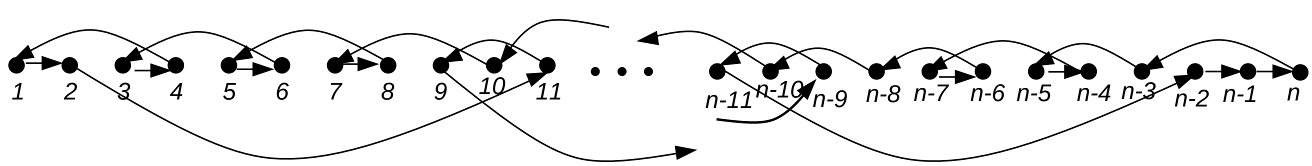

Figure 69: Hamiltonian circuit in $T_n\langle 1,9;2,3\rangle$ when $n \equiv 7 (\bmod 9)$

Case 9

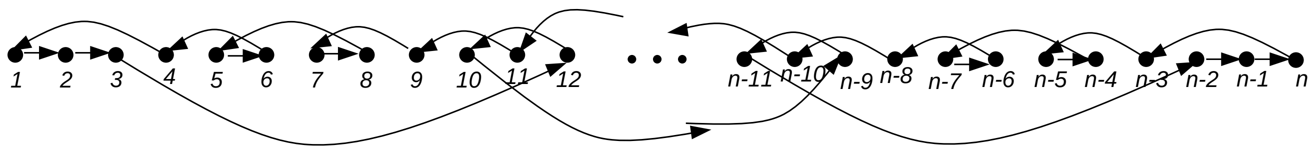

Figure 70: Hamiltonian circuit in $T_n\langle 1,9;2,3\rangle$ when $n \equiv 8(\bmod 9)$

If $n \equiv 8(\bmod 9)$ then we have the following hamiltonian circuit $(1,7,5,11,9,15,13,\ldots,n-8,n-10,n-4,n-6,n,n-2,n-1,n-3,n-5,n-7,\ldots,6,4,2,3,1)$.

Theorem 3.7. *$T_n\langle 1,10;2,3\rangle$ is hamiltonian for all $n>10$.*

Proof

To show that $T_n\langle 1,10;2,3\rangle$ possess a hamiltonian circuit for all $n>10$ we have following ten cases.

Case 1

If $n\equiv 0(\bmod 10)$ then we have the following hamiltonian circuit $(1,11,21,\ldots,n-9,n-11,n-1,n,n-2,n-4,n-3,n-6,n-5,n-8,n-7,n-10,n-12,n-14,n-13,\ldots,12,13,10,8,9,6,7,4,5,2,3,1)$.

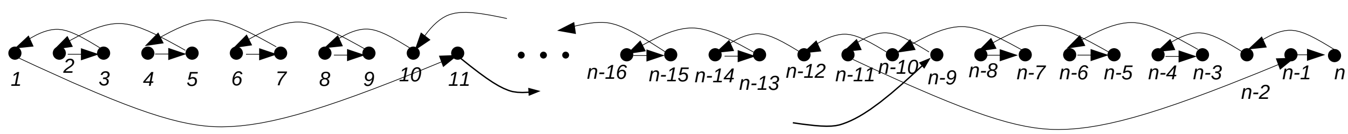

Figure 71: Hamiltonian circuit in $T_n\langle 1,10;2,3\rangle$ when $n\equiv 0(\bmod 10)$

Case 2

If $n\equiv 1(\bmod 10)$ then we have the following hamiltonian circuit $(1,11,21,\ldots,n-10,n,n-2,n-1,n-4,n-3,n-6,n-5,n-8,n-7,n-9,n-11,n-13,\ldots,12,13,10,8,9,6,7,4,5,2,3,1)$.

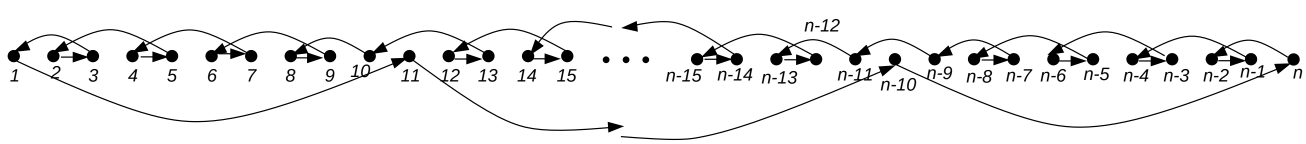

Figure 72: Hamiltonian circuit in $T_n\langle 1,10;2,3\rangle$ when $n\equiv 1(\bmod 10)$

Case 3

If $n\equiv 2(\bmod 10)$ then we have the following hamiltonian circuit $(1,11,21,\ldots,n-11,n-1,n,n-2,n-4,n-3,n-6,n-5,n-8,n-7,n-10,n-9,n-12,\ldots,12,13,10,8,9,6,7,4,5,2,3,1)$.

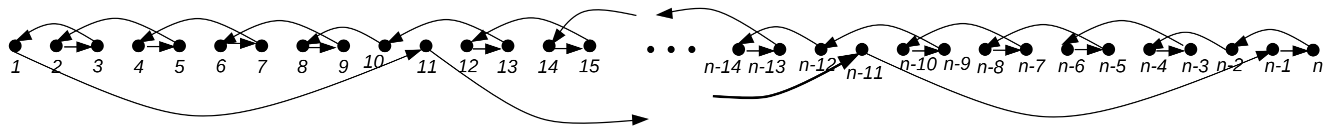

Figure 73: Hamiltonian circuit in $T_n\langle 1,10;2,3\rangle$ when $n \equiv 2 (\mathrm{mod}\, 10)$

Case 4

If $n \equiv 3 (\mathrm{mod}\, 10)$ then we have the following hamiltonian circuit $(1,2,12,\ldots,n-11,n-1,n,n-2,n-4,n-3,n-6,n-5,n-8,n-7,n-10,n-9,n-14,n-13,\ldots,11,9,10,7,8,5,6,3,4,1)$.

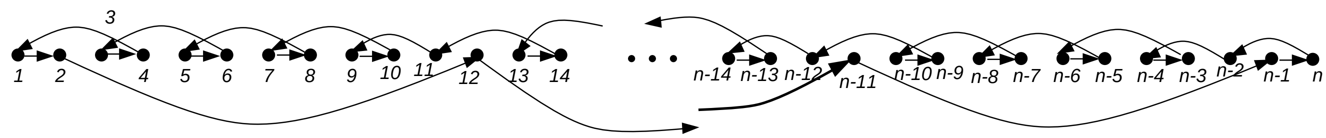

Figure 74: Hamiltonian circuit in $T_n\langle 1,10;2,3\rangle$ when $n \equiv 3 (\mathrm{mod}\, 10)$

Case 5

If $n \equiv 4 (\mathrm{mod}\, 10)$ then we have the following hamiltonian circuit $(1,2,12,\ldots,n-12,n-2,n-1,n,n-3,n-5,n-4,n-7,n-6,n-9,n-8,n-11,n-10,n-13,\ldots,11,9,10,7,8,5,6,3,4,1)$.

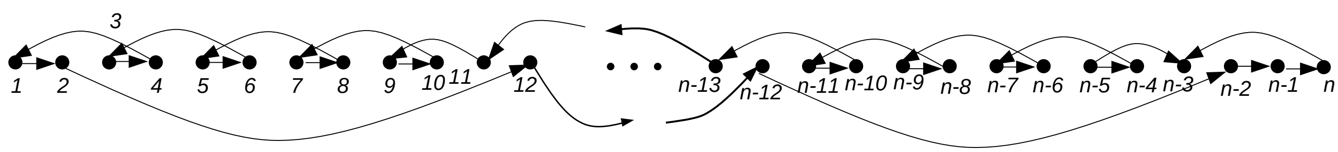

Figure 75: Hamiltonian circuit in $T_n\langle 1,10;2,3\rangle$ when $n \equiv 4 (\mathrm{mod}\, 10)$

Case 6

If $n \equiv 5 (\mathrm{mod}\, 10)$ then we have the following hamiltonian circuit $(1,11,9,\ldots,n-8,n-10,n,n-2,n-1,n-4,n-3,n-6,n-5,n-7,n-9,n-11,\ldots,10,8,6,7,4,5,2,3,1)$.

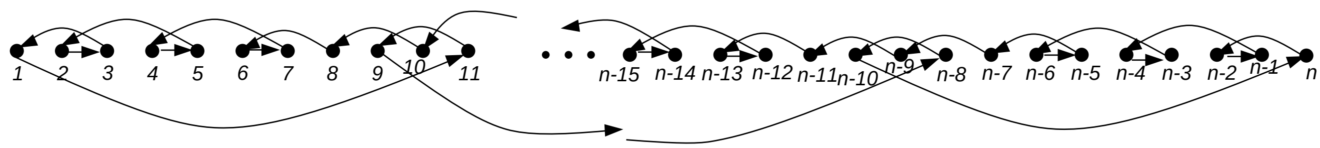

Figure 76: Hamiltonian circuit in $T_n\langle 1,10;2,3\rangle$ when $n \equiv 5 (\mathrm{mod}\, 10)$

Case 7

If $n \equiv 6 (\mathrm{mod}\, 10)$ then we have the following hamiltonian circuit $(1, 11, 9, \ldots, n-9, n-11, n-1, n, n-2, n-4, n-3, n-6, n-5, n-8, n-7, n-10, n-12, \ldots, 10, 8, 6, 7, 4, 5, 2, 3, 1)$.

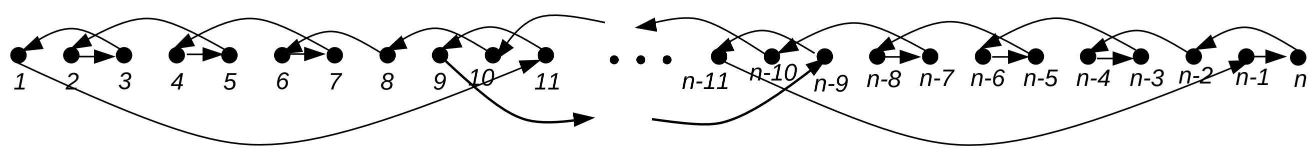

Figure 77: Hamiltonian circuit in $T_n\langle 1,10;2,3\rangle$ when $n \equiv 6 (\mathrm{mod}\, 10)$

Case 8

If $n \equiv 7 (\mathrm{mod}\, 10)$ then we have the following hamiltonian circuit $(1, 2, 12, 10, \ldots, n-8, n-10, n-4, n-6, n, n-2, n-1, n-3, n-5, n-7, \ldots, 11, 9, 10, 7, 8, 5, 6, 3, 4, 1)$.

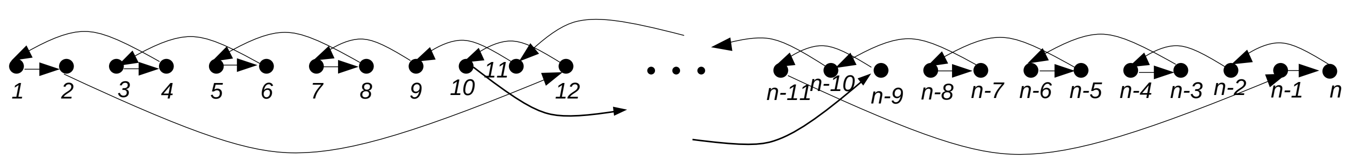

Figure 78: Hamiltonian circuit in $T_n\langle 1,10;2,3\rangle$ when $n \equiv 7 (\mathrm{mod}\, 10)$

Case 9

If $n \equiv 8 (\bmod 10)$ then we have the following hamiltonian circuit $(1,2,12,\ldots,n-10,n-12,n-2,n-1,n,n-3,n-5,n-4,n-7,n-6,n-10,n-9,n-11,n-14,n-13,\ldots,11,9,10,7,8,5,6,3,4,1)$.

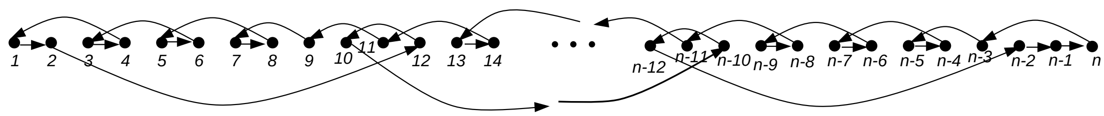

Figure 79: Hamiltonian circuit in $T_n\langle 1,10;2,3\rangle$ when $n \equiv 8 (\bmod 10)$

Case 10

If $n \equiv 9 (\bmod 10)$ then we have the following hamiltonian circuit $(1,11,9,\ldots,n-12,n-2,n-1,n,n-3,n-5,n-4,n-7,n-6,n-9,n-8,n-11,n-10,n-13,n-15,n-14,\ldots,10,8,6,7,4,5,2,3,1)$.

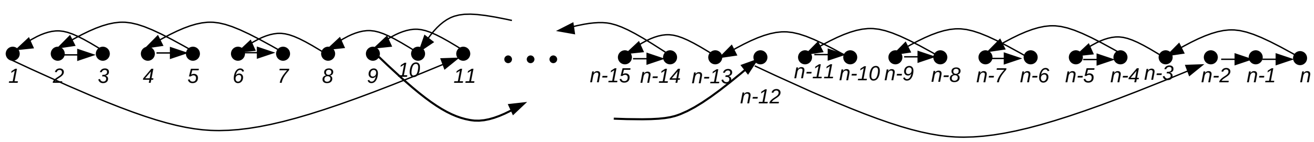

Figure 80: Hamiltonian circuit in $T_n\langle 1,10;2,3\rangle$

when $n \equiv 9 (\bmod 10)$

Theorem 3.8. *$T_n\langle 1,11;2,3\rangle$ is hamiltonian for all $n > 11$.*

Proof

To show $T_n\langle 1,11;2,3\rangle$ contains a hamiltonian circuit we the have following eleven cases.

Case 1

If $n \equiv 0 (\bmod 11)$ then we have the following hamiltonian circuit $(1,2,13,9,\ldots,n-9,n-13,n-2,n-1,n,n-3,n-5,n-4,n-7,n-6,n-8,n-10,\ldots,12,10,7,8,5,6,3,4,1)$.

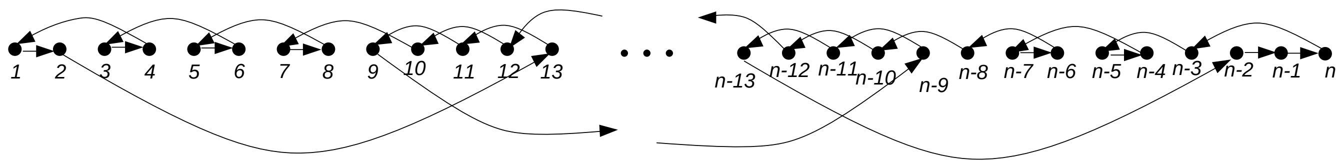

Figure 81: Hamiltonian circuit in $T_n\langle 1,11;2,3\rangle$ when $n \equiv 0(\bmod 11)$

Case 2

If $n \equiv 1(\bmod 11)$ then we have the following hamiltonian circuit $(1,12,\ldots,n-22,n-11,n,n-2,n-1,n-4,n-3,n-6,n-5,n-8,n-7,n-10,n-9,n-12,\ldots,13,10,11,8,9,6,7,4,5,2,3,1)$.

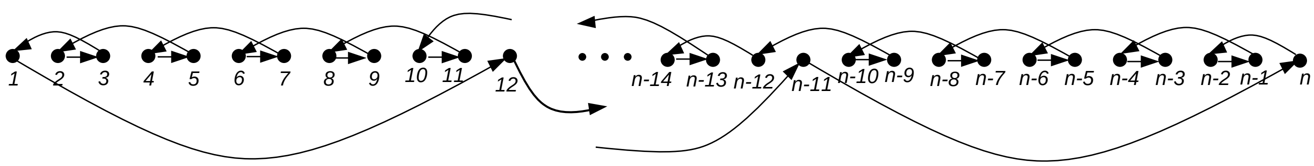

Figure 82: Hamiltonian circuit in $T_n\langle 1,11;2,3\rangle$ when $n \equiv 1(\bmod 11)$

Case 3

If $n \equiv 2(\bmod 11)$ then we have the following hamiltonian circuit $(1,2,13,\ldots,n-22,n-11,n,n-2,n-1,n-4,n-3,n-6,n-5,n-8,n-7,n-10,n-9,n-12,\ldots,14,11,12,9,10,7,8,5,6,3,4,1)$.

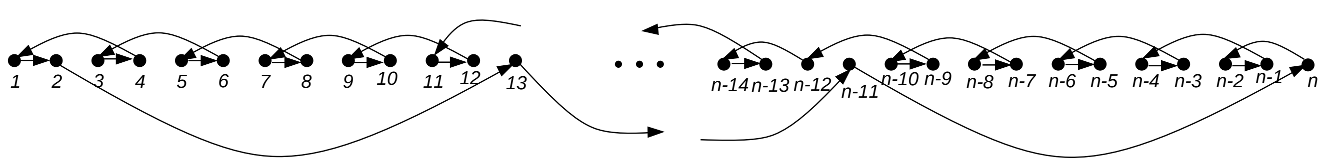

Figure 83: Hamiltonian circuit in $T_n\langle 1,11;2,3\rangle$ when $n \equiv 2(\bmod 11)$

Case 4

If $n \equiv 3 (\bmod 11)$ then we have the following hamiltonian circuit $(1,2,13,\ldots,n-12,n-1,n,n-2,n-4,n-3,n-6,n-5,n-8,n-7,n-10,n-9,n-11,n-14,\ldots,12,10,11,8,9,6,7,4,5,3,1)$.

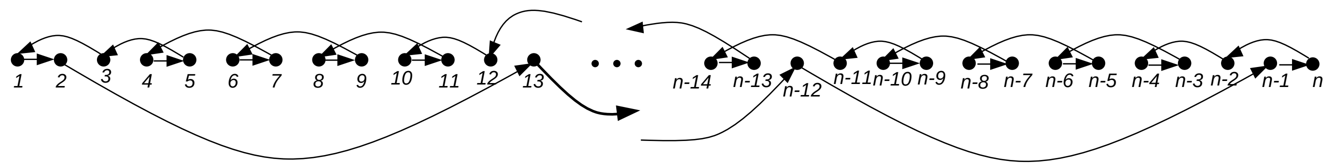

Figure 84: Hamiltonian circuit in $T_n\langle 1,11;2,3\rangle$ when $n \equiv 3 (\bmod 11)$

Case 5

If $n \equiv 4 (\bmod 11)$ then we have the following hamiltonian circuit $(1,2,13,\ldots,n-13,n-2,n-1,n,n-3,n-5,n-4,n-7,n-6,n-9,n-8,n-11,n-10,n-12,\ldots,12,10,11,8,9,6,7,4,5,3,1)$.

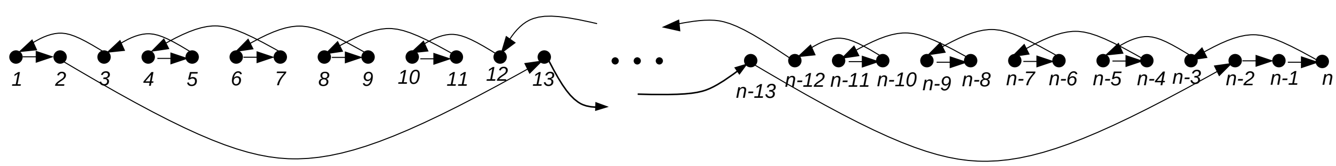

Figure 85: Hamiltonian circuit in $T_n\langle 1,11;2,3\rangle$ when $n \equiv 4 (\bmod 11)$

Case 6

If $n \equiv 5 (\bmod 11)$ then we have the following hamiltonian circuit $(1,2,3,14,\ldots,n-13,n-2,n-1,n,n-3,n-5,n-4,n-7,n-6,n-9,n-8,n-11,n-10,n-12,\ldots,13,11,12,9,10,7,8,5,6,4,1)$.

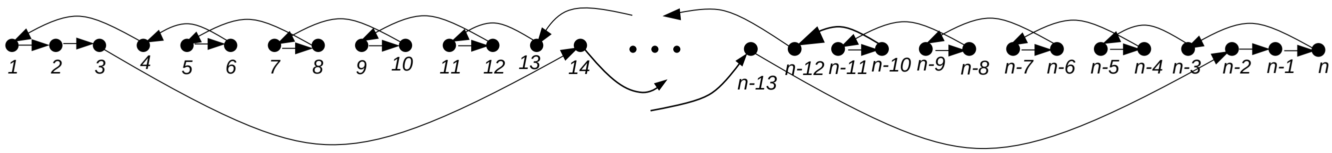

Figure 86: Hamiltonian circuit in $T_n\langle 1,11;2,3\rangle$ when $n \equiv 5(\bmod 11)$

Case 7

If $n \equiv 6(\bmod 11)$ then we have the following hamiltonian circuit $(1,12,8,\ldots,n-6,n-11,n,n-2,n-1,n-4,n-5,n-7,n-10,n-9,n-12,\ldots,12,14,11,9,6,7,4,5,2,3,1)$.

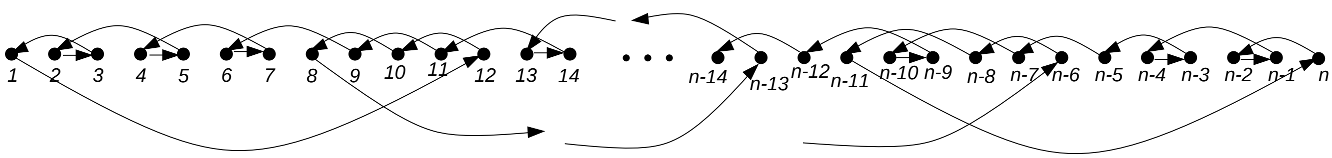

Figure 87: Hamiltonian circuit in $T_n\langle 1,11;2,3\rangle$ when $n \equiv 6(\bmod 11)$

Case 8

If $n \equiv 7(\bmod 11)$ then we have the following hamiltonian circuit $(1,2,13,9,\ldots,n-6,n-11,n,n-2,n-1,n-4,n-3,n-5,n-7,n-10,n-9,n-12,n-14,\ldots,12,10,7,8,5,6,3,4,1)$.

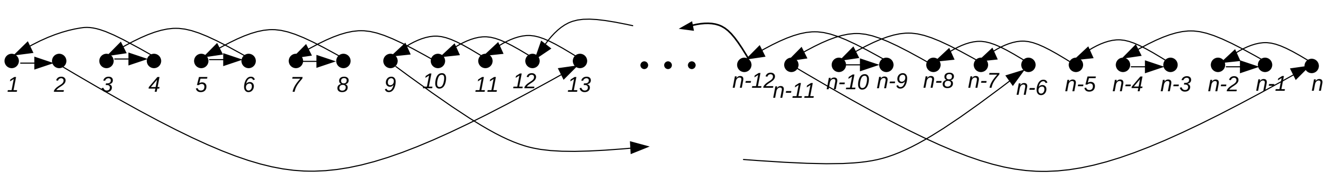

Figure 88: Hamiltonian circuit in $T_n\langle 1,11;2,3\rangle$ when $n \equiv 7(\bmod 11)$

Case 9

If $n \equiv 8(\bmod 11)$ then we have the following hamiltonian circuit $(1,2,13,9,\ldots,n-7,n-11,n,n-2,n-1,n-4,n-3,n-6,n-5,n-8,n-10,n-13,n-12,\ldots,12,10,7,8,5,6,3,4,1)$.

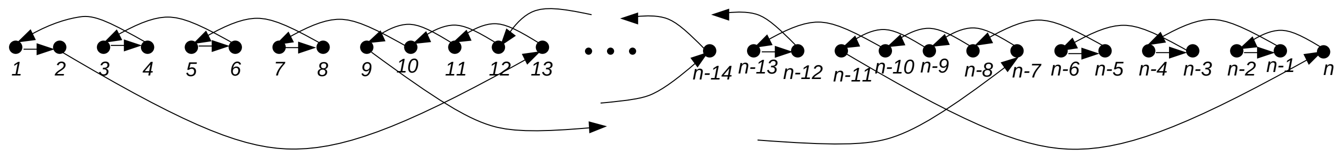

Figure 89: Hamiltonian circuit in $T_n\langle 1,11;2,3\rangle$ when $n \equiv 8(\bmod 11)$

Case 10

If $n \equiv 9(\bmod 11)$ then we have the following hamiltonian circuit $(1,2,13,9,\ldots,n-8,n-12,n-1,n,n-2,n-4,n-3,n-6,n-5,n-7,n-9,n-14,n-13\ldots,12,10,7,8,5,6,3,4,1)$.

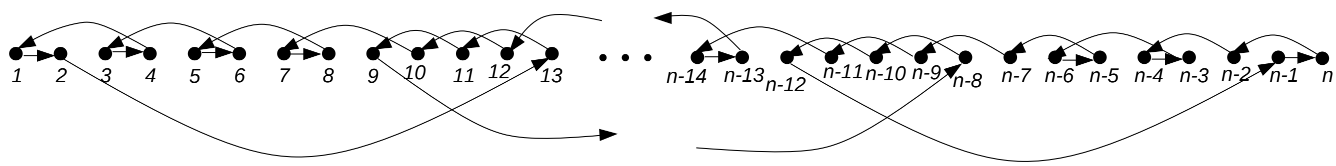

Figure 90: Hamiltonian circuit in $T_n\langle 1,11;2,3\rangle$ when $n \equiv 9(\bmod 11)$

Case 11

If $n \equiv 10(\bmod 11)$ then we have the following hamiltonian circuit $(1,2,13,9,\ldots,n-9,n-13,n-2,n-1,n,n-3,n-5,n-4,n-7,n-6,n-8,n-10,n-12,\ldots,12,10,7,8,5,6,3,4,1)$.

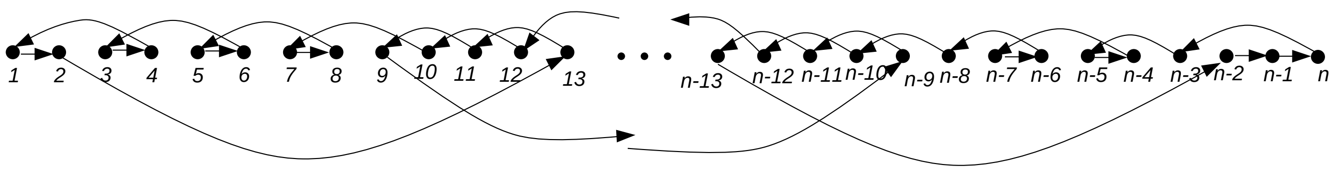

Figure 91: Hamiltonian circuit in $T_n\langle 1,11;2,3\rangle$ when $n \equiv 10 (\bmod 11)$

Theorem 3.9. *$T_n\langle 1,12;2,3\rangle$ is hamiltonian for all n.*

Proof

To show $T_n\langle 1,12;2,3\rangle$ contains a hamiltonian circuit we have the following twelve cases.

Case 1

If $n \equiv 0 (\bmod 12)$ then we have the following hamiltonian circuit $(1,2,14,12,\ldots,n-12,n,n-2,n-1,n-4,n-3,n-6,n-5,n-8,n-7,n-6,n-10,n-9,n-11,n-13,\ldots,13,11,9,10,7,8,5,6,3,4,1)$.

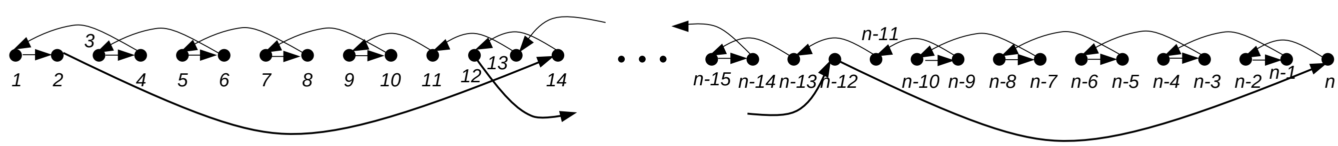

Figure 92: Hamiltonian circuit in $T_n\langle 1,12;2,3\rangle$ when $n \equiv 0 (\bmod 12)$

Case 2

If $n \equiv 1 (\bmod 12)$ then we have the following hamiltonian circuit $(1,13,25,\ldots,n-12,n,n-2,n-1,n-4,n-3,n-6,n-5,n-8,n-7,n-6,n-10,n-9,n-11,n-13,\ldots,14,15,12,10,11,8,9,7,4,5,2,3,1)$.

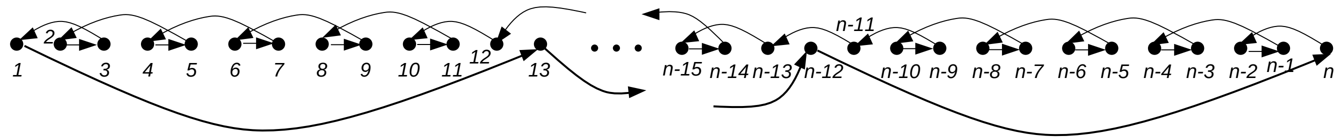

Figure 93: Hamiltonian circuit in $T_n\langle 1,12;2,3\rangle$ when $n \equiv 1 (\mathrm{mod}\, 12)$

Case 3

If $n \equiv 2 (\mathrm{mod}\, 12)$ then we have the following hamiltonian circuit $(1,2,14,26,\ldots,n,n-2,n-1,n-4,n-3,n-6,n-5,n-8,n-7,n-10,n-9,n-12,n-14,n-13,\ldots,15,16,13,11,12,9,10,7,8,5,6,3,4,1)$.

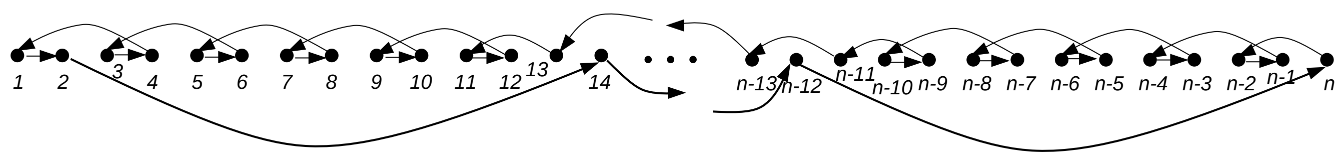

Figure 94: Hamiltonian circuit in $T_n\langle 1,12;2,3\rangle$ when $n \equiv 2 (\mathrm{mod}\, 12)$

Case 4

If $n \equiv 3 (\mathrm{mod}\, 12)$ then we have the following hamiltonian circuit $(1,2,14,26,\ldots,n-1,n,n-2,n-4,n-3,n-6,n-5,n-8,n-7,n-10,n-9,n-12,n-11,n-14,n-16,n-15,\ldots,15,16,11,12,9,10,7,8,5,6,3,4,1)$.

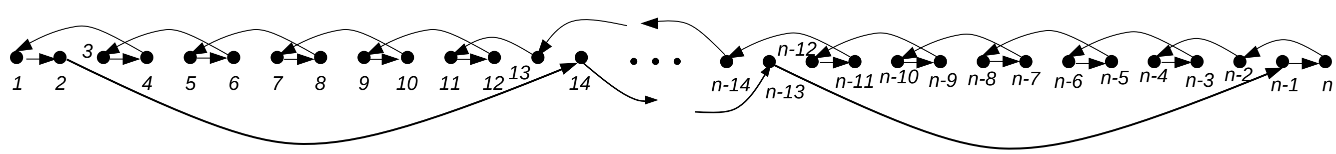

Figure 95: Hamiltonian circuit in $T_n\langle 1,12;2,3\rangle$ when $n \equiv 3 (\mathrm{mod}\, 12)$

Case 5

If $n \equiv 4 (\bmod 12)$ then we have the following hamiltonian circuit $(1, 2, 14, 26, \ldots, n-14, n-2, n-1, n, n-3, n-5, n-4, n-7, n-6, n-9, n-8, n-11, n-10, n-13, n-12, n-15, \ldots, 15, 12, 13, 10, 11, 8, 9, 6, 7, 4, 5, 3, 1)$.

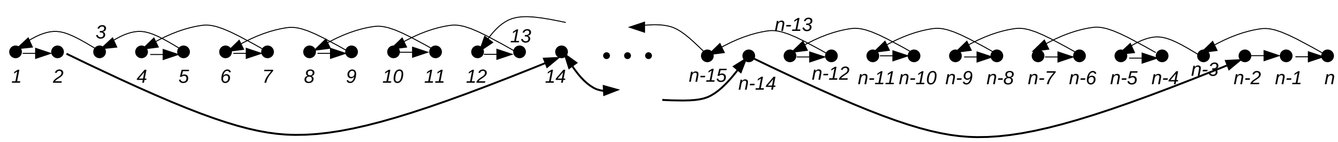

Figure 96: Hamiltonian circuit in $T_n\langle 1, 12; 2, 3\rangle$ when $n \equiv 4 (\bmod 12)$

Case 6

If $n \equiv 5 (\bmod 12)$ then we have the following hamiltonian circuit $(1, 2, 3, 15, \ldots, n-14, n-2, n-1, n, n-3, n-6, n-5, n-4, n-7, n-10, n-9, n-8, n-11, n-13, n-12, n-15, \ldots, 16, 17, 14, 11, 12, 13, 10, 8, 9, 7, 5, 6, 4, 1)$.

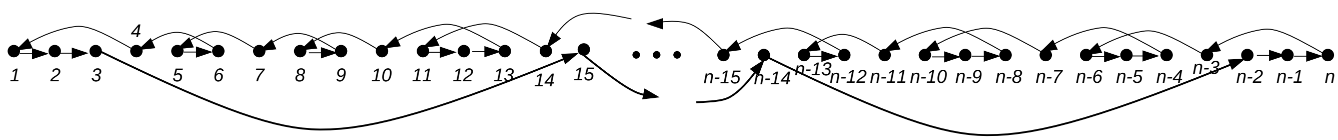

Figure 97: Hamiltonian circuit in $T_n\langle 1, 12; 2, 3\rangle$ when $n \equiv 5 (\bmod 12)$

Case 7

If $n \equiv 6 (\bmod 12)$ then we have the following hamiltonian circuit $(1, 13, 11, \ldots, n-9, n-12, n, n-3, n-2, n-1, n-4, n-7, n-6, n-5, n-8, n-11, n-10, n-13, \ldots, 14, 15, 12, 10, 8, 9, 6, 7, 4, 5, 2, 3, 1)$.

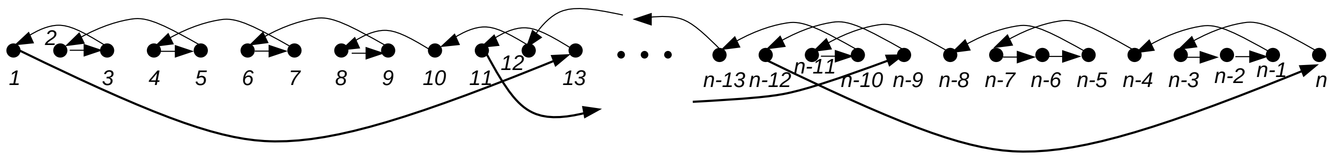

Figure 98: Hamiltonian circuit in $T_n\langle 1,12;2,3\rangle$ when $n \equiv 6 (\mathrm{mod}\, 12)$

Case 8

If $n \equiv 7 (\mathrm{mod}\, 12)$ then we have the following hamiltonian circuit $(1,2,14,12,22,24,\ldots,n-1,n,n-2,n-4,n-3,n-6,n-5,n-8,n-7,n-10,n-12,n-14,\ldots,15,16,13,11,9,10,7,8,9,5,6,3,4,1)$.

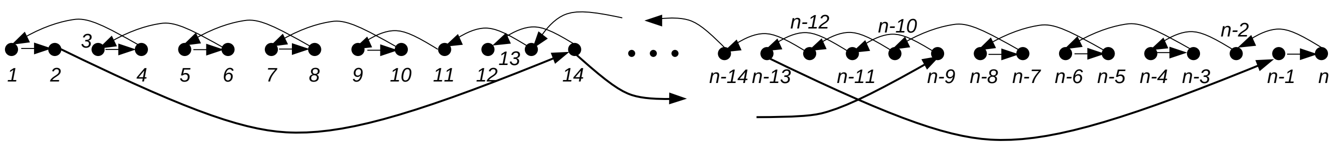

Figure 99: Hamiltonian circuit in $T_n\langle 1,12;2,3\rangle$ when $n \equiv 7 (\mathrm{mod}\, 12)$

Case 9

If $n \equiv 8 (\mathrm{mod}\, 12)$ then we have the following hamiltonian circuit $(1,2,14,12,24,22,\ldots,n-11,n-13,n-1,n,n-2,n-4,n-3,n-6,n-5,n-8,n-7,n-10,n-9,n-12,n-14,\ldots,15,16,13,11,9,10,7,8,5,6,3,4,1)$.

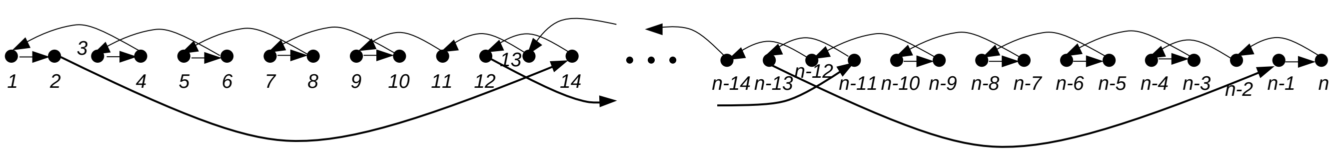

Figure 100: Hamiltonian circuit in $T_n\langle 1,12;2,3\rangle$ when $n \equiv 8 (\mathrm{mod}\, 12)$

Case 10

If $n \equiv 9 \pmod{12}$ then we have the following hamiltonian circuit $(1,2,14,24,22,\ldots,n-10,n-12,n,n-2,n-1,n-4,n-3,n-6,n-5,n-8,n-7,n-9,n-11,n-13,\ldots,15,13,10,11,8,9,6,7,4,5,3,1)$.

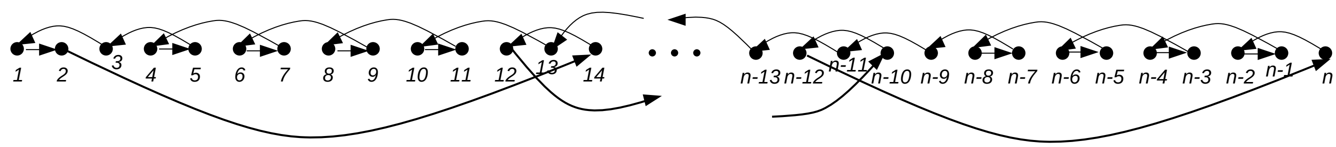

Figure 101: Hamiltonian circuit in $T_n\langle 1,12;2,3\rangle$ when $n \equiv 9 \pmod{12}$

Case 11

If $n \equiv 10 \pmod{12}$ then we have the following hamiltonian circuit $(1,13,10,22,34,\ldots,n,n-2,n-1,n-4,n-3,n-6,n-5,n-8,n-7,n-10,n-9,n-11,n-13,\ldots,14,11,12,9,6,7,8,4,5,2,3,1)$.

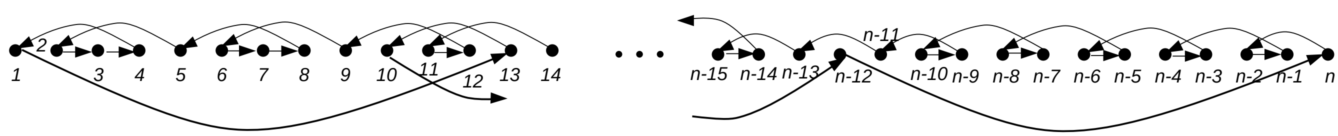

Figure 102: Hamiltonian circuit in $T_n\langle 1,12;2,3\rangle$ when $n \equiv 10 \pmod{12}$

Case 11

If $n \equiv 11 \pmod{12}$ then we have the following hamiltonian circuit $(1,2,14,11,23,\ldots,n-12,n,n-2,n-1,n-4,n-3,n-6,n-5,n-8,n-7,n-10,n-9,n-11,n-14,n-13,\ldots,15,12,13,10,8,9,6,7,4,5,3,1)$.

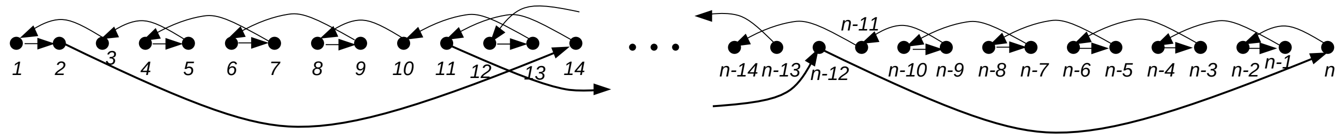

Figure 103: Hamiltonian circuit in $T_n\langle 1,12;2,3\rangle$ when $n \equiv 11 (\bmod 12)$

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Chapter 4

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