New Measures of Intuitionistic Inclusion and Similarity with Applications

By

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DEDICATION

To my Mother and Daughter

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ABSTRACT

New Measures of Intuitionistic Inclusions and Similarity with Applications

Similarity and inclusion are two most important conceptual ways for looking at any possible relationships between objects (sets). A similarity measure is used for estimating the degree of resemblance/similarity between two objects while an inclusion measure expresses the degree to which one of the two is covered by the other one.

In this dissertation, we have presented two different yet equally important approaches of defining similarity and inclusion measures for intuitionistic fuzzy sets. The first approach can be regarded as a logical approach implanting intuitionistic fuzzy implication, bi-implications and t-norms while the second one is a purely set theoretic approach having cardinalities and sets operations involved in its construction.

In the logical approach, the degree of inclusion/ similarity was obtained by composing the intuitionistic logical operators (implications and bi-implications) with fuzzy measures on intuitionistic fuzzy sets. For this purpose we initially defined some new normal fuzzy measures on intuitionistic fuzzy sets along with a class of scalar cardinality measure for intuitionistic fuzzy sets. Later, we introduced different classes of intuitionistic fuzzy bi-implication operators having axiomatic as well as constructive approaches. A study on the properties of these bi-implication operators by utilizing Lukasiewicz intuitionistic fuzzy implicator revealed some remarkable results. The intuitionistic fuzzy bi-implication operators along with new defined fuzzy measures gave rise to multiple classes of intuitionistic fuzzy bi-implicator based similarity measures for intuitionistic fuzzy sets. Also the same normal fuzzy measures were employed to obtain the degree of inclusion between two intuitionistic fuzzy sets when composed with intuitionistic fuzzy implication operators. The new implication based classes of inclusion and similarity measures fulfilled almost all of the universally accepted criteria's.

In the set theoretic approach, we employed one of the members of new introduced scalar cardinality of intuitionistic fuzzy sets to construct a four parametric family of cardinality based similarity and inclusion measures. Both of these parametric families,

for different combinations of parameters, generated intuitionistic fuzzy versions of some of the famous crisp measures of time.

Lastly, the utility of the new measures (similarity, inclusion, cardinality and others) of intuitionistic fuzzy sets introduced in this work is exhibited by utilizing them as a part of solution techniques to the problems arising in real life situations.

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LIST OF ABBREVIATIONS

FS	Fuzzy Set
IFS	Intuitionistic Fuzzy Set
IVFS	Interval Valued Fuzzy Set
VS	Vague Set
GS	Grey Set
$F_L S$	L-fuzzy Set
IBI	Intuitionistic Fuzzy Bi-Implicator
IFBSM	Intuitionistic Fuzzy Bi-Implicator based Similarity Measure
IFβSM	Intuitionistic Fuzzy β −Bi-Implicator based Similarity Measure
IFIM	Intuitionistic Fuzzy Inclusion Measure
IIFIM	Implication based Intuitionistic Fuzzy Inclusion Measure
MCDM	Multi-Criteria Decision Making
TOPIIS	Technique to Order Preferences by Inclusion of Ideal Solution

Chapter 1

Introduction

According to Zadeh, the ultimate goal of fuzzy set (FS) was to provide the best possible explanation of how a human mind acquires and manipulates a piece of information. It was soon realized that the fuzzy concepts existed and were extensively used by a human brain, which can processes many hedges such as: good, very good, long, tall, very tall, brilliant, more brilliant and many more alike, rather conveniently than the numbers. Thus, one of the most important fact of human thinking is its capability of summarizing an information into a FS that possesses an approximation relation to the primary data. In a FS, a membership function allocates to each element of the universe of discourse X a value from the unit interval [0,1], specifying its level/degree of belongingness to that set. A non-belongingness of an element in a FS is naturally defined as the complement of its belongingness value which is 1 minus the degree of membership. This relationship of an element's membership degree and a non membership degree to FS was argued by many who considered it to be too restrictive for a human mind that can change its decisions under many influences. Moreover, there is a fair chance that a human being who expresses the level of membership of an element in a FS may not consider its non membership to be the complement of its membership. This psychological fact that the linguistic negation not always identifies with logical negation led to different generalizations of fuzzy sets (FS's) namely, Interval valued fuzzy sets (IVFS's) , Vague sets (VS's) , Grey sets (GS's) and Intuitionistic fuzzy sets (IFS's).

The interval valued fuzzy set was introduced by Zadeh [172] himself while Gau [70] proposed the notion of Vague set. The intuitionistic fuzzy set was introduced by Atanassov [7], enriched the fuzzy set with a notion of indeterminacy expressing hesitation or abstention. An intuitionistic fuzzy set is specified by two degrees of freedoms, namely, the membership degree and a non membership degree such that the sum of two degrees should remain less than 1. Among all the other higher order fuzzy sets, the intuitionistic fuzzy sets and intuitionistic fuzzy logic have motivated the research interest of scientific community for the past few decades. They have been successfully applied to scenarios where the description of a problem by a linguistic variable defined in terms of a membership function only, seemed too rough. Such problems appear when we come across with the human opinions involving two or more answers such as "Yes", "No", "I don't know", "I am not sure" etc., indicating an inadequate knowledge or lack of a background information. Voting is a reasonable example of such a situation, as voters may be classified into following three categories:

(1). Those who vote in favour;

(2). Those who vote against;

(3). Those who abstain or give invalid votes.

The third group of voters is of great interest since they exhibit the true ambiguities of real life situation that is very well represented by an IFS. Thus, an intuitionistic fuzzy set theory (IFS-theory) enjoys a special interest which is evidenced by the recent books [11, 13]. Moreover a study on inter-relationships between different higher order sets revealed a close connection between IFS and other generalizations of FS's. For instance, in [31], it was proved that a vague set is an intuitionistic fuzzy set. Similarly, a strong connection between intuitionistic fuzzy set and interval valued set was claimed to exist in [61, 154].

The emergence of all different types of higher order fuzzy sets tended to a significant trend in mathematics, a trend towards generalizations of existing mathematical concepts and theories. Each generalization not only enriched our insight but also developed our abilities to model the intricacies of the real world in the best possible way. The search for the most appropriate generalizations of logical operators as well as measures on intuitionistic fuzzy sets can be regarded as one of the core area of interest for those who have been following this trend.

Measure on a set is one of the most fundamental concepts of mathematics. In classical measure theory, a measure on an ordinary set (crip set) is characterized by its additivity property. This characterization is very effective and convenient, but in many real life situations have proved to be too rigid and thus unrealistic. As a solution to this rigidness, the fuzzy measure was proposed. A fuzzy measure is a generalization of the classical measure which is obtained by replacing the additivity axiom of a classical measure with a weaker axiom of monotonicity related to the inclusion of sets.

By the origin of fuzzy measure theory, fuzzy measures like similarity, inclusion, entropy, distance, compatibility, cardinality and many more have been proposed and applied in different fields such as: multicriteria decision making, decision making under uncertainty, risk analysis, game theory, welfare theory and combinatorics etc [74]. Moreover, the concept of fuzzy measure has not been restricted to the fuzzy sets only. Many different forms of measures have been developed and generalized to other higher order fuzzy sets. For instance, in case of intuitionistic fuzzy set, the concept of measure has evolved into two directions namely, fuzzy measures on intuitionistic fuzzy sets and intuitionistic fuzzy measures on intuitionistic fuzzy sets. Both of these measures are defined on sigma algebra of intuitionistic fuzzy sets. A detailed study on this area is made in [18].

Although, wider applicability and mathematical rigor of this area has developed a substantial scientific literature, yet the search for the most appropriate measure remained open as most of these measures were developed according to the context of their applications. In this thesis, we have firstly introduced in chapter 3, titled as 'New Measures on Intuitionistic Fuzzy Sets', some new normal fuzzy measures on intuitionistic fuzzy sets. Also, under the same heading a class of a new scalar cardinality measure on intuitionistic fuzzy set called the weighted average cardinality measure is proposed and studied for its properties. The cardinality and the other fuzzy measures introduced in this chapter are extensively used throughout the dissertation in accomplishing the core target of this research; i.e., the development of new classes of intuitionistic fuzzy similarity and inclusion measures between intuitionistic fuzzy sets.

Measuring similarity and inclusion between objects have been two basic concepts in human cognition. Similarity plays a vital role in our daily life. It defines an organizing principle for the classification of objects, formation of human concepts and generalization of ideas. Numerous definitions of similarity measure have already been proposed in literature as according to Zadeh, "Formulation of a valid, general purpose definition of similarity is a challenging problem". However, a general understanding of its ordinal interpretation is: greater the value, the more similar the objects are. In fuzzy literature, the similarity measures are classified into three categories:

(a) Set theoretic based measures;

(b) Metric based measures;

(c) Implication based measures.

The first two approaches of fuzzy similarity measures root back to the technique which was originally adopted and applied in the field of numerical taxonomy for developing suitable measures (similarity and inclusion) between ordinary sets. It was in fact the simplest way of comparing two objects. Based on an appropriate set of features typical for those two objects, a binary vector for each of the two objects was constructed which encoded the presence (1) or absence (0) of each of the features. Any comparison (similarity or inclusion) between these feature vectors was simply equal to comparing the objects themselves. The degree of similarity or inclusion of two objects is often expressed in terms of the cardinalities of the later sets. These measures used set theoretic operations and they were called set theoretic based measures. In case of fuzzy set theory, the cardinality based set theoretic fuzzy measures (similarity or inclusion) used the feature vectors scaled to real unit vector [0,1] as the representatives of the fuzzy sets involved. The fuzzy similarity measures developed by this technique originated as genuine extensions of some of the most popular similarity measures of ordinary sets like Jaccard coefficient [30, 39, 85, 118], the Simple Matching coefficient [30, 87, 136], the Rogers and Tanimoto coefficient [130], the Sokal & Sneath coefficient [87, 137], the Dice coefficient [39, 64, 118], the cosine coefficient [113, 115], Restle's, Gregson's and Enta's Model [179] and most importantly the Tversky's contrast model [30, 131, 146, 151].

The distance (metric) based fuzzy similarity measures also utilized the representative vectors for the fuzzy sets. In the first step the representative vectors for the fuzzy sets were used to obtain a distance measure for the fuzzy sets and in the next step one of the relationship between similarity and distance was utilized to define a degree of similarity between the fuzzy sets [94, 131, 158].

The last approach of measuring a similarity between fuzzy sets was introduced in [155]. They used fuzzy logical operators such as bi-implicators to formulate the similarity relations between objects (sets). It was extensively studied by Beg [25, 26], who experimented in its structure with different fuzzy measures, implicators and t-norms.

The bi-implication/bi-implicator operator is an important concept of fuzzy logic which is not only related in defining a similarity relation between objects but has also been utilized in introducing new algebras along with the interpretation of fuzzy logical phenomenons. A bi-implication operation has appeared with many different names in fuzzy literature [32, 77, 108, 111], however, due to the structural constraints of the underlying lattice, not much of it has been extended to intuitionistic fuzzy environment [109]. Thus, any new construction of a similarity relation between sets based on bi-implicator formally requires, firstly, development of a new bi-implicator operator.

Working on these lines, the Chapter 4 of this dissertation is reserved for an intuitionistic fuzzy version of the classical bi-implication, that is, an extension of classical bi-implication to the canonical domain of mathematical intuitionistic fuzzy logics, the intuitionistic fuzzy lattice L^*. Our approach to intuitionistic fuzzy bi-implications may be summarized as follows: Firstly, we present intuitionistic fuzzy extensions of a well known approach to bi-implication by Fodor-Roubens via a direct axiomatization of the properties of the corresponding class of operators supported by an evident example. Next, we turn towards an investigation of a particular defining standard of classical bi-implication $(p \implies q \wedge q \implies p)$ in terms of intuitionistic fuzzy t-norms and intuitionistic fuzzy R-implicators, and we proposed three different classes of intuitionistic fuzzy bi-implications having constructive approach.

Later, the axiomatic definition proposed earlier is further utilized to classify these new constructive classes of intuitionistic fuzzy bi-implicators depending upon their level of obedience to the presented set of axioms. It was found that some of the constructive classes completely obeyed while other slightly disobeyed the given set of axioms. We believe that such a classification of intuitionistic fuzzy bi-implicator operators can lay a background for constructing new intuitionistic fuzzy logical phenomenons. Moreover, in order to have a deep insight into the structural characteristic of one of the classes, we have fixed and chosen a specific combination of the intuitionistic fuzzy connectives involved in its definition, which disclosed its properties closer to those proposed by Sinha for fuzzy inclusion. Thus, we provide a suitable framework towards defining any similarity relation between intuitionistic fuzzy sets.

The researchers working on intuitionistic fuzzy set theory have tried to capture the essence of most of the approaches of defining a similarity measure of fuzzy sets. For instance in [58, 81, 82, 140, 143, 156, 164], the authors have defined different expressions for distances and distance based similarity measures for intuitionistic fuzzy sets along with methods to demonstrate their applications in medical diagnosis. A full guide on these measures is provided in [117]. Likewise, in [83], we find some extensions of set theoretic based similarity measures of crisp set proposed in [51] to their intuitionistic version. Moreover, in [166] a generalization of Ochiai/Cosine coefficient to IFS is attempted . An other possibility to compute the similarity between two intuitionistic fuzzy sets was proposed in [80] based on J-divergence. Lie et al [101] made comparative analysis for similarity measures between VS's/IFS's and found inadequate conditions for similarity measures. These measures were said to be inefficient although they had involved in their construction both membership and a non-membership degrees. As a remedy to the problem, the degree of indeterminacy was introduced to these similarity measures between VS's/IFS's [139, 165, 174]. Zhang [175] introduced some new similarity measures for IFS's, fuzzy rough sets and rough fuzzy sets. He also worked with the degree of indeterminacy in the similarity measure for IFS's in order to improve distinguished precision. Numerical examples given in [175] showed the effectiveness of proposed method however, in certain cases, this method failed and needed modifications for acceptable results. Such modifications were made by [169] in order to obtain the best results in accordance to the usage of degree of indeterminacy in a measure. Recent papers have brought attention of researchers to the new results on intuitionistic fuzzy similarity measures [15, 27, 40, 41, 42, 57, 106, 110].

Although, many different ideas of defining similarity measures between IFS have been presented, yet a discussion and a comparison between these measures reveals that there is not a single similarity measure existing so far which is capable of recognizing every pattern provided to it. Moreover, a historical preview of these measures unveils that in contrast to similarity measures for fuzzy sets, the similarity measures for IFS's are mostly based on distances and rarely on set theoretic approach. However, the concept of similarity measure between any two intuitionistic fuzzy sets based on intuitionistic fuzzy bi-implications is not introduced so far. Therefore, in Chapter 5 of this thesis we have attempted to define two different classes/ families of similarity measures for intuitionistic fuzzy sets which are structured by utilizing bi-implications and set theoretic operations. The first class of similarity measures called the intuitionistic fuzzy bi-implicator based similarity measures and are constructed by composing the new defined fuzzy measures in Chapter 3 with the different classes of intuitionistic fuzzy bi-implicators presented in Chapter 4. The proofs of the properties of this new similarity measure between IFS's utilizes the set of properties of intuitionistic fuzzy bi-implication operators explored in Chapter 4. Moreover, a particular choice of intuitionistic fuzzy connectives used in its definition uncovers the fact that this new similarity measure defines a fuzzy equivalence on set of all intuitionistic fuzzy sets on X ($IFS(X)$). The second family of similarity measures for IFS's is based on the set theoretic approach, having four parameters involved in its definition. A different combination of these parameters generates a variety of reasonable similarity measures suitable in any intuitionistic fuzzy environment.

Similarity, however, is only one of the conceptual ways for looking at a possible relationship between the objects. It is natural to ask for a measure expressing the degree to which one of the two given objects is contained in another, or in other words, the degree to which one of the two is covered by the other. In an attempt to answer this question, one is naturally drawn to the concept of inclusion measure. The search for a chemical compound, the query about a data base or even an understanding of how well a phylogenetic tree is included in a tree of life are few of the many applications of measuring inclusion between objects.

Fuzzy inclusion/Fuzzy subsethood is considered as one of the major concepts of fuzzy set theory. The foremost attempt to define a fuzzy subsethood was made by Zadeh [173]. For two fuzzy sets A and B, he defined a fuzzy subset hood as: $A \subseteq B \Leftrightarrow (\forall x \in X,\ A(x) \leq B(x))$. Graphical interpretation of this definition is stated as: a FS 'A' is a subset of another FS 'B', if the graph of A never lies above the graph of B. Many researchers tried to relax the rigidity of Zadeh's definition of inclusion to perceive a softer approach that could have a better compatibility with the spirit of FS-theory. Thus, instead of giving crisp decisions about subsethood of sets they proposed that since they are dealing with fuzzy sets that assigns a graded membership to their elements therefore, the subsethood/inclusion between them should also be allowed a graded assignment of somewhat being a subset of other.

A typical approach adopted by the majority of the researchers to define a fuzzy subsethood was the fuzzification of the three distinct yet, equivalent definitions of crisp subsethood stated as:

$$A \subseteq B \Leftrightarrow (\forall x \in X)(x \in A \Rightarrow x \in B) \tag{1.1}$$

$$\Leftrightarrow A = \phi \text{ or } \frac{|A \cap B|}{|A|} = 1 \tag{1.2}$$

$$\Leftrightarrow \frac{|A^c \cup B|}{|X|} = 1 \tag{1.3}$$

The first one is purely logical in terms while the other two are cardinality based and therefore it has a probabilistic touch in them. Unlike their crisp version, it was observed that the new fuzzified version of these definitions ceased to be equivalent. This observation has led to a research of defining an appropriate fuzzy subsethood to the following two different tracks:

(a) logical based approach;

(b) frequency based approach.

One can construct various inclusion indicators using different tools and approaches, however, we could impose some elementary conditions on this indicator in order to come up with a genuine implementation of inclusion. The first contribution in this direction was made by Kitainik [91] in 1987. Moreover, the two more axiom systems that seem to prevail notably in literature are the ones proposed by Young [168] and Sinha [134]. Both of these axiom systems required a fulfillment of the reflexive property by an inclusion indicator $Inc(A, B)$ stated as: $A \subseteq B$ implies $Inc(A, B) = 1$. The major difference between these systems lie on how each of them treated inclusion of crisp sets. While Sinha and Dougherty [134] demanded that $Inc(A, B) \in \{0, 1\}$ for crisp sets A and B, Young [168] did not impose it.

In case of intuitionistic fuzzy sets, an inclusion measure is defined to be a pair wise relation between two IFS's that indicates the degree to which one IFS is contained in another and represents a generalized inclusion relation. The first and direct generalization of an inclusion relation between intuitionistic fuzzy sets may be defined as:

$$\begin{aligned} A &\subseteq B \Leftrightarrow (\forall x \in X,\ A(x) \leq_{L^*} B(x)) \\ &\Leftrightarrow (\forall x \in X,\ \mu_A(x) \leq \mu_B(x) \text{ and } \nu_A(x) \geq \nu_B(x)) \end{aligned}$$

Some authors have investigated inclusion measures of IFS's. Cornelis and Kerre [49] introduced the intuitionistic-valued inclusion measures of IFS's. Grzgorzewski [75] proposed the axioms characterizing the inclusion measures (subsethood measure) of IFS, which are similar to those given by Young [168] about fuzzy sets. Park [120], defined a kind of sophisticated inclusion measure of IFS and applied it to interval valued fuzzy topological spaces. Xie [163], proposed couple of cardinality (frequency) based inclusion measures for IFS along with some other measures. A similar effort is made in [176, 177], while [90] gave a generalized vision of an IFS inclusion measure by defining L-valued inclusion indicators for all types of higher order fuzzy set.

One of the major targets of this research thesis is the development of new ideas for defining reasonable inclusion measure (the one satisfying most of the axioms of inclusion) of intuitionistic fuzzy sets and which could be successfully and economically applied in real life situations. Therefore, in contrast to an intuitionistic valued measure we believe that a fuzzy valued measure of inclusion for intuitionistic fuzzy set would serve the purpose well. Thus, in Chapter 6, of this thesis we have attempted to define two different classes of fuzzy measures of intuitionistic fuzzy inclusions. The first class of inclusion measures introduced in this chapter can be regarded as an effort put forward in defining a direct and most appropriate generalization of (1.1). The basic working technique behind such a generalization is a two step method: In step one, we formulate the concept of a local intuitionistic fuzzy inclusion indicator $IInc$ as an $IFS(X) \times IFS(X) \longrightarrow IFS(X)$ mapping which allocates to every pair of $A, B \in IFS(X)$ an intuitionistic fuzzy set of inclusion $IInc(A, B)$, based on intuitionistic fuzzy implication operator. The set $IInc(A, B)$ provides, an information about a local degree of inclusion, as well as, a non inclusion of every element of A into B. In the next step we obtain a global evaluation of IFS A into B by composing the operator $IInc(A, B)$ with the new defined normal fuzzy measures of intuitionistic fuzzy sets formally introduced in Chapter 3. This will result into a mapping $m_{IInc} : IFS(X) \times IFS(X) \longrightarrow [0, 1]$ which will assign a global (single) inclusion degree of one IFS into other. The properties of the new fuzzy inclusion indicator m_{IInc} depends upon the properties of intuitionistic fuzzy implications implanted in $IInc(A, B)$. Since a lot of variations is possible in the properties of implications, instead of working with general implication, it seems appropriate to particularize the implication in use. Therefore, in this thesis we have restricted our selves within the frame work of intuitionistic fuzzy Lukasiewicz implicator. The study of the properties of this new fuzzy inclusion indicator m_{IInc} shows that it enjoys the fulfillment of most of the set of axioms as stated by Sinha and Young [134, 168]. Our second approach, of defining an inclusion indicator of intuitionistic fuzzy sets, captures the essence of inclusion by taking into consideration the other two definitions (1.2,1.3) of subsethood for ordinary sets. We have called this the weighted average cardinality based parametric family of intuitionistic fuzzy inclusion measure. This family also generates multiple fuzzy inclusion indicators for two intuitionistic fuzzy sets. A study on the properties of different members of this family indicates their obedience to the set of criterias as set by Young [168].

In Chapter 7, the fuzzy measures on IFS's introduced in Chapter 3,5, and 6 are applied to real life situations, such as, the diagnosis of a disease in a patient or the selection of the best employee in an organization. Various techniques in this regard are constructed and demonstrated there.

Lastly, taking into consideration the close relation between IFS's and the other generalized fuzzy sets such as IVFS's and VS's, we are in a position to claim that, all the results on intuitionistic fuzzy set theory and logic produced in this thesis can be easily modified and adapted to the extended frame works of any of the mentioned higher order fuzzy sets.

Before we proceed, let me provide to the reader an outline of this dissertation.

Outline

This thesis targets to develop different classes of intuitionistic fuzzy similarity and inclusion measures between intuitionistic fuzzy sets and studies their application in real life situations.

- Chapter 2 gives preliminaries for intuitionistic fuzzy set theory and logical operators, t-norms, conorms, negators, implicators etc.
- Chapter 3 presents different new normal fuzzy measures on intuitionistic fuzzy sets along with a new class of scalar cardinality measure for intuitionistic fuzzy set
- Chapter 4 introduces multiple new classes of intuitionistic fuzzy bi-implicator operators having axiomatic as well as constructive approaches. It also investigates the relation between these classes.
- Chapter 5 works on constructing new classes of intuitionistic fuzzy bi-implicator based similarity measures that utilize intuitionistic fuzzy bi-implicator operators and measures introduced in previous two chapters along with a new cardinality based parametric family of similarity measure.
- Chapter 6 targets towards proposing diverse classes of intuitionistic fuzzy inclusion measures based on intuitionistic fuzzy implications and fuzzy measures on intuitionistic fuzzy sets. Moreover, it also presents a new cardinality based parametric family of inclusion measure for intuitionistic fuzzy sets.
- Chapter 7 exhibits the utility of the measures introduced in this thesis in the fields of medical diagnosis and organizational management.
- The last chapter concludes the research conducted in this thesis and provides an indication for a future research in these directions.

The List of Research Papers Derived From Thesis

This thesis is based upon the following published, accepted and submitted papers:

- Measure of intuitionistic fuzzy inclusion. Comptes Rendus Del' Academie Bulgare Des Sciences, 69(8), 971-980.

- The intuitionistic fuzzy multicriteria decision making based on inclusion degree. Comptes Rendus Del' Academie Bulgare Des Sciences, 70(7), 925-934.

- Intuitionistic fuzzy bi-implicator and properties of Lukasiewicz intuitionistic fuzzy bi-implicator, accepted for publication in TWMS Journal of Applied and Engineering Mathematics.

- Weighted average cardinality measure of IFS and its application, submitted.

- A novel intuitionistic fuzzy bi-implicator based similarity measure and its application to student evaluation, submitted.

- A weighted average cardinality based parametric family of intuitionistic fuzzy inclusion measures, submitted.

- Intuitionistic Jaccard similarity measure and its application to Medical Diagnosis, submitted.

Chapter 2

Preliminaries

Several mathematical models have been presented in history to dissolve uncertainties in various types of systems. The inception of Fuzzy Logic or Fuzzy Set Theory is one such attempt which is based on the extension of ordinary set theory. In 1965, Zadeh [173] introduced the notion of fuzzy sets as a method for representing uncertainties. Then, the theory of fuzzy sets has become a vigorous area of research in different disciplines including logic, topology, algebra, analysis, information theory, artificial intelligence, operations research, neural networks and planning etc.

The intuitionistic fuzzy sets (IFS's) and interval valued fuzzy sets (IVFS's) appeared independently as appropriate generalizations of fuzzy sets. The IVFS's reflected the ambiguous situations unanswered by fuzzy sets in the form of closed interval membership function $[\mu_1, \mu_2]$ such that $\mu_1, \mu_2 \in [0, 1]$ and $\mu_1 \leq \mu_2$. The intuitionistic fuzzy sets however, are equipped with a nonmembership degree ν along with the membership degree μ such that $\mu, \nu \in [0, 1]$ and $\mu + \nu \leq 1$. Though the equivalence of these two approaches has been addressed in [61], but each of these generalizations have successfully generated an extensive literature covering multiple aspects of their applications and the possible extensions of fuzzy logical operators along with the set theoretic concepts [6, 10, 16, 45, 46, 60, 61, 62, 102].

For the past two decades, fuzzy connectives such as fuzzy triangular norm, conorms, negators and implicators have been extensively employed within the research focused on fuzzy sets and fuzzy logic. The t-norms, conorms and negators have played a prominent role in defining a generalized intersection, union and complementation of fuzzy sets while fuzzy implicators have proved their usefulness in a rule based (expert) systems. These connectives have served many different purposes like defining measures such as measure of inclusion and the measure of guaranteed possibility and necessity. Also, they have been utilized for characterizing fuzzy rough sets and linguistic modifiers and for extending the composition of fuzzy relations. Not only this but also they have been implanted in preference modelling, aggregation, decision making, pattern classification, fuzzy logic programming and for many other purposes.

With the interest in intuitionistic fuzzy relational calculus in particular and intuitionistic fuzzy set theory in general, the scientific communities have invested time and efforts in expanding these connectives to an intuitionistic fuzzy framework so that these connectives, just as their fuzzy counterpart, could be formerly investigated for their effectiveness in different applications and constructions.

In this chapter, we have presented the necessary concepts of intuitionistic fuzzy set theory along with an overview of fuzzy connectives and their possible intuitionistic fuzzy extensions that are extensively used throughout this research work.

However, before we proceed to relevant definitions, let us familiarize our reader with important notations associated to certain general terminologies particularly specified for this dissertation only given in Table 2.1:

Notation	Terminologies
T	Fuzzy t-norm
S	Fuzzy t-conorm
$T^{'}$	Fuzzy t-norm different from T
$S^{'}$	Fuzzy t-conorm different from S
N	Fuzzy negator/negation
I	Fuzzy implicator/implication
$\check{T}$	Intuitionistic Fuzzy t-norm
$\check{S}$	Intuitionistic Fuzzy t-conorm
$\check{T}'$	Intuitionistic Fuzzy t-norm different from $\check{T}$
$\check{S}'$	Intuitionistic Fuzzy t-conorm different from $\check{S}$
$\check{N}$	Intuitionistic Fuzzy negator/negation
$\check{I}$	Intuitionistic Fuzzy implicator/implication
$\check{T}^*$	Dual of Intuitionistic Fuzzy t-norm $\check{T}$
$\check{S}^*$	Dual of Intuitionistic Fuzzy t-conorm $\check{S}$
$\overline{T}$	t-representable Intuitionistic Fuzzy t-norm
$\overline{S}$	t-representable Intuitionistic Fuzzy t-conorm
$\widetilde{T}$	Non t-representable Intuitionistic Fuzzy t-norm
$\widetilde{S}$	Non t-representable Intuitionistic Fuzzy t-conorm

Table 2.1:General terminologies and notations

2.1 A Review of Fuzzy Logic

The study on fuzzy logic is classified into two main branches, namely, *The fuzzy logic in the narrow sense* and *The fuzzy logic in the broad sense* [77, 112].

Constructed in the spirit of classical binary logic, the first branch (*The fuzzy logic in the narrow sense*) is formed by many valued logic [129] and is more or less taken as a symbolic logic concerned with syntax, semantics, axiomatization, soundness, completeness, etc [72, 77].

The second branch (*The fuzzy logic in the broad sense*) is usually perceived as an extension of the fuzzy logic in the narrow sense as it taken to be a way of interpreting natural language to model human reasoning [112].

One of the main focuses of the research in fuzzy logic (both in the narrow sense and in the broad sense) was an extension of the classical binary logical operators *negation* (denoted as $\neg$), *conjunction* (denoted as $\wedge$), *disjunction* (denoted as $\vee$) and *implication* (denoted as $\rightarrow$) to fuzzy logic operators.

Now, before proceeding towards an overview of the fuzzy extensions of above mentioned binary logical connectives, let us formally define the term of fuzzy set, complete lattice and an L-fuzzy set. These definitions are stated in [46, 71, 173].

Throughout this dissertation, we shall use the notation X for the crisp set (universe of discourse) and ϕ for the crisp empty set.

Definition 2.1.1 "A *fuzzy set* (FS) on a universe X is an object of the form $A = \{(x, A(x)) \mid x \in X\}$, where the function $A(x) \in [0, 1]$ defines the degree of membership of x in the set A. The class of all fuzzy sets on X is denoted by $FS(X)$. A fuzzy set is called finite if universe X if finite, writing it as $X = \{x_1, x_2, ...x_n\}$.

For a fuzzy set $A = \{(x, A(x)) \mid x \in X\}$ we define the *complement* of A in X as $A^c = \{(x, 1 - A(x)) \mid x \in X\}$, the *Support* of A in X as a subset of X given by $Supp(A) = \{x \in X \mid A(x) \neq 0\}$, the *Kernel* of A in X as $Ker(A) = \{x \in X \mid A(x) = 1\}$. Moreover, we denote $1_X = \{(x, 1) \mid x \in X\}$ and $0_X = \{(x, 0) \mid x \in X\}$ the fuzzy sets corresponding to entire universe X and empty set ϕ respectively. Also, for $A \in FS(X)$, we define an $\alpha - cut$ of A as $A^\alpha = \{x \in X \mid A(x) \geq \alpha\}$."

Definition 2.1.2 "A *complete lattice* is a partially ordered set $(L, \leq_L)$ such that every non empty subset of L has a supremum and infimum. A *complete lattice with negator* N is a triplet $(L, \leq_L, N)$, where $(L, \leq_L)$ is a complete lattice and N

is a unary, involutive and order reversing operator on L."

Definition 2.1.3 "An L-*fuzzy set* on a universe X is an $X \longrightarrow L$ mapping, where $(L, \leq_L, N)$ is a complete lattice with negator N. The class of all L-*fuzzy sets* on a universe X is denoted by $F_L(X)$."

2.1.1 Fuzzy Negation, Conjunction and Disjunction

All the definitions, results and examples stated in this section are taken from [92, 133].

In fuzzy logic, the t-norms, t-conorms and the negators defined on complete bounded lattice [0,1] are widely used to model fuzzy conjunction, disjunction and negation operations.

In many-valued logic the classical binary negation/negator is extended to the unit interval as $[0, 1] \longrightarrow [0, 1]$ mapping given as follows:

Definition 2.1.1.1 "A fuzzy *negator* is a decreasing $[0, 1] \longrightarrow [0, 1]$ mapping N that satisfies $N(0) = 1$ and $N(1) = 0$.

Moreover, a fuzzy negator N is said to be strict if N is continuous and strictly decreasing mapping. A fuzzy negator is said to be strong (or at times involutive) if $N(N(x)) = x$, $\forall x \in [0, 1]$."

Remark 2.1.1.2 Notice that a strong/involutive fuzzy negator is strict and is continuous mapping. We now present examples of non-continuous fuzzy negator, a continuous but non strict fuzzy negator, a strict but non strong fuzzy negator and a strong fuzzy negator.

Example 2.1.1.3

(a). "The fuzzy negator N_{1a} :

$$N_{1a}(x) = \left\{ \begin{array}{c} 0 \text{ if } x = 1, \\ 1 \text{ otherwise} \end{array} , x \in [0, 1] \right\}$$

and N_{1b} :

$$N_{1b}(x) = \left\{ \begin{array}{c} 1 \text{ if } x = 0, \\ 0 \text{ otherwise} \end{array} , x \in [0, 1] \right\}$$

are not continuous.

(b). The fuzzy negator N_2 :

$$N_2(x) = \left\{ \begin{array}{c} 1-x \text{ if } x \in [0,0.4], \\ 0.6 \text{ if } x \in [0.4,0.6] \quad , x \in [0,1] \\ -1.5x+1.5 \text{ otherwise} \end{array} \right\}$$

is continuous but not strict.

(c). The fuzzy negator N_3 :

$$N_3(x) = 1 - x^2, \quad x \in [0,1]$$

is strict but not strong.

(d). The fuzzy negator N_s :

$$N_s(x) = 1 - x, \quad x \in [0,1]$$

is strong (involutive). The mapping N_s is called the standard fuzzy negator."

In many-valued logic the classical binary conjunction is extended to the unit interval as a $[0,1]^2 \longrightarrow [0,1]$ mapping called t-norm defined as follows:

Definition 2.1.1.4 "A fuzzy *t-norm* is an increasing, commutative, associative $[0,1]^2 \longrightarrow [0,1]$ mapping T satisfying $T(1,x) = x$ for all $x \in [0,1]$."

Proposition 2.1.1.5 "A t-norm T satisfies $T(x,x) \leq x$, for all $x \in [0,1]$."

Example 2.1.1.6 "The following are the basic t-norms: For $x, y \in [0,1]$,

(i). *Minimum* t-norm (T_M) is given by $T_M(x,y) = \min(x,y)$;

(ii). *Product* t-norm (T_P) is given by $T_P(x,y) = xy$;

(iii). *Lukasiewicz* t-norm (T_L) is given by $T_L(x,y) = \max(x+y-1,0)$;

(iv). *Drastic product* t-norm (T_W) is given by

$$T_W(x,y) = \left\{ \begin{array}{c} \min(x,y), \ if \ \ \max(x,y) = 1 \\ 0, \quad\quad otherwise \end{array} \right\}.$$

With respect to pointwise order T_M and T_W are the largest and the smallest t-norms respectively."

Example 2.1.1.7 "The following are some of the most important parametric families of t-norms:

(i). Family of Frank t-norms:

$$T_\omega^F(x,y) = \left\{ \begin{array}{c} T_M(x,y), \quad \text{if } \omega = 0 \\ T_P(x,y), \quad \text{if } \omega = 1 \\ T_L(x,y), \quad \text{if } \omega = +\infty \\ \log_\omega \left(1 + \frac{(\omega^x-1)(\omega^y-1)}{\omega-1}\right), \quad \text{otherwise} \end{array} \right\}.$$

(ii). Family of Hamacher t-norms:

$$T_\omega^H(x,y) = \frac{xy}{\omega + (1-\omega)(x+y-xy)}, \ \omega \in [0,+\infty[.$$

(iii). Family of Yager t-norms:

$$T_\omega^Y(x,y) = \left\{ \begin{array}{c} T_W(x,y), \quad \text{if } \omega = 0 \\ T_M(x,y), \quad \text{if } \omega = +\infty \\ \max\left(0, 1 - ((1-x)^\omega + (1-y)^\omega)^{\frac{1}{\omega}}\right), \quad \text{otherwise} \end{array} \right\}.$$

(iv). Family of Sugeno t-norms:

$$T_\omega^S(x,y) = \max((x+y-1)(1+\omega) - \omega xy, 0), \ \omega \in]-1,\infty[.\text{"}$$

The disjunction in fuzzy logic is often modelled as follows:

Definition 2.1.1.8 "A fuzzy *t-conorm* is an increasing, commutative, associative $[0,1]^2 \longrightarrow [0,1]$ mapping S satisfying $S(0,x) = x$ for all $x \in [0,1]$."

Proposition 2.1.1.9 "A t-conorm S satisfies $S(x,x) \geq x$, for all $x \in [0,1]$."

Example 2.1.1.10 The following are the basic t-conorms: For $x, y \in [0,1]$,

(i). *Maximum* t-conorm (S_M) is given by $S_M(x,y) = \max(x,y)$;

(ii). *Probabilistic sum* t-conorm (S_P) is given by $S_P(x,y) = x + y - xy$;

(iii). *Lukasiewicz* t-conorm (S_L) is given by $S_L(x,y) = \min(x+y, 1)$;

(iv). *Drastic sum* t-conorm (S_W) is given by:

$$S_W(x,y) = \left\{ \begin{array}{c} \max(x,y), \ if \ \min(x,y) = 0 \\ 1, \quad otherwise \end{array} \right\}.$$

With respect to pointwise order S_M and S_W are the smallest and the largest t-conorms respectively.

Definition 2.1.1.11 "The dual of t-norm $T(t-conorm\ S)$ w.r.t. a negator N is a mapping T^*[respectively S^*] defined by, for $x, y \in [0,1]$,

$$\begin{aligned} T^*(x,y) &= N(T(N(x),N(y))) = 1 - T(1-x, 1-y) \\ \text{[respectively } S^*(x,y) &= N(S(N(x),N(y))) = 1 - S(1-x, 1-y)] \end{aligned}$$

Where T^*[respectively S^*] is a t-conorm (respectively t-norm) on [0,1]."

We denote by (T, S) the pair formed by t-norm T and its dual S. Obviously, $(T_M, S_M), (T_P, S_P), (T_L, S_L)$ and (T_W, S_W) are pairs of t-norms and t-conorms which are mutually dual to each other.

Example 2.1.1.12 "The following are some of the most important parametric families of t-conorms:

(i). Family of Frank t-conorms:

$$S_\omega^F(x,y) = \left\{ \begin{array}{c} S_M(x,y), \quad \text{if } \omega = 0 \\ S_P(x,y), \quad \text{if } \omega = 1 \\ S_L(x,y), \quad \text{if } \omega = +\infty \\ 1 - \log_\omega \left(1 + \frac{(\omega^{1-x}-1)(\omega^{1-y}-1)}{\omega - 1}\right), \quad \text{otherwise} \end{array} \right\}.$$

It is easy to verify that S_ω^F is the dual of T_ω^F for every $\omega \in [0, +\infty[$;

(ii). Family of Hamacher t-conorms:

$$S_\omega^H(x,y) = \frac{x + y + (\omega - 2)xy}{1 + (\omega - 1)xy}, \ \omega \in [0, +\infty[.$$

(iii). Family of Yager t-norms:

$$S_\omega^Y(x,y) = \left\{ \begin{array}{c} S_W(x,y), \quad \text{if } \omega = 0 \\ S_M(x,y), \quad \text{if } \omega = +\infty \\ \min\left(1, (x^\omega + y^\omega)^{\frac{1}{\omega}}\right), \quad \text{otherwise} \end{array} \right\}.$$

(iv). Family of Sugeno t-norms:

$$S_\omega^S(x,y) = \min(x + y - \omega xy, 1), \ \omega \in]-1, \infty[.$$

Clearly, S_ω^H is the dual of T_ω^H, S_ω^Y is the dual of T_ω^Y and S_ω^S is the dual of T_ω^S for every $\omega \in [0, +\infty[$."

Theorem 2.1.1.13 "Let $(T_\alpha)_{\alpha\in\Lambda}$ be a family of t-norms and $(]a_\alpha, b_\alpha[)_{\alpha\in\Lambda}$ be a family of pair wise disjoint open subintervals of [0,1]. The function $T : [0,1]^2 \to [0,1]$ defined by:

$$T(x,y) = \left\{ \begin{array}{c} a_\alpha + (b_\alpha - a_\alpha)T_\alpha(\frac{x-a_\alpha}{b_\alpha-a_\alpha}, \frac{y-a_\alpha}{b_\alpha-a_\alpha}) \text{ if } (x,y) \in [a_\alpha, b_\alpha]^2 \\ \min(x,y), \quad \text{otherwise} \end{array} \right\}$$

is a t-norm called the ordinal sum of the summands $(a_\alpha, b_\alpha, T_\alpha)_{\alpha\in\Lambda}$."

Theorem 2.1.1.14 "The family of Frank t-norms T_ω^F and the family of Frank t-conorms S_ω^F together with ordinal sums of these are the unique t-norms and t-conorms which satisfy the relation:

$$T_\omega^F(x,y) + S_\omega^F(x,y) = x + y$$

for every $x, y \in [0,1]$."

2.1.2 Fuzzy Implication

The implication/implicator operator in the classical binary logic worked on two truth values 0 and 1 while a fuzzy implication is a $[0,1]^2 \longrightarrow [0,1]$. In early literature, different authors have proposed several individual definitions of fuzzy implicators. Among others individual definitions, Trillas et al. [147, 148] proposed two classes of fuzzy implicators (*The Strong implicators* also called the $S-implicators$ and *The Residuated implicators*, commonly known as $R-implicators$) that were generated from the three basic fuzzy logic connectives namely, negation, conjunctions and disjunction. The S-implicators were based on classical binary logical statement $p \to q = \neg p \vee q$ where p and q are two propositions while the R-implicators were defined on the basis of the fact that an implication in classical binary logic is, residuated with an '*and*' in it. Besides these two classes of fuzzy implicators, there was another class of fuzzy implicators called the Quantum Logic implicators ($QL-implicators$), which originated from quantum logic and involved the fuzzy logical operators negation, conjunction and disjunction in their definition.

All the definitions, results and examples stated in this section are taken from [133, 135].

Definition 2.1.2.1 "A fuzzy *implicator* is an $[0,1]^2 \longrightarrow [0,1]$ mapping I satisfying $I(0,0) = 1, I(0,1) = 1, I(1,0) = 0, I(1,1) = 1$. Moreover, we require I to be decreasing in its first and increasing in its second component."

Remark 2.1.2.2 (Axioms of Smetz and Margrez for a fuzzy implicator)

In [135], Smetz and Margrez proposed a set of six axiom scheme for implicators on [0,1]; by driving a number of tautologies from classical logic and translating them into four algebraic axioms along with the two additional conditions of monotonicity and continuity imposed on fuzzy implicators. This set of axioms is given as follows:

(a.1). $(\forall y \in [0,1])(I(.,y)$ is decreasing)
$(\forall x \in [0,1])(I(x,.)$ is increasing);
(monotonicity law)

(a.2). $(\forall x \in [0,1])(I(1,x) = x)$;
(neutrality principle)

(a.3). $(\forall x, y \in [0,1]))(I(x,y) = I(N(x), N(y)))$;
(contrapositivity w.r.t. a fuzzy negator N)

(a.4). $(\forall x, y, z \in [0,1])(I(x, I(y,z)) = I(y, I(x,z)))$;
(interchageability principle)

(a.5). $(\forall x, y \in [0,1])(\ x \leq y \Longleftrightarrow I(x,y) = 1)$;
(confinement principle)

(a.6). I is continuous.
(continuity)

Definition 2.1.2.3 "Let S be a t-conorm and N a negator on $[0,1]$ The S-*implicator* generated by S and N is the mapping $I_{S,N} : [0,1]^2 \longrightarrow [0,1]$ defined as follows: for all $x, y \in [0,1]$

$$I_{S,N}(x,y) = S(N(x), y).\text{''}$$

Remark 2.1.2.4 Here N is supposed to be a strong negation [33], however it has been shown in [17, 92] that N is not necessary supposed to be strong, not even continuous.

Definition 2.1.2.5 "Let T be a t-norm on $[0,1]$. The *R-implicator* generated by T is a mapping I_T defined as follows: for all $x, y \in [0,1]$

$$I_T(x,y) = \sup\{\gamma \in [0,1] \mid T(x,\gamma) \leq y\}."$$

Definition 2.1.2.6 "Let S be a t-conorm, N be a strong negator and T be a t-norm. A $QL-implication$ is defined as follows: for all $x, y \in [0,1]$

$$I_{S,N_s,T}(x,y) = S(N(x), T(x,y))."$$

Example 2.1.2.7 "Some of the famous fuzzy implicators are as follows: for $x, y \in [0,1]$

(i). Kleen-Dienes (I_b) is given by $I_b(x,y) = \max(1-x, y)$;

(ii). *Reichenbach* (I_r) is given by $I_r(x,y) = 1 - x + xy$;

(iii). *Lukasiewicz* (I_L) is given by $I_L(x,y) = \min(1-x+y, 1)$;

(iv). Gödel (I_g) is given by $I_g(x,y) = \left\{\begin{array}{ll} 1, & \text{if } x \leq y \\ y, & \text{if } x > y \end{array}\right\}$;

(v). Goguen (I_Δ) is given by $I_\Delta(x,y) = \left\{\begin{array}{ll} 1, & \text{if } x \leq y \\ y/x, & \text{if } x > y \end{array}\right\}$;

(vi). Yager (I_E) is given by $I_E(x,y) = y^x$."

Remark 2.1.2.8 Unlike the other proposed fuzzy implicators the Lukasiewicz implicator I_L satisfies all the above mentioned six axioms (a.1 to a.6). Moreover, it is an R-implicator with respect to t-norm T_L, and an S-implicator with respect to conorm S_L and negator N_s. It is also a QL-implicator with respect to a combination of T_M, S_L and N_s.

2.1.3 Operations and Relations Between Fuzzy Sets Induced by Fuzzy Connectives

All the definitions and results stated in this section are taken from [25, 86].

Triangular norms were first introduced in the context of probabilistic metric spaces [132]. In fuzzy literature, the fuzzy t-norms (T), t-conorms (S), negators (N) and implicators (I) have been successfully employed in defining fuzzy relations such as transitivity and equivalence. Moreover, the first three of the mentioned connectives have modeled the basic set theoretic operations namely, the generalized intersection ($\cap$),union ($\cup$) and complementation operations between fuzzy sets. Thus, for $A, B \in FS(X)$ and for all $x \in X$, we define:

$$\begin{aligned} A \cap_T B(x) &= T(A(x), B(x)); && (2.1) \\ A \cup_S B(x) &= S(A(x), B(x)); && (2.2) \\ coA(x) &= N(A(x)); && (2.3) \end{aligned}$$

where if we fix $N = N_s$ in (2.3), then we obtain $coA(x) = A^c(x)$ as given in Definition 2.1.1.

Definition 2.1.3.1 "A fuzzy relation R between two universes X and Y is defined as a fuzzy set on $X \times Y$. Thus, we can write

$$R = \{((x, y), R(x, y)) \mid (x, y) \in X \times Y\}$$

where the function $R(x, y) \in [0, 1]$ defines the degree to which x is in relation R to y."

Definition 2.1.3.2 "Given a t-norm T, a $T-$*equivalence relation* on a set X is a fuzzy relation E on X that satisfies:

(a). $E(x, x) = 1$ for all $x \in X$ (Reflexivity);

(b). $E(x, y) = E(y, x)$ for all $x, y \in X$ (Symmetry);

(c). $T(E(x, y), E(y, z)) \leq E(x, z)$ for all $x, y, z \in X$ (T-transitivity)."

Definition 2.1.3.3 "Let R be a fuzzy relation on X. The fuzzy set of *transitivity* $tr^{I,T}(R)$ is a fuzzy relation on X defined as:

$$tr^{I,T}(R)(x, z) = \inf_{y \in X} I(T(R(x, y), R(y, z)), R(x, z)). \qquad (2.4)$$

The transitivity function so defined, assigns a degree of transitivity to the relation at each point of $X \times X$. The superscripts I and T highlight the dependence of pointwisely transitivity upon the implicator I and the t-norm T, where R denotes the fuzzy relation for which the fuzzy relation of transitivity is being constructed."

Definition 2.1.3.4 "The measure of fuzzy transitivity is a mapping $Tr^{I,T} : F(X \times$

$X) \longrightarrow [0,1]$ defined as:

$$Tr^{I,T}(R) = \inf(tr^{I,T}(R)),$$

where $F(X \times X)$ denotes the set of all fuzzy relations on X. A fuzzy relation R is called an $\epsilon - fuzzy\ transitive$ if $Tr^{I,T}(R) = \epsilon$.

A fuzzy relation R is called fuzzy transitive if $Tr^{I,T}(R) > 0$, strong fuzzy transitive if $\epsilon \geq 0.5$ and R is non-transitive if $Tr^{I,T}(R) = 0$."

Remark 2.1.3.5 It has been proved in [24] that if Lukasiewicz implicator I_L is used in the calculation of degree of transitivity then the T_L-transitive relation enjoy the degree of transitivity equal to 1. Therefore, in this work we have specified I and T of (2.4) by I_L and T_L.

Definition 2.1.3.6 "A fuzzy relation E on a universe X is local fuzzy equivalence relation on $FS(X)$, if the following conditions hold for all $x \in X$:

(a). $E(A(x), A(x)) = 1$ (local fuzzy reflexivity);

(b). $E(A(x), B(x)) = E(B(x), A(x))$ (local fuzzy symmetry);

(c). $I_L(T_L(E(A(x), B(x)), E(B(x), C(x))), E(A(x), C(x))) = \epsilon > 0$ (local fuzzy transitivity)."

Definition 2.1.3.7 "A fuzzy relation R on $FS(X)$ is called a *fuzzy equivalence* relation on $FS(X)$, if for all $x \in X$ and $A, B, C \in FS(X)$, the following conditions hold:

(a). $R(A, A) = 1$ (fuzzy reflexivity);

(b). $R(A, B) = R(B, A)$ (fuzzy symmetry);

(c). $I_L(T_L(R(A, B), R(B, C)), R(A, C)) = \epsilon > 0$ (fuzzy transitivity)."

Definition 2.1.3.8 "A fuzzy relation R on $FS(X)$ is called a fuzzy *quasi-order* relation on $FS(X)$, if for all $x \in X$ and $A, B, C \in FS(X)$, the following properties hold:

(a). $R(A, A) = 1$ (fuzzy reflexivity);

(b). $I_L(T_L(R(A, B), R(B, C)), R(A, C)) = \epsilon > 0$ (fuzzy transitivity)."

Now we move towards the relevant topics of intuitionistic fuzzy set theory and logic which are extensively used throughout this dissertation.

2.2 The Basics of Intuitionistic Fuzzy Set Theory

All the definitions, results and examples stated in this section are taken from [7, 8, 49, 63].

Definition 2.2.1 "An *intuitionistic fuzzy set* (*IFS*) on a universe X is an object of the form $A = \Big\{(x, \mu_A(x), \nu_A(x)) \mid x \in X\Big\}$, where the functions $\mu_A(x)$ and $\nu_A(x) \in [0,1]$ define respectively the degree of membership and the degree of non membership of x in the set A, while μ_A and ν_A satisfy $(\forall x \in X)(\mu_A(x) + \nu_A(x) \leq 1)$. The class of all intuitionistic fuzzy sets on X is denoted by $IFS(X)$. An IFS is called finite if universe X is finite, writing it $X = \{x_1, x_2, ...x_n\}$.

Note that a fuzzy set in X is then just an intuitionistic fuzzy set for which $\mu_A(x) + \nu_A(x) = 1$ holds for every $x \in X$. For an intuitionistic fuzzy set $A = \Big\{(x, \mu_A(x), \nu_A(x)) \mid x \in X\Big\}$ we define the *complement* of A in X as:
$A^c = \Big\{(x, \nu_A(x), \mu_A(x)) \mid x \in X\Big\}$, the *Support* of A in X as a subset of X given by $Supp(A) = \{x \in X \mid \mu_A(x) \neq 0 \text{ or } \nu_A(x) \neq 1\}$, the *Kernel* of A in X as $Ker(A) = \{x \in X \mid \mu_A(x) = 1 \text{ and } \nu_A(x) = 0\}$."

Moreover, we denote by $\widetilde{1}_X = \{(x, 1, 0) \mid x \in X\}$ and $\widetilde{0}_X = \{(x, 0, 1) \mid x \in X\}$ the intuitionistic fuzzy sets corresponding to entire universe X and empty set ϕ respectively.

"Also, for all $A, B \in IFS(X)$, the some expressions and operators are defined as follows:

$$
\begin{aligned}
A &\subseteq B \text{ if and only if } (\forall x \in X)(\mu_A(x) \leq \mu_B(x) \text{ and } \nu_A(x) \geq \nu_B(x));\\
A &= B \text{ if and only if } A \subseteq B \text{ and } B \subseteq A;\\
A &\preceq B \text{ if and only if } (\forall x \in X)(\mu_A(x) \leq \mu_B(x) \text{ and } \nu_A(x) \leq \nu_B(x));\\
A^{(n)} &= \{(x, (\mu_A(x))^n, 1 - (1 - (\nu_A(x))^n)) \mid x \in X\};\\
(A^c)^{(n)} &= \{(x, (\nu_A(x))^n, 1 - (1 - (\mu_A(x))^n)) \mid x \in X\};\\
(n)A &= \{(x, 1 - (1 - (\mu_A(x))^n, (\nu_A(x))^n)) \mid x \in X\};\\
(n)A^c &= \{(x, 1 - (1 - (\nu_A(x))^n), (\mu_A(x))^n) \mid x \in X\}.\text{''}
\end{aligned}
$$

Definition 2.2.2 "The set $L^* = \{(x_1, x_2) \in [0,1]^2 \mid x_1 + x_2 \leq 1\}$ is a complete and bounded lattice $(L^*, \leq_{L^*})$ equipped with order $\leq_{L^*}$, which is defined as: $(x_1, x_2) \leq_{L^*} (y_1, y_2)$ if and only if $x_1 \leq y_1$ and $x_2 \geq y_2$. The elements $1_{L^*} = (1, 0)$ and $0_{L^*} = (0, 1)$ are the greatest and the smallest elements of the lattice L^* respectively."

Note 2.2.3 From now on, we will assume that if $x \in L^*$, then x_1 and x_2 denote respectively the first and the second component of x, i.e. $x = (x_1, x_2)$.

Remark 2.2.4 The intuitionistic fuzzy set $A = \Big\{(x, \mu_A(x), \nu_A(x)) \mid x \in X\Big\}$ is thus a special case of $L-$fuzzy set [see Definition 2.1.3] in the sense of [71], where $L = L^*$ i.e. a mapping $A : X \longrightarrow L^*$: $x \longrightarrow (\mu_A(x), \nu_A(x))$. In the sequel we will use the same notation for an intuitionistic fuzzy set and its associated L^*-fuzzy set. So for the intuitionistic fuzzy set A we will also use the notation $A(x) = (\mu_A(x), \nu_A(x))$.

Interpreting intuitionistic fuzzy sets as L^*-fuzzy sets gives a way to greater flexibility in calculating with membership and non membership degree. The pair formed by the two degrees is an element of L^*, and often allows to obtain more compact formulas and well build operations between intuitionistic fuzzy sets.

Definition 2.2.5 For any $A, B \in IFS(X)$, A is said to be point wise comparable with B if for all $x \in X$ either $A(x) \leq_{L^*} B(x)$ or $B(x) \leq_{L^*} A(x)$. Moreover, it may be noted that:

1. A is point wise comparable to A for all $A \in IFS(X)$
2. If A is pointwise comparable with B then B is pointwise comparable to A.
3. If A is pointwise comparable to B and B is point wise comparable to C then A is comparable to C.

Definition 2.2.6 "An intuitionistic fuzzy relation R between two universes X and Y is defined as an intuitionistic fuzzy set on $X \times Y$. Thus, we can write $R = \{((x, y), \mu_R(x, y), \nu_R(x, y)) \mid (x, y) \in X \times Y\}$, where the functions $\mu_R(x, y)$ and $\nu_R(x, y) \in [0, 1]$ define respectively the degree to which x is in relation R to y and the degree to which x is in non relation R to y, while $\mu_R(x, y)$ and $\nu_R(x, y)$ satisfy $(\forall (x, y) \in X \times Y)(\mu_R(x, y) + \nu_R(x, y) \leq 1)$."

Since $\leq_{L^*}$ is a partial order on L^*, an order theoretic extension of classical negation, conjunction, disjunction and implication on L^* under the notion of intuitionistic fuzzy $(IF) - negators$, $IF - t - norms$, $IF - t - conorms$ and $IF - implicators$ respectively arises quite naturally. Thus, to have a better understanding of the work presented in this thesis let us review these connectives in detail.

2.2.1 The Intuitionistic Fuzzy Negation, Conjunction and Disjunction

All the definitions, results and examples stated in this section are taken from [46, 60].

Definition 2.2.1.1 "An intuitionistic fuzzy *negator* is a decreasing $L^* \longrightarrow L^*$ mapping $\check{N}$ that satisfies $\check{N}(0_{L^*}) = 1_{L^*}$ and $\check{N}(1_{L^*}) = 0_{L^*}$. If $\check{N}(\check{N}(x)) = x$, $\forall x \in L^*$, $\check{N}$ is called an involutive/strong intuitionistic fuzzy negator. The mapping $\check{N}_s$ defined as: $\check{N}_s(x_1, x_2) = (x_2, x_1)$ $\forall (x_1, x_2) \in L^*$ will be called the standard intuitionistic fuzzy negator. An involutive negator on L^* can always be related to an involutive negator on $[0, 1]$."

In [60] Deschrijver, Cornelis and Kerre have established a representation theorem for involutive intuitionistic fuzzy negators: any involutive intuitionistic fuzzy negator can be represented using involutive fuzzy negator N satisfying $N(0) = 1$ and $N(1) = 0$.

Theorem 2.2.1.2 "Let $\check{N}$ be an involutive intuitionistic fuzzy negator, and let $[0, 1] \to [0, 1]$ mapping N be defined by, for all $a \in [0, 1]$, $N(a) = pr_1\check{N}(a, 1 - a)$. Then for all $x \in L^*$, $\check{N}(x) = (N(1 - x_2), 1 - N(x_1))$. Moreover, N is an involutive fuzzy negator. Conversely, if N is an involutive fuzzy negator, then the $L^* \to L^*$ mapping $\check{N}$ defined by, for all $x \in L^*$, $\check{N}(x) = (N(1 - x_2), 1 - N(x_1))$ is an involutive intuitionistic fuzzy negator."

For instance if $N = N_s$ where N_s denotes the fuzzy standard negator defined as, for all $x \in [0, 1]$, $N_s = 1 - x$, then we obtain the intuitionistic fuzzy standard negator $\check{N}_s$.

Definition 2.2.1.3 "An intuitionistic fuzzy *t-norm* is an increasing, commutative, associative $(L^*)^2 \longrightarrow L^*$ mapping $\check{T}$ satisfying $\check{T}(1_{L^*}, x) = x$ for all $x \in L^*$."

Definition 2.2.1.4 "An intuitionistic fuzzy *t-conorm* is an increasing, commutative, associative $(L^*)^2 \longrightarrow L^*$ mapping $\check{S}$ satisfying $\check{S}(0_{L^*}, x) = x$ for all $x \in L^*$."

Definition 2.2.1.5 "The dual of an intuitionistic fuzzy t-norm $\check{T}(t - conorm\ \check{S})$

w.r.t. a negator $\check{N}$ is a mapping $\check{T}^*$[respectively $\check{S}^*$] defined by, for $x, y \in L^*$,

$$\begin{aligned}\check{T}^*(x,y) &= \check{N}(\check{T}(\check{N}(x),\check{N}(y)))\\ [\text{respectively } \check{S}^*(x,y) &= \check{N}(\check{S}(\check{N}(x),\check{N}(y)))].\end{aligned}$$

It can be verified that $\check{T}^*$ is an intuitionistic fuzzy conorm and $\check{S}^*$ is an intuitionistic fuzzy t-norm."

Definition 2.2.1.6 "An intuitionistic fuzzy t-norm $\check{T} = \overline{T}$ (respectively t-conorm $\check{S} = \overline{S}$) is called t-representable if there exists a t-norm T and a t-conorm S on [0,1] (respectively a t-conorm S' and a t-norm T' on [0,1]) such that, for $x, y \in L^*$,

$$\begin{aligned}\overline{T}(x,y) &= (T(x_1,y_1), S(x_2,y_2))\\ [\text{respectively } \overline{S}(x,y) &= (S'(x_1,y_1), T'(x_2,y_2))]\end{aligned}$$

where T and S (respectively S' and T') are called the representants of $\check{T}$ (respectively $\check{S}$)."

Next, we state two very important theorems related to t-representability of intuitionstic fuzzy t-norms (respectively, t-conorms). The first theorem states the condition under of which a pair of connectives on [0,1] gives rise to a t-representable intuitionistic t-norm (t-conorm) while, the second theorem assures the t-representability of the dual of a given t-representable intuitionistic fuzzy t-norm or t-conorm.

Theorem 2.2.1.7 "Given a t-norm T and t-conorm S satisfying $T(a,b) \leq 1 - S(1-a, 1-b)$ for all $a, b \in [0,1]$, the mappings $\check{T}$ (respectively $\check{S}$) defined by, for $x, y \in L^*$,

$$\begin{aligned}\check{T}(x,y) &= (T(x_1,y_1), S(x_2,y_2))\\ [\text{respectively } \check{S}(x,y) &= (S(x_1,y_1), T(x_2,y_2))]\end{aligned}$$

is intuitionistic fuzzy t-norm (respectively t-conorm)."

Remark 2.2.1.8 If S is the dual of T (or equivalently T is the dual of S) then $T(a,b) = 1 - S(1-a, 1-b)$ for all $a, b \in [0,1]$ and therefore, the hypothesis in the above theorem is satisfied.

We obtain the most important t-representable intuitionistic fuzzy t-norms:

$$\begin{aligned}
\overline{T}_M(x,y) &= (T_M(x_1,y_1), S_M(x_2,y_2)) \\
&= (\min(x_1,y_1), \max(x_2,y_2)); \\
\overline{T}_P(x,y) &= (T_P(x_1,y_1), S_P(x_2,y_2)) \\
&= (x_1y_1, x_2+y_2-x_2y_2); \\
\overline{T}_L(x,y) &= (T_L(x_1,y_1), S_L(x_2,y_2)) \\
&= (\max(0,x_1+y_1-1), \min(1,x_2+y_2));
\end{aligned}$$

the most important intuitionistic fuzzy t-conorms:

$$\begin{aligned}
\overline{S}_M(x,y) &= (S_M(x_1,y_1), T_M(x_2,y_2)) \\
&= (\max(x_1,y_1), \min(x_2,y_2)); \\
\overline{S}_P(x,y) &= (S_P(x_1,y_1), T_P(x_2,y_2)) \\
&= (x_1+y_1-x_1y_1, x_2y_2); \\
\overline{S}_L(x,y) &= (S_L(x_1,y_1), T_L(x_2,y_2)) \\
&= (\min(1,x_1+y_1), \max(0,x_2+y_2-1));
\end{aligned}$$

and the families of intuitionistic fuzzy t-norms and t-conorms:

$$\begin{aligned}
\overline{T}^F_\omega(x,y) &= (T^F_\omega(x_1,y_1), S^F_\omega(x_2,y_2)); \\
\overline{T}^H_\omega(x,y) &= (T^H_\omega(x_1,y_1), S^H_\omega(x_2,y_2)); \\
\overline{T}^Y_\omega(x,y) &= (T^Y_\omega(x_1,y_1), S^Y_\omega(x_2,y_2)); \\
\overline{T}^S_\omega(x,y) &= (T^S_\omega(x_1,y_1), S^S_\omega(x_2,y_2));
\end{aligned}$$

$$\begin{aligned}
\overline{S}^F_\omega(x,y) &= (S^F_\omega(x_1,y_1), T^F_\omega(x_2,y_2)); \\
\overline{S}^H_\omega(x,y) &= (S^H_\omega(x_1,y_1), T^H_\omega(x_2,y_2)); \\
\overline{S}^Y_\omega(x,y) &= (S^Y_\omega(x_1,y_1), T^Y_\omega(x_2,y_2)); \\
\overline{S}^S_\omega(x,y) &= (S^S_\omega(x_1,y_1), T^S_\omega(x_2,y_2));
\end{aligned}$$

respectively that are based on the families of t-norms and t-conorms given in Example 2.1.1.7 and Example 2.1.1.12.

It must be noted that the greatest t-norm with respect to ordering $\leq_{L^*}$ is $\overline{T}_M(x,y) = (\min(x_1,y_1), \max(x_2,y_2))$ which is an t-representable extension of T_M to L^*. Moreover, $\overline{T}_M$ has this property that if $z \leq_{L^*} x$ and $z \leq_{L^*} y$ then $z \leq \overline{T}_M(x,y)$ for all $x,y,z \in L^*$. Similarly, the smallest t-conorm with respect to ordering $\leq_{L^*}$ is $\overline{S}_M(x,y) = (\max(x_1,y_1), \min(x_2,y_2))$ which is an extension of S_M to L^*.

Theorem 2.2.1.9 "The dual intuitionistic fuzzy t-norm with respect to an involutive negator $\check{N}$ on L^* of t-representable intuitionistic fuzzy t-conorm is t-representable. The dual intuitionstic fuzzy t-conorm with respect to an involutive negator $\check{N}$ on L^* of a t-representable intuitionistic fuzzy t-norm is t-representable."

Remark 2.2.1.10 It must be noted that not all intuitionistic fuzzy t-norms (t-conorms) are t-representable, i.e., the converse of Theorem 2.2.1.7 is not always true. In other words, we don't have that for every intuitionistic fuzzy t-norm $\check{T}$ or a t-conorm $\check{S}$ there exist a t-norm T and a t-conorm S such that $\check{T} = (T,S)$ or $\check{S} = (S,T)$.

For instance the following extension of probabilistic sum S_P to L^*

$$\widetilde{S}_P(x,y) = \left\{ \begin{array}{cc} x & \text{if } y = 0_{L^*} \\ y & \text{if } x = 0_{L^*} \\ (\max(1-x_2, 1-y_2), \min(x_2,y_2)), & \text{otherwise} \end{array} \right\}$$

was proved to violate the converse of Theorem 2.2.1.7 hence is a non t-representable extension of S_P to L^*, for all $x,y \in L^*$.

Its dual is given by:

$$\widetilde{T}_P(x,y) = \left\{ \begin{array}{cc} x & \text{if } y = 1_{L^*} \\ y & \text{if } x = 1_{L^*} \\ (\min(x_1,y_1), \max(1-x_1, 1-y_1)), & \text{otherwise} \end{array} \right\}$$

which is a non t-representable extension of T_P to L^*.

Moreover, following are some of the examples of non t-representable Lukasiewicz intuitionistic fuzzy t-norms and their dual intuitionistic fuzzy t-conorms given in [59].

(a). $\widetilde{T}_L(x,y) = (\max(0, x_1 + y_1 - 1), \min(1, x_2 + 1 - y_1, y_2 + 1 - x_1))$

$\widetilde{S}_L(x,y) = (\min(1, x_1 + 1 - y_2, y_1 + 1 - x_2), \max(0, x_2 + y_2 - 1))$ are one of the extension of Lukasiewicz t-norm and t-conorm to L^*. It is interesting to note that for all $x, y \in L^*, \check{S}_L(x, 1_{L^*}) = \check{S}_L(1_{L^*}, y) = 1_{L^*}$.

The other non t-representable extensions of Lukasiewicz t-norm and t-conorm on [0,1] to $L^*$are:

(b). $\widetilde{T}'_L(x,y) = (\max(0, x_1 + y_1 - x_2y_2 - 1), \min(1, x_2 + y_2))$;

$\widetilde{S}'_L(x,y) = (\min(1, x_1 + y_1), \max(0, x_2 + y_2 - x_1y_1 - 1))$.

(c). $\widetilde{T}''_L(x,y) = (\max(0, \min(x_1 - y_2, y_1 - x_2)), \min(1, x_2 + y_2))$;

$\widetilde{S}''_L(x,y) = (\min(1, x_1 + y_1), \max(0, \min(x_2 - y_1, y_2 - x_1)))$.

(d). $\widetilde{T}'''_L(x,y) = (\max(0, x_1 + y_1 - 1), \min(1, y_2 + 2(1 - x_1), x_2 + 2(1 - y_1), 1 - x_1 + 1 - y_1))$;

$\widetilde{S}'''_L(x,y) = (\min(1, y_1 + 2(1 - x_2), x_1 + 2(1 - y_2), 1 - x_2 + 1 - y_2), \max(0, x_2 + y_2 - 1))$.

(e). $\widetilde{T}''''_L(x,y) = (\max(0, x_1 + y_1 - 1), \min(1, x_2 + y_2 + \frac{1}{2}, 1 - x_1 + y_2, 1 - y_1 + x_2))$;

$\widetilde{S}''''_L(x,y) = (\min(1, x_1 + y_1 + \frac{1}{2}, 1 - x_2 + y_1, 1 - y_2 + x_1), \max(0, x_2 + y_2 - 1))$.

(f). $\widetilde{T}'''''_L(x,y) = (\max(0, x_1 + y_1 - 1), \min(1, x_2 + y_2))$;

$\widetilde{S}'''''_L(x,y) = (\min(1, x_1 + y_1), \max(0, x_2 + y_2 - 1))$.

Theorem 2.2.1.11 "Let $\check{T}$ be an intuitionistic fuzzy t-norm. If $\sup\limits_{z\in Z}\check{T}(x,z) = \check{T}(x, \sup\limits_{z\in Z} z)$, for all non-empty subsets Z of L^*, then $\check{T}$ is intuitionistic fuzzy left continuous."

2.2.2 The Intuitionistic Fuzzy Implication

All the definitions, results and examples stated in this section are taken from [46, 60].

Definition 2.2.2.1 "An intuitionistic fuzzy *implicator* is an $(L^*)^2 \longrightarrow L^*$ mapping $\check{I}$ satisfying $\check{I}(0_{L^*}, 0_{L^*}) = 1_{L^*}, \check{I}(0_{L^*}, 1_{L^*}) = 1_{L^*}, \check{I}(1_{L^*}, 0_{L^*}) = 0_{L^*}, \check{I}(1_{L^*}, 1_{L^*}) = 1_{L^*}$. Moreover, we require $\check{I}$ to be decreasing in its first and increasing in its second component."

Definition 2.2.2.2 "The intuitionistic fuzzy implicator $\check{I}$ is said to satisfy the left ordering property (LOP), if $x \leq_{L^*} y$, then $\check{I}(x,y) = 1_{L^*}$ for all $x, y \in L^*$."

Definition 2.2.2.3 "Let $\check{S}$ be a t-conorm and $\check{N}$ a negator on L^*. The *S-implicator* generated by $\check{S}$ and $\check{N}$ is the mapping $\check{I}_{\check{S},\check{N}} : (L^*)^2 \longrightarrow L^*$ defined as, for all $x, y \in L^*$:

$$\check{I}_{\check{S},\check{N}}(x,y) = \check{S}(\check{N}(x), y).\text{''} \tag{2.5}$$

Example 2.2.2.4 Let $\check{S} = \overline{S}_M$ and $\check{N} = \check{N}_s$. Then

$$\check{I}_{\overline{S}_M,\check{N}_s}(x,y) = (\max(x_2,y_1), \min(x_1,y_2))$$

is a S-implicator and an extension of the Kleen Dienes $(I(x,y) = \max(1-x,y))$ implicator on [0,1] to L^*.

Example 2.2.2.5 Let $\check{S} = \widetilde{S}_L''''''$ and $\check{N} = \check{N}_s$. Then

$$\check{I}_{\widetilde{S}_L'''''',\check{N}_s}(x,y) = (\min(1, x_2 + y_1), \max(0, x_1 + y_2 - 1))$$

is a S-implicator and an extension of the Lukasiewicz $(I(x,y) = \min(1, 1-x+y))$ implicator to L^*.

Definition 2.2.2.6 "Let $\check{T}$ be a t-norm on L^*. The *R-implicator* generated by $\check{T}$ is the mapping $\check{I}_{\check{T}}$ defined as follows: for all $x, y \in L^*$

$$\check{I}_{\check{T}}(x,y) = \sup\{\gamma \in L^* \mid \check{T}(x,\gamma) \leq_{L^*} y\}.\text{"} \tag{2.6}$$

Example 2.2.2.7 Choosing $\check{T} = \overline{T}_M$ in (2.6) we get the following intuitionistic fuzzy R-implicator:

$$\check{I}_{\overline{T}_M}(x,y) = \left\{ \begin{array}{ll} 1_{L^*} & \text{if } x_1 \leq y_1 \text{ and } x_2 \geq y_2 \\ (1-y_2, y_1) & \text{if } x_1 \leq y_1 \text{ and } x_2 < y_2 \\ (y_1, 0) & \text{if } x_1 > y_1 \text{ and } x_2 \geq y_2 \\ (y_1, y_2) & \text{if } x_1 > y_1 \text{ and } x_2 < y_2 \end{array} \right\},$$

which is an extension of Gödel implicator to L^*.

Example 2.2.2.8 Choosing $\check{T} = \widetilde{T}_L''''''$ in (2.6) we get the following intuitionistic fuzzy R-implicator:

$$\breve{I}_{\widetilde{T}_L'''''}(x,y) = (\min(1 - y_2 + x_2, 1 + y_1 - x_1), y_2 - x_2),$$

which is an extension of Lukasiewicz implicator to L^*.

Example 2.2.2.9 If we take $\breve{S} = \widetilde{S}_L$ and $\breve{N} = \breve{N}_s$ in (2.5), then, $\breve{I}_{\widetilde{S}_L,\breve{N}_s}(x,y) = (\min(1, y_1 + 1 - x_1, x_2 + 1 - y_2), \max(0, x_1 + y_2 - 1))$ is an extension of Lukasiewicz implicator on [0,1] to L^* and is a S-implicator on L^*. Also this extension can be obtained by taking $\breve{T} = \widetilde{T}_L$ in (2.6) which makes it a R-implicator extension on L^*. Thus we have $\breve{I}_{\widetilde{S}_L,\breve{N}_s}(x,y) = \breve{I}_{\widetilde{T}_L}(x,y) = (\min(1, y_1 + 1 - x_1, x_2 + 1 - y_2), \max(0, x_1 + y_2 - 1))$. It is a contrapositive intuitionistic fuzzy extension of Lukasiewicz implicator to L^*.

Remark 2.2.2.10 In order to check the suitability of intuitionistic fuzzy implicators for a variety of purpose the criteria Smetz and Margrez were generalized for intuitionistic fuzzy implicators in [45]. Also it was shown that $\breve{I}_{\widetilde{T}_L}(x,y) = (\min(1, y_1 + 1 - x_1, x_2 + 1 - y_2), \max(0, x_1 + y_2 - 1))$ is the only intuitionistic fuzzy implicator that satisfies all the axioms of Smetz and Margrez developed for IFS.

Remark 2.2.2.11(Axioms of Smetz and Margrez for an intuitionistic fuzzy implicator)

Taking into consideration the importance of Smetz-Margrez axioms, that stood as a yardstick to test the suitability of fuzzy implicators, [45], generalized these axioms for intuitionistic fuzzy implicators as follows:

(A.1). $(\forall y \in L^*)(\breve{I}(.,y)$ is decreasing)

$(\forall x \in L^*)(\breve{I}(x,.)$ is increasing);

(monotonicity law)

(A.2). $(\forall x \in L^*)(\breve{I}(1_{L^*}, x) = x)$;

(neutrality principle)

(A.3). $(\forall (x,y) \in (L^*)^2)(\breve{I}(x,y) = \breve{I}(\breve{N}(x), \breve{N}(y))$;

(contrapositivity w.r.t. an intuitionistic fuzzy negator $\breve{N}$)

(A.4). $(\forall (x,y,z) \in (L^*)^3)(\breve{I}(x, \breve{I}(y,z)) = \breve{I}(y, \breve{I}(x,z)))$;

(interchageability principle)

(A.5). $(\forall (x,y) \in (L^*)^2)(\ x \leq_{L^*} y \Longleftrightarrow \breve{I}(x,y) = 1_{L^*})$;

(confinement principle)

(A.6). $\check{I}$ is continuous.

(continuity)

Definition 2.2.2.12 (Residuation principle) "An intuitionistic fuzzy t-norm $\check{T}$ satisfies the residuation principle if and only if, for all $x, y, z \in L^*$, $\check{T}(x, y) \leq_{L^*} z \Longleftrightarrow y \leq_{L^*} \check{I}_{\check{T}}(x, z)$ where $\check{I}$ denotes the residual implicator generated by $\check{T}$."

Theorem 2.2.2.13 "Let T be a left continuous t-norm and $t \in [0, 1]$. Define an $(L^*)^2 \to L^*$ mapping $\check{T}_{T,t}$ by, for all $x, y \in L^*$,

$$\begin{aligned}\check{T}_{T,t}(x, y) &= (T(x_1, y_1), \min\{1 - T(1 - t, T(1 - x_2, 1 - y_2)), \\ &\quad 1 - T(1 - y_2, x_1), 1 - T(1 - x_2, y_1)\}).\end{aligned}$$

Then $\check{T}_{T,t}$ is an intuitionistic fuzzy t-norm satisfying the residuation principle."

Note 2.2.2.14 Note that $\check{T}_{T,t}$ is t-representable if and only if $t = 0$. Then, clearly $\check{T}_{T,0}(x, y) = (T(x_1, y_1), 1-T(1-x_2, 1-y_2))$. If $t \neq 0$, then $\check{T}_{T,t}((0, 0), (0, 0)) = (0, t)$. So $\check{T}_{T,t}$ is not t-representable since for an intuitionistic fuzzy t-representable t-norm $\overline{T}((0, 0)(0, 0)) = (0, 0)$.

2.2.3 Operations on Intuitionistic Fuzzy Sets Induced by Intuitionistic Fuzzy Connectives

Likewise to FS-theory, in IFS -theory the generalized intersection ($\cap$),union ($\cup$) and complementation operations between IFS's is modeled by intuitionstic fuzzy t-norms $\check{T}$, intuitionstic fuzzy t-conorms $\check{S}$ and intuitionistic fuzzy negator $\check{N}$. Thus for $A, B \in IFS(X)$ and for all $x \in X$, we define:

$$\begin{aligned}A \cap_{\check{T}} B(x) &= \check{T}(A(x), B(x)); &(2.7)\\ A \cup_{\check{S}} B(x) &= \check{S}(A(x), B(x)); &(2.8)\\ coA &= \check{N}(A(x)); &(2.9)\end{aligned}$$

where $\check{T}$ (respectively $\check{S}$) used in (2.7) and (2.8) could be t-representable or non t-representable intuitionistic fuzzy t-norm (respectively conorm). In case, we define generalized $\cap$ and $\cup$ between IFS's by t-representable intuitionstic fuzzy t-norm (respectively dual t-conorm), expressions (2.7) and (2.8) can be redefined as follows:

$$\begin{aligned} A \cap_{\overline{T}} B &= \overline{T}(A(x), B(x)) && (2.10)\\ &= \left\{(x, T(\mu_A(x), \mu_B(x)), S(\nu_A(x), \nu_B(x))) \mid x \in X\right\}; && (2.11)\\ A \cup_{\overline{S}} B &= \overline{S}(A(x), B(x)) && (2.12)\\ &= \left\{(x, S(\mu_A(x), \mu_B(x)), T(\nu_A(x), \nu_B(x))) \mid x \in X\right\}. && (2.13) \end{aligned}$$

such that $A(x) = (\mu_A(x), \nu_A(x))$ and $B(x) = (\mu_B(x), \nu_B(x))$ and T and S are t-norm and t-conorm on $[0, 1]$,respectively, satisfying the following two relations:

$$\begin{aligned} T(\mu_A(x), \mu_B(x)) + S(1 - \mu_A(x), 1 - \mu_B(x)) &\leq 1; && (2.14)\\ S(\mu_A(x), \mu_B(x)) + T(1 - \mu_A(x), 1 - \mu_B(x)) &\leq 1. && (2.15) \end{aligned}$$

Now if we work with the pair (T, S) which are duals of each other then the relations (2.14) and (2.15) are verified by equality [Remark 2.2.1.7].
In this dissertation, we have fixed the t-representable intuitionstic fuzzy pair $\left(\overline{T}, \overline{S}\right)$ in (2.10) and (2.12) by the family of intuitionistic fuzzy frank t-norms $\overline{T}(x, y) = (T_\omega^F(x_1, y_1), S_\omega^F(x_2, y_2))$ and their dual intuitionistic fuzzy frank conorms $\overline{S}(x, y) = (S_\omega^F(x_1, y_1), T_\omega^F(x_2, y_2))$ where $x, y \in L^*$, such that their representant family of t-norms T_ω^F and their dual t-conorms S_ω^F satisfy the valuation property stated as: $T_\omega^F(a, b) + S_\omega^F(a, b) = a + b$ for all $a, b \in [0, 1]$ and $\omega = 0, 1, +\infty$ [Theorem 2.1.1.14].

Chapter 3

Fuzzy Measures on Intuitionistic Fuzzy Sets

In the late of 19^{th} century, there was a growing need for more precise mathematical analysis, induced primarily by rapidly advancing science and technology. As a result, new questions regarding measurement emerged. Considering for example, the set of real numbers between 0 and 1, which may be viewed as points on a real line, mathematicians asked what was the measure of the remaining set (or the length of the remaining interval of real line). Questions like these and many more were carefully examined by a french mathematician Emile Boral (1871-1956). He developed a theory to deal with questions which were an important step towards a more general theory that we now refer to as the classical measure theory.

A classical measure is a set function [78] satisfying the additivity property. The additivity property of classical measure although is convenient to apply it as yet is too restrictive. Consequently, unrealistic results in many reasoning environments of real world have been given. For example, if we want to measure the efficiency of a set of workers then the efficiency of the same group of people doing team work is not the addition of the efficiency of each individual working on his own. The solution to the above problem was appeared in the late of 70's and 80's of the previous century with a notion of fuzzy measure theory. A fuzzy measure is a non additive set function having additivity axioms of classical measure replaced with the axioms of monotonicity related to the inclusion of sets.

Since an intuitionistic fuzzy set is a special case of $F_LS(X)$ in the sense of Gougen [71], where $L = L^*$, an attempt to generalize the concept of set function for intuitionistic fuzzy sets resulted into two diversions namely, a real valued intuitionistic fuzzy set function which laid the background of fuzzy measure on intuitionistic fuzzy sets and an intuitionistic valued set function that played the key role in defining an intuitionistic fuzzy measure on intuitionistic fuzzy set. For a detailed study we refer to [18].

Among other different measures on a set, the cardinality of a set(crisp) is an elementary measure on a set referring to a number giving the count of the elements belonging to the set. In the case of fuzzy sets, the partial belongingness of elements to a fuzzy set leds to different approaches presented in fuzzy literature which were presented from time to time in order to define the most appropriate generalization of this measure. The proposals presented in this regard can be roughly classified into the following three groups:

(i) scalar cardinalities;

(ii) fuzzy cardinalities;

(iii) integer cardinalities.

The first type of cardinality is a set function which is mapped to the set of positive real numbers or in other words the cardinality of fuzzy set is defined to be a positive real number [37, 38, 56, 73, 161]. In the second kind, cardinality of a fuzzy set is described by some fuzzy quantity which may not be convex in nature [29, 55, 66, 162, 171]. The last type of cardinality of a fuzzy set is obtained by applying some defuzzification method on the fuzzy cardinality of that set and can be equivalently defined as nothing more than counting the elements of $0.5-$ cut of that fuzzy set [128] [see Definition 2.1.1, Section 2.1.1, Chapter 2].

Among all these three types the scalar cardinality of a fuzzy set is the most acceptable due to its convenience in real life applications. Therefore, in this work we shall restrict our self to the discussion of the scalar cardinality of fuzzy and higher order fuzzy sets. The cardinality of fuzzy set was initially proposed by De Luca and Termini who named it the power of finite fuzzy set. According to them the scalar cardinality of a finite fuzzy set A is the sum of the membership degrees of its elements, symbolically defined by $|.|$, and is given as:

$$|A| = \sum_{x \in X} \mu_A(x) \tag{3.1}$$

where $X = \{x_1, x_2, ...x_n\}$ and is called the *Sigma count of A*.

In fuzzy literature, the scalar cardinalities defined had axiomatic as well as constructive approaches. A detailed study on scalar cardinalities of fuzzy sets that covers all the historical approached had been made by Wygralak [160]. According to him, a mapping $card(A) : FS(X) \longrightarrow [0, \infty)$ is called the scalar cardinality of fuzzy sets if it satisfies the following postulates:

(i). Coincidence: For all $x \in X$, $card(A(x)) = 1$ when $A(x) = 1 \in [0, 1]$.

(ii). Monotonicity: For all $A, B \in FS(X)$,

$$A \subseteq B \text{ implies } card(A) \leq card(B). \tag{3.2}$$

(iii). Additivity: For all $A, B \in FS(X)$,

$$Supp(A) \cap Supp(B) = \phi \text{ implies } card(A \cup B) = card(A) + card(B). \qquad (3.3)$$

In 1995, Atanassova [14] introduced the concept of a scalar cardinality of an intuitionistic fuzzy set while Szmidt [141], presented it in the form of a numerical interval with membership degree as lower bound and the sum of hesitancy and membership degree as an upper bound. Tripathy [149], re-explored different properties of this numerical interval cardinality and further proposed the concept of relative intuitionistic fuzzy count. The mean value of the cardinality interval was defined as a scalar cardinality of IFS by Vlachos [153] while Král [96, 97], developed an axiomatic approach towards the extensions of scalar cardinalities of fuzzy sets to their intuitionistic versions. Moreover, some other related ideas to this concept can be found in [44, 104, 150].

In this chapter, we have introduced four new normal fuzzy measures on intuitionistic fuzzy sets along with a new cardinality measure called weighted average cardinality measure for intuitionistic fuzzy sets. We also studied in detail some important properties of this new measure like valuation property, complementary rule and its behavior with the famous logical model operator of necessity and possibility as introduced by [7]. The measures in this chapter will be extensively used in accomplishing the core target of this dissertation, i.e., construction of new measures of similarity and inclusions between intuitionistic fuzzy sets.

3.1 Some New Fuzzy Measures on Intuitionistic Fuzzy Sets

The results presented in this section are a part of our published paper [127].

In intuitionistic fuzzy literature, the notion of measure for the IFS has been extended mainly by Ban [18] in two ways: as a fuzzy measure of intuitionistic fuzzy sets and intuitionistic fuzzy measure of intuitionistic fuzzy sets. Both of these measures were based on $\sigma-algebra$ of intuitionistic fuzzy sets in a crisp universe X called *an intuitionistic fuzzy* $\sigma-algebra$ *on* X.

Definition 3.1.1 [19] An *intuitionistic fuzzy $\sigma-$algebra* on X is a family $\pounds$ of IFS's on X satisfying the properties:

(i) $\widetilde{1}_X \in \pounds$;

(ii) If $A \in \pounds$ then it implies $A^c \in \pounds$;

(iii) $\cup_{n\in N} A_n \in \pounds$ for every sequence $(A_n)_{n\in N}$ of IFS's in $\pounds$;

where $\widetilde{1}_X = \{(x, 1, 0) \mid x \in X\}$ is the intuitionistic fuzzy set corresponding to the universe X.

The sets in $\pounds$ are called the *IF measurable sets* and the pair $(X, \pounds)$ is called an *intuitionistic fuzzy measurable space.*

Definition 3.1.2 [18] Let $(X, \pounds)$ be an intuitionistic fuzzy measurable space. A function $m : \pounds \to [0, \infty]$ is called a *fuzzy measure of intuitionistic fuzzy sets* if it satisfies the following conditions:

(i) $m(\widetilde{0}_X) = 0$;

(ii) For any $A, B \in \pounds$, $A \subseteq B$ implies $m(A) \leq m(B)$.

where $\widetilde{0}_X = \{(x, 0, 1) \mid x \in X\}$ is the intuitionistic fuzzy set corresponding to the empty set ϕ. The measure m with the boundary condition $m(\widetilde{1}_X) = 1$ is called a *normalized or normal fuzzy measure.*

The following normal fuzzy measure of intuitionistic fuzzy sets was introduced in [18] : For any $A \in IFS(X)$,

$$m(A) = \left(\frac{\sum_{x\in X} \mu_A(x) + (1 - \nu_A(x))}{2n} \right)^2.$$

In this first section, we present some new fuzzy measures on intuitionistic fuzzy sets. The first two measures are for any universe X while the next two are defined on $X = \{x_1, x_2, ...x_n\}$ with $|X| = n$.

(a). For all $A \in IFS(X)$,

$$m_1(A) = \frac{1}{2}(\inf_{x\in X} \mu_A(x) + (1 - \sup_{x\in X} \nu_A(x))). \tag{3.4}$$

We can verify that $m_1(A)$ satisfies the conditions given in Definition 3.1.2:

(i). $m_1(\widetilde{0}_X) = \frac{1}{2}(\inf_{x\in X}(0) + (1 - \sup_{x\in X}(1)) = 0.$

(ii). For any $A, B \in \pounds$, where $(X, \pounds)$ be an intuitionistic fuzzy measurable space

$A \subseteq B$ implies that $\mu_A(x) \leq \mu_B(x)$ and $\nu_A(x) \geq \nu_B(x)$ for all $x \in X$

implies that $\inf_{x\in X}(\mu_A(x)) \leq \inf_{x\in X}(\mu_B(x))$ and $-\sup_{x\in X}(\nu_A(x)) \leq -\sup_{x\in X}(\nu_B(x))$

implies that $\frac{1}{2}(\inf_{x\in X}\mu_A(x) + (1 - \sup_{x\in X}\nu_A(x))) \leq \frac{1}{2}(\inf_{x\in X}\mu_B(x) + (1 - \sup_{x\in X}\nu_B(x)))$

implies that $m_1(A) \leq m_1(B)$.

In addition it can be easily verified that $m_1(\widetilde{1}_X) = 1$.

(b). For all $A \in IFS(X)$,

$$m_2(A) = \frac{1}{4}(\inf_{x\in X}\mu_A(x) + (1 - \sup_{x\in X}\nu_A(x)) + \sup_{x\in X}\mu_A(x) + (1 - \inf_{x\in X}\nu_A(x))). \qquad (3.5)$$

Similarly to m_1, we can verify that measure m_2 satisfies the conditions of Definition 3.1.2, i.e.,

(i). $m_2(\widetilde{0}_X) = \frac{1}{4}(\inf_{x\in X}(0) + (1 - \sup_{x\in X}(1)) + \sup_{x\in X}(0) + (1 - \inf_{x\in X}(1))) = 0$.

(ii). For any $A, B \in \pounds$, where $(X, \pounds)$ be an intuitionistic fuzzy measurable space

$A \subseteq B$ implies that $\mu_A(x) \leq \mu_B(x)$ and $\nu_A(x) \geq \nu_B(x)$ for all $x \in X$

implies that $\inf_{x\in X}(\mu_A(x)) \leq \inf_{x\in X}(\mu_B(x))$ and $\sup_{x\in X}(\nu_A(x)) \geq \sup_{x\in X}(\nu_B(x))$

and $\sup_{x\in X}(\mu_A(x)) \leq \sup_{x\in X}(\mu_B(x))$ and $\inf_{x\in X}(\nu_A(x)) \geq \inf_{x\in X}(\nu_B(x))$

implies that $\inf_{x\in X}(\mu_A(x)) \leq \inf_{x\in X}(\mu_B(x))$ and $1 - \sup_{x\in X}(\nu_A(x)) \leq 1 - \sup_{x\in X}(\nu_B(x))$

and $\sup_{x\in X}(\mu_A(x)) \leq \sup_{x\in X}(\mu_B(x))$ and $1 - \inf_{x\in X}(\nu_A(x)) \leq 1 - \inf_{x\in X}(\nu_B(x))$

implies that $\frac{1}{4}(\inf_{x\in X}\mu_A(x) + (1 - \sup_{x\in X}\nu_A(x)) + \sup_{x\in X}\mu_A(x) + (1 - \inf_{x\in X}\nu_A(x)))$

$\leq \frac{1}{4}(\inf_{x\in X}\mu_B(x) + (1 - \sup_{x\in X}\nu_B(x)) + \sup_{x\in X}\mu_B(x) + (1 - \inf_{x\in X}\nu_B(x)))$

implies that $m_2(A) \leq m_2(B)$.

Moreover, $m_2(\widetilde{1}_X) = \frac{1}{4}(\inf_{x\in X}(1) + (1 - \sup_{x\in X}(0)) + \sup_{x\in X}(1) + (1 - \inf_{x\in X}(0))) = 1$.

(c). For all $A \in IFS(X)$,

$$m_3(A) = Card_\theta(A) = \sum_{x\in X} \theta\mu_A(x) + (1-\theta)(1 - \nu_A(x)) \qquad (3.6)$$

$$m_3^{'}(A) = \frac{Card_\theta(A)}{|X|} = \frac{\sum_{x\in X} \theta\mu_A(x) + (1-\theta)(1 - \nu_A(x))}{n} \qquad (3.7)$$

where $\theta \in [0.5, 1]$.

$Card_\theta(A)$ is the weighted average cardinality measure of intuitionistic fuzzy set

A which is family of cardinality measures of intuitionistic fuzzy sets for different choices of $\theta \in [0.5, 1]$.

The difference between m_3' and m_3 is that the first one has been normalized which we focus on as a normal fuzzy measure.

(i). Using Definition 3.1.2 we obtain $m_3'(\widetilde{0}_X) = \frac{\sum_{x\in X} \theta(0)+(1-\theta)(1-(1))}{n} = 0$.

(ii). For any $A, B \in IFS(X)$ such that $A \subseteq B$

implies that $\mu_A(x) \leq \mu_B(x)$ and $\nu_A(x) \geq \nu_B(x)$ for all $x \in X$

implies that $\theta\mu_A(x) \leq \theta\mu_B(x)$ and $(1-\theta)(1-\nu_A(x)) \leq (1-\theta)(1-\nu_B(x))$ for all $x \in X$

implies that $\frac{\sum_{x\in X} \theta\mu_A(x)+(1-\theta)(1-\nu_A(x))}{n} \leq \frac{\sum_{x\in X} \theta\mu_B(x)+(1-\theta)(1-\nu_B(x))}{n}$

implies that $m_3'(A) \leq m_3'(B)$.

Similarly, $m_3'(\widetilde{1}_X) = \frac{\sum_{x\in X} \theta(1)+(1-\theta)(1-(0))}{n} = 1$.

Next, we introduce the last normal fuzzy measure of intuitionistic fuzzy set that also satisfies the conditions given in Definition 3.1.2.

(d). For all $A \in IFS(X)$,

$$m_4(A) = \frac{\sum_{x\in X} \sqrt{\mu_A(x)(1-\nu_A(x))}}{n}. \tag{3.8}$$

(i). $m_4(\widetilde{0}_X) = \frac{\sum_{x\in X} \sqrt{(0)(1-1)}}{n} = 0$.

(ii). For any $A, B \in IFS(X)$ such that $A \subseteq B$

implies that $\mu_A(x) \leq \mu_B(x)$ and $\nu_A(x) \geq \nu_B(x)$ for all $x \in X$

implies that $\sqrt{\mu_A(x)(1-\nu_A(x))} \leq \sqrt{\mu_B(x)(1-\nu_B(x))}$

implies that $\frac{\sum_{x\in X} \sqrt{\mu_A(x)(1-\nu_A(x))}}{n} \leq \frac{\sum_{x\in X} \sqrt{\mu_B(x)(1-\nu_B(x))}}{n}$

implies that $m_4(A) \leq m_4(B)$.

Likewise, $m_4(\widetilde{1}_X) = \frac{\sum_{x\in X} \sqrt{1(1-0)}}{n} = \frac{n}{n} = 1$.

Remark 3.1.3 It is sometimes useful to require the fulfillment of the following properties by a fuzzy measure $m(A)$ of $A \in IFS(X)$ i.e.,

(a). $m(A) = 1$ implies that $A = \widetilde{1}_X$;

(b). $m(A) = 0$ implies that $A = \widetilde{0}_X$.

We shall verify that new defined measures $m_2(A)$ and $m_3(A)$ satisfy both of these properties while the measure $m_1(A)$ satisfies only the first property.

(1-a). Let $m_2(A) = 1$

implies that $(\inf_{x\in X}\mu_A(x) + (1 - \sup_{x\in X}\nu_A(x)) + \sup_{x\in X}\mu_A(x) + (1 - \inf_{x\in X}\nu_A(x))) = 4$

implies that $\inf_{x\in X}\mu_A(x) = \sup_{x\in X}\mu_A(x) = 1$ and $\sup_{x\in X}\nu_A(x) = \inf_{x\in X}\nu_A(x) = 0$

implies that $A = \widetilde{1}_X$.

(1-b). Let $m_2(A) = 0$

implies that $(\inf_{x\in X}\mu_A(x) + (1 - \sup_{x\in X}\nu_A(x)) + \sup_{x\in X}\mu_A(x) + (1 - \inf_{x\in X}\nu_A(x))) = 0$

implies that $\inf_{x\in X}\mu_A(x) = \sup_{x\in X}\nu_A(x) = 0$ and $(1 - \sup_{x\in X}\nu_A(x)) = (1 - \inf_{x\in X}\nu_A(x))) = 0$

implies that $\mu_A(x) = 0$ and $\nu_A(x) = 1$ for all $x \in X$

implies that $A = \widetilde{0}_X$.

(2-a). Let $m_3(A) = 1$

implies that $\sum_{x\in X} \theta\mu_A(x) + (1-\theta)(1-\nu_A(x)) = n$

implies that $\theta\mu_A(x) + (1-\theta)(1-\nu_A(x)) = 1$ for all $x \in X$

implies that $\theta\mu_A(x) + (1-\theta)\nu_A(x) = 1 - (1-\theta)$ for all $x \in X$

implies that $\theta\mu_A(x) + (1-\theta)\nu_A(x) = \theta$ for all $x \in X$

implies that $\mu_A(x) = 1$ and $\nu_A(x) = 0$ for all $x \in X$

implies that $A = \widetilde{1}_X$.

(2-b). Let $m_3(A) = 0$

implies that $\sum_{x\in X} \theta\mu_A(x) + (1-\theta)(1-\nu_A(x)) = 0$

now as $\mu_A(x), (1-\nu_A(x)), \theta$ and $(1-\theta) \in [0,1]$ for all $x \in X$

implies that $\theta\mu_A(x) + (1-\theta)(1-\nu_A(x)) = 0$ for all $x \in X$

implies that $\theta\mu_A(x) - (1-\theta)\nu_A(x) = -(1-\theta)$ for all $x \in X$

implies that $\mu_A(x) = 0$ and $\nu_A(x) = 1$ for all $x \in X$

implies that $A = \widetilde{0}_X$.

(3-a). Let $m_1(A) = 1$

implies that $\inf_{x\in X}\mu_A(x) + (1 - \sup_{x\in X}\nu_A(x)) = 2$

implies that $\inf_{x\in X}\mu_A(x) = 1$ and $\sup_{x\in X}\nu_A(x) = 0$

implies that $\mu_A(x) = 1$ and $\nu_A(x) = 0$ for all $x \in X$

implies that $A = \widetilde{1}_X$.

Next, we show that m_1 does not satisfy the second property i.e.

$m_1(A) = 0$

implies that $\inf_{x\in X}(\mu_A(x)) + (1 - \sup_{x\in X}\nu_A(x)) = 0$

implies that $\inf_{x\in X}(\mu_A(x)) + 1 = \sup_{x\in X}(\nu_A(x))$

implies that $\inf_{x\in X}(\mu_A(x)) = 0$ and $\sup_{x\in X}(\nu_A(x)) = 1$

it further does not implies that $\mu_A(x) = 1$ and $\nu_A(x) = 0$ for all $x \in X$.

Hence $A \neq \widetilde{0}_X$.

3.2 A New Measure of Cardinality on Intuitionistic Fuzzy Sets

The results produced in this section as well as the one following it are a part of our submitted paper [124].

Since the main part of this thesis is based on cardinality based constructions, it seems appropriate to study in detail the properties of m_3 as the measure of cardinality. We observe that m_3 satisfies properties similar to the ones introduced for fuzzy sets in [160]. Moreover, we have studied some basic characteristics of this family of cardinality measure especially the valuation property, the subadditivity property, the complementary rule defined using t-norms and negations on L^*.

Definition 3.2.1 A mapping $Card_\theta(A) : IFS(X) \longrightarrow [0, \infty)$ is called the *weighted average cardinality of intuitionistic fuzzy sets* given as:

$$\begin{aligned} Card_\theta(A) &= \sum_{x\in X} \theta\mu_A(x) + (1-\theta)(1-\nu_A(x)); \\ \text{where } \theta &\in [0.5, 1]. \end{aligned} \tag{3.9}$$

It is easy to observe that $Card_\theta(A)$ satisfies the following properties for all $\theta \in [0.5, 1]$:

(i). Coincidence: For all $x \in X$, $Card_\theta(A(x)) = 1$ when $A(x) = 1_{L^*} \in L^*$.

(ii). Monotonicity: For all $A, B \in IFS(X)$,

$$A \subseteq B \text{ implies } Card_\theta(A) \leq Card_\theta(B). \tag{3.10}$$

(iii). Additivity: For all $A, B \in IFS(X)$,

$Supp(A) \cap Supp(B) = \phi$ implies $Card_\theta(A \cup B) = Card_\theta(A) + Card_\theta(B)$.

Since we have, already, mentioned in Section 2.2.3, Chapter 2, for $A, B \in IFS(X)$, we shall model the generalized $\cup$ and $\cap$ between two IFS by t-representable intuitionistic fuzzy Frank t-norm $\overline{T}(x, y) = (T^F_\omega(x_1, y_1), S^F_\omega(x_2, y_2))$ and their dual intuitionistic fuzzy Frank conorms $\overline{S}(x, y) = (S^F_\omega(x_1, y_1), T^F_\omega(x_2, y_2))$ [see Example 2.11, 2.15] where $x, y \in L^*$, such that their representant family of t-norms $T^F_\omega(a, b)$ and their dual conorms $S^F_\omega(a, b)$ satisfy the valuation property stated as: $T^F_\omega(a, b) + S^F_\omega(a, b) = a + b$ for all $a, b \in [0, 1]$ and $\omega = 0, 1, +\infty$. Therefore, this combination of t-norm $\overline{T}$ and conorm $\overline{S}$ becomes the most suitable choice to study the properties of the new defined weighted average cardinality measure for intuitionistic fuzzy sets.

Also, for an intuitionistic fuzzy t-norm $\overline{T}$ with no zero divisor, i.e., $\overline{T}(x, y) = 0_{L^*}$ implies $x = 0_{L^*}$ or $y = 0_{L^*}$ and its dual t-conorm $\overline{S}$, the third property of $Card_\theta$ of IFS can be equivalently defined by the following statements:

(i). For $A, B \in IFS(X)$, $Supp(A) \cap Supp(B) = \phi$ implies $Card_\theta(A \cup_{\overline{S}} B) = Card_\theta(A) + Card_\theta(B)$;

(ii). For $A, B \in IFS(X)$, $A \cap_{\overline{T}} B = \widetilde{0}_X$ implies $Card_\theta(A \cup_{\overline{S}} B) = Card_\theta(A) + Card_\theta(B)$.

Proposition 3.2.2 Let $A, B \in IFS(X)$. The following properties hold for $Card_\theta$

(a). Valuation property: $Card_\theta(A \cup_{\overline{S}} B) + Card_\theta(A \cap_{\overline{T}} B) = Card_\theta(A) + Card_\theta(B)$;

(b). Complementary rule: $Card_\theta(A) + Card_\theta(A^c) = Card_\theta(X)$ if and only if $\theta = 0.5$;

(c). Coincidence: $Card_\theta(A) = |Ker(A)|$, if A is a crisp set;

where generalized intersection $\cap_{\overline{T}}$, union $\cup_{\overline{S}}$, complement A^c of intuitionistic fuzzy sets are defined using three basic intuitionistic fuzzy Frank t-norms their dual conorms and the standard negations operator defined on L^*.

Proof

(a). We shall construct three independent results of valuation property using t-representable intuitionistic fuzzy frank t-norms:

$\overline{T}_M(A(x), B(x)) = (T_M(\mu_A(x), \mu_B(x)), S_M(\nu_A(x), \nu_B(x)))$,

$\overline{T}_P(A(x), B(x)) = (T_P(\mu_A(x), \mu_B(x)), S_P(\nu_A(x), \nu_B(x)))),$

$\overline{T}_L(A(x), B(x)) = (T_L(\mu_A(x), \mu_B(x)), S_L(\nu_A(x), \nu_B(x))))$ and, their dual conorms

$\overline{S}_M(A(x), B(x)) = (S_M(\mu_A(x), \mu_B(x)), T_M(\nu_A(x), \nu_B(x)))),$

$\overline{S}_P(A(x), B(x)) = (S_P(\mu_A(x), \mu_B(x)), T_P(\nu_A(x), \nu_B(x))))$ and

$\overline{S}_L(A(x), B(x)) = (T_L(\mu_A(x), \mu_B(x)), S_L(\nu_A(x), \nu_B(x))))$

such that $A(x) = (\mu_A(x), \nu_A(x))$ and $B(x) = (\mu_B(x), \nu_B(x))$, i.e.; we shall prove:

(a-i). $Card_\theta(A \cup_{\overline{S}_M} B) + Card_\theta(A \cap_{\overline{T}_M} B) = Card_\theta(A) + Card_\theta(B)$.

(a-ii). $Card_\theta(A \cup_{\overline{S}_P} B) + Card_\theta(A \cap_{\overline{T}_P} B) = Card_\theta(A) + Card_\theta(B)$.

(a-iii). $Card_\theta(A \cup_{\overline{S}_L} B) + Card_\theta(A \cap_{\overline{T}_L} B) = Card_\theta(A) + Card_\theta(B)$.

(a-i). For all $A, B \in IFS(X)$, $(A \cap_{\overline{T}_M} B)(x) =$
$(T_M(\mu_A(x), \mu_B(x)), S_M(\nu_A(x), \nu_B(x)))$ and
$(A \cup_{\overline{S}_M} B)(x) = (S_M(\mu_A(x), \mu_B(x)), T_M(\nu_A(x), \nu_B(x)))$, implies that

$$Card_\theta(A \cap_{\overline{T}_M} B) = \sum_{x_i \in X} \theta \min(\mu_A(x_i), \mu_B(x_i)) + (1-\theta) \min((1-\nu_A(x_i)), (1-\nu_B(x_i)));$$

$$Card_\theta(A \cup_{\overline{S}_M} B) = \sum_{x_i \in X} \theta \max(\mu_A(x_i), \mu_B(x_i)) + (1-\theta) \max((1-\nu_A(x_i)), (1-\nu_B(x_i)));$$

which together implies that $Card_\theta(A \cup_{\overline{S}_M} B) + Card_\theta(A \cap_{\overline{T}_M} B) = Card_\theta(A) + Card_\theta(B)$.

(a-ii). For all $A, B \in IFS(X)$, $(A \cap_{\overline{T}_P} B)(x) =$
$(T_P(\mu_A(x), \mu_B(x)), S_P(\nu_A(x), \nu_B(x)))$
and $(A \cup_{\overline{S}_P} B)(x) = (S_P(\mu_A(x), \mu_B(x)), T_P(\nu_A(x), \nu_B(x)))$, implies that

$$Card_\theta(A \cap_{\overline{T}_P} B) = \sum_{x_i \in X} \theta(\mu_A(x_i)\mu_B(x_i)) + (1-\theta)(1 - \nu_A(x_i) - \nu_B(x_i) + \nu_A(x_i)\nu_B(x_i));$$

$$Card_\theta(A \cup_{\overline{S}_P} B) = \sum_{x_i \in X} \theta(\mu_A(x_i) + \mu_B(x_i) - \mu_A(x_i)\mu_B(x_i)) + (1-\theta)(1 - \nu_A(x_i)\nu_B(x_i)).$$

Hence $Card_\theta(A \cup_{\overline{S}_P} B) + Card_\theta(A \cap_{\overline{T}_P} B) = Card_\theta(A) + Card_\theta(B)$.

(a-iii). For all $A, B \in IFS(X)$, $(A \cap_{\overline{T}_L} B)(x) = (T_L(\mu_A(x), \mu_B(x)), S_L(\nu_A(x), \nu_B(x)))$ and $(A \cup_{\overline{S}_L} B)(x) = (S_L(\mu_A(x), \mu_B(x)), T_L(\nu_A(x), \nu_B(x)))$, consider the right hand side:

$$Card_\theta(A \cup_{\overline{S}_L} B) + Card_\theta(A \cap_{\overline{T}_L} B) = \sum_{x_i \in X} \theta \min(1, \mu_A(x_i) + \mu_B(x_i))) + (1-\theta) \max(0, \mu_A(x_i) + \mu_B(x_i) - 1))$$

$$+ \sum_{x_i \in X} \theta \min(1, (1-\nu_A(x_i)) + (1-\nu_B(x_i))) + (1-\theta) \max(0, (1-\nu_A(x_i)) + (1-\nu_B(x_i)) - 1)).$$

Now without the loss of generality we take

$$\min(1, \mu_A(x_i) + \mu_B(x_i)) = \mu_A(x_i) + \mu_B(x_i) \text{ and } \max(0, \mu_A(x_i) + \mu_B(x_i) - 1) = 0.$$

This choice leads to:
$\min(1, (1 - \nu_A(x_i)) + (1 - \nu_B(x_i))) = 1$ and
$\max(0, (1 - \nu_A(x_i)) + (1 - \nu_B(x_i)) - 1) = (1 - \nu_A(x_i)) + (1 - \nu_B(x_i)) - 1$
which leads to $Card_\theta(A \cup_{\overline{S}_L} B) + Card_\theta(A \cap_{\overline{T}_L} B) = Card_\theta(A) + Card_\theta(B)$.
The other choice also leads to the same results.
(b). Let $Card_\theta(A) + Card_\theta(A^c) = Card_\theta(X)$
if and only if $\sum_{x \in X} \theta\mu_A(x) + (1-\theta)(1 - \nu_A(x)) + \sum_{x \in X} \theta\nu_A(x) + (1-\theta)(1 - \mu_A(x))$
$= \sum_{x \in X} \theta + (1-\theta) = \sum_{x \in X} 1$
if and only if $\theta\mu_A(x) + (1-\theta)(1 - \nu_A(x)) + \theta\nu_A(x) + (1-\theta)(1 - \mu_A(x)) = 1$ for all $x \in X$
if and only if $2(1-\theta) = 1$
if and only if $1 - \theta = 0.5$
if and only if $\theta = 0.5$.
(c). Let A be crisp set. Then in terms of intuitionistic fuzzy set
$A = \{(x, 1, 0) \mid x \in X\} = Ker(A)$
implies $Card_\theta(A) = |Ker(A)|$.
Theorem 3.2.3 For any $A, B \in IFS(X)$,
(a). $Card_\theta((A \cup_{\overline{S}_M} B)^c) + Card_\theta((A \cap_{\overline{T}_M} B)^c) = Card_\theta(A^c) + Card_\theta(B^c)$;
(b). $Card_\theta((A \cup_{\overline{S}_P} B)^c) + Card_\theta((A \cap_{\overline{T}_P} B)^c) = Card_\theta(A^c) + Card_\theta(B^c)$;
(c). $Card_\theta((A \cup_{\overline{S}_L} B)^c) + Card_\theta((A \cap_{\overline{T}_L} B)^c) = Card_\theta(A^c) + Card_\theta(B^c)$.
Proof
(a). For all $A, B \in IFS(X)$, $(A \cap_{\overline{T}_M} B)^c(x) = (S_M(\nu_A(x), \nu_B(x)), T_M(\mu_A(x), \mu_B(x)))$ and
$(A \cup_{\overline{S}_M} B)^c(x) = (T_M(\nu_A(x), \nu_B(x)), S_M(\mu_A(x), \mu_B(x)))$ implies that
$Card_\theta(A \cup_{\overline{S}_M} B)^c = \sum_{x_i \in X} \theta \min(\nu_A(x_i), \nu_B(x_i)) + (1-\theta) \min((1-\mu_A(x_i)), (1-\mu_B(x_i)))$;
$Card_\theta(A \cap_{\overline{T}_M} B)^c = \sum_{x_i \in X} \theta \max(\nu_A(x_i), \nu_B(x_i)) + (1-\theta) \max((1-\mu_A(x_i)), (1-\mu_B(x_i)))$;

which together implies that $Card_\theta(A \cup_{\overline{S}_M} B)^c + Card_\theta(A \cap_{\overline{T}_M} B)^c = Card_\theta(A^c) + Card_\theta(B^c)$.

(b). For all $A, B \in IFS(X)$, $(A \cap_{\overline{T}_P} B)^c(x) = (S_P(\nu_A(x), \nu_B(x)), T_P(\mu_A(x), \mu_B(x)))$ and $(A \cup_{\overline{S}_P} B)^c = (T_P(\nu_A(x), \nu_B(x)), S_P(\mu_A(x), \mu_B(x)))$ implies that

$Card_\theta(A \cup_{\overline{S}_P} B)^c = \sum_{x_i \in X} \theta(\nu_A(x_i)\nu_B(x_i)) + (1-\theta)(1-\mu_A(x_i)-\mu_B(x_i)+\mu_A(x_i)\mu_B(x_i))$;

$Card_\theta(A \cap_{\overline{T}_P} B)^c = \sum_{x_i \in X} \theta(\nu_A(x_i)+\nu_B(x_i)-\nu_A(x_i)\nu_B(x_i)) + (1-\theta)(1-\mu_A(x_i)\mu_B(x_i))$.

Hence $Card_\theta(A \cup_{\overline{S}_P} B)^c + Card_\theta(A \cap_{\overline{T}_P} B)^c = Card_\theta(A^c) + Card_\theta(B^c)$.

(c). For all $A, B \in IFS(X)$, $(A \cup_{\overline{S}_L} B)^c(x) = (T_L(\nu_A(x), \nu_B(x)), S_L(\mu_A(x), \mu_B(x)))$ and $(A \cap_{\overline{T}_L} B)^c(x) = (S_L(\nu_A(x), \nu_B(x)), T_L(\mu_A(x), \mu_B(x)))$, consider right hand side:

$Card_\theta(A \cap_{\overline{T}_L} B)^c + Card_\theta(A \cup_{\overline{S}_L} B)^c = \sum_{x_i \in X} \theta \min(1, \nu_A(x_i)+\nu_B(x_i)) + (1-\theta)\max(0, (1-\mu_A(x_i)) + (1-\mu_B(x_i)) - 1))$

$+ \sum_{x_i \in X} \theta \max(0, \nu_A(x_i)+\nu_B(x_i)-1)) + (1-\theta)\min(1, (1-\mu_A(x_i)) + (1-\mu_B(x_i)))$.

Now without the loss of generality we take

$$\min(1, \nu_A(x_i)+\nu_B(x_i)) = \nu_A(x_i)+\nu_B(x_i) \text{ and } \max(0, \nu_A(x_i)+\nu_B(x_i)-1)) = 0.$$

This choice leads to:

$\min(1, (1-\mu_A(x_i)) + (1-\mu_B(x_i))) = 1$ and

$\max(0, (1-\mu_A(x_i)) + (1-\mu_B(x_i)) - 1) = (1-\mu_A(x_i)) + (1-\mu_B(x_i)) - 1$

which leads to $Card_\theta(A \cap_{\overline{T}_L} B)^c + Card_\theta(A \cup_{\overline{S}_L} B)^c = Card_\theta(A^c) + Card_\theta(B^c)$.

The other choice also leads to the same results.

3.3 Results of $Card_\theta(A)$ on Possibility and Necessity Operators

Among several interesting operators introduced in the family of all intuitionistic fuzzy sets [7], the two operators called by Atanassov the necessity and possibility operators have gained a substantial attention of the research community. For any $A \in IFS(X)$, the necessity operator $\Box : IFS(X) \longrightarrow IFS(X)$ is defined as:

$$\Box A = \Big\{(x_i, \mu_A(x_i), 1-\mu_A(x_i)) \mid x_i \in X\Big\} \tag{3.11}$$

and the possibility operator $\Diamond : IFS(X) \longrightarrow IFS(X)$ is defined as:

$$\Diamond A = \Big\{(x_i, 1 - \nu_A(x_i), \nu_A(x_i)) \mid x_i \in X\Big\}. \tag{3.12}$$

Both of these operators had exhibited many interesting properties. Besides their connection with the well known possibility theory, the terms possibility and necessity are clear even to those practitioners who are not expert in this field. A natural and straightforward interpretation of these two ideas made them useful in many different fields. For example, let us mention the problem of ranking fuzzy numbers, where hundreds of methods have been proposed but its formulation in the setting of possibility theory have resulted in generally accepted tool; the possibility/necessity indices of dominance; which are successfully applied in many situations.

In this subsection, we shall explore different aspects of the new defined weighted average cardinality measure $Card_\theta(A) = \sum_{x\in X} \theta\mu_A(x) + (1-\theta)(1-\nu_A(x))$ when applied to modal operators $\Box A$ and $\Diamond A$.

Remark 3.3.1 For any $A \in IFS(X)$, the following hold:

(a). $Card_\theta(\Box A) = Card_\theta((\Diamond A^c)^c)$;

(b). $Card_\theta(\Diamond A) = Card_\theta((\Box A^c)^c)$;

(c). $Card_\theta(\Box\Box A) = Card_\theta(\Box A)$;

(d). $Card_\theta(\Box\Diamond A) = Card_\theta(\Diamond A)$;

(e). $Card_\theta(\Diamond\Box A) = Card_\theta(\Box A)$;

(f). $Card_\theta(\Diamond\Diamond A) = Card_\theta(\Diamond A)$;

(g). $Card_\theta(\Diamond A^c) = Card_\theta((\Box A)^c)$;

(h). $Card_\theta(\Box A^c) = Card_\theta((\Diamond A)^c)$;

(i). $Card_\theta(\Box A) \leq Card_\theta(\Diamond A)$.

Theorem 3.3.2 For any $A, B \in IFS(X)$, we have:

(a). $Card_\theta\Box(A \cup_{\overline{S}_M} B) = Card_\theta(\Box A \cup_{\overline{S}_M} \Box B)$;

(b). $Card_\theta\Diamond(A \cup_{\overline{S}_M} B) = Card_\theta(\Diamond A \cup_{\overline{S}_M} \Diamond B)$;

(c). $Card_\theta\Box(A \cap_{\overline{T}_M} B) = Card_\theta(\Box A \cap_{\overline{T}_M} \Box B)$;

(d). $Card_\theta\Diamond(A \cap_{\overline{T}_M} B) = Card_\theta(\Diamond A \cap_{\overline{T}_M} \Diamond B)$;

(e). $Card_\theta\Box(A \cup_{\overline{S}_P} B) = Card_\theta(\Box A \cup_{\overline{S}_P} \Box B)$;

(f). $Card_\theta \Diamond(A \cup_{\overline{S}_P} B) = Card_\theta(\Diamond A \cup_{\overline{S}_P} \Diamond B)$;

(g). $Card_\theta \Box(A \cap_{\overline{T}_P} B) = Card_\theta(\Box A \cap_{\overline{T}_P} \Box B)$;

(h). $Card_\theta \Diamond(A \cap_{\overline{T}_P} B) = Card_\theta(\Diamond A \cap_{\overline{T}_P} \Diamond B)$;

(i). $Card_\theta \Box(A \cup_{\overline{S}_L} B) = Card_\theta(\Box A \cup_{\overline{S}_L} \Box B)$;

(j). $Card_\theta \Diamond(A \cup_{\overline{S}_L} B) = Card_\theta(\Diamond A \cup_{\overline{S}_L} \Diamond B)$;

(k). $Card_\theta \Box(A \cap_{\overline{T}_L} B) = Card_\theta(\Box A \cap_{\overline{T}_L} \Box B)$;

(l). $Card_\theta \Diamond(A \cap_{\overline{T}_L} B) = Card_\theta(\Diamond A \cap_{\overline{T}_L} \Diamond B)$.

Proof

(a). For any $A, B \in IFS(X)$,

$$\begin{aligned}
\Box(A \cup_{\overline{S}_M} B) &= \Big\{(x, \max(\mu_A(x_i), \mu_B(x_i)), 1 - \max(\mu_A(x_i), \mu_B(x_i))) \mid x_i \in X\Big\} \\
&= \Big\{(x, \max(\mu_A(x_i), \mu_B(x_i)), \min(1 - \mu_A(x_i), 1 - \mu_B(x_i))) \mid x_i \in X\Big\} \\
&= \Box A \cup_{\overline{S}_M} \Box B.
\end{aligned}$$

Since the two sets are equal, by the basic monotonicity property of $Card_\theta$

$Card_\theta(\Box(A \cup_{\overline{S}_M} B)) = Card_\theta(\Box A \cup_{\overline{S}_M} \Box B)$.

(b). For any $A, B \in IFS(X)$,

$$\begin{aligned}
\Diamond(A \cup_{\overline{S}_M} B) &= \Big\{(x, 1 - \min(\nu_A(x_i), \nu_B(x_i)), \min(\nu_A(x_i), \nu_B(x_i))) \mid x_i \in X\Big\} \\
&= \Big\{(x, \max(1 - \nu_A(x_i), 1 - \nu_B(x_i)), \min(\nu_A(x_i), \nu_B(x_i))) \mid x_i \in X\Big\} \\
& \Diamond A \cup_{\overline{S}_M} \Diamond B.
\end{aligned}$$

Since the two sets are equal, by the basic monotonicity property of $Card_\theta$

$Card_\theta \Diamond(A \cup_{\overline{S}_M} B) = Card_\theta(\Diamond A \cup_{\overline{S}_M} \Diamond B)$.

The proofs of the remaining parts can be constructed in similar manner.

Theorem 3.3.3 For any $A, B \in IFS(X)$,

(a). $Card_\theta \Box(A \cup_{\overline{S}_M} B) + Card_\theta \Box(A \cap_{\overline{T}_M} B) = Card_\theta \Box(A) + Card_\theta \Box(B)$;

(b). $Card_\theta \Diamond(A \cup_{\overline{S}_M} B) + Card_\theta \Diamond(A \cap_{\overline{T}_M} B) = Card_\theta \Diamond(A) + Card_\theta \Diamond(B)$;

(c). $Card_\theta \Box(A \cup_{\overline{S}_P} B) + Card_\theta \Box(A \cap_{\overline{T}_P} B) = Card_\theta \Box(A) + Card_\theta \Box(B)$;

(d). $Card_\theta \Diamond(A \cup_{\overline{S}_P} B) + Card_\theta \Diamond(A \cap_{\overline{T}_P} B) = Card_\theta \Diamond(A) + Card_\theta \Diamond(B)$;

(e). $Card_\theta \Box(A \cup_{\overline{S}_L} B) + Card_\theta \Box(A \cap_{\overline{T}_L} B) = Card_\theta \Box(A) + Card_\theta \Box(B)$;

(f). $Card_\theta \Diamond(A \cup_{\overline{S}_L} B) + Card_\theta \Diamond(A \cap_{\overline{T}_L} B) = Card_\theta \Diamond(A) + Card_\theta \Diamond(B)$.

Proof

(a). From Theorem 3.3.2 (a) and (c) we have that, for any $A, B \in IFS(X)$,

$Card_\theta \Box(A \cup_{\overline{S}_M} B) + Card_\theta \Box(A \cap_{\overline{T}_M} B) = Card_\theta(\Box A \cup_{\overline{S}_M} \Box B) + Card_\theta(\Box A \cap_{\overline{T}_M} \Box B)$

$$= \sum_{x_i \in X} \theta \max(\mu_A(x_i), \mu_B(x_i)) + (1-\theta)(1 - \min((1-\mu_A(x_i)), (1-\mu_B(x_i))))$$
$$+ \sum_{x_i \in X} \theta \min(\mu_A(x_i), \mu_B(x_i)) + (1-\theta)(1 - \max((1-\mu_A(x_i)), (1-\mu_B(x_i))))$$
$$= \sum_{x_i \in X} \theta \max(\mu_A(x_i), \mu_B(x_i)) + (1-\theta) \max(\mu_A(x_i), \mu_B(x_i))$$
$$+ \sum_{x_i \in X} \theta \min(\mu_A(x_i), \mu_B(x_i)) + (1-\theta) \min(\mu_A(x_i), \mu_B(x_i))$$
$$= Card_\theta \Box(A) + Card_\theta \Box(B).$$

(b). From Theorem 3.3.2 (b) and (d) we have that:

$Card_\theta \Diamond(A \cup_{\overline{S}_M} B) + Card_\theta \Diamond(A \cap_{\overline{T}_M} B) = Card_\theta(\Diamond A \cup_{\overline{S}_M} \Diamond B) + Card_\theta(\Diamond A \cap_{\overline{T}_M} \Diamond B)$

$$= \sum_{x_i \in X} \theta \max(1-\nu_A(x_i), 1-\nu_B(x_i)) + (1-\theta)(1 - \min(\nu_A(x_i), \nu_B(x_i)))$$
$$+ \sum_{x_i \in X} \theta(\min(1-\nu_A(x_i), 1-\nu_B(x_i))) + (1-\theta)(1 - \max(\nu_A(x_i), \nu_B(x_i)))$$
$$= \sum_{x_i \in X} \theta \max(1-\nu_A(x_i), 1-\nu_B(x_i)) + (1-\theta)(\max(1-\nu_A(x_i), 1-\nu_B(x_i)))$$
$$+ \sum_{x_i \in X} \theta(\min(1-\nu_A(x_i), 1-\nu_B(x_i))) + (1-\theta)(\min(1-\nu_A(x_i), 1-\nu_B(x_i)))$$
$$= Card_\theta \Diamond(A) + Card_\theta \Diamond(B).$$

(c). From Theorem 3.3.2 (e) and (g) we have that:

$Card_\theta \Box(A \cup_{\overline{S}_P} B) + Card_\theta \Box(A \cap_{\overline{T}_P} B) = Card_\theta(\Box A \cup_{\overline{S}_P} \Box B) + Card_\theta(\Box A \cap_{\overline{T}_P} \Box B)$

$$= \sum_{x_i \in X} \theta(\mu_A(x_i) + \mu_B(x_i) - \mu_A(x_i)\mu_B(x_i)) + (1-\theta)(1 - ((1-\mu_A(x_i))(1-\mu_B(x_i))))$$
$$+ \sum_{x_i \in X} \theta(\mu_A(x_i)\mu_B(x_i)) + (1-\theta)(1 - ((1-\mu_A(x_i)) + (1-\mu_B(x_i)) - (1-\mu_A(x_i)(1-\mu_A(x_i)))$$
$$= \sum_{x_i \in X} \theta\mu_A(x_i) + \theta\mu_B(x_i) + (1-\theta)(1 - ((1-\mu_A(x_i))) + (1-\theta)(1 - ((1-\mu_B(x_i)))$$
$$= Card_\theta \Box(A) + Card_\theta \Box(B).$$

(d). From Theorem 3.3.2 (f) and (h) we have that:

$Card_\theta \Diamond(A \cup_{\overline{S}_P} B) + Card_\theta \Diamond(A \cap_{\overline{T}_P} B) = Card_\theta(\Diamond A \cup_{\overline{S}_P} \Diamond B) + Card_\theta(\Diamond A \cap_{\overline{T}_P} \Diamond B)$

$= \sum_{x_i \in X} \theta((1-\nu_A(x_i)) + (1-\nu_B(x_i)) - (1-\nu_A(x_i))(1-\nu_B(x_i))) + (1-\theta)(1-$
$\nu_A(x_i)\nu_B(x_i))$

$+ \sum_{x_i \in X} \theta((1-\nu_A(x_i))(1-\nu_B(x_i))) + (1-\theta)(1-(\nu_A(x_i)+\nu_B(x_i)-\nu_A(x_i)\nu_B(x_i)))$

$= \sum_{x_i \in X} \theta((1-\nu_A(x_i)) + (1-\nu_B(x_i)) - (1-\nu_A(x_i))(1-\nu_B(x_i)))$

$+(1-\theta)((1-\nu_A(x_i)) + (1-\nu_B(x_i)) - (1-\nu_A(x_i))(1-\nu_B(x_i)))$

$+ \sum_{x_i \in X} \theta(1-\nu_A(x_i))(1-\nu_B(x_i)) + (1-\theta)(1-\nu_A(x_i))(1-\nu_B(x_i))$

$= Card_\theta \Diamond(A) + Card_\theta \Diamond(B).$

(e). From Theorem 3.3.2 (i) and (k) we have that:

$Card_\theta \Box(A \cup_{\overline{S}_L} B) + Card_\theta \Box(A \cap_{\overline{T}_L} B) = Card_\theta(\Box A \cup_{\overline{S}_L} \Box B) + Card_\theta(\Box A \cap_{\overline{T}_L}$
$\Box B)$

$= \sum_{x_i \in X} \theta \min(1, \mu_A(x_i)+\mu_B(x_i)) + (1-\theta)(1-\max(0,(1-\mu_A(x_i))+(1-\mu_B(x_i))-1))$

$+ \sum_{x_i \in X} \theta \max(0, \mu_A(x_i)+\mu_B(x_i)-1) + (1-\theta)(1-\min(1,(1-\mu_A(x_i))+(1-\mu_B(x_i))))$

$= \sum_{x_i \in X} \theta \min(1, \mu_A(x_i)+\mu_B(x_i)) + (1-\theta)\min(1, \mu_A(x_i)+\mu_B(x_i))$

$+ \sum_{x_i \in X} \theta \max(0, \mu_A(x_i)+\mu_B(x_i)-1) + (1-\theta)\max(0, \mu_A(x_i)+\mu_B(x_i)-1)$

$= Card_\theta \Box(A) + Card_\theta \Box(B).$

(f). From Theorem 3.3.2 (j) and (l) we have that:

$Card_\theta \Diamond(A \cup_{\overline{S}_L} B) + Card_\theta \Diamond(A \cap_{\overline{T}_L} B) = Card_\theta(\Diamond A \cup_{\overline{S}_L} \Diamond B) + Card_\theta(\Diamond A \cap_{\overline{T}_L}$
$\Diamond B)$

$= \sum_{x_i \in X} \theta \max(0,(1-\nu_A(x_i))+(1-\nu_B(x_i))-1) + (1-\theta)(1-\min(1, \nu_A(x_i)+\nu_B(x_i)))$

$+ \sum_{x_i \in X} \theta \min(1,(1-\nu_A(x_i))+(1-\nu_B(x_i))) + (1-\theta)(1-\max(0, \nu_A(x_i)+\nu_B(x_i)-1))$

$= \sum_{x_i \in X} \theta \max(0,(1-\nu_A(x_i))+(1-\nu_B(x_i))-1) + (1-\theta)\max(0,(1-\nu_A(x_i))+$
$(1-\nu_B(x_i))-1)$

$+ \sum_{x_i \in X} \theta \min(1,(1-\nu_A(x_i))+(1-\nu_B(x_i))) + (1-\theta)\min(1,(1-\nu_A(x_i))+(1-\nu_B(x_i)))$

$= Card_\theta \Diamond(A) + Card_\theta \Diamond(B).$

Chapter 4

Intuitionistic Fuzzy Bi-implicator Operators

The investigation of fuzzy logic in a narrow sense, as subfield of multi-valued logic as initiated by Hajek [77], opened a lot of debate about 'the most reasonable way' of extending the most common operators from the classical domain {0,1} into the continuum of [0,1]. In the mean while, if some classical connectives appeared as their well established fuzzy counterparts, others remained largely as a matter of contention. For instance, the fuzzy interpretation of conjunction was well settled in the framework of triangular norms, similar to the conjunction which was dealt by its dual conorm. Likewise, the classical negation, in its both weak or strong form owns a reasonably well-studied associated fuzzy logical operator. Furthermore, fuzzy literature is rich in many different competing fuzzy versions of implication operators as well.

The classical bi-implicator operator also does not fall short of fuzzy interpretation, and we may find it appearing in fuzzy literature under the appellations of T-equivalence [108], fuzzy similarity [77], fuzzy equality [111] and restricted equivalence functions [32]. Some of these approaches had axiomatic definitions while other were constructive in structure. However, in all of these approaches the operators defined were more or less required to satisfy reflexivity, symmetry and even T-transitivity. Moreover, in some cases they were bound to be comparable with the notion of metric in [0,1] like in [111], where the concept of fuzzy bi-implicator appeared with the notion of fuzzy equality which was utilized as a basic connective in defining a class of algebras called EQ-algebras. Furthermore, these connectives were later utilized in the development of a generalization of fuzzy logic called Fuzzy Type Theory. The condition of T-transitivity was relaxed under the notion of fuzzy resemblance relation [53, 54].

In intuitionistic fuzzy theory, the structural constraints of the underlying lattice such as the existence of incomparable elements has resulted into a large hollow space of unanswered questions existing in this particular area of logic. The fact that besides its wide range of applicability, in contrast to its fuzzy counterparts, one rarely finds any development of intuitionistic fuzzy bi-implication operator [109] which has attracted our motivation towards, developing a generalized vision of *Intuitionistic Fuzzy Bi-implicators (IBI).*

The aim of this chapter is a two fold: The foremost is to convey to our reader, a comprehensive picture of some new generalized classes of intuitionistic fuzzy bi-implicators having axiomatic and constructive definitions along with their mutual relationship. To be more specific, we have introduced three different classes of intuitionistic fuzzy bi-implicators called the class of *intuitionistic fuzzy* $\kappa-$*bi-implicator,* the class of *intuitionistic fuzzy* $\beta-$*bi-implicator* and a larger class of *intuitionistic fuzzy* $\breve{T}'\breve{S}'-$*bi-implicator* based on different intuitionistic fuzzy t-norms, R-implicators and intuitionistic fuzzy conorms. A mutual relation between these constructive classes has reveals that formally mentioned two classes fall in the third class making it a more generalized class of intuitionistic fuzzy bi-implicators. Such a study is expected to lay a groundwork for the development of new intuitionistic fuzzy logic or algebra having different intuitionistic fuzzy bi-implicators as basic connective operators. The second and an equally important aim of the above mentioned study is to develop a firm structural bi-implication framework for constructing some new classes of bi-implication based similarity measures for intuitionistic fuzzy sets. For that we have studied the properties and characteristics of one of the newly defined constructive bi-implicator called intuitionistic fuzzy $\beta-$bi-implicator by utilizing the intuitionistic fuzzy Lukasiewicz implicator $\breve{I}_{\widetilde{T}_L}$ [see Example 2.2.2.9, Chapter 2] along with a t-representable intuitionistic fuzzy Min t-norm $(\overline{T}_M)$ in its definition. The properties exhibited by this particular combination of intuitionistic fuzzy t-norm and R-implicator has been found to resemble the properties of inclusion introduced by Sinha [134]. In fuzzy literature, these properties have been categorized as a set of most reasonable properties for defining not only an inclusion indicator but also a stable similarity relation between fuzzy sets. We believe that these can be reworked for their intuitionistic versions in defining a new measure of similarity between intuitionistic fuzzy sets which is one of the core targets of this dissertation.

The results produced in this chapter are a part of our accepted for publication paper [5].

4.1 Some New Approaches of Intuitionistic Fuzzy Bi-implicators

This section is divided into two subsections. In the first subsection, a deep study of an intuitionistic fuzzy bi-implicator is presented by defining three different classes of intuitionistic fuzzy bi-implicator operators having intuitionistic fuzzy logical connectives such as different intuitionistic fuzzy t-norms, their residual intuitionistic fuzzy implications and t-conorms implanted in their definitions. Moreover, their mutual relation is also presented there.

In the second subsection, we have build a more generalized view of intuitionistic fuzzy bi-implicators by giving an axiomatic approach of an intuitionistic fuzzy bi-implicator. This axiomatic definition is further utilized to classify the new classes of intuitionistic fuzzy bi-implicators presented in the first subsection depending upon their level of obedience to the presented set of axioms. It was found that some of the constructive classes completely obeyed while other slightly disobeyed the given set of axioms. We believe that such a classification of intuitionistic bi-implicator operators can lay a background for constructing new intuitionistic fuzzy logical phenomenons and some new algebras.

4.1.1 Classes of Intuitionistic Fuzzy Bi-implicators having Constructive Approaches and their Mutual Relation

Inspired by the classical (in fact, intuitionistic) equivalence between $p \iff q$ and $(p \implies q) \wedge (q \implies p)$, in this subsection, we have proposed several new constructive approaches for defining an intuitionistic fuzzy bi-implicator using intuitionistic fuzzy t-norms and their residual implicators. Moreover, their mutual relationship and properties are studied and it is proved that there exists a non hull intersection between these classes.

Definition 4.1.1.1 Let $\check{T}$ be a left continuous intuitionistic fuzzy t-norm and $\check{I}_{\check{T}}$ be the corresponding intuitionistic fuzzy R-implicator. Then the *intuitionistic fuzzy $\kappa-$bi-implicator* is the $(L^*)^2 \longrightarrow L^*$ mapping IBI_κ defined as:

$$IBI_\kappa(x, y) = \check{T}(\check{I}_{\check{T}}(x, y), \check{I}_{\check{T}}(y, x)).$$

Definition 4.1.1.2 Let $\check{T}'$ be an intuitionistic fuzzy t-norm, $\check{T}$ a left continuous intuitionistic fuzzy t-norm and $\check{I}_{\check{T}}$ be the corresponding intuitionistic fuzzy R-implicator. Then the *intuitionistic fuzzy $\beta-$bi-implicator* is the $(L^*)^2 \longrightarrow L^*$ mapping IBI_β defined as:

$$IBI_\beta(x, y) = \check{T}'(\check{I}_{\check{T}}(x, y), \check{I}_{\check{T}}(y, x)).$$

Definition 4.1.1.3 Let $\check{T}'$ be an intuitionistic fuzzy t-norm, $\check{T}$ a left continuous intuitionistic fuzzy t-norm, $\check{I}_{\check{T}}$ be the corresponding intuitionistic fuzzy R-implicator and $\check{S}'$ be an intuitionistic fuzzy conorm. Then the *intuitionistic fuzzy $\check{T}'\check{S}'-$bi-implicator* is the $(L^*)^2 \longrightarrow L^*$ mapping $IBI_{\check{T}'\check{S}'}$ defined as:

$$IBI_{\check{T}'\check{S}'}(x, y) = \check{I}_{\check{T}}(\check{S}'(x, y), \check{T}'(x, y)).$$

Proposition 4.1.1.4 Let $\check{T}$ be a left continuous intuitionistic fuzzy t-norm and $\check{I}_{\check{T}}$ be the corresponding intuitionistic fuzzy R-implicator then it holds:

$$x \leq_{L^*} y \Longrightarrow \check{T}_M(\check{I}_{\check{T}}(x, y), \check{I}_{\check{T}}(y, x)) = \check{I}_{\check{T}}(\check{S}_M(x, y), \check{T}_M(x, y)).$$

Proof Suppose $x \leq_{L^*} y$. Then we obtain,
$\check{I}_{\check{T}}(x, y) = 1_{L^*}, \check{T}_M(x, y) = x$ and $\check{S}_M(x, y) = y$.
Hence, $\check{T}_M(\check{I}_{\check{T}}(x, y), \check{I}_{\check{T}}(y, x)) = \check{T}_M(1_{L^*}, \check{I}_{\check{T}}(y, x))$
$= \check{I}_{\check{T}}(y, x) = \check{I}_{\check{T}}(\check{S}_M(x, y), \check{T}_M(x, y))$.

Proposition 4.1.1.5 Let $\check{T}'$ be an intuitionistic fuzzy t-norm, $\check{T}$ a left continuous intuitionistic fuzzy t-norm and $\check{I}_{\check{T}}$ be the corresponding intuitionistic fuzzy R-implicator, then the *intuitionistic fuzzy $\beta-$bi-implicator* IBI_β satisfies the following properties for all $x, y \in L^*$:

$(b'1)$. $IBI_\beta(x, y) = 1_{L^*}$ if $x = y$ (reflexivity).

$(b'2)$. $IBI_\beta(x, y) = IBI_\beta(y, x)$ (symmetry).

$(b'3)$. $IBI_\beta(x, y) = \check{T}'(\check{I}_{\check{T}}(x, y), \check{I}_{\check{T}}(y, x)) = \overline{T}_M(\check{I}_{\check{T}}(x, y), \check{I}_{\check{T}}(y, x))$ provided $x \leq_{L^*} y$

$(b'4)$. $IBI_\beta(x, y) = \check{I}_{\check{T}}(\overline{S}_M(x, y), \overline{T}_M(x, y))$ provided $x \leq_{L^*} y$.

Proof

$(b'1)$. $IBI_\beta(x,x) = \check{T}'(\breve{I}_{\check{T}}(x,x), \breve{I}_{\check{T}}(x,x)) = \check{T}'(1_{L^*}, 1_{L^*})) = 1_{L^*}$.

$(b'2)$. $IBI_\beta(x,y) = \check{T}'(\breve{I}_{\check{T}}(x,y), \breve{I}_{\check{T}}(y,x)) = \check{T}'(\breve{I}_{\check{T}}(y,x), \breve{I}_{\check{T}}(x,y)) = IBI_\beta(y,x)$.

$(b'3)$. From $x \leq_{L^*} y$ we get $\breve{I}_{\check{T}}(x,y) = 1_{L^*}$
and hence $IBI_\beta(x,y) = \check{T}'(\breve{I}_{\check{T}}(x,y), \breve{I}_{\check{T}}(y,x)) = \check{T}'(1_{L^*}, \breve{I}_{\check{T}}(y,x))$
$= \breve{I}_{\check{T}}(y,x) = \overline{T}_M(\breve{I}_{\check{T}}(y,x), 1_{L^*}) = \overline{T}_M(1_{L^*}, \breve{I}_{\check{T}}(y,x)) = \overline{T}_M(\breve{I}_{\check{T}}(x,y), \breve{I}_{\check{T}}(y,x))$.

$(b'4)$. Let $x \leq_{L^*} y$.
Then $IBI_\beta(x,y) = \overline{T}_M(\breve{I}_{\check{T}}(x,y), \breve{I}_{\check{T}}(y,x)) = \breve{I}_{\check{T}}(\overline{S}_M(x,y), \overline{T}_M(x,y))$ (By Proposition 4.1.1.4).

Proposition 4.1.1.6 An intuitionistic fuzzy bi-implicator IBI is an intuitionistic fuzzy $\kappa-$bi-implicator if and only if it is an intuitionistic fuzzy $\beta-$bi-implicator.

Proof By taking $\check{T}' = \check{T}$ in the definition of intuitionistic fuzzy $\beta-$bi-implicator we can get an intuitionistic fuzzy $\kappa-$bi-implicator. Conversely, we need to show that a $\kappa-$bi-implicator is an intuitionistic fuzzy $\beta-$bi-implicator. Let IBI_κ be an intuitionistic fuzzy $\kappa-$bi-implicator. Then we show that it satisfies all the axioms of Proposition 4.1.1.5 to become a $\beta-$bi-implicator.

$(b'1)$. Suppose $x = y$.
Then we obtain $IBI_\kappa(x,x) = \check{T}(\breve{I}_{\check{T}}(x,x), \breve{I}_{\check{T}}(x,x)) = \check{T}(1_{L^*}, 1_{L^*})) = 1_{L^*}$[Definition 2.2.2.2 and Definition 2.2.1.3]

$(b'2)$. $IBI_\kappa(x,y) = \check{T}(\breve{I}_{\check{T}}(x,y), \breve{I}_{\check{T}}(y,x))$
$= \check{T}(\breve{I}_{\check{T}}(y,x), \breve{I}_{\check{T}}(x,y)) = IBI_\kappa(y,x)$.

$(b'3)$. Supposing $x \leq_{L^*} y$ we get $\breve{I}_{\check{T}}(x,y) = 1_{L^*}$
and hence $IBI_\kappa(x,y) = \check{T}(\breve{I}_{\check{T}}(x,y), \breve{I}_{\check{T}}(y,x)) = \check{T}(1_{L^*}, \breve{I}_{\check{T}}(y,x))$
$= \breve{I}_{\check{T}}(y,x) = \overline{T}_M(\breve{I}_{\check{T}}(y,x), 1_{L^*}) = \overline{T}_M(1_{L^*}, \breve{I}_{\check{T}}(y,x))$
$= \overline{T}_M(\breve{I}_{\check{T}}(x,y), \breve{I}_{\check{T}}(y,x))$.

$(b'4)$. Suppose $x \leq_{L^*} y$.
Then it follows: $IBI_\kappa(x,y) = \overline{T}_M(\breve{I}_{\check{T}}(x,y), \breve{I}_{\check{T}}(y,x))$
$= \breve{I}_{\check{T}}(\overline{S}_M(x,y), \overline{T}_M(x,y))$ (by Proposition 4.1.1.4).

Remark 4.1.1.7 It must be noted that if we restrict ourself to the choice of all those $x, y \in L^*$ such that either $x \leq_{L^*} y$ or $y \leq_{L^*} x$ and $\check{T}$ to be a left continuous intuitionistic fuzzy t-norm and $\breve{I}_{\check{T}}$ be the corresponding intuitionistic fuzzy R-implicator then, the class of all intuitionistic fuzzy $\kappa-$bi-implicators and the class of all intuitionistic fuzzy $\beta-$bi-implicators satisfies $(b'4)$. Thus they become the subclasses of the class of all intuitionistic fuzzy $\check{T}'\check{S}'-$bi-implicators.

4.1.2 An Axiomatic Approach of Intuitionistic Fuzzy Bi-implicator and its Relation with Constructive Classes

As it happens with implication, for bi-implication $\Leftrightarrow$ there is no universal agreement on what should constitute its fuzzy counterparts. The most well known class of fuzzy bi-implication was investigated by Fodor and Roubens [68]. In this section, we have presented an axiomatic definition of intuitionistic fuzzy bi-implicator operator which can be regarded as intuitionistic fuzzy generalization of *Fodor-Roubens fuzzy bi-implicator*. Furthermore, we have studied its relation with the different classes of intuitionistic fuzzy bi-implicators having constructive approaches as presented in the previous subsection.

Definition 4.1.2.1 An *intuitionistic fuzzy bi-implicator* is an $(L^*)^2 \longrightarrow L^*$mapping IBI satisfying for all $w, x, y, z \in L^*$:

$(b1)$. $IBI(x, y) = IBI(y, x)$;

$(b2)$. $IBI(0_{L^*}, 1_{L^*}) = 0_{L^*}$;

$(b3)$. $IBI(x, x) = 1_{L^*}$;

$(b4)$. If $w \leq_{L^*} x \leq_{L^*} y \leq_{L^*} z$, then $IBI(w, z) \leq_{L^*} IBI(x, y)$.

Example 4.1.2.2 Let for all $w, x, y, z \in L^*$ such that $w = (w_1, w_2), x = (x_1, x_2), y = (y_1, y_2)$ and $z = (z_1, z_2)$. Then the operator defined as:

$$IBI(x, y) = \left\{ \begin{array}{c} 1_{L^*} \text{ if } x = y \\ (\min(1 - x_2, 1 - y_2), \max(x_2, y_2)) \text{ if } x \neq y \end{array} \right\}$$

is an intuitionistic fuzzy bi-implicator.

Indeed we will show that $IBI(x, y)$ satisfies the four axioms of Definition 4.1.2.1, i.e.,

$(b1)$. $IBI(x, y) = IBI(y, x)$. Straightforward.

$(b2)$. $IBI(0_{L^*}, 1_{L^*}) = IBI((0, 1), (1, 0)) = (\min(1-1, 1-0), \max(1, 0)) = (0, 1) = 0_{L^*}$.

$(b3)$. $IBI(x, x) = 1_{L^*}$, by its definition.

$(b4)$. Let $w \leq_{L^*} x \leq_{L^*} y \leq_{L^*} z$. This

implies $w_1 \leq x_1 \leq y_1 \leq z_1$ and $w_2 \geq x_2 \geq y_2 \geq z_2$

implies $1 - w_2 \leq 1 - x_2 \leq 1 - y_2 \leq 1 - z_2$

implies $IBI(w, z) = (\min(1 - w_2, 1 - z_2), \max(w_2, z_2)) = (1 - w_2, w_2)$

implies $\min(1 - w_2, 1 - z_2) \leq \min(1 - y_2, 1 - x_2)$ and $\max(w_2, z_2) \geq \max(y_2, x_2)$

implies $(\min(1 - w_2, 1 - z_2), \max(w_2, z_2)) \leq_{L^*} (\min(1 - y_2, 1 - x_2), \max(y_2, x_2))$

and hence $IBI(w, z) \leq_{L^*} IBI(x, y)$.

Remark 4.1.2.3 If we take $w = x$ in axiom $(b4)$ of Definition 4.1.2.1 then axiom $(b4)$ can be equivalently replaced by axiom $(b^\diamond 4)$ provided IBI satisfies $(b1)$:

$(b^\diamond 4)$. If $x \leq_{L^*} y \leq_{L^*} z$, then $IBI(x, z) \leq_{L^*} IBI(x, y)$ and $IBI(x, z) \leq_{L^*} IBI(y, z)$.

Proof

(i). $(b4) \Longrightarrow (b^\diamond 4)$.

Indeed putting $w = x$ in $(b4)$ implies

$IBI(x, z) \leq_{L^*} IBI(x, y)$ i.e. $IBI(x, .)$ is decreasing for all $x \in L^*$.

From $x \leq_{L^*} y \leq_{L^*} z \leq_{L^*} z$ it follows with $(b4)$

$IBI(x, z) \leq_{L^*} IBI(y, z)$ i.e. $IBI(., z)$ is increasing for all $z \in L^*$.

(ii). $(b^\diamond 4) \Longrightarrow (b4)$.

Suppose $w \leq_{L^*} x \leq_{L^*} y \leq_{L^*} z$.

From $IBI(w, .)$ being decreasing and $x \leq_{L^*} z$ we get:

$$IBI(w, z) \leq_{L^*} IBI(w, x). \tag{4.1}$$

From $IBI(., x)$ being increasing and $w \leq_{L^*} y$ we get:

$$IBI(w, x) \leq_{L^*} IBI(y, x). \tag{4.2}$$

From (4.1) and (4.2) we get:

$$IBI(w, z) \leq_{L^*} IBI(y, x)$$

and hence if IBI satisfies $(b1)$:

$$IBI(w, z) \leq_{L^*} IBI(x, y).$$

Proposition 4.1.2.4 An intuitionistic fuzzy $\kappa-$bi-implicator satisfies the axioms of definition 4.1.2.1.

Proof Let $\check{T}$ be a left continuous intuitionistic fuzzy t-norm and $\check{I}_{\check{T}}$ be its intuitionistic fuzzy R-implicator and IBI_κ be the intuitionistic fuzzy $\kappa-$ bi-implicator based on $\check{T}$ and $\check{I}_{\check{T}}$. Then we only have to prove that an intuitionistic fuzzy $\kappa-$ bi-implicator satisfies the axioms $(b2)$ and $(b4)$, as $(b1) = (b^{'}2)$ and $(b3) = (b^{'}1)$ have already been proved in Proposition 4.1.1.6.

$(b2)$. $IBI_\kappa(0_{L^*}, 1_{L^*}) = \check{T}(\check{I}_{\check{T}}(0_{L^*}, 1_{L^*}), \check{I}_{\check{T}}(1_{L^*}, 0_{L^*})) = \check{T}(1_{L^*}, 0_{L^*}) = 0_{L^*}$.

$(b4)$. Suppose $w \leq_{L^*} x \leq_{L^*} y \leq_{L^*} z$. Then we obtain:

$IBI_\kappa(w,z) = \overline{T}_M(\check{I}_{\check{T}}(w,z), \check{I}_{\check{T}}(z,w))$ by $(b^{'}3)$

implies that $IBI_\kappa(w,z) = \overline{T}_M(1_{L^*}, \check{I}_{\check{T}}(z,w)) = \check{I}_{\check{T}}(z,w)$

implies that $IBI_\kappa(w,z) = \check{I}_{\check{T}}(z,w) \leq_{L^*} \check{I}_{\check{T}}(y,w) \leq_{L^*} \check{I}_{\check{T}}(y,x)$ as $y \leq_{L^*} z$ and $w \leq_{L^*} x$

implies that $IBI_\kappa(w,z) \leq_{L^*} \check{I}_{\check{T}}(y,x) = \overline{T}_M(1_{L^*}, \check{I}_{\check{T}}(y,x)) = \overline{T}_M(\check{I}_{\check{T}}(x,y), \check{I}_{\check{T}}(y,x))$

implies that $IBI_\kappa(w,z) \leq_{L^*} IBI_\kappa(x,y)$.

Proposition 4.1.2.5 An intuitionistic fuzzy $\check{T}^{'}\check{S}^{'}-$bi-implicator satisfies the properties $(b1)$ and $(b2)$ but may fail to satisfy the properties $(b3)$ and $(b4)$.

Proof Let $\check{T}^{'}$ be an intuitionistic fuzzy t-norm, $\check{T}$ a left continuous intuitionistic fuzzy t-norm, $\check{I}_{\check{T}}$ be its intuitionistic fuzzy R-implicator and $\check{S}^{'}$ be an intuitionistic fuzzy conorm. Let $IBI_{\check{T}^{'}\check{S}^{'}}$ be the intuitionistic fuzzy $\check{T}^{'}\check{S}^{'}-$*bi-implicator* based on $\check{T}$, $\check{I}_{\check{T}}$ and $\check{S}^{'}$. Then we show that $IBI_{\check{T}^{'}\check{S}^{'}}$satisfies the properties $(b1)$ and $(b2)$ but may fail to satisfy the properties $(b3)$ and $(b4)$. For all $x, y \in L^*$

$(b1)$. $IBI_{\check{T}^{'}\check{S}^{'}}(x,y) = \check{I}_{\check{T}}(\check{S}^{'}(x,y), \check{T}^{'}(x,y)) = \check{I}_{\check{T}}(\check{S}^{'}(y,x), \check{T}^{'}(y,x)) = IBI_{\check{T}^{'}\check{S}^{'}}(y,x)$.

$(b2)$. $IBI_{\check{T}^{'}\check{S}^{'}}(0_{L^*}, 1_{L^*}) = \check{I}_{\check{T}}(\check{S}^{'}(0_{L^*}, 1_{L^*}), \check{T}^{'}(0_{L^*}, 1_{L^*})) = \check{I}_{\check{T}}(1_{L^*}, 0_{L^*}) = 0_{L^*}$.

In order to show that $IBI_{\check{T}^{'}\check{S}^{'}}$ fails to satisfy $(b3)$ and $(b4)$ we shall present the following counter examples:

Let $x = (0.7, 0.1) \in L^*$. Then by Remark 2.2.1.7 and Example 2.2.2.7 we take $\check{S}^{'}(x,x) = \overline{S}_P(x,x) = (0.91, 0.01)$

and $\check{T}^{'}(x,x) = \overline{T}_L(x,x) = (0.4, 0.2)$,

which implies that $IBI_{\overline{T}_L\overline{S}_P}(x,x) = \check{I}_{\overline{T}_M}(\overline{S}_P(x,x), \overline{T}_L(x,x)) = (0.4, 0.2) \neq 1_{L^*}$.

Hence $IBI_{\check{T}^{'}\check{S}^{'}}$ fails to satisfy $(b3)$.

Let $w = (0.2, 0.7), x = (0.3, 0.4)$, $y = (0.5, 0.3)$, $z = (0.8, 0.1) \in L^*$. Then $\overline{S}_L(w,z) = (1,0), \overline{T}_P(w,z) = (0.16, 0.73)$ and $\overline{S}_L(x,y) = (0.8, 0), \overline{T}_P(x,y) =$

$(0.15, 0.58)$.

Which implies that $IBI_{\overline{T}_P \overline{S}_L}(w, z) = \breve{I}_{\overline{T}_M}(\overline{S}_L(w, z), \overline{T}_P(w, z)) = (0.16, 0.73)$ and $IBI_{\overline{T}_P \overline{S}_L}(x, y) = \breve{I}_{\overline{T}_M}(\overline{S}_L(x, y), \overline{T}_P(x, y)) = (0.15, 0.58)$.

Clearly, we see that $IBI_{\overline{T}_P \overline{S}_L}(w, z) \nleq_{L^*} IBI_{\overline{T}_P \overline{S}_L}(x, y)$ as $0.16 > 0.15$ and $0.58 < 0.73$.

Hence, $IBI_{\overline{T}_P \overline{S}_L}$ fails to satisfy the property $(b4)$.

Remark 4.1.2.6 Since it has been already proved in Proposition 4.1.1.6 that an intuitionistic fuzzy bi-implicator IBI is an intuitionistic fuzzy $\kappa-$bi-implicator if and only if it is an intuitionistic fuzzy $\beta-$bi-implicator, we can claim that intuitionistic fuzzy $\beta-$bi-implicator also fulfill the requirements of Definition 4.1.2.1.

4.2 Lukasiewicz Intuitionistic Fuzzy Bi-implicator

In the previous section we have presented a broader view of intuitionistic fuzzy bi-implicator operators. In this section we shall have a deep look into properties of one of the new bi-implicator classes presented here called intuitionistic fuzzy $\beta-$bi-implicator by utilizing intuitionistic fuzzy Min t-norm $(\overline{T}_M)$ and intuitionistic fuzzy Lukasiewicz implicator $\breve{I}_{\tilde{T}_L}$ in its definition. The motivation for this particular combination is a fact that $\overline{T}_M$ is the largest t-norm and intuitionistic fuzzy Lukasiewicz implicator $\breve{I}_{\tilde{T}_L}$ serves as the basic connector for intuitionistic fuzzy Lukasiewicz logic. The properties exhibited had close resemblance with the general properties of any similarity relation between intuitionistic fuzzy sets. Such a knowledge not only provides a better understanding about the structural details of the particular class but also signifies the role of a bi-implicator in defining any similarity relation between two intuitionistic fuzzy sets. Thus for all $A, B \in IFS(X)$ and $x \in X$ we have:

$$\begin{aligned} IBI_\beta(A, B)(x) &= \overline{T}_M(\breve{I}_{\tilde{T}_L}(A(x), B(x)), \breve{I}_{\tilde{T}_L}(B(x), A(x))) && (4.3) \\ &= (\min(1, b_1 - a_1 + 1, a_2 - b_2 + 1, a_1 - b_1 + 1, b_2 - a_2 + 1), \\ &\quad \max(0, a_1 + b_2 - 1, b_1 + a_2 - 1)) && (4.4) \end{aligned}$$

where $A(x) = (a_1, a_2) = (\mu_A(x), \nu_A(x))$, $B(x) = (b_1, b_2) = (\mu_B(x), \nu_B(x)) \in L^*$.

Proposition 4.2.1 For all $A, B \in IFS(X)$,

(a). $IBI_\beta(A,B)(x) = 1_{L^*}$ if and only if $A(x) = B(x)$;

(b). $IBI_\beta(A,B) = \widetilde{1}_X$ if and only if $A = B$;

(c). $IBI_\beta(A,B)(x) = 0_{L^*}$ if and only if $x \in Ker(A) \cap (Supp(B))^c$or $x \in Ker(B) \cap (Supp(A))^c$;

(d). $IBI_\beta(A,B) = \widetilde{0}_X$ implies $Ker(A) \cap (Supp(B))^c \neq \phi$ or $Ker(B) \cap (Supp(A))^c \neq \phi$.

Proof Let $A, B \in IFS(X)$

(a). $IBI_\beta(A,B)(x) = 1_{L^*}$ for any $x \in X$,

if and only if $\overline{T}_M(\breve{I}_{\widetilde{T}_L}(A(x), B(x)), \breve{I}_{\widetilde{T}_L}(B(x), A(x))) = 1_{L^*}$

if and only if $(\min(1, b_1 - a_1 + 1, a_2 - b_2 + 1, a_1 - b_1 + 1, b_2 - a_2 + 1), \max(0, a_1 + b_2 - 1, b_1 + a_2 - 1)) = 1_{L^*}$

if and only if $\min(1, b_1 - a_1 + 1, a_2 - b_2 + 1, a_1 - b_1 + 1, b_2 - a_2 + 1) = 1$ and $\max(0, a_1 + b_2 - 1, b_1 + a_2 - 1) = 0$

if and only if $b_1 - a_1 + 1 \geq 1, a_2 - b_2 + 1 \geq 1,\ a_1 - b_1 + 1 \geq 1, b_2 - a_2 + 1 \geq 1$ and $a_1 + b_2 - 1 \leq 0, b_1 + a_2 - 1 \leq 0$

if and only if $b_1 \geq a_1, a_2 \geq b_2$ and $a_1 \geq b_1, b_2 \geq a_2$

if and only if $a_1 = b_1$ and $a_2 = b_2$

if and only if $A(x) = B(x)$.

(b). $IBI_\beta(A,B) = \widetilde{1}_X$

if and only if $IBI_\beta(A,B)(x) = 1_{L^*}$ for all $x \in X$

if and only if $A(x) = B(x)$ for all $x \in X$

if and only if $A = B$.

(c). $IBI_\beta(A,B)(x) = 0_{L^*}$ for any $x \in X$,

if and only if $\overline{T}_M(\breve{I}_{\widetilde{T}_L}(A(x), B(x)), \breve{I}_{\widetilde{T}_L}(B(x), A(x))) = 0_{L^*}$

if and only if $(\min(1, b_1 - a_1 + 1, a_2 - b_2 + 1, a_1 - b_1 + 1, b_2 - a_2 + 1), \max(0, a_1 + b_2 - 1, b_1 + a_2 - 1)) = 0_{L^*}$

if and only if $\min(1, b_1 - a_1 + 1, a_2 - b_2 + 1, a_1 - b_1 + 1, b_2 - a_2 + 1) = 0$ and $\max(0, a_1 + b_2 - 1, b_1 + a_2 - 1) = 1$

if and only if either $\{b_1 - a_1 + 1 = 0$ and $\max(0, a_1 + b_2 - 1, b_1 + a_2 - 1) = 1\}$

or $\{a_2 - b_2 + 1 = 0$ and $\max(0, a_1 + b_2 - 1, b_1 + a_2 - 1) = 1\}$

or $\{a_1 - b_1 + 1 = 0$ and $\max(0, a_1 + b_2 - 1, b_1 + a_2 - 1) = 1\}$

or $\{b_2 - a_2 + 1 = 0$ and $\max(0, a_1 + b_2 - 1, b_1 + a_2 - 1) = 1\}$.

Next, we discuss all these cases one by one such that they have a mutual relation of "or" between them.

Case 1: If $b_1 - a_1 + 1 = 0$ then we have $b_1 = 0$, $a_1 = 1$, $a_2 = 0$ and $b_2 \leq 1$. However, the condition $\max(0, a_1 + b_2 - 1, b_1 + a_2 - 1) = 1$ will enforce $b_2 = 1$.

Thus, we have $(a_1, a_2) = (1, 0)$ and $(b_1, b_2) = (0, 1)$ and hence $x \in Ker(A) \cap (Supp(B))^c$.

Case 2: If $a_2 - b_2 + 1 = 0$ then we get $a_2 = 0$, $b_2 = 1$, $b_1 = 0$ and $a_1 \leq 1$. However, the condition $\max(0, a_1 + b_2 - 1, b_1 + a_2 - 1) = 1$ will enforce $a_1 = 1$.

Thus, we have $(a_1, a_2) = (1, 0)$ and $(b_1, b_2) = (0, 1)$ and hence $x \in Ker(A) \cap (Supp(B))^c$.

Case 3: If $a_1 - b_1 + 1 = 0$ then we have $a_1 = 0$, $b_1 = 1$, $b_2 = 0$ and $a_2 \leq 1$. Likewise, to above two cases the condition $\max(0, a_1 + b_2 - 1, b_1 + a_2 - 1) = 1$ will enforce $a_2 = 1$.

Thus, we have $(a_1, a_2) = (0, 1)$ and $(b_1, b_2) = (1, 0)$ and hence $x \in Ker(B) \cap (Supp(A))^c$.

Case 4: If we choose $b_2 - a_2 + 1 = 0$ then we get $b_2 = 0$, $a_2 = 1$, $a_1 = 0$ and $b_1 \leq 1$. The condition $\max(0, a_1 + b_2 - 1, b_1 + a_2 - 1) = 1$ will enforces $b_1 = 1$.

Thus, we have $(a_1, a_2) = (0, 1)$ and $(b_1, b_2) = (1, 0)$ and hence $x \in Ker(B) \cap (Supp(A))^c$.

Thus, all of these situations lead to the result:

$IBI_\beta(A, B)(x) = 0_{L^*}$ implies $x \in Ker(A) \cap (Supp(B))^c$or $x \in Ker(B) \cap (Supp(A))^c$.

Conversely,

let $x \in Ker(A) \cap (Supp(B))^c$

implies that$x \in Ker(A)$ and $x \in (Supp(B))^c$

implies that $(a_1, a_2) = (1, 0)$ and $x \in (Supp(B))^c$.

Now, $x \in Supp(B) \Leftrightarrow (b_1 \neq 0$ or $b_2 \neq 1)$ and hence $x \in (Supp(B))^c \Leftrightarrow (b_1 = 0$ and $b_2 = 1) \Leftrightarrow (b_1, b_2) = (0, 1)$

implies that $[(a_1, a_2) = (1, 0)$ and $(b_1, b_2) = (0, 1)]$

implies that $\min(1, b_1 - a_1 + 1, a_2 - b_2 + 1, a_1 - b_1 + 1, b_2 - a_2 + 1) = 0$ and $\max(0, a_1 + b_2 - 1, b_1 + a_2 - 1) = 1$

implies that$(\min(1, b_1 - a_1 + 1, a_2 - b_2 + 1, a_1 - b_1 + 1, b_2 - a_2 + 1)$, $\max(0, a_1 +$

$b_2 - 1, b_1 + a_2 - 1)) = (0, 1)$

implies that $IBI_\beta(A, B)(x) = 0_{L^*}$.

Similarly, for $x \in Ker(B) \cap (Supp(A))^c$ we get $IBI_\beta(A, B)(x) = 0_{L^*}$.

Thus, $x \in Ker(A) \cap (Supp(B))^c$or $x \in Ker(B) \cap (Supp(A))^c$ implies $IBI_\beta(A, B)(x) = 0_{L^*}$.

(d). $IBI_\beta(A, B) = \widetilde{0}_X$

implies that $IBI_\beta(A, B)(x) = 0_{L^*}$ for all $x \in X$

implies that either $[x \in Ker(A) \cap (Supp(B))^c]$ or $[x \in Ker(B) \cap (Supp(A))^c]$ for all $x \in X$

implies that $Ker(A) \cap (Supp(B))^c \neq \phi$ or $Ker(B) \cap (Supp(A))^c \neq \phi$.

Proposition 4.2.2 For all $A, B \in IFS(X)$,

$IBI_\beta(A, B) = IBI_\beta(B, A)$.

Proof The result holds due to commutativity of $\overline{T}_M$.

Corollary 4.2.3 For $A \in IFS(X)$,

(a). $IBI_\beta(A, A^c)(x) = 1_{L^*}$if and only if $A(x) = (a_1, a_2)$ such that $a_1, a_2 \leq 0.5$;

(b). $IBI_\beta(A, A^c) = \widetilde{1}_X$ if and only if $A(x) = (a_1, a_2)$ such that $a_1, a_2 \leq 0.5$ for all $x \in X$;

(c). $IBI_\beta(A, A^c)(x) = 0_{L^*}$ if and only if either $A(x) = 1_{L^*}$ or $A(x) = 0_{L^*}$;

(d). $IBI_\beta(A, A^c) = \widetilde{0}_X$ if and only if $A = \widetilde{1}_X$ or $A = \widetilde{0}_X$.

Proof Let $A \in IFS(X)$,

(a). $IBI_\beta(A, A^c)(x) = 1_{L^*}$ for any $x \in X$

if and only if $\overline{T}_M(\breve{I}_{\widetilde{T}_L}(A(x), A^c(x)), \breve{I}_{\widetilde{T}_L}(A^c(x), A(x))) = 1_{L^*}$

if and only if $(\min(1, a_2 - a_1 + 1, a_2 - a_1 + 1, a_1 - a_2 + 1, a_1 - a_2 + 1), \max(0, a_1 + a_1 - 1, a_2 + a_2 - 1)) = 1_{L^*}$

if and only if $\min(1, a_2 - a_1 + 1, a_1 - a_2 + 1) = 1$ and $\max(0, 2a_1 - 1, 2a_2 - 1) = 0$

if and only if $a_2 - a_1 + 1 \geq 1$ and $a_1 - a_2 + 1 \geq 1$ with $2a_1 - 1 \leq 0$ and $2a_2 - 1 \leq 0$

if and only if $a_1 = a_2$ with $a_1 \leq 0.5$ and $a_2 \leq 0.5$

if and only if $A(x) = (a_1, a_2)$ such that $a_1 = a_2$ and $a_1, a_2 \leq 0.5$.

(b). $IBI_\beta(A, A^c) = \widetilde{1}_X$

if and only if $IBI_\beta(A, A^c)(x) = 1_{L^*}$ for all $x \in X$

if and only if $A(x) = (a_1, a_2)$ such that $a_1 = a_2$ and $a_1, a_2 \leq 0.5$ for all $x \in X$.

(c). $IBI_\beta(A, A^c)(x) = 0_{L^*}$ for any $x \in X$

if and only if $\overline{T}_M(\breve{I}_{\widetilde{T}_L}(A(x), A^c(x)), \breve{I}_{\widetilde{T}_L}(A^c(x), A(x))) = 0_{L^*}$

if and only if $(\min(1, a_2 - a_1 + 1, a_2 - a_1 + 1, a_1 - a_2 + 1, a_1 - a_2 + 1), \max(0, a_1 + a_1 - 1, a_2 + a_2 - 1)) = 0_{L^*}$

if and only if $\min(1, a_2 - a_1 + 1, a_1 - a_2 + 1) = 0$ and $\max(0, 2a_1 - 1, 2a_2 - 1) = 1$

if and only if $a_2 - a_1 + 1 = 0$ or $a_1 - a_2 + 1 = 0$ and $2a_1 - 1 = 1$ or $2a_2 - 1 = 1$

if and only if $a_2 = 0$ or $a_1 = 0$ and $a_1 = 1$ or $a_2 = 1$

if and only if either $A(x) = 1_{L^*}$ or $A(x) = 0_{L^*}$.

(d). $IBI_\beta(A, A^c) = \widetilde{0}_X$

if and only if $IBI_\beta(A, A^c)(x) = 0_{L^*}$ for all $x \in X$

if and only if either $A(x) = 1_{L^*}$or $A(x) = 0_{L^*}$ for all $x \in X$

if and only if either $A = \widetilde{1}_X$ or $A = \widetilde{0}_X$.

Proposition 4.2.4 For $A, B \in IFS(X)$,

$IBI_\beta(A, B) = IBI_\beta(B^c, A^c)$.

Proof The result holds due to contrapositivity of the Lukasiewicz intuitionistic fuzzy implicator $\breve{I}_{\overline{T}_L}$ [see Remarks 2.2.2.10,2.2.2.11, Chapter 2] used in (4.3).

Proposition 4.2.5 For any $A, B, C \in IFS(X)$ such that $A \subseteq B \subseteq C$ we have:

(a). $IBI_\beta(A, C) \subseteq \left\{ \begin{array}{c} IBI_\beta(A, B) \\ IBI_\beta(B, C) \end{array} \right\}$ i.e., the first partial mapping $IBI_\beta(\cdot, C)$ of IBI_β is increasing and the second partial mapping $IBI_\beta(A, \cdot)$ is decreasing;

(b). $IBI_\beta(A, C) \subseteq \overline{T}_M(IBI_\beta(A, B), IBI_\beta(B, C))$.

Proof Let $A, B, C \in IFS(X)$ such that $A \subseteq B \subseteq C$.

(a). $A(x) \leq_{L^*} B(x) \leq_{L^*} C(x)$ for all $x \in X$

implies that $a_1 \leq b_1 \leq c_1$ and $a_2 \geq b_2 \geq c_2$.

Now as, $[a_1 \leq b_1$ and $a_2 \geq b_2$ and $a_1 + a_2 \leq 1]$

implies that$1 \leq b_1 - a_1 + 1$ and $a_2 - b_2 + 1 \geq 1$

implies that $\min(1, b_1 - a_1 + 1, a_2 - b_2 + 1) = 1$ and $\max(0, a_1 + b_2 - 1) = 0$

implies that $\breve{I}_{\widetilde{T}_L}(A(x), B(x)) = (\min(1, b_1 - a_1 + 1, a_2 - b_2 + 1), \max(0, a_1 + b_2 - 1))$
$= (1, 0) = 1_{L^*}$

implies that $IBI_\beta(A, B)(x) = \overline{T}_M(\breve{I}_{\widetilde{T}_L}(A(x), B(x)), \breve{I}_{\widetilde{T}_L}(B(x), A(x)))$
$= \breve{I}_{\widetilde{T}_L}(B(x), A(x)) = (\min(1, a_1 - b_1 + 1, b_2 - a_2 + 1), \max(0, b_1 + a_2 - 1))$.

Similarly, we have $IBI_\beta(B, C)(x) = (\min(1, b_1 - c_1 + 1, c_2 - b_2 + 1), \max(0, c_1 + b_2 - 1))$ and $IBI_\beta(A, C)(x) = (\min(1, a_1 - c_1 + 1, c_2 - a_2 + 1), \max(0, c_1 + a_2 - 1))$.

Now as, $[a_1 - b_1 + 1 \geq a_1 - c_1 + 1$ and $b_2 - a_2 + 1 \geq c_2 - a_2 + 1$ also $c_1 + a_2 - 1 \geq b_1 + a_2 - 1]$

implies that $[\min(1, a_1 - c_1 + 1, c_2 - a_2 + 1) \leq \min(1, a_1 - b_1 + 1, b_2 - a_2 + 1)$
and $\max(0, c_1 + a_2 - 1) \geq \max(0, b_1 + a_2 - 1)]$
implies that $(\min(1, a_1 - c_1 + 1, c_2 - a_2 + 1), \max(0, c_1 + a_2 - 1))$
$\leq_{L^*} (\min(1, a_1 - b_1 + 1, b_2 - a_2 + 1), \max(0, b_1 + a_2 - 1))$
implies that $IBI_\beta(A, C)(x) \leq_{L^*} IBI_\beta(A, B)(x)$ for all $x \in X$
implies that $IBI_\beta(A, C) \subseteq IBI_\beta(A, B)$.
Moreover, $[a_1 - c_1 + 1 \leq b_1 - c_1 + 1$, $c_2 - a_2 + 1 \leq c_2 - b_2 + 1$ and $c_1 + a_2 - 1 \geq c_1 + b_2 - 1]$
implies that $\min(1, a_1 - c_1 + 1, c_2 - a_2 + 1) \leq \min(1, b_1 - c_1 + 1, c_2 - b_2 + 1)$
and $\max(0, c_1 + a_2 - 1) \geq \max(0, c_1 + b_2 - 1)$
implies that $(\min(1, a_1 - c_1 + 1, c_2 - a_2 + 1), \max(0, c_1 + a_2 - 1))$
$\leq_{L^*} (\min(1, b_1 - c_1 + 1, c_2 - b_2 + 1), \max(0, c_1 + b_2 - 1))$
implies that $IBI_\beta(A, C)(x) \leq_{L^*} IBI_\beta(B, C)(x)$ for all $x \in X$
implies that $IBI_\beta(A, C) \subseteq IBI_\beta(B, C)$.
(b). $A(x) \leq_{L^*} B(x) \leq_{L^*} C(x)$ for all $x \in X$
implies that $a_1 \leq b_1 \leq c_1$ and $a_2 \geq b_2 \geq c_2$
implies that $\breve{I}_{\widetilde{T}_L}(A(x), B(x)) = 1_{L^*}$, $\breve{I}_{\widetilde{T}_L}(A(x), C(x)) = 1_{L^*}$and $\breve{I}_{\widetilde{T}_L}(B(x), C(x)) = 1_{L^*}$
implies that $IBI_\beta(A, B)(x) = (\min(1, a_1 - b_1 + 1, b_2 - a_2 + 1), \max(0, b_1 + a_2 - 1))$, $IBI_\beta(B, C)(x) = (\min(1, b_1 - c_1 + 1, c_2 - b_2 + 1), \max(0, c_1 + b_2 - 1))$ and $IBI_\beta(A, C)(x) = (\min(1, a_1 - c_1 + 1, c_2 - a_2 + 1), \max(0, c_1 + a_2 - 1))$.
Now $[a_1 - b_1 + 1 \geq a_1 - c_1 + 1$, $b_2 - a_2 + 1 \geq c_2 - a_2 + 1, b_1 - c_1 + 1 \geq a_1 - c_1 + 1$, $c_2 - b_2 + 1 \geq c_2 - a_2 + 1$ and $c_1 + b_2 - 1 \leq c_1 + a_2 - 1$, $b_1 + a_2 - 1 \leq c_1 + a_2 - 1]$
implies that $\min(1, a_1 - c_1 + 1, c_2 - a_2 + 1) \leq \min(1, b_2 - a_2 + 1, a_1 - b_1 + 1, b_1 - c_1 + 1, c_2 - b_2 + 1)$ and $\max(0, c_1 + a_2 - 1) \geq \max(0, c_1 + b_2 - 1, b_1 + a_2 - 1)$
implies that $\min(1, a_1 - c_1 + 1, c_2 - a_2 + 1) \leq \min(\min(1, b_2 - a_2 + 1, a_1 - b_1 + 1), \min(1, b_1 - c_1 + 1, c_2 - b_2 + 1))$ and $\max(0, c_1 + a_2 - 1) \geq \max(\max(0, c_1 + b_2 - 1), \max(0, b_1 + a_2 - 1))$
implies that $(\min(1, a_1 - c_1 + 1, c_2 - a_2 + 1), \max(0, c_1 + a_2 - 1))$
$\leq_{L^*} (\min(\min(1, b_2 - a_2 + 1, a_1 - b_1 + 1), \min(1, b_1 - c_1 + 1, c_2 - b_2 + 1)), \max(\max(0, c_1 + b_2 - 1), \max(0, b_1 + a_2 - 1)))$
implies that $IBI_\beta(A, C)(x) \leq_{L^*} \overline{T}_M(IBI_\beta(A, B)(x), IBI_\beta(B, C)(x))$ for all $x \in X$
implies that $IBI_\beta(A, C) \subseteq \overline{T}_M(IBI_\beta(A, B), IBI_\beta(B, C))$.

Proposition 4.2.6 For any $A, B \in IFS(X)$, such that A and B are pointwise comparable (see Definition 2.2.5, Chapter 2):

(a). $IBI_\beta(A, \overline{T}_M(A,B)) = IBI_\beta(B, \overline{S}_M(A,B))$;

(b). $IBI_\beta(A, \overline{S}_M(A,B)) = IBI_\beta(B, \overline{T}_M(A,B))$.

Proof

(a). Let $A, B \in IFS(X)$, such that A and B are pointwise comparable. Then for all $x \in X$, either $A(x) \leq_{L^*} B(x)$ or $B(x) \leq_{L^*} A(x)$

implies that either $\overline{T}_M(A(x), B(x)) = A(x)$ and $\overline{S}_M(A(x), B(x)) = B(x)$

or $\overline{T}_M(A(x), B(x)) = B(x)$ and $\overline{S}_M(A(x), B(x)) = A(x)$

implies that $IBI_\beta(A(x), \overline{T}_M(A(x), B(x))) = IBI_\beta(A(x), A(x)) = 1_{L^*}$

and $IBI_\beta(B(x), \overline{S}_M(A(x), B(x))) = IBI_\beta(B(x), B(x)) = 1_{L*}$

or $IBI_\beta(A(x), \overline{T}_M(A(x), B(x))) = IBI_\beta(A(x), B(x))$

and $IBI_\beta(B(x), \overline{S}_M(A(x), B(x))) = IBI_\beta(B(x), A(x)) = IBI_\beta(A(x), B(x))$

implies that $IBI_\beta(A, \overline{T}_M(A,B))(x) = IBI_\beta(B, \overline{S}_M(A,B))(x)$

implies that $IBI_\beta(A, \overline{T}_M(A,B)) = IBI_\beta(B, \overline{S}_M(A,B))$.

(b). The proof can be constructed in a similar way as part (a).

Proposition 4.2.7 For any $A, B \in IFS(X)$, such that A and B are pointwise comparable, the following intuitionistic fuzzy sets are equal:

(a). $IBI_\beta(A, B)$;

(b). $\overline{T}_M(IBI_\beta(\overline{T}_M(A,B), A), IBI_\beta(A, \overline{S}_M(A,B)))$;

(c). $\overline{T}_M(IBI_\beta(\overline{T}_M(A,B), B), IBI_\beta(B, \overline{S}_M(A,B)))$;

(d). $IBI_\beta(\overline{T}_M(A,B), \overline{S}_M(A,B))$;

(e). $\overline{T}_M(IBI_\beta(A, \overline{S}_M(A,B)), IBI_\beta(B, \overline{S}_M(A,B)))$;

(f). $\overline{T}_M(IBI_\beta(A, \overline{T}_M(A,B)), IBI_\beta(B, \overline{T}_M(A,B)))$.

Proof Let $A, B \in IFS(X)$, such that A and B are pointwise comparable. Then, for all $x \in X$, either $A(x) \leq_{L^*} B(x)$ or $B(x) \leq_{L^*} A(x)$

implies that for any $x \in X$, either $\overline{T}_M(A,B)(x) = A(x)$ and $\overline{S}_M(A,B) = B(x)$

or $\overline{T}_M(A,B)(x) = B(x)$ and $\overline{S}_M(A,B) = A(x)$.

For simplicity of proofs we consider the cases of all those $x \in X$ for which $A(x) \leq_{L^*}$ $B(x)$

i.e., $\overline{T}_M(A,B)(x) = A(x)$ and $\overline{S}_M(A,B) = B(x)$ then,

(b)=(a). $\overline{T}_M(IBI_\beta(\overline{T}_M(A(x), B(x)), A(x)), IBI_\beta(A(x), \overline{S}_M(A(x), B(x))))$

$= \overline{T}_M(IBI_\beta(A(x), A(x)), IBI_\beta(A(x), B(x))))$

$= \overline{T}_M(1_{L^*}, IBI_\beta(A, B)(x)) = IBI_\beta(A, B)(x)$.

(c)=(a). $\overline{T}_M(IBI_\beta(\overline{T}_M(A(x), B(x)), B(x)), IBI_\beta(B(x), \overline{S}_M(A(x), B(x))))$

$= \overline{T}_M(IBI_\beta(A(x), B(x)), IBI_\beta(B(x), B(x)))$

$= \overline{T}_M(IBI_\beta(A, B)(x), 1_{L^*}) = IBI_\beta(A, B)(x)$.

(d)=(a). $IBI_\beta(\overline{T}_M(A, B), \overline{S}_M(A, B))(x)$

$= IBI_\beta(A(x), B(x)) = IBI_\beta(A, B)(x)$.

(e)=(a). $\overline{T}_M(IBI_\beta(A, \overline{S}_M(A, B)), IBI_\beta(B, \overline{S}_M(A, B)))(x)$

$= \overline{T}_M(IBI_\beta(A(x), B(x)), 1_{L*}) = IBI_\beta(A, B)(x)$.

(f)=(a). $\overline{T}_M(IBI_\beta(A, \overline{T}_M(A, B)), IBI_\beta(B, \overline{T}_M(A, B)))(x)$

$= \overline{T}_M(IBI_\beta(A(x), A(x)), IBI_\beta(B(x), A(x)))$

$= \overline{T}_M(1_{L^*}, IBI_\beta(B(x), A(x)))$

$= IBI_\beta(B(x), A(x)) = IBI_\beta(A(x), B(x)) = IBI_\beta(A, B)(x)$ (because of the symmetry of IBI_β).

The above results also hold for all those $x \in X$, for which $B(x) \leq_{L^*} A(x)$.

Chapter 5

Intuitionistic Fuzzy Similarity Measures

Similarity is a core element in achieving an understanding of variables that motivate behavior and mediate affect. In fact, in terms of human psychology it acts as a basic organizing principle by which a human mind not only classifies objects, but also, forms conceptual thinking and build different generalizations. Therefore, its importance has been not only been realized in the field of psychology, but also, in many scientific domains such as decision making, pattern recognition, information retrieval, biology, chemistry, machine learning and market prediction. In mathematics, geometric methods for assessing similarity are used in studies of congruence as well as in allied fields such as trigonometry. Topological methods are applied in fields such as semantics. Graph theory is widely used for assessing cladistic similarities in taxonomy. Thus, similarity of objects can be regarded as a central concept in data mining and knowledge discovery, which is applicable in all those fields that require a clear analyzation of data sets, in order to get patterns and regularities containing crucial information about the given set of data. Besides, its wide range of applicability in many real life problems, there are certain aspects of the concept of similarity that have eluded its formalization; resulting into its numerous different forms and definitions that are still appearing in scientific literature. Some of these proposed similarity measures were limited to their use in a particular domain while others were designed to capture their maximum utility in wider spread areas and disciplines. Thus, it is a worth mentioning fact that so far there does not exist a general purpose, mathematically valid, definition of similarity, rather; it is at times, specified by many different set of axioms exhibiting its properties as a measure.

In fuzzy literature, fuzzy similarity measure has appeared in several ways and in various domains. The two most popular techniques of defining any new similarity measure for FS's that have evolved by the passage of time, and been adopted by most of the researchers in this area, are either; derivation of an axiomatic structured fuzzy similarity measure from the scratch [25, 26, 94, 131, 155, 158], or by starting from a similarity measure for ordinary sets and then establishing a set of fuzzification rules by which this measure is transformed in a consistent manner from an ordinary to a fuzzy measure [64, 85, 130, 136, 137].

In the setting of IFS-theory, similarity has drawn the attention of many researchers who have proposed multiple similarity measures based on distances [58, 81, 82, 140, 143, 156, 164], a few with set theoretic operations [83], or the ones having their own independent axiomatic structures [15, 42, 57, 65, 89, 106, 119], yet, it is hard to find any with implications based constructions. In this chapter, we have tried to fill this gap and have defined two different classes of similarity measures for IFS's; the first one involving the intuitionistic fuzzy bi-implications and the second one having cardinality based structures.

As far as the bi-implicator based similarity measures are concerned, in [20] a generalization of the classical equivalence/bi-implication relation $A = B \iff A \subseteq B$ and $B \subseteq A$, to a class of fuzzy similarity measures was defined as:

$$E_I(A, B) = \min(\inf_{x \in X} I(A(x), B(x)), \inf_{x \in X} I(B(x), A(x))) \tag{5.1}$$

where I stands for fuzzy implications and $A, B \in FS(X)$. A different version of the above similarity measure between two fuzzy sets was established and studied in [25]. Then, an extensive study on fuzzy similarity measures have been done with bi-implicators serving as a back bone of several new theories on measuring a similarity between two fuzzy sets [25, 26].

As a foremost attempt towards defining a similarity measure for IFS's, in the first section of this chapter, we have generalized the above mentioned idea from fuzzy to intuitionistic fuzzy measure. The new defined class of similarity measures has been studied in the perspective of axioms presented in [134].

Next, we worked on the extension of cardinality based set theoretic similarity measures for ordinary sets existing in the field of numerical taxonomy, where the comparable objects are represented by a set of binary feature vectors representing the presence or absence of a feature in an object. In fuzzy environment, the same measures are constructed using feature vectors with components scaled to real unit interval [0,1]. Such a feature vector can be described as vector in which the decision about the presence or absence of a feature in any object is not given in a clear cut manner, rather, it is characterized by a membership degree in [0,1]. Higher the degree of membership the more the feature is present.

Since an intuitionistic fuzzy set (IFS) is one of the higher order fuzzy sets having each element characterized by its membership and a nonmembership degree respectively, any feature comparison vector representing an intuitionistic fuzzy set must have components that are naturally scaled to lattice L^* i.e., each feature of this comparison vector is assigned a degree of presence and a degree of non presence (absence) from [0,1] with a condition that the sum of these two degrees should not exceed one. Any similarity measure constructed using these feature vectors will be regarded as a similarity measure for IFS's.

In second section of this chapter, we have constructed a parametric family of cardinality based similarity measures for IFS's using the intuitionistic fuzzy feature vector approach. The new measures can be regarded as an extension of set theoretic approach similarity measures for ordinary of fuzzy sets given in [87]. The results of this chapter are submitted as [4, 125].

5.1 Intuitionistic Fuzzy Bi-implicator based Similarity Measure (IFBSM)

This section is reserved for a composition of the new classes of intuitionistic fuzzy bi-implicator operators and the new normal fuzzy measures on IFS's introduced in chapter 3 and 4, to create different [0,1]-valued classes of *Intuitionistic Fuzzy Bi-implicator based Similarity Measures* (IFBSM).

Since each of the intuitionistic fuzzy bi-implicator operator represented a class of bi-implicators, fixing a class of bi-implicator operator and any of the new fuzzy measures in IFBSM will generate a single family of IFBSM. Moreover, after introducing a generalized vision of the classes of IFBSMs, we shall present to our reader an insight view of the properties exhibited by one specific family of IFBSM called *Intuitionistic Fuzzy $\beta-$Bi-implicator Based Similarity Measure* (IFβBSM). Among many different choices of intuitionistic fuzzy t-norms and intuitionistic fuzzy R-implicators, the properties of IFβBSM are constructed by implanting Lukasiewicz intuitionistic fuzzy implicator $\breve{I}_{\widetilde{T}_L}$ [see Example 2.2.2.9, Chapter 2] and Min intuitionistic fuzzy t-norm ($\overline{T}_M$), which is believed to be the most suitable choice for modeling an implication operation (fuzzy or its extension). A study on the properties of IFβBSM with this particular combination of intuitionistic fuzzy t-norm and intuitionistic fuzzy implicator has generated some remarkable results.

Definition 5.1.1 An *intuitionistic fuzzy bi-implicator based similarity measure* (IFBSM) is a mapping $\Omega_{IBI} : IFS(X) \times IFS(X) \longrightarrow [0,1]$ that assigns to all $A, B \in IFS(X)$ a value in $[0,1]$ defined as:

$$\Omega_{IBI}(A,B) = m(IBI(A,B)) \tag{5.2}$$

where $IBI(A,B)$ is the intuitionistic fuzzy of bi-implicator from A to B and m is the normal fuzzy measure on intuitionistic fuzzy set.

Now using the new normal fuzzy measures given in Chapter 3 and Definition 5.1.1 we get:

(a).

$$\Omega^1_{IBI}(A,B) = \frac{1}{2}[\inf_{x\in X}\mu_{IBI(A,B)}(x) + (1 - \sup_{x\in X}\nu_{IBI(A,B)}(x))].$$

(b).

$$\begin{aligned}\Omega^2_{IBI}(A,B) \quad = \quad & \frac{1}{4}[\inf_{x\in X}\mu_{IBI(A,B)}(x) + (1 - \sup_{x\in X}\nu_{IBI(A,B)}(x)) \\ & +\sup_{x\in X}\mu_{IBI(A,B)}(x) + (1 - \inf_{x\in X}\nu_{IBI(A,B)}(x))].\end{aligned}$$

(c).

$$\begin{aligned}\Omega^3_{IBI}(A,B) \quad &= \quad \frac{|IBI(A,B)|}{|X \times X|} = \frac{\sum_{x\in X}\theta\mu_{IBI(A,B)}(x) + (1-\theta)(1-\nu_{IBI(A,B)}(x))}{n^2} \\ \text{where } \theta \quad &\in \quad [0.5, 1].\end{aligned}$$

(d).

$$\Omega^4_{IBI}(A,B) = \frac{1}{n^2}\sum_{x\in X}\sqrt{\mu_{IBI(A,B)}(x)(1-\nu_{IBI(A,B)}(x))}.$$

In order to have a better look into the properties of the above new similarity measures we shall restrict our self to the choice of the class of intuitionistic fuzzy $\beta-$bi-implicator in (5.2).

Definition 5.1.2 An *intuitionistic fuzzy $\beta-$bi-implicator based similarity measure* (IFβBSM) is a mapping $\Omega_{IBI_\beta} : IFS(X) \times IFS(X) \longrightarrow [0,1]$ that allocates to all

$A, B \in IFS(X)$ a value in $[0,1]$ defined as:

$$\Omega_{IBI_\beta}(A,B) = m(IBI_\beta(A,B)) \tag{5.3}$$

where $IBI_\beta(A,B) = \check{T}'(\check{I}_{\check{T}}(A,B), \check{I}_{\check{T}}(B,A))$ such that $\check{T}'$ is any intuitionistic fuzzy t-norm, $\check{T}$ is a left continuous intuitionistic fuzzy t-norms, $\check{I}_{\check{T}}$ is the corresponding intuitionistic fuzzy R-implicator and m is the normal fuzzy measure on intuitionistic fuzzy set.

Clearly, from the above construction one can deduce that different combinations of $\check{T}'$and $\check{I}_{\check{T}}$ will generate different $IBI_\beta-$operators. Thus, Ω_{IBI_β} will represent different families of IFβBSMs. Therefore, for a deep examination of the properties of one of the family of IFβBSM, let us restrict our self to the combination $\check{T}' = \overline{T}_M$ and $\check{I}_{\check{T}} = \check{I}_{\widetilde{T}_L}$ in (5.3). leading us to the following properties of Ω_{IBI_β}:

Proposition 5.1.3 If the fuzzy measure m on IFS possesses the property stated in Remark 3.1.3, then for all $A, B \in IFS(X)$ we have:

(a). $\Omega_{IBI_\beta}(A,B) = 1$ if and only if $A = B$;

(b). $\Omega_{IBI_\beta}(A,B) = 0$ implies $Ker(A) \cap (Supp(B))^c \neq \phi$ or $Ker(B) \cap (Supp(A))^c \neq \phi$;

(c). $\Omega_{IBI_\beta}(A,A^c) = 1$ if and only if for all $x \in X$, $A(x) = (a_1, a_2)$ such that $a_1, a_2 \leq 0.5$;

(d). $\Omega_{IBI_\beta}(A,A^c) = 0$ if and only if $A = \widetilde{1}_X$ or $A = \widetilde{0}_X$.

Proof Let $A, B \in IFS(X)$,

(a). $A = B$

if and only if $A(x) = B(x)$ for all $x \in X$

if and only if $a_1 = b_1$ and $a_2 = b_2$

if and only if $b_1 \geq a_1$, $a_2 \geq b_2$,and $a_1 \geq b_1$, $b_2 \geq a_2$

if and only if $[b_1 - a_1 + 1 \geq 1, a_2 - b_2 + 1 \geq 1,\ a_1 - b_1 + 1 \geq 1, b_2 - a_2 + 1 \geq 1$ and $a_1 + b_2 - 1 \leq 0, b_1 + a_2 - 1 \leq 0]$

if and only if $\min(1, b_1 - a_1 + 1, a_2 - b_2 + 1, a_1 - b_1 + 1, b_2 - a_2 + 1) = 1$ and $\max(0, a_1 + b_2 - 1, b_1 + a_2 - 1) = 0$

if and only if $(\min(1, b_1 - a_1 + 1, a_2 - b_2 + 1, a_1 - b_1 + 1, b_2 - a_2 + 1), \max(0, a_1 + b_2 - 1, b_1 + a_2 - 1)) = 1_{L^*}$

if and only if $\overline{T}_M(\breve{I}_{\widetilde{T}_L}(A(x),B(x)),\breve{I}_{\widetilde{T}_L}(B(x),A(x)))=1_{L^*}$ for all $x\in X$

if and only if $IBI_\beta(A,B)(x)=1_{L^*}$ for all $x\in X$

if and if $IBI_\beta(A,B)=\widetilde{1}_X$

if and only if $m(IBI_\beta(A,B))=m(\widetilde{1}_X)$

if and only if $\Omega_{IBI_\beta}(A,B)=1$.

(b). $\Omega_{IBI_\beta}(A,B)=0$

implies that $m(IBI_\beta(A,B))=m(\widetilde{0}_X)$

implies that $IBI_\beta(A,B)=\widetilde{0}_X$

implies that $IBI_\beta(A,B)(x)=0_{L^*}$ for all $x\in X$

implies that either $x\in Ker(A)\cap(Supp(B))^c$ or $x\in Ker(B)\cap(Supp(A))^c$ for all $x\in X$ [by Proposition 4.2.1 (d)]

implies that $Ker(A)\cap(Supp(B))^c\neq\phi$ or $Ker(B)\cap(Supp(A))^c\neq\phi$

(c). $A(x)=(a_1,a_2)$ such that $a_1,a_2\le 0.5$ for all $x\in X$

if and only if $a_2-a_1+1\ge 1$and $a_1-a_2+1\ge 1$ with $2a_1-1\le 0$ and $2a_2-1\le 0$

if and only if $\min(1,a_2-a_1+1,a_1-a_2+1)=1$ and $\max(0,2a_1-1,2a_2-1)=0$

if and only if $(\min(1,a_2-a_1+1,a_2-a_1+1,a_1-a_2+1,a_1-a_2+1),\max(0,a_1+a_1-1,a_2+a_2-1))=1_{L^*}$

if and only if $\breve{T}_M(\breve{I}_{\widetilde{T}_L}(A(x),A^c(x)),\breve{I}_{\widetilde{T}_L}(A^c(x),A(x)))=1_{L^*}$ for all $x\in X$

if and only if $IBI_\beta(A,A^c)(x)=1_{L^*}$ for all $x\in X$

if and only if $IBI_\beta(A,A^c)=\widetilde{1}_X$

if and only if $m(IBI_\beta(A,A^c))=m(\widetilde{1}_X)$

if and only if $\Omega_{IBI_\beta}(A,A^c)=1$.

(d). $A=\widetilde{1}_X$ or $A=\widetilde{0}_X$

if and only if either $A(x)=1_{L^*}$or $A(x)=0_{L^*}$ for all $x\in X$

if and only if $a_2=0$ or $a_1=0$ and $a_1=1$ or $a_2=1$

if and only if $\min(1,a_2-a_1+1,a_1-a_2+1)=0$ and $\max(0,2a_1-1,2a_2-1)=1$

if and only if $(\min(1,a_2-a_1+1,a_2-a_1+1,a_1-a_2+1,a_1-a_2+1),\max(0,a_1+a_1-1,a_2+a_2-1))=0_{L^*}$

if and only if $\breve{T}_M(\breve{I}_{\widetilde{T}_L}(A(x),A^c(x)),\breve{I}_{\widetilde{T}_L}(A^c(x),A(x)))=0_{L^*}$ for all $x\in X$

if and only if $IBI_\beta(A,A^c)(x)=0_{L^*}$ for all $x\in X$

if and only if $IBI_\beta(A,A^c)=\widetilde{0}_X$

if and only if $m(IBI_\beta(A,A^c))=m(\widetilde{0}_X)$

if and only if $\Omega_{IBI_\beta}(A, A^c) = 0$.

Proposition 5.1.4 For all $A, B \in IFS(X)$ we have:

(a). $\Omega_{IBI_\beta}(A, B) = \Omega_{IBI_\beta}(B, A)$;

(b). $\Omega_{IBI_\beta}(A, B) = \Omega_{IBI_\beta}(B^c, A^c)$.

Proof

(a). By Proposition 4.2.2 and Definition 5.1.2 we get $\Omega_{IBI_\beta}(A, B) = \Omega_{IBI_\beta}(B, A)$.

(b). By Proposition 4.2.4 and Definition 5.1.2 we get $\Omega_{IBI_\beta}(A, B) = \Omega_{IBI_\beta}(B^c, A^c)$.

Proposition 5.1.5 For any $A, B, C \in IFS(X)$ such that $A \subseteq B \subseteq C$ we have:

(a). $\Omega_{IBI_\beta}(A, C) \leq \left\{ \begin{array}{c} \Omega_{IBI_\beta}(A, B) \\ \Omega_{IBI_\beta}(B, C) \end{array} \right\}$;

(b). $\Omega_{IBI_\beta}(A, C) \leq \min(\Omega_{IBI_\beta}(A, B), \Omega_{IBI_\beta}(B, C))$.

Proof

(a). Let $A \subseteq B \subseteq C$. Then from Proposition 4.2.5 we have $IBI_\beta(A, C) \subseteq \left\{ \begin{array}{c} IBI_\beta(A, B) \\ IBI_\beta(B, C) \end{array} \right\}$. Then by applying the measure m on both sides we get $\Omega_{IBI_\beta}(A, C) \leq \left\{ \begin{array}{c} \Omega_{IBI_\beta}(A, B) \\ \Omega_{IBI_\beta}(B, C) \end{array} \right\}$.

(b). Let $A, B, C \in IFS(X)$ such that $A \subseteq B \subseteq C$

implies that $A(x) \leq_{L^*} B(x) \leq_{L^*} C(x)$ for all $x \in X$

implies that $a_1 \leq b_1 \leq c_1$ and $a_2 \geq b_2 \geq c_2$.

Now as, $[a_1 \leq b_1$ and $a_2 \geq b_2$ and $a_1 + a_2 \leq 1]$

implies that $1 \leq b_1 - a_1 + 1$ and $a_2 - b_2 + 1 \geq 1$

implies that $\min(1, b_1 - a_1 + 1, a_2 - b_2 + 1) = 1$ and $\max(0, a_1 + b_2 - 1) = 0$

implies that $\breve{I}_{\widetilde{T}_L}(A(x), B(x)) = (\min(1, b_1 - a_1 + 1, a_2 - b_2 + 1), \max(0, a_1 + b_2 - 1))$ $= (1, 0) = 1_{L^*}$

implies that $IBI(A, B)(x) = \breve{T}_M(\breve{I}_{\widetilde{T}_L}(A(x), B(x)), \breve{I}_{\widetilde{T}_L}(B(x), A(x)))$ $= \breve{I}_{\widetilde{T}_L}(B(x), A(x)) = (\min(1, a_1 - b_1 + 1, b_2 - a_2 + 1), \max(0, b_1 + a_2 - 1))$.

Similarly, we have $IBI(B, C)(x) = (\min(1, b_1 - c_1 + 1, c_2 - b_2 + 1), \max(0, c_1 + b_2 - 1))$ and $IBI(A, C)(x) = (\min(1, a_1 - c_1 + 1, c_2 - a_2 + 1), \max(0, c_1 + a_2 - 1))$.

Now $[a_1 - b_1 + 1 \geq a_1 - c_1 + 1,\ b_2 - a_2 + 1 \geq c_2 - a_2 + 1, b_1 - c_1 + 1 \geq a_1 - c_1 + 1$ and $c_2 - b_2 + 1 \geq c_2 - a_2 + 1$ also $c_1 + b_2 - 1 \leq c_1 + a_2 - 1,\ b_1 + a_2 - 1 \leq c_1 + a_2 - 1]$

implies that $[\min(1, a_1 - c_1 + 1, c_2 - a_2 + 1) \leq \min(1, b_2 - a_2 + 1, a_1 - b_1 + 1, b_1 - c_1 + 1, c_2 - b_2 + 1)$

and $\max(0, c_1 + a_2 - 1) \geq \max(0, c_1 + b_2 - 1, b_1 + a_2 - 1)]$

implies that $[\min(1, a_1 - c_1 + 1, c_2 - a_2 + 1) \leq \min(1, b_2 - a_2 + 1, a_1 - b_1 + 1)$ and $\min(1, a_1 - c_1 + 1, c_2 - a_2 + 1) \leq \min(1, b_1 - c_1 + 1, c_2 - b_2 + 1))]$

and $[\max(0, c_1 + a_2 - 1) \geq \max(0, c_1 + b_2 - 1)$ and $\max(0, c_1 + a_2 - 1) \geq \max(0, b_1 + a_2 - 1))]$

implies that $IBI_\beta(A, C)(x) \leq_{L^*} IBI_\beta(A, B)(x)$

and $IBI_\beta(A, C)(x) \leq_{L^*} IBI_\beta(B, C)(x)$ for all $x \in X$

implies that $IBI_\beta(A, C) \subseteq IBI_\beta(A, B)$

and $IBI_\beta(A, C) \subseteq IBI_\beta(B, C)$

implies that $\Omega_{IBI_\beta}(A, C) \leq \Omega_{IBI_\beta}(A, B)$ and $\Omega_{IBI_\beta}(A, C) \leq \Omega_{IBI_\beta}(B, C)$

implies that $\Omega_{IBI_\beta}(A, C) \leq \min(\Omega_{IBI_\beta}(A, B), \Omega_{IBI_\beta}(B, C))$.

Proposition 5.1.6 For any $A, B \in IFS(X)$, such that A and B are pointwise comparable [see Definition 2.7]:

(a). $\Omega_{IBI_\beta}(A, \overline{T}_M(A, B)) = \Omega_{IBI_\beta}(B, \overline{S}_M(A, B))$;

(b). $\Omega_{IBI_\beta}(A, \overline{S}_M(A, B)) = \Omega_{IBI_\beta}(B, \overline{T}_M(A, B))$.

Proof For any $A, B \in IFS(X)$,

(a). By Proposition 4.2.6 (a) and Definition 5.1.2 we get $\Omega_{IBI_\beta}(A, \overline{T}_M(A, B)) = \Omega_{IBI_\beta}(B, \overline{S}_M(A, B))$.

(b). By Proposition 4.2.6 (b) and Definition 5.1.2 we get $\Omega_{IBI_\beta}(A, \overline{S}_M(A, B)) = \Omega_{IBI_\beta}(B, \overline{T}_M(A, B))$.

Proposition 5.1.7 For any $A, B \in IFS(X)$, such that A and B are pointwise comparable. Then for all $x \in X$, the following fuzzy sets are equal:

(a). $\Omega_{IBI_\beta}(A, B)$;

(b). $\min(\Omega_{IBI_\beta}(\overline{T}_M(A, B), A), \Omega_{IBI_\beta}(A, \overline{S}_M(A, B)))$;

(c). $\min(\Omega_{IBI_\beta}(\overline{T}_M(A, B), B), \Omega_{IBI_\beta}(B, \overline{S}_M(A, B)))$;

(d). $\Omega_{IBI_\beta}(\overline{T}_M(A, B), \overline{S}_M(A, B))$;

(e). $\min(\Omega_{IBI_\beta}(A, \overline{S}_M(A, B)), \Omega_{IBI_\beta}(B, \overline{S}_M(A, B)))$;

(f). $\min(\Omega_{IBI_\beta}(A, \overline{T}_M(A, B)), \Omega_{IBI_\beta}(B, \overline{T}_M(A, B)))$.

Proof

Let $A, B \in IFS(X)$, such that A and B are pointwise comparable then, for all $x \in X$, either $A(x) \leq_{L^*} B(x)$ or $B(x) \leq_{L^*} A(x)$ implies that for any $x \in X$, either $[\overline{T}_M(A, B)(x) = A(x)$ and $\overline{S}_M(A, B) = B(x)]$ or $[\overline{T}_M(A, B)(x) = B(x)$ and

$\overline{S}_M(A,B) = A(x)]$. For simplicity of proofs we consider the cases of all those $x \in X$ for which $A(x) \leq_{L^*} B(x)$ i.e., $\overline{T}_M(A,B)(x) = A(x)$ and $\overline{S}_M(A,B) = B(x)$ then,

(b)=(a). $\min(\Omega_{IBI_\beta}(\overline{T}_M(A,B),A), \Omega_{IBI_\beta}(A,\overline{S}_M(A,B)))$

$= \min(\Omega_{IBI_\beta}(A,A), \Omega_{IBI_\beta}(A,B)) = \min(1, \Omega_{IBI_\beta}(A,B)) = \Omega_{IBI_\beta}(A,B)$.

(c)=(a). $\min(\Omega_{IBI_\beta}(\overline{T}_M(A,B),B), \Omega_{IBI_\beta}(B,\overline{S}_M(A,B)))$

$= \min(\Omega_{IBI_\beta}(A,B), \Omega_{IBI_\beta}(B,B)) = \min(\Omega_{IBI_\beta}(A,B), 1) = \Omega_{IBI_\beta}(A,B)$.

(d)=(a). $\Omega_{IBI_\beta}(\overline{T}_M(A,B), \overline{S}_M(A,B)) = \Omega_{IBI_\beta}(A,B)$.

(e)=(a). $\min(\Omega_{IBI_\beta}(A,\overline{S}_M(A,B)), \Omega_{IBI_\beta}(B,\overline{S}_M(A,B)))$

$= \min(\Omega_{IBI_\beta}(A,B), \Omega_{IBI_\beta}(B,B)) = \min(\Omega_{IBI_\beta}(A,B), 1) = \Omega_{IBI_\beta}(A,B)$.

(f)=(a). $\min(\Omega_{IBI_\beta}(A,\overline{T}_M(A,B)), \Omega_{IBI_\beta}(B,\overline{T}_M(A,B)))$

$= \min(\Omega_{IBI_\beta}(A,A), \Omega_{IBI_\beta}(B,A))$

$= \min(1, \Omega_{IBI_\beta}(B,A)) = \Omega_{IBI_\beta}(B,A) = \Omega_{IBI_\beta}(A,B)$.

The above results also hold for all those $x \in X$ for which $B(x) \leq_{L^*} A(x)$.

Theorem 5.1.8 If the fuzzy measure m satisfies the property stated in Remark 3.1.3, then Ω_{IBI_β} defines a fuzzy equivalence relation [see Definition 2.1.3.7, Chapter 2] on $IFS(X)$.

Proof

(a). For all $A \in IFS(X)$, $\Omega_{IBI_\beta}(A,A) = 1$ by Proposition 5.1.3.

(b). For all $A, B \in IFS(X)$, $\Omega_{IBI_\beta}(A,B) = \Omega_{IBI_\beta}(A,B)$.

(c). On contrary let us suppose that Ω_{IBI_β} is not fuzzy transitive. Then,

$$I_L(T_L(\Omega_{IBI_\beta}(A,B), \Omega_{IBI_\beta}(B,C)), \Omega_{IBI_\beta}(A,C)) = 0.$$

It implies that $\Omega_{IBI_\beta}(A,B) = 1, \Omega_{IBI_\beta}(B,C) = 1$ and, $\Omega_{IBI_\beta}(A,C) = 0$.

It further implies that $A = B = C$ but $\Omega_{IBI_\beta}(A,C) = 0$ which is a contradiction. So Ω_{IBI_β} is fuzzy transitive.

5.2 Weighted Average Cardinality based Parametric family of Intuitionistic Fuzzy Similarity Measures

Some of the results presented in this section are a part of our submitted paper [4].

Using the feature vector based approach, [87] presented a parametric family of cardinality based similarity measures for ordinary sets. Some of the famous similarity measures like Jaccard coefficient [85], Simple matching coefficient [136], Dice coefficient [64], Rogers and Tanimoto [130], Sokal and Sneath family of measures [137] were the members of this family.

Getting our inspiration from the work done of [87], in this section, we have defined a new *Weighted Average Cardinality based Parametric family of Intuitionistic Fuzzy Similarity Measure* for IFS's involving family of intuitionistic fuzzy frank t-norms in their basic constructions. Also we have studied their properties, which were found closer to the set axioms satisfied by any similarity measure in general.

Before proceeding towards the main definition of the new similarity measure, we want to first provide certain specifications which are important in order to understand the constructions of new introduced family of similarity measure i.e., **(a).** Through out this section we shall specify the generalized $\cup$ and $\cap$ between $A, B \in IFS$ by three basic t-representable intuitionistic fuzzy frank t-norms $\overline{T}_j$ and conorms $\overline{S}_j$ given as:

$$
\begin{aligned}
A \cup_{\overline{S_j}} B &= \Big\{(x, S_j(\mu_A(x), \mu_B(x)), T_j(\nu_A(x), \nu_B(x))) \mid x \in X\Big\} \\
&= \Big\{(x, \mu_A(x) + \mu_B(x) - T_j(\mu_A(x), \mu_B(x)), T_j(\nu_A(x), \nu_B(x))) \mid x \in X\Big\} \\
& \quad \Big\{(x, S_j(\mu_A(x), \mu_B(x)), 1 - S_j(1 - \nu_A(x), 1 - \nu_B(x))) \mid x \in X\Big\} \\
A \cap_{\overline{T_j}} B &= \Big\{(x, T_j(\mu_A(x), \mu_B(x)), S_j(\nu_A(x), \nu_B(x))) \mid x \in X\Big\} \\
&= \Big\{(x, T_j(\mu_A(x), \mu_B(x)), \nu_A(x) + \nu_B(x) - T_j(\nu_A(x), \nu_B(x))) \mid x \in X\Big\} \\
&= \Big\{(x, T_j(\mu_A(x), \mu_B(x)), 1 - T_j(1 - \nu_A(x), 1 - \nu_B(x))) \mid x \in X\Big\}
\end{aligned}
$$

where $j = M, P, L$.

(b). For the scalar cardinality of intuitionistic fuzzy set A where $X = \{x_1, x_2, ...x_n\}$, we shall use $|A| = Card_{0.5}(A)$ where $Card_{0.5}(A) = \frac{1}{2}\sum\limits_{x_i \in X} \mu_A(x_i) + (1 - \nu_A(x_i))$.

(c). Moreover, we shall use the following classical set theoretic identities which will be converted to their intuitionistic fuzzy versions as follows:

(i). $|A^c| = n - |A|$ will be modelled by

$$Card_{0.5}(A^c) = n - Card_{0.5}(A) = \frac{n}{2} + \frac{1}{2}\sum_{x_i \in X} \mu_A(x_i) - \nu_A(x_i).$$

(ii). $|A \backslash B| = |A| - |A \cap B|$ will be modeled by

$$Card_{0.5}(A \backslash B) = Card_{0.5}(A) - Card_{0.5}(A \cap B)$$

$$= \frac{\sum\limits_{x_i \in X} \mu_A(x_i) - \nu_A(x_i) - T_j(\mu_A(x_i), \mu_B(x_i)) + S_j(\nu_A(x_i), \nu_B(x_i))}{2}$$

$$= \frac{\sum\limits_{x_i \in X} \mu_A(x_i) + \nu_B(x_i) - T_j(\mu_A(x_i), \mu_B(x_i)) - T_j(\nu_A(x_i), \nu_B(x_i))}{2}$$

$$= \frac{\sum\limits_{x_i \in X} \mu_A(x_i) + 1 - \nu_A(x_i) - T_j(\mu_A(x_i), \mu_B(x_i)) - T_j(1 - \nu_A(x_i), 1 - \nu_B(x_i))}{2}.$$

(iii). $|A \triangle B| = |A \backslash B| + |B \backslash A| = |A| + |B| - 2\,|A \cap B|$ will be modelled by

$$Card_{0.5}(A \triangle B) = Card_{0.5}(A) + Card_{0.5}(B) - 2Card_{0.5}(A \cap B)$$

$$= \frac{\sum\limits_{x_i \in X} \mu_A(x_i) + \mu_B(x_i) - \nu_A(x_i) - \nu_B(x_i) - 2T_j(\mu_A(x_i), \mu_B(x_i)) + 2S_j(\nu_A(x_i), \nu_B(x_i))}{2}$$

$$= \frac{\sum\limits_{x_i \in X} \mu_A(x_i) + \mu_B(x_i) + \nu_A(x_i) + \nu_B(x_i) - 2T_j(\mu_A(x_i), \mu_B(x_i)) - 2T_j(\nu_A(x_i), \nu_B(x_i))}{2}$$

$$= \frac{\sum\limits_{x_i \in X} \mu_A(x_i) + \mu_B(x_i) + (1 - \nu_A(x_i)) + (1 - \nu_B(x_i)) - 2T_j(\mu_A(x_i), \mu_B(x_i)) - 2T_j(1 - \nu_A(x_i), 1 - \nu_B(x_i))}{2}.$$

Definition 5.2.1 A *weighted average cardinality based parametric family of intuitionistic fuzzy similarity measure* is a binary relation $\delta : IFS(X) \times IFS(X) \longrightarrow [0, 1]$ defined as:

$$\delta(A, B) = \frac{x(Card_{0.5}(A \triangle B)) + y(Card_{0.5}(A \cap B)) + z(Card_{0.5}((A \cup B)^c))}{x'(Card_{0.5}(A \triangle B)) + y(Card_{0.5}(A \cap B)) + z(Card_{0.5}((A \cup B)^c))}$$

$$= \frac{x(Card_{0.5}(A) + Card_{0.5}(B) - 2Card_{0.5}(A \cap B)) + y(Card_{0.5}(A \cap B)) + z(n - Card_{0.5}(A \cup B))}{x'(Card_{0.5}(A) + Card_{0.5}(B) - 2Card_{0.5}(A \cap B)) + y(Card_{0.5}(A \cap B)) + z(n - Card_{0.5}(A \cup B))}$$

where x, x', y, z are positive real numbers.

Remark 5.2.2 Since $\delta(A, B) \in [0, 1]$, $0 \leq x \leq x'$, which makes $\delta(A, B)$ define a class of $[0, 1]-$valued similarity measures that are rational expressions in the cardinalities of the sets involved. In case of ordinary sets, it was proved in [87] that some of the famous similarity measures known in literature are a member of this parametric family. They obtained the expressions for these similarity measures by setting different combinations of the parameter involved. The results obtained can be represented as follows:

Measure	Expression	x	x'	y	z
Jaccard (δ_J)	$\frac{\lvert A\cap B\rvert}{\lvert A\cup B\rvert}$	0	1	1	0
Simple Matching (δ_{SM})	$1-\frac{\lvert A\triangle B\rvert}{n}$	0	1	1	1
Dice Coefficient (δ_D)	$\frac{2\lvert A\cap B\rvert}{\lvert A\triangle B\rvert+2\lvert A\cap B\rvert}$	0	1	2	0
Rogers and Tanimoto (δ_{RT})	$\frac{n-\lvert A\triangle B\rvert}{n+\lvert A\triangle B\rvert}$	0	2	1	1
Sokal and Sneath 1 (δ_{SK1})	$\frac{\lvert A\cap B\rvert}{\lvert A\cap B\rvert+2\lvert A\triangle B\rvert}$	0	2	1	0
Sokal and Sneath 2 (δ_{SK2})	$1-\frac{\lvert A\triangle B\rvert}{2n-\lvert A\triangle B\rvert}$	0	1	2	2

Table 5.2.1: Some well known similarity measures for ordinary sets

Now for intuitionistic fuzzy sets we have transformed all the similarity measures for ordinary sets given in Table 5.2.1 as follows:

(1). Jaccard:

$$\delta_J^j(A,B)=\frac{Card_{0.5}(A\cap B)}{Card_{0.5}(A\cup B)}$$

$$=\frac{\sum\limits_{x_i\in X} T_j(\mu_A(x_i),\mu_B(x_i))+1-S_j(\nu_A(x_i),\nu_B(x_i))}{\sum\limits_{x_i\in X} S_j(\mu_A(x_i),\mu_B(x_i))+1-T_j(\nu_A(x_i),\nu_B(x_i))}$$

$$=\frac{\sum\limits_{x_i\in X} T_j(\mu_A(x_i),\mu_B(x_i))+1-\nu_A(x_i)-\nu_B(x_i)+T_j(\nu_A(x_i),\nu_B(x_i))}{\sum\limits_{x_i\in X} \mu_A(x)+\mu_B(x)-T_j(\mu_A(x_i),\mu_B(x_i))+1-T_j(\nu_A(x_i),\nu_B(x_i))}$$

$$=\frac{\sum\limits_{x_i\in X} T_j(\mu_A(x_i),\mu_B(x_i))+T_j(1-\nu_A(x_i),1-\nu_B(x_i))}{\sum\limits_{x_i\in X} S_j(\mu_A(x_i),\mu_B(x_i))+S_j(1-\nu_A(x_i),1-\nu_B(x_i))};$$

(2). Simple Matching:

$$\delta_{SM}^j(A,B)=1-\frac{Card_{0.5}(A\triangle B)}{n}$$

$$=\frac{2n-\sum\limits_{x_i\in X} \mu_A(x_i)+\mu_B(x_i)-\nu_A(x_i)-\nu_B(x_i)-2T_j(\mu_A(x_i),\mu_B(x_i))+2S_j(\nu_A(x_i),\nu_B(x_i))}{2n}$$

$$=\frac{2n-\sum\limits_{x_i\in X} \mu_A(x_i)+\mu_B(x_i)+\nu_A(x_i)+\nu_B(x_i)-2T_j(\mu_A(x_i),\mu_B(x_i))-2T_j(\nu_A(x_i),\nu_B(x_i))}{2n}$$

$$= \frac{2n - \sum_{x_i \in X} \mu_A(x_i)+\mu_B(x_i)+(1-\nu_A(x_i))+(1-\nu_B(x_i))-2T_j(\mu_A(x_i),\mu_B(x_i))-2T_j(1-\nu_A(x_i),1-\nu_B(x_i))}{2n};$$

(3). Dice Coefficient:

$$\delta_D^j(A,B) = \frac{2Card_{0.5}(A\cap B)}{Card_{0.5}(A\triangle B)+2Card_{0.5}(A\cap B)}$$

$$= \frac{2\sum_{x_i \in X} T_j(\mu_A(x_i),\mu_B(x_i))+1-S_j(\nu_A(x_i),\nu_B(x_i))}{\sum_{x_i \in X} \mu_A(x_i)+\mu_B(x_i)+(1-\nu_A(x_i))+(1-\nu_B(x_i))}$$

$$= \frac{2\sum_{x_i \in X} T_j(\mu_A(x_i),\mu_B(x_i))+1-\nu_A(x_i)-\nu_B(x_i)+T_j(\nu_A(x_i),\nu_B(x_i))}{\sum_{x_i \in X} \mu_A(x_i)+\mu_B(x_i)+(1-\nu_A(x_i))+(1-\nu_B(x_i))}$$

$$= \frac{2\sum_{x_i \in X} T_j(\mu_A(x_i),\mu_B(x_i))+T_j(1-\nu_A(x_i),1-\nu_B(x_i))}{\sum_{x_i \in X} \mu_A(x_i)+\mu_B(x_i)+(1-\nu_A(x_i))+(1-\nu_B(x_i))};$$

(4). Rogers and Tanimoto:

$$\delta_{RT}^j(A,B) = \frac{n-Card_{0.5}(A\triangle B)}{n+Card_{0.5}(A\triangle B)}$$

$$= \frac{2n-\sum_{x_i \in X} \mu_A(x_i)+\mu_B(x_i)-\nu_A(x_i)-\nu_B(x_i)-2T_j(\mu_A(x_i),\mu_B(x_i))+2S_j(\nu_A(x_i),\nu_B(x_i))}{2n+\sum_{x_i \in X} \mu_A(x_i)+\mu_B(x_i)-\nu_A(x_i)-\nu_B(x_i)-2T_j(\mu_A(x_i),\mu_B(x_i))+2S_j(\nu_A(x_i),\nu_B(x_i))}$$

$$= \frac{2n-\sum_{x_i \in X} \mu_A(x_i)+\mu_B(x_i)+\nu_A(x_i)+\nu_B(x_i)-2T_j(\mu_A(x_i),\mu_B(x_i))-2T_j(\nu_A(x_i),\nu_B(x_i))}{2n+\sum_{x_i \in X} \mu_A(x_i)+\mu_B(x_i)+\nu_A(x_i)+\nu_B(x_i)-2T_j(\mu_A(x_i),\mu_B(x_i))-2T_j(\nu_A(x_i),\nu_B(x_i))}$$

$$= \frac{2n-\sum_{x_i \in X} \mu_A(x_i)+\mu_B(x_i)+(1-\nu_A(x_i))+(1-\nu_B(x_i))-2T_j(\mu_A(x_i),\mu_B(x_i))-2T_j(1-\nu_A(x_i),1-\nu_B(x_i))}{2n+\sum_{x_i \in X} \mu_A(x_i)+\mu_B(x_i)+(1-\nu_A(x_i))+(1-\nu_B(x_i))-2T_j(\mu_A(x_i),\mu_B(x_i))-2T_j(1-\nu_A(x_i),1-\nu_B(x_i))};$$

(5). Sokal and Sneath 1:

$$\delta_{SK1}^j(A,B) = \frac{Card_{0.5}(A\cap B)}{Card_{0.5}(A\cap B)+2Card_{0.5}(A\triangle B)}$$

$$= \frac{\sum_{x_i \in X} T_j(\mu_A(x_i),\mu_B(x_i))+1-S_j(\nu_A(x_i),\nu_B(x_i))}{\sum_{x_i \in X} 2\mu_A(x_i)+2\mu_B(x_i)+2(1-\nu_A(x_i))+2(1-\nu_B(x_i))-3T_j(\mu_A(x_i),\mu_B(x_i))-3+3S_j(\nu_A(x_i),\nu_B(x_i))}$$

$$= \frac{\sum_{x_i \in X} T_j(\mu_A(x_i),\mu_B(x_i))+1-\nu_A(x_i)-\nu_B(x_i)+T_j(\nu_A(x_i),\nu_B(x_i))}{\sum_{x_i \in X} 2\mu_A(x_i)+2\mu_B(x_i)+2(1-\nu_A(x_i))+2(1-\nu_B(x_i))-3T_j(\mu_A(x_i),\mu_B(x_i))-3+3\nu_A(x)+3\nu_B(x)-3T_j(\nu_A(x_i),\nu_B(x_i))}$$

$$= \frac{\sum_{x_i\in X} T_j(\mu_A(x_i),\mu_B(x_i))+T_j(1-\nu_A(x_i),1-\nu_B(x_i))}{\sum_{x_i\in X} 2\mu_A(x_i)+2\mu_B(x_i)+2(1-\nu_A(x_i))+2(1-\nu_B(x_i))-3T_j(\mu_A(x_i),\mu_B(x_i))-3T_j(1-\nu_A(x_i),1-\nu_B(x_i))};$$

(6). Sokal and Sneath 2:

$\delta^j_{SK2}(A,B) = 1 - \frac{Card_{0.5}(A\triangle B)}{2n-Card_{0.5}(A\triangle B)}$

$$= \frac{4n-2\sum_{x_i\in X} \mu_A(x_i)+\mu_B(x_i)-\nu_A(x_i)-\nu_B(x_i)-2T_j(\mu_A(x_i),\mu_B(x_i))+2S_j(\nu_A(x_i),\nu_B(x_i))}{4n-\sum_{x_i\in X} \mu_A(x_i)+\mu_B(x_i)-\nu_A(x_i)-\nu_B(x_i)-2T_j(\mu_A(x_i),\mu_B(x_i))+2S_j(\nu_A(x_i),\nu_B(x_i))}$$

$$= \frac{4n-2\sum_{x_i\in X} \mu_A(x_i)+\mu_B(x_i)+\nu_A(x_i)+\nu_B(x_i))-2T_j(\mu_A(x_i),\mu_B(x_i))-2T_j(\nu_A(x_i),\nu_B(x_i))}{4n-\sum_{x_i\in X} \mu_A(x_i)+\mu_B(x_i)+\nu_A(x_i)+\nu_B(x_i))-2T_j(\mu_A(x_i),\mu_B(x_i))-2T_j(\nu_A(x_i),\nu_B(x_i))}$$

$$= \frac{4n-2\sum_{x_i\in X} \mu_A(x_i)+\mu_B(x_i)+(1-\nu_A(x_i))+(1-\nu_B(x_i))-2T_j(\mu_A(x_i),\mu_B(x_i))-2T_j(1-\nu_A(x_i),1-\nu_B(x_i))}{4n-\sum_{x_i\in X} \mu_A(x_i)+\mu_B(x_i)+(1-\nu_A(x_i))+(1-\nu_B(x_i))-2T_j(\mu_A(x_i),\mu_B(x_i))-2T_j(1-\nu_A(x_i),1-\nu_B(x_i))}$$

such that $j = M, L, P$.

Moreover, from here onward we shall use the notation δ^j_k, $k = J, SM, D, RT, SK1, SK2$ to specify the above transformed six members of the weighted average cardinality based parametric family of intuitionistic fuzzy similarity measure (δ).

Theorem 5.2.3 The members δ^j_k, $k = J, SM, D, RT, SK1, SK2$ of δ satisfy the following properties: For all $A, B, C \in IFS(X)$,

(a). $\delta^j_k(A,A) = 1$ holds for $j = M$;

(b-1). $\delta^j_k(A,A^c) = 0$ iff either $A = \widetilde{1}_X$ or $A = \widetilde{0}_X$ for $j = M, P$;

(b-2). $\delta^j_k(A,A^c) = 0$ iff A is a fuzzy set where $j = L$;

(c). $\delta^j_k(A,A^c) = 1$ if and only if $\mu_A(x_i) = \nu_A(x_i)$ for all $x_i \in X$, where $j = M$;

(d). $\delta^j_k(A,B) = \delta^j_k(B,A)$ holds for $j = M, P, L$;

(e). If $A \subseteq B \subseteq C$, then $\delta^j_k(A,C) \le \delta^j_k(A,B)$ and $\delta^j_k(A,C) \le \delta^j_k(B,C)$ holds for $k = J$ and $j = M$.

Proof

(a-i). $\delta^M_J(A,A) =$

$$\frac{\sum_{x_i\in X} \min(\mu_A(x_i),\mu_A(x_i))+\min(1-\nu_A(x_i),1-\nu_A(x_i))}{\sum_{x_i\in X} \max(\mu_A(x_i),\mu_A(x_i))+\max(1-\nu_A(x_i),1-\nu_A(x_i))} = 1.$$

(a-ii). $\delta^M_{SM}(A,A) =$

$$\frac{2n-\sum\limits_{x_i\in X}\mu_A(x_i)+\mu_A(x_i)+(1-\nu_A(x_i))+(1-\nu_A(x_i))-2\min(\mu_A(x_i),\mu_A(x_i))-2\min(1-\nu_A(x_i)),(1-\nu_A(x_i)))}{2n}=1.$$

(a-iii). $\delta_D^M(A,A)=$

$$\frac{2\sum\limits_{x_i\in X}\min(\mu_A(x_i),\mu_A(x_i))+\min(1-\nu_A(x_i),1-\nu_A(x_i))}{\sum\limits_{x_i\in X}\mu_A(x_i)+\mu_A(x_i)+(1-\nu_A(x_i))+(1-\nu_A(x_i))}=1.$$

(a-iv). $\delta_{RT}^M(A,A)=$

$$\frac{2n-\sum\limits_{x_i\in X}\mu_A(x_i)+\mu_A(x_i)+(1-\nu_A(x_i))+(1-\nu_A(x_i))-2\min(\mu_A(x_i),\mu_A(x_i))-2\min(1-\nu_A(x_i),1-\nu_A(x_i))}{2n+\sum\limits_{x_i\in X}\mu_A(x_i)+\mu_A(x_i)+(1-\nu_A(x_i))+(1-\nu_A(x_i))-2\min(\mu_A(x_i),\mu_A(x_i))-2\min(1-\nu_A(x_i),1-\nu_A(x_i))}=1.$$

(a-v). $\delta_{SK1}^M(A,A)=$

$$\frac{\sum\limits_{x_i\in X}\min(\mu_A(x_i),\mu_A(x_i))+\min(1-\nu_A(x_i),1-\nu_A(x_i))}{\sum\limits_{x_i\in X}2\mu_A(x_i)+2\mu_A(x_i)+2(1-\nu_A(x_i))+2(1-\nu_A(x_i))-3\min(\mu_A(x_i),\mu_A(x_i))-3\min(1-\nu_A(x_i),1-\nu_A(x_i))}=1.$$

(a-vi). $\delta_{SK2}^M(A,A)=$

$$\frac{4n-2\sum\limits_{x_i\in X}\mu_A(x_i)+\mu_A(x_i)+(1-\nu_A(x_i))+(1-\nu_A(x_i))-2\min(\mu_A(x_i),\mu_A(x_i))-2\min(1-\nu_A(x_i),1-\nu_A(x_i))}{4n-\sum\limits_{x_i\in X}\mu_A(x_i)+\mu_A(x_i)+(1-\nu_A(x_i))+(1-\nu_A(x_i))-2\min(\mu_A(x_i),\mu_A(x_i))-2\min(1-\nu_A(x_i),1-\nu_A(x_i))}=1.$$

(b-1-i). $\delta_J^j(A,A^c)=0$

if and only if $\sum\limits_{x_i\in X}T_j(\mu_A(x_i),\nu_A(x_i))+T_j(1-\nu_A(x_i),1-\mu_A(x_i))=0$

if and only if

$$T_j(\mu_A(x_i),\nu_A(x_i))+T_j(1-\nu_A(x_i),1-\mu_A(x_i))=0 \text{ for all } x_i\in X. \tag{5.4}$$

(Case 1): Let us fix $j=M$ in (5.4).

Then $\min(\mu_A(x_i),\nu_A(x_i))=0$ and $\min(1-\nu_A(x_i),1-\mu_A(x_i))=0$ for all $x_i\in X$

if and only if either $[\mu_A(x_i)=0$ and $1-\nu_A(x_i)=0]$ or $[\nu_A(x_i)=0$ and $1-\mu_A(x_i)=0]$ for all $x_i\in X$

if and only if either $[\mu_A(x_i)=0$ and $\nu_A(x_i)=1]$ or $[\nu_A(x_i)=0$ and $\mu_A(x_i)=1]$ for all $x_i\in X$

if and only if either $A=\widetilde{1}_X$ or $A=\widetilde{0}_X$.

(Case 2): Let us fix $j=P$ in (5.4).

Then $T_P(\mu_A(x_i),\nu_A(x_i))+T_P(1-\nu_A(x_i),1-\mu_A(x_i))=0$ for all $x_i\in X$

if and only if $(\mu_A(x_i)\nu_A(x_i)) = 0$ and $(1 - \nu_A(x_i))(1 - \mu_A(x_i)) = 0$ for all $x_i \in X$

if and only if either $[\mu_A(x_i) = 0$ and $1-\nu_A(x_i) = 0]$ or $[\nu_A(x_i) = 0$ and $1-\mu_A(x_i) = 0]$ for all $x_i \in X$

if and only if either $[\mu_A(x_i) = 0$ and $\nu_A(x_i) = 1]$ or $[\nu_A(x_i) = 0$ and $\mu_A(x_i) = 1]$ for all $x_i \in X$

if and only if either $A = \widetilde{1}_X$ or $A = \widetilde{0}_X$.

(b-1-ii). $\delta^j_{SM}(A, A^c) = 0$

if and only if $2n - \sum_{x_i \in X} [\mu_A(x_i) + \nu_A(x_i) + (1 - \nu_A(x_i)) + (1 - \mu_A(x_i))$
$-2T_j(\mu_A(x_i), \nu_A(x_i)) - 2T_j(1 - \nu_A(x_i)), (1 - \mu_A(x_i)))] = 0$

if and only if $\sum_{x_i \in X} [\mu_A(x_i) + \nu_A(x_i) + (1 - \nu_A(x_i)) + (1 - \mu_A(x_i))$
$-2T_j(\mu_A(x_i), \nu_A(x_i)) - 2T_j(1 - \nu_A(x_i)), (1 - \mu_A(x_i)))] = 2n$

if and only if $[\mu_A(x_i) + \nu_A(x_i) + (1 - \nu_A(x_i)) + (1 - \mu_A(x_i))$
$-2T_j(\mu_A(x_i), \nu_A(x_i)) - 2T_j(1 - \nu_A(x_i)), (1 - \mu_A(x_i)))] = 2$ for all $x_i \in X$

if and only if $T_j(\mu_A(x_i), \nu_A(x_i)) + T_j(1 - \nu_A(x_i), 1 - \mu_A(x_i)) = 0$ for all $x_i \in X$

which is equation (5.4). The rest of the proof form part (b-1-i) gives $\delta^j_{SM}(A, A^c) = 0$ if and only if either $A = \widetilde{1}_X$ or $A = \widetilde{0}_X$; $j = M, P$.

(b-1-iii). $\delta^j_D(A, A^c) = 0$

if and only $2\sum_{x_i \in X} T_j(\mu_A(x_i), \nu_A(x_i)) + T_j(1 - \nu_A(x_i), 1 - \mu_A(x_i)) = 0$

if and only if $T_j(\mu_A(x_i), \nu_A(x_i)) + T_j(1 - \nu_A(x_i), 1 - \mu_A(x_i)) = 0$ for all $x_i \in X$

which is equation (5.4). The rest of the proof is similar to (b-1-i) and gives $\delta^j_D(A, A^c) = 0$ if and only if either $A = \widetilde{1}_X$ or $A = \widetilde{0}_X$; $j = M, P$.

(b-1-iv). $\delta^M_{RT}(A, A^c) = 0$

if and only if $2n - \sum_{x_i \in X} [\mu_A(x_i) + \nu_A(x_i) + (1 - \nu_A(x_i)) + (1 - \mu_A(x_i))$
$-2T_j(\mu_A(x_i), \nu_A(x_i)) - 2T_j(1 - \nu_A(x_i), 1 - \mu_A(x_i))] = 0$

if and only if $\sum_{x_i \in X} [\mu_A(x_i) + \nu_A(x_i) + (1 - \nu_A(x_i)) + (1 - \mu_A(x_i))$
$-2T_j(\mu_A(x_i), \nu_A(x_i)) - 2T_j(1 - \nu_A(x_i)), (1 - \mu_A(x_i)))] = 2n$

if and only if $[\mu_A(x_i) + \nu_A(x_i) + (1 - \nu_A(x_i)) + (1 - \mu_A(x_i))$
$-2T_j(\mu_A(x_i), \nu_A(x_i)) - 2T_j(1 - \nu_A(x_i)), (1 - \mu_A(x_i)))] = 2$ for all $x_i \in X$

if and only if $T_j(\mu_A(x_i), \nu_A(x_i)) + T_j(1 - \nu_A(x_i), 1 - \mu_A(x_i)) = 0$ for all $x_i \in X$

which is equation (5.4) and the rest of the proof is similar to part (b-1-i) and gives

$\delta^j_{RT}(A, A^c) = 0$ if and only if either $A = \widetilde{1}_X$ or $A = \widetilde{0}_X$; $j = M, P$.

(b-1-v). $\delta^M_{SK1}(A, A^c) = 0$

if and only if $\sum_{x_i \in X} T_j(\mu_A(x_i), \nu_A(x_i)) + T_j(1 - \nu_A(x_i), 1 - \mu_A(x_i)) = 0$

if and only if $T_j(\mu_A(x_i), \nu_A(x_i)) + T_j(1 - \nu_A(x_i), 1 - \mu_A(x_i)) = 0$ for all $x_i \in X$

which is again equation (5.4) giving

$\delta^j_{SK1}(A, A^c) = 0$ if and only if either $A = \widetilde{1}_X$ or $A = \widetilde{0}_X$; $j = M, P$.

(b-1-vi). $\delta^M_{SK2}(A, A^c) = 0$

if and only if $4n - 2\sum_{x_i \in X} [\mu_A(x_i) + \nu_A(x_i) + (1 - \nu_A(x_i)) + (1 - \mu_A(x_i))$

$-2T_j(\mu_A(x_i), \nu_A(x_i)) - 2T_j(1 - \nu_A(x_i), 1 - \mu_A(x_i))] = 0$

if and only if $\sum_{x_i \in X} [\mu_A(x_i) + \nu_A(x_i) + (1 - \nu_A(x_i)) + (1 - \mu_A(x_i))$

$-2T_j(\mu_A(x_i), \nu_A(x_i)) - 2T_j(1 - \nu_A(x_i)), (1 - \mu_A(x_i)))] = 2n$

if and only if $[\mu_A(x_i) + \nu_A(x_i) + (1 - \nu_A(x_i)) + (1 - \mu_A(x_i))$

$-2T_j(\mu_A(x_i), \nu_A(x_i)) - 2T_j(1 - \nu_A(x_i)), (1 - \mu_A(x_i)))] = 2$ for all $x_i \in X$

if and only if $T_j(\mu_A(x_i), \nu_A(x_i)) + T_j(1 - \nu_A(x_i), 1 - \mu_A(x_i)) = 0$ for all $x_i \in X$

which is equation (5.4) and the rest of the proof is the same as done in part (b-1-i) which gives $\delta^j_{SK2}(A, A^c) = 0$ if and only if either $A = \widetilde{1}_X$ or $A = \widetilde{0}_X$; $j = M, P$.

(b-2-i). $\delta^L_J(A, A^c) = 0$

if and only if $\sum_{x_i \in X} T_L(\mu_A(x_i), \nu_A(x_i)) + T_L(1 - \nu_A(x_i), 1 - \mu_A(x_i)) = 0$

if and only if

$$T_L(\mu_A(x_i), \nu_A(x_i)) + T_L(1 - \nu_A(x_i), 1 - \mu_A(x_i)) = 0 \text{ for all } x_i \in X \tag{5.5}$$

if and only if $\max(0, \mu_A(x_i) + \nu_A(x_i) - 1) + \max(0, (1 - \nu_A(x_i) + (1 - \mu_A(x_i)) - 1) = 0$ for all $x_i \in X$

if and only if either $[\max(0, \mu_A(x_i) + \nu_A(x_i) - 1) = 0$

and $\max(0, (1 - \nu_A(x_i) + (1 - \mu_A(x_i)) - 1) = 0]$

or $[\max(0, \mu_A(x_i) + \nu_A(x_i) - 1) = -\max(0, (1 - \nu_A(x_i) + (1 - \mu_A(x_i)) - 1) = 0]$

if and only if either $[\mu_A(x_i) + \nu_A(x_i) - 1 \leq 0$ and $(1 - \nu_A(x_i) + (1 - \mu_A(x_i)) - 1) \leq 0]$

or $[\mu_A(x_i) + \nu_A(x_i) - 1 = -((1 - \nu_A(x_i) + (1 - \mu_A(x_i)) - 1))]$

if and only if either $[\mu_A(x_i) + \nu_A(x_i) \leq 1$ and $\mu_A(x_i) + \nu_A(x_i) \geq 1]$

or $[\mu_A(x_i) + \nu_A(x_i) - 1 = -1 + \nu_A(x_i) - 1 + \mu_A(x_i) + 1]$

if and only if $[\mu_A(x_i) + \nu_A(x_i) = 1]$ for all $x_i \in X$

if and only if A is fuzzy set.

(b-2-ii). $\delta^L_{SM}(A, A^c) = 0$

if and only if $2n - \sum_{x_i \in X} [\mu_A(x_i) + \nu_A(x_i) + (1 - \nu_A(x_i)) + (1 - \mu_A(x_i))$

$-2T_L(\mu_A(x_i), \nu_A(x_i)) - 2T_L(1 - \nu_A(x_i)), (1 - \mu_A(x_i)))] = 0$

if and only if $\sum_{x_i \in X} [\mu_A(x_i) + \nu_A(x_i) + (1 - \nu_A(x_i)) + (1 - \mu_A(x_i))$

$-2T_L(\mu_A(x_i), \nu_A(x_i)) - 2T_L(1 - \nu_A(x_i)), (1 - \mu_A(x_i)))] = 2n$

if and only if $[\mu_A(x_i) + \nu_A(x_i) + (1 - \nu_A(x_i)) + (1 - \mu_A(x_i))$

$-2T_L(\mu_A(x_i), \nu_A(x_i)) - 2T_L(1 - \nu_A(x_i)), (1 - \mu_A(x_i)))] = 2$ for all $x_i \in X$

if and only if $T_L(\mu_A(x_i), \nu_A(x_i)) + T_L(1 - \nu_A(x_i), 1 - \mu_A(x_i)) = 0$ for all $x_i \in X$

which is equation (5.5). The rest of the proof is straight forward from part (b-2-i) giving $\delta^L_{SM}(A, A^c) = 0$ if and only if A is a fuzzy set.

(b-2-iii). $\delta^L_D(A, A^c) = 0$

if and only $2 \sum_{x_i \in X} T_L(\mu_A(x_i), \nu_A(x_i)) + T_L(1 - \nu_A(x_i), 1 - \mu_A(x_i)) = 0$

if and only if $T_L(\mu_A(x_i), \nu_A(x_i)) + T_L(1 - \nu_A(x_i), 1 - \mu_A(x_i)) = 0$ for all $x_i \in X$

which is equation (5.5) and the rest of the proof is the same as done in part (b-2-i) giving

$\delta^L_D(A, A^c) = 0$ if and only if A is a fuzzy set.

(b-2-iv). $\delta^M_{RT}(A, A^c) = 0$

if and only if $2n - \sum_{x_i \in X} [\mu_A(x_i) + \nu_A(x_i) + (1 - \nu_A(x_i)) + (1 - \mu_A(x_i))$

$-2T_L(\mu_A(x_i), \nu_A(x_i)) - 2T_L(1 - \nu_A(x_i), 1 - \mu_A(x_i))] = 0$

if and only if $\sum_{x_i \in X} [\mu_A(x_i) + \nu_A(x_i) + (1 - \nu_A(x_i)) + (1 - \mu_A(x_i))$

$-2T_L(\mu_A(x_i), \nu_A(x_i)) - 2T_L(1 - \nu_A(x_i)), (1 - \mu_A(x_i)))] = 2n$

if and only if $[\mu_A(x_i) + \nu_A(x_i) + (1 - \nu_A(x_i)) + (1 - \mu_A(x_i))$

$-2T_L(\mu_A(x_i), \nu_A(x_i)) - 2T_L(1 - \nu_A(x_i)), (1 - \mu_A(x_i)))] = 2$ for all $x_i \in X$

if and only if $T_L(\mu_A(x_i), \nu_A(x_i)) + T_L(1 - \nu_A(x_i), 1 - \mu_A(x_i)) = 0$ for all $x_i \in X$

which is again equation (5.5) giving

$\delta^L_{RT}(A, A^c) = 0$ if and only if A is a fuzzy set.

(b-2-v). $\delta^M_{SK1}(A, A^c) = 0$

if and only if $\sum_{x_i \in X} T_L(\mu_A(x_i), \nu_A(x_i)) + T_L(1 - \nu_A(x_i), 1 - \mu_A(x_i)) = 0$

if and only if $T_L(\mu_A(x_i), \nu_A(x_i)) + T_L(1-\nu_A(x_i), 1-\mu_A(x_i)) = 0$ for all $x_i \in X$

which is equation (5.5).

Thus, $\delta^L_{SK1}(A, A^c) = 0$ if and only if A is a fuzzy set.

(b-2-vi). $\delta^M_{SK2}(A, A^c) = 0$

if and only if $4n - 2\sum_{x_i \in X} [\mu_A(x_i) + \nu_A(x_i) + (1-\nu_A(x_i)) + (1-\mu_A(x_i))$

$-2T_L(\mu_A(x_i), \nu_A(x_i)) - 2T_L(1-\nu_A(x_i), 1-\mu_A(x_i))] = 0$

if and only if $\sum_{x_i \in X} [\mu_A(x_i) + \nu_A(x_i) + (1-\nu_A(x_i)) +$

$(1-\mu_A(x_i)) - 2T_L(\mu_A(x_i), \nu_A(x_i)) - 2T_L(1-\nu_A(x_i)), (1-\mu_A(x_i)))] = 2n$

if and only if $[\mu_A(x_i) + \nu_A(x_i) + (1-\nu_A(x_i)) + (1-\mu_A(x_i))$

$-2T_L(\mu_A(x_i), \nu_A(x_i)) - 2T_L(1-\nu_A(x_i)), (1-\mu_A(x_i)))] = 2$ for all $x_i \in X$

if and only if $T_L(\mu_A(x_i), \nu_A(x_i)) + T_L(1-\nu_A(x_i), 1-\mu_A(x_i)) = 0$ for all $x_i \in X$

which is equation (5.5) and the rest of the proof is similar to part (b-2-i) and gives $\delta^L_{SK2}(A, A^c) = 0$ if and only if A is a fuzzy set.

(c-i). $\delta^M_J(A, A^c) = 1$

if and only if $\min(\mu_A(x_i), \nu_A(x_i)) + \min(1-\nu_A(x_i), 1-\mu_A(x_i))$

$= \max(\mu_A(x_i), \nu_A(x_i)) + \max(1-\nu_A(x_i), 1-\mu_A(x_i))$ for all $x_i \in X$

if and only if $\mu_A(x_i) = \nu_A(x_i)$ for all $x_i \in X$.

(c-ii). $\delta^M_{SM}(A, A^c) = 1$

if and only if $[\mu_A(x_i) + \nu_A(x_i) + (1-\nu_A(x_i)) + (1-\nu_A(x_i))$

$-2\min(\mu_A(x_i), \nu_A(x_i)) - 2\min(1-\mu_A(x_i)), (1-\nu_A(x_i)))] = 0$ for all $x_i \in X$

if and only if $\min(\mu_A(x_i), \nu_A(x_i)) + \min(1-\nu_A(x_i), 1-\mu_A(x_i)) = 1$ for all $x_i \in X$

if and only if either $[\min(\mu_A(x_i), \nu_A(x_i)) = \mu_A(x_i)$ and $\min(1-\nu_A(x_i), 1-\mu_A(x_i)) = 1-\nu_A(x_i)]$ or $[\min(\mu_A(x_i), \nu_A(x_i)) = \nu_A(x_i)$ and $\min(1-\nu_A(x_i), 1-\mu_A(x_i)) = 1-\mu_A(x_i)]$

if and only if $\mu_A(x_i) = \nu_A(x_i)$ for all $x_i \in X$.

(c-iii). $\delta^M_D(A, A^c) = 1$

if and only if $\sum_{x_i \in X} 2\min(\mu_A(x_i), \nu_A(x_i)) + 2\min(1-\nu_A(x_i), 1-\mu_A(x_i))$

$= \sum_{x_i \in X} \mu_A(x_i) + \nu_A(x_i) + (1-\nu_A(x_i)) + (1-\mu_A(x_i))$

if and only if $[\mu_A(x_i) + \nu_A(x_i) + (1-\nu_A(x_i)) + (1-\nu_A(x_i))$

$-2\min(\mu_A(x_i), \nu_A(x_i)) - 2\min(1-\mu_A(x_i)), (1-\nu_A(x_i)))] = 0$ for all $x_i \in X$

if and only if $\min(\mu_A(x_i), \nu_A(x_i)) + \min(1-\nu_A(x_i), 1-\mu_A(x_i)) = 1$ for all $x_i \in X$

if and only if either $[\min(\mu_A(x_i), \nu_A(x_i)) = \mu_A(x_i)$

and $\min(1-\nu_A(x_i), 1-\mu_A(x_i)) = 1-\nu_A(x_i)]$ or $[\min(\mu_A(x_i), \nu_A(x_i)) = \nu_A(x_i)$ and $\min(1-\nu_A(x_i), 1-\mu_A(x_i)) = 1-\mu_A(x_i)]$

if and only if $\mu_A(x_i) = \nu_A(x_i)$ for all $x_i \in X$.

(c-iv). $\delta^M_{RT}(A, A^c) = 1$

if and only if $2n - \sum_{x_i \in X}[\mu_A(x_i) + \nu_A(x_i) + (1-\nu_A(x_i)) + (1-\mu_A(x_i))$

$-2\min(\mu_A(x_i), \nu_A(x_i)) - 2\min(1-\nu_A(x_i), 1-\mu_A(x_i))]$

$= 2n + \sum_{x_i \in X}[\mu_A(x_i) + \nu_A(x_i) + (1-\nu_A(x_i)) + (1-\mu_A(x_i))$

$-2\min(\mu_A(x_i), \nu_A(x_i)) - 2\min(1-\nu_A(x_i), 1-\nu_A(x_i))]$

if and only if $[\mu_A(x_i) + \nu_A(x_i) + (1-\nu_A(x_i)) + (1-\nu_A(x_i))$

$-2\min(\mu_A(x_i), \nu_A(x_i)) - 2\min(1-\mu_A(x_i)), (1-\nu_A(x_i)))] = 0$ for all $x_i \in X$

if and only if $\min(\mu_A(x_i), \nu_A(x_i)) + \min(1-\nu_A(x_i), 1-\mu_A(x_i)) = 1$ for all $x_i \in X$

if and only if either $[\min(\mu_A(x_i), \nu_A(x_i)) = \mu_A(x_i)$

and $\min(1-\nu_A(x_i), 1-\mu_A(x_i)) = 1-\nu_A(x_i)]$ or $[\min(\mu_A(x_i), \nu_A(x_i)) = \nu_A(x_i)$ and $\min(1-\nu_A(x_i), 1-\mu_A(x_i)) = 1-\mu_A(x_i)]$

if and only if $\mu_A(x_i) = \nu_A(x_i)$ for all $x_i \in X$.

(c-v). $\delta^M_{SK1}(A, A^c) = 1$

if and only if $\sum_{x_i \in X}\min(\mu_A(x_i), \mu_A(x_i)) + \min(1-\nu_A(x_i), 1-\nu_A(x_i))$

$= \sum_{x_i \in X}[2\mu_A(x_i) + 2\nu_A(x_i) + 2(1-\nu_A(x_i)) + 2(1-\mu_A(x_i))$

$-3\min(\mu_A(x_i), \nu_A(x_i)) - 3\min(1-\nu_A(x_i), 1-\mu_A(x_i))]$

if and only if $[\mu_A(x_i) + \nu_A(x_i) + (1-\nu_A(x_i)) + (1-\nu_A(x_i))$

$-2\min(\mu_A(x_i), \nu_A(x_i)) - 2\min(1-\mu_A(x_i)), (1-\nu_A(x_i)))] = 0$ for all $x_i \in X$

if and only if $\min(\mu_A(x_i), \nu_A(x_i)) + \min(1-\nu_A(x_i), 1-\mu_A(x_i)) = 1$ for all $x_i \in X$

if and only if either $[\min(\mu_A(x_i), \nu_A(x_i)) = \mu_A(x_i)$

and $\min(1-\nu_A(x_i), 1-\mu_A(x_i)) = 1-\nu_A(x_i)]$ or $[\min(\mu_A(x_i), \nu_A(x_i)) = \nu_A(x_i)$ and $\min(1-\nu_A(x_i), 1-\mu_A(x_i)) = 1-\mu_A(x_i)]$

if and only if $\mu_A(x_i) = \nu_A(x_i)$ for all $x_i \in X$.

(c-vi). $\delta^M_{SK2}(A, A^c) = 1$

if and only if $4n - 2\sum_{x_i \in X}[\mu_A(x_i) + \nu_A(x_i) + (1-\nu_A(x_i)) + (1-\mu_A(x_i))$

$-2\min(\mu_A(x_i), \nu_A(x_i)) - 2\min(1-\nu_A(x_i), 1-\mu_A(x_i))]$

$= 4n - \sum_{x_i \in X}[\mu_A(x_i) + \nu_A(x_i) + (1-\nu_A(x_i)) + (1-\mu_A(x_i))$

$-2\min(\mu_A(x_i), \nu_A(x_i)) - 2\min(1-\nu_A(x_i), 1-\mu_A(x_i))]$

if and only if $\mu_A(x_i) + \nu_A(x_i) + (1-\nu_A(x_i)) + (1-\nu_A(x_i))$

$-2\min(\mu_A(x_i), \nu_A(x_i)) - 2\min(1-\mu_A(x_i)), (1-\nu_A(x_i))) = 0$ for all $x_i \in X$

if and only if $\min(\mu_A(x_i), \nu_A(x_i)) + \min(1-\nu_A(x_i), 1-\mu_A(x_i)) = 1$ for all $x_i \in X$

if and only if either $[\min(\mu_A(x_i), \nu_A(x_i)) = \mu_A(x_i)$

and $\min(1-\nu_A(x_i), 1-\mu_A(x_i)) = 1-\nu_A(x_i)]$ or $[\min(\mu_A(x_i), \nu_A(x_i)) = \nu_A(x_i)$ and $\min(1-\nu_A(x_i), 1-\mu_A(x_i)) = 1-\mu_A(x_i)]$

if and only if $\mu_A(x_i) = \nu_A(x_i)$ for all $x_i \in X$.

(d). From Definition 5.2.1, clearly, $\delta_k^j(A,B) = \delta_k^j(B,A)$ holds for all $j = M, P, L$.

(e). $A \subseteq B \subseteq C$,

implies that $\mu_A(x_i) \leq \mu_B(x_i) \leq \mu_C(x_i)$ and $\nu_A(x_i) \geq \nu_B(x_i) \geq \nu_C(x_i)$ for all $x_i \in X$

implies that $\min(\mu_A(x_i), \mu_B(x_i)) = \min(\mu_A(x_i), \mu_C(x_i))$

and $\min(1-\nu_A(x_i), 1-\nu_B(x_i)) = \min(1-\nu_A(x_i), 1-\nu_C(x_i))$

implies that $\max(\mu_A(x_i), \mu_B(x_i)) \leq \max(\mu_A(x_i), \mu_C(x_i))$

and $\max(1-\nu_A(x_i), 1-\nu_B(x_i)) \leq \max(1-\nu_A(x_i), 1-\nu_C(x_i))$

implies that $\frac{\min(\mu_A(x_i),\mu_C(x_i))+\min(1-\nu_A(x_i),1-\nu_C(x_i))}{\max(\mu_A(x_i),\mu_C(x_i))+\max(1-\nu_A(x_i),1-\nu_C(x_i))}$

$\leq \frac{\min(\mu_A(x_i),\mu_B(x_i))+\min(1-\nu_A(x_i),1-\nu_B(x_i))}{\max(\mu_A(x_i),\mu_B(x_i))+\max(1-\nu_A(x_i),1-\nu_B(x_i))}$ for all $x_i \in X$

implies that $\delta_J^M(A,C) \leq \delta_J^M(A,B)$

similarly we can obtain $\delta_J^M(A,C) \leq \delta_J^M(B,C)$.

Theorem 5.2.4 The intuitionistic fuzzy similarity measure δ_J^j, $j = M$ defines a fuzzy equivalence relation on $IFS(X)$, i.e., for all $A, B, C \in IFS(X)$,

(a). $\delta_J^j(A,B) = 1$ if and only if $A = B$ (reflexivity);

(b). $\delta_J^j(A,B) = \delta_J^j(B,A)$ (symmetry);

(c). $I_L(T_L(\delta_J^j(A,B), \delta_J^j(B,C)), \delta_J^j(A,C)) > 0$ (T_L-fuzzy transitivity).

Proof

(a). $\delta_J^j(A,B) = 1$; $j = M$

implies that

$$\begin{aligned} T_M(\mu_A(x_i), \mu_B(x_i)) &= S_M(\mu_A(x_i), \mu_B(x_i)) \text{ and} \\ T_M(1-\nu_A(x_i), 1-\nu_B(x_i)) &= S_M(1-\nu_A(x_i), 1-\nu_B(x_i)), \forall x_i \in X. \end{aligned} \tag{5.6}$$

implies that $\mu_A(x_i) = \mu_B(x_i)$ and $\nu_A(x_i) = \nu_B(x_i)$ for all $x_i \in X$

implies that $A = B$

Conversely, for $A = B$ we get $T_M(\mu_A(x_i), \mu_A(x_i)) + T_M(1 - \nu_A(x_i), 1 - \nu_A(x_i))$

$= S_M(\mu_A(x_i), \mu_B(x_i)) + S_M(1 - \nu_A(x_i), 1 - \nu_B(x_i))$ for all $x_i \in X$

implies that $\delta_J^j(A, B) = 1,\ j = M$.

(b). $\delta_J^j(A, B) = \dfrac{\sum\limits_{x_i \in X} T_j(\mu_A(x_i), \mu_B(x_i)) + T_j(1-\nu_A(x_i), 1-\nu_B(x_i))}{\sum\limits_{x_i \in X} S_j(\mu_A(x_i), \mu_B(x_i)) + S_j(1-\nu_A(x_i), 1-\nu_B(x_i))},\ j = M.$

Interchanging the role of A and B will yield the same expression thus,

$\delta_J^j(A, B) = \delta_J^j(B, A);\ j = M$.

(c). On contrary, let suppose that $\delta_J^j;\ j = M$ is not fuzzy transitive. Then,

$$I_L(T_L(\delta_J^j(A, B), \delta_J^j(B, C)), \delta_J^j(A, C)) = 0.$$

It implies that $\delta_J^j(A, B) = 1, \delta_J^j(B, C) = 1$ and, $\delta_J^j(A, C) = 0;\ j = M$.

It further implies that $A = B = C$ but $\delta_J^j(A, C) = 0;\ j = M$,a contradiction. Hence $\delta_J^j;\ j = M$ is T_L-fuzzy transitive.

Chapter 6

Intuitionistic Fuzzy Inclusion Measures

In fuzzy set theory, subsethood/inclusion is, by default, defined by Zadeh himself as: for two $A_1, A_2 \in FS(X)$, $A_1 \subseteq A_2 \Leftrightarrow \mu_{A_1}(x) \leq \mu_{A_2}(x)$ for all $x \in X$ where μ_{A_1} and μ_{A_2} are membership functions of A_1 and A_2 respectively. A direct implication of this definition states that: the subsethood (inclusion) of two fuzzy sets is described by a bivalent function; i.e. a set *is* or *is not* a subset of another set. The restrictive nature of this definition of subsethood of fuzzy sets was challenged by many researchers [20, 34, 67, 69, 95, 159], who proposed different alternative inclusion operators Inc that takes value in [0,1] giving the level of inclusion of FSs; and introduced different systems of axioms [50, 134, 168].

The plethora and importance of the potential applications of IFS's have drawn the attention of many researchers who treated subsethood in various ways [49, 75, 90, 120, 163, 168, 176, 177], however, the search for the most reasonable, truly comparable and computationaly economical inclusion indicator of intuitionstic fuzzy sets is still opened. Keeping all these factors in mind, we believe that a two degree inclusion indicator for an intuitionistic fuzzy set as stated by many searchers [49, 75, 90, 120, 163, 176] may not fulfill these requirements, and hence, we direct our attention towards the search of a single degree (fuzzy) inclusion indicator of IFS. We believe that a fuzzy inclusion measure in an intuitionistic environment is capable of capturing the true essence of vagueness as well as the delicacy of mathematical computation. In this chapter, we have extended the existing knowledge of subsethood/inclusion of sets to intuitionistic fuzzy sets (IFS's). These extensions are based on two different approaches of inclusion existing in fuzzy literature.

The first approach is based on a well known hypothesis that: an implicator plays a fundamental role in defining an inclusion between sets. In case of fuzzy set theory, the first effort in this regard was made by Bandler and Kohout [20]. They defined the inclusion degree of A into B by an inclusion indicator given as:

$$Inc(A, B) = \inf_{x \in X}(I(A, B)) \tag{6.1}$$

where I stands for fuzzy implicator and $A, B \in FS(X)$. This definition gave rise to an extensive research in the area of graded fuzzy inclusion. A valuable contribution in this regard has been made by [22, 23], in which the authors have modified the definition (6.1) by replacing the quantifier 'inf' with different fuzzy measures. The resulting inclusion measure satisfied the axioms of Sinha and Dougherty [134].

In intuitionistic fuzzy set theory we rarely find any inclusion indicator which is based on the above approach of intuitionistic fuzzy implication and hence it remained an open question until now. Thus, as a first attempt towards defining an inclusion measure for IFS's we have generalized the above mentioned idea from fuzzy to intuitionistic fuzzy measure. The new defined class of inclusion measures for intuitionistic fuzzy sets possessed properties similar to the axioms of Sinha and Dougherty [134].

The second approach is based on a parallel idea of inclusion existing in literature where inclusion between sets (ordinary) is defined by cardinalities of sets and different set theoretic operations. This approach specifies the two comparable ordinary sets (objects) by their feature binary vectors. It then formulate different techniques involving the count of the elements of sets (cardinality of ordinary set) to get the degree of inclusion between these representative binary vectors of sets. Getting our inspiration from one of the idea presented in [87], we have constructed a weighted average cardinality based parametric family of intuitionistic fuzzy inclusion measure for IFS's. The new family have properties similar to those proposed by Young [168].

6.1 An Implication based Intuitionistic Fuzzy Inclusions Measure (IIFIM)

The results produced in this section are a part of our published paper [127].

In intuitionistic fuzzy literature, we may find some approaches to IFIM (Intuitionistic Fuzzy Inclusion Measures) but very little is available about the intuitionistic inclusion measures based on intuitionistic fuzzy implicators.

In this first section of our chapter, we have utilized the results presented in [45, 46, 60, 102] about the intuitionistic fuzzy implicators and intuitionistic fuzzy t-norms to define an implication based intuitionistic fuzzy inclusion mapping which is an extension of the idea given in [22].

Next, we have defined the degree of inclusion between two intuitionistic fuzzy sets by composing the new fuzzy measures of intuitionistic fuzzy sets presented in chapter 3 with the new defined intuitionistic fuzzy inclusion maps. This will result into a measure called *Implication based Intuitionistic Fuzzy Inclusion Measure* (IIFIM). The objective behind such a construction is that the new measure of inclusion should be well equipped with the capablility of capturing not the amount of overlap but also the orientation of one IFS with respect to the other IFS. This, surely is, in accordances with the true spirit of IFS-theory.

The properties exhibited by these measures are familiar to the well established axioms of inclusion measures existing in literature.

In this work we have used Lukasiewicz intuitionistic fuzzy implicator $\breve{I}_{\widetilde{T}_L}$ [see Example 2.2.2.9, Chapter 2] to explore these properties. Since a Lukasiewicz intuitionistic fuzzy implicator chooses between the values depending on the region of its domain, i.e., for $A, B \in IFS(X)$, if A is a subset of B in the sense of Zadeh ($A \subseteq B$ if and only if $A(x) \leq_{L^*} B(x)$ i.e., $\mu_A(x) \leq \mu_B(x)$ and $\nu_A(x) \geq \nu_B(x)$ for all $x \in X$) then this implicator will assign at each point of the domain an inclusion value equal to 1_{L^*}. Next, when measure m is applied to obtain the complete inclusion degree, it results into the degree of inclusion of A into B equals to 1. Clearly, in this case the new measure completely agrees to the idea of Zadeh's inclusion. It however, differs from Zadeh's inclusion on points of the domain where $A(x) \nleq_{L^*} B(x)$. In such cases, Zadeh's idea of inclusion fails while the new defined inclusion measure does capture inclusion of sets involved. Thus, the new measure is capable of tackling the real life situations and uncertainties in more practical manner.

Definition 6.1.1 An intuitionistic fuzzy *inclusion* $IInc$ is an $IFS(X) \times IFS(X) \rightarrow IFS(X)$ mapping that assigns to all $A, B \in IFS(X)$ an intuitionistic fuzzy set $IInc(A, B)$ on X defined as:

$$IInc(A, B)(x) = (\mu_{IInc(A,B)}(x), \nu_{IInc(A,B)}(x)) = \breve{I}(A(x), B(x)) \qquad (6.2)$$

where $\breve{I}$ stands for intuitionistic fuzzy implicator and for all $x \in X$, $A(x) = (a_1, a_2), B(x) = (b_1, b_2) \in L^*$. The set of all intuitionistic fuzzy inclusions on X will be denoted by $IInc(X)$.

Definition 6.1.2 An *implication based intuitionistic fuzzy inclusion measure* (IIFIM) is a mapping $m_{IInc} : IFS(X) \times IFS(X) \longrightarrow [0, 1]$, which assigns to all $A, B \in$

$IFS(X)$ a value in $[0,1]$ defined as:

$$m_{IInc}(A,B) = m(IInc(A,B)) \tag{6.3}$$

where $IInc(A,B)$ is the intuitionistic fuzzy inclusion of A into B and m is a normal fuzzy measure of intuitionistic fuzzy sets. Now using the new normal fuzzy measures on IFS given in chapter 3 and Definitions (6.1.1, 6.1.2) we get :

(i). For all $A, B \in IFS(X)$

$$m_{1IInc}(A,B) = \frac{1}{2}(\inf_{x\in X}\mu_{IInc(A,B)}(x) + (1 - \sup_{x\in X}\nu_{IInc(A,B)}(x))).$$

(ii). For all $A, B \in IFS(X)$

$$\begin{aligned} m_{2IInc}(A,B) &= \frac{1}{4}(\inf_{x\in X}\mu_{IInc(A,B)}(x) + (1 - \sup_{x\in X}\nu_{IInc(A,B)}(x)) \\ &\quad + \sup_{x\in X}\mu_{IInc(A,B)}(x) + (1 - \inf_{x\in X}\nu_{IInc(A,B)}(x))). \end{aligned}$$

(iii). For all $A, B \in IFS(X)$

$$\begin{aligned} m'_{3IInc}(A,B) &= \frac{|IInc(A,B)|}{|X\times X|} = \frac{\sum_{x\in X}\theta\mu_{IInc(A,B)}(x) + (1-\theta)(1-\nu_{IInc(A,B)}(x))}{n^2} \\ \text{where } \theta &\in [0.5, 1]. \end{aligned}$$

(iv). For all $A, B \in IFS(X)$

$$m_{4IInc}(A,B) = \frac{1}{n^2}\sum_{x\in X}\sqrt{\mu_{IInc(A,B)}(x)(1-\nu_{IInc(A,B)}(x))}.$$

Remark 6.1.3 Now if we take $\breve{I} = \breve{I}_{\widetilde{T}_L}$ in (6.2) we get

$$IInc(A,B)(x) = (\min(1, b_1 - a_1 + 1, a_2 - b_2 + 1), \max(0, a_1 + b_2 - 1)). \tag{6.4}$$

Moreover, for $A, B \in F(X)$, it is easy to verify that: $IInc(A,B)(x) = (\min(1, b_1 - a_1 + 1), \max(0, a_1 - b_1))$. Hence the degree of being included is: $\min(1, b_1 - a_1 + 1)$ and degree of not being included is: $\max(0, a_1 - b_1)$.

Next, we shall establish certain results on intuitionistic fuzzy inclusion measure defined in (6.3) by taking $IInc(A,B)(x) = (\min(1, b_1 - a_1 + 1, a_2 - b_2 + 1), \max(0, a_1 + b_2 - 1))$.

Proposition 6.1.4 If the measure m satisfies the additional property stated in Remark 3.1.3, then for all $A, B \in IFS(X)$ we have:

(a). $m_{IInc}(A,B) = 1$ if and only if $A \subseteq B$;

(b). $m_{IInc}(A,A^c) = 1$ if and only if A is a fuzzy set with constant membership degree $\mu_A(x) = 0.5$;

(c). $m_{IInc}(A,B) = 0$ if and only if $A = \widetilde{1}_X$ and $B = \widetilde{0}_X$;

(d). $m_{IInc}(A,A^c) = 0$ if and only if $A = \widetilde{1}_X$.

Proof Let $A, B \in IFS(X)$,

(a). $A \subseteq B$

if and only if $A(x) \leq_{L^*} B(x)$ for all $x \in X$

if and only if $a_1 \leq b_1$ and $a_2 \geq b_2$ and $a_1 + a_2 \leq 1$

if and only if $1 \leq b_1 - a_1 + 1$ and $a_2 - b_2 + 1 \geq 1$

if and only if $\min(1, b_1 - a_1 + 1, a_2 - b_2 + 1) = 1$ and $\max(0, a_1 + b_2 - 1) = 0$

if and only if $IInc(A,B)(x) = (\min(1, b_1 - a_1 + 1, a_2 - b_2 + 1), \max(0, a_1 + b_2 - 1))$ $= (1,0) = 1_{L^*}$ for all $x \in X$

if and only if $IInc(A,B) = \widetilde{1}_X$

if and only if $m_{IInc}(A,B) = 1$.

(b). $m_{IInc}(A,A^c) = 1$

if and only if $IInc(A,A^c) = \widetilde{1}_X$

if and only if $IInc(A,A^c)(x) = 1_{L^*}$, $\forall x \in X$

if and only if $(\min(1, a_2 - a_1 + 1, a_2 - a_1 + 1), \max(0, a_1 + a_1 - 1)) = 1_{L^*}$

if and only if $a_2 - a_1 + 1 = 1$ and $2a_1 - 1 = 0$

if and only if $A(x) = (0.5, 0.5)$

if and only if A is a fuzzy set with constant degree of membership $\mu_A(x) = 0.5$.

(c). $A = \widetilde{1}_X$ and $B = \widetilde{0}_X$

if and only if $A(x) = 1_{L^*}$ and $B(x) = 0_{L^*}$ for all $\forall x \in X$

if and only if $a_1 = 1$, $b_1 = 0$, $a_2 = 0$ and $b_2 = 1$

if and only if $b_1 - a_1 + 1 = 0$, $a_2 - b_2 + 1 = 0$ and $a_1 + b_2 - 1 = 1$

if and only if $\min(1, b_1 - a_1 + 1, a_2 - b_2 + 1) = 0$ and $\max(0, a_1 + b_2 - 1) = 1$

if and only if $(\min(1, b_1 - a_1 + 1, a_2 - b_2 + 1), \max(0, a_1 + b_2 - 1)) = (0,1)$

if and only if $IInc(A, B)(x) = 0_{L^*}$ for all $x \in X$

if and only if $IInc(A, B) = \widetilde{0}_X$

if and only if $m_{IInc}(A, B) = 0$.

(d). $A = \widetilde{1}_X$

if and only if $A(x) = 1_{L^*}$ for all $x \in X$

if and only if $IInc(A, A^c)(x) = 0_{L^*}$ for all $x \in X$

if and only if $IInc(A, A^c) = \widetilde{0}_X$

if and only if $m_{IInc}(A, A^c) = 0$.

Proposition 6.1.5 For all $A, B \in IFS(X)$ we have:

$m_{IInc}(A, B) = m_{IInc}(B^c, A^c)$.

Proof

Let $A, B \in IFS(X)$.

For $m_{IInc}(A, B) = m_{IInc}(B^c, A^c)$ we simply need to prove that $IInc(A, B) = IInc(B^c, A^c)$.

Then, for any $x \in X$ if $A(x) = (a_1, a_2)$ then $A^c(x) = (a_2, a_1)$ and if $B(x) = (b_1, b_2)$ then $B^c(x) = (b_2, b_1)$.

Thus using (6.4), we have:

$$\begin{aligned} IInc(A, B)(x) &= (\min(1, b_1 - a_1 + 1, a_2 - b_2 + 1), \\ &\quad \max(0, a_1 + b_2 - 1)) \qquad (6.5) \\ IInc(B^c, A^c)(x) &= (\min(1, a_2 - b_2 + 1, b_1 - a_1 + 1), \\ &\quad \max(0, b_2 + a_1 - 1)). \qquad (6.6) \end{aligned}$$

Thus from (6.5), and (6.6), we get $IInc(A, B) = IInc(B^c, A^c)$.

Proposition 6.1.6 For any $A, B, C \in IFS(X)$

(a). $B \subseteq C \Longrightarrow m_{IInc}(A, B) \leq m_{IInc}(A, C)$;

(b). $B \subseteq C \Longrightarrow m_{IInc}(C, A) \leq m_{IInc}(B, A)$.

Proof

(a). $B \subseteq C$

implies that $B(x) \leq_{L^*} C(x)$ for $x \in X$

implies that $(b_1, b_2) \leq_{L^*} (c_1, c_2)$

implies that $b_1 \leq c_1$ and $b_2 \geq c_2$.

Now $IInc(A,B)(x) = IInc(A(x),B(x)) = (\min(1, b_1-a_1+1, a_2-b_2+1), \max(0, a_1+b_2-1))$
and $IInc(A,C)(x) = IInc(A(x),C(x)) = (\min(1, c_1-a_1+1, a_2-c_2+1), \max(0, a_1+c_2-1))$.
Thus, if $b_1 \leq c_1$ and $b_2 \geq c_2$ then $a_2-b_2+1 \leq a_2-c_2+1$, $b_1-a_1+1 \leq c_1-a_1+1$ and $a_1+b_2-1 \geq a_1+c_2-1$
implies that $\min(1, b_1-a_1+1, a_2-b_2+1) \leq \min(1, c_1-a_1+1, a_2-c_2+1)$ and $\max(0, a_1+b_2-1) \geq \max(0, a_1+c_2-1)$
implies that $(\min(1, b_1-a_1+1, a_2-b_2+1), \max(0, a_1+b_2-1)) \leq_{L^*} (\min(1, c_1-a_1+1, a_2-c_2+1), \max(0, a_1+c_2-1)$
implies that $IInc(A,B)(x) \leq_{L^*} IInc(A,C)(x)$, for all $x \in X$
and hence $IInc(A,B) \subseteq IInc(A,C)$ and $m_{IInc}(A,B) \leq m_{IInc}(A,C)$.
(b). $B \subseteq C$
implies that $B(x) \leq_{L^*} C(x)$ for $x \in X$
implies that $(b_1, b_2) \leq_{L^*} (c_1, c_2)$
implies that $b_1 \leq c_1$ and $b_2 \geq c_2$.
Now $IInc(C,A)(x) = IInc(C(x),A(x)) = (\min(1, a_1-c_1+1, c_2-a_2+1), \max(0, c_1+a_2-1))$
and $IInc(B,A)(x) = IInc(B(x),A(x)) = (\min(1, b_1-c_1+1, c_2-b_2+1), \max(0, c_1+b_2-1))$.
Thus, if $b_1 \leq c_1$ and $b_2 \geq c_2$ then $a_1-b_1+1 \geq a_1-c_1+1$, $b_2-a_2+1 \geq c_2-a_2+1$and $b_1+a_2-1 \leq c_1+a_2-1$
implies that $\min(1, a_1-c_1+1, c_2-a_2+1) \leq \min(1, b_1-c_1+1, c_2-b_2+1)$ and $\max(0, c_1+b_2-1) \leq \max(0, c_1+a_2-1)$
implies that $(\min(1, a_1-c_1+1, c_2-a_2+1), \max(0, c_1+a_2-1)) \leq_{L^*} (\min(1, b_1-c_1+1, c_2-b_2+1), \max(0, c_1+b_2-1))$
implies that $IInc(C,A)(x) \leq_{L^*} IInc(B,A)(x)$ for all $x \in X$
and hence $IInc(C,A) \subseteq IInc(B,A)$ and $m_{IInc}(C,A) \leq m_{IInc}(B,A)$.

Proposition 6.1.7 For all $A, B \in IFS(X)$,

$S_L(m_{IInc}(A,B), m_{IInc}(B,A)) = 1$ if and only if either $A \subseteq B$ or $B \subseteq A$.

Proof For all $A, B \in IFS(X)$, let $S_L(m_{IInc}(A,B), m_{IInc}(B,A)) = 1$
if and only if either $m_{IInc}(A,B) = 1$ or $m_{IInc}(B,A) = 1$

if and only if either $A \subseteq B$ or $B \subseteq A$ (by Proposition 6.1.4 (a)).

Theorem 6.1.8 If a fuzzy measure of intuitionistic inclusion m possesses the additional property stated in Remark 3.1.3, then it is a fuzzy quasi order [see Definition 2.1.3.8, Chapter 2] on $IFS(X)$.

Proof

(a). For all $A \in IFS(X)$, $m_{IInc}(A, A) = 1$ by Proposition 6.1.4 (a).

(b). Suppose on the contrary that m_{IInc} is not fuzzy transitive so,

$$I_L(T_L(m_{IInc}(A, B), m_{IInc}(B, C)), m_{IInc}(A, C)) = 0 \text{ for some } A, B, C \in IFS(X).$$

It implies that $m_{IInc}(A, B) = 1, m_{IInc}(B, C) = 1$ and $m_{IInc}(A, C) = 0$.

It further implies that $A \subseteq B \subseteq C$ but $m_{IInc}(A, C) = 0$ leading to a contradiction. Hence, m_{IInc} is fuzzy transitive.

6.2 Weighted Average Cardinality based Parametric family of Intuitionistic Fuzzy Inclusion Measure

The results produced in this section are a part of our submitted paper [126].

In [52], a rational family of cardinality based inclusion measures for ordinary sets was proposed. It was parametric in nature and was based on eight parameters. This family was modified by [87], who reduced the number of parameters from eight to four, but still satisfying the basic requirement of inclusion i.e., $A \subseteq B \Rightarrow Inc(A, B) = 1$. Both of these families involved some basic set theoretic operations.

In this section, we have defined a *Weighted Average Cardinality based Parametric family of Intuitionistic Fuzzy Inclusion Measure* for intuitionistic fuzzy sets that involves intuitionistic fuzzy frank t-norms as their basic structural tool.

Likewise to Section 5.2 (Chapter 5), in this section also, we have used $|A| = Card_{0.5}(A) = \frac{1}{2}\sum\limits_{x_i \in X} \mu_A(x_i) + (1 - \nu_A(x_i))$ where $X = \{x_1, x_{2,} ... x_n\}$ as scalar cardinality of an $A \in IFS(X)$, and the expressions for generalized $\cup$ and $\cap$ between $A, B \in IFS(X)$ when the pair $(\overline{T}, \overline{S})$ is modelled by three basic intuitionistic fuzzy frank t-norms and its dual conorms $(\overline{T_j}, \overline{S_j})$ such that $j = M, P, L$ respectively:

$$
\begin{aligned}
A \cup_{\overline{S_j}} B &= \Big\{(x, S_j(\mu_A(x), \mu_B(x)), T_j(\nu_A(x), \nu_B(x))) \mid x \in X\Big\} \\
&= \Big\{(x, \mu_A(x) + \mu_B(x) - T_j(\mu_A(x), \mu_B(x_i)), T_j(\nu_A(x), \nu_B(x))) \mid x \in X\Big\} \\
&\quad \Big\{(x, S_j(\mu_A(x), \mu_B(x)), 1 - S_j(1 - \nu_A(x), 1 - \nu_B(x))) \mid x \in X\Big\}; \\
A \cap_{\overline{T_j}} B &= \Big\{(x, T_j(\mu_A(x), \mu_B(x)), S_j(\nu_A(x), \nu_B(x))) \mid x \in X\Big\} \\
&= \Big\{(x, T_j(\mu_A(x), \mu_B(x)), \nu_A(x) + \nu_B(x) - T_j(\nu_A(x), \nu_B(x))) \mid x \in X\Big\} \\
&= \Big\{(x, T_j(\mu_A(x), \mu_B(x)), 1 - T_j(1 - \nu_A(x), 1 - \nu_B(x))) \mid x \in X\Big\}.
\end{aligned}
$$

Moreover, in Section 5.2, using the above expressions we converted some of the classical set theoretic identities to their intuitionistic fuzzy versions. In this section also, we shall use the same identities given as follows:

(i). $Card_{0.5}(A^c) = n - Card_{0.5}(A) = \frac{n}{2} + \frac{1}{2}\sum\limits_{x_i \in X} \mu_A(x_i) - \nu_A(x_i)$;

(ii). $Card_{0.5}(A \backslash B) = Card_{0.5}(A) - Card_{0.5}(A \cap B)$

$$= \frac{\sum\limits_{x_i \in X} \mu_A(x_i) - \nu_A(x_i) - T_j(\mu_A(x_i), \mu_B(x_i)) + S_j(\nu_A(x_i), \nu_B(x_i))}{2}$$

$$= \frac{\sum\limits_{x_i \in X} \mu_A(x_i) + \nu_B(x_i) - T_j(\mu_A(x_i), \mu_B(x_i)) - T_j(\nu_A(x_i), \nu_B(x_i))}{2}$$

$$= \frac{\sum\limits_{x_i \in X} \mu_A(x_i) + 1 - \nu_A(x_i) - T_j(\mu_A(x_i), \mu_B(x_i)) - T_j(1 - \nu_A(x_i), 1 - \nu_B(x_i))}{2}.$$

(iii). $Card_{0.5}(A \triangle B) = Card_{0.5}(A) + Card_{0.5}(B) - 2Card_{0.5}(A \cap B)$

$$= \frac{\sum\limits_{x_i \in X} \mu_A(x_i) + \mu_B(x_i) - \nu_A(x_i) - \nu_B(x_i) - 2T_j(\mu_A(x_i), \mu_B(x_i)) + 2S_j(\nu_A(x_i), \nu_B(x_i))}{2}$$

$$= \frac{\sum\limits_{x_i \in X} \mu_A(x_i) + \mu_B(x_i) + \nu_A(x_i) + \nu_B(x_i) - 2T_j(\mu_A(x_i), \mu_B(x_i)) - 2T_j(\nu_A(x_i), \nu_B(x_i))}{2}$$

$$= \frac{\sum\limits_{x_i \in X} \mu_A(x_i) + \mu_B(x_i) + (1 - \nu_A(x_i)) + (1 - \nu_B(x_i)) - 2T_j(\mu_A(x_i), \mu_B(x_i)) - 2T_j(1 - \nu_A(x_i), 1 - \nu_B(x_i))}{2}.$$

Definition 6.2.1 A *weighted average cardinality based parametric family of intuitionistic fuzzy inclusion measure* is a binary relation $\Delta : IFS(X) \times IFS(X) \longrightarrow [0, 1]$ defined as:

$$\Delta(A, B) = \frac{x(Card_{0.5}(A \backslash B)) + x'(Card_{0.5}(B \backslash A)) + y(Card_{0.5}(A \cap B)) + z(Card_{0.5}((A \cup B)^c))}{x'(Card_{0.5}(A \triangle B)) + y(Card_{0.5}(A \cap B)) + z(Card_{0.5}((A \cup B)^c))}$$

where x, x', y, z are positive real numbers.

Remark 6.2.2 Since $\Delta(A,B) \in [0,1]$, $0 \le x \le x'$, which $\Delta(A,B)$ defines a class of $[0,1]-$valued inclusion measures on IFSs that are rational expressions in the cardinalities of the sets involved.

In case of ordinary sets, it was proved in [87] that some of the famous inclusion measures known in literature were members of this parametric family. They obtained the expressions for these inclusion measures by setting different combinations of the parameter involved. The results obtained can be represented as follows:

Measure	Expression	x	x'	y	z
$\Delta_1(A,B)$	$\frac{\lvert B\backslash A\rvert}{\lvert A\triangle B\rvert}$	0	1	0	0
$\Delta_2(A,B)$	$\frac{\lvert A^c\rvert}{\lvert (A\cap B)^c\rvert}$	0	1	0	1
$\Delta_3(A,B)$	$\frac{\lvert B\rvert}{\lvert (A\cup B)\rvert}$	0	1	1	0
$\Delta_4(A,B)$	$\frac{\lvert (A\backslash B)^c\rvert}{n}$	0	1	1	1
$\Delta_5(A,B)$	$1-\frac{2\lvert A\backslash B\rvert}{n+\lvert (A\triangle B)\rvert}$	0	2	1	1

Table 6.2.1: Some well-known inclusion measures for ordinary sets

Now for intuitionistic fuzzy sets we have transformed all the inclusion measures for ordinary sets given in Table 6.2.1 as follows:

(i). $\Delta_1^j(A,B) = \frac{Card_{0.5}(B\backslash A)}{Card_{0.5}(A\triangle B)}$

$$= \frac{\sum\limits_{x_i\in X} \mu_B(x_i)-\nu_B(x_i)-T_j(\mu_A(x_i),\mu_B(x_i))+S_j(\nu_A(x_i),\nu_B(x_i))}{\sum\limits_{x_i\in X} \mu_A(x_i)+\mu_B(x_i)-\nu_A(x_i)-\nu_B(x_i)-2T_j(\mu_A(x_i),\mu_B(x_i))+2S_j(\nu_A(x_i),\nu_B(x_i))}$$

$$= \frac{\sum\limits_{x_i\in X} \mu_B(x_i)+\nu_A(x_i)-T_j(\mu_A(x_i),\mu_B(x_i))-T_j(\nu_A(x_i),\nu_B(x_i))}{\sum\limits_{x_i\in X} \mu_A(x_i)+\mu_B(x_i)+\nu_A(x_i)+\nu_B(x_i)-2T_j(\mu_A(x_i),\mu_B(x_i))-2T_j(\nu_A(x_i),\nu_B(x_i))}$$

$$= \frac{\sum\limits_{x_i\in X} \mu_B(x_i)+(1-\nu_B(x_i))-T_j(\mu_A(x_i),\mu_B(x_i))-T_j(1-\nu_A(x_i),1-\nu_B(x_i))}{\sum\limits_{x_i\in X} \mu_A(x_i)+\mu_B(x_i)+(1-\nu_A(x_i))+(1-\nu_B(x_i))-2T_j(\mu_A(x_i),\mu_B(x_i))-2T_j(1-\nu_A(x_i),1-\nu_B(x_i))};$$

(ii). $\Delta_2^j(A,B) = \frac{Card_{0.5}(A^c)}{Card_{0.5}((A\cap B)^c)}$

$$= \frac{2n-\sum\limits_{x_i\in X} \mu_A(x_i)+1-\nu_A(x_i)}{2n-\sum\limits_{x_i\in X} T_j(\mu_A(x_i),\mu_B(x_i))+1-S_j(\nu_A(x_i),\nu_B(x_i))}$$

$$= \frac{2n - \sum_{x_i \in X} \mu_A(x_i) + 1 - \nu_A(x_i)}{2n - \sum_{x_i \in X} T_j(\mu_A(x_i), \mu_B(x_i)) + 1 - \nu_A(x_i) - \nu_B(x_i) + T_j(\nu_A(x_i), \nu_B(x_i))}$$

$$= \frac{2n - \sum_{x_i \in X} \mu_A(x_i) + 1 - \nu_A(x_i)}{2n - \sum_{x_i \in X} T_j(\mu_A(x_i), \mu_B(x_i)) + T_j(1 - \nu_A(x_i), 1 - \nu_B(x_i))};$$

(iii). $\Delta_3^j(A, B) = \frac{Card_{0.5}(B)}{Card_{0.5}(A \cup B)}$

$$= \frac{\sum_{x_i \in X} \mu_B(x_i) + 1 - \nu_B(x_i)}{\sum_{x_i \in X} S_j(\mu_A(x_i), \mu_B(x_i)) + 1 - T_j(\nu_A(x_i), \nu_B(x_i))}$$

$$= \frac{\sum_{x_i \in X} \mu_B(x_i) + 1 - \nu_B(x_i)}{\sum_{x_i \in X} \mu_A(x_i) + \mu_B(x_i) - T_j(\mu_A(x_i), \mu_B(x_i)) + 1 - T_j(\nu_A(x_i), \nu_B(x_i))}$$

$$= \frac{\sum_{x_i \in X} \mu_B(x_i) + 1 - \nu_B(x_i)}{\sum_{x_i \in X} T_j(\mu_A(x_i), \mu_B(x_i)) + T_j(1 - \nu_A(x_i), 1 - \nu_B(x_i))};$$

(iv). $\Delta_4^j(A, B) = \frac{Card_{0.5}((A \backslash B)^c)}{n}$

$$= \frac{2n - \sum_{x_i \in X} \mu_A(x_i) - \nu_A(x_i) - T_j(\mu_A(x_i), \mu_B(x_i)) + S_j(\nu_A(x_i), \nu_B(x_i))}{2n}$$

$$= \frac{2n - \sum_{x_i \in X} \mu_A(x_i) + \nu_B(x_i) - T_j(\mu_A(x_i), \mu_B(x_i)) - T_j(\nu_A(x_i), \nu_B(x_i))}{2n}$$

$$= \frac{2n - \sum_{x_i \in X} \mu_A(x_i) + (1 - \nu_A(x_i)) - T_j(\mu_A(x_i), \mu_B(x_i)) - T_j(1 - \nu_A(x_i), 1 - \nu_B(x_i))}{2n};$$

(v). $\Delta_5^j(A, B) = 1 - \frac{2Card_{0.5}(A \backslash B)}{n + Card_{0.5}(A \triangle B)}$

$$= \frac{2n + \sum_{x_i \in X} \mu_B(x_i) + (1 - \nu_B(x_i)) - \mu_A(x_i) - (1 - \nu_A(x_i))}{2n + \sum_{x_i \in X} \mu_A(x_i) + \mu_B(x_i) - \nu_A(x_i) - \nu_B(x_i) - 2T_j(\mu_A(x_i), \mu_B(x_i)) + 2S_j(\nu_A(x_i), \nu_B(x_i))}$$

$$= \frac{2n + \sum_{x_i \in X} \mu_B(x_i) + (1 - \nu_B(x_i)) - \mu_A(x_i) - (1 - \nu_A(x_i))}{2n + \sum_{x_i \in X} \mu_A(x_i) + \mu_B(x_i) + \nu_A(x_i) + \nu_B(x_i) - 2T_j(\mu_A(x_i), \mu_B(x_i)) - 2T_j(\nu_A(x_i), \nu_B(x_i))}$$

$$= \frac{2n+\sum_{x_i\in X} \mu_B(x_i)+(1-\nu_B(x_i))-\mu_A(x_i)-(1-\nu_A(x_i))}{2n+\sum_{x_i\in X} \mu_A(x_i)+\mu_B(x_i)+(1-\nu_A(x_i))+(1-\nu_B(x_i))-2T_j(\mu_A(x_i),\mu_B(x_i))-2T_j(1-\nu_A(x_i),1-\nu_B(x_i))}$$

where $j = M, L, P$.

Also, throughout this work we have used the notation Δ_k^j, $k = 1, 2, 3, 4, 5$ to specify the above transformed five members of the weighted average cardinality based parametric family of intuitionistic fuzzy inclusion measure (Δ).

Theorem 6.2.3 The members Δ_k^j, $k = 1, 2, 3, 4, 5$ of Δ satisfy the following properties: For all $A, B, C \in IFS(X)$,

(a). $\Delta_k^j(A, B) = 1$ if and only if $A \subseteq B$, holds for $j = M$;

(b). $\Delta_k^j(A, A^c) = 0$, $j = M$ implies either $A = \widetilde{1}_X$ or $\mu_A(x_i) \geq \nu_A(x_i)$, for all $x_i \in X$;

(c). If $B \subseteq C$, then $\Delta_k^j(A, B) \leq \Delta_k^j(A, C)$, $k = 2, 4$, $j = M$ holds for all $A, B, C \in IFS(X)$;

(d). If $B \subseteq C$, then $\Delta_k^j(B, A) \geq \Delta_k^j(C, A), k = 3$, $j = M$ holds for all $A, B, C \in IFS(X)$.

Proof Let $A, B \in IFS(X)$,

(a-i). $\Delta_1^M(A, B) = 1$

if and only if $\frac{Card_{0.5}(B\backslash A)}{Card_{0.5}(A\triangle B)}$

if and only if $Card_{0.5}(B\backslash A) = Card_{0.5}(A \triangle B)$

if and only if $Card_{0.5}(A\backslash B) = 0$

if and only if $Card_{0.5}(A) = Card_{0.5}(A \cap B)$

if and only if $\frac{1}{2}\sum_{x_i\in X} \mu_A(x_i) + 1 - \nu_A(x_i)$

$= \frac{1}{2}\sum_{x_i\in X} \min(\mu_A(x_i), \mu_B(x_i)) + 1 - \max(\nu_A(x_i), \nu_B(x_i))$

if and only if $\mu_A(x_i) = \min(\mu_A(x_i), \mu_B(x_i))$ and $\nu_A(x_i) = \max(\nu_A(x_i), \nu_B(x_i))$ for all $x_i \in X$

if and only if $\mu_A(x_i) \leq \mu_B(x_i)$ and $\nu_A(x_i) \geq \nu_B(x_i)$ for all $x_i \in X$

if and only if $A(x_i) \leq B(x_i)$ for all $x_i \in X$

if and only if $A \subseteq B$.

(a-ii). $\Delta_2^M(A, B) = 1$

if and only if $\frac{Card_{0.5}(A^c)}{Card_{0.5}((A\cap B)^c)} = 1$

if and only if $n - Card_{0.5}(A) = n - Card_{0.5}(A \cap B)$

if and only if and only if $A \subseteq B$.

(a-iii). $\Delta_3^M(A, B) = 1$

if and only if $\frac{Card_{0.5}(B)}{Card_{0.5}(A \cup B)} = 1$

if and only if $Card_{0.5}(B) = Card_{0.5}(A \cup B)$

if and only if $\frac{1}{2} \sum\limits_{x_i \in X} \mu_B(x_i) + 1 - \nu_B(x_i) = \frac{1}{2} \sum\limits_{x_i \in X} \max(\mu_A(x_i), \mu_B(x_i)) + 1 - \min(\nu_A(x_i), \nu_B(x_i))$

if and only if $\mu_B(x_i) = \max(\mu_A(x_i), \mu_B(x_i))$ and $\nu_B(x_i) = \min(\nu_A(x_i), \nu_B(x_i))$ for all $x_i \in X$

if and only if $\mu_A(x_i) \leq \mu_B(x_i)$ and $\nu_A(x_i) \geq \nu_B(x_i)$ for all $x_i \in X$

if and only if $A(x_i) \leq B(x_i)$ for all $x_i \in X$

if and only if $A \subseteq B$.

(a-iv). $\Delta_4^M(A, B) = 1$

if and only if $\frac{Card_{0.5}((A \backslash B)^c)}{n} = 1$

if and only if $n - Card_{0.5}(A \backslash B) = n$

if and only if $Card_{0.5}(A \backslash B) = 0$

if and only if $A(x_i) \leq B(x_i)$ for all $x_i \in X$

if and only if $A \subseteq B$.

(a-v). $\Delta_5^M(A, B) = 1$

if and only if $1 - \frac{2Card_{0.5}(A \backslash B)}{n + Card_{0.5}(A \triangle B)} = 1$

if and only if $Card_{0.5}(A \backslash B) = 0$

if and only if $A \subseteq B$.

(b-i). $\Delta_1^M(A, A^c) = 0$

implies that $\frac{Card_{0.5}(A^c \backslash A)}{Card_{0.5}(A \triangle A^c)} = 0$

implies that $Card_{0.5}(A^c \backslash A) = 0$

implies that $Card_{0.5}(A^c) - Card_{0.5}(A \cap A^c) = 0$

implies that $\frac{1}{2} \sum\limits_{x_i \in X} \nu_A(x_i) + 1 - \mu_A(x_i)$

$= \frac{1}{2} \sum\limits_{x_i \in X} \min(\nu_A(x_i), \mu_A(x_i)) + 1 - \max(\nu_A(x_i), \mu_A(x_i))$

implies that $\nu_A(x_i) = \min(\nu_A(x_i), \mu_A(x_i))$ and $\mu_A(x_i) = \max(\mu_A(x_i), \nu_A(x_i))$ for all $x_i \in X$

implies that $\mu_A(x_i) \geq \nu_A(x_i)$ for all $x_i \in X$.

(b-ii). $\Delta_2^M(A, A^c) = 0$

implies that $\frac{Card_{0.5}(A^c)}{Card_{0.5}((A \cap A^c)^c)} = 0$

implies that $n - Card_{0.5}(A) = 0$

implies that $Card_{0.5}(A) = n$

implies that $\frac{1}{2} \sum_{x_i \in X} \mu_A(x_i) + 1 - \nu_A(x_i) = n$

implies that $\sum_{x_i \in X} \mu_A(x_i) + 1 - \nu_A(x_i) = 2n$

implies that $\mu_A(x_i) + 1 - \nu_A(x_i) = 2$ for all $x_i \in X$

implies that $\mu_A(x_i) = 1 + \nu_A(x_i)$ for all $x_i \in X$

implies that $\mu_A(x_i) = 1$ and $\nu_A(x_i) = 0$ for all $x_i \in X$

implies that $A = \widetilde{1}_X$.

(b-iii). $\Delta_3^M(A, A^c) = 0$

implies that $\frac{Card_{0.5}(A^c)}{Card_{0.5}(A \cup A^c)} = 0$

implies that $n - Card_{0.5}(A) = 0$

implies that $Card_{0.5}(A) = n$

implies that $\frac{1}{2} \sum_{x_i \in X} \mu_A(x_i) + 1 - \nu_A(x_i) = n$

implies that $\sum_{x_i \in X} \mu_A(x_i) + 1 - \nu_A(x_i) = 2n$

implies that $\mu_A(x_i) + 1 - \nu_A(x_i) = 2$ for all $x_i \in X$

implies that $\mu_A(x_i) = 1 + \nu_A(x_i)$ for all $x_i \in X$

implies that $\mu_A(x_i) = 1$ and $\nu_A(x_i) = 0$ for all $x_i \in X$

implies that $A = \widetilde{1}_X$.

(b-iv). $\Delta_4^M(A, A^c) = 0$

implies that $\frac{Card_{0.5}((A \backslash A^c)^c)}{n} = 0$

implies that $Card_{0.5}(A \backslash A^c) = n$

implies that $Card_{0.5}(A) - Card_{0.5}(A \cap A^c) = n$

implies that $\sum_{x_i \in X} \mu_A(x_i) + 1 - \nu_A(x_i) -$
$\sum_{x_i \in X} \min(\nu_A(x_i), \mu_A(x_i)) + 1 - \max(\nu_A(x_i), \mu_A(x_i)) = 2n$

implies that $\mu_A(x_i) + 1 - \nu_A(x_i) - \min(\nu_A(x_i), \mu_A(x_i)) - 1 + \max(\nu_A(x_i), \mu_A(x_i)) = 2$ for all $x_i \in X$

implies that $\mu_A(x_i) - \nu_A(x_i) = 2 + \min(\nu_A(x_i), \mu_A(x_i)) - \max(\nu_A(x_i), \mu_A(x_i))$ for all $x_i \in X$

implies that $\max(\nu_A(x_i), \mu_A(x_i)) = \mu_A(x_i)$ and $\min(\nu_A(x_i), \mu_A(x_i)) = \nu_A(x_i)$ for all $x_i \in X$

implies that $\mu_A(x_i) = 1 + \nu_A(x_i)$ for all $x_i \in X$

implies that $\mu_A(x_i) = 1$ and $\nu_A(x_i) = 0$ for all $x_i \in X$

implies that $A = \widetilde{1}_X$.

(b-v). $\Delta_5^M(A, A^c) = 0$

implies that $1 - \frac{2Card_{0.5}(A\backslash A^c)}{n+Card_{0.5}(A\triangle A^c)} = 0$

implies that $n + Card_{0.5}(A^c\backslash A) - Card_{0.5}(A\backslash A^c) = 0$

implies that $n + Card_{0.5}(A^c) = Card_{0.5}(A)$

implies that $\sum\limits_{x_i \in X} \nu_A(x_i) + 1 - \mu_A(x_i) = 0$ for all $x \in X$

implies that $\mu_A(x_i) = 1$ and $\nu_A(x_i) = 0$ for all $x_i \in X$

implies that $A = \widetilde{1}_X$.

(c-i). $B \subseteq C$

implies that $\mu_B(x_i) \leq \mu_C(x_i)$ and $\nu_B(x_i) \geq \nu_C(x_i)$ for all $x_i \in X$

implies that $\min(\mu_A(x_i), \mu_B(x_i)) \leq \min(\mu_A(x_i), \mu_C(x_i))$

and $\max(\nu_A(x_i), \nu_B(x_i)) \geq \max(\nu_A(x_i), \nu_C(x_i))$ for all $x_i \in X$.

Case 1: If $\min(\mu_A(x_i), \mu_B(x_i)) = \mu_B(x_i)$, then $\mu_B(x_i) \leq \mu_A(x_i)$.

Now if $\min(\mu_A(x_i), \mu_C(x_i)) = \mu_A(x_i)$ or $\min(\mu_A(x_i), \mu_C(x_i)) = \mu_C(x_i)$

then $\min(\mu_A(x_i), \mu_B(x_i)) \leq \min(\mu_A(x_i), \mu_C(x_i))$.

Case 2: If $\min(\mu_A(x_i), \mu_B(x_i)) = \mu_A(x_i)$, then $\mu_A(x_i) \leq \mu_B(x_i)$.

Also if $\min(\mu_A(x_i), \mu_C(x_i)) = \mu_A(x_i)$ as $\mu_B(x_i) \leq \mu_C(x_i)$,

then $\min(\mu_A(x_i), \mu_B(x_i)) = \min(\mu_A(x_i), \mu_C(x_i))$;

implies that $A \cap B \subseteq A \cap C$

implies that $Card_{0.5}(A \cap B) \leq Card_{0.5}(A \cap C)$

implies that $n - Card_{0.5}(A \cap B) \geq n - Card_{0.5}(A \cap C)$

implies that $\frac{n-Card_{0.5}(A)}{n-Card_{0.5}(A\cap B)} \leq \frac{n-Card_{0.5}(A)}{n-Card_{0.5}(A\cap C)}$

implies that $\frac{Card_{0.5}(A^c)}{Card_{0.5}((A\cap B)^c)} \leq \frac{Card_{0.5}(A^c)}{Card_{0.5}((A\cap C)^c)}$

implies that $\Delta_2^M(A, B) \leq \Delta_2^M(A, C)$.

(c-ii). $B \subseteq C$

implies that $\mu_B(x_i) \leq \mu_C(x_i)$ and $\nu_B(x_i) \geq \nu_C(x_i)$ for all $x_i \in X$

implies that $A \cap B \subseteq A \cap C$

implies that $Card_{0.5}(A \cap B) \leq Card_{0.5}(A \cap C)$

implies that $Card_{0.5}(A) - Card_{0.5}(A \cap B) \geq Card_{0.5}(A) - Card_{0.5}(A \cap C)$
implies that $Card_{0.5}(A \backslash B) \geq Card_{0.5}(A \backslash C)$
implies that $\frac{Card_{0.5}((A \backslash B)^c)}{n} \leq \frac{Card_{0.5}((A \backslash C)^c)}{n}$
implies that $\Delta_4^M(A, B) \leq \Delta_4^M(A, C)$.
(d). Let $B \subseteq C$
implies that $\mu_B(x_i) \leq \mu_C(x_i)$ and $\nu_B(x_i) \geq \nu_C(x_i)$ for all $x_i \in X$
implies that $\max(\mu_A(x_i), \mu_B(x_i)) \leq \max(\mu_A(x_i), \mu_C(x_i))$
and $\min(\nu_A(x_i), \nu_B(x_i)) \geq \min(\nu_A(x_i), \nu_C(x_i))$ for all $x_i \in X$.
Case 1: if $\max(\mu_A(x_i), \mu_B(x_i)) = \mu_A(x_i)$, then $\mu_B(x_i) \leq \mu_A(x_i)$.
If $\max(\mu_A(x_i), \mu_C(x_i)) = \mu_A(x_i)$ as $\mu_B(x_i) \leq \mu_C(x_i)$,
then $\max(\mu_A(x_i), \mu_B(x_i)) \leq \max(\mu_A(x_i), \mu_C(x_i))$.
Case 2: If $\max(\mu_A(x_i), \mu_B(x_i)) = \mu_B(x_i)$, then $\mu_A(x_i) \leq \mu_B(x_i)$.
Also if $\max(\mu_A(x_i), \mu_C(x_i)) = \mu_A(x_i)$ as $\mu_B(x_i) \leq \mu_C(x_i)$
implies $\max(\mu_A(x_i), \mu_B(x_i)) = \max(\mu_A(x_i), \mu_C(x_i))$;
implies that $B \cup A \subseteq C \cup A$
implies that $Card_{0.5}(B \cup A) \leq Card_{0.5}(C \cup A)$
implies that $\frac{Card_{0.5}(A)}{Card_{0.5}(B \cup A)} \geq \frac{Card_{0.5}(A)}{Card_{0.5}(C \cup A)}$
implies that $\Delta_3^M(B, A) \geq \Delta_3^M(C, A)$.

Theorem 6.2.4 The members Δ_k^j, $k = 1, 2, 3, 4, 5$ and $j = M$ of Δ define a fuzzy quasi order on $IFS(X)$.

Proof

(a). $\Delta_k^j(A, A) = 1$ for all $k = 1, 2, 3, 4, 5$ and $j = M$ holds by Theorem 6.2.3 part (a).

(b). Next we need to show that Δ_k^j for all $k = 1, 2, 3, 4, 5$ and $j = M$ is fuzzy transitive i.e.,

$I_L(T_L(\Delta_k^M(A, B), \Delta_k^M(B, C)), \Delta_k^M(A, C)) > 0$ holds for all $A, B, C \in IFS(X)$.

On the contrary, let us assume that Δ_k^j is not fuzzy transitive thus,

$$I_L(T_L(\Delta_k^M(A, B), \Delta_k^M(B, C)), \Delta_k^M(A, C)) = 0 \text{ for some } A, B, C \in IFS(X).$$

It implies that $\Delta_k^M(A, B) = 1, \Delta_k^M(B, C) = 1$ and $\Delta_k^M(A, C) = 0$.
It further implies that $A \subseteq B \subseteq C$ but $\Delta_k^M(A, C) = 0$ a contradiction.
So Δ_k^M, $k = 1, 2, 3, 4, 5$ is fuzzy transitive.

Chapter 7

Applications

In human cognition, measures are the basic concepts. They play an essential role in taxonomy, multi-criteria decision making (MCDM), recognition, case-based reasoning, diagnosis, artificial intelligence and many other fields. Due to their wide acceptability and mathematical rigor, the literature on these areas have been developed substantially. In this chapter we have demonstrated the real life applicability of the classes of different intuitionistic fuzzy inclusion, similarity and cardinality measures proposed in this thesis.

The results produced in the first two sections of this chapter are a part of our published paper [3] and the ones introduced in third and fourth sections are contained in our submitted papers [124] and [4] respectively.

7.1 The Intuitionistic Fuzzy Multi-Criteria Decision Making based on Inclusion Degree (TOPIIS, Technique to Order Preference by Inclusion of Ideal Solution)

In this section, we shall introduce an intuitionistic fuzzy inclusion based technique for solving MCDM problems similar to the TOPSIS (The Technique for Order Preferences by Similarity of Ideal Solution) method. The technique of TOPSIS was introduced in 1981, by Huang Qinglai for the sorting of the optimal alternative closest to the ideal solution.

The idea behind this technique can be summarized as follows: Taking into consideration the weighted decision matrix, we initially search for the optimal scheme and the worst one in the decision matrix. Next, we obtain the distances between the object of evaluation and the optimal and the worse scheme respectively. Then, we compute the degree of relative proximity of the evaluated object to the optimal scheme, in order to judge the feasibility of each evaluation object.

Working on the similar lines, and instead of using distances as the basic tool of comparisons of alternative schemes from ideal and non ideal solution, we wish to use the concept of inclusion degree of intuitionstic fuzzy sets to obtain the best scheme. Our technique will also create an *Ideal solution* and a *Negative ideal solu-*

tion that can be regarded as the most excellent or worst solutions not existing in the given set of alternative schemes that are to be judged. Then each alternative scheme under consideration is judged according to its degree of inclusion in the non ideal solution and the degree of inclusion of ideal solution in it. The most optimal solution is the highest ranked alternative scheme which simultaneously contains the ideal solution maximally and is contained in the negative ideal solution minimally. We shall call this technique TOPIIS(Technique to Order Preference by Inclusion of Ideal Solution).

Let us consider a scheme set $A = \{A_1, A_2, A_3, ..., A_m\}$ which is consisting of m evaluated objects or alternatives. The indexes (criteria) of evaluation for the evaluated objects can be given by index set $U = \{U_1, U_2, U_3, ..., U_n\}$. An evaluation index system is a logical and comprehensive system of a sequence of mutually connected and restricted indexes, which is designed to elaborate the features and regularity of multilevel and multifactor complex phenomena . Moreover, we shall define the weight of an evaluation index U_t as w_t, such that $w_t \geq 0$ and $\sum\limits_{t=1}^{n} w_t = 1$. The procedure can be outlined in the following steps:

(i). Establish an intuitionistic fuzzy relation between A and U represented by the following evaluation matrix:

$$R = \begin{bmatrix} (U_1, \mu_{11}, \nu_{11}) & . & . & (U_s, \mu_{1s}, \nu_{1s}) & . & . & (U_n, \mu_{1n}, \nu_{1n}) \\ . & & & . & & & . \\ . & & & . & & & . \\ . & & & . & & & . \\ . & & & . & & & . \\ (U_1, \mu_{m1}, \nu_{m1}) & . & . & (U_s, \mu_{ms}, \nu_{ms}) & . & . & (U_n, \mu_{mn}, \nu_{mn}) \end{bmatrix}$$

The intuitionistic fuzzy relation matrix R clearly elaborates that each scheme A_r with respect to the evaluation index U_t can be repressented by the intuitionistic fuzzy set $A_{rt} = (U_t, \mu_{rt}, \nu_{rt})$ where μ_{rt} and ν_{rt} indicate the degree of importance and the degree of unimportance of $U_t \in U$ to the scheme $A_r \in A$ respectively, satisfying $0 \leq \mu_{rt} + \nu_{rt} \leq 1$ where $r = 1, 2, ...m$ and $t = 1, 2, ...n$. The value of evaluation of the scheme A_r w.r.t. n indexes can be denoted by:

$A_r = \{(U_1, \mu_{r1}, \nu_{r1}), (U_2, \mu_{r2}, \nu_{r2}), ..., (U_n, \mu_{rn}, \nu_{rn})\}$.

(ii). Employ the intuitionistic fuzzy entropy method to calculate the weight of an evaluation index:

For an intuitionistic fuzzy set $A = \Big\{(x, \mu_A(x), \nu_A(x)) \mid x \in X\Big\}$, the degree hesitancy of x in A is defined as: $\pi_A(x) = 1 - \mu_A(x) - \nu_A(x)$. Also $\lambda_A(x) = 1 - \Big|\mu_A(x) - \nu_A(x)\Big|$ is taken to be the fuzzy degree of x in the set A. Then we define for an evaluation index U_t its intuitionistic fuzzy entropy as:

$$E(U_t) = \frac{1}{m}\sum_{r=1}^{m}\sqrt{\frac{\pi_{rt}^2 + \lambda_{rt}^2}{2}}.$$

The greater value of the intuitionistic fuzzy entropy $E(U_t)$ means the uncertainty level of index U_t is higher which results in a smaller weight for index U_t. Thus the weight of index U_t is given as:

$$w_t = \frac{1 - E(U_t)}{\sum\limits_{t=1}^{n}(1 - E(U_t))}.$$

The above formula provides the weight vector $w = (w_1, w_2..., w_n)^T$ of the evaluation index set.

(iii). Extract the intuitionistic fuzzy set positive ideal solution A^+ (the best scheme) and the intuitionistic fuzzy negative ideal solution (worst scheme) A^- given as:

$$\begin{aligned} A^+ &= \{A_t^+\} = \big\{(U_t, \mu_t^+, \nu_t^+)\big\} = \left\{(U_t, \bigvee_{r=1}^{m} \mu_{rt}, \bigwedge_{r=1}^{m} \nu_{rt})\right\} \\ A^- &= \{A_t^-\} = \big\{(U_t, \mu_t^-, \nu_t^-)\big\} = \left\{(U_t, \bigwedge_{r=1}^{m} \mu_{rt}, \bigvee_{r=1}^{m} \nu_{rt})\right\} \end{aligned}$$

where the pair$\left(\bigwedge, \bigvee\right)$ stands for fuzzy conjunction and disjunction respectively which can be modeled by t-norms and their dual conorms respectively.

(iv). Calculate the weighted degree $D(A^+, A_r)$ of inclusion of positive ideal solution A^+ in alternative A_r and the weighted degree $d(A_r, A^-)$ of inclusion of the alternative A_r in negative ideal solution by formulas given below:

$$\begin{aligned} D(A^+, A_r) &= \bigvee_{t=1}^{n} (w_t m_{IInc}(A_t^+, A_{rt})) \\ d(A_r, A^-) &= \bigwedge_{t=1}^{n} (w_t m_{IInc}(A_{rt}, A_t^-)) \end{aligned}$$

where we have defined $m_{IInc}(A, B)(x) = m(IInc(A, B)(x))$ and $IInc(A, B)(x) = (\min(1, b_1 - a_1 + 1, a_2 - b_2 + 1), \max(0, a_1 + b_2 - 1))$ for all $A, B \in IFS(X)$ such that $A(x) = (a_1, a_2), B(x) = (b_1, b_2) \in L^*$. Moreover, as mentioned in step (3) $\bigwedge$ stands for fuzzy conjunction and $\bigvee$ stands for fuzzy disjunction which can be modeled by t-norms and their dual t-conorms.

(v). Define a ranking index of alternative scheme A_r as:

$$\gamma_r = \frac{D(A^+, A_r)}{D(A^+, A_r) + d(A_r, A^-)}$$

where $r = 1, 2, 3..., m$. The best scheme among A_r, $r = 1, 2, 3...m$ is chosen on the basis of its ranking index value γ_r. The highest ranking value γ_r assigns the scheme A_r as the most optimal solution of the multicriteria decision making problem since the degree of inclusion of positive ideal solution in the respective scheme will be maximum and the degree of inclusion of this scheme in negative ideal solution will be minimal. However, in a situation when two schemes have the same degree of inclusion with respect to ideal solution, the negative ideal solution plays its part to discriminate which alternative is better than other. In such a situation, the scheme with smaller degree of inclusion with respect to negative ideal solution is the suitable choice. Similarly, if we may come across with the situation when two or more schemes have equal $d(A_r, A^-)$, then the scheme with greater $D(A^+, A_r)$ will be the better choice.

7.2 The Application of TOPIIS to Employee's Performance Appraisal

The rapid globalization and developing economy has mounted the competition among the enterprises. Performance appraisal is a core aspect of Human Resource (HR) management. The effective performance appraisal of an employee in an enterprise not only generates a better internal management with less flaws but also improves its market competitiveness by retaining attracting the best employee. The HR appraisal is a comprehensive evaluation with multiple levels, dimensions and factors implicated in it and with a lot of fuzziness involved in quantifying the performance indexes of the employees. Thus, it has gained and sustained the interest of researchers of management theory for the past few decades [76, 123, 178]. This subsection is reserved for a rework on the case study originally made in [167]. In [167], the authors designed a HR appraisal index system and utilized the TOPSIS technique to rank the best employee based on the evaluation indexes that they have developed. This subsection will elaborate and compare the results obtained from the new TOPIIS method with the results appearing in [167]. The new results were supportive of the study done in [167]. However, our new technique uses general fuzzy conjunction and disjunction that could be modeled by different t-norms and their dual conorms. Moreover, a variety of fuzzy measures of intuitionistic fuzzy sets can provide multiple choices for the degree of inclusion of intuitionistic fuzzy sets used in the technique. This provides a more broader and flexible framework to the decision maker who has to deal with any type of data in a complex multiple attribute decision making environment of actual economic management and similar fields.

But, firstly to build a background of the problem we present a short review of the index system of employee appraisal developed by [167] in the form Table 7.1.1;

The index system	The first class index	The second class index
of employee performance	Talent quality U_1	Enterprise U_{11}
		Responsibility U_{12}
		Integrity U_{13}
		Professional responsibility U_{14}
	Personal ability U_2	Professional skill U_{21}
		Coordination ability U_{22}
		Learning ability U_{23}
		Decision making ability U_{24}
	Work performance U_3	The quantity of work U_{31}
		The quality of work U_{32}
		The efficiency of work U_{33}
		The effect of work U_{34}
		The innovation of work U_{35}
	Work attitude U_4	Active learning and training U_{41}
		The enthusiasm of work U_{42}
		Execution U_{43}
		Attendance rate U_{44}
		Discipline U_{45}

Table 7.2.1: The index system of employee performance

Clearly, from Table 7.2.1 we see that authors have proposed an employee evaluation index system based on the four first class of indexes namely, talent quality U_1, personal ability U_2, work performance U_3, and work attitude U_4 . Each of these first class of indexes is further subclassified into the second class sub indexes. For a detailed study about the selection and this particular classifications of the first class and the second class indexes we refer the reader to [167]. Next, we present the case study as follows:

Let us assume that an enterprise selects five of its employees denoted by the scheme set $A = \{A_1, A_2, A_3, A_4, A_5\}$ for their performance evaluation. Taking into consideration the characteristics of performance appraisal, we choose only the first class indexes, which are: Talent quality U_1, Personal ability U_2, Work performance U_3, and Work attitude U_4. These can be combined into a set of attributes $U = \{U_1, U_2, U_3, U_4\}$ to judge the employees work performance. Now according to the above mentioned technique we shall first construct the intuitionistic fuzzy evaluation matrix of schemes, then we determine the weight of every index by entropy weight method, we define the optimal and worst schemes and then for each alternative scheme we calculate the degree of inclusion of optimal scheme in the given alternative scheme and the degree of inclusion of the same alternative scheme in the worst scheme. Finally, we calculate the ranking index for each alternative scheme. In particular:

(i). Using the present knowledge, research and experience of HR- managers we can construct the intuitionistic fuzzy evaluation matrix as follows:

$$R = \begin{bmatrix} (U_1, 0.6, 0.3) & (U_2, 0.4, 0.3) & (U_3, 0.3, 0.5) & (U_4, 0.6, 0.3) \\ (U_1, 0.5, 0.3) & (U_2, 0.5, 0.3) & (U_3, 0.4, 0.1) & (U_4, 0.4, 0.1) \\ (U_1, 0.8, 0.1) & (U_2, 0.3, 0.2) & (U_3, 0.6, 0.1) & (U_4, 0.5, 0.4) \\ (U_1, 0.7, 0.2) & (U_2, 0.1, 0.6) & (U_3, 0.5, 0.3) & (U_4, 0.2, 0.5) \\ (U_1, 0.6, 0.2) & (U_2, 0.5, 0.1) & (U_3, 0.3, 0.4) & (U_4, 0.6, 0.2) \end{bmatrix}$$

(ii). Determine the weights of each evaluation index by intuitionistic fuzzy entropy method i.e.,

$$\begin{aligned} E(U_1) &= \frac{1}{m}\sum_{r=1}^{m}\sqrt{\frac{\pi_{r1}^2 + \lambda_{r1}^2}{2}} = \frac{1}{5}\Big(\sqrt{\frac{0.1^2 + 0.7^2}{2}} + \sqrt{\frac{0.2^2 + 0.8^2}{2}} \\ &\quad + \sqrt{\frac{0.1^2 + 0.3^2}{2}} + \sqrt{\frac{0.1^2 + 0.5^2}{2}} + \sqrt{\frac{0.2^2 + 0.6^2}{2}}\Big) \\ &= 0.423. \end{aligned}$$

Similarly, we obtain $E(U_2) = 0.581, E(U_3) = 0.571, E(U_4) = 0.547$. From the given values the intuitionistic fuzzy entropy $E(U_1) = 0.423$ is the smallest of all which implies that the weight of U_1 should be biggest. Then since the entropy $E(U_4) = 0.547$ which is slightly bigger than $E(U_1)$, hence the weight of U_2 should be smaller than U_1. The entropies $E(U_2) = 0.581$ and $E(U_3) = 0.571$ are very close to each other and yet they have a greater value than $E(U_4)$, thus the weights of U_2 and U_3 should be big and closer to each other, but they should not exceed in value the weights of U_4 and U_1 respectively. Next, we use these entropies to calculate the weight vector $w = (w_1, w_2, w_3, w_4)^T$ of the evaluation indexes where

$$\begin{aligned} w_1 &= \frac{1 - E(U_1)}{\sum_{t=1}^{5}(1 - E(U_t))} = \frac{0.577}{1.878} = 0.307. \\ w_2 &= \frac{1 - E(U_2)}{\sum_{t=1}^{5}(1 - E(U_t))} = \frac{0.419}{1.878} = 0.223. \\ w_3 &= \frac{1 - E(U_3)}{\sum_{t=1}^{5}(1 - E(U_t))} = \frac{0.429}{1.878} = 0.229. \\ w_4 &= \frac{1 - E(U_4)}{\sum_{t=1}^{5}(1 - E(U_t))} = \frac{0.453}{1.878} = 0.241. \end{aligned}$$

According to the above index system of any enterprise, talent quality of the employee (U_1) has the greatest weight $w_1 = 0.307$ that makes it the prior index of evaluation in different departments. Moreover, the employee's cognition of the enterprise and his motivation to work is shown by the index work attitude (U_4), and an enterprise designates a larger value during any performance appraisal. However, the entropies of personal ability (U_2) and work performance (U_3) are large and the difference among them is small so their weight value in the performance appraisal are closer and smaller than the other two indexes.

(iii). Next, we find the intuitionistic fuzzy positive ideal solution A^+ and the intuitionistic fuzzy negative ideal solution A^- by formulas:

$$\begin{aligned} A^+ &= \{A_t^+\} = \{(U_t, \mu_t^+, \nu_t^+)\} = \left\{(U_t, \bigvee_{r=1}^{m} \mu_{rt}, \bigwedge_{r=1}^{m} \nu_{rt})\right\}; \\ A^- &= \{A_t^-\} = \{(U_t, \mu_t^-, \nu_t^-)\} = \left\{(U_t, \bigwedge_{r=1}^{m} \mu_{rt}, \bigvee_{r=1}^{m} \nu_{rt})\right\}. \end{aligned}$$

Now, among different choices for t-norm and their dual t-conorm we fix the pair$\left(\bigwedge, \bigvee\right)$ by (T_M, S_M) where $T_M(x, y) = \min(x, y)$ and $S_M(x, y) = \max(x, y)$ for all $x, y \in [0, 1]$. The main motivation for this particular choice is the fact that we wish to compare our results with the result presented in [167].

$$\begin{aligned} A^+ &= \{A_1^+, A_2^+, A_3^+, A_4^+\} = \{(U_1, 0.8, 0.1), (U_2, 0.5, 0.1), (U_3, 0.6, 0.1), (U_4, 0.6, 0.1)\}\,; \\ A^- &= \{A_1^-, A_2^-, A_3^-, A_4^-\} = \{(U_1, 0.5, 0.3), (U_2, 0.1, 0.6), (U_3, 0.3, 0.5), (U_4, 0.2, 0.5)\}\,. \end{aligned}$$

(iv). Next, we calculate the weighted degree $D(A^+, A_r)$ of inclusion of positive ideal solution A^+ in alternative A_r and the weighted degree $d(A_r, A^-)$ of inclusion of the alternative A_r in negative ideal solution by formulas given below:

$$\begin{aligned} D(A^+, A_r) &= \bigvee_{t=1}^{n} (w_t m_{IInc}(A_t^+, A_{rt})); \\ d(A_r, A^-) &= \bigwedge_{t=1}^{n} (w_t m_{IInc}(A_{rt}, A_t^-)). \end{aligned}$$

Now, $m_{IInc}(A_t^+, A_{rt}) = m(IInc(A_t^+, A_{rt})(U_t))$ and $IInc(A_t^+, A_{rt})(U_t) = (\min(1, \nu_t^+ - \mu_t^+ + 1, \mu_{rt} - \nu_{rt} + 1), \max(0, \mu_t^+ + \nu_{rt} - 1))$.
Similarly, $m_{IInc}(A_{rt}, A_t^-) = m(IInc(A_{rt}, A_t^-)(U_t))$ and $IInc(A_{rt}, A_t^-)(U_t) = (\min(1, \nu_{rt} - \mu_{rt} + 1, \mu_t^- - \nu_t^- + 1), \max(0, \mu_{rt} + \nu_t^- - 1))$. Moreover, we may choose any fuzzy measure m introduced in [Section 3.1, Chapter 3]. However, for now we restrict our calculations to the choice $m_1(A) = \frac{1}{2}(\inf_{x\in X} \mu_A(x) + (1 - \sup_{x\in X} \nu_A(x)))$ where $A \in IFS(X)$ which leads to the expressions:

$$\begin{aligned} D(A^+, A_r) &= \bigvee_{t=1}^{n} [\frac{w_t}{2}(\min(1, \nu_t^+ - \mu_t^+ + 1, \mu_{rt} - \nu_{rt} + 1) \\ &\quad +1 - \max(0, \mu_t^+ + \nu_{rt} - 1))]. \\ d(A_r, A^-) &= \bigwedge_{t=1}^{n} [\frac{w_t}{2}(\min(1, \nu_{rt} - \mu_{rt} + 1, \mu_t^- - \nu_t^- + 1) \\ &\quad +1 - \max(0, \mu_{rt} + \nu_t^- - 1))]. \end{aligned}$$

Here, we have multiple choices for the pair of fuzzy $\left(\bigwedge, \bigvee\right)$ due to the existence of different pairs of t-norm and their dual t-conorm. In this case study, we shall discuss three different scenarios for the degrees $D(A^+, A_r)$ and $d(A_r, A^-)$ when we model $\left(\bigwedge, \bigvee\right)$ by three basic t-norms namely, Min (T_M), Product (T_P) and Lukasiewicz (T_L) and their dual t-conorms. The results of these calculations will be exhibited by three different tables showing the degrees of inclusions $D(A^+, A_r)$

and $d(A_r, A^-)$ for each employee A_r where $r = 1, 2, 3, 4, 5$.

Table 7.2.2 : The Inclusion degree of employee w.r.t. (T_M, S_M) :

	A_1	A_2	A_3	A_4	A_5
$D(A^+, A_r)$	0.2609	0.2456	0.8330	0.6112	0.2763
$d(A_r, A^-)$	0.1895	0.1672	0.1717	0.223	0.1561

Table 7.2.3 : The Inclusion degree of employee w.r.t. (T_P, S_P) :

	A_1	A_2	A_3	A_4	A_5
$D(A^+, A_r)$	0.6168	0.6198	0.8330	0.6112	0.6508
$d(A_r, A^-)$	0.0024	0.0018	0.0015	0.0033	0.0017

Table 7.2.4 : The Inclusion degree of employee w.r.t. (T_L, S_L):

	A_1	A_2	A_3	A_4	A_5
$D(A^+, A_r)$	0.8558	0.8581	0.9415	0.8346	0.9229
$d(A_r, A^-)$	0	0	0	0	0

(v). Next, corresponding to each Table in step (4) we calculate ranking index of alternative scheme A_r by formula:

$$\gamma_r = \frac{D(A^+, A_r)}{D(A^+, A_r) + d(A_r, A^-)}.$$

Now, using Table 7.2.2 we get the following set of ranking values:

$$\gamma_1 = 0.579, \gamma_2 = 0.595, \gamma_3 = 0.829, \gamma_4 = 0.733, \gamma_5 = 0.639;$$

and the ranking of employees as $A_3 \succ A_4 \succ A_5 \succ A_2 \succ A_1$.
Using Table 7.2.3, we get the ranking values:

$$\gamma_1 = 0.9961, \gamma_2 = 0.9971, \gamma_3 = 0.9982, \gamma_4 = 0.9946, \gamma_5 = 0.9974;$$

and the ranking of employees as $A_3 \succ A_5 \succ A_2 \succ A_1 \succ A_4$.

The data from Table 7.2.4 reveals that we are in a situation in which there is no discrimination existing among all the alternatives A_r on the basis of their inclusions in the negative ideal solution. In such a case, we will rank the employees on the basis of the degree of inclusion of positive ideal solution in them. Thus, from the given data we get the ranking of employees as: $A_3 \succ A_5 \succ A_2 \succ A_1 \succ A_4$, which is the same as obtained from Table 7.2.3.

The above rankings reveal that the performance of employee A_3 is highest among his competitors, conforming to the calculation results of his inclusions. That is to say that the data from the three tables show that the employee A_3 has maximum degree of inclusion of positive ideal solution in it. Not only this, its inclusion degrees in negative ideal solution are negligible and even minimum as appearing in Table 7.2.3 and Table 7.2.4. The enterprise should prioritize the employee A_3 when giving any promotions and rewards. Now, as far as the identification of the weakest performer is concerned, we see that there is a tie between A_1 and A_4. Although, A_4 has been ranked the weakest twice yet we confirm his position in the given chain of performance appraisal by deeply analyzing the given data. This analysis will also result in a final ranking chain. Now, in both the chains we see that performance of the employee A_2 has been better than A_1. Thus, A_2 must be ranked higher than A_1 in any ranking chain. Moreover, the employee A_5 has been a good performer in both analysis and it is acceptable if he or she is assigned a second or third ranking position in the given situation. Now, as far as the decision about the positions of A_1 and A_4 is concerned, we see that in both Tables (7.2.3,7.2.4), the candidate A_4 has the same degree of inclusion of positive ideal in him or her, while in both situations he or she has a maximum containment in negative ideal solution than any of the five candidates. Thus, beside his or her abilities, there are a few hidden factors which do not allow him or her to position better than A_1 who is a moderate to weak performer. Finally, we conclude that the ranking chain $A_3 \succ A_5 \succ A_2 \succ A_1 \succ A_4$ is more realistic and thus the candidate A_4 is judged as the lowest performer among all the five employees. The management should work as a team with the concerned employee for the improvement of his/her self quality as well as of the work level, so as to contribute towards the improvement of his or her performance as a whole. The given ranking also matches with the result produced in [167].

7.3 The Applications of Weighted Average Cardinality Measure to Ranking of Intuitionistic Fuzzy Sets

In this section, we shall employ the new proposed weighted average cardinality measure $Card_\theta(A)$ for an intuitionistic fuzzy set A to define a new definition of relative sigma count for intuitionistic fuzzy sets and then based on this concept we will present a ranking technique for intuitionistic fuzzy sets. The notion of *relative sigma count* of fuzzy sets was introduced by Zadeh [170] as: $\sum_{x_i \in X} Count(A/B) = \frac{\sum_{x_i \in X} Count(A \cap B)}{\sum_{x_i \in X} Count(B)} = \frac{\sum_{x_i \in X} \min(\mu_A(x_i), \mu_B(x_i))}{\sum_{x_i \in X} \mu_B(x_i)}$. The relative sigma count of an intuitionistic fuzzy set was introduced as a numerical interval in [149]. We have presented a new definition of this concept and quantified it to a scalar number which makes it a more feasible object that could be worked upon in any situation.

Definition 7.3.1 Let $A, B \in IFS(X)$. The relative sigma count of set A with respect to set B is defined as:

$$\begin{aligned} Card_\theta(A/B) &= \frac{Card_\theta(A \cap B)}{Card_\theta(B)} \\ &= \frac{\sum_{x_i \in X} \theta \min(\mu_A(x_i), \mu_B(x_i)) + (1-\theta)\min(1-\nu_A(x_i), 1-\nu_B(x_i))}{\sum_{x_i \in X} \theta \mu_B(x_i) + (1-\theta)(1-\nu_B(x_i))} \\ \text{where } \theta &\in [0.5, 1]. \end{aligned}$$

Suppose that A and B are two fuzzy sets. Then:

$$Card_\theta(A/B) = \frac{\sum_{x_i \in X} \min(\mu_A(x_i), \mu_B(x_i))}{\sum_{x_i \in X} \mu_B(x_i)}.$$

Now if A is an intuitionistic fuzzy set and B is a fuzzy set then:

$$Card_\theta(A/B) = \frac{\sum_{x_i \in X} \theta \min(\mu_A(x_i), \mu_B(x_i)) + (1-\theta)\min(1-\nu_A(x_i), \mu_B(x_i))}{\sum_{x_i \in X} \mu_B(x_i)};$$

and if A is a fuzzy set and $B = X$ then:

$$Card_\theta(A/B) = \frac{\sum_{x_i \in X} \mu_A(x_i)}{n} = \frac{1}{n} \sum_{x_i \in X} \mu_A(x_i).$$

Clearly, $Card_\theta(A/B) \in [0, 1]$ and

$$Card_\theta(A/A) = \frac{\sum_{x_i \in X} \theta\mu_A(x_i) + (1-\theta)(1-\nu_A(x_i))}{\sum_{x_i \in X} \theta\mu_A(x_i) + (1-\theta)(1-\nu_A(x_i))} = 1.$$

Definition 7.3.2 For a given class $A_l \in IFS(X), l \in L$ and for any fixed $B \in IFS(X)$, we say A_p dominates A_q written as $A_p \rhd A_q$ if $Card_\theta(A_p/B) \geq Card_\theta(A_q/B)$ where $\theta \in [0.5, 1]$.

Definition 7.3.3 For a given class $A_l \in IFS(X), l \in L$, the set $\psi \in IFS(X)$ is said to be the *super* IFS if

$$\psi = \left\{(x_i, \mu_\psi(x_i), \nu_\psi(x_i)) \mid x_i \in X\right\}$$

where $\mu_\psi(x_i) = \max_{l \in L}(\mu_{A_l}(x_i))$ and $\nu_\psi(x_i) = \min_{l \in L}(\nu_{A_l}(x_i))$.

7.3.1 Application of Relative Sigma Count to Medical Diagnosis and Organizational Management

In this subsection, we shall demonstrate the efficiency and simplicity of the proposed ranking procedure of intuitionistic fuzzy sets by presenting two case studies. The first one is related to the field of medical diagnosis while the second one addresses the employee appraisal problem of organizational management. In the first case study we shall directly utilize the Definition 7.3.2 in the ranking procedure of intuitionistic fuzzy sets. However, in the second situation we need firstly, construct the *super* set ψ as defined in Definition 7.3.3 and then we rank the given class of intuitionistic fuzzy sets with respect to the set ψ. It must be noted that through out this subsection we shall specify $\theta = 0.5$ in the definition of relative sigma count for intuitionistic fuzzy sets.

Case Study 7.3.1.1: (Medical Diagnosis)

Suppose that a doctor has to make an judgement about a patient disease, who is claiming the presence of symptoms such as Temperature, Cough, Chest pain, Headache, Stomach pain that are collective symptoms of diseases like Viral fever, Malaria, Chest problem, Typhoid, Stomach problem. We propose a diagnosis procedure comprising of the following three simple steps:

(i). Determination of symptoms;

(ii). Formulation of medical knowledge based on intuitionstic fuzzy sets;

(iii). Determination of diagnosis on the basis of ranking procedure proposed in Definition 7.3.2.

In the above situation the doctor will, firstly, formulate a crisp set of symptoms σ ={s_1(temperature), s_2(cough), s_3(chest pain), s_4(headache), s_5(Stomach pain)}. As a next step he will assign to the patient P a degree of membership and a degree of non membership with respect to each of the symptom $s_i \in \sigma, i = 1, 2...5$ on the basis of his/her medical examination and tests. Thus, for the doctor a patient is now an intuitionistic fuzzy set P on the set of symptoms σ given as say:

$P(Patient)$	$(0.8, 0.0)$	$(0.6, 0.1)$	$(0.2, 0.8)$	$(0.6, 0.1)$	$(0.1, 0.6)$

Table 7.3.1: Patient w.r.t. symptons-1

Moreover, on the basis of his medical knowledge he will view each of the possible diagnosis like, Viral Fever(VF), Malaria(M), Chest Problem(CP), Typhoid(T), Stomach Problem(SP) as intuitionstic fuzzy set over the set of symptoms σ.

In the following case study, we formulate a matrix representing all these five diseases as the intuitionistic fuzzy sets over the set of symptoms σ.

	s_1	s_2	s_3	s_4	s_5
VF	$(0.4, 0.0)$	$(0.3, 0.5)$	$(0.1, 0.7)$	$(0.4, 0.3)$	$(0.1, 0.7)$
M	$(0.7, 0.0)$	$(0.2, 0.6)$	$(0.0, 0.9)$	$(0.7, 0.0)$	$(0.1, 0.8)$
CP	$(0.3, 0.3)$	$(0.6, 0.1)$	$(0.2, 0.7)$	$(0.2, 0.6)$	$(0.1, 0.9)$
T	$(0.1, 0.7)$	$(0.2, 0.7)$	$(0.2, 0.4)$	$(0.8, 0.0)$	$(0.2, 0.7)$
SP	$(0.1, 0.8)$	$(0.0, 0.8)$	$(0.2, 0.8)$	$(0.2, 0.8)$	$(0.8, 0.1)$

Table 7.3.2: Diseases w.r.t. symptoms-1

Then as the final stage he will find the relative sigma count for all of the diseases with respect to patient P and adopt the ranking technique defined in Definition 7.3.2 to make his final judgement. The mathematical calculation regarding this scenario is as following:

$$\begin{aligned} Card_{0.5}(VF/P) &= 0.707; \\ Card_{0.5}(M/P) &= 0.745; \\ Card_{0.5}(CP/P) &= 0.672; \\ Card_{0.5}(T/P) &= 0.490; \\ Card_{0.5}(SP/P) &= 0.327. \end{aligned}$$

Thus, we obtain a ranking as $M \triangleright VF \triangleright CP \triangleright T \triangleright SP$ which indicates that the patient is most likely facing the problem of Malaria.

Case Study 7.3.1.2 :(Organizational management) Consider the problem of annual selection of the best employee in an organization. The characteristics, which are used to determine this selection are:

$$\begin{aligned} C_1 &: \textit{Progressiveness;} \\ C_2 &: \textit{Team work;} \\ C_3 &: \textit{Discipline;} \\ C_4 &: \textit{Punctuality;} \\ C_5 &: \textit{Efficiency;} \\ C_6 &: \textit{Stress management.} \end{aligned}$$

The people under consideration are: John, David, Billy and Brown.

We formulate a matrix representing the four employee who are evaluated on the basis of the above mentioned six characteristics. A team of experts has allocated the following scores to the candidates with respect to corresponding criteria. For example experts have allocated John 0.8 with respect to discipline while 0.1 is given to him for non discipline.

	C_1	C_2	C_3	C_4	C_5	C_6
John	$(0.2, 0.7)$	$(0.5, 0.2)$	$(0.8, 0.1)$	$(0.6, 0.3)$	$(0.4, 0.5)$	$(0.3, 0.6)$
David	$(0.6, 0.2)$	$(0.2, 0.7)$	$(0.7, 0.3)$	$(0.8, 0.2)$	$(0.5, 0.3)$	$(0.9, 0.1)$
Billy	$(0.2, 0.7)$	$(0.4, 0.5)$	$(0.8, 0.2)$	$(0.9, 0.1)$	$(0.6, 0.3)$	$(0.5, 0.2)$
Brown	$(0.5, 0.4)$	$(0.3, 0.5)$	$(0.6, 0.3)$	$(0.5, 0.3)$	$(0.7, 0.2)$	$(0.9, 0.0)$

Table 7.3.3: Employees w.r.t. selection characteristics

To apply the proposed technique we deal the data of John as a single IFS. So we are dealing with a class of four IFS's: John, David, Billy and Brown.

First of all we construct the super intuitionistic fuzzy set ψ of the above mentioned data which is also as an IFS given in the form of a row matrix:

ψ	$(0.6, 0.2)$	$(0.5, 0.2)$	$(0.8, 0.1)$	$(0.9, 0.1)$	$(0.7, 0.2)$	$(0.9, 0.0)$

Table 7.3.4: Super IFS w.r.t. selection characteristics

A selector may have to consider the above characteristics and formulate their judgement for each employee in a organization utilizing all the given information. We shall use the technique of dominance defined in Definition 7.3.2 as the factor of performance appraisal. So we observe that:

$$\begin{aligned}
Card_{0.5}(\text{John}/\psi) &= 0.66; \\
Card_{0.5}(\text{David}/\psi) &= 0.822; \\
Card_{0.5}(\text{Billy}/\psi) &= 0.77; \\
Card_{0.5}(\text{Brown}/\psi) &= 0.812.
\end{aligned}$$

Clearly,

$$Card_{0.5}(\text{David}/\psi) \geq Card_{0.5}(\text{Brown}/\psi) \geq Card_{0.5}(\text{Billy}/\psi) \geq Card_{0.5}(\text{John}/\psi).$$

Thus, we have a ranking:

$$\text{David} \triangleright \text{Brown} \triangleright \text{Billy} \triangleright \text{John}.$$

7.3.2 Application of Relative Sigma Count in Defining Quantification Rules

In this subsection, we shall once again illustrate some applications of relative sigma count for intuitionistic fuzzy sets by defining extended quantified intuitionistic fuzzy propositions and studying their truthfulness with the aid of practical examples. We shall demonstrate the applications by taking $\theta = 0.5$ in the definition of relative sigma count for intuitionistic fuzzy sets.

Definition 7.3.2.1 For $A, B \in IFS(X)$, we define a *quantified intuitionistic fuzzy proposition* as '$Qx's\ are\ A's$' and an *extended quantified intuitionistic fuzzy proposition* by '$QA's\ are\ B's$', where the *intuitionistic quantifier* Q is specified by a membership function $\mu_Q(x)$ and a non membership function $\nu_Q(x)$ for any particular $x \in X$. Mathematically speaking, the truth value of the extended quantified intuitionistic fuzzy proposition ($QA's\ are\ B's$) in a finite universe X is given by $(\mu_Q(r), \nu_Q(r))$ such that:

$$r = Card_{0.5}(B/A) = \frac{Card_{0.5}(A \cap B)}{Card_{0.5}(A)}.$$

In particular if $A, B \in FS(X)$, then the proposition '$QA's\ are\ B's$' is reduced to Zadeh's sense [170].

Moreover, as a special case when $A = X$ and $B \in IFS(X)$, the extended quantified intuitionistic fuzzy proposition '$QA's\ are\ B's$' becomes a simple quantified intuitionistic fuzzy proposition '$Qx's\ are\ B's$' and, then the truth value of this intuitionistic quantified proposition is modelled by a situation:

$truth(Qx's\ are\ B's) = (\mu_Q(r_0), \nu_Q(r_0))$ such that:

$$r_0 = Card_{0.5}(B/X) = \frac{Card_{0.5}(B \cap X)}{Card_{0.5}(X)} = \frac{\frac{1}{2}\sum_{x_i \in X} \mu_B(x_i) + (1 - \nu_B(x_i))}{n}.$$

We demonstrate both the situations by the following case studies.

Case Study 7.3.2.2

Let us consider the proposition '*most cars are fast.* We want to check its truthfulness. Clearly it is a quantified intuitionistic fuzzy proposition of the form '$Qx's\ are\ B's$'.

To illustrate the problem we may restrict our self to the given situation where $X = cars = \{x_1, x_2, x_3\}$ is a finite universe of discourse and $B = fast = \{(x_1, 0.1, 0.8), (x_2, 0.6, 0.2), (x_3, 0.8, 0.2)\}$ is the intuitionistic fuzzy set on X giving the idea of fastness of each car ($x_i \in X$) and $Q = most$ is the intuitionistic fuzzy quantifier with membership and non membership functions defined as:

$$\mu_Q(t) = \left\{ \begin{array}{cc} 0 & 0 \le t \le 0.3 \\ 1 - (1 + (2t - .6)^2)^{-1} & 0.3 \le t \le 0.7 \\ 1 & t \ge 0.7 \end{array} \right\}; \tag{7.1}$$

$$\nu_Q(t) = \left\{ \begin{array}{cc} 1 & 0 \le t \le 0.4 \\ (1 + (2t - .8)^2)^{-1} & 0.4 \le t \le 0.8 \\ 0 & t \ge 0.8 \end{array} \right\} \tag{7.2}$$

Now this is the case where the truth value of a quantified intuitionistic fuzzy proposition '$Qx's\ are\ B's$' is modelled by a situation in which $A = X$ and $B \in IFS(X)$, i.e.,

$truth(Qx's\ are\ B's) = (\mu_Q(r_0), \nu_Q(r_0))$ where,

$r_0 = Card_{0.5}(B/X) = \dfrac{\frac{1}{2}\sum_{x_i \in X} \mu_B(x_i) + (1 - \nu_B(x_i))}{n}$.

Thus $r_0 = Card_{0.5}(B/X) = \frac{3.3}{6} = 0.55$ where r_0 defines the proposition of B in X.

Now substituting $r_0 = 0.55$ in (7.1), (7.2) we have $\mu_Q(0.55) = 0.2$ and $\nu_Q(0.55) = 0.53$.

The truth value may vary depending on how the quantifier $Q(most)$ and the set $B(fast)$ are defined.

Case Study 7.3.2.3

Let us consider a more general proposition, '*Most fast cars are dangerous*'. Clearly this proposition is an extended quantified intuitionistic fuzzy proposition of the form '$QB's\ are\ A's$'. Now here we take the data for $Q = most$, $B = fast$ and $X = cars$ from the previous example, but we do need to define the intuitionistic fuzzy set the *dangerous* over X. Let it be: $A = dangerous =$

$\{(x_1, 0.2, 0.7), (x_2, 0.5, 0.4), (x_3, 0.6, 0.4)\}$. Then the truth value of this extended quantified intuitionistic fuzzy proposition is calculated by:

$truth(QB's\ are\ A's) = (\mu_Q(r), \nu_Q(r))$ where,

$$r = Card_{0.5}(A/B) = \frac{\frac{1}{2}\sum_{x_i \in X} \min(\mu_A(x_i), \mu_B(x_i)) + \min(1-\nu_A(x_i), 1-\nu_B(x_i))}{\frac{1}{2}\sum_{x_i \in X} \mu_B(x_i) + (1-\nu_B(x_i))} = 0.787.$$

Finally, substituting the value of r in (7.1) and (7.2) we get $\mu_Q(0.787) = 0.48$ and $\nu_Q(0.787) = 0.46$

The final results may vary depending on how the quantifier $Q(most)$, the set $B(fast)$, and the set $A(dangerous)$ are defined.

7.4 The Application of Weighted Average Cardinality Based Parametric Family of Intuitionistic Fuzzy Similarity Measure to Medical Diagnosis

Fuzzy set theory and fuzzy logic [173] after their development were highly acceptable in medical industry for the development of knowledge based systems in medicine, and for objectives such as interpretation of sets of medical findings, syndrome differentiations in medicine, disease diagnosis in modern medicine, the most appropriate selection of medical procedure involving western and eastern medicine and for actual and timely monitoring of patients record [79, 114, 121, 152]. However, with further development of knowledge the real time situations occurred in which the description of a problem by fuzzy linguistic variable, given in terms of membership function only seemed too rough. Thus, among many generalizations of fuzzy sets *Intuitionistic fuzzy set* (*IFS*) introduced by Atanassov, dealt with the situation more flexibly particularly in the case of decision making problem of medical diagnosis, where there is a fair chance of presence of a non null hesitation part at each moment of evaluation of an unknown object [7, 6].

Intuitionistic fuzzy similarity measures play a vital role in the field of medical diagnosis as they serve as organizing tool by which one classifies and analyses the medical data sets in order to find the similar patterns and regularities that contain important knowledge of the given data. The simplest class of similarity measures is the one in which we construct a feature vector for each of the two objects under consideration representing a collection of appropriate set of features typical to both these objects. The feature vector takes its entries from the set $L^* = \{(a_1, a_2) \in L^2 \mid L = [0,1], a_1 + a_2 \leq 1\}$, encoding the degree of presence and degree of absence. The two objects are then compared by constructing different intuitionistic fuzzy similarity measures using their vector representations.

In this section, we demonstrate an effective and easy method of diagnose of a disease by the pattern recognition technique of the members $\delta_k^j, k = J, SM, D,$ $RT, SK1, SK2$ and $j = M, P, L,$ of the new weighted average cardinality based parametric family of intuitionistic fuzzy similarity measure (δ) proposed in Section 5.2, Chapter 5.

The method constructs a data base i.e., a description of a set of symptoms σ, a set of diagnosis Q and, a state of patient knowing results of his/her medical tests by using the notion of intuitionstic fuzzy sets and then applies the pattern recognition technique to diagnose of the correct disease.

This technique can be summarized to the following three simple steps:

(i). Determination of symptoms;

(ii). Formulation of medical knowledge based on intuitionstic fuzzy sets;

(iii). Determination of diagnosis on the basis of pattern recognition technique involving different intuitionistic fuzzy similarity measures δ_k^j where $k = J, SM, D, RT, SK1, SK2$ and $j = M, P, L$.

To be more specific, we suppose that there is a patient P whose initial medical examination reveals the presence of symptoms like Temperature, Headache, Stomach pain, Cough and Chest pain. Thus as a fist step, a medical expert will formulate a set σ of symptoms which will be an ordinary set i.e., $\sigma =\{s_1$*(Temperature)*, s_2*(Headache)*, s_3*(Stomach Pain)*, s_4*(Cough)*, s_5*(Chest Pain)*$\}$. As a second stage he will use his knowledge to list out all the possible diagnosis and formulate a set of diagnosis say $Q =\{Q_1$*(Viral fever)*, Q_2*(Malaria)*, Q_3*(Typhoid)*, Q_4*(Stomach Problem)*, Q_5*(Chest Problem)*$\}$ such that, each of these diagnosis are intuitionistic fuzzy sets on σ which in this situation can be viewed as follows:

	s_1	s_2	s_3	s_4	s_5
Q_1	$(0.4, 0.0)$	$(0.3, 0.5)$	$(0.1, 0.7)$	$(0.4, 0.3)$	$(0.1, 0.7)$
Q_2	$(0.7, 0.0)$	$(0.2, 0.6)$	$(0.0, 0.9)$	$(0.7, 0.0)$	$(0.1, 0.8)$
Q_3	$(0.3, 0.3)$	$(0.6, 0.1)$	$(0.2, 0.7)$	$(0.2, 0.6)$	$(0.1, 0.9)$
Q_4	$(0.1, 0.7)$	$(0.2, 0.4)$	$(0.8, 0.0)$	$(0.2, 0.7)$	$(0.2, 0.7)$
Q_5	$(0.1, 0.8)$	$(0.0, 0.8)$	$(0.2, 0.8)$	$(0.2, 0.8)$	$(0.8, 0.1)$

Table 7.4.1: Diseases w.r.t. symptoms-2

Moreover, according to the given technique the state of the patient P is also given by an intuitionstic fuzzy set on the set of symptoms σ i.e. in our case it may be represented by:

$P(Patient)$	$(0.8, 0.1)$	$(0.6, 0.1)$	$(0.2, 0.8)$	$(0.6, 0.1)$	$(0.1, 0.6)$

Table 7.4.2: Patient w.r.t. symptoms-2

As a final stage of this diagnosis problem he will classify pattern P in one of the classes Q_1, Q_2, Q_3, Q_4, and Q_5. The proper diagnosis Q_t is derived according to the formula:

$$\tau = \arg \underset{1 \le t \le 5}{Max} \left\{ \delta_k^j(P, Q_t) \right\} \text{ where } k = J, SM, D, RT, SK1, SK2 \text{ and } j = M, P, L. \tag{7.3}$$

Now particularly, for this example the diagnosis table representing the pattern recognition results of different $\delta_k^j(P, Q_t)$, $k = J, SM, D, RT, SK1, SK2$ and $j = M, P, L$ are given in the following tables:

$\delta_J^j(P, Q_t)$	$t = 1$	$t = 2$	$t = 3$	$t = 4$	$t = 5$
$j = M$	0.6724	0.6949	0.6491	0.3521	0.3143
$j = P$	0.4328	0.5083	0.3967	0.2468	0.2137
$j = L$	0.2763	0.4085	0.2703	0.1294	0.1220

Table 7.4.3: Result with similarity δ_J^j, $j = M, P, L$

$\delta_{SM}^j(P, Q_t)$	$t = 1$	$t = 2$	$t = 3$	$t = 4$	$t = 5$
$j = M$	0.81	0.82	0.8	0.54	0.52
$j = P$	0.616	0.674	0.564	0.420	0.404
$j = L$	0.4500	0.5800	0.4600	0.2600	0.2800

Table 7.4.4: Result with similarity δ_{SM}^j, $j = M, P, L$

$\delta^j_D(P,Q_t)$	$t=1$	$t=2$	$t=3$	$t=4$	$t=5$
$j=M$	0.8041	0.82	0.7872	0.5208	0.4783
$j=P$	0.6091	0.674	0.5681	0.3958	0.3522
$j=L$	0.4330	0.5800	0.4255	0.2292	0.2174

Table 7.4.5: Result with similarity δ^j_D, $j=M,P,L$

$\delta^j_{RT}(P,Q_t)$	$t=1$	$t=2$	$t=3$	$t=4$	$t=5$
$j=M$	0.649	0.6807	0.6667	0.3699	0.3514
$j=P$	0.4451	0.5023	0.4225	0.2658	0.2531
$j=L$	0.2903	0.4085	0.2987	0.1494	0.1628

Table 7.4.6: Result with similarity δ^j_{RT}, $j=M,P,L$

$\delta^j_{SK1}(P,Q_t)$	$t=1$	$t=2$	$t=3$	$t=4$	$t=5$
$j=M$	0.5065	0.5325	0.4805	0.2136	0.1864
$j=P$	0.2762	0.3407	0.2475	0.1407	0.1196
$j=L$	0.1603	0.2566	0.1563	0.0692	0.0649

Table 7.4.7: Result with similarity δ^j_{SK1}, $j=M,P,L$

$\delta^j_{SK2}(P,Q_t)$	$t=1$	$t=2$	$t=3$	$t=4$	$t=5$
$j=M$	0.8950	0.9011	0.8889	0.7013	0.6842
$j=P$	0.7624	0.8053	0.7453	0.5915	0.5755
$j=L$	0.6207	0.7342	0.6301	0.4127	0.4375

Table 7.4.8: Result with similarity δ^j_{SK2}, $j=M,P,L$

Applying the formula (7.3) to the above Tables we deduce that all the eighteen intuitionistic fuzzy similarity measures remarkably assigned the patient P the same diagnosis $Q_2(Malaria)$ according to recognition principal. It is clear that our diagnosis results for this case study are identical to those obtained by [116, 138, 157], however, a deep look into the data table reveals the fact that a use of three different t-norms within the same similarity measure provides a variety in the measure of similarity between the same pair of sets. For instance, considering the Jaccard measure δ^j_J a variety of three basic t-norms could be more practically utilizable in the sense that a decision maker has this choice of a selection among all of these three measures depending upon the situation in which he is placed, and his personal preference of being optimistic, pessimistic, or maintaining a neutral behavior. Thus, we can claim that different intuitionistic Jaccard coefficients (and any of the other proposed measures) provides a more flexible, reliable and a better workable environment for decision making situations.

Chapter 8

Conclusion and Indication for Future Research

This final chapter will pointwise conclude the research conducted in this dissertation along with the possible indications of any future research within the scientific domain of this area.

The core target of this research activity was to formulate different measures of similarity and inclusion between objects existing in an intuitionistic environment and to implement these measures in designing techniques that can act as an effective solution to many real world problems.

The journey for the search of new measures of similarity and inclusions between intuitionistic fuzzy sets took us to the point where we required some new fuzzy measures on intuitionistic fuzzy sets, especially a scalar cardinality for intuitionistic fuzzy sets and some other logical operators such as intuitionistic fuzzy bi-implicators. Thus, in order to accomplish our main interest we systematically proposed some new normal fuzzy measures on intuitionistic fuzzy sets, a class of scalar cardinality measure followed by different classes of intuitionistic fuzzy bi-implication operators. Afterwards, we used these new fuzzy measures and intuitionistic logical operators (implicators and bi-implicators) to construct for both (similarity and inclusion measure on IFS's) two different classes/families, generally called implication/bi-implication based classes and the cardinality based parametric families.

Let us summarize the results presented in this thesis and some open problems in the form of following listed points:

(1). The main target of our research was the formation of new similarity and inclusion measures for intuitionistic fuzzy sets. Therefore, as an attempt to do so we adopted a technique which composed the measures on a set with the logical operators to define inclusion or similarity measures between sets. Thus in chapter 3, we formally introduced four new normal fuzzy measures on intuitionistic fuzzy sets and a new class of weighted average cardinality measure for IFS's. Moreover, this new defined weighted average cardinality was studied for the valuation property by specifying the $\cap$ and $\cup$ between two IFS with the t-representable intuitionistic fuzzy Frank t-norms and their dual conorms. Similarly, its behavior with model operators as defined by Atanassov [7] was also studied.

Since, an intuitionstic fuzzy sets is regarded as an $L^{*}-$ fuzzy sets in the sense of

Gougen [71], therefore, in [18] it was proved that measures on intuitionistic fuzzy sets can be classified as fuzzy valued measures on intuitionistic fuzzy sets and the intuitionistic valued measures on intuitionistic fuzzy sets. In this thesis, we have restricted our self to the study of the first class of measures i.e., the fuzzy measures on intuitionistic fuzzy sets also called the fuzzy valued measures on intuitionistic fuzzy sets. A broader view towards defining new measures on intuitionistic fuzzy sets may lead to the following open problems:

- A similar or extended version of the fuzzy measures on intuitionstic fuzzy set can be developed under the notion of intuitionistic fuzzy measures for intuitionistic fuzzy sets as introduced in [18] and it is believed to be a challenging new area of research in this direction.

- Another variation that is possible with the fuzzy measures described in this work is to assume their values in the set of normal fuzzy numbers or even normal intuitionistic fuzzy numbers in stead of real interval [0,1]. A similar effort in this direction for other fuzzy measures was attempted in [93].

- Within the framework of this thesis an investigation of different properties especially the valuation property of the new defined weighted average cardinality measure by modeling the $\cap$ and $\cup$ between IFS's by other t-representable intuitionstic fuzzy t-norms and their dual conorms can give interesting results.

- The same properties of this new scalar cardinality could be checked for a pair of non t-representable intuitionistic fuzzy t-norms and their dual conorms.

- In [87], the authors showed that the Bell inequalities were of particular interest in the context of cardinalities of fuzzy sets. Historically speaking, the Bell type inequalities were proposed by Bell [28] in 1964, as an attempt to answer the issue of existence of hidden variables in a quantum mechanics which was believed to be incomplete as claimed by Einstein, Podolsky and Rosen in their famous EPR-paradox. Later, Pitowsky [122] showed that the original Bell inequalities could be derived in purely mathematical context without any reference to physics. He showed that the classical

Kolmonorrovain probability model related to the situations where, for a set of probabilities connected to the outcome of experiments are studied can equivalently demonstrate the situations(environments) where the bell type inequalities are satisfied i.e.,

If the probabilities of two random events A_i, A_j and their intersection $A_i \cap A_j$ are given as $p_i = P(A_i)$, $p_j = P(A_j)$ and $p_{ij} = P(A_i \cap A_j)$ then the following double inequality

$$\begin{aligned} p_i + p_j - 1 &\leq p_{ij} \leq \min(p_i, p_j) \\ &\text{or} \\ T_L(p_i, p_j) &\leq p_{ij} \leq T_M(p_i, p_j) \end{aligned}$$

is regarded as the first Bell type inequality. Later, the following set of inequalities were found by Pitowsky by experimenting with three or four random events such that at most two events were intersected at a time.

$$\begin{aligned} 0 &\leq p_i - p_{ij} - p_{ik} + p_{jk}, \\ p_i + p_j + p_k - p_{ij} - p_{ik} - p_{jk} &\leq 1, \\ 0 &\leq p_i + p_k - p_{ik} - p_{il} - p_{jk} + p_{jl} \leq 1, \end{aligned}$$

where i, j, k and l are different variables.

(a). We believe that all these Bell type inequalities could be rewritten in the context of the new defined scalar weighted average cardinality of intuitionistic fuzzy sets such that the intersection between the events is modelled by intuitionistic fuzzy t-norms which may or may not be t-representable.

(b). Moreover, these inequalities can be extended to four or more events in which almost two events are intersected at a time. A similar effort for fuzzy sets in this regard has been attempted in [87]

(c). Also, if we choose to model the intersection between events by parametric intuitionistic fuzzy t-norm families (especially the intuitionistic Frank t-norm family) then we feel that it can be a fruitful effort to calculate the range of the parameter for which each of the Bell inequality is satisfied since we believe that an intuitionistic fuzzified Bell-inequalities can be exploited to develop a framework in which the validity of more general inequality on the other cardinalities of intuitionistic fuzzy sets can be checked.

(2). For the construction of an intuitionistic fuzzy bi-implicator based similarity measure we introduced in chapter 4 of our thesis several new classes of intuitionistic fuzzy bi-implicators having axiomatic as well as constructive approaches. A study on these classes revealed the existence of non hull intersection between these classes. Moreover, one of the class called intuitionistic fuzzy $\beta-$bi-implicator was investigated for its properties by implanting the intuitionistic fuzzy Lukasiewicz implicator $\breve{I}_{\widetilde{T}_L}$ and the t-representable intuitionistic fuzzy t-norm $\overline{T}_M$ in its definition. The properties exhibited by this choice in intuitionistic fuzzy $\beta-$bi-implicator were close to the set of criteria given in [134].

We now list down, some out of many ideas, of possible research that may be conducted in order to broaden the outlook of different intuitionistic fuzzy bi-implication operators presented in this dissertation:

- By employing any of the constructive intuitionistic fuzzy bi-implicator and some other operators we can study the defining standards for basic intuitionistic operators such as intuitionistic fuzzy t-norms, t-conorms , implicators and negators. For instance we can try to obtain a intuitionistic fuzzy t-norm by the use of an intuitionistic fuzzy R-implicator and the *intuitionistic fuzzy* $\check{T}'\check{S}'-$bi-implicator operator. In case, if failed to obtain some of those intuitionistic fuzzy operators then this *intuitionistic fuzzy* $\check{T}'\check{S}'-$bi-implicator can be regarded as a restricted intuitionistic fuzzy operator.

- If by use of any of the intuitionistic fuzzy bi-implicator presented in this thesis, we succeed in constructing all the above mentioned operators, then this could lead us to a possibility of building a new sound and complete intuitionistic fuzzy logic with respects to all those obtained operators.

- Moreover, a deep study into the properties all these intuitionistic fuzzy bi-implicators especially the intuitionistic fuzzy $\check{T}'\check{S}'-$bi-implicator by use of different intuitionistic fuzzy t-representable or non t-representable t-norms and conorms can conclude to which properties are special cases or consequences of others. For instance, we may want to analyze whether the constructive classes of intuitionistic fuzzy bi-implicators proposed in this work are continuous for any combination of intuitionistic fuzzy t-norm $\check{T}'$,a left continuous intuitionistic fuzzy t-norm $\check{T}$, and an intuitionistic fuzzy conorm $\check{S}'$.

- Axiomatization of the intuitionistic fuzzy $\check{T}'\check{S}'-$bi-implicator in order to obtain the definitions of a primitive connective could be a fruitful effort. If concluded successfully, we can generate an extensive research activity with a goal of deriving new axioms for other intuitionistic fuzzy connectives say t-norms, conorms, implicators and negators that were previously constructed with the help of intuitionistic fuzzy $\check{T}'\check{S}'-$bi-implicator.
- In [107] a study on different concepts of operations fitting the $KMP-$fuzzy bi-implicator was made. A similar effort can be made in an intuitionistic fuzzy frame work by changing the role of a $KMP-$fuzzy bi-implicator with any of the new intuitionistic fuzzy bi-implicator presented in this thesis.

(3). The fuzzy measures on intuitionistic fuzzy sets introduced in chapter 3 and the classes of intuitionistic fuzzy bi-implicators presented in chapter 4 were combined to define classes of intuitionistic fuzzy bi-implicator based similarity measures in section 5.1 of chapter 5. The properties of one of the class of intuitionistic fuzzy bi-implicator based similarity measure were studied by specifying in its definition the intuitionistic fuzzy $\beta-$bi-implicator based on intuitionistic fuzzy Lukasiewicz implicator $\check{I}_{\tilde{T}_L}$ and the t-representable intuitionistic fuzzy t-norm $\overline{T}_M$. These properties were found to be similar to ones proposed in [134]. Moreover, this particular class of bi-implicator based similarity measures was proved to be T_L- transitive and defined an equivalence relation on $IFS(X)$. Now, before proceeding to the conclusion of section 5.2, we believe that a list of open problems related to the intuitionistic fuzzy bi-implicator based similarity measures can be of great interest to our reader. An intuitionistic fuzzy bi-implicator based similarity measure is constructed by composition of two types of operators (a fuzzy measure and a bi-implicator operator) on IFS's and therefore a variation in any of the two operators can bring about some great changes.

For instance,

- In this thesis, we have restricted our self to the study of fuzzy measures on IFS only. If we shift these fuzzy measures with intuitionistic fuzzy measures on intuitionistic fuzzy sets then the resulting similarity measure will be also be an intuitionistic fuzzy valued similarity measure.

- If, successful, such a construction of similarity measure for IFS can generate rigorous research on the study of its properties say in particular a $\check{T}-$transitivity or symmetry, monotonicity and other famous set of axioms and criteria required by any reasonable similarity measure to be satisfied.

- Moreover, as mentioned earlier, in this thesis we have studied the properties of only one of the members of the intuitionistic fuzzy $\beta-$bi-implicator based similarity measure with the fuzzy measures working in its background. This particular class was proved to be T_L -transitive. An investigation of its T_M and T_P transitivity is still undiscovered.

- Likewise, it is still an open problem to examine the properties of other classes of $\beta-$bi-implicator based similarity measures by choosing different combinations of intuitionistic fuzzy t-norms $\check{T}'$, $\check{T}$, and an intuitionistic fuzzy R-implicator $\check{I}_{\check{T}}$ other than the Lukasiewicz implicator $\check{I}_{\widetilde{T}_L}$ and the t-representable intuitionistic fuzzy t-norm $\overline{T}_M$. Also it will be interesting to study the properties by choosing non-t-representable intuitionstic fuzzy t-norms and their R-implicator.

- A fixation of fuzzy measures and a variation of different classes of new defined intuitionstic fuzzy bi-implicators in the definition of an intuitionistic fuzzy bi-implicator based similarity measure can provide different classes of intuitionistic fuzzy bi-implicator based similarity measures and therefore we can claim that this area of research has more to offer to all those who seek knowledge in this dimension.

In section 5.2 of chapter 5, we introduced a *weighted average cardinality based parametric family of intuitionistic fuzzy similarity measure*. This family was based on set theoretic operations and the identities involved were converted to their intuitionistic versions by the use of weighted average cardinality measure as introduced in chapter 3 and the t-representable intuitionistic fuzzy Frank t-norms along with their dual conorms. This parametric family involved four parameters and for different combinations of these parameters the intuitionistic versions of some of the famous crisp measures of time were generated. A study of their properties showed some remarkable results. For instance, the intuitionistic fuzzy Jaccard coefficient was found to satisfy most of all the axioms of a similarity measure as mentioned in [134]. Moreover, it was proved to be T_L-transitive and it defined an equivalence relation on $IFS(X)$. Let us list down some of the open problems that could be addressed:

- In this dissertation, we had only considered the family of similarity measures that were based on the addition (or subtraction) of scalar cardinalities. In mathematical literature, we find measures that were based on multiplications of cardinalities like the cosine coefficient. Clearly, they cannot be a member of the suggested family of similarity measures. Thus, a design of a new family of similarity measures which could cover all such or similar measures can definitely be regarded as an open challenge.

- Moreover, this parametric family of similarity measures was constructed using scalar cardinality for intuitionstic fuzzy sets. A choice of other types of cardinality measures for intuitionstic fuzzy sets can lead to some remarkable constructions of cardinality based similarity measures for intuitionistic fuzzy sets

- Likewise, the choice of only four parameters in this family of similarity measures is appeared as an other constraint that cause the disappearance of an intuitionistic version of some of the famous crisp measures (for example, the first coefficient of Kulczynski [98]). We believe that this situation can be tackled by considering six parameters instead of four. However, this new direction definitely needs further research.

- Lastly, within the context of this thesis we have confined our self to the choice of t-representable intuitionstic fuzzy Frank t-norms and their dual conorms for modelling the generalized $\cap$ and $\cup$ between intuitionistic fuzzy sets. We believe that a variation of other intuitionistic fuzzy t-norms (t-representable or non t-representable) and their dual conorms for modelling generalized intersection and union can generate some interesting results on $T-$ transitivity or even monotonicity of this new parametric family of similarity measures for IFS's. Even with intuitionistic fuzzy Frank t-norms used in the definition, a test of T_P and T_M- transitivity of the parametric family of similarity measure is still an open problem.

(4). In chapter 6, we have worked on defining two separate classes/families of inclusion measures for intuitionstic fuzzy sets based on two different approaches. The first approach was based on the fact that an implication acts as a backbone of any inclusion relation between sets. Thus, we introduced multiple classes of inclusion measures for intuitionstic fuzzy sets by composing the fuzzy measures of chapter 3 with different intuitionistic fuzzy implication operators. A study on the properties of one of the classes was made by specifying the intuitionistic fuzzy Lukasiewicz implicator $\breve{I}_{\tilde{T}_L}$ in its definition. The new class of inclusion measures satisfied all the axioms of inclusion proposed by [134]. Moreover, it was proved to be T_L-transitive and it also defined a *quasi-order* relation on $IFS(X)$. Likewise to chapter 5, in the second section of this chapter 6, we have introduced a four parametric family of weighted average cardinality based intuitionistic fuzzy inclusion measure. Different inclusion measures of this family were extracted using the set theoretic identities for intuitionistic fuzzy set that were previously defined in chapter 5. The inclusion measures obeyed majority of the axioms presented in [168]. Moreover, these measures also defined a *quasi-order* relation on $IFS(X)$ when they involved t-representable intuitionistic fuzzy Frank t-norms for defining the generalized intersection between two intuitionistic fuzzy sets.

- As far as open problems related to this area are concerned, we can surely restate all the points that are previously stated under the conclusion of chapter 5 with the role of similarity interchanged with inclusion.

- Moreover, we feel that the intuitionistic fuzzy similarity and inclusion measure presented in chapter 5 and chapter 6 of this dissertation can be reworked for interval valued fuzzy sets which in itself is a new direction and definitely requires more research.

(5). Chapter 7 of this thesis was reserved for the study of the applications of these new defined measures in different fields of scientific and social domains. Like in section 7.1, we developed a new technique to cater the MCDM problem. The new technique involved intuitionistic fuzzy inclusion measure from section 5.1 and was called TOPIIS (Technique to Order Preference by Inclusion of Ideal Solution). Likewise, to the famous TOPSIS (The Technique for Order Preferences by Similarity of Ideal Solution) method, our new technique also formed an ideal and a non ideal solution that could be regarded as the best and worst solutions respectively not existing in the set of alternatives to be judged. It then generates a degree of inclusion of each of the alternative in the non ideal solution and a degree of inclusion of ideal solution in each of the alternative. The best alternative is ranked on the basis of its minimum inclusion in non ideal solution and the maximum inclusion of ideal solution in it. The technique was elaborated by the help of a case study which was originally conducted in [167]. The new results were supportive of the study done in [167]. However, our new technique used general fuzzy conjunction and disjunctions that could be modeled by a different family of t-norms and their dual conorms. Moreover, a variety of fuzzy measures of intuitionistic fuzzy sets could provide multiple choices for the degree of inclusion of intuitionistic fuzzy sets used in the technique. This provided a more broader and flexible framework to the decision maker who had to deal with any type of data in a complex multiple attribute decision making environment of actual economic management and similar fields.

In section 7.2, we utilized the weighted average cardinality measures introduced in chapter 3 to formulate a scalar valued relative sigma count for intuitionistic fuzzy sets which was then employed in formulating a new ranking technique for intuitionistic fuzzy sets and defining the intuitionistic fuzzy quantifiers. The case studies conducted in the fields of medical diagnosis and organizational management elaborated the simplicity yet effectiveness of the new ranking technique.

In the last section we demonstrated an effective and easy method of diagnose of a disease by the pattern recognition technique of the members of the new parametric family of cardinality based intuitionistic fuzzy similarity measures proposed in Section 5.2. We constructed six tables demonstrating the results of 18 cardinality based intuitionistic fuzzy similarity measures. All of the similarity measures diagnosed to the patient the same disease which in itself is a proof of their efficiency.

- Therefore, developing a further research on similar, to the main subject of this thesis, results, open problems will be generated by studying applications of new results to the real life world, as presented in the chapter 7.

Chapter 9

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