Valuations and Metric Dimensions on Graphs

By

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DEDICATION

To my Parents,

Wife and Daughters

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All praises to ALLAH Almighty who created the universe and knows whatever is in it hidden or evident, who bestowed upon me the intellectual ability and wisdom to search for it's secrets and whose blessings flourished my thoughts and thrived my ambitions and enabled me to contribute sincerely to the sacred wealth of knowledge. Special praise and regards to his last messenger Muhammad (PBUH) who is a tower of guidance and knowledge for humanity.

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ABSTRACT

Valuations and Metric Dimensions on Graphs

Labeling of graph is a beautiful and rich technique of Mathematics. It has extensive usage in applications such as data transmission, x-ray, circuit design, astronomy, communication links and coding theory. Labeling of graph is defined as a correspondence from graph elements to non-negative numbers. A labeling of type $(1,1,1)$ assigns labels from the set $\{1,2,3,\ldots,v+e+f\}$ to the vertices, edges and faces of a plane graph G in such a way that each vertex, edge and face receives exactly one label and each number is used exactly once as a label. The weight of a face under a labeling of type $(1,1,1)$ is the sum of labels carried by that face and all the edges and vertices surrounding it. A labeling of type $(1,1,1)$ of a plane graph G is called *d*-antimagic if for every positive integer s the set of weights of all s-sided faces is $\{a, a+d, a+2d, \ldots, a+(f-1)d\}$ for some integer a and $d \geq 0$, where f is the number of s-sided faces. In particular when $d=0$ then the *d*-antimagic labeling of type $(1,1,1)$ is called magic labeling of type $(1,1,1)$. Moreover, if the vertices are labeled by the smallest possible number. It is known as super magic and anti-magic accordingly.

Furthermore, let G be a connected graph and $d(x,y)$ be the distance between the vertices x and y in *G*. A set of vertices W resolves a graph *G* if every vertex is uniquely determined by its vector of distances to the vertices in *W*. A metric dimension of *G* is the minimum cardinality of a resolving set of G and is denoted by $\dim(G)$.

This dissertation studies about the construction of super face anti-magic labeling of type $(1,1,1)$ for disjoint union of toroidal fullerenes, magic and anti-magic labeling of type $(1,1,1)$ for union of generalized friendship graphs. Metric dimension of Jahangir related graphs, generalized prism related graphs and graphs obtained by rooted product has been also calculated.

List of Publications from Thesis

[1] Imran, S., Hussain, M., Siddiqui, M. K., & Numan, M. (2017). Super face d-antimagic labeling for disjoint union of toroidal fullerenes. *Journal of Mathematical Chemistry*, **55(3)**, 849-863.

[2] Imran, S., Tabraiz, A., Bilal, H., M., Cheema, I. Z., Saleem, Z., & Hussain, M. (2018). Computing the metric dimension of Gear graphs, *Symmetry*, 10(6), 209-220.

[3] Imran, S., & Hussain, M. (2016). On Constant Metric Dimension of Generalized Prism Related Graphs. *Sci. Int*., **28(5)**, 4229-4332.

[4] Imran, S., Hussain, M., & Siddiqui, M. K., On metric dimension of graphs obtained by rooted product, *IET Control Theory & Application*, (submitted).

[5] Imran, S., Hussain, M., Tabraiz, A., Union of generalized friendship graph with(a,b,c) labelings, *Electronic Journal of Graph Theory and Applications*, (submitted).

Further Work During Ph.D.

[1] Imran, S., Hussain, M., & Siddiqui, M. K., Super d-antimagic labeling of subdivision of generalized prism, *Miskolc Math. Notes*, (submitted).

[2] Imran, S., Hussain, M., & Tabraiz, A., Face d-antimagic labeling of type (1, 1, 1) for union of subdivided prism, *Optoelectronics and Advanced Materials-Rapid Communications*, (Submitted).

[3] Imran, S., & Hussain, M., Upper bound for the metric dimension of cellulose network, *Nanoscience and Nanotechnology Letters*, (Submitted).

[4] Tabraiz, A., Imran, S., & Hussain, M., Edge magic deficiency of subdivided trees, *Sci. Int*., (in press).

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Chapter 1

Introduction

In recent years, Graph Theory is used in many fields of science that is why it is gaining more popularity. It is being used in biochemistry, electrical engineering, mathematics, computer science, technology, operation research and many other fields. Its vital role is in the areas of chemical bonds in chemistry, a number of algorithms which are important in computer applications, some arrangements of cell samples in biochemistry, for managing train and airline networks in order to connect large number of cities and ports, for reducing traffic in rail, air and road networks, to form migration networks of species in biology and disease spread models in epidemiology.

Graphs are also being utilized in other branches of sciences like mathematics and statistics. The best planning and schedule of large and complex projects in operation research is done by suitable networks. Graph theory plays a very important role by using graph labeling so that graph could be understood easily. Part played by networks representation cannot be under-estimated in different fields like communication network, computer science, data base management and circuit design. Graph theory helps in database management as tables can be taken as nodes and then connections can be drawn between the nodes and relationships between the nodes can be evaluated. On the other hand in pure mathematics to prove fundamental results the vital combinatorial methods found in graph theory have also been used. Thus, graph theory has helped in solving a wide variety of problems in a large number of different fields of life.

1.1 Practical uses of graph in our life

Internet can be considered as an application of graph theory all around us. We can consider each page on internet as a vertex of a graph with the links between pages as edges among the adjacent vertices. The resulting graph is very large and complex. Indexing of these pages was a huge task for search engines because a single search query on the internet can result in thousands of web pages. A computer network in which a single computer is connected to a large number of computers directly, can be considered as a

bipartite graph. On the other hand, a complete graph is formed when each computer is directly connected to all the computers in the network. The problem of sorting these pages according to authenticity and usefulness is highly desirable. As an example, consider the pages return after querying ”Islamabad” on a search engines. The resulting pages may be related to shops, parks, offices. The information may arise from a variety of different sources and may not be from official pages. One possible way of solving this problem was provided by Google. Google considers any page authentic if a large number of pages have links to this page. Similarly, a page that is visited a large number of times, is also ranked higher. These pages can then be arranged in descending order with respect to their rank to provide the most suitable page on top.
In a similar manner, Facebook also presents us with an application of Graph Theory. In this graph, the vertices are the facebook users and an the friendship between any two users is the edge. The working of social networking websites can be improved greatly using graph theory models. Another example of a large graph is Google maps which includes a large number of edges and vertices. It is impressive that Google algorithms can make calculations in such short time despite the large number of vertices and edges. The processing power required to make millions of these calculations simultaneously must be very high. Time table management in educational institutions is also a very big task as no student or teacher cannot be assigned more than one lecture at any given time. Similarly, same student cannot have two exams at the same time and a teacher cannot be assigned invigilation duty at two different places at the same time. Graph theory can facilitate in this painstaking job due to its utility and adaptiveness. In a nut shell, Graph Theory has its unique impact in different walks of life, which is progressing day by day.

1.2 Artificial neural networks and graph theory

Artificial neural networks have gained considerable attention in solving various problems of graph theory. Graph theory also shows promise in many areas of neural networks, for example design of algorithms for artificial neural networks, pattern classification problems on discrete feedforward networks and separability of Boolean functions. The conjectures of graph theory can also be justified with the help of artificial neural networks. Artificial neural networks have solved many issues of graph theory like graph isomorphism problem, graph vertex coverage problems, graph coloring problem, maximum clique problem and maximum independent set problem. Graph coloring is one of the fundamental problems of graph theory which has further applications in different fields. The algorithms used in ANNs can help simplify energy function, speed up convergence of networks and lessen local minimum value points. Mathematics and engineering researchers give a lot of importance to graph isomorphism problem. In this respect ANNs can help to improve and simplify energy functions in order to reduce complexity and boost convergent speed of networks. Researchers in graph theory convert their graph into artificial neural networks to solve different problems.

1.3 Labeling and metric dimension of graphs with their applications

Graph labeling and metric dimensions can be applied to design many different problems in Graph Theory. Graph Theory relies heavily on labeling. Labeling is a technique in which different symbols are assigned to different elements of the graph. Under some circumstances, it might not be necessary to assign a unique label to each and every member of the graph and only certain elements of the graph may be labeled as per the requirement. There is a large number of applications of graph labeling in the real life. Some of these applications are discussed here. Its various applications are helpful in myriad fields of life as well as in science. Labeling can help in smooth flow of traffic in road signals networks and to avoid obstacles. It can also help solve different problems in industries and provide optimal solutions of different problems. It has been applied to make efficient internet networking systems for institutions, offices, and commercial settings. Such networks can provide fix or desirable data transfer rates for the user. It may also help to control air traffic in civil aviation. Vertex coloring or metric dimension can be used to design different circuits for network analysis or circuit analysis and to find shortest path between locations. Graph theory has been used in model flow charts to solve problems in engineering. Mobile network technology for commercial and academic purposes has also benefited from graph theory by allowing effective flow of traffic signals. Briefly, it can be concluded that graph labeling is a versatile field. I have discussed some new results of d-antimagic labeling of type (1,1,1) and metric dimensions of different graphs.

Consider a planar graph H which is finite, simple and undirected. "A graph labeling or valuation is a correspondence from graph elements to numbers". "If we assign distinct labels to all the elements(vertices, edges and faces) of a planar graph H from $\{1,2,3,...|V|+|E|+|F|\}$, it is known as labeling of type $(1,1,1)$". "Face weight is a number that can be obtained by adding the labels of vertices and edges surrounding that face as well as the label assigned to the face itself". "Labeling of type $(1,1,1)$ is known as d- antimagic if an arithmetic progression $b_r, b_r+d, b_r+2d, ..., b_r+(F_r-1)d$ is obtained from weights of equal sided faces (say r sided), where F_r represents the number of faces having r sides". "Magic labeling of type $(1,1,1)$ is a labeling in which weight of all r sided faces is same but different weights for different values of r are allowed". "If vertices take least possible values from the set $\{1,2,3,...|V|+|E|+|F|\}$, edges and faces receive the remaining values then that labeling is known as super d-antimagic labeling of type $(1,1,1)$". "For planar graphs Lih introduce the idea of magic labeling" in [41]. Later on, Lih's magic labeling was extended as d-antimagic labeling of plane graphs in [13].

Now we are going to discuss metric dimension of graph which is useful on commercial levels in biotechnology, chemistry, commercial geography etc. Through metric dimension, not only investment but also time can be saved in this rapid and busy world of industrialism and commercialism. As it is the age of robots, the labor can be saved through applying metric dimension methodology. A chemical compound can be represented by more than one suggested structures but only one of them expresses the physical and chemical properties of compounds, is acceptable. Chemical compounds are represented using mathematical expressions by chemists to distinctly identify different compounds. These representations can be investigated by metric dimension. Metric dimensions can be elaborated by the following fundamental concepts." Suppose G is a connected graph, the length of path between two vertices p and q represented by $d(p,q)$ is known as distance between them when it is shortest". For a connected graph G the distance $d(x,y)$ between two vertices of G satisfies all the axioms of a metric as follows:

1. $d(x,y) \geq 0 \; \forall x,y \in V(G)$.
2. $d(x,y) = 0$ if and only if $x = y$
3. $d(x,y) = d(y,x) \; \forall x,y \in V(G)$
4. $d(x,z) = d(x,y) + d(y,z) \; \forall x,y,z \in V(G)$

As d satisfies all the axioms of metric so $(V(G),d)$ is a metric space.

"Suppose $W = \{p_1, p_2, ...p_k\}$ be an order set of vertices of a connected graph G, $r(q/W)$ represents any fixed vertex q of G with respect to the set W, is the k-tuple $(d(q,p_1), d(q,p_2), ..., d(q,p_k))$. The set W is a resolving set for G if every two vertices of G have distinct representations". "The metric dimension of G is the minimum cardinality of the resolving set for G, denoted by $dim(G)$. A resolving set containing a minimum number of vertices is called a basis for G". For example, the set $W = \{p,q\}$ is basis for the graph G of Figure 1.1 and the unique representations for the vertices p,q,r,u and v with respect to the resolving set W are $(0,1)$, $(1,0)$, $(1,1)$, $(1,2)$ and $(2,1)$ respectively. Hence G is a 2-dimensional graph. Slater introduced the notion of metric dimension of a connected graph G in [51].

The contents of this dissertation can be conveniently divided as:

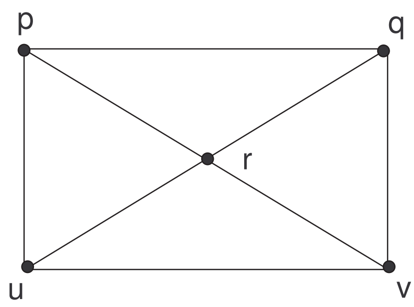

Figure 1.1: Example.

In 2nd Chapter, preliminaries and some well known graphs are studied.

In 3rd Chapter, well known results for d-antimagic labeling and metric dimension of graphs are presented.

In 4th Chapter, Super face antimagic labelings for n copies of toroidal fullerenes are discussed.

In 5th Chapter, (a, b, c) labelings for union of generalized friendship graph are studied for $a = b = c = 1$.

In 6th Chapter, metric dimension of m-level Jahangir graph $J_{2n,m}$ and generalized Jahangir graph J_{3n} is calculated.

In 7th Chapter, metric dimension of different families of graphs obtained by rooted product is presented.

In 8th Chapter, constant metric dimension of two families related to generalized prism graph is studied.

In 9th Chapter, conclusion is stated and some open problems are presented for future investigations.

Chapter 2

Basic Concepts and Terminologies

In the present chapter some fundamental notions, concepts and terminologies regarding Graph Theory are recalled, which are useful for the current study. Basic concepts regarding graph theory are elaborated in [55, 54].

2.1 Basic Definitions

2.1.1 Preliminaries

A *graph* G contains a finite non empty set $V = V(G)$ of *vertices* and a collection of two elements subsets of V denoted by $E = E(G)$ called *edges*. A graph can be represented as $G = G(V,E)$ and any vertex q of a graph can be denoted as $q \in V$. While dealing with graphs, a 2-element set $\{p,q\}$ can be written as pq or qp. "Let x and y be any two vertices of a graph G having an edge between them, then these vertices are known as adjacent vertices $p,q \in V$ are known as *incident* with that edge". "If two different edges have a common vertex then they are known as *adjacent*". "Degree of the vertex represents the number of edges connected with that vertex in the graph G". "The cardinality of the set of vertices is known as *order* of graph denoted by $|V(G)|$ and *size* of graph is the cardinality of $E(G)$ represented as $|E(G)|$". "Furthermore G is known as *finite graph* when cardinalities of both $V(G)$ and $E(G)$ are finite". "For $p,q \in V(G)$ and $p \neq q$, if there exist two or more edges connecting p and q then those edges are known as *parallel edges*". "A non distinct 2-elements subset $\{u,u\}$ of V is known as *loop*". "A graph without loops and parallel edges is known as *simple graph*". For a connected graph G there exists a path from a to b, $\forall a,b \in V(G)$ otherwise G is a disconnected graph. Let $V(G) = \{b_1,b_2,b_3,...,b_n\}$ be the vertex set of the graph G and $d_G(b_i)$ represents the degree of each b_i. By *Handshaking lemma* we have.

$$2|E(G)| = \sum_{i=1}^{n} d_G(b_i)$$

Hence sum of degrees of vertices of a graph is always even. Let G_1 and G_2 be two graphs, then G_1 will be isomorphic to G_2 if there exist an isomorphism $f : G_1 \to G_2$

such that for all $x, y \in V(G_1)$ if $xy \in E(G_1)$ then $f(x)f(y) \in E(G_2)$.

2.1.2 Some well known graphs

A graph can be expressed by many different ways but usually it is represented by a diagram. Let y and w be two vertices in a graph G with following vertex set and edge set.

$$V(G) = \{y = y_0, y_1, y_2, \ldots, y_n = w\}, \;\; E(G) = \{x_1, x_2, x_3, \ldots, x_n\}$$

Suppose $x_i = y_{i-1}y_i$ then a finite alternating sequence $y = y_0, x_1, y_1, x_2, y_2, \ldots, y_{n-1}, x_n, y_n = w$ of vertices and edges of the graph is called a walk in G. In a walk repetition of vertices and edges is also allowed. A path is a walk with all distinct vertices and is represented by P_n(see Figure 2.1). If $y_0 = y_n$ then it is said to be closed. If all the vertices in a closed walk are distinct except $y_0 = y_n$ is known as cycle and is expressed by C_n (see Figure 2.1). “Tree is a connected graph that does not contain any cycle ”(see Figure 2.1).

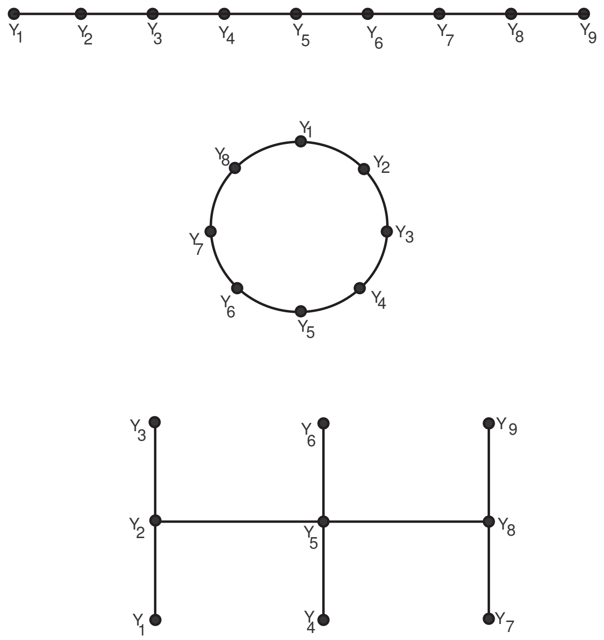

Figure 2.1: The graph of a path P_9, cycle C_8 and a tree.

If every pair of vertices are adjacent in a simple graph then that graph is known as a *complete graph*.It is represented by K_m having $\frac{m(m-1)}{2}$ edges and m vertices(see Figure 2.2). “If we divide the set of vertices $V(G)$ into two disjoint sets D_1 and D_2 according

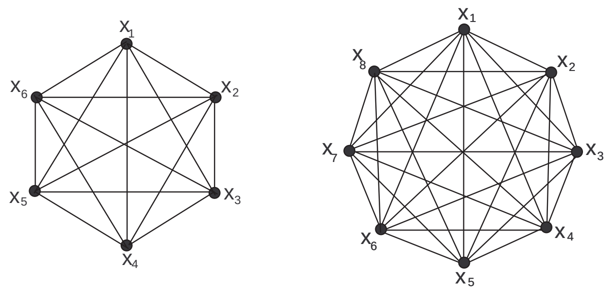

Figure 2.2: Complete graphs K_6 and K_8.

to the condition that every edge of G joins a vertex of D_1 as well as a vertex of D_2 is known as *Bipartite graph*”. It is a simple graph. To construct *complete bipartite graph* $K_{m,n}$every vertex in D_1 will be joined by every vertex in D_2 by exactly one edge where $m = |D_1|$ and $n = |D_2|$ (see Figure 2.3).

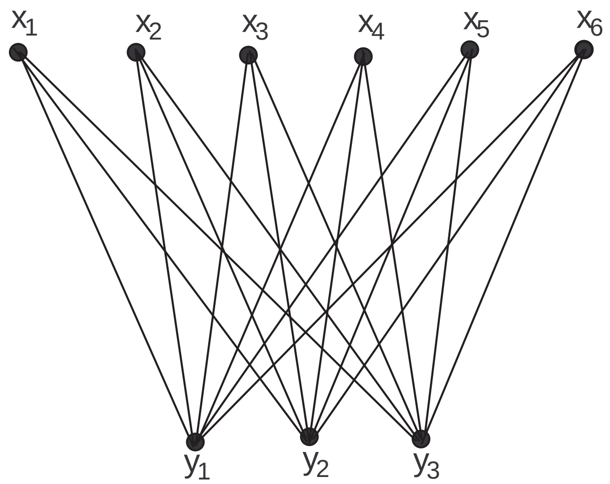

Figure 2.3: Complete bipartite graph $K_{6,3}$.

Harary graph $H_{p,q}$ can be constructed as follows (see [56]): Given $p \leq q$, place equally spaced n vertices around a circle. If p is even, form $H_{p,q}$ by making each vertex adjacent to the nearest $\frac{p}{2}$ vertices in each direction around the circle. If p is odd and q is even, form $H_{p,q}$ by making each vertex adjacent to the nearest $\frac{p-1}{2}$ vertices in each direction around the circle and to the diametrically opposite vertex. In each case $H_{p,q}$ is p-regular. When p and q are both odd, index the vertices with the integers modulo q. Construct $H_{p,q}$ from $H_{p-1,q}$ by adding the edges $r \leftrightarrow r + \frac{q-1}{2}$ for $0 \leq r \leq \frac{q-1}{2}$.(see Figure 2.4)

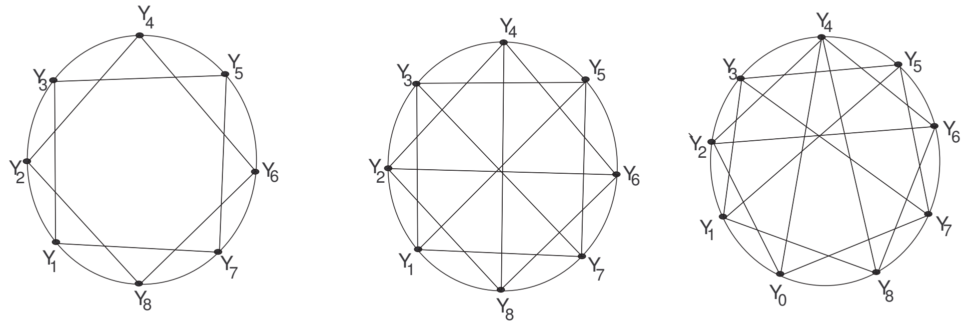

Figure 2.4: Graphs $H_{4,8}$, $H_{5,8}$ and $H_{5,9}$.

2.1.3 Graphs generated by different operations

To make a larger graph we can combine two graphs. If $K_1 = (V(K_1), E(K_1))$ and $K_2 = (V(K_2), E(K_2))$ be two graphs then their disjoint union is a graph $K_1 \cup K_2$ having $V(K_1 \cup K_2)$ and $E(K_1 \cup K_2)$ as vertex and edge set respectively. $K_1 + K_2$ is a graph obtained by the join of two graphs $K_1 = (V(K_1), E(K_1))$ and $K_2 = (V(K_2), E(K_2))$ having vertex set $V(K_1 + K_2)$ and edge set $E(K_1 + K_2) = E(K_1) \cup E(K_2) \cup \{xy; x \in V(K_1), y \in V(K_2)\}$. Wheel graph represented by $W_n = C_n + K_1$ is the join of C_n and K_1. Another example of join of two graphs is given in Figure 2.5. Let $K_1 = (V(K_1), E(K_1))$ and $K_2 = (V(K_2), E(K_2))$ be two graphs then their Cartesian product is a graph $K_1 \Box K_2$ with vertex set $V(K_1) \times V(K_2)$ and vertex (x, y) is adjacent to the vertex (w, z) if and only if y is adjacent to z and $x = w$ or x is adjacent to w and $y = z$. Prism graph represented by $C_n \Box P_2$ is the Cartesian product of C_n and P_2. Another example of Cartesian product of two graphs is given in Figure 2.6.

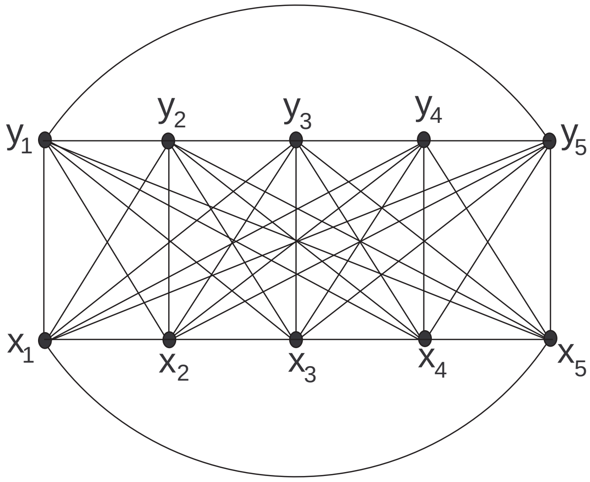

Figure 2.5: Join of two cycles $C_5 + C_5$.

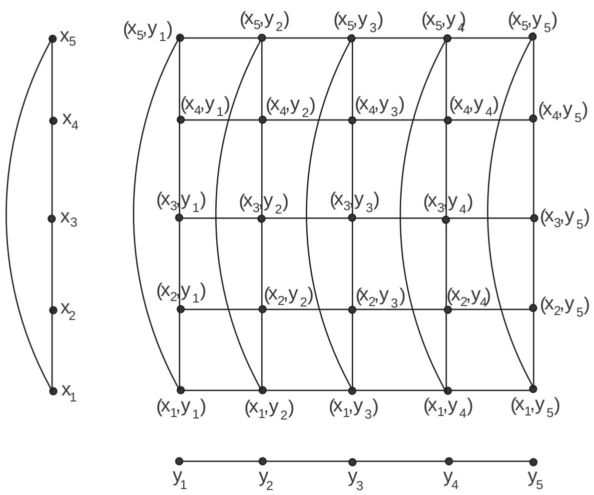

Figure 2.6: $C_5 \square P_5$.

Chapter 3

Labeling and Metric dimensions of graphs

3.1 Labelings

The concept of graph labeling was introduced after the middle of 20^{th} century. *Labeling* (or *valuation*) of graph is a correspondence from graph elements to numbers. If both the edge set and vertex set are domain of correspondence then it is known as *total* labeling. In *face labeling* symbols are assigned to faces only, *edge labeling* assign symbols to edges only. Similarly, if symbols are assigned to vertices only then it is called *vertex labeling*. Other kinds of labeling are Cordial labeling, α-labeling, graceful labeling and d-antimagic labeling. Detailed literature survey is presented in this chapter.

3.2 d-AntiMagic valuations of type (λ, μ, ν).

Consider a planar graph H which is finite, simple and undirected. “A graph labeling or valuation is a correspondence from graph elements to numbers”. “If we assign distinct labels to all the elements(vertices, edges and faces) of a planar graph H from $\{1,2,3,...|V|+|E|+|F|\}$, it is known as labeling of type $(1,1,1)$”. “Face weight is a number that can be obtained by adding the labels of vertices and edges surrounding that face as well as the label assigned to the face itself”. Labeling of type $(1,1,1)$ is known as d- antimagic if an arithmetic progression $b_r, b_r+d, b_r+2d, ..., b_r+(F_r-1)d$ is obtained from weights of equal sided faces (say r sided), where F_r represents the number of faces having r sides.“Magic labeling of type $(1,1,1)$ is a labeling in which weight of all r sided faces is same but different weights for different values of r are allowed”. “If vertices take least possible values from the set $\{1,2,3,...|V|+|E|+|F|\}$, edges and faces receive the remaining values then that labeling is known as super d-antimagic labeling of type $(1,1,1)$”.

3.2.1 Literature survey.

Notion of “magic labeling” was intoduced by Ko-Wei Lih in [41]. Magic labelings of certain planar graphs have been discussed by Katheresan and Gokulakrishnan [36]. Face antimagic labeling for disjoint union of antiprism has been determined by Bača in [16]. Super face antimagic labeling of subdivided wheel graph has been formulated by Imran et al. [35]. Toroidal fullerenes graphs has been valuated by Bača in [15]. Labeling of type $(1,1,1)$ for disjoint union of anti prisms is given in [11]. Face antimagic labeling was defined by Bača in [10]. Face antimagic labeling of disjoint union of prisms is given in [4]. Super d-antimagic labeling of type $(1,1,1)$ of gear graph is given in [48]. In [1] evaluation of a subdivision of the ladder graph has been studied. Super magic and antimagic labelings of disjoint union of plane graphs have been formulated by Ahmad et al. [2]. Magic Labelings of type (a,b,c) of families of wheels have been discussed in [5]. Group distance magic labeling of direct product of graphs was determined by Anholcer et al.[7]. In [8] odd sum labeling of graphs obtained by duplicating any edge of some graphs has been studied. Antimagic labelings for some families of regular graphs have been constructed by Bača et al. [14]. In [33] super d-antimagic labeling of subdivided kC_5 has been formulated. Antimagic labeling of regular graphs was determined by Chang et.al [19]. In [35] Super d-antimagic labeling of uniform subdivision of wheel has been studied. Antimagic labelling of Cartesian product of graphs has been studied by Liang et al. [40]. In [44] antimagic labeling of generalized sausage graphs has been formulated. Supermagic labeling of finite copies of Cartesian product of cycles has been discussed by Singhun et al. [49]. In [52] Magic and antimagic total labeling on subdivision of grid graphs has been studied. A general survey of graph labelings has been discussed in [28].

3.3 Distances in Graphs

In facility location problem, our target is the selection of suitable sites subject to some conditions, for example, to determine the location of an emergency facility such as rescue point, fire stations and hospitals. It is most important requirement to minimize the response time between emergency point and the facilities. For this purpose we need to study the concept of centrality. In this section, center of a graph will be defined with the help of distance between vertices of a graph and some relevant definitions will be given.

"Let G be a connected graph then length of shortest path between two vertices p and

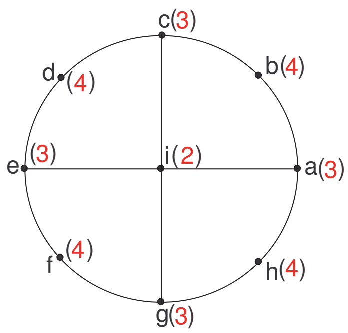

Figure 3.1: Eccentricities of vertices.

q is the distance $d(p,q)$ between them". "The maximum value of $d(p,q)$ is known as eccentricity of the vertex p where q varies through all the vertices of G". It is denoted by $e(p)$. Thus $e(p) = max\{d(p,q); \forall\ q \in G\}$. If a vertex p has minimum eccentricity then the eccentricity of this vertex p is called radius of the graph G denoted by $r(G)$. Similarly if a vertex q has maximum eccentricity then the eccentricity of this vertex q is called diameter of the graph G denoted by $diam(G)$. "A vertex p is known as central vertex if it has minimum eccentricity". In other words when $e(p) = r(G)$ then p is called central vertex. "The set of all central vertices is known as center of G denoted by $C(G)$". These concepts are explained in Figure 3.1 where values shown in parenthesis represent the eccentricity of the vertices.

3.4 Resolving Set and Metric Dimensions

Assume that a building has five offices as shown in Figure 3.2. Each office is denoted by X_1, X_2, X_3, X_4 and X_5 such that X_1 is surrounded by the other four offices. A fire sensor S_1 is placed in one of the peripheral offices, say X_5. The fire sensor is capable of detecting the distance of office in case of a fire incident, so that it shows a distance 2 if fire breaks out in X_3 and a distance 0 if fire breaks out in X_5. However, in case of a fire in rooms X_1, X_2 and X_4, the sensor detects a distance 1 in each case, which is not sufficient to determine the location of fire. To overcome this problem, we need to place another sensor S_2 in one of the other offices, say X_2. Now, in case of a fire in one of the offices, each sensor displays a distance. Both readings can be decoded to determine the exact location of the fire with the help of understanding of metric dimension. This problem can be shown in the form of a graph given in Figure 3.3 in which nods represent the offices and two nods are adjacent if two offices are adjacent.

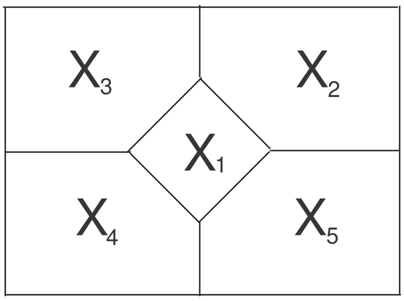

Figure 3.2: A facility consisting of five offices.

"Let G be a connected graph and $W = \{p_1, p_2, ...p_k\}$ be an order set of vertices of G, $r(q/W)$ represents any fixed vertex q of G with respect to the set W is the k-tuple$(d(q, p_1), d(q, p_2), ..., d(q, p_k))$". "The set W is a resolving set for G if every two vertices of G have distinct representations". "The metric dimension of G is the minimum cardinality of the resolving set for G". "A resolving set containing a minimum number of vertices is called a basis for G".

The set $W = \{x_4, x_5\}$ is basis for the graph G of Figure 3.3 and the unique representa-

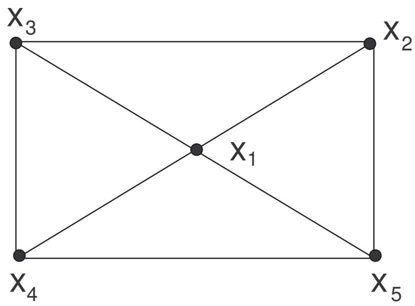

Figure 3.3: Graph representing five offices.

tions for the vertices x_1, x_2, x_3, x_4 and x_5 with respect to the resolving set W are $(1,1)$, $(2,1)$, $(1,2)$, $(0,1)$ and $(1,0)$ respectively.

3.4.1 Some Well Known Results on Metric Dimensions

Slater introduced the notion of metric dimension of a connected graph G in [51] and used locating set for what was called resolving set in [17] and [20]. Harary and Melter studied independently metric dimension of a connected graph in [31]. Zhang, Chartrand and Poisson found formula of metric dimensions of wheel graphs in [17]. In [32] Approximation complexity of metric dimension problem has been studied. Metric dimension of a graph from primary subgraphs was determined by Kuziak et al. [39]. Cruz et al.[23] calculated the simultaneous metric dimension of different families of graphs. In [24] complexity of metric dimension on planar graphs has been studied. Metric dimension of graphs obtained by corona product has been determined by Yero in [58]. Saputro et al. [45] “studied the lexicographic product of graphs in the sense of metric dimension”. In [30] optimal approximability results for computing the strong metric dimension has been studied. Eroh et al. [26] determined a comparison between the metric dimension and zero forcing number of trees and unicyclic graphs. In [27] identification, location-domination and metric dimension on interval and permutation graphs. II. algorithms and complexity has been studied. Chartrand studied metric dimension and introduced the idea of sharp lower bound and greatest degree in [21]. Metric dimension of Mobious Ladder has been studied by Ali in [3]. Ba$\check{c}$a calculated the metric dimension of regular bipartite graphs in [12]. In [34] metric dimension of circulant graphs has been studied. Minimum metric dimension of honeycomb networks has been studied by Manuel in [42]. In [43] Melter and Tomescu studied metric basis in digital geometry, Computer vision, Graphics, and Image processing. Metric dimension of Jahangir graph has been studied by Tomescu in [53].

Chapter 4

Super face d-antimagic labeling for disjoint union of toroidal fullerenes

4.1 Introduction

The classical fullerenes in its trivalent molecular graph has only hexagonal and twelve pentagonal faces entrenched on spherical surface. Deza reflected in [25] that four surface namely torus, sphere, projective plane and Klein bottle are possible by the extension of fulleren to other closed surface. The fulleren of toroidal and Klein bottles has only hexagonal surface regarded as tesselation. A toroidal fullerene is a graph trenched on the surface of torus (See Figure 4.1) in which each vertex has degree three and each face is hexagonal as well as it is bipartite graph, detailed study is given in [22, 37, 50]. By applying different techniques like transfer matrix, a few work in the enumeration of perfect matching of toroidal polyhexes have been studied in [38, 46].

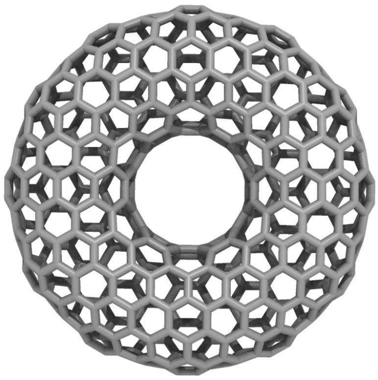

Figure 4.1: Toroidal polyhex.

More accurately, "consider a regular hexagonal lattice represented by L and an $m \times n$ quadrilateral section cut from the regular hexagonal lattice L denoted by P_m^n. To form a cylinder two lateral sides of P_m^n will be detected and then finally the upper and lower sides of P_m^n at their respective points will be identified, consequently a toroidal polyhex $\mathbb{H}_m^n$ will be obtained". More affectedly, Figure 4.2 is the k^{th} copy (where $1 \leq k \leq t$) from disjoint union of t copies of a toroidal polyhex $\mathbb{H}_m^n$ having $2mnt$ vertices, $3mnt$ edges and mnt faces.

This chapter observes the reality of $(1,1,1)$ labeling for a family of cubic graphs

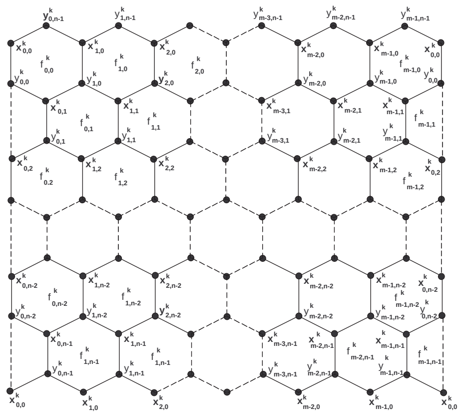

Figure 4.2: k^{th} copy in t copies of a regular hexagonal lattice $\mathbb{P}_m^n$.

$\mathbb{G}$ entrenched on the surface of a torus such that every face form a hexagon. It is shown that weights of all hexagonal faces of disjoint union of toroidal fullerenes form an arithmetic sequence with common difference $d \in \{1,3,5\}$.

4.2 Necessary conditions on bounds

In this section, bounds for the value of common difference d for super face d-antimagic labeling of a disjoint union of t copies of a toroidal fullerene $\mathbb{H}_m^n$ have been calculated.

Theorem 4.2.1. *For $t \geq 1$ copies of a toroidal fullerene $\mathbb{H}_m^n$, super d-antimagic labeling of type $(1,1,1)$ cannot be calculated for $d \geq 40$ where n is even and $m,n \geq 2$.*

Proof.

Let $t\mathbb{H}_m^n$ be a toroidal fullerene having $2mnt$, $3mnt$ and mnt vertices edges and faces respectively. Suppose $\mu : V(t\mathbb{H}_m^n) \cup E(t\mathbb{H}_m^n) \cup F(t\mathbb{H}_m^n) \to \{1,2,\ldots,6mnt\}$ is a bijection and $t\mathbb{H}_m^n$ has super face antimagic then weights of faces form an arithmetic sequence $b, b+d, b+2d, \ldots, b+(mnt-1)d$. The weight of hexagonal face will give smallest possible value when six vertex labels viz $1,2,\ldots,6$ of smallest possible values are added and smallest possible values assigned to six edges and one face from the set $\{2mnt+$

$1, 2mnt+2, \dots, 6mnt\}$. Thus the following inequality holds

$$14mnt+49 = \sum_{r=1}^{6} r + \sum_{s=1}^{7} (2mnt+s) \leq b \tag{4.2.1}$$

Every face label is used one time, each label assigned to edge will be used two times and each label assigned to vertex will be used three times to determine weights of all the faces. In this case, smallest possible labels $1, 2, \dots, 2mnt$ will be assigned to vertices. From the set $\{2mnt+1, 2mnt+2, \dots, 6mnt\}$ smallest $3mnt$ labels will be assigned to edges and remaining labels will be assigned to faces, on the other hand, from the set $\{2mnt+1, 2mnt+2, \dots, 6mnt\}$ smallest $3mnt$ labels will be assigned to faces and largest $3mnt$ labels will be assigned to edges or anything in between. So we get

$$\sum_{f\in F} \mu(f) + 2\sum_{e\in E} \mu(e) + 3\sum_{v\in V} \mu(v) \geq \frac{1}{2}(65m^2n^2t^2 + 13mnt) \tag{4.2.2}$$

$$\Rightarrow \frac{1}{2}(71m^2n^2t^2 + 13mnt) \geq \frac{1}{2}(65m^2n^2t^2 + 13mnt) \tag{4.2.3}$$

When all the face-weights are summed up we get

$$b + (d+b) + \cdots + (d(mnt-1)+b) = mntb + \frac{(-1+mnt)mntd}{2} \tag{4.2.4}$$

Since $14mnt+49$ is the smallest possible face weight, thus from (4.2.3) and (4.2.4) the following inequality is obtained

$$(mnt-1)d \leq 71mnt + 13 - 2b \tag{4.2.5}$$

and

$$d \leq 43 - \frac{42}{mnt-1} < 43 \tag{4.2.6}$$

Conversely, to obtain face weight which is maximum the six largest possible vertex labels, namely $2mnt-5, 2mnt-4, \dots, 2mnt$ together with six edges and one face labels which take maximum values from the set $\{2mnt+1, 2mnt+2, \ldots, 6mnt\}$ will be added.

$$\Rightarrow b+(mnt-1)d \leq 54mnt-36 \tag{4.2.7}$$

and

$$d \leq 40-\frac{45}{mnt-1} < 40 \tag{4.2.8}$$

Hence the required result has been achieved. □

4.3 On some results about super face d-antimagic labelings of $t\mathbb{H}_m^n$

As $t\mathbb{H}_m^n$ is the cubic graph which is also 2-colorable, therefore there exist a perfect matching 1-factor and a 2-factor (t copies of n cycles on $2m$ vertices each). $x_{i,j}^k y_{i,j}^k$ will represent edges in 1-factor, where $0 \leq j \leq n-1$, $1 \leq k \leq t$, and $0 \leq i \leq m-1$. Let j is even and $0 \leq j \leq n-2$ then $2m$-cycles will be represented by

$$x_{0,j}^k y_{0,j-1}^k \ x_{1,j}^k y_{1,j-1}^k \ x_{2,j}^k y_{2,j-1}^k \cdots y_{m-2,j-1}^k x_{m-1,j}^k \ y_{m-1,j-1}^k x_{0,j}^k$$

and

$$y_{0,j}^k x_{0,j+1}^k \ y_{1,j}^k x_{1,j+1}^k \ y_{2,j}^k x_{2,j+1}^k \cdots x_{m-2,j+1}^k y_{m-1,j}^k \ x_{m-1,j+1}^k y_{0,j}^k$$

in 2-factor.

Let $f_{i,j}^k$, $t \geq k \geq 1$, $n-1 \geq j \geq 0$ and $m-1 \geq i \geq 0$ be hexagonal face of $t\mathbb{H}_m^n$, see Figure 4.2. First we create the correspondence of vertices $\psi : V(t\mathbb{H}_m^n) \rightarrow \{1, 2, \ldots, 2mnt\}$ in such a way

$\psi(x_{i,j}^k) = mtj + ti + k$,

$\psi(y_{i,j}^k) = (n+j)mt + ti + k$ where $t \geq k \geq 1$, $n-1 \geq j \geq 0$ and $m-1 \geq i \geq 0$.

Edges of $2m$-cycles will be labeled by the successive values $3mnt+1, 3mnt+2, \ldots, 5mnt$ as follows:

$$\varphi(x^k_{i,j}y^k_{i,j-1}) = (5n-j)mt - ti + 1 - k,\ if\ \ j\ \ is\ \ even, \tag{4.3.1}$$

$$\varphi(x^k_{i,j}y^k_{i+1,j-1}) = (5n-j)mt - ti + 1 - k,\ if\ \ j\ \ is\ \ odd, \tag{4.3.2}$$

$$\varphi(y^k_{i,j}x^k_{i+1,j+1}) = (4n-j)mt - ti + 1 - k,\ if\ \ j\ \ is\ \ odd, \tag{4.3.3}$$

$$\varphi(y^k_{i,j}x^k_{i,j+1}) = (4n-j)mt - ti + 1 - k,\ if\ \ j\ \ is\ \ even. \tag{4.3.4}$$

Where $t \geq k \geq 1$, $n-1 \geq j \geq 0$ and $m-1 \geq i \geq 0$, i and j are taken over modulo m and n respectively. The following lemmas can be proved by the aforementioned labeling of vertices and edges.

Lemma 4.3.1. *For j odd and $1 \leq j \leq n-1$ the partial weight of the face $f^k_{i,j}$ is given as*

$$\begin{aligned} w(f^k_{i,j}) &= \psi(x^k_{i,j}) + \varphi(x^k_{i,j}y^k_{i+1,j-1}) + \psi(y^k_{i+1,j-1}) + \varphi(y^k_{i+1,j-1}x^k_{i+1,j}) \\ &+ \psi(x^k_{i+1,j}) + \psi(y^k_{i,j}) + \varphi(y^k_{i,j}x^k_{i+1,j+1}) + \psi(x^k_{i+1,j+1}) \\ &+ \varphi(x^k_{i+1,j+1}y^k_{i+1,j}) + \psi(y^k_{i+1,j}) \end{aligned}$$

where $1 \leq k \leq t$ and $0 \leq i \leq m-1$. The partial weight for $t \geq 1$ and $m,n \geq 2$, where n is even is given as

$$w(f^k_{i,j}) = \begin{cases} 2ti + 2k + 4 + 2t + (21n + 2j)tm, & 0 \leq i \leq m-2 \\ 2k + 4 + (21n + 2j)tm, & i = m-1. \end{cases} \tag{4.3.5}$$

Proof.

For odd value of j, weight of the face $f^k_{i,j}$ by direct computation can be obtained as follows: when $0 \leq i \leq m-2$

$$\begin{aligned} w(f_{i,j}^k) &= mtj+ti+k+(5n-j)tm-ti+1-k+(n+j-1)tm+ti \\ &+ t(i+1)+k+(4n-j)tm-ti+1-k+(4n-j)tm+mtj+t \\ &- ti-t+1-k+(j+n)tm+ti+k+(5n-j)tm-t(i+1) \\ &+ 1-k+mtj+ti+t+k+(j+1)tm+t(i+1)+k+mnt+k \\ &= (21n+2j)tm+2ti+2k+4+2t \end{aligned}$$

and

$$\begin{aligned} w(f_{m-1,j}^k) &= mtj+t(m-1)+k+(5n-j)tm-t(m-1)+1-k+k+mnt \\ &+ (n+j-1)tm+mt+k+(5n-j-1)tm+1-k+mtj+mt \\ &+ (n+j)tm+mt+k+mtj+mt+mt+k+mtj+tm-t+k \\ &+ (4n-j-1)tm+t+1-k+4mnt-mtj-mt+1-k \\ &= (21n+2j)tm+2k+4. \end{aligned}$$

□

Lemma 4.3.2. *For j even and $0 \leq j \leq n-2$ partial weight of the face $f_{i,j}^k$ is given as*

$$\begin{aligned} w(f_{i,j}^k) &= \psi(x_{i,j}^k)+\varphi(x_{i,j}^k y_{i,j-1}^k)+\psi(y_{i,j-1}^k)+\varphi(y_{i,j-1}^k x_{i+1,j}^k) \\ &+ \psi(x_{i+1,j}^k)+\psi(y_{i,j}^k)+\varphi(y_{i,j}^k x_{i,j+1}^k)+\psi(x_{i,j+1}^k) \\ &+ \varphi(x_{i,j+1}^k y_{i+1,j}^k)+\psi(y_{i+1,j}^k) \end{aligned}$$

where $0 \leq i \leq m-1$ and $1 \leq k \leq t$. The partial weight for $t \geq 1$ and $m,n \geq 2$, where *n is even is given as*

$$w(f_{i,j}^{k}) = \begin{cases} (21n+2j)tm+2ti+2k+4+2t, & 0 \le i \le m-2 \\ (21n+2j)tm+2k+4, & i = m-1. \end{cases} \tag{4.3.6}$$

Proof. For even value of j one can easily calculate that partial weight of the face $f_{i,j}^{k}$ take value as follows

$$\begin{aligned} w(f_{i,j}^{k}) &= mtj+ti+k+(5n-j)tm-ti+1-k+(n+j-1)tm+ti+k \\ &+ (4n-j)tm-ti+1-k+(4n-j)tm+mt-it+1-k+1-k \\ &+ mtj+ti+t+k+(j+n)tm+ti+k+(5n-j)tm-mt-ti \\ &+ mtj+mt+ti+k+(j+n)tm+t(i+1)+k \\ &= (21n+2j)tm+2ti+2k+4+2t \end{aligned}$$

for $m-2 \ge i \ge 0$ and

$$\begin{aligned} w(f_{m-1,j}^{k}) &= mtj+k-(m-1)t+1-k \\ &+ k+tm(5n-j-1)+1-k+k+mnt \\ &+ (n+j)tm+k+mtj+mt+k+mtj+tm-t+tm(n+j-1)+k \\ &+ (4n-j+1)tm+tm(5n-j)+t+1-k+4mnt-mtj-mt+1-k \\ &= (21n+2j)tm+2k+4. \end{aligned}$$

□

Hence Lemmas 4.3.1 and 4.3.2 give

Theorem 4.3.3. *"Super d- antimagic labeling of type $(1,1,1)$" for the graph $t\mathbb{H}_m^n$ can be formulated for two different values of d that is $d \in \{1,3\}$ where $t \ge 1$ and $m,n \ge 2$, n is even.*

Proof.

Vertices of $t\mathbb{H}_m^n$ will be labeled by the mapping ψ and labels of the edges of $2m$-cycles will be assigned by the correspondence φ defined above. We can conclud that weights of faces $f_{i,j}^k$, $0 \leq i \leq m-1$, $0 \leq j \leq n-1$, $1 \leq k \leq t$ form an arithmetic progression with common difference 2 that is the progression $21mnt+6, 21mnt+8, \ldots, 23mnt+2, 23mnt+4$ by using Lemmas 4.3.2 and 4.3.1. Now we discuss two different cases.

Case 1. $d = 1$. If 1-factor's edges take values from the set $\{2mnt+1, 2mnt+2, \ldots, 3mnt\}$ in such a way that

$$\varphi(x_{i,j}^k y_{i,j}^k) = (3n-j)tm - ti + 1 - k \;\; for \;\; 0 \leq j \leq n-1, \;\; 1 \leq k \leq t, \;\; 0 \leq i \leq m-1 \tag{4.3.7}$$

then for $n-1 \geq j \geq 0, t \geq k \geq 1$, $m-2 \geq i \geq 0$

$w'(f_{i,j}^k) = \varphi(x_{i,j}^k y_{i,j}^k) + \varphi(x_{i+1,j}^k y_{i+1,j}^k) + w(f_{i,j}^k) = (21n+2j)tm + 2ti + 2k + 2t + 4 + (3n-j)tm - ti + 1 - k + (3n-j)tm - (i+1) + (1-k) = 27mnt + 6 + t$

and for $n-1 \geq j \geq 0$

$w'(f_{m-1,j}^k) = \varphi(x_{m-1,j}^k y_{m-1,j}^k) + \varphi(x_{0,j}^k y_{0,j}^k) + w(f_{m-1,j}^k) = (21n+2j)tm + 2k + 4 + tm(-j+3n) - (-1+m)t + 1 - k + tm(-j+3n) + 1 - k = tm(-1+27n) + 6 + t$

If the faces $f_{i,j}^k$, $0 \leq j \leq n-1$, $1 \leq k \leq t$, $0 \leq i \leq m-1$, receive labels from the set $\{5mnt+1, 5mnt+2, \ldots, 6mnt\}$ as $\xi(f_{i,j}^k) = (5n+j)tm + ti + k$ then the face weight for $0 \leq j \leq n-1$, $1 \leq k \leq t$, $0 \leq i \leq m-2$ is given as

$wt(f_{i,j}^k) = w'(f_{i,j}^k) + \xi(f_{i,j}^k) = 27mnt + 6 + t + (5n+j)tm + ti + k = (32n+j)tm + t(i+1) + k + 6$,

and for $n-1 \geq j \geq 0$

$wt(f_{m-1,j}^k) = \xi(f_{m-1,j}^k) + w'(f_{m-1,j}^k) = (27n-1)tm + tm(j+5n) + 6 + t + (-1+m)t + k = 6 + k + mt(j+32n)$,

Thus an arithmetic sequence can be obtained from weights of the faces starting from $32mnt+7$ and the last term $33mnt + t(m-1) + 6$ having common difference 1. This complete the proof of case 1.

Case 2. when $d = 3$. Labels of edges in 1-factor will be completed by the labeling

φ as:

$$\varphi(x^k_{i,j}y^k_{i,j}) = (2n+j)tm+ti+k \quad 0 \le j \le n-1, \quad 1 \le k \le t, \quad 0 \le i \le m-1. \quad (4.3.8)$$

In this case labeling of faces can be defined as,

$$\xi(f^k_{i,j}) = (6n-j)tm-ti+1-k, for \ \ 0 \le j \le n-1, 1 \le k \le t, 0 \le i \le m-1 \quad (4.3.9)$$

Note that in 1-factor labels of edges will be assigned from $\{2mnt+1, 2mnt+2, \ldots, 3mnt\}$ and labels of faces will be assigned from $\{5mnt+1, 5mnt+2, \ldots,$
$6mnt\}$. Thus face weights can be calculated as

$wt(f^k_{i,j}) = w(f^k_{i,j}) + \varphi(x^k_{i,j}y^k_{i,j}) + \varphi(x^k_{i+1,j}y^k_{i+1,j}) + \xi(f^k_{i,j})$
$= (21n+2j)tm + 2ti + 2k + mt(j+2n) + 2t + 4 + (2n+j)tm + ti + k + (1+i)t + k + mt(-j+6n) + 1 - k - it = (3j+31n)tm + 3ti + 5 + 3t + 3k$, for $n-1 \ge j \ge 0$, $m-2 \ge i \ge 0, t \ge k \ge 1$, and

$wt(f^k_{m-1,j}) = \varphi(x^k_{m-1,j}y^k_{m-1,j}) + \varphi(x^k_{0,j}y^k_{0,j}) + \xi(f^k_{m-1,j}) + w(f^k_{m-1,j}) = 2mt(j+2n) + (21n+2j)tm + 2k + 4 + (-1+m)t + 2k + mt(-j+6n) - (-1+m)t + 1 - k = mt(3j + 31n) + 5 + 3k$, for $1 \le k \le t$, $0 \le j \le n-1$. Thus face weights form an arithmetic sequence $31mnt+8, 31mnt+11, \ldots,$
$34mnt + 3t(1-m) + 5$ with common difference 3. Hence a-super face 3-antimagic labeling is obtained. □

In the next theorem, a super face antimagic labeling for disjoint union of t copies of a toroidal fullerene $\mathbb{H}^n_m$ will be formulated. For odd value of j edges in a 1-factor of $\mathbb{H}^n_m$ will be represented by $x^k_{i,j}y^k_{i+1,j-1}$ and $y^k_{i,j}x^k_{i+1,j+1}$ where $1 \le j \le n-1$, $1 \le k \le t$, $0 \le i \le m-1$. Let $2n$-cycles in a 2-factor of $t\mathbb{H}^n_m$ denoted as
$x^k_{i,0}y^k_{i,0}\ x^k_{i,1}y^k_{i,1}\ x^k_{i,2}y^k_{i,2}\ \ldots\ x^k_{i,n-2}y^k_{i,n-2}\ x^k_{i,n-1}y^k_{i,n-1}$, for $0 \le i \le m-1$, $1 \le k \le t$.
Suppose $\psi : V(t\mathbb{H}^n_m) \to \{1, 2, \ldots, 2mnt\}$ for $m-1 \ge i \ge 0$, $t \ge k \ge 1$ and $n-1 \ge j \ge 0$ represents a correspondence as follows:

$$\psi(x^k_{i,j}) = 2nti + 2tj + t + k \qquad (4.3.10)$$

$$\psi(y_{i,j}^{k}) = \begin{cases} 2nit + 2tj + 2t + k, & 0 \leq i \leq n-2 \\ 2nit + k, & j = n-1. \end{cases} \tag{4.3.11}$$

Consecutive integers $2mnt+1, 2mnt+2, \ldots, 4mnt$ are assigned to the edges of $2n$-cycles as labels, for $1 \leq k \leq t$,$0 \leq i \leq m-1$.

$$\varphi(x_{i,j}^{k} y_{i,j}^{k}) = 2nt(2m-i) - 2tj - t + 1 - k, if \;\; 0 \leq j \leq n-1, \tag{4.3.12}$$

$$\varphi(y_{i,j}^{k} x_{i,j+1}^{k}) = 2nt(2m-i) - 2tj - 2t + 1 - k, if \;\; 0 \leq j \leq n-2 \tag{4.3.13}$$

$$\varphi(y_{i,n-1}^{k} x_{i,0}^{k}) = 2nt(2m-i) + 1 - k \tag{4.3.14}$$

where j is considered over modulo n.

Lemma 4.3.4. *Let* $w(f_{i,j}^{k}) = \psi(x_{i,j}^{k}) + \varphi(x_{i,j}^{k} y_{i,j}^{k}) + \psi(y_{i,j}^{k}) + \varphi(y_{i,j}^{k} x_{i,j+1}^{k}) + \psi(x_{i,j+1}^{k}) + \psi(y_{i+1,j}^{k}) + \varphi(y_{i+1,j}^{k} x_{i+1,j}^{k}) + \psi(x_{i+1,j}^{k}) + \psi(y_{i,j-1}^{k}) + \varphi(y_{i,j-1}^{k} x_{i,j}^{k})$ *for even value of* j, $0 \leq j \leq n-2, 0 \leq i \leq m-1$ *and* $1 \leq k \leq t$ *"be a partial weight of the face* $f_{i,j}^{k}$ *of* $t\mathbb{H}_m^n$*". For* $t \geq 1$ *and* $m,n \geq 2$ *where n is even, then partial weight of the face* $f_{i,j}^{k}$ *is* $w(f_{i,j}^{k}) = 2(2i+1+8m)nt + 4tj + 5t + 2k + 4$, $m-2 \geq i \geq 0$, $1 \leq k \leq t$, $0 \leq j \leq n-2$, *and* $w(f_{m-1,j}^{k}) = 2nt(9m-1) + 4tj + 5t + 2k + 4$, $0 \leq j \leq n-2$, $1 \leq k \leq t$.

Proof.

Let j takes even values, $n-2 \geq j \geq 0$, $t \geq k \geq 1$ and $m-2 \geq i \geq 0$ the faces $f_{i,j}^{k}$ have partial weights as follows:

$w(f_{i,j}^{k}) = 2nti + 2tj + t + k + 2nt(2m-i) - 2tj - t + 1 - k + 2nti + 2tj + 2t + k + 2nt(2m-i) - 2tj - 2t + 1 - k + 2nti + 2t(j+1) + t + k + 2nt(i+1) + 2tj + 2t + k + 2nt(2m-i-1) - 2tj - t + 1 - k + 2nt(i+1) + 2tj + t + k + 2nti + (j-1)2t + 2t + k + (2m-i)2nt - (j-1)2t - 2t + 1 - k = 2nt(8m+2i+1) + 4tj + 5t + 2k + 4.$

and

$w(f_{m-1,j}^{k}) = (m-1)2nt + 2tj + t + k + 2nt(2m-m+1) - 2tj - t + 1 - k + 2nt(m-1) + 2tj + 2t + k + 2nt(2m-m+1) - 2tj - 2t + 1 - k + 2nt(m-1) + 2t(j+1) + t + k + 2tj + 2t + k + 4mnt - 2tj - t + 1 - k + 2tj + t + k + 2nt(m-1) + 2t(j-1) + 2t +$

$k+2nt(2m-m+1)-2t(j-1)-2t+1-k=2nt(9m-1)+4tj+5t+2k+4.$ $\square$

Lemma 4.3.5. *For j odd and $n-1\geq j\geq 1$ the partial weight of the face $f^k_{i,j}$ is given as $w(f^k_{i,j})=\psi(x^k_{i,j})+\varphi(x^k_{i,j}y^k_{i,j})+\psi(y^k_{i,j})+\psi(x^k_{i+1,j+1})+\varphi(x^k_{i+1,j+1}y^k_{i+1,j})$ $+\psi(y^k_{i+1,j})+\varphi(x^k_{i+1,j}y^k_{i+1,j})+\psi(x^k_{i+1,j})+\varphi(x^k_{i+1,j}y^k_{i+1,j-1})+\psi(y^k_{i+1,j-1})$ where $m-1\geq i\geq 0$ and $t\geq k\geq 1$. For $t\geq 1$ and $m,n\geq 2$ where n is even, the partial weight for $1\leq k\leq t$ will be:*

$$w(f^k_{i,j})=2nt(8m+2i+1)+4tj+5t+2k+4,\ \ n-3\geq j\geq 1,\ \ m-2\geq i\geq 0 \tag{4.3.15}$$

$$w(f^k_{m-1,j})=2nt(9m-1)+4tj+5t+2k+4,\ \ 1\leq j\leq n-3 \tag{4.3.16}$$

$$w(f^k_{i,n-1})=(8m+2i+1)(2nt)+2k+t+4,\ \ 0\leq i\leq m-2 \tag{4.3.17}$$

$$w(f^k_{m-1,n-1})=(9m-1)(2nt)+t+2k+4. \tag{4.3.18}$$

Proof.

Suppose j takes odd values. To calculate partial weights of the face $f^k_{i,j}$ we have

$w(f^k_{i,j})=2nti+2tj+t+k+(2m-i)(2nt)-2tj-t+1-k+2nti+2tj+2t+k+(i+1)(2nt)+(j+1)(2t)+t+k+2nt(2m-i-1)-2tj-2t+1-k+2nt(i+1)+2tj+2t+k+2nt(2m-i-1)-2tj-t+1-k+2nt(i+1)+2tj+t+k+2nt(2m-i-1)-2t(j-1)-2t+1-k+(i+1)(2nt)+(j-1)(2t)+2t+k=2nt(8m+2i+1)+4tj+5t+2k+4$, for $m-2\geq i\geq 0$, $n-3\geq j\geq 1$, $t\geq k\geq 1$

$w(f^k_{m-1,j})=2tj+t+k+2nt(2m-m+1)-2tj-t+1-k+2nt(m-1)+2tj+2(-1+m)nt+2t+k+2t(j+1)+t+k+4mnt-2tj-2t+1-k+2tj+2t+k+4mnt-2tj-t+1-k+2tj+t+k+4mnt-2t(j-1)-2t+1-k+2t(j-1)+2t+k=2nt(9m-1)+4tj+5t+2k+4$, for $n-3\geq j\geq 1$,

$w(f^k_{i,n-1})=t+2t(-1+n)+k+2nt(2m-i)-2t(n-1)-t+2nti+1-k+2nti+k+2nt(i+1)+t+k+4mnt-2nt(i+1)-2nt+2t-2t+1-k+2nt(i+1)+k+$

$2nt(2m-i-1)-2t(n-1)-t+1-k+2nt(i+1)+2t(n-1)+t+k+2nt(2m-i-1)-2t(n-2)-2t+1-k+2nt(i+1)+2t(n-2)+2t+k=2nt(8m+2i+1)+2k+t+4,$ for $m-2\geq i\geq 0$ and

$w(f^k_{-1+m,-1+n})=t+k+2nt(-1+m)+2nt(2m-m+1)-2t(n-1)-t+1-k+2nt(m-1)+k+t+k+4mnt-2t(n-1)-2t+1-k+k+4mnt-2t(n-1)-t+1-k+2t(n-1)+t+k+4mnt-2nt+4t-2t+1-k+2nt-4t+2t+k+2t(-1+n)$
$=2nt(9m-1)+t+2k+4.$ $\square$

From Lemmas 4.3.4 and 4.3.5 it can be observed that partial weights of the faces $f^k_{i,j}$ form an arithmetic progression having common difference 4.

Theorem 4.3.6. *For $t\geq 1$ and $m,n\geq 2$ where n is even the graph $t\mathbb{H}^n_m$ admits a super d-antimagic labeling of type $(1,1,1)$ for $d=5$.*

Proof.

Vertices of $t\mathbb{H}^n_m$ will be labeled by the mapping ψ and labels of the edges of $2n$-cycles will be assigned by the correspondence φ. For $1\leq k\leq t$ edge labeling φ can be completed by the numbers assigned to edges in 1-factor such that

$$\varphi(x^k_{i+1,j}y^k_{i,j-1})=nt(4m+i)+tj+k,\ \ 1\leq j\leq n-3\ \ and\ \ 1\leq i\leq m-2 \tag{4.3.19}$$

$$\varphi(x^k_{i+1,0}y^k_{i,n-1})=(4m+i)(nt)+k,\ \ 0\leq i\leq m-2 \tag{4.3.20}$$

$$\varphi(x^k_{i,j+1}y^k_{i+1,j})=nt(4m+i)+tj+t+k,\ \ 1\leq j\leq n-1\ \ and\ \ 0\leq i\leq m-2 \tag{4.3.21}$$

$$\varphi(x^k_{0,0}y^k_{m-1,n-1})=nt(6m-1)+k \tag{4.3.22}$$

$$\varphi(x^k_{0,j}y^k_{m-1,j-1})=nt(6m-1)+tj+k,\ \ 0\leq j\leq n-2 \tag{4.3.23}$$

$$\varphi(x^k_{m-1,j+1}y^k_{0,j})=nt(6m-1)+tj+t+k,\ \ 0\leq j\leq n-2. \tag{4.3.24}$$

The values of faces can be arranged in such a way that $\xi(f^k_{i,j})=-jt-t+k+nt(-i-1+6m)$, for $1\leq k\leq t$, $0\leq i\leq m-1, 0\leq j\leq n-1$.

One can easily check the face labels and edge labels take values from the set of consecutive integers $\{4mnt+1, 4mnt+2, \ldots, 6mnt\}$. For even values of j face-weights are calculated as $wt(f_{i,j}^k) = w(f_{i,j}^k) + \varphi(x_{i+1,j}^k y_{i,j-1}^k) + \varphi(y_{i+1,j}^k x_{i,j+1}^k) + \xi(f_{i,j}^k) = 2nt(8m+2i+1) + 4tj + 5t + 2k + 4 + nt(4m+i) + tj + k + nt(4m+i) + tj + t + k + nt(6m-i-1) - tj - t + k = nt(30m+5i+1) + 5tj + 5t + 5k + 4$, for $n-2 \geq j \geq 0$, $m-2 \geq i \geq 0$, $t \geq k \geq 1$,

$wt(f_{m-1,j}^k) = \varphi(x_{0,j}^k y_{m-1,j-1}^k) + \varphi(y_{0,j}^k x_{m-1,j+1}^k) + \xi(f_{m-1,j}^k) + w(f_{m-1,j}^k) = 2nt(9m-1) + 4tj + 5t + 2k + 4 + nt(6m-1) + tj + k + nt(6m-1) + tj + t + k + nt(6m-(m-1)-1) - tj - t + k = nt(35m-4) + 5tj + 5t + 5k + 4$, for $0 \leq j \leq n-2$, $1 \leq k \leq t$.

Face-weights in case of odd values of j is given as

$wt(f_{i,j}^k) = w(f_{i,j}^k) + \varphi(x_{i,j}^k y_{i+1,j-1}^k) + \varphi(y_{i,j}^k x_{i+1,j+1}^k) + \xi(f_{i,j}^k) = 2nt(8m+2i+1) + 4tj + 5t + 2k + 4 + nt(4m+i) + t(j-1) + t + k + nt(4m+i) + t(j+1) + k + nt(6m-i-1) - tj - t + k = nt(30m+5i+1) + 5tj + 5t + 5k + 4$ for $0 \leq i \leq m-2$ and $1 \leq j \leq n-3$,

$wt(f_{m-1,j}^k) = w(f_{m-1,j}^k) + \varphi(x_{m-1,j}^k y_{0,j-1}^k) + \varphi(y_{m-1,j}^k x_{0,j+1}^k) + \xi(f_{m-1,j}^k) = 2nt(9m-1) + 4tj + 5t + 2k + 4 + nt(6m-1) + t(j-1) + t + k + nt(6m-1) + t(j+1) + k + nt(6m-(m-1)-1) - tj - t + k = nt(35m-4) + 5tj + 5t + 5k + 4$ for $n-3 \geq j \geq 1$,

$wt(f_{i,-1+n}^k) = \varphi(x_{i,-1+n}^k y_{1+i,-2+n}^k) + \varphi(y_{i,-1+n}^k x_{1+i,0}^k) + \xi(f_{i,-1+n}^k) + w(f_{i,-1+n}^k) = 2nt(8m+2i+1) + 2k + 4 + t + nt(-i-1+6m) + nt(4m+i) + t(n-2) + t + k + nt(4m+i) + k - t(-1+n) - t + k = 5k + 4 + nt(5i+30m+1)$ for $m-2 \geq i \geq 0$,

$wt(f_{-1+m,-1+n}^k) = w(f_{m-1,n-1}^k) + \varphi(x_{m-1,n-1}^k y_{0,n-2}^k) + \varphi(y_{m-1,n-1}^k x_{0,0}^k) + \xi(f_{m-1,n-1}^k) = 2nt(9m-1) + 2k + 4 + t + nt(6m-1) + t(n-2) + nt(6m-(m-1)) + t + k + nt(6m-1) + k - 1 - t(-1+n) - t + k = nt(-4+35m) + 5k + 4$. One can easily check that face-weights form an arithmetic sequence starting from $nt(30m+1) + 5t + 9$ with common difference 5, hence the above labeling is a super face 5-antimagic labeling. □

Chapter 5

Union of generalized friendship graph with (a, b, c) labelings

5.1 Introduction

In this chapter, disjoint union of generalized (subdivided) friendship graph has been studied in the sense of super d- antimagic labeling for various values of common difference d. "Lih constructed the friendship graph as the join of np_2 and k_1" in [41]. "Generalized (subdivided) friendship graph can be constructed as the join of np_p and k_1 will be represented by $f_{p,n}$ $p,n \geq 2$."

5.2 Face antimagic labeling for disjoint union of generalized friendship graph

In this section m isomorphic copies of subdivided friendship graph denoted by $mf_{p,n}$ in "the perspective of face antimagic labeling for different values of d" will be discussed. Throughout this section, following representation for the set of vertices, edges and faces of $mf_{p,n}$ will be used. For $1 \leq k \leq p, 1 \leq j \leq m, 1 \leq i \leq n$

$$V(mf_{p,n}) = \{a^j\} \cup \{v^j_{ik}\}, \ \ F(mf_{p,n}) = \{f^j_i\}$$

$$E(mf_{p,n}) = \{a^j v^j_{i1}\} \cup \{a^j v^j_{ip}\} \cup \{v^j_{ik} v^j_{ik+1}\}$$

Set of vertices, edges and faces of $f_{p,n}$ can be obtained by using $m = 1$. Graph of $2f_{3,4}$ is given in Figure 5.1.

Theorem 5.2.1. *For every odd $n \geq 3$ and $m \geq 1, p \geq 2$ the graph $G \cong mf_{p,n}$ admits d-antimagic labeling of type $(1,1,1)$ with common difference $d = 1$.*

Proof.
For $m \geq 1, p \geq 2$ and $n \geq 3$ where n is odd, consider a correspondence λ which mappes $\{1,2,3...,2mn(p+1)+m\}$ to elements of graph $mf_{p,n}$.

$$\lambda(a^j) = \ 2mn + m - j + 1, \qquad 1 \leq j \leq m \tag{5.2.1}$$

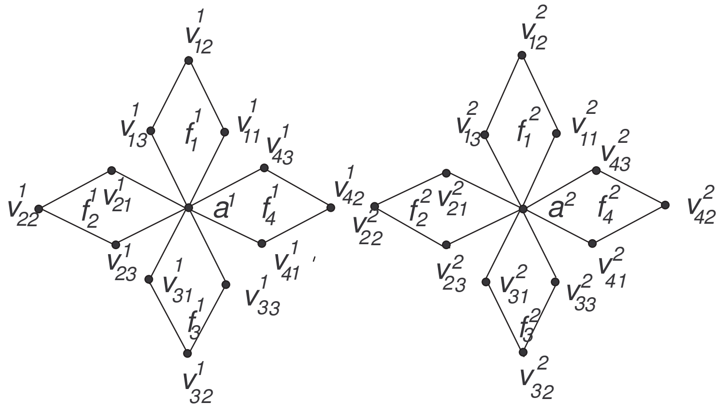

Figure 5.1: Graph of $mf_{p,n}$ for m=2, p=3, and n=4.

$$\lambda(v_{i1}^j) = \begin{cases} (i-1)m+j, & 1 \le j \le m, 1 \le i \le n, \ i \ \text{is odd} \\ im-j+1, & 1 \le j \le m, 1 \le i \le n, \ i \ \text{is even} \end{cases} \tag{5.2.2}$$

$$\lambda(v_{ip}^j) = \begin{cases} \frac{1}{2}(3n-i)m+j, & 1 \le j \le m, 1 \le i \le n, \ i \ \text{is odd} \\ \frac{1}{2}(4n-i)m+j, & 1 \le j \le m, 1 \le i \le n, \ i \ \text{is even} \end{cases} \tag{5.2.3}$$

$$\lambda(a^j v_{ip}^j) = \ 3mn+im+j, \ \ 1 \le j \le m, 1 \le i \le n \tag{5.2.4}$$

$$\lambda(a^j v_{i1}^j) = \ 4mn+(n+2-i)m-j+1, \ \ 1 \le i \le n, 1 \le j \le m \tag{5.2.5}$$

$$\lambda(f_i^j) = \begin{cases} \frac{1}{2}(5n+1-i)m+j, & 1 \le j \le m, 1 \le i \le n, i \ \text{is even} \\ \frac{1}{2}(6n-1-i)m+2m-j+1, & 1 \le j \le m, 1 \le i \le n, \ i \ \text{is odd} \end{cases} \tag{5.2.6}$$

$$\lambda(v_{ik}^j v_{ik+1}^j) = \ (2k+3)mn+im+j, \ \ 1 \le j \le m, 1 \le i \le n, 1 \le k \le p-1 \tag{5.2.7}$$

For $1 \le j \le m, 1 \le i \le n, \ 2 \le k \le p-1$

$$\lambda(v_{ik}^j) = \ (2k+3)mn+(2-i)m-j+1 \tag{5.2.8}$$

Hence the weights of all p+1-sided faces form an arithmetic sequence $\frac{21}{2}mn+\frac{5}{2}m-pmn+(p-1)(2pmn+7mn+2m+1)+3$,
$\frac{21}{2}mn+\frac{5}{2}m-pmn+(p-1)(2pmn+7mn+2m+1)+4, \ldots, \frac{23}{2}mn+\frac{5}{2}m-pmn+(p-$

$1)(2pmn+7mn+2m+1)+2.$

Hence proved. $\square$

Theorem 5.2.2. *For $m \geq 1, p \geq 2$ and $n \geq 3$ where n is odd the graph $G \cong mf_{p,n}$ admits d-antimagic labeling of type $(1,1,1)$ with common difference $d = (2p-1)$.*

Proof.

For $m \geq 1, p \geq 2$ and $n \geq 3$ where n is odd, consider the correspondence λ which mappes $\{$1,2,3...,2mn(p+1)+m$\}$ to elements of graph $mf_{p,n}$.

$$\lambda(a^j) = \ 2mn+m-j+1, \ \ 1 \leq j \leq m \tag{5.2.9}$$

$$\lambda(v_{i1}^j) = \begin{cases} (i-1)m+j, \ \ 1 \leq j \leq m, 1 \leq i \leq n, \ i \ \ is \ \ odd \\ im-j+1, \ \ 1 \leq j \leq m, 1 \leq i \leq n, \ i \ \ is \ \ even \end{cases} \tag{5.2.10}$$

$$\lambda(v_{ip}^j) = \begin{cases} \frac{1}{2}(3n-i)m+j, \ \ 1 \leq j \leq m, 1 \leq i \leq n, \ i \ \ is \ \ odd \\ \frac{1}{2}(4n-i)m+j, \ \ 1 \leq j \leq m, 1 \leq i \leq n, \ i \ \ is \ \ even \end{cases} \tag{5.2.11}$$

$$\lambda(a^j v_{ip}^j) = \begin{cases} 2mn+(\frac{i+1}{2})m+j, \ \ 1 \leq j \leq m, 1 \leq i \leq n, \ i \ \ is \ \ odd \\ \frac{5}{2}mn+(\frac{i+3}{2})m-j+1, \ \ 1 \leq j \leq m, 1 \leq i \leq n, \ i \ \ is \ \ even \end{cases} \tag{5.2.12}$$

$$\lambda(a^j v_{i1}^j) = \begin{cases} 4mn+(\frac{3-i}{2})m-j+1, \ \ 1 \leq j \leq m, 1 \leq i \leq n, \ i \ \ is \ \ odd \\ 3mn+(\frac{n+1-i}{2})m+j, \ \ 1 \leq j \leq m, 1 \leq i \leq n, \ i \ \ is \ \ even \end{cases} \tag{5.2.13}$$

$$\lambda(f_i^j) = \begin{cases} 4mn+(\frac{i+1}{2})m+j, \ \ 1 \leq j \leq m, 1 \leq i \leq n, i \ \ is \ \ odd \\ \frac{9}{2}mn+\frac{m}{2}+(\frac{i+2}{2})m-j+1, \ \ 1 \leq j \leq m, 1 \leq i \leq n, \ i \ \ is \ \ even \end{cases} \tag{5.2.14}$$

$$\lambda(v_{ik}^j v_{ik+1}^j) = \begin{cases} (2k+3)mn+(\frac{i+1}{2})m+j, \\ \qquad 1 \leq j \leq m, 1 \leq i \leq n, 1 \leq k \leq p-1, i \ \ is \ \ odd \\ (2k+\frac{7}{2})mn+(\frac{i+3}{2})m-j+1, \\ \qquad 1 \leq j \leq m, 1 \leq i \leq n, 1 \leq k \leq p-1, \ i \ \ is \ \ even \end{cases} \tag{5.2.15}$$

$$\lambda(v_{ik}^{j}) = \begin{cases} 2(k+1)mn + (\frac{i+1}{2})m + j, \\ \quad 2 \le k \le p-1, 1 \le j \le m, 1 \le i \le n, i \ \text{is odd} \\ (2k+\frac{5}{2})mn + (\frac{i+3}{2})m - j + 1, \\ \quad 2 \le k \le p-1, 1 \le j \le m, 1 \le i \le n, i \ \text{is even} \end{cases} \tag{5.2.16}$$

Hence the weights of all p+1-sided faces form an arithmetic sequence $\frac{21}{2}mn + \frac{5}{2}m - pmn + 2(p-1)(pmn+3mn+m+1)+3$,
$\frac{21}{2}mn + \frac{5}{2}m - pmn + 2(p-1)(pmn+3mn+m+1)+2(p+1),...,\frac{21}{2}mn + \frac{5}{2}m - pmn + 2(p-1)(pmn+3mn+m+1)+(2p-1)(mn-1)+3$.
This completes the proof. □

Theorem 5.2.3. *For every even $m,n \ge 2$ and $p \ge 2$ the graph $G \cong mf_{p,n}$ admits d-antimagic labeling of type $(1,1,1)$ with common difference $d = 1$.*

Proof.
For every even $m,n \ge 2$ and $p \ge 2$, consider the correspondence λ which mappes {1,2,3...,2mn(p+1)+m} to elements of graph $mf_{p,n}$.

$$\lambda(a^j) = \ 2mn + j, \quad 1 \le j \le m \tag{5.2.17}$$

$$\lambda(v_{i1}^{j}) = \begin{cases} (i-1)m + j, \ \ 1 \le j \le m, 1 \le i \le n, \ i \ \text{is odd} \\ im - j + 1, \ \ 1 \le j \le m, 1 \le i \le n, \ i \ \text{is even} \end{cases} \tag{5.2.18}$$

$$\lambda(v_{ip}^{j}) = \begin{cases} 2mn + (1-i)m - j + 1, \ \ 1 \le j \le m, 1 \le i \le n, \ i \ \text{is odd} \\ 2mn - im + j, \ \ 1 \le j \le m, 1 \le i \le n, \ i \ \text{is even} \end{cases} \tag{5.2.19}$$

$$\lambda(a^j v_{ip}^{j}) = \begin{cases} 2mn + \frac{1}{2}((2+i)m - (j-1)), \\ \quad 1 \le j \le m, 1 \le i \le n, \ j \ \text{is odd} \\ \frac{5}{2}mn + (\frac{i+2}{2})m - \frac{j}{2} + 1, \\ \quad 1 \le j \le m, 1 \le i \le n, \ j \ \text{is even} \end{cases} \tag{5.2.20}$$

$$\lambda(a^j v_{i1}^j) = \begin{cases} 3mn + (\frac{i+1}{2})m + \frac{j}{2}, & 1 \le j \le m, 1 \le i \le n, \ j \ \ is \ \ even \\ \frac{7}{2}mn + (\frac{i+1}{2})m + \frac{j+1}{2}, & 1 \le j \le m, 1 \le i \le n, \ j \ \ is \ \ odd \end{cases} \tag{5.2.21}$$

For $1 \le i \le n, 1 \le j \le m$

$$\lambda(f_i^j) = \ 4mn + im + j \tag{5.2.22}$$

For $1 \le j \le m, 1 \le i \le n, 1 \le k \le p-1$

$$\lambda(v_{ik}^j v_{ik+1}^j) = \text{“}m(2-i) - j + 2mn(k+2) + 1\text{”} \tag{5.2.23}$$

For $p-1 \ge k \ge 2, 1 \le j \le m, 1 \le i \le n$

$$\lambda(v_{ik}^j) = \ 2(k+1)mn + im + j \tag{5.2.24}$$

Hence the weights of all p+1-sided faces form an arithmetic sequence $\frac{21}{2}mn + \frac{5}{2}m - pmn + (p-1)(2pmn + 7mn + 2m + 1) + 3$,
$\frac{21}{2}mn + \frac{5}{2}m - pmn + (p-1)(2pmn + 7mn + 2m + 1) + 4, ..., \frac{23}{2}mn + \frac{5}{2}m - pmn + (p-1)(2pmn + 7mn + 2m + 1) + 2$. This completes the proof. □

Theorem 5.2.4. *For every even $m, n \ge 2$ and $p \ge 2$ the graph $G \cong mf_{p,n}$ admits $(2p-1)$-antimagic labeling of type $(1,1,1)$.*

Proof.

For every even $m, n \ge 2$ and $p \ge 2$, consider the correspondence λ which mappes {1,2,3...,2mn(p+1)+m} to elements of graph $mf_{p,n}$.

$$\lambda(a^j) = \ 2mn + j, \ \ 1 \le j \le m \tag{5.2.25}$$

$$\lambda(v_{i1}^j) = \begin{cases} (i-1)m + j, & 1 \le j \le m, 1 \le i \le n, \ i \ \ is \ \ odd \\ im - j + 1, & 1 \le j \le m, 1 \le i \le n, \ i \ \ is \ \ even \end{cases} \tag{5.2.26}$$

$$\lambda(v_{ip}^{j}) = \begin{cases} 2mn+(1-i)m-j+1, & 1 \leq j \leq m, 1 \leq i \leq n,\ i\ \ is\ \ odd \\ 2mn-im+j, & 1 \leq j \leq m, 1 \leq i \leq n,\ i\ \ is\ \ even \end{cases} \tag{5.2.27}$$

$$\lambda(a^{j}v_{ip}^{j}) = \begin{cases} 2mn+\frac{1}{2}((2+i)m-(j-1)), \\ \qquad\qquad 1 \leq j \leq m, 1 \leq i \leq n,\ j\ \ is\ \ odd \\ \frac{5}{2}mn+(\frac{i+2}{2})m-\frac{j}{2}+1, \\ \qquad\qquad 1 \leq j \leq m, 1 \leq i \leq n,\ j\ \ is\ \ even \end{cases} \tag{5.2.28}$$

$$\lambda(a^{j}v_{i1}^{j}) = \begin{cases} 3mn+(\frac{i+1}{2})m+\frac{j}{2}, \\ \quad 1 \leq j \leq m, 1 \leq i \leq n,\ j\ \ is\ \ even \\ \frac{7}{2}mn+(\frac{i+1}{2})m+\frac{j+1}{2}, \\ \quad 1 \leq j \leq m, 1 \leq i \leq n,\ j\ \ is\ \ odd \end{cases} \tag{5.2.29}$$

For $p-1 \geq k \geq 1, m \geq j \geq 1, n \geq i \geq 1$

$$\lambda(v_{ik}^{j}v_{ik+1}^{j}) = \ 2kmn+3mn+im+j \tag{5.2.30}$$

For $1 \leq j \leq m, 1 \leq i \leq n, 2 \leq k \leq p-1$

$$\lambda(v_{ik}^{j}) = \ 2(k-1)mn+4mn+im+j \tag{5.2.31}$$

$$\lambda(f_{i}^{j}) = \text{“}j+4mn+im\text{”} \ \ 1 \leq i \leq n, 1 \leq j \leq m \tag{5.2.32}$$

Hence the weights of all p+1-sided faces form an arithmetic sequence $\frac{21}{2}mn+\frac{5}{2}m-pmn+2(p-1)(pmn+3mn+m+1)+3$,
$\frac{21}{2}mn+\frac{5}{2}m-pmn+2(p-1)(pmn+3mn+m+1)+2(p+1)$,...
,$\frac{23}{2}mn+\frac{5}{2}m-pmn+2(p-1)(pmn+3mn+m+1)+(2p-1)(mn-1)+3$.
This completes the proof. $\square$

Theorem 5.2.5. *For every odd $m \geq 1$ and $p,n \geq 2$ the graph $G \cong mf_{p,n}$ admits face magic labeling.*

Proof.

For $p,n \geq 2$ and for every odd $m \geq 1$, consider the correspondence λ which mappes {1,2,3...,2mn(p+1)+m} to elements of graph $mf_{p,n}$.

$$\lambda(a^j) = \begin{cases} (n+1)m-2j+2, & 1 \leq j \leq \frac{m+1}{2} \\ (n+2)m-2j+2, & \frac{m+3}{2} \leq j \leq m \end{cases} \tag{5.2.33}$$

$$\lambda(v_{i1}^j) = \ (i-1)m+j, \ \ 1 \leq j \leq m, 1 \leq i \leq n \tag{5.2.34}$$

$$\lambda(v_{ip}^j) = \begin{cases} 2mn+(1-i)m+\frac{m-1}{2}+j, & 1 \leq j \leq \frac{m+1}{2} \\ 2mn+(1-i)m-\frac{m+1}{2}+j, & \frac{m+3}{2} \leq j \leq m \end{cases} \tag{5.2.35}$$

$$\lambda(a^jv_{ip}^j) = \ 2mn+im+j, \quad 1 \leq j \leq m, 1 \leq i \leq n \tag{5.2.36}$$

$$\lambda(a^jv_{i1}^j) = \ 3mn+(n+2-i)m-j+1, \ \ m \geq j \geq 1, n \geq i \geq 1 \tag{5.2.37}$$

For $1 \leq j \leq m, 1 \leq i \leq n, 1 \leq k \leq p-1$

$$\lambda(v_{ik}^jv_{ik+1}^j) = \ 2(k+2)mn+(2-i)m-j+1 \tag{5.2.38}$$

For $1 \leq j \leq m, 1 \leq i \leq n, 2 \leq k \leq p-1$

$$\lambda(v_{ik}^j) = \ 2(k+1)mn+im+j \tag{5.2.39}$$

$$\lambda(f_i^j) = \ im+j+4mn, \quad m \geq j \geq 1, 1 \leq i \leq n \tag{5.2.40}$$

the weight of p+1-sided faces is $2p^2mn+4pmn+3mn+2pmn+p+\frac{3}{2}(m+1)$. This completes the proof. $\square$

Chapter 6

Computing the Metric dimension of Jahangir related graphs

6.1 Introduction

In this chapter, metric dimension of of m-level Jahangir graph $J_{2n,m}$ and generalized Jahangir graph J_{3n} has been studied . In [53] Jahangir graph has been defined by Tomescu using definition of wheel graph [17].Metric dimension of m-level wheel graphs has been studied by siddique in [47]. Suppose $W_{2n,m} \cong mC_{2n} + K_1$ be m-level wheel graph which contains m copies of C_{2n}, by following the idea of Tomescu given in [53] m-level Jahangir graph denoted by $J_{2n,m}$ can be constructed by alternatively deleting n spokes of each copy of C_{2n} in $W_{2n,m}$ and J_{3n} be a generalized Jahangir graph obtained by alternately deleting $2n$ spokes of the wheel graph W_{3n}. It is shown that for every $n \geq 4$, $dim(J_{2n,m}) = dim(J_{2n,1}) + (m-1)\lceil \frac{2n}{3} \rceil$ and $dim(J_{3n}) = \lfloor \frac{n}{2} \rfloor$ for $n \geq 6$.

6.2 Metric dimension of m-level Jahangir graph $J_{2n,m}$

As defined in [47], the join of $2C_n + K_1$ yields a double wheel graph represented by $W_{n,2}$ and the join of $mC_n + k_1$ yields an m-level wheel graph inductively expressed by $W_{n,m}$. In the similar manner, a double Jahangir graph denoted by $J_{2n,2}$ can be obtained from double wheel $W_{2n,2} = 2C_{2n} + k_1$ by alternatively deleting n spokes of each copy of C_{2n} and inductively an m-level Jahangir graph denoted by $J_{2n,m}$ can be constructed from m-level wheel $W_{2n,m} \cong mC_{2n} + k_1$ by alternatively deleting n spokes of each C_{2n}. Independently a double Jahangir graph $J_{2n,2}$ is defined as follows : consider two even cycles $C_{2n,1}$ and $C_{2n,2}$: $v_1^1, v_2^1, v_3^1, \ldots, v_{2n}^1, v_1^1$ and $v_1^2, v_2^2, v_3^2, \ldots, v_{2n}^2, v_1^2$ with $n \geq 2$, take $v \in V(G)$ and connect it to n vertices of $C_{2n,1}$:$v_2^1, v_4^1, \ldots, v_{2n}^1$ by n distinct paths of length 1, also connect v to n vertices of $C_{2n,2}$:$v_2^2, v_4^2, \ldots, v_{2n}^2$ by n distinct paths of length 1. An m-level Jahangir graph can be constructed inductively, represented by $J_{2n,m}$, by taking m even cycles $C_{2n,1}, C_{2n,2}, \ldots, C_{2n,m}$. □

The vertices of $C_{2n,i}$, $1 \leq i \leq 2$ in the double Jahangir graph can be divided into two sets on degree based: all the vertices in one set have degree two and vertices in second set have degree three. All the vertices of degree two will be named as minor vertices and vertices of degree three will be considered as major vertices. One can easily check that $dim(J_{4,2}) = 3+2$ (central vertex v with one major and minor vertex of each $C_{2n,i}, 1 \leq i \leq 2$ form basis), $dim(J_{6,2}) = 3+2$ (central vertex v with two minor vertices of each $C_{2n,i}, 1 \leq i \leq 2$ form basis) , $dim(J_{8,2}) = 2+3$(two minor vertices u^1, w^1 such that $d(u^1, w^1) = 2$ of $C_{2n,1}$ with one minor vertex u^2 and two major vertices w^2, x^2 of $C_{2n,2}$ such that $d(u^2, w^2) = d(u^2, x^2) = 3$ form basis), $dim(J_{10,2}) = 3+4$ (three minor vertices u^1, w^1, x^1 satisfying $d(u^1, w^1) = d(w^1, x^1) = 2$, $d(u^1, x^1) = 4$ of $C_{2n,1}$ with three minor vertices u^2, w^2, x^2 and one major vertex z^2 of $C_{2n,2}$ satisfying $d(u^2, w^2) = d(w^2, x^2) = 2$, $d(u^2, x^2) = 4$ and $d(u^2, z^2) = d(w^2, z^2) = d(x^2, z^2) = 3$ form metric basis of $J_{10,2}$).

Let us consider an outer cycle $C_{2n,1}$ of $J_{2n,1}$ having length $2n$. Suppose B represents basis set of $J_{2n,1}$ then B consists of $r \geq 2$ vertices of $C_{2n,1}$ for $n \geq 6$. Suppose $B = \{v_{i_1}, v_{i_2}, ..., v_{i_r}\}$ then vertices of B can be ordered as $v_{i_1} < v_{i_2} ... < v_{i_r}$ such that $\{v_{i_t}, v_{i_{t+1}}\}$ for $1 \leq t \leq r-1$ and $\{v_{i_r}, v_{i_1}\}$ are called neighboring vertices. Vertices of $C_{2n,1}$ lying between any two neighboring vertices of B is called a gap and their cardinality is called size of gap will be denoted by G_{i_t} for $1 \leq t \leq r-1$ and G_{i_r}. One can easily observe that every vertex of B has two neighboring vertices, gaps generated by these three vertices are called neighboring gaps. Such a concept already exists in [17] and [53]. In a situation where $deg(v_i) = \nu$ and $deg(v_j) = \mu$ or $deg(v_i) = \mu$ and $deg(v_j) = \nu$, a gap between two neighboring vertices of basis say v_i and v_j will be called $\nu - \mu$ with $\mu \geq \nu$. Thus there can be three kinds of gaps $2-2$,$2-3$ and $3-3$.
For the graph $J_{2n,2}, n \geq 4$ basis set B does not contain central vertex v. Since $d(v_i^j, v) \leq 2$ $\forall$, $1 \leq i \leq 2n$, $1 \leq j \leq 2$ and $diam(J_{2n,2}) = 4$ if central vertex $v \in B$ then $\exists$ two vertices u_i and u_j, $u_i \neq u_j$ for $1 \leq i \neq j \leq 2n$ such that $r(u_i/B) = r(u_J/B)$. Therefore, basis set B of $J_{2n,2}$ contains vertices of the cycles induced by $C_{2n,1}$ and $C_{2n,2}$. It is given in [53] that basis set B of $J_{2n,1}$ contains vertices of the cycle induced by $C_{2n,1}$ that fulfil the following axioms.

(1) For $n \geq 6$, basis set of $J_{2n,1}$ generate $2-2$ gap of size at most 5, $2-3$ gap of size at most 4 and $3-3$ gap of size at most 3.

(2) Every basis set B of $J_{2n,1}$ for $n \geq 6$ consists at most one major gap.
(3) For each basis set B of $J_{2n,1}$ for $n \geq 6$ two neighboring gaps constitute together six vertices at most in which one gap is a major gap .
(4) For each basis set B of $J_{2n,1}$ for $n \geq 6$ two minor neighboring gaps constitute together four vertices at most.

Lemma 6.2.1. *For each basis set* B *of* $J_{2n,2}$ *for* $n \geq 6$ *every* $3-3$, $2-3$ *and* $2-2$ *gap of* B *induced by* $C_{2n,1}$ *and* $C_{2n,2}$ *contains* $3,4$*and* 5 *vertices at most respectively.*

Proof.
Suppose the result is false and there exists $2-2$ gap of size 7 say $u_1, u_2, u_3, u_4, u_5, u_6$, u_7 consisting consecutive vertices of $C_{2n,1}$ or $C_{2n,2}$ with

$$deg(u_1) = deg(u_7) = 3 \tag{6.2.1}$$

then

$$r(u_3/B) = r(u_5/B) \tag{6.2.2}$$

which is a contradiction . If there exists a $2-3$ gap of size 6 then exists a path $u_1, u_2, u_3, u_4, u_5, u_6$ consisting consecutive vertices of $C_{2n,1}$ or $C_{2n,2}$ with

$$deg(u_1) = 3 \tag{6.2.3}$$

$$deg(u_6) = 2 \tag{6.2.4}$$

then

$$r(u_3/B) = r(u_5/B) \tag{6.2.5}$$

which is again a contradiction . The existence of a $3-3$ gap of size 5 say u_1, u_2, u_3, u_4, u_5 induced by $C_{2n,1}$ or $C_{2n,2}$ with

$$deg(u_1) = deg(u_5) = 2 \tag{6.2.6}$$

would imply

$$r(u_2/B) = r(u_4/B) \tag{6.2.7}$$

a contradiction. □

The $3-3$, $2-3$ and $2-2$ gaps containing 3,4 and 5 vertices respectively will be referred as major gaps and remaining gaps are called minor gaps. In the proof of lemma $2-4$ sign of star stand for major vertices.

Lemma 6.2.2. *Every basis set* B *of* $J_{2n,2}$, $n \geq 6$ *contains at most one major gap induced by* $C_{2n,1}$ *and* $C_{2n,2}$.

Proof.

Suppose B contains two distinct major gaps induced by $C_{2n,1}$ or $C_{2n,2}$.

Case-I: when both gaps are $3-3$ then there exist two distinct paths consisting consecutive vertices u_1, u_2^*, u_3 and w_1, w_2^*, w_3 of $C_{2n,1}$ and $C_{2n,2}$ respectively in this case

$$r(u_3^*/B) = r(w_3^*/B) \tag{6.2.8}$$

a contradiction .

Case-II: when both gaps are $2-2$ then there exist two distinct paths consisting of consecutive vertices $u_1^*, u_2, u_3^*, u_4, u_5^*$ and $w_1^*, w_2, w_3^*, w_4, w_5^*$ of $C_{2n,1}$ and $C_{2n,2}$ respectively but

$$r(u_3^*/B) = r(w_3^*/B) \tag{6.2.9}$$

a contradiction .

Case-III: when both gaps are $2-3$ then there exist two distinct paths consisting of consecutive vertices u_1^*, u_2, u_3^*, u_4 and w_1^*, w_2, w_3^*, w_4 of $C_{2n,1}$ and $C_{2n,2}$ respectively in this case

$$r(u_3^*/B) = r(w_3^*/B) \tag{6.2.10}$$

a contradiction .

Case-IV: when one gap is $3-3$ and other is $2-2$ then there exist two distinct paths

u_1, u_2^*, u_3 and $w_1^*, w_2, w_3^*, w_4, w_5^*$ induced by $C_{2n,1}$ and $C_{2n,2}$ respectively but

$$r(u_2^*/B) = r(w_3^*/B) \tag{6.2.11}$$

a contradiction .

Case-V: when one gap is $3-3$ and other is $2-3$ then there exist two distinct paths consisting of consecutive vertices u_1, u_2^*, u_3 and w_1^*, w_2, w_3^*, w_4 of $C_{2n,1}$ and $C_{2n,2}$ respectively but

$$r(u_2^*/B) = r(w_3^*/B) \tag{6.2.12}$$

a contradiction .

Case-VI: when one gap is $2-2$ and other is $2-3$ then there exist two distinct paths consisting of consecutive vertices $u_1^*, u_2, u_3^*, u_4, u_5^*$ and w_1^*, w_2, w_3^*, w_4 of $C_{2n,1}$ and $C_{2n,2}$ respectively in this case

$$r(u_3^*/B) = r(w_3^*/B) \tag{6.2.13}$$

a contradiction .

Similarly if both major gaps are induced by $C_{2n,1}$ then a contradiction arises and a similar contradiction arises if $C_{2n,2}$ induced both major gaps. □

Lemma 6.2.3. *For each basis set* B *of* $J_{2n,2}$ *for* $n \geq 6$ *two neighboring gaps in which one gap is a major gap induced by exactly one of two cycles* $C_{2n,1}$ *or* $C_{2n,2}$ *contain together six vertices at most.*

Proof.

In case of $3-3$ gap there is nothing to prove by Lemma 6.2.2. Without loss of any generality we can say that only $C_{2n,1}$ induced major gap by Lemma 6.2.2. If the major gap is $2-2$ gap having five vertices then its neighboring minor gap contains at most one vertex. If this statement is false and $2-2$, $2-3$ minor gaps having three and two vertices respectively are neighboring gaps of $2-2$ major gap then there exist two paths consisting of consecutive vertices of $C_{2n,1}$:$u_9^*, u_8, u_7^*, u_6, u_5^*, u_4, u_3^*, u_2, u_1^*$ and

$w_1^*, w_2, w_3^*, w_4, w_5^*, w_6, w_7^*, w_8$ where $u_4, w_6 \in B$ induced by $2-2$ major, $2-2$ minor gaps and $2-2$ major, $2-3$ minor gaps respectively. In this case

$$r(u_3^*/B) = r(u_5^*/B) \tag{6.2.14}$$

$$r(w_5^*/B) = r(w_7^*/B) \tag{6.2.15}$$

a contradiction. The existence of $2-3$ major gap having 4 vertices is not possible if its neighboring minor gap is $2-2$ gap having 3 vertices. In this case the following path:$u_8, u_7^*, u_6, u_5^*, u_4, u_3^*, u_2, u_1^*$ can be considered where $u_4 \in B$ then

$$r(u_4^*/B) = r(u_5^*/B) \tag{6.2.16}$$

a contradiction. □

Lemma 6.2.4. *For each basis set* B *of $J_{2n,2}$ for $n \geq 6$ two minor neighboring gaps induced by $C_{2n,1}$ or $C_{2n,2}$ contain together four vertices at most.*

Proof.

To prove the statement, it is sufficient to prove two cases .

(i) $2-2$ minor gap with 3 vertices cannot be neighboring gap of $2-2$ minor gap having 3 vertices otherwise there exists a path containing consecutive vertices of $C_{2n,1}$ or $C_{2n,2}$:$u_1^*, u_2, u_3^*, u_4, u_5^*, u_6, u_7^*$ where $u_4 \in B$ in this case

$$r(u_3^*/B) = r(u_5^*/B) \tag{6.2.17}$$

(ii)$2-2$ minor gap with three vertices cannot be neighboring gap of $2-3$ minor gap having two vertices otherwise there exists a path consisting of consecutive vertices of $C_{2n,1}$ or $C_{2n,2}$:$w_1^*, w_2, w_3^*, w_4, w_5^*, w_6$ where $w_4 \in B$ in this case

$$r(w_3^*/B) = r(w_5^*/B) \tag{6.2.18}$$

a contradiction. $\square$

Theorem 6.2.5. *If $n \geq 4$, then $dim(J_{2n,2}) = dim(J_{2n,1}) + \lceil \frac{2n}{3} \rceil$*

Proof.

It has been discussed that

$$dim(J_{8,2}) = 5 = dim(J_{8,1}) + \lceil \frac{8}{3} \rceil \tag{6.2.19}$$

$$dim(J_{10,2}) = 7 = dim(J_{10,1}) + \lceil \frac{10}{3} \rceil \tag{6.2.20}$$

and the "central vertex v does not belong to any basis B" of $J_{2n,2}$. Suppose two outer cycles $C_{2n,1}$: $v_1^1, v_2^1, v_3^1, ..., v_{2n}^1, v_1^1$ and $C_{2n,2}$: $v_1^2, v_2^2, v_3^2, ..., v_{2n}^2, v_1^2$ of $J_{2n,2}$ at level 1 and 2 respectively. First, it will be shown that

$$dim(J_{2n,1}) + \lceil \frac{2n}{3} \rceil \geq dim(J_{2n,2}) \tag{6.2.21}$$

For this purpose resolving set W having $dim(J_{2n,1}) + \lceil \frac{2n}{3} \rceil$ vertices will be created in $J_{2n,2}$. By using residue class modulo 3, three cases to which n belongs will be considered.

Case-I: When $2n = 3k$, $k \geq 4$ where k is even that is $n \equiv 0(mod3)$ and

$$2k = dim(J_{2n,1}) + \lceil \frac{2n}{3} \rceil \tag{6.2.22}$$

in this case W can be considered as

$$W = \{v_1^j, v_{2n-1}^j; 1 \leq j \leq 2\} \cup \{v_{6i+1}^1, v_{6i+3}^1, v_{6i-1}^2, v_{6i+1}^2; 1 \leq i \leq \frac{k}{2} - 1\} \tag{6.2.23}$$

Case-II: When $n \equiv 1(mod3)$, then it can be stated that $2n = 3k + 2$, where $k \geq 4$ is even and

$$dim(J_{2n,1}) + \lceil \frac{2n}{3} \rceil = 2k + 1 \tag{6.2.24}$$

in this case W can be considered as

$$W=\{v_1^1,v_{2n-1}^1,v_1^2\}\cup\{v_{6i+1}^1,v_{6i+3}^1;1\leq i\leq \frac{k}{2}-1\}\cup\{v_{6i-1}^2,v_{6i+1}^2;1\leq i\leq \frac{k}{2}\} \quad (6.2.25)$$

Case-III: When $2n=3k+1$, $k\geq 5$ where k is even that is $n\equiv 2(mod3)$ and $2k+1=dim(J_{2n,1})+\lceil\frac{2n}{3}\rceil$ thus W can be considered as

$$W=\{v_1^1,v_1^2,v_{2n-1}^2\}\cup\{v_{6i+1}^1,v_{6i+3}^1;1\leq i\leq \frac{k-1}{2}\}\cup\{v_{6i-1}^2,v_{6i+1}^2;1\leq i\leq \frac{k-1}{2}\} \quad (6.2.26)$$

The set W contains a unique $2-2$ major gap having at most five vertices and all other gaps are $2-2$ minor gaps which contain at most three vertices. As two major or two minor vertices falling in same gap or separate gap are differentiated by at least 1 vertex in the set of 3 vertices of W producing these neighboring gaps, when gaps are not neighboring gaps then the set of 4 vertices of W which produce 2 gaps make the representation unique of every vertex of these 2 gaps so W is a resolving set of $J_{2n,2}$. Representation of central vertex v is $(2,2,2,...,2)$ which is different from the representation of all other vertices of $J_{2n,2}$. Hence

$$dim(J_{2n,2})\leq dim(J_{2n,1})+\lceil\frac{2n}{3}\rceil \quad (6.2.27)$$

Now we show that

$$dim(J_{2n,2})\geq dim(J_{2n,1})+\lceil\frac{2n}{3}\rceil. \quad (6.2.28)$$

Suppose B represents basis set of $J_{2n,2}$ such that $|B|=r$ then there exist r gaps and the central vertex v is not member of B. By Lemma 6.2.2, B contains at most one major gap, without loss of generality it can be said that major gap lies on $C_{2n,1}$. Hence B induces $\lfloor\frac{r}{2}\rfloor$ gaps on $C_{2n,1}$ and $\lceil\frac{r}{2}\rceil$ gaps on $C_{2n,2}$. The gaps on $C_{2n,1}$ will be denoted by $G_1^1,G_2^1,G_3^1,...,G_{\lfloor\frac{r}{2}\rfloor}^1$ where G_i^1 and G_{i+1}^1 are called neighboring gaps for $1\leq i\leq\lfloor\frac{r}{2}\rfloor-1$ as well as $G_{\lfloor\frac{r}{2}\rfloor}^1$ is also neighboring gap of G_1^1 and the gaps on $C_{2n,2}$ will be denoted by $G_1^2,G_2^2,G_3^2,...,G_{\lceil\frac{r}{2}\rceil}^2$ where G_i^2 and G_{i+1}^2 are called neighboring gaps for $1\leq i\leq\lceil\frac{r}{2}\rceil-1$

as well as $G^2_{\lceil \frac{r}{2} \rceil}$ is also neighboring gap of G^2_1. By Lemma 6.2.2 suppose G^1_1 is a major gap. By Lemma 6.2.3 and 6.2.4 it can be stated that for $2 \leq i \leq \lfloor \frac{r}{2} \rfloor - 1$ the following inequalities hold

$$|G^1_1 + G^1_2| \leq 6 \tag{6.2.29}$$

$$|G^1_1 + G^1_{\lfloor \frac{r}{2} \rfloor}| \leq 6 \tag{6.2.30}$$

$$|G^1_i + G^1_{i+1}| \leq 4 \tag{6.2.31}$$

and for $2 \leq i \leq \lceil \frac{r}{2} \rceil - 1$.

$$|G^2_1 + G^2_2| \leq 4 \tag{6.2.32}$$

$$|G^2_1 + G^2_{\lceil \frac{r}{2} \rceil}| \leq 4 \tag{6.2.33}$$

$$|G^2_i + G^2_{i+1}| \leq 4 \tag{6.2.34}$$

By using residue class modulo 2, two cases to which r belongs will be considered.

Case-I: When $r \equiv 0 (mod 2)$: In this case

$$\lfloor \frac{r}{2} \rfloor = \lceil \frac{r}{2} \rceil = \frac{r}{2} \tag{6.2.35}$$

By summing the above inequalities (6.2.29) to (6.2.31) the following results can be obtained

$$2(2n - \frac{r}{2}) = 2 \sum_{i=1}^{\frac{r}{2}} |G^1_i| \leq 2r + 4 \tag{6.2.36}$$

$$\Rightarrow \quad \frac{r}{2} \geq \frac{2n-2}{3} \tag{6.2.37}$$

$$\Rightarrow \quad \frac{r}{2} \geq \lfloor \frac{2n}{3} \rfloor \tag{6.2.38}$$

Again from (6.2.32) to (6.2.34)

$$2(2n - \frac{r}{2}) = 2 \sum_{i=1}^{\frac{r}{2}} |G^2_i| \leq 2r \tag{6.2.39}$$

$$\Rightarrow \quad \frac{r}{2} \geq \frac{2n}{3} \tag{6.2.40}$$

$$\Rightarrow \quad \frac{r}{2} \geq \lceil \frac{2n}{3} \rceil \tag{6.2.41}$$

from (6.2.38) and (6.2.41) the following inequality exists

$$r \geq \lfloor \frac{2n}{3} \rfloor + \lceil \frac{2n}{3} \rceil \tag{6.2.42}$$

$$\Rightarrow \quad dim(J_{2n,2}) \geq dim(J_{2n,1}) + \lceil \frac{2n}{3} \rceil \tag{6.2.43}$$

Case-II: When $r \equiv 1(mod2)$: In this case $\lfloor \frac{r}{2} \rfloor = \frac{r-1}{2}$ and $\lceil \frac{r}{2} \rceil = \frac{r+1}{2}$

By summing the inequalities (6.2.29) to (6.2.31) the following inequality can be determined

$$2(2n - \frac{r-1}{2}) = 2\sum_{i=1}^{\frac{r-1}{2}} |G_i^1| \leq 4 + 4(\frac{r-1}{2}) \tag{6.2.44}$$

$$\Rightarrow \quad \frac{r-1}{2} \geq \frac{2n-2}{3} \tag{6.2.45}$$

$$\Rightarrow \quad \frac{r-1}{2} \geq \lfloor \frac{2n}{3} \rfloor \tag{6.2.46}$$

and from (6.2.32) to (6.2.34) the following inequality exists

$$2(2n - \frac{r+1}{2}) = 2\sum_{i=1}^{\frac{r+}{2}} |G_i^2| \leq 4(\frac{r+1}{2}) \tag{6.2.47}$$

$$\Rightarrow \quad \frac{r+1}{2} \geq \frac{2n}{3} \tag{6.2.48}$$

$$\Rightarrow \quad \frac{r+1}{2} \geq \lceil \frac{2n}{3} \rceil \tag{6.2.49}$$

from (6.2.46) and (6.2.49) the following inequality can be obtained

$$r \geq \lfloor \frac{2n}{3} \rfloor + \lceil \frac{2n}{3} \rceil \tag{6.2.50}$$

$$\Rightarrow \quad dim(J_{2n,2}) \geq dim(J_{2n,1}) + \lceil \frac{2n}{3} \rceil \tag{6.2.51}$$

which complete the proof. $\square$

Theorem 6.2.6. If $n \geq 4, m \geq 3$ then $dim(J_{2n,m}) = dim(J_{2n,1}) + (m-1)\lceil \frac{2n}{3} \rceil$.

Proof.

Mathematical induction on level m of Jahangir graph will be used to prove the result. For $m = 1$

$$\lfloor \frac{2n}{3} \rfloor = dim(J_{2n,1}) \tag{6.2.52}$$

is given in [53]. For $m = 2$

$$dim(J_{2n,2}) = dim(J_{2n,1}) + \lceil \frac{2n}{3} \rceil \tag{6.2.53}$$

By Theorem 6.2.1. Suppose for $m = k$, the result is true

$$dim(J_{2n,k}) = dim(J_{2n,1}) + (k-1)\lceil \frac{2n}{3} \rceil. \tag{6.2.54}$$

it will be shown that the result for $m = k+1$ is also true, by using the concept of Theorem 6.2.1

$$dim(J_{2n,k+1}) = dim(J_{2n,k}) + \lceil \frac{2n}{3} \rceil \tag{6.2.55}$$

$$dim(J_{2n,k+1}) = dim(J_{2n,k}) + \lceil \frac{2n}{3} \rceil = dim(J_{2n,1}) + (k-1)\lceil \frac{2n}{3} \rceil + \lceil \frac{2n}{3} \rceil \tag{6.2.56}$$

$$\Rightarrow \quad dim(J_{2n,k+1}) = dim(J_{2n,1}) + k\lceil \frac{2n}{3} \rceil. \tag{6.2.57}$$

This completes the proof. $\square$

6.3 Metric dimension of generalized Jahangir graph J_{3n}

To define the generalized Jahangir ghraph J_{3n}; consider a cycle C_{3n} having vertices $v_1, v_2, v_3, ..., v_{3n}, v_1$ with $n \geq 2$, "join the vertices $v_3, v_6, v_9, ..., v_{3n}$ with a new vertex v".

J_{3n} has order $3n+1$ and size $4n$. It can be obtained from wheel graph W_{3n} by alternately deleting $2n$ spokes. $\square$

The vertices of C_{3n} in the graph J_{3n} are of two kinds: vertices of degree 2 and 3. Vertices of degree 2 and 3 will be considered as minor and major vertices respectively. The graph J_{3n} is a bipartite graph in which one bipartition class contains minor vertices together with central vertex v and the second bipartition class contains major vertices. In the proof of lemma $6.2.2-6.2.5$ major vertices will be represented by star. One can easily check that $dim(J_6)=2$(one minor vertex of C_6 together with central vertex v form basis), $dim(J_9)=2=dim(J_{12})$ (two minor vertices w_1 and w_2 such that $d(w_1,w_2)=3$ form basis), $dim(J_{15})=3$(three minor vertices w_1, w_2 and w_3 such that $d(w_1,w_2)=d(w_2,w_3)=d(w_3,w_4)=4$ form basis).

For the graph $J_{3n}, n\geq 4$ central vertex v is not member of any basis set B. Since $d(v_i,v)\leq 2$ $\forall$, $1\leq i\leq 3n$, and $diam(J_{3n})=4$ if central vertex $v\in B$ then $\exists$ two vertices u_i and u_j, $u_i\neq u_j$ for $1\leq i\neq j\leq 3n$ such that $r(u_i/B)=r(u_J/B)$. If B is a basis of J_{3n} and central vertex v does not belong to B then by using the concept of gap given in section 6.2, there exist gaps of three types $2-2$ gap, $2-3$ gap and $3-3$ gap.

Lemma 6.3.1. *If* B *is a basis of* J_{3n}, $n\geq 6$ *then every* $2-2, 2-3$ *and* $3-3$ *gap of* B *contains at most 8, 7 and 5 points respectively.*

Proof.

Suppose the basis set B contains a $2-2$ gap of nine consecutive vertices $u_1,u_2,u_3,$ u_4,u_5,u_6,u_7,u_8,u_9 of C_{3n} such that

$$deg(u_1)=deg(u_9) \tag{6.3.1}$$

Then

$$r(u_4/B)=r(u_6/B) \tag{6.3.2}$$

in this case. If $2-3$ gap contains more then 7 vertices then it contains 9 consecutive vertices $u_1, u_2, u_3, u_4, u_5, u_6, u_7, u_8, u_9$ of C_{3n} such that

$$deg(u_1) = 3 \tag{6.3.3}$$

and

$$deg(u_9) = 2 \tag{6.3.4}$$

Thus

$$r(u_4/B) = r(u_7/B) \tag{6.3.5}$$

a contradiction in this case. If a $3-3$ gap contains more then 5 vertices then it contains 8 consecutive vertices $u_1, u_2, u_3, u_4, u_5, u_6, u_7, u_8$ such that

$$deg(u_1) = deg(u_8) = 2 \tag{6.3.6}$$

then

$$r(u_3/B) = r(u_6/B) \tag{6.3.7}$$

which is again a contradiction. □

We will call $2-2$ gap, $2-3$ gap and $3-3$ gap having 8, 7 and 5 vertices respectively as major gaps and all the other gaps are minor gaps.

Lemma 6.3.2. *For $n \geq 6$, every basis set* B *of J_{3n} , containing one major gap either $2-2$ or $2-3$ then it does not contain $2-2$ and $2-3$ minor gap having 7 and 6 vertices respectively.*

Proof.

Case-I: when one gap is $2-2$ major gap and other is $2-2$ minor gap having 7 vertices then there exist two distinct paths $u_1, u_2^*, u_3, u_4, u_5^*, u_6, u_7, u_8^*$ and $w_1^*, w_2, w_3, w_4^*, w_5, w_6, w_7^*$ but

$$r(u_5^*/B) = r(w_4^*/B) \tag{6.3.8}$$

Case-II: when one gap is $2-2$ major gap and other is $2-3$ minor gap having 6 vertices then there exist two distinct paths $u_1,u_2^*,u_3,u_4,u_5^*,u_6,u_7,u_8^*$ and $w_1^*,w_2,w_3,w_4^*,w_5,w_6$ but

$$r(u_5^*/B) = r(w_4^*/B) \tag{6.3.9}$$

Case-III: when one gap is $2-3$ major gap and other is $2-2$ minor gap having 7 vertices then there exist two distinct paths $u_1,u_2^*,u_3,u_4,u_5^*,u_6,u_7$ and $w_1^*,w_2,w_3,w_4^*,w_5,w_6,w_7^*$ but

$$r(u_5^*/B) = r(w_4^*/B) \tag{6.3.10}$$

Case-IV: when one gap is $2-3$ major gap and other is $2-3$ minor gap having 6 vertices then there exist two distinct paths $u_1,u_2^*,u_3,u_4,u_5^*,u_6,u_7$ and $w_1^*,w_2,w_3,w_4^*,w_5,w_6$ but

$$r(u_5^*/B) = r(w_4^*/B) \tag{6.3.11}$$

□

Lemma 6.3.3. *For each basis set* B *of J_{3n} for $n \geq 6$ two neighboring gaps consist together thirteen vertices at most in which one gap is a major gap.*

Proof.
To show the statement, it is sufficient to show that a $2-2$ major gap with 8 vertices has a neighboring $2-2$ minor gap with 6 vertices cannot occur. If it holds then there exists the path $u_1,u_2,u_3^*,u_4,u_5,u_6^*,u_7,u_8,u_9^*,u_{10},u_{11},u_{12}^*,u_{13},u_{14},u_{15}^*,$ u_{16},u_{17} with $u_1,u_{10},u_{17} \in B$ in this case

$$r(u_7/B) = r(w_{13}/B) \tag{6.3.12}$$

a contradiction. □

Lemma 6.3.4. *For each basis set* B *of J_{3n} for $n \geq 6$ two minor neighboring gaps consist together thirteen vertices at most.*

Proof.

To show the statement, it is sufficient to show that a $2-2$ gap with 6 vertices has a neighboring $2-2$ gap with 6 vertices cannot occur. As gap is $2-2$ so both the base element must have degree 2. For two consecutive $2-2$ gap having 6 vertices there exist two possible paths.

(i) First possible path is

$u_1, u_2, u_3^*, u_4, u_5, u_6^*, u_7, u_8, u_9^*, u_{10}, u_{11}, u_{12}^*, u_{13}, u_{14}, u_{15}^*$ with $u_1, u_8, u_{15}^* \in B$ which is not possible as

$$d(u_1) = 2 = d(u_8) \tag{6.3.13}$$

but

$$d(u_{15}^*) = 3 \neq 2 \tag{6.3.14}$$

(i) Second possible path is

$u_2, u_3^*, u_4, u_5, u_6^*, u_7, u_8, u_9^*, u_{10}, u_{11}, u_{12}^*, u_{13}, u_{14}, u_{15}^*, u_{16}$ with $u_2, u_9^*, u_{16} \in B$ which is not possible as

$$d(u_2) = 2 = d(u_{16}) \tag{6.3.15}$$

but

$$d(u_9^*) = 3 \neq 2 \tag{6.3.16}$$

Hence two minor gap contain at most 11 vertices. □

Theorem 6.3.5. *If $n \geq 6$, then $dim(J_{3n}) = \lfloor \frac{n}{2} \rfloor$.*

Proof.

First, it will be shown that $dim(J_{3n}) \leq \lfloor \frac{n}{2} \rfloor$ for this purpose resolving set W having $\lfloor \frac{n}{2} \rfloor$ vertices will be created in J_{3n}. By using residue class modulo 2, two cases to which n belongs will be considered.

Case-I: For $n \equiv 0(mod2)$, W can be considered as

$$W = \{v_1, v_{10}, v_{16}\} \cup \{v_{6i+5}; 3 \leq i \leq \frac{n}{2} - 1\} \tag{6.3.17}$$

Case-II: When $n \equiv 1(mod2)$ then W can be considered as

$$W = \{v_1, v_{10}, v_{16}\} \cup \{v_{6i+5}; 3 \leq i \leq \frac{n-1}{2} - 1\} \tag{6.3.18}$$

The set W contains a unique $2-2$ major gap and all other gaps are $2-2$ minor gaps which contain at most five vertices, only one $2-2$ minor gap contains six vertices. As two minor or two major vertices falling in same gap or separate gap are differentiated by at least 1 vertex in the set of 3 vertices of W producing these neighboring gaps, when gaps are not neighboring gaps then the set of 4 vertices of W which produce 2 gaps make the representation unique of every vertex of these 2 gaps so W is a resolving set of J_{3n}. Representation of central vertex $(2,2,2,...,2)$ which is different from the representation of all other vertices of J_{3n}. Hence

$$dim(J_{3n}) \leq \lfloor \frac{n}{2} \rfloor \tag{6.3.19}$$

Now it will be shown that

$$dim(J_{3n}) \geq \lfloor \frac{n}{2} \rfloor. \tag{6.3.20}$$

Suppose B be a basis set of J_{3n} with $|B| = r$ such that central vertex v is not member of B then there exist r gaps on C_{3n} produced by elements of B. These gaps will be denoted by $G_1, G_2, G_3, ..., G_r$ where G_i and G_{i+1}, known as neighboring gaps for $1 \leq i \leq r-1$ as well as G_r is also neighboring gap of G_1. By Lemma 6.3.3 at most one of them say G_1 is a major gap. Lemmas 6.3.2, 6.3.3 and 6.3.5 give

$$|G_1 + G_2| \leq 13 \tag{6.3.21}$$

$$|G_1 + G_r| \leq 13 \tag{6.3.22}$$

and Lemma 6.3.4 gives

$$|G_2 + G_3| \leq 11 \tag{6.3.23}$$

$$|G_3 + G_4| \leq 11 \tag{6.3.24}$$

and

$$|G_i + G_{i+1}| \leq 10 \;\; \forall \;\; 4 \leq i \leq r-1. \tag{6.3.25}$$

By summing these inequalities the following results can be obtained

$$2(3n-r) = 2\sum_{i=1}^{r} |G_i| \leq 8 + 10r \tag{6.3.26}$$

$$\Rightarrow \quad 6n - 2r \leq 8 + 10r \tag{6.3.27}$$

$$\Rightarrow \quad 6n - 8 \leq 12r \tag{6.3.28}$$

$$\Rightarrow r \geq \frac{n}{2} - \frac{2}{3} \tag{6.3.29}$$

Hence

$$r = \lfloor \frac{n}{2} \rfloor \tag{6.3.30}$$

which complete the proof. □

Chapter 7

On metric dimension of graphs obtained by rooted product

7.1 Introduction

In this chapter, cycle graph, Harary graphs and their rooted product in the perspective of metric dimension are studied. It is proved that all the graphs have unbounded metric dimensions. In [34] constant metric dimension of Harary graph has been calculated and rooted product with its spectrum has been studied by Godsil in [29].

7.2 Metric dimension of graphs obtained by rooted product of Harary graph with cycle.

As defined in [29], the rooted product of a labeled graph H on n vertices with a sequence G of n rooted graphs $G_1, G_2, ...G_n$ denoted by $H(G)$ can be obtained by identifying the root of G_i with the ith vertex of H. The graphs $H_{m,n}(C_3)$, $H_{m,n}(C_4)$ and $H_{m,n}(C_5)$ are the rooted product of Harary graphs $H_{m,n}$ by cycles C_3, C_4 and C_5 respectively. Suppose $C_3^i, 1 \leq i \leq n$, $C_4^i, 1 \leq i \leq n$ and $C_5^i, 1 \leq i \leq n$ be n copies of C_3, $C_4$and C_5 having vertices $\{v_i^j, 1 \leq i \leq n, 1 \leq j \leq 3\}$, $\{v_i^j, 1 \leq i \leq n, 1 \leq j \leq 4\}$ and $\{v_i^j, 1 \leq i \leq n, 1 \leq j \leq 5\}$ respectively and $\{v_1, v_2, v_3, ..., v_n\}$ be the set of vertices of $H_{m,n}$. By definition of rooted product $\{v_i^j, 1 \leq i \leq n, 1 \leq j \leq 3\}$, $\{v_i^j, 1 \leq i \leq n, 1 \leq j \leq 4\}$ and $\{v_i^j, 1 \leq i \leq n, 1 \leq j \leq 5\}$ will be sets of vertices of $H_{m,n}(C_3)$,$H_{m,n}(C_4)$ and $H_{m,n}(C_5)$ respectively with indices taken modulo n. After rooted product, it is considered that all the cycles shares $\{v_i^1, 1 \leq i \leq n\}$ with $H_{m,n}$. Now the results related to metric dimension of $H_{m,n}(C_3)$, $H_{m,n}(C_4)$ and $H_{m,n}(C_5)$ are presented.

Theorem 7.2.1. *If $G_1 \cong H_{m,n}(C_3)$ then for every resolving set W of G_1 there exist $\{v_i^2; 1 \leq i \leq n\} \subseteq W(G_1)$, $\{v_i^3; 1 \leq i \leq n\} \nsubseteq W(G_1)$ and $|W| \geq n$.*

Proof.

$$d(v_i^2, v_i^1) = d(v_i^3, v_i^1) \quad \forall \quad 1 \leq i \leq n \tag{7.2.1}$$

and

$$d(v_i^2, v_k^j) = d(v_i^3, v_k^j) \quad \forall \quad 1 \leq i,k \leq n, \quad k \neq i, \quad j = 1,2,3 \tag{7.2.2}$$

$\Rightarrow$ either $v_i^2 \in W(G_1)$ or $v_i^3 \in W(G_1)$.

To minimize the cardinality of $W(G_1)$ it can be said without loss of any generality $\{v_i^2; 1 \leq i \leq n\} \subseteq W(G_1)$ and $\{v_i^3; 1 \leq i \leq n\} \nsubseteq W(G_1)$.

$$\Rightarrow |W(G_1)| \geqslant n \tag{7.2.3}$$

□

Theorem 7.2.2. *If $G_1 \cong H_{m,n}(C_3)$ and W be a minimum resolving set of G_1 then $|W| = n$.*

Proof.

From Theorem 7.2.1 $|W(G_1)| \geqslant n$ to prove that $|W| = n$, we shall prove that $W \setminus \{v_i^1; 1 \leq i \leq n\}$ is also the resolving set.

Now take $\{v_i^1, v_i^3; 1 \leq i \leq n\} \cap W = \phi$ then W is also resolving set.

By Theorem 1 $v_i^2 \in W$ for all $1 \leq i \leq n$

and

$$d(v_i^2, v_i^1) \neq d(v_i^2, v_k^1), \quad \forall \quad 1 \leq i,k \leq n, \quad i \neq k \tag{7.2.4}$$

$$\Rightarrow \quad r(v_i^1/W) \neq r(v_i^1/W), \quad \forall \quad 1 \leq i,k \leq n, \quad i \neq k \tag{7.2.5}$$

As $v_i^2 \in W \Rightarrow v_{i+1}^2 \in W$ and

$$d(v_{i+1}^2, v_i^1) \neq d(v_{i+1}^2, v_i^3), \quad \forall \quad 1 \leq i \leq n \tag{7.2.6}$$

$$\Rightarrow \quad r(v_i^1/W) \neq r(v_i^3/W), \ \forall \ 1 \leq i \leq n \tag{7.2.7}$$

$$r(v_i^1/W) \neq r(v_i^2/W) \ \ As \ \ \{v_i^2; 1 \leq i \leq n\} \subseteq W \tag{7.2.8}$$

also

$$r(v_i^3/W) \neq r(v_i^2/W) \ \ As \ \ \{v_i^2; \ \ 1 \leq i \leq n\} \subseteq W \tag{7.2.9}$$

$$d(v_i^3, v_i^2) \neq d(v_k^3, v_i^2), \ \ \forall \ \ 1 \leq i, k \leq n, \ \ i \neq k. \tag{7.2.10}$$

$$\Rightarrow \ \ r(v_i^3/W) \neq r(v_k^3/W), \ \ \forall \ \ 1 \leq i, k \leq n, \ \ i \neq k. \tag{7.2.11}$$

Hence the result. □

Theorem 7.2.3. *If $G_2 \cong H_{m,n}(C_4)$ then for every resolving set W of G_2 there exist $\{v_i^2; 1 \leq i \leq n\} \subseteq W(G_2)$, $\{v_i^4; 1 \leq i \leq n\} \nsubseteq W(G_2)$ and $|W| \geq n$.*

Proof.

Similar to the proof of Theorem 7.2.1.

Theorem 7.2.4. *If $G_2 \cong H_{m,n}(C_4)$ and W be a minimum resolving set of G_2 then $|W| = n$.*

Proof.

From Theorem 7.2.3 $|W(G_2)| \geqslant n$ to prove that $|W| = n$, we shall prove that
$W \setminus \{v_i^1, v_i^3; 1 \leq i \leq n\}$ is also the resolving set.
Then take

$$\{v_i^1, v_i^3, v_i^4; 1 \leq i \leq n\} \cap W = \phi \tag{7.2.12}$$

then W is also resolving set.
By Theorem 7.2.3 $v_i^2 \in W$ for all $1 \leq i \leq n$ and

$$d(v_i^2, v_i^1) \neq d(v_i^2, v_k^1), \ \ \forall \ \ 1 \leq i, k \leq n, \ \ i \neq k \tag{7.2.13}$$

$$\Rightarrow \ \ r(v_i^1/W) \neq r(v_k^1/W), \ \ \forall \ \ 1 \leq i, k \leq n, \ \ i \neq k \tag{7.2.14}$$

By definition of resolving set

$$r(v_i^1/W) \neq r(v_i^2/W) \ \ \forall \ \ 1 \leq i \leq n \tag{7.2.15}$$

$$d(v_i^2, v_i^1) \neq d(v_i^2, v_i^4), \;\; \forall \;\; 1 \leq i \leq n \tag{7.2.16}$$

$$\Rightarrow \;\; r(v_i^1/W) \neq r(v_i^4/W), \;\; \forall \;\; 1 \leq i \leq n \tag{7.2.17}$$

As $v_i^2 \in W$ for all $1 \leq i \leq n \Rightarrow v_{i+1}^2 \in W$

$$d(v_{i+1}^2, v_i^3) \neq d(v_{i+1}^2, v_i^1), \;\; \forall \;\; 1 \leq i \leq n \tag{7.2.18}$$

$$\Rightarrow \;\; r(v_i^1/W) \neq r(v_i^3/W), \;\; \forall \;\; 1 \leq i \leq n \tag{7.2.19}$$

By definition of resolving set

$$r(v_i^2/W) \neq r(v_k^2/W) \;\; \forall \;\; 1 \leq i,k \leq n, \;\; i \neq k \tag{7.2.20}$$

$$r(v_i^2/W) \neq r(v_k^3/W) \;\; \forall \;\; 1 \leq i,k \leq n, \;\; i \neq k \tag{7.2.21}$$

$$r(v_i^2/W) \neq r(v_k^4/W) \;\; \forall \;\; 1 \leq i,k \leq n, \;\; i \neq k \tag{7.2.22}$$

As

$$d(v_i^2, v_i^3) \neq d(v_i^2, v_k^3), \;\; \forall \;\; 1 \leq i,k \leq n, \;\; i \neq k \tag{7.2.23}$$

$$\Rightarrow \;\; r(v_i^3/W) \neq r(v_k^3/W), \;\; \forall \;\; 1 \leq i,k \leq n, \;\; i \neq k \tag{7.2.24}$$

$$d(v_i^2, v_i^3) \neq d(v_i^2, v_i^4), \;\; \forall \;\; 1 \leq i \leq n \tag{7.2.25}$$

$$\Rightarrow \;\; r(v_i^4/W) \neq r(v_i^3/W), \;\; \forall \;\; 1 \leq i \leq n \tag{7.2.26}$$

$$d(v_i^2, v_i^4) \neq d(v_i^2, v_k^4), \;\; \forall \;\; 1 \leq i,k \leq n, \;\; i \neq k \tag{7.2.27}$$

$$\Rightarrow \;\; r(v_i^4/W) \neq r(v_k^4/W), \;\; \forall \;\; 1 \leq i,k \leq n, \;\; i \neq k \tag{7.2.28}$$

$\Rightarrow$ representation of all the vertices is unique if $\{v_i^1, v_i^3; 1 \leq i \leq n\} \nsubseteq W(G_2)$
Also from theorem 7.2.3 $\{v_i^4; 1 \leq i \leq n\} \nsubseteq W(G_2)$ and $\{v_i^2; 1 \leq i \leq n\} \subseteq W(G_2)$.
Hence $W = \{v_i^2; 1 \leq i \leq n\}$ is the minimum resolving set of G_2 and $|W| = n$. □

Theorem 7.2.5. *If $G_3 \cong H_{m,n}(C_5)$ then for every resolving set W of G_3 there exists $\{v_i^2; 1 \leq i \leq n\} \subseteq W(G_3)$ and $|W| \geq n$.*

Proof.

As

$$d(v_i^2, v_k^j) = d(v_i^5, v_k^j) \ \forall \ 1 \leq i,k \leq n, \ \ i \neq k \ \ and \ \ 1 \leq j \leq 5 \tag{7.2.29}$$

and

$$d(v_i^2, v_i^1) = d(v_i^5, v_i^1) \ \forall \ 1 \leq i \leq n \tag{7.2.30}$$

$$d(v_i^2, v_i^3) \neq d(v_i^5, v_i^3) \ \forall \ 1 \leq i \leq n \tag{7.2.31}$$

and

$$d(v_i^2, v_i^4) \neq d(v_i^5, v_i^4) \ \forall \ 1 \leq i \leq n \tag{7.2.32}$$

$\Rightarrow \ \forall \quad 1 \leq i \leq n$ and $\forall$ resolving sets W in which $\{v_i^2, v_i^3, v_i^4, v_i^5\} \cap W = \phi$.

$$r(v_i^2/W) = r(v_i^5/W) \tag{7.2.33}$$

also $\forall \ 1 \leq i \leq n$ and $\forall$ resolving sets W in which
$\{v_i^2, v_i^3, v_i^4, v_i^5\} \cap W = \phi$.

$$r(v_i^3/W) = r(v_i^4/W) \tag{7.2.34}$$

To make the representation unique it can be said that $\{v_i^2, v_i^3, v_i^4, v_i^5\} \cap W \neq \phi$. Without loss of any generality it can be assumed that $\{v_i^2; 1 \leq i \leq n\} \subseteq W(G_3) \Rightarrow |W| \geq n$. $\square$

Theorem 7.2.6. *If $G_3 \cong H_{m,n}(C_5)$ and W be a minimum resolving set of G_3 then $|W| = n$.*

Proof.

From theorem 7.2.5 $|W(G_3)| \geqslant n$ to prove that $|W| = n$, we shall prove that
$W \setminus \{v_i^1, v_i^3, v_i^4, v_i^5; 1 \leq i \leq n\}$ is also the resolving set.

Then take

$$\{v_i^1, v_i^3, v_i^4, v_i^5; 1 \leq i \leq n\} \cap W = \phi \tag{7.2.35}$$

then W is also resolving set. By Theorem 7.2.5 $v_i^2 \in W$ for all $1 \leq i \leq n$

and

$$d(v_i^2, v_i^1) \neq d(v_i^2, v_k^1), \;\; \forall \;\; 1 \leq i,k \leq n, \;\; i \neq k \tag{7.2.36}$$

$$\Rightarrow \;\; r(v_i^1/W) \neq r(v_k^1/W), \;\; \forall \;\; 1 \leq i,k \leq n, \;\; i \neq k \tag{7.2.37}$$

and

$$r(v_k^1/W) \neq r(v_i^2/W) \;\; \forall \;\; 1 \leq i,k \leq n \tag{7.2.38}$$

By definition of resolving set

$$d(v_i^1, v_{i+1}^2) \neq d(v_i^3, v_{i+1}^2), \;\; \forall \;\; 1 \leq i \leq n \tag{7.2.39}$$

and

$$d(v_i^1, v_i^2) \neq d(v_k^3, v_i^2), \;\; \forall \;\; 1 \leq i,k \leq n, \;\; i \neq k \tag{7.2.40}$$

$$r(v_i^1/W) \neq r(v_k^3/W) \;\; \forall \;\; 1 \leq i,k \leq n \tag{7.2.41}$$

As

$$d(v_i^1, v_i^2) \neq d(v_k^4, v_i^2) \;\; \forall \;\; 1 \leq i, \;\; k \leq n \tag{7.2.42}$$

$$\Rightarrow \;\; r(v_i^1/W) \neq r(v_k^4/W) \;\; \forall \;\; 1 \leq i,k \leq n \tag{7.2.43}$$

As

$$d(v_i^1, v_i^2) \neq d(v_k^5, v_i^2) \;\; \forall \;\; 1 \leq i,k \leq n \tag{7.2.44}$$

$$\Rightarrow \;\; r(v_i^1/W) \neq r(v_k^5/W) \;\; \forall \;\; 1 \leq i,k \leq n \tag{7.2.45}$$

By definition of resolving set

$$r(v_i^2/W) \neq r(v_k^j/W) \;\; \forall \;\; 1 \leq i,k \leq n \;\; 1 \leq j \leq 5 \tag{7.2.46}$$

As

$$d(v_i^3, v_i^2) \neq d(v_k^3, v_i^2) \;\; \forall \;\; 1 \leq i,k \leq n \;\; i \neq k \tag{7.2.47}$$

$$\Rightarrow \;\; r(v_i^3/W) \neq r(v_k^3/W) \;\; \forall \;\; 1 \leq i,k \leq n \;\; i \neq k \tag{7.2.48}$$

$$d(v_i^3, v_i^2) \neq d(v_k^4, v_i^2) \;\; \forall \;\; 1 \leq i,k \leq n \tag{7.2.49}$$

$$\Rightarrow \;\; r(v_i^3/W) \neq r(v_k^4/W) \;\; \forall \;\; 1 \leq i,k \leq n \tag{7.2.50}$$

$$d(v_i^3, v_i^2) \neq d(v_k^5, v_i^2) \;\; \forall \;\; 1 \leq i,k \leq n \tag{7.2.51}$$

$$\Rightarrow \;\; r(v_i^3/W) \neq r(v_k^5/W) \;\; \forall \;\; 1 \leq i,k \leq n \tag{7.2.52}$$

As

$$d(v_i^4, v_i^2) \neq d(v_k^4, v_i^2), \;\; \forall \;\; 1 \leq i,k \leq n \;\; i \neq k \tag{7.2.53}$$

$$\Rightarrow \;\; r(v_i^4/W) \neq r(v_k^4/W), \;\; \forall \;\; 1 \leq i,k \leq n \;\; i \neq k \tag{7.2.54}$$

$$d(v_i^4, v_{i+1}^2) \neq d(v_i^5, v_{i+1}^2), \;\; \forall \;\; 1 \leq i \leq n \tag{7.2.55}$$

and

$$d(v_i^4, v_i^2) \neq d(v_k^5, v_i^2), \;\; \forall \;\; 1 \leq i,k \leq n, \;\; i \neq k \tag{7.2.56}$$

$$r(v_i^4/W) \neq r(v_k^5/W), \;\; \forall \;\; 1 \leq i,k \leq n \tag{7.2.57}$$

$$d(v_i^5, v_i^2) \neq d(v_k^5, v_i^2), \;\; \forall \;\; 1 \leq i,k \leq n \;\; i \neq k \tag{7.2.58}$$

$$\Rightarrow \;\; r(v_i^5/W) \neq r(v_k^5/W), \;\; \forall \;\; 1 \leq i,k \leq n \;\; i \neq k \tag{7.2.59}$$

Hence $W \setminus \{v_i^1, v_i^3, v_i^4, v_i^5; 1 \leq i \leq n\}$ is the resolving set.

$$\Rightarrow \;\; |W| = n \tag{7.2.60}$$

□

Chapter 8

On constant metric dimension of generalized prism related graphs

8.1 Introduction

In this chapter, generalized prism related graphs are studied and their metric dimensions have been calculated. It is proved that metric dimensions of each graph is either two or three in certain cases. In [18] constant metric dimension of generalized prism graph has been calculated and metric dimensions of two families of graphs related to prism graph have been studied by Ali in [6].

8.2 Main Results

Generalized prism related graphs can be constructed with the help of generalized prism $C_n \times P_m$. Consider $C_n \times P_{m-1}$ and put a pendent on each vertex of last cycle consequently one of the generalized prism related graphs is obtained, for second graph put two pendants on each vertex of last cycle and take three consecutive pendent vertices of cycle then connect one pendent of the middle pendent vertex with one pendent of the next vertex as well as one pendent of the middle pendent vertex with one pendent of the previous vertex, repeating the procedure on every triplet(three consecutive pendent vertices of cycle) of last cycle, second generalized prism related graph is obtained as shown in Figure 8.1. Graph containing n pendant will be denoted by GP^n_{m-1} and graph containing $2n$ pendant will be denoted by GP^{2n}_{m-1}. In particular when $m = 3$ a graph as described in [6] is obtained.

Generalized prism related graphs can be used in networking [57]. Vertices of generalized prism related graphs GP^n_{m-1} and GP^{2n}_{m-1} can be represented by a^j_i where $1 \leq i \leq n$ and $1 \leq j \leq m$. The graphs GP^n_{m-1} and GP^{2n}_{m-1} have $2(mn-n)$ and $2mn-n$ edges respectively but these graphs have equal number of mn vertices. Now the main results on metric dimension of generalized prism related graphs GP^n_{m-1} and GP^{2n}_{m-1} are presented.

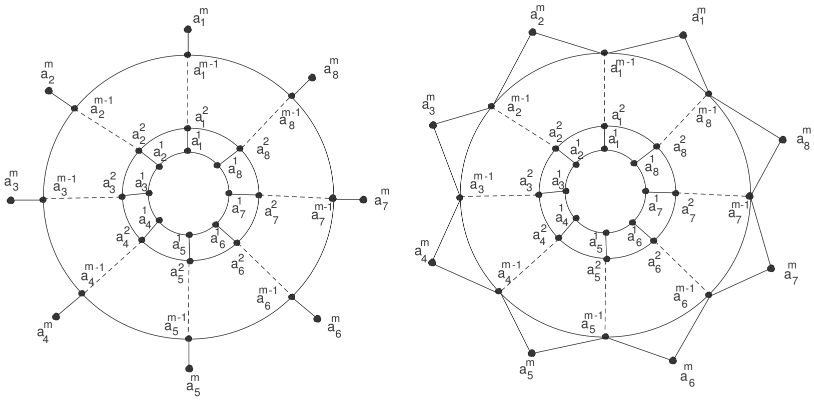

Figure 8.1: Graph of GP^{8}_{m-1} and GP^{16}_{m-1}.

Theorem 8.2.1. For $G \cong GP^{n}_{m-1}$

$$dim(G) = \begin{cases} 2, & n = 2p+1, p \in N \\ 3, & n = 2p+2, p \in N \end{cases}$$

where $m, n > 2$

Proof.

Case-I

When $n = 2k+1, k \in N$ let $W = \{a^1_1, a^1_{k+1}\}$ be the resolving set of G then

$$r(a^j_i/W) = \begin{cases} (i+j-2, k+j-i) \\ \qquad\qquad 1 \le i \le k+1, 1 \le j \le m, m \in N \\ (2k+j-i+1, i-k+j-2) \\ \qquad\qquad k+2 \le i \le n, 1 \le j \le m, m \in N \end{cases} \tag{8.2.1}$$

Since G is not a path graph so $dim(G) = 2$.

Case-II

When $n = 2k, k \in N$ let $W = \{a^1_1, a^1_2, a^1_{k+1}\}$ be the resolving set of G then

$$r(a_i^j/W) = \begin{cases} (j-1, j, k+j-1) \\ \qquad\qquad i = 1, 1 \leq j \leq m, m \in N \\ (i+j-2, i+j-3, k+j-i) \\ \qquad\qquad 2 \leq i \leq k+1, 1 \leq j \leq m, m \in N \\ (2k+j-i, 2k+j-i+1, i+j-k-2) \\ \qquad\qquad k+2 \leq i \leq n, 1 \leq j \leq m, m \in N \end{cases} \tag{8.2.2}$$

Since distinct vertices have distinct representation so $dim(G) \leq 3$.

Now, $dim(G) \neq 2$ due to the following contradictions which arise in all possible cases.

Without loss of generality it can be said that

$W = \{a_1^p, a_t^j\}$ where $1 \leq p \leq m, p \leq j \leq m, 1 \leq t \leq k+1$ and $t \neq 1$ when $j = p$ represent all possible resolving sets in which $|W| = 2$.

Now take $W = \{a_1^p, a_t^j\}$ where $1 \leq p \leq m-1, p \leq j \leq m-1, 1 \leq t \leq k+1$ and $t \neq 1$ when $j = p$ and $1 \leq t \leq k$

$$r(a_n^j/W) = r(a_1^{j+1}/W) = (j-p+1, t) \tag{8.2.3}$$

a contradiction.

When $t = k+1$

$$r(a_n^j/W) = r(a_2^{j+1}/W) = (j-p+1, k-1) \tag{8.2.4}$$

a contradiction.

Take $W = \{a_1^p, a_t^m\}$ where $m-1 \geq p \geq 1, k+1 \geq t \geq 1$

When "$k-1 \geq t \geq 1$"

$$r(a_n^m/W) = (m-p+1, t+2) = r(a_{n-1}^{m-1}/W) \tag{8.2.5}$$

a contradiction.

For $t = k$ the following contradiction arises.

$$r(a_1^{p+1}/W) = r(a_2^p/W) = (1, k+m-p-2) \tag{8.2.6}$$

When $t = k+1$

$$r(a_1^{p+1}/W) = r(a_2^p/W) = (1, k+m-p-1) \tag{8.2.7}$$

a contradiction.

Take $W = \{a_1^m, a_t^m\}$ where $2 \le t \le k+1$
When $2 \le t \le k$ the following contradiction arises.

$$r(a_n^{m-1}/W) = r(a_1^{m-2}/W) = (2, t+1) \tag{8.2.8}$$

For $t = k+1$ the following contradiction arises.

$$r(a_n^{m-1}/W) = r(a_2^{m-1}/W) = (2, k) \tag{8.2.9}$$

Hence $dim(G) = 3$. This completes the proof. □

Theorem 8.2.2. *For $G \cong GP_{m-1}^{2n}$ and $m \ge 3, n \ge 6$ then $dim(G) = 3$*

Proof.
when $n = 2k, k \in N$ let $W = \{a_1^1, a_2^1, a_{k+1}^1\}$ be the resolving set of G then

$$r(a_i^j/W) = \begin{cases} (2k+j-i, 2k+j-i+1, i+j-k-2) \\ \qquad k+2 \le i \le n, 1 \le j \le m-1, m \in N \\ (i+j-2, i+j-3, k+j-i) \\ \qquad 2 \le i \le k+1, 1 \le j \le m-1, m \in N \\ (j-1, j, k+j-1) \\ \qquad i = 1, 1 \le j \le m-1, m \in N \end{cases} \tag{8.2.10}$$

$$r(a_i^m/W)=\begin{cases}(k+m-2,k+m-2,m-1), & k+2=i\\(-i+2k+m,m+2k-i+1,-k+i+m-3), & n\geq i\geq k+3\\(m+i-3,m+i-4,m+k-i), & 3\leq i\leq k+1\\(m-1,m,m+k-2), & i=1\\(m-1,m-1,m+k-2), & i=2\end{cases} \tag{8.2.11}$$

Since distinct vertices have distinct representation so $dim(G)\leq 3$.

Now, $dim(G)\neq 2$ due to the following contradictions which arise in all possible cases.

Without loss of generality it can be said that

$W=\{a_1^p,a_t^j\}$ where $1\leq p\leq m, p\leq j\leq m, 1\leq t\leq k+1$ and $t\neq 1$ when $j=p$ represent all possible resolving sets in which $|W|=2$.

Now take $W=\{a_1^p,a_t^j\}$ where $1\leq p\leq m-1, p\leq j\leq m-1, 1\leq t\leq k+1$ and $t\neq 1$ when $j=p$ and $1\leq t\leq k$

$$r(a_n^j/W)=r(a_1^{j+1}/W)=(j-p+1,t) \tag{8.2.12}$$

a contradiction.

when $j=p$ and $t=k+1$

$$r(a_n^j/W)=r(a_2^j/W)=(j-p+1,k-1) \tag{8.2.13}$$

a contradiction.

Take $W=\{a_1^j,a_t^m\}$ where $1\leq j\leq m-2, k+1\geq t\geq 1$

When $k-1\geq t\geq 1$

$$r(a_n^m/W)=(m-j+1,t+1)=r(a_{n-1}^{m-1}/W) \tag{8.2.14}$$

a contradiction.

For $t = k$ the following contradiction arises.

$$r(a_1^{j+1}/W) = r(a_2^j/W) = (1, k+m-j-3) \tag{8.2.15}$$

a contradiction.

When $t = k+1$

$$r(a_1^{j+1}/W) = r(a_2^j/W) = (1, k+m-j-2) \tag{8.2.16}$$

a contradiction.

Take $W = \{a_1^{m-1}, a_t^m\}$

When $k \geq t \geq 1$

$$\text{“}(r(a_n^{m-2}/W) = 2, t+1) = r(a_n^m/W)\text{”} \tag{8.2.17}$$

a contradiction.

When “$t = k+1$”

$$\text{“}r(a_{k+2}^m/W) = (k, 2) = r(a_k^{m-2}/W)\text{”} \tag{8.2.18}$$

a contradiction.

Take $W = \{a_1^m, a_t^m\}$ where $k+1 \geq t \geq 2$

When $k \geq t \geq 2$

$$r(a_n^m/W) = r(a_n^{m-2}/W) = (2, t+1) \tag{8.2.19}$$

a contradiction.

For $t = k+1$ the following contradiction arises.

$$r(a_n^{m-1}/W) = r(a_1^{m-1}/W) = (1, k) \tag{8.2.20}$$

Hence $dim(G) = 3$.

Case-II

when $n = 2k+1, k \in N$ let $W = \{a_1^1, a_2^1, a_{k+1}^1\}$ be the resolving set of G then

$$r(a_i^j/W) = \begin{cases} (j-1, j, k+j-1) \\ \qquad\qquad i = 1, 1 \le j \le m-1, m \in N \\ (i+j-2, i+j-3, k+j-i) \\ \qquad\qquad 2 \le i \le k+1, 1 \le j \le m-1, m \in N \\ (K+j-1, K+j-1, j) \\ \qquad\qquad i = k+2, 1 \le j \le m-1, m \in N \\ (2k+j-i+1, 2k+j-i+2, i+j-k-2) \\ \qquad\qquad k+3 \le i \le n, 1 \le j \le m-1, m \in N \end{cases} \tag{8.2.21}$$

.

$$r(a_i^m/W) = \begin{cases} (2k+m-i+1, 2k+m-i+2, i-k+m-3) & k+3 \le i \le n \\ (m-1, m-1, m+k-2) & i = 2 \\ (m+i-3, m+i-4, m+k-i) & 3 \le i \le k+1 \\ (m+k-1, m+k-2, m-1) & i = k+2 \\ (m-1, m, m+k-1) & i = 1 \end{cases} \tag{8.2.22}$$

Since distinct vertices have distinct representation so $dim(G) \le 3$.

Now, $dim(G) \neq 2$ due to the following contradictions which arise in all possible cases. Without loss of generality it can be assumed that

$W = \{a_1^p, a_t^j\}$ where $1 \le p \le m, p \le j \le m, 1 \le t \le k+1$ and $t \neq 1$ when $j = p$ represent all possible resolving sets in which $|W| = 2$.

Now take $W = \{a_1^p, a_t^j\}$ where $1 \le p \le m-1, p \le j \le m-1, 1 \le t \le k+1$ and $t \neq 1$ when $j = p$ and $1 \le t \le k$

$$r(a_n^j/W) = r(a_1^{j+1}/W) = (j-p+1, t) \tag{8.2.23}$$

a contradiction.

For $t = k+1$ the following contradiction arises.

$$r(a_n^{m-1}/W) = r(a_2^m/W) = (m-p, m+k-j-1) \tag{8.2.24}$$

a contradiction.

Take $W = \{a_1^j, a_t^m\}$ where $1 \leq j \leq m-2, 1 \leq t \leq k+1$

When $1 \leq t \leq k$

$$r(a_n^m/W) = r(a_{n-1}^{m-1}/W) = (m-j+1, t+1) \tag{8.2.25}$$

a contradiction.

For $t = k+1$ the following contradiction arises.

$$r(a_1^{j+1}/W) = r(a_2^j/W) = (1, k+m-j-2) \tag{8.2.26}$$

Take $W = \{a_1^{m-1}, a_t^m\}$ where

When $k \geq t \geq 1$

$$r(a_n^m/W) = (2, t+1) = r(a_n^{m-2}/W) \tag{8.2.27}$$

a contradiction.

For $t = k+1$ the following contradiction arises.

$$r(a_{k+2}^m/W) = r(a_{k+1}^{m-2}/W) = (k+1, 2) \tag{8.2.28}$$

Take $W = \{a_1^m, a_t^m\}$

When $k \geq t \geq 2$

$$r(a_n^m/W) = r(a_n^{m-2}/W) = (2, t+1) \tag{8.2.29}$$

a contradiction.

For $t = k+1$ the following contradiction arises.

$$r(a_k^m/W) = r(a_{k+2}^{m-1}/W) = (k,2) \tag{8.2.30}$$

Hence $dim(G) = 3$. This completes the proof. $\square$

Chapter 9

Conclusions and Open Problems

9.1 Conclusions

This dissertation contained the formulation for disjoint union of toroidal fullerenes $\mathbb{H}_m^n$ when it admits super d-antimagic labeling of type $(1,1,1)$. Formulation for disjoin union of generalized (subdivided) friendship graph $f_{p,n}$ when it admits "d-antimagic labeling of type $(1,1,1)$" has been presented. Metric dimension of m-level Jahangir graph $J_{2n,m}$ and generalized Jahangir graph J_{3n} has been calculated. It is proved that both the families of graphs have unbounded metric dimension. Metric dimensions of the rooted product of Harary graph $H_{m,n}$ with cycles C_3, C_4 and C_5 that is $H_{m,n}(C_3)$, $H_{m,n}(C_4)$ and $H_{m,n}(C_5)$ has also been discussed. It is shown that all the graphs have unbounded metric dimensions. Metric dimension of generalized prism related graphs has been studied. It is concluded that both the families have constant metric dimensions. Artificial neural networks approach can be applied to find minimum number of colors of graphs discussed in our thesis such as generalized Jahangir graphs and Toroidal Fullerenes graphs. But we did not use artificial neural networks methods to solve the problems in this thesis. However, this approach can be basis for further research in this field in the future. "Now we give open problem for advance study in this field".

9.2 Open Problems

Open Problem 1. "Find other possible values of the parameter d and corresponding super d-antimagic labeling of type $(1,1,1)$ for a disjoint union of t copies of a toroidal fullerene $\mathbb{H}_m^n$".

Open Problem 2. Determine whether for $m \geq 1, p \geq 2$ and $n \geq 2$ the graph $G \cong mf_{p,n}$ admits super face-antimagic labeling of type (1, 1, 1).

Open Problem 3. Determine the formulation for super d-antimagic labeling of type (1,1,1) for generalized (subdivided) friendship graph $f_{p,n}$ under certain conditions.

Open Problem 4. Determine metric dimension of m-level generalized Jahangir graph

$J_{2n,k,m}$.

Open Problem 5. Determine metric dimension of rooted product of Harary graph $H_{m,n}$ with cycle C_n for $n \geq 6$.

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