Dispersion Free Finite Difference Scheme for Time Harmonic Wave Equation

By

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CIIT/FA10-MSMATH-005/LHR

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MS (Master of Science in Mathematics)

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A Post Graduate Thesis submitted to the Department of Mathematics as partial fulfillment of the requirement for the award of Degree of M.S (Master of Science in Mathematics).

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Final Approval

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Declaration

I M. Sarmad Arshad with registration number CIIT/FA10-MSMATH-005/LHR hereby declare that I have produced the work presented in this thesis, during the scheduled period of study. I also declare that I have not taken any material from any source except referred to wherever due, that amount of plagiarism is within acceptable range. If a violation of HEC rules on research has occurred in this thesis, I shall be liable to punishable action under the plagiarism rules of the HEC.

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It is certified that M. Sarmad Arshad & CIIT/FA10-MSMATH-005/LHR has carried out all the work related to this thesis under my supervision at the Department of Mathematics, COMSATS Institute of Information Technology, Lahore campus and the work fulfills the requirement for award of MS degree.

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To my loving parents
and
Family

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Abstract

Dispersion Free Finite Difference Scheme for Time Harmonic Wave Equation

In this work, we construct novel central finite difference schemes for solving the Helmholtz equation by following the idea of Nehrbass and Yau Shu in one and two dimensions. We present an alternative construction by which not only central finite difference schemes but all types of schemes (forward, backward) can be made exact for Helmholtz equation. We give explicit expressions for the coefficient of the central node in the standard central finite difference scheme of any order which makes the novel scheme dispersion free in one dimension and optimal in two dimensions. Such schemes add no cost for the implementation and one needs not to write brand new code instead just need to change the central node coefficient with the new one presented in this work to have highly accurate results. Comparison of numerical solutions obtained using novel schemes with standard schemes for Helmholtz equation in single and higher dimensions are given.

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Chapter 1

Introduction

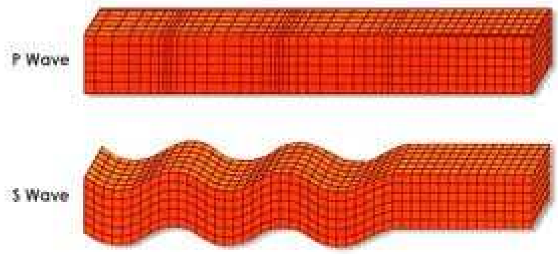
1.1 Wave Propagation

In the old days, the time required to deliver a message depended on the speed of a horse. In the end of 19th century, both distance and time limitations were largely overcome. The invention of the telegraph made possible instantaneous communication over long wires. Then after a short time, man discovered how to transmit messages in the form of RADIO WAVES.

In solids, sound waves can propagate in four modes that are based on the way the particles oscillate. Sound can propagate as longitudinal waves, shear waves, surface waves and as plate waves. Longitudinal and shear waves are the two modes of propagation most widely used in ultrasonic testing. Now the other type of the wave propagation is seismic wave propagation and commonly generated by the earthquakes. Mostly artificial sources also generate such type of waves like chemical explosions, surface vibrators or weight dropping devices.

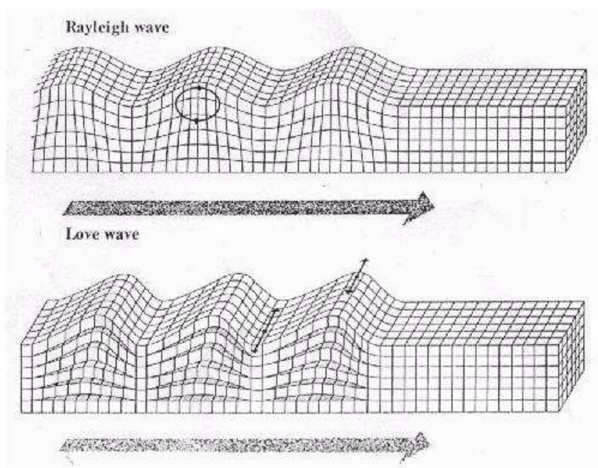
Body waves: There are two types of body waves: P waves and S waves.

Surface Waves: Surface waves are seismic waves that are guided along the surface of the Earth and the layers near the surface. These waves do not penetrate the deep interior of the earth, and are normally generated by shallow earthquakes, named as Rayleigh waves and Love waves.



and S waves.jpg

Figure 1.1: Primary and Secondary waves



and R waves.jpg

Figure 1.2: Love and Rayleigh waves

Atmospheric Propagation: Within the atmosphere, radio waves can be reflected, refracted, and diffracted like light and heat waves.

The theory of wave propagation that we discussed in this introduction applies to a wide range of electronic equipment, such as radar, navigation, detection, antenna theory and in different communication equipments. The wave propagation is also using in the Ground penetrating radar and Nondestructive testing There are two major areas of research in computational wave propagation and they are as follows: Direct scattering problem and Inverse scattering problem.

Firstly, we should know about scattering, and then we will come on direct and

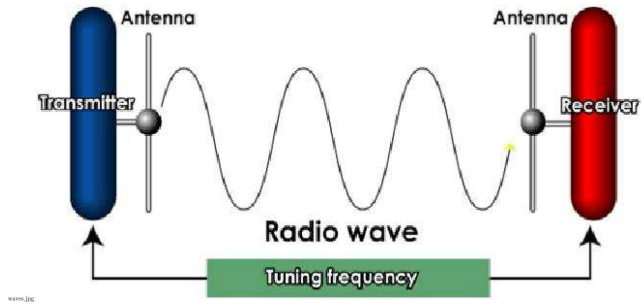


Figure 1.3: Radio waves

inverse scattering problem. So, what is scattering? It is study and understanding of scattering of waves and particles. Wave scattering corresponds to the collision and scattering of a wave with some material object. For example sunlight scattered by rain drops to form a rainbow.

Now come on the direct scattering problem and inverse scattering problem. They are the following:

Direct scattering problem:

It determines the distribution of scattered waves or particle flux basing on the characteristics of the scattered.

Inverse scattering problem:

It determines the characteristic of an object (e.g, its shape, internal constitution) from measurement data of waves or particles scattered from the object.

In the following section, we study the Helmholtz equation named for Hermann von Helmholtz (1821 - 1894). Helmholtz equation also called *Time harmonic wave equation.*

1.2 Model Problem

First of all we study the Helmholtz equation

$$u''(x) + k^2 u(x) = 0. \tag{1.1}$$

whose general solution is

$$u(x) = A_1 e^{ikx} + A_2 e^{-ikx} \tag{1.2}$$

where A_1 and $A_2 \in \mathbb{C}$ are constants to be determined.

In this problem, we consider the following boundary conditions

$$u(0) = 1 \quad (Drichlet\ \ boundary\ \ condition) \tag{1.3}$$

$$u'(1) - iku(1) = 0 \quad (Non-reflecting\ \ boundary\ \ condition).$$

From (1.2) and (1.3), we get

$$1 = A_1 + A_2$$

$$u'(x) = ik\left(A_1 e^{ikx} - A_2 e^{-ikx}\right)$$

$$u'(1) = ik\left(A_1 e^{ik} - A_2 e^{-ik}\right) \tag{1.4}$$

$$iku(1) = ik\left(A_1 e^{ik} + A_2 e^{-ik}\right) \tag{1.5}$$

from (1.4) and (1.5), it becomes

$$u'(1) - iku(1) = 2ike^{ik} A_1 = 0$$

$$A_1 = 0\ \ and\ \ A_2 = 1.$$

Finally,

$$u(x) = e^{-ikx}$$

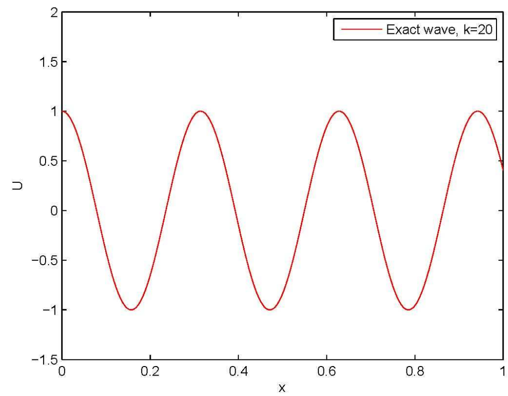


Figure 1.4: Exact wave with $k = 20$.

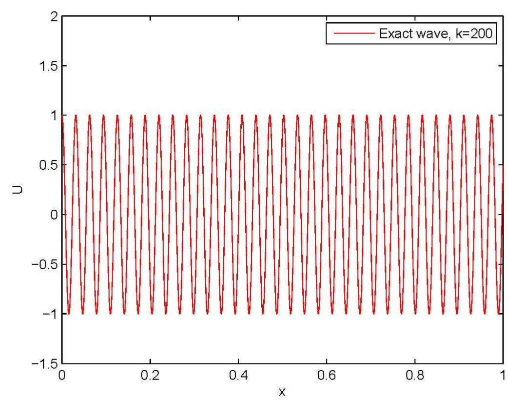


Figure 1.5: Exact wave with wave number $k = 200$. The number of occilations increase as we increase the wave number.

In the Figures 1.4, 1.5, 1.6, we can see the solution of Helmholtz equation with real and complex wave numbers. In Figure 1.5, the number of oscillations increase as we increase the wave number. In Figure 1.6, if we use the pure complex wave number in which real part is greater than imaginary part then wave will dissipate after a slight propagation. when we use the pure imaginary wave number then wave will dissipate immediately as shown in Figure 1.7.

In the following section, we repeat the construction of different finite difference schemes of first and second order derivatives which is done by many [4–6] for the

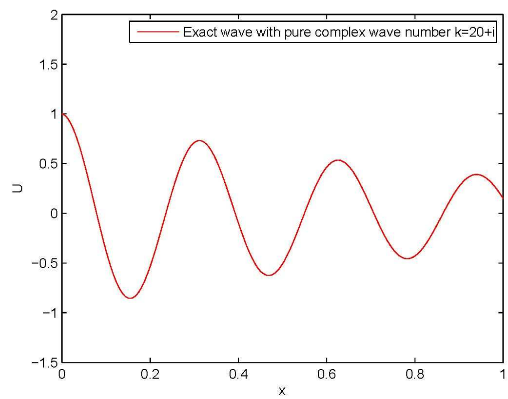


Figure 1.6: Exact wave with pure complex wave number $k = 20 + i$. If the real part is greater than the imaginary part then wave will dissipate after a slight propagation.

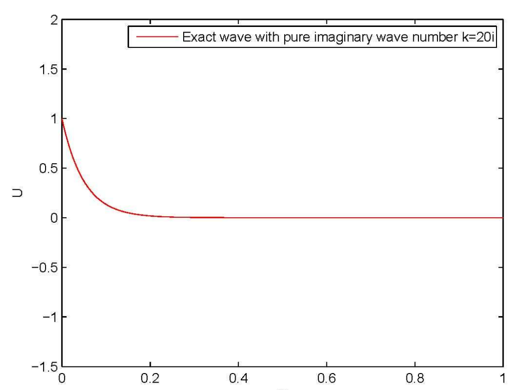


Figure 1.7: Exact wave with pure imaginary number $k = 20i$. The wave will dissipate immediately.

sake of better understanding and clarity of our work.

1.3 Approximations of first and second order derivatives for uniform mesh size

1.3.1 Construction of forward difference approximation of first order derivative

We start with the Taylor series approximation of $u(x+h)$ given by

$$u(x+h) = u(x) + hu'(x) + \frac{h^2}{2}u''(x) + \frac{h^3}{6}u'''(x) + \cdots$$

which on rewriting, has the following form

$$u(x+h) - u(x) = hu'(x) + \frac{h^2}{2}u''(x) + \frac{h^3}{6}u'''(x) + \frac{h^4}{24}u^4(x) + \cdots .$$

Dividing by h on both sides

$$\frac{u(x+h) - u(x)}{h} = u'(x) + \frac{h}{2}u''(x) + \frac{h^2}{6}u'''(x) + \cdots .$$

Solving above for $u'(x)$, we get

$$u'(x) = \frac{u(x+h) - u(x)}{h} - \frac{h}{2}u''(x) + \cdots .$$

This is called *first order accurate forward difference approximation of first order derivative* [9].

1.3.2 Construction of second order accurate central finite difference approximation of second order derivative for uniform discritization

Again we start with the Taylor series expansion of $u(x-h)$ and $u(x+h)$. We must consider three nodal grid with uniform mesh size in the following central finite difference approximation of second order derivative to obtain the desired accuracy.

$$u(x-h) = u(x) - hu'(x) + \frac{h^2}{2}u''(x) - \frac{h^3}{6}u'''(x) + \frac{h^4}{24}u^4(x) - \cdots$$

$$u(x+h) = u(x) + hu'(x) + \frac{h^2}{2}u''(x) + \frac{h^3}{6}u'''(x) + \frac{h^4}{24}u^4(x) + \cdots .$$

Rearranging

$$u(x-h) - 2u(x) + u(x+h) = h^2u''(x) + \frac{h^4}{12}u^4(x) + \cdots .$$

Dividing by h^2 on both sides, we have

$$\frac{u(x-h) - 2u(x) + u(x+h)}{h^2} = u''(x) + \frac{h^2}{12}u^4(x) + \cdots$$

$$u''(x) = \frac{u(x-h) - 2u(x) + u(x+h)}{h^2} - \frac{h^2}{12}u^4(x) + \cdots . \quad (1.6)$$

Hence the expression (1.6) is known as *second order accurate central finite difference approximation of the second derivative* [9].

1.3.3 Code for second order accurate central finite difference approximation

We now present code for second order accurate central finite difference scheme for the following problem

$$u''(x) + k^2u(x) = 0$$

where $u(0) = 1$ and $u'(1) - iku(1) = 0$

$$\frac{u(x-h) - 2u(x) + u(x+h)}{h^2} + k^2u(x) = 0$$

$$u(x-h) + (-2 + h^2k^2)u(x) + u(x+h) = 0$$

$$u_{j-1} + (-2 + h^2k^2)u_j + u_{j+1} = 0$$

where $j = 1, 2, 3, ..., N-1$

$$u_0 = 1 \quad \text{for } j = 0.$$

Now when we put $j = N$, we will have the fictitious node u_{N+1} which will not in our domain

$$u_{N-1} + u_N(-2 + k^2h^2) + u_{N+1} = 0 \quad \text{for } j = N. \tag{1.7}$$

In order to find the value at fictitious node, we use central finite difference approximation of first derivative up to second order accurate in the non-reflecting boundary condition $u'(1) - iku(1) = 0$

$$\frac{u(x+h) - u(x-h)}{2h} - iku(x) = 0$$

$$u(x+h) - u(x-h) - 2ikhu(x) = 0$$

$$u_{j+1} - u_{j-1} - 2iku_j = 0.$$

For $j = N$

$$u_{N+1} - u_{N-1} - 2ikhu_N = 0$$

$$u_{N+1} = u_{N-1} + 2ikhu_N.$$

On substituting the value of u_{N+1} in (1.7)

$$u_{N-1} + u_N(-2 + k^2h^2) + u_{N-1} + 2ikhu_N = 0$$

$$u_{N-1} + u_N(-2 + k^2h^2 + 2ikh) + u_{N-1} = 0$$

$$2u_{N-1} + u_N(-2 + k^2h^2 + 2ikh) = 0.$$

Finally, we have the system of equations

$$u_0 = 1 \quad \text{for } j = 0;$$

$$u_{j-1} + (-2 + h^2k^2)u_j + u_{j+1} = 0, \quad \forall j = 1, 2, \cdots, N-1$$

$$2u_{N-1} + u_N(-2 + k^2h^2 + 2ikh) = 0 \quad \text{for } j = N.$$

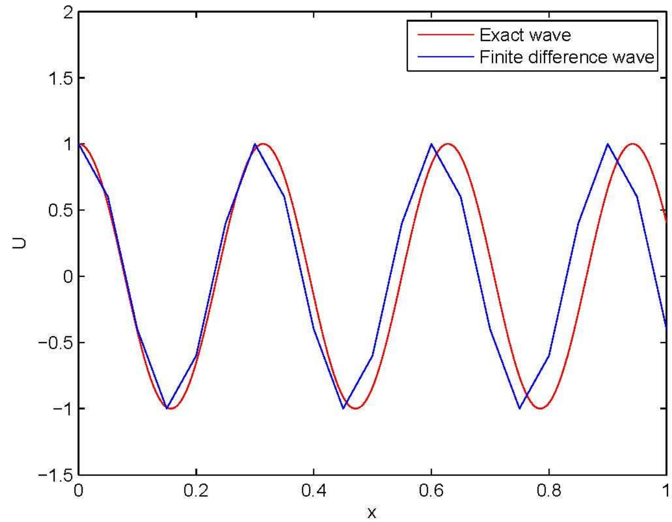


Figure 1.8: Comparison of exact wave and the wave obtained by second order accurate standard central finite difference scheme with the wave number $k = 20$ and number of elements $N = 20$. One can easily see the (phase lag) dispersion.

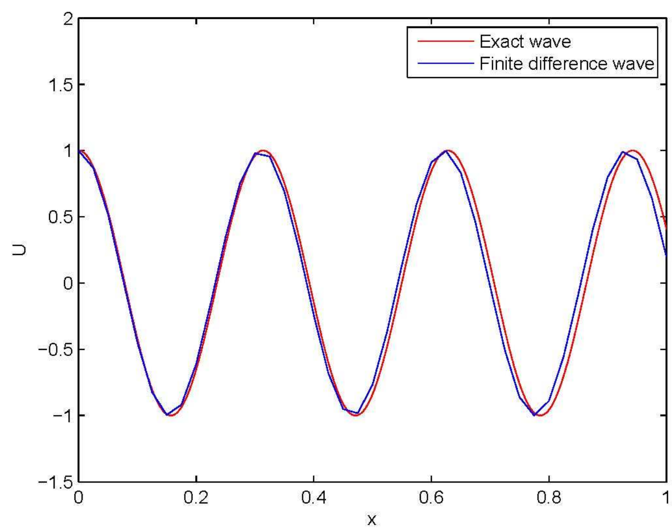


Figure 1.9: Comparison of exact wave and the wave obtained by second order accurate Standard C.F.D scheme. Dispersion is reduced when we increase number of elements from $N = 20$ to $N = 40$ with $k = 20$.

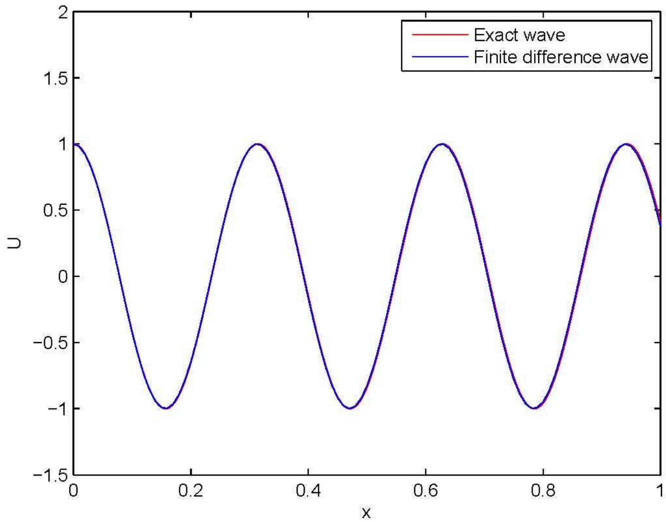


Figure 1.10: Dispersion is almost eliminated with $k = 20$ and $N = 100$.

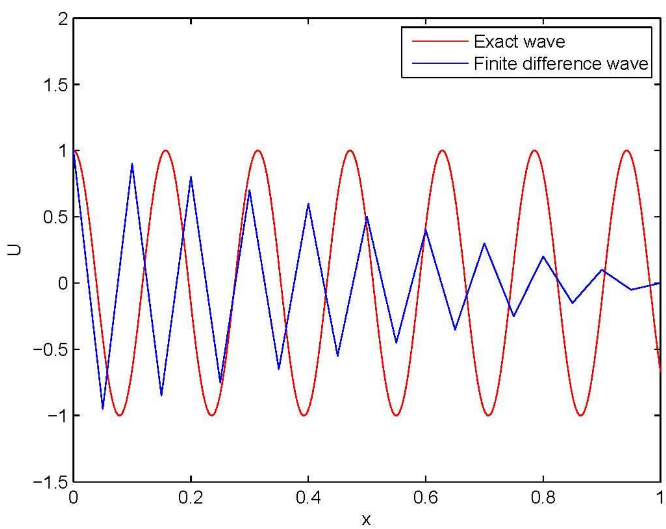


Figure 1.11: Comparison of exact wave and the wave obtained by second order accurate Standard C.F.D scheme with high wave number $k = 40$ and number of elements $N = 20$. Here, Standard C.F.D dissipates.

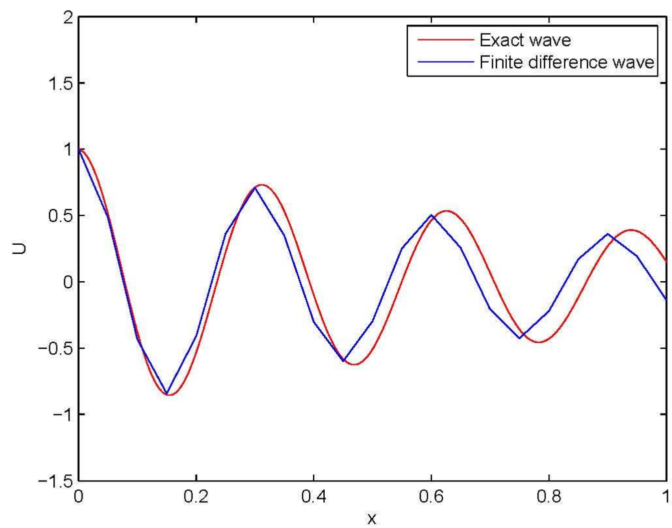


Figure 1.12: Both exact and S.F.D wave dissipates when $k = 20 + i$ and $N = 20$ after a slight propagation.

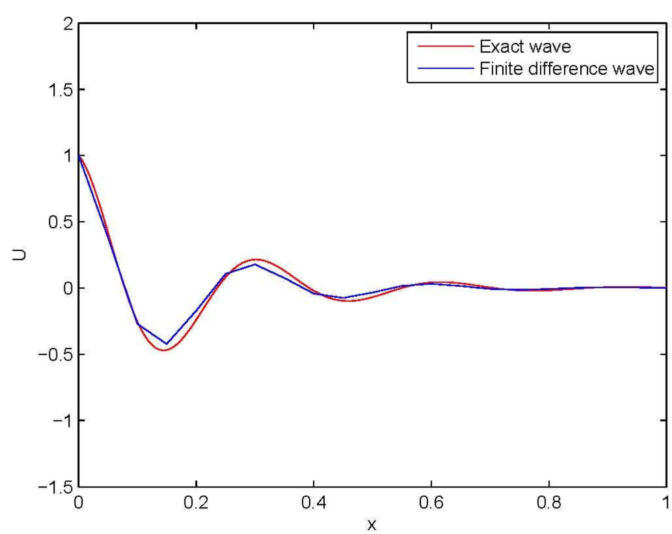


Figure 1.13: Increasing the imaginary part of the complex number reduces the propagation of exact wave and wave obtained by second order accurate S.C.F.D scheme. Here, $k = 20 + 5i$ and $N = 20$.

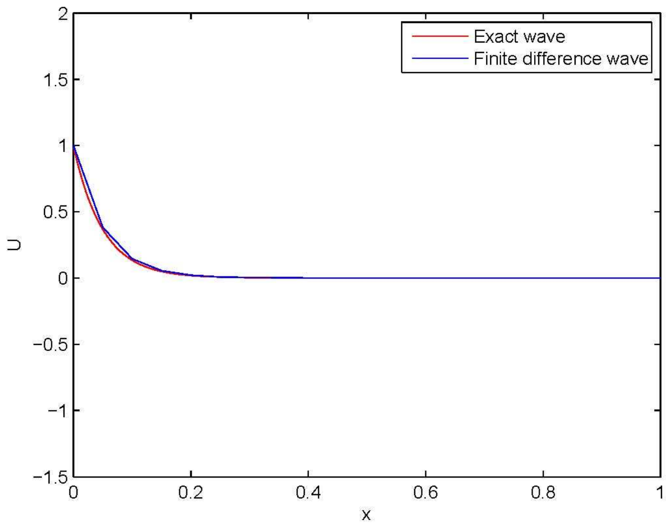


Figure 1.14: The exact and second order accurate Standard C.F.D wave dissipates immidiately when wave number is pure imaginary.

1.3.4 Construction of fourth order accurate central finite difference approximation of second order derivative for uniform discritization

As we take three nodal grid in the construction of second order accurate central finite difference approximation of second order derivative, similarly we must take five nodal grid to obtain fourth order accurate central finite difference approximation of the second order derivative. Consider the Taylor series approximations of $u(x-2h)$, $u(x-h)$, $u(x+h)$ and $u(x+2h)$

$$u(x-2h) = u(x) - 2hu'(x) + 2h^2u''(x) - \frac{4h^3}{3}u'''(x) + \frac{2h^4}{3}u^{(4)}(x) - \frac{4h^5}{15}u^{(5)}(x) + \frac{4h^6}{45}u^{(6)}(x) + \cdots$$

$$u(x-h) = u(x) - hu'(x) + \frac{h^2}{2}u''(x) - \frac{h^3}{6}u'''(x) + \frac{h^4}{24}u^{(4)}(x) - \frac{h^5}{120}u^{(5)}(x) + \frac{h^6}{720}u^{(6)}(x) + \cdots$$

$$u(x+h) = u(x) + hu'(x) + \frac{h^2}{2}u''(x) + \frac{h^3}{6}u'''(x) + \frac{h^4}{24}u^{(4)}(x) + \frac{h^5}{120}u^{(5)}(x) + \frac{h^6}{720}u^{(6)}(x) + \cdots$$

$$u(x+2h) = u(x) + 2hu'(x) + 2h^2u''(x) + \frac{4h^3}{3}u'''(x) + \frac{2h^4}{3}u^{(4)}(x) + \frac{4h^5}{15}u^{(5)}(x) + \frac{4h^6}{45}u^{(6)}(x) + \cdots .$$

Rearranging the above Taylor series approximations as follows.

$$\frac{u(x-h) - 2u(x) + u(x+h)}{h^2} = u''(x) + \frac{h^2}{12}u^{(4)}(x) + \frac{h^4}{360}u^{(6)}(x) + \cdots$$

$$\frac{u(x-2h) - 2u(x) + u(x+2h)}{4h^2} = u''(x) + \frac{h^2}{3}u^{(4)}(x) + \frac{h^4}{360}u^{(6)}(x) + \cdots \quad .$$

Let

$$A = u(x-h) - 2u(x) + u(x+h)$$

$$B = u(x-2h) - 2u(x) + u(x+2h)$$

$$\frac{A}{h^2} = u''(x) + \frac{h^2}{12}u^{(4)}(x) + \frac{h^4}{360}u^{(6)}(x) + \cdots \tag{1.8}$$

$$\frac{B}{4h^2} = u''(x) + \frac{h^2}{3}u^{(4)}(x) + \frac{2h^4}{45}u^{(6)}(x) + \cdots . \tag{1.9}$$

Multiplying (1.8) by α and (1.9) by β and adding, we have

$$\alpha\left(\frac{A}{h^2}\right) + \beta\left(\frac{B}{4h^2}\right) = (\alpha+\beta)u''(x) + \left(\frac{\alpha}{12} + \frac{\beta}{3}\right)h^2u^{(4)}(x) + \left(\frac{\alpha}{360} + \frac{2\beta}{45}\right)h^4u^{(6)}(x) + \cdots \quad . \tag{1.10}$$

In order to obtain approximation of second derivative with accuracy of order four, it is necessary to have

$$\alpha + \beta = 1$$

$$\frac{\alpha}{12} + \frac{\beta}{3} = 0.$$

Solving the above equations simultaneously, gives

$$\alpha = \frac{4}{3} \quad \text{and} \quad \beta = \frac{-1}{3}.$$

Substituting these values of α and β in (1.10)

$$\frac{4}{3}\left(\frac{A}{h^2}\right) - \frac{1}{3}\left(\frac{B}{4h^2}\right) = u''(x) - 4h^4u^{(6)}(x) + \cdots$$

$$u''(x) = \frac{4}{3}\left(\frac{u(x-h) - 2u(x) + u(x+h)}{h^2}\right) - \frac{1}{3}\left(\frac{u(x-2h) - 2u(x) + u(x+2h)}{4h^2}\right) + 4h^4u^{(6)}(x) + \cdots$$

or, alternatively

$$u''(x) = \frac{-u(x-2h) + 16u(x-h) - 30u(x) + 16u(x+h) - u(x+2h)}{12h^2} + 4h^4u^{(6)}(x) + \cdots .$$

which is the required *fourth order accurate central finite difference approximation of the second order derivative*. This is already done in Gray Cohen [8]

1.3.5 Implementation of fourth order accurate central finite difference scheme

We reconsider (1.1), Replacing second order derivative $u''(x)$ in (1.1) with fourth order accurate central finite difference scheme, we have

$$\frac{-u(x-2h) + 16u(x-h) - 30u(x) + 16u(x+h) - u(x+2h)}{12h^2} + k^2u(x) = 0$$

or rearranging

$$u(x-2h) - 16u(x-h) + 30u(x) - 16u(x+h) + u(x+2h) + 12h^2k^2 = 0$$

$$u_{j-2} - 16u_{j-1} + (30 - 12h^2k^2)u_j - 16u_{j+1} + u_{j+2} = 0.$$

For $j = 1$,

Here we use fourth order accurate seven point forward difference scheme presented in [3].

$$-9u_0 + 9u_1 + 19u_2 - 34u_3 + 21u_4 - 7u_5 + u_6 - 12h^2k^2u_1 = 0$$

$$u_1(9-12h^2k^2)+19u_2-34u_3+21u_4-7u_5+u_6=9.$$

For $j = N-1$, we use fourth order accurate seven point backward difference scheme presented in [3].

$$u_{N-6}-7u_{N-5}+21u_{N-4}-34u_{N-3}+19u_{N-2}+9u_{N-1}-9u_N-12h^2k^2u_{N-1}=0$$

$$u_{N-6}-7u_{N-5}+21u_{N-4}-34u_{N-3}+19u_{N-2}+u_{N-1}(9-12h^2k^2)-9u_N=0.$$

Now for $j = N$,

we need fourth order accurate finite difference approximation of first order derivative presented in [3], given by

$$u'(1)=\frac{1}{12h}(3u_{N-4}-16u_{N-3}+36u_{N-2}-48u_{N-1}+25u_N)$$

which on inserting into the following

$$u'(1)-iku(1)=0$$

gives

$$3u_{N-4}-16u_{N-3}+36u_{N-2}-48u_{N-1}+(25-12ikh)u_N=0$$

finally, we have the following system

$$u_0=1 \quad \text{for } j=0;$$

$$u_1(9-12h^2k^2)+19u_2-34u_3+21u_4-7u_5+u_6=9 \quad \text{for } j=1;$$

$$-16u_1+(30-12h^2k^2)u_2-16u_3+u_4=-1 \quad \text{for } j=2;$$

$$u_{j-4}-16u_{j-3}+(30-12h^2k^2)u_{j-2}-16u_{j-1}+u_j=0 \quad \text{for } 3\le j\le N-2;$$

$$u_{N-6}-7u_{N-5}+21u_{N-4}-34u_{N-3}+19u_{N-2}+u_{N-1}(9-12h^2k^2)-9u_N=0 \quad \text{for } j=N-1;$$

$$3u_{N-4}-16u_{N-3}+36u_{N-2}-48u_{N-1}+(25-12ikh)u_N=0 \quad \text{for } j=N.$$

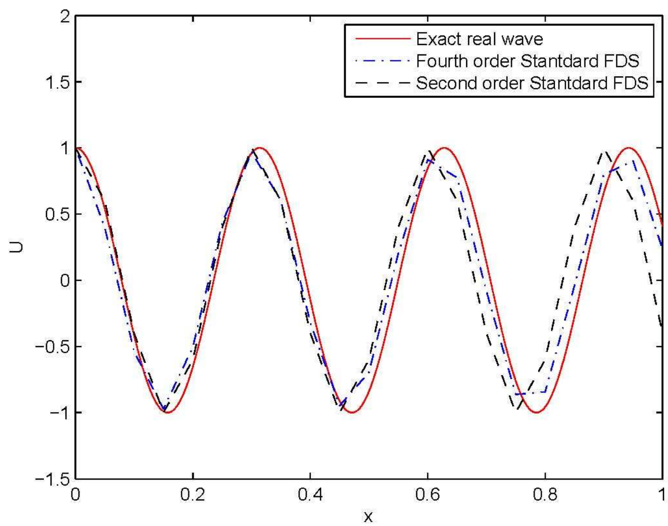


Figure 1.15: Comaprison of exact wave, numerical wave obtained by second order accurate Standard C.F.D scheme and numerical wave obtained by fourth order accurate Standard C.F.D scheme with wave number $k = 20$ and number of elements $N = 20$. One can see that wave obtained by fourth order accurate Standard C.F.D scheme performs better than second order accurate Standard C.F.D scheme.

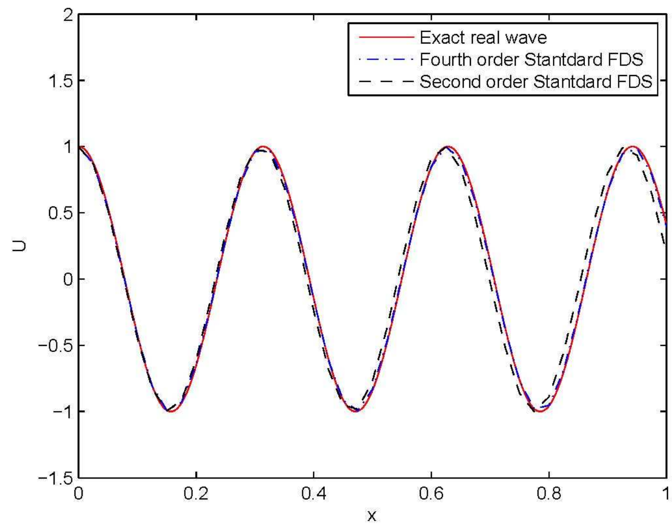


Figure 1.16: Comparison with wave number $k = 20$ and number of elements $N = 40$. Fourth order accurate Standard C.F.D scheme behaves like an exact in a very less computational cost as compare to second order accurate Standard C.F.D scheme.

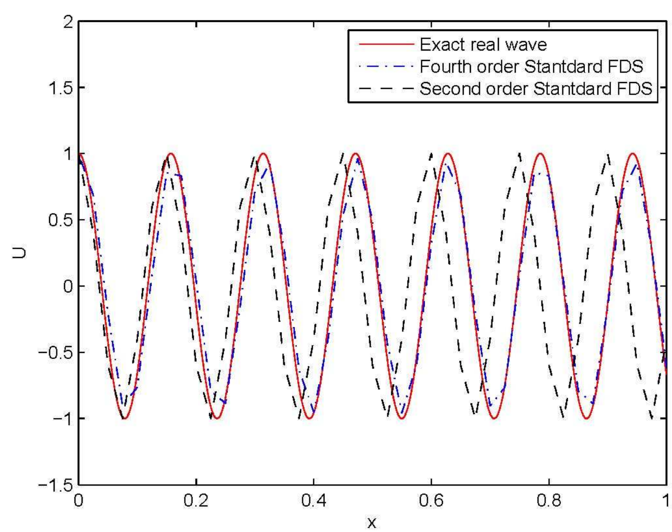


Figure 1.17: Comparison with $k = 40$ and $N = 40$. Again fourth order accurate Standard C.F.D scheme performs better as compare to second order accurate Standard C.F.D scheme at high wave number.

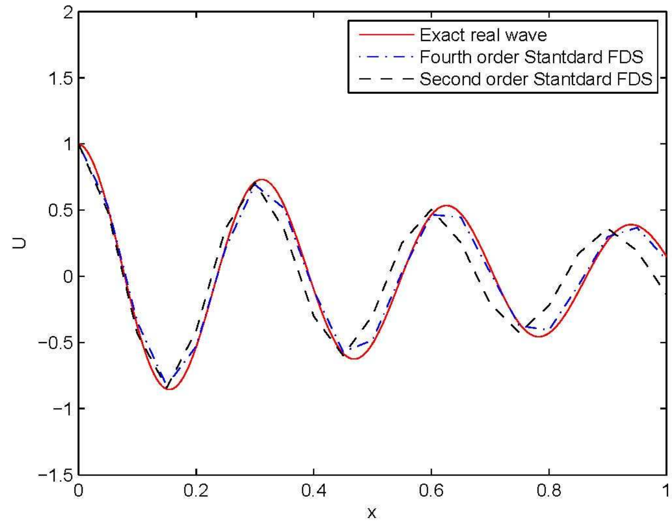


Figure 1.18: Comparison with complex wave number $k = 20 + i$ at number of elements $N = 20$. Still fourth order accurate Standard C.F.D scheme performs superior than second order accurate Standard C.F.D scheme.

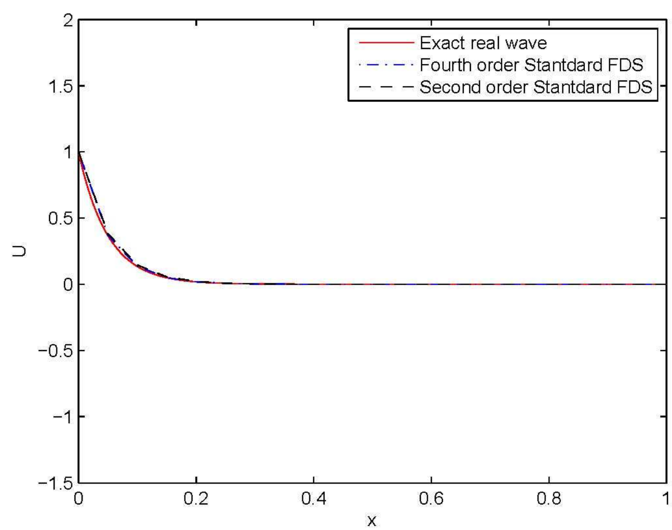


Figure 1.19: Comparison with pure imaginary wave number $k = 20i$ at number of elements $N = 20$

1.3.6 Construction of sixth order accurate central finite difference approximation of second order derivative for uniform discritization

As we discussed in the above sections, similarly we must take seven nodal grid to obtain sixth order accurate central difference approximation of second order derivative. Again consider the Taylor series approximations of $u(x-3h)$, $u(x-2h)$, $u(x-h)$, $u(x+h)$, $u(x+2h)$ and $u(x+3h)$

$$\begin{aligned} u(x-3h) &= u(x) - 3hu'(x) + \frac{9h^2}{2}u''(x) - \frac{9h^3}{2}u'''(x) + \frac{27h^4}{8}u^{(4)}(x) \\ &\quad - \frac{8h^5}{40}u^{(5)}(x) + \frac{81h^6}{80}u^{(6)}(x) - \frac{243h^7}{560}u^{(7)}(x) + \frac{729h^8}{4480}u^{(8)}(x) - \cdots \end{aligned}$$

$$\begin{aligned} u(x-2h) &= u(x) - 2hu'(x) + 2h^2u''(x) - \frac{4h^3}{3}u'''(x) + \frac{2h^4}{3}u^{(4)}(x) \\ &\quad - \frac{4h^5}{15}u^{(5)}(x) + \frac{4h^6}{45}u^{(6)}(x) - \frac{8h^7}{315}u^{(7)}(x) + \frac{2h^8}{315}u^{(8)}(x) - \cdots \end{aligned}$$

$$\begin{aligned} u(x-h) &= u(x) - hu'(x) + \frac{h^2}{2}u''(x) - \frac{h^3}{6}u'''(x) + \frac{h^4}{24}u^{(4)}(x) - \frac{h^5}{120}u^{(5)}(x) \\ &\quad + \frac{h^6}{720}u^{(6)}(x) - \frac{h^7}{5040}u^{(7)}(x) + \frac{h^8}{40320}u^{(8)}(x) - \cdots \end{aligned}$$

$$\begin{aligned} u(x+h) &= u(x) + hu'(x) + \frac{h^2}{2}u''(x) + \frac{h^3}{6}u'''(x) + \frac{h^4}{24}u^{(4)}(x) + \frac{h^5}{120}u^{(5)}(x) \\ &\quad + \frac{h^6}{720}u^{(6)}(x) + \frac{h^7}{5040}u^{(7)}(x) + \frac{h^8}{40320}u^{(8)}(x) + \cdots \end{aligned}$$

$$\begin{aligned} u(x+2h) &= u(x) + 2hu'(x) + 2h^2u''(x) + \frac{4h^3}{3}u'''(x) + \frac{2h^4}{3}u^{(4)}(x) \\ &\quad + \frac{4h^5}{15}u^{(5)}(x) + \frac{4h^6}{45}u^{(6)}(x) + \frac{8h^7}{315}u^{(7)}(x) + \frac{2h^8}{315}u^{(8)}(x) + \cdots \end{aligned}$$

$$
\begin{aligned}
u(x+3h) \quad = \quad & u(x) + 3hu'(x) + \frac{9h^2}{2}u''(x) + \frac{9h^3}{2}u'''(x) + \frac{27h^4}{8}u^{(4)}(x) \\
& + \frac{8h^5}{40}u^{(5)}(x) + \frac{81h^6}{80}u^{(6)}(x) + \frac{243h^7}{560}u^{(7)}(x) + \frac{729h^8}{4480}u^{(8)}(x) + \cdots .
\end{aligned}
$$

After rearranging, we have following three expressions

$$
\begin{aligned}
A = u(x-3h) - 2u(x) + u(x+3h) \quad = \quad & 9h^2u''(x) + \frac{27h^4}{4}u^{(4)}(x) + \frac{81h^6}{40}u^{(6)}(x) \\
& + \frac{729h^8}{2240}u^{(8)}(x) + \cdots
\end{aligned} \tag{1.11}
$$

$$
\begin{aligned}
B = u(x-2h) - 2u(x) + u(x+2h) \quad = \quad & 4h^2u''(x) + \frac{4h^4}{3}u^{(4)}(x) + \frac{8h^6}{45}u^{(6)}(x) \\
& + \frac{4h^8}{315}u^{(8)}(x) + \cdots
\end{aligned} \tag{1.12}
$$

$$
\begin{aligned}
C = u(x-h) - 2u(x) + u(x+h) \quad = \quad & h^2u''(x) + \frac{h^4}{12}u^{(4)}(x) + \frac{h^6}{360}u^{(6)}(x) \\
& + \frac{h^8}{20160}u^{(8)}(x) + \cdots .
\end{aligned} \tag{1.13}
$$

Dividing (1.11),(1.12) and (1.13) by $9h^2$,$4h^2$ and h^2 respectively

$$
\frac{A}{9h^2} \quad = \quad u''(x) + \frac{3h^2}{4}u^{(4)}(x) + \frac{9h^4}{40}u^{(6)}(x) + \frac{81h^6}{2240}u^{(8)}(x) + \cdots \tag{1.14}
$$

$$
\frac{B}{4h^2} \quad = \quad u''(x) + \frac{h^2}{3}u^{(4)}(x) + \frac{2h^4}{45}u^{(6)}(x) + \frac{h^6}{315}u^{(8)}(x) + \cdots \tag{1.15}
$$

$$
\frac{C}{h^2} \quad = \quad u''(x) + \frac{h^2}{12}u^{(4)}(x) + \frac{h^4}{360}u^{(6)}(x) + \frac{h^6}{20160}u^{(8)}(x) + \cdots . \tag{1.16}
$$

Multiplying (1.14) by α , (1.15) by β and (1.16) by γ and adding, gives

$$
\begin{aligned}
\alpha\left(\frac{A}{9h^2}\right) + \beta\left(\frac{B}{4h^2}\right) + \gamma\left(\frac{C}{h^2}\right) = & (\alpha+\beta+\gamma)u''(x) + \left(\frac{3\alpha}{4} + \frac{\beta}{3} + \frac{\gamma}{12}\right)h^2u^{(4)}(x) \\
& + \left(\frac{9\alpha}{40} + \frac{2\beta}{45} + \frac{\gamma}{360}\right)h^4u^{(6)}(x) \\
& + \left(\frac{81\alpha}{2240} + \frac{\beta}{315} + \frac{\gamma}{20160}\right)h^6u^{(8)}(x) + \cdots \; .
\end{aligned} \tag{1.17}
$$

In order to obtain approximation of second order derivative with accuracy of order six, it is necessary to have

$$\alpha + \beta + \gamma = 1$$

$$9\alpha + 4\beta + \gamma = 0$$

$$81\alpha + 16\beta + \gamma = 0.$$

Solving above system, we get the following values of α, β and γ

$$\alpha = \frac{1}{10}, \quad \beta = \frac{-3}{5}, \quad \gamma = \frac{3}{2}.$$

Inserting these values of α, β and γ in (1.17), gives

$$\frac{1}{10}\left(\frac{A}{9h^2}\right) - \frac{3}{5}\left(\frac{B}{4h^2}\right) + \frac{3}{2}\left(\frac{C}{h^2}\right) = u''(x) + \frac{1}{728}h^6 u^{(8)}(x) + \cdots$$

$$u''(x) = \frac{1}{10}\left(\frac{A}{9h^2}\right) - \frac{3}{5}\left(\frac{B}{4h^2}\right) + \frac{3}{2}\left(\frac{C}{h^2}\right) - \frac{1}{728}h^6 u^{(8)}(x) + \cdots .$$

$$u''(x) = \frac{1}{180h^2}\Big\{2u(x-3h) - 27u(x-2h) + 270u(x-h) - 490u(x) + 270u(x+h) - 27u(x+2h) + 2u(x+3h)\Big\} - \frac{1}{728}h^6 u^{(8)}(x) + \cdots .$$

which is the required *sixth order accurate central finite difference approximation of the second order derivative.*

1.4 Numerical dispersion

1.4.1 Dispersion relation of second order accurate central finite difference approximation for the Helmholtz equation

Reconsider the Helmholtz equation (1.1)

$$u''(x) + k^2 u(x) = 0.$$

Replacing second order derivative by second order accurate central finite difference approximation in (1.1), gives

$$\frac{u(x-h)-2u(x)+u(x+h)}{h^2}+k^2u(x)=0. \tag{1.18}$$

Now, we consider a plane wave solution of the form

$$u(x)=Ae^{i\tilde{k}x}+Be^{-i\tilde{k}x}$$

$$u(x+h)=Ae^{i\tilde{k}(x+h)}+Be^{-i\tilde{k}(x+h)}$$

$$u(x-h)=Ae^{i\tilde{k}(x-h)}+Be^{-i\tilde{k}(x-h)}.$$

Adding the above equations, we have

$$\begin{aligned} u(x+h)+u(x-h) &= Ae^{ikx}(e^{ikh}+e^{-ikh})+Be^{-ikx}(e^{ikh}+e^{-ikh}) \\ &= u(x)(e^{i\tilde{k}h}+e^{-i\tilde{k}h}) \\ &= 2\cos\tilde{k}hu(x) \end{aligned}$$

which on inserting into (1.18), gives

$$\left(\frac{2cos\tilde{k}hu(x)-2u(x)}{h^2}\right)+k^2u(x)=0$$

$$u(x)\left(\frac{2}{h^2}(cos\tilde{k}h-1)+k^2\right)=0$$

where $u(x)\neq 0$, so

$$\frac{2}{h^2}(cos\tilde{k}h-1)+k^2=0$$

$$cos\tilde{k}h=1-\frac{k^2h^2}{2}$$

$$\tilde{k}h=\arccos\left(1-\frac{k^2h^2}{2}\right)$$

$$\tilde{k}h=kh+\frac{k^3h^3}{24}+\cdots$$

$$\tilde{k}h-kh=+\frac{k^3h^3}{24}+\cdots$$

$$\tilde{k} - k = +\frac{k^3h^2}{24} + \cdots . \tag{1.19}$$

In the following section, we discuss the idea presented in [1] in which dispersion free central finite difference scheme was obtained by just modifying the coefficient of the central node in the standard central finite difference scheme.

1.4.2 Modified central finite difference scheme proposed by Nehrbass [1]

Reconsider the Helmholtz equation (1.1)

$$u''(x) + k^2u(x) = 0.$$

Replace coefficient of the middle node 2 by α in the central finite difference scheme, then the scheme will become

$$\frac{u(x-h) - \alpha u(x) + u(x+h)}{h^2} + k^2u(x) = 0. \tag{1.20}$$

We know that

$$\begin{aligned} u(x+h) + u(x-h) &= 2\cos khu(x) \\ \left(\frac{2\cos kh - \alpha}{h^2}\right) u(x) + k^2u(x) &= 0 \\ \frac{2\cos kh - \alpha}{h^2} + k^2 &= 0 \\ \alpha &= 2\cos kh + h^2k^2. \end{aligned}$$

Substituting the value of α in (1.20) and performing ordinary simplification, we have

$$u(x-h) - 2\cos khu(x) + u(x+h) = 0. \tag{1.21}$$

Inserting plane wave solution of the form

$$u(x) = e^{i\tilde{k}x}$$

$$u(x-h) + u(x+h) = 2\cos\tilde{k}hu(x).$$

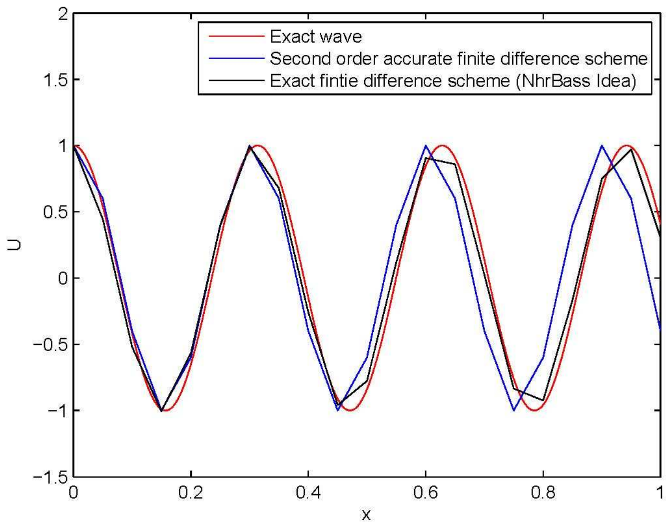


Figure 1.20: Comaprison of Exact, second order accurate Standard C.F.D scheme and Modified finite difference scheme proposed by Nehrbass, with wave number $k = 20$ and number of elements is $N = 20$. Nehrbass claims that this is an exact finite difference scheme but one can see that it is not an exact. It just reduces dispersion rapidly.

Equation (1.21) becomes

$$
\begin{aligned}
(2\cos \tilde{k}h - 2\cos kh)u(x) &= 0 \\
\tilde{k}h &= kh \\
\tilde{k} - k &= 0. \qquad (1.22)
\end{aligned}
$$

Nehrbass claims that there is no dispersion when the difference between exact wave number and discrete wave number is zero but here we can see dispersion when we use the idea of Nehrbass. This is because of inexact derivative boundary conditions.

Now we consider expression (1.19) obtained for standard finite difference scheme in which the difference of exact wave number and discrete wave number is not zero.

$$\tilde{k} - k = +\frac{k^3h^2}{24} + \cdots .$$

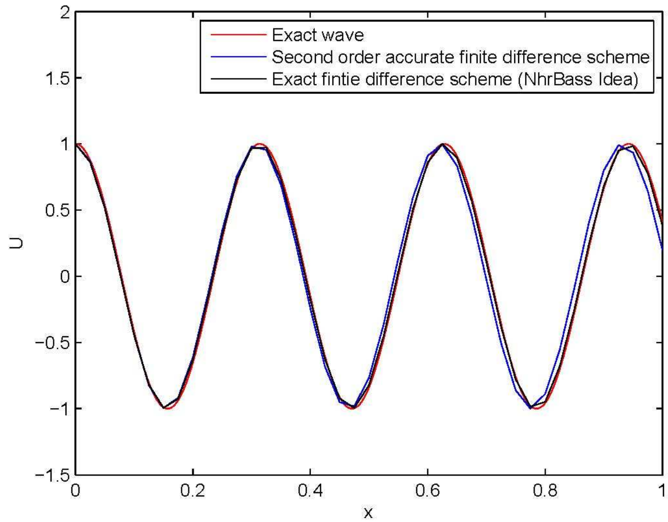


Figure 1.21: Comaprison with fine mesh size, where $k = 20$ and $N = 40$

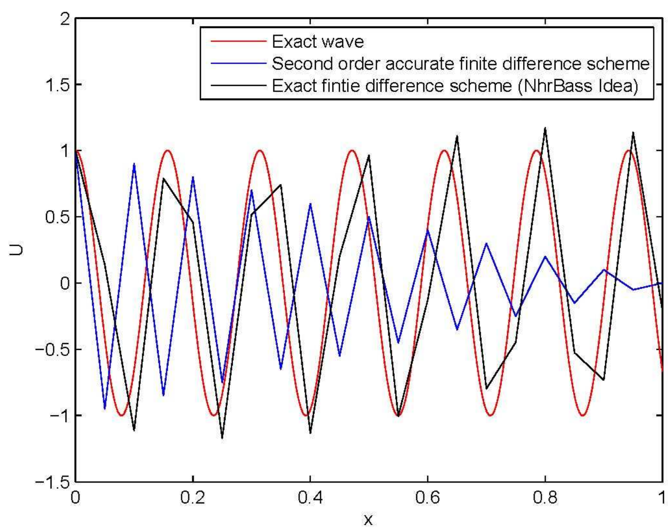


Figure 1.22: Standard C.F.D wave dissipates when we increase the wave number but scheme proposed by Nehrbass performs better at high wave number $k = 40$ $N = 20$

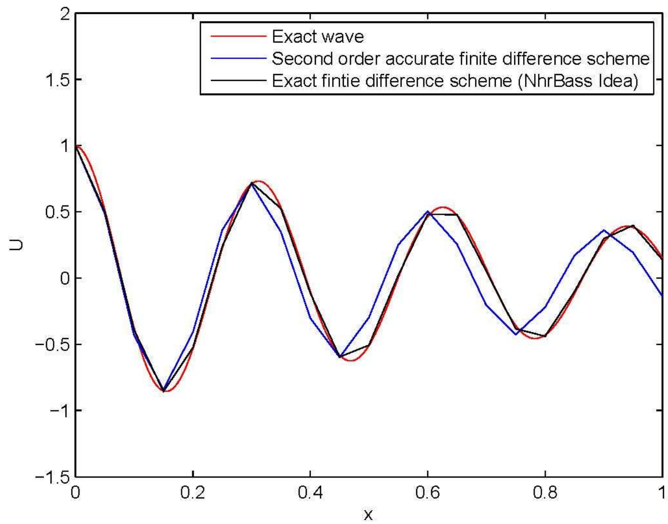


Figure 1.23: Comparison with complex wave number $k = 20 + i$ and $N = 20$

For this difference to be zero required very fine mesh size and results into prohibitive computational cost. Which is much more problematic when one deals with problems posed in higher dimensions.

As we know dispersion is the difference between exact wave number and discrete wave number. Hence dispersion relation (1.22) shows that there is no dispersion. This idea is presented in [1] just for second order accurate central finite difference approximation but not for fourth and sixth order accurate central finite difference approximation. Modification of the coefficient of the middle node in fourth and sixth order accurate central finite difference approximation will be different from the second order accurate central finite difference approximation. In the following section, we extend the idea presented in [1] for fourth and sixth order accurate central finite difference schemes and finally we will make efforts to generalize this concept.

1.4.3 Construction of fourth order accurate novel central finite difference scheme

Reconsider the Helmholtz equation (1.1)

$$u''(x) + k^2 u(x) = 0.$$

Replace coefficient of the central difference scheme at the middle node 2 by α, then the scheme will become

$$u''(x) = \frac{4}{3}\left(\frac{u(x-h) - \alpha u(x) + u(x+h)}{h^2}\right) - \frac{1}{3}\left(\frac{u(x-2h) - \alpha u(x) + u(x+2h)}{4h^2}\right).$$

Helmholtz equation becomes

$$\frac{4}{3}\left(\frac{u(x-h) - \alpha u(x) + u(x+h)}{h^2}\right) - \frac{1}{3}\left(\frac{u(x-2h) - \alpha u(x) + u(x+2h)}{4h^2}\right) + k^2 u(x) = 0. \tag{1.23}$$

The following solution satisfies the Helmholtz equation

$$u(x) = e^{ikx}.$$

We know that

$$u(x+h) + u(x-h) = 2\cos kh u(x)$$

$$u(x+2h) + u(x-2h) = 2\cos 2kh u(x)$$

$$\frac{4}{3}\left(\frac{2\cos kh - \alpha}{h^2}\right) - \frac{1}{3}\left(\frac{2\cos 2kh - \alpha}{4h^2}\right) + k^2 = 0$$

$$16(2\cos kh - \alpha) - (2\cos 2kh - \alpha) + 12k^2h^2 = 0$$

$$32\cos kh - 16\alpha - 2\cos 2kh + \alpha + 12k^2h^2 = 0$$

$$15\alpha = 32\cos kh - 2\cos 2kh + 12k^2h^2$$

$$\alpha = \frac{32}{15}\cos kh - \frac{2}{15}\cos 2kh + \frac{12}{15}k^2h^2.$$

Substituting the value of α in (1.23) and performing ordinary simplification, we have

$$16u(x-h)+16u(x+h)-u(x-2h)-u(x+2h)-32\cos khu(x)+2\cos 2khu(x)=0. \tag{1.24}$$

Inserting plane wave solution $u(x)=e^{i\tilde{k}x}$ in (1.24)

$$16\left(e^{-i\tilde{k}h}+e^{i\tilde{k}h}\right)u(x)-\left(e^{-2i\tilde{k}x}-e^{2i\tilde{k}x}\right)u(x)-32\cos khu(x)+32\cos 2khu(x)=0$$

$$\left(32\cos\tilde{k}h-2\cos 2\tilde{k}h-32\cos kh+2\cos 2kh\right)u(x)=0$$

$$32\cos\tilde{k}h-2\cos 2\tilde{k}h-32\cos kh+2\cos 2kh=0$$

$$32\cos\tilde{k}h-2\cos 2\tilde{k}h=32\cos kh-2\cos 2kh$$

$$\tilde{k}h=kh$$

$$\tilde{k}-k=0.$$

We conclude that there is no dispersion after replacing the coefficient of the middle node.

1.4.4 Construction of sixth order accurate novel central finite difference scheme

Reconsider the Helmholtz equation (1.1)

$$u''(x)+k^2u(x)=0.$$

Replace coefficient of the central difference scheme at the middle node 2 by α, then the scheme will become

$$u''(x)=\frac{3}{2}\left\{\frac{u(x-h)-\alpha u(x)+u(x+h)}{h^2}\right\}-\frac{3}{5}\left\{\frac{u(x-2h)-\alpha u(x)+u(x+2h)}{4h^2}\right\}$$
$$+\frac{1}{10}\left\{\frac{u(x-3h)-\alpha u(x)+u(x+3h)}{9h^2}\right\}.$$

Helmholtz equation becomes

$$\frac{3}{2}\left\{\frac{u(x-h)-\alpha u(x)+u(x+h)}{h^2}\right\}-\frac{3}{5}\left\{\frac{u(x-2h)-\alpha u(x)+u(x+2h)}{4h^2}\right\}$$
$$+\frac{1}{10}\left\{\frac{u(x-3h)-\alpha u(x)+u(x+3h)}{9h^2}\right\}+k^2u(x)=0. \quad (1.25)$$

Consider the solution of the Helmholtz equation

$$u(x)=e^{ikx}.$$

We know that

$$u(x+h)+u(x-h)=2\cos khu(x)$$

$$u(x+2h)+u(x-2h)=2\cos 2khu(x)$$

$$u(x+3h)+u(x-3h)=2\cos 3khu(x).$$

Substituting the values in (1.25)

$$\frac{3}{2}\left(\frac{2\cos kh-\alpha}{h^2}\right)-\frac{3}{5}\left(\frac{2\cos 2kh-\alpha}{4h^2}\right)+\frac{1}{10}\left(\frac{2\cos 3kh-\alpha}{9h^2}\right)+k^2=0$$

$$270(2\cos kh-\alpha)-27(2\cos 2kh-\alpha)+2(2\cos 3kh-\alpha)+180k^2h^2=0 \quad (1.26)$$

$$540\cos kh-270\alpha-54\cos 2kh+27\alpha+4\cos 3kh-2\alpha+180k^2h^2=0$$

$$245\alpha=540\cos kh-54\cos 2kh+4\cos 3kh+10h^2k^2$$

$$\alpha=\frac{108}{49}\cos kh-\frac{54}{245}\cos 2kh+\frac{4}{245}\cos 3kh+\frac{36}{49}k^2h^2.$$

Substituting the value of α in (1.26), we have

$$270\left\{u(x-h)+u(x+h)-\left(\frac{108}{49}\cos kh-\frac{54}{245}\cos 2kh+\frac{4}{245}\cos 3kh+\frac{36}{49}k^2h^2\right)u(x)\right\}$$
$$-27\left\{u(x-2h)+u(x+2h)-\left(\frac{108}{49}\cos kh-\frac{54}{245}\cos 2kh+\frac{4}{245}\cos 3kh+\frac{36}{49}k^2h^2\right)u(x)\right\}$$
$$+2\left\{u(x-3h)+u(x+3h)-\left(\frac{108}{49}\cos kh-\frac{54}{245}\cos 2kh+\frac{4}{245}\cos 3kh+\frac{36}{49}k^2h^2\right)u(x)\right\}$$
$$+180k^2h^2u(x)=0.$$

on performing ordinary simplification, we arrive

$$270u(x-h)+270u(x+h)-27u(x-2h)-27u(x+2h)+2u(x-3h)+2u(x+3h)-\\ 540\cos khu(x)+54\cos 2khu(x)-4\cos 3khu(x)=0 \quad (1.27)$$

Inserting plane wave solution $u(x)=e^{i\tilde{k}x}$ in (1.27), we have

$$270\left(e^{i\tilde{k}x}.e^{-i\tilde{k}h}+e^{i\tilde{k}x}.e^{i\tilde{k}h}\right)-27\left(e^{i\tilde{k}x}.e^{-2i\tilde{k}h}+e^{i\tilde{k}x}.e^{2i\tilde{k}h}\right)+2\left(e^{i\tilde{k}x}.e^{-3i\tilde{k}h}+e^{i\tilde{k}x}.e^{3i\tilde{k}h}\right)\\ -540\cos khe^{i\tilde{k}x}+54\cos 2khe^{i\tilde{k}x}-4\cos 3khe^{i\tilde{k}x}=0$$

$$270(2\cos\tilde{k}h-27(2\cos 2\tilde{k}h)+2(2\cos 3\tilde{k}h)-540\cos kh+54\cos 2kh-4\cos 3kh=0$$

$$540\cos\tilde{k}h-54\cos 2\tilde{k}h+4\cos 3\tilde{k}h=540\cos kh-54\cos 2kh+4\cos 3kh$$

$$\tilde{k}h=kh\Rightarrow\tilde{k}-k=0.$$

Hence by dispersion relation, we conclude that there is no dispersion.

1.4.5 An alternative way for the construction of exact central finite difference scheme for the Helmholtz equation proposed by Yau Shu Wong [2]

In this section we discuss the alternative way presented in the paper [2] for the construction of exact central finite difference scheme for the Helmholtz Equation. Reconsider the Helmholtz Equation,

$$u''(x)+k^2u(x)=0$$

$$u''(x)=-k^2u(x) \quad (1.28)$$

$$u^{(4)}(x)=-k^2u''(x)$$

$$u^{(4)}(x)=-k^2(-k^2u(x))$$

$$u^{(4)}(x)=k^4u(x).$$

Therefore, we have the following recursive relation

$$u^{(2n)}(x) = (-1)^n (k^2)^n u(x) \qquad where \;\; n = 1, 2, 3, \ldots$$

$$u(x+h) = u(x) + hu'(x) + \frac{h^2}{2!}u''(x) + \frac{h^3}{3!}u'''(x) + \frac{h^4}{4!}u^4(x) + \cdots$$

$$u(x-h) = u(x) - hu'(x) + \frac{h^2}{2!}u''(x) - \frac{h^3}{3!}u'''(x) + \frac{h^4}{4!}u^4(x) - \cdots$$

$$\begin{aligned}
u(x+h) + u(x-h) &= 2u(x) + \frac{2h^2}{2!}u''(x) + \frac{2h^4}{4!}u^{(4)}(x) + \frac{2h^6}{6!}u^{(6)}(x) + \cdots \\
&= 2\left(u(x) + \frac{h^2}{2!}u''(x) + \frac{h^4}{4!}u^{(4)}(x) + \frac{h^6}{6!}u^{(6)}(x) + \cdots\right) \\
&= 2\left(u(x) - \frac{h^2k^2}{2!}u(x) + \frac{h^4k^4}{4!}u(x) - \frac{h^6k^6}{6!}u(x) + \cdots\right) \\
&= 2u(x)\left(1 - \frac{(kh)^2}{2!} + \frac{(kh)^4}{4!} - \frac{(kh)^6}{6!} + \cdots\right).
\end{aligned}$$

Now consider the modified central finite difference scheme presented in [1]

$$\begin{aligned}
u''(x) &= \frac{u(x-h) - \alpha u(x) + u(x+h)}{h^2} \\
&= \frac{2\cos khu(x) - \alpha u(x)}{h^2}.
\end{aligned}$$

Inserting the value of $u''(x)$ in (1.28), then we will get

$$\frac{2\cos khu(x) - \alpha u(x)}{h^2} = -k^2 u(x)$$

$$\frac{2\cos kh - \alpha}{h^2} = -k^2$$

$$2\cos kh - \alpha = -h^2k^2$$

$$\alpha = 2\cos kh + h^2k^2.$$

This is the coefficient of the middle node in the central finite difference scheme same as the α presented in [1].

Now Substituting α in the central finite difference scheme,

$$u(x+h) - 2\cos khu(x) + u(x-h) = 0$$

$$u_{N+1} - 2\cos khu_N + u_{N-1} = 0. \tag{1.29}$$

In [1] the solution of discretized equations satisfies the solution of Helmholtz equation exactly only at the interior nodes, but in [2], Yau Shu Wong introduce a new finite difference approach which satisfies the solution of Helmholtz equation exactly the interior nodes of the Helmholtz equation at any wavenumber. He also derive the finite difference for the non-reflecting boundary conditions so that the solution of discretized equations satisfies the solution of Helmholtz equation exactly at the interior nodes as well as at boundary nodes which is discussed in the following section.

1.4.6 Finite difference scheme for Non-Reflecting boundary condition

Consider the non-reflecting boundary condition

$$u'(x) - iku(x) = 0.$$

Using the second order accurate central finite difference scheme of first derivative

$$\frac{u(x+h) - u(x-h)}{2h} - iku(x) = 0.$$

Working on same steps which we have already discussed

$$\begin{aligned}
u(x+h) &= u(x) + hu'(x) + \frac{h^2}{2!}u''(x) + \frac{h^3}{3!}u'''(x) + \frac{h^4}{4!}u^4(x) + \cdots \\
u(x-h) &= u(x) - hu'(x) + \frac{h^2}{2!}u''(x) - \frac{h^3}{3!}u'''(x) + \frac{h^4}{4!}u^4(x) - \cdots \\
u(x+h) - u(x-h) &= 2hu'(x) + \frac{2h^3}{3!}u'''(x) + \frac{2h^5}{5!}u^{(5)}(x) + \cdots \\
\frac{u(x+h) - u(x-h)}{2h} &= u'(x) + \frac{h^2}{3!}u'''(x) + \frac{h^4}{5!}u^{(5)}(x) + \cdots .
\end{aligned}$$

Rewrite the non-reflecting boundary condition

$$u'(x) = iku(x)$$

$$
\begin{aligned}
u''(x) &= iku'(x) \\
&= ik\,(iku(x)) \\
&= -k^2u(x)
\end{aligned}
$$

$$
\begin{aligned}
u'''(x) &= (-k^2)u'(x) \\
&= (-k^2)(iku(x)) \\
&= -ik^3u(x)
\end{aligned}
$$

similarly,

$$u^{(4)}(x) = k^4u(x)$$

$$u^{(5)}(x) = ik^5u(x).$$

Therefore, we have the following recursive relation of the odd order derivatives

$$u^{(2n+1)}(x) = (-1)^n ik^{2n+1}u(x) \quad where \quad n = 1, 2, 3, \ldots$$

$$
\begin{aligned}
\frac{u(x+h) - u(x-h)}{2h} &= u'(x) + \frac{h^2}{3!}u'''(x) + \frac{h^4}{5!}u^{(5)}(x) + \cdots \\
&= iku(x) - \frac{ih^2k^3}{3!}u(x) + \frac{ih^4k^5}{5!}u(x) - \cdots \\
&= iku(x)\left\{1 - \frac{(kh)^2}{3!} + \frac{(kh)^4}{5!} - \cdots\right\} \\
&= \frac{i}{h}\left\{kh - \frac{(kh)^3}{3!} + \frac{(kh)^5}{5!} - \cdots\right\}u(x) \\
&= \frac{i}{h}\sin khu(x).
\end{aligned}
$$

Hence

$$\frac{u(x+h) - u(x-h)}{2h} = \frac{i}{h}\sin khu(x)$$

$$u(x+h) - u(x-h) = 2i\sin khu(x)$$

$$u_{N+1} - u_{N-1} = 2i\sin khu_N.$$

From this equation, putting the value of u_{N+1} in (1.21), finally we get

$$2(-\cos kh + i\sin kh)u_N + 2u_{N-1} = 0.$$

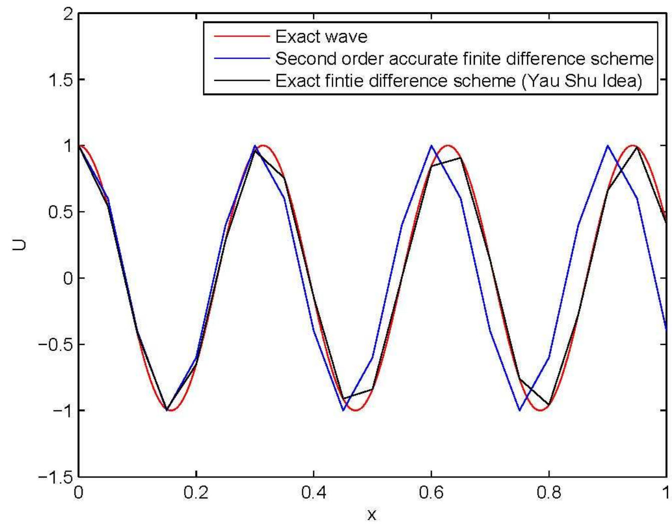


Figure 1.24: Comaprison of exact, second order accurate Standard C.F.D scheme and Exact C.F.D scheme proposed by Yau Shu Wong at wave number $k = 20$ and number of elements $N = 20$. One can easily see that the wave generated by the scheme proposed by Yau Shu satisfies the exact wave at interior nodes as well as boundary nodes.

1.4.7 Implementation of exact central finite difference scheme

$$u_0 = 1 \qquad \text{for } \ j = 0;$$

$$-2\cos khu_1 + u_2 = -1 \qquad \text{for } \ j = 1;$$

$$u_{j-1} - 2\cos khu_j + u_{j+1} = 0 \qquad \forall \ j = 2, 3, \ldots, N-1;$$

$$u_{N-1} - 2\cos khu_N + u_{N+1} = 0 \qquad \text{for } \ j = N.$$

We know the value of u_{N+1} from (1.29) by using the second order central finite difference scheme of first derivative in the non-reflecting boundary condition

$$2u_{N-1} - 2(\cos kh - i\sin kh)u_N = 0 \qquad \text{for } \ j = N.$$

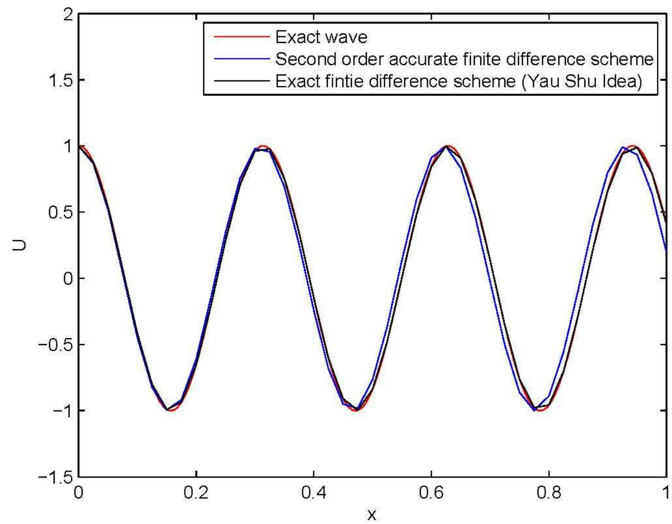


Figure 1.25: Comaprison with wave number $k = 20$ and $N = 40$

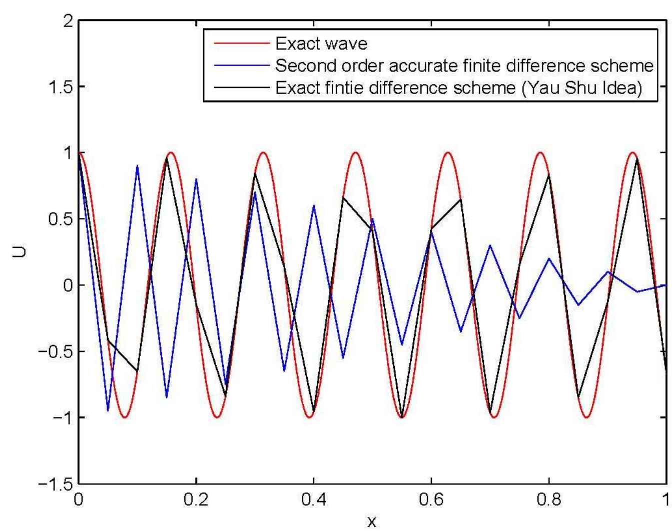


Figure 1.26: Exact finite difference scheme also satisfies the solution of Helmholtz equation exactly at interior and boundary nodes at high wave number, that is $k = 40$ and $N = 20$.

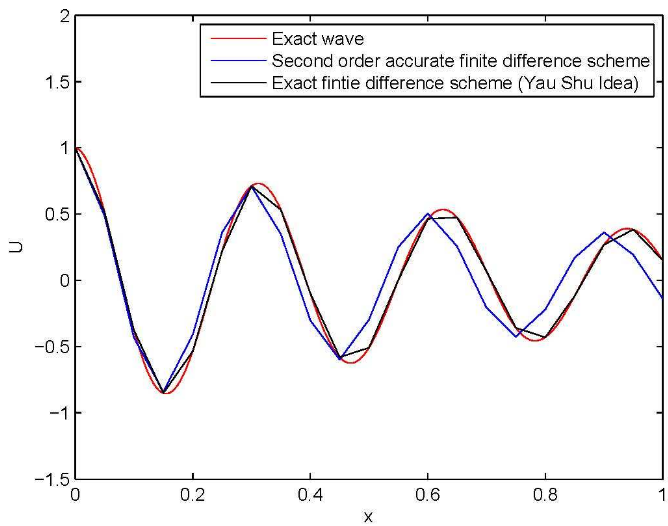


Figure 1.27: Comparison with complex wave number $k = 20 + i$ at number of elements $N = 20$

Chapter 2
Dispersion free finite difference schemes for Helmholtz equations

Chapter 2

Dispersion free finite difference schemes for Helmholtz equation

Wave propagation phenomena is very important in various fields of engineering sciences. Helmholtz equation is the time harmonic case of wave equation. Solution of Helmholtz equation required in different applied sciences such as acoustic wave scattering, geophysical problems, radar theory, antenna theory, archaeology, civil engineering and many more. Finite difference methods are important in simulations of waves. Analytic methods to find the solution are not much efficient so we are interested in finding the numerical solution by using finite difference methods which helps us to simulate such wave problems.

Other methods such as finite element method [5, 6], finite volume method [15], boundary element method [7] and spectral method [12, 13, 15, 16] are also commonly used to simulate the wave in time-dependent as well as in time-harmonic problems. Many people working to develop finite difference methods from decades [1, 2, 4–6, 11, 13, 15, 16, 19], especially working to reduce dispersion between exact wave and wave obtained by finite difference schemes [1, 2, 5–7, 16]. Most of them reduce dispersion by refining mesh size but it increases computational cost significantly.

Nehrbass [1] constructed the finite difference scheme by modifying the coefficient of the central node in the standard central finite difference scheme which makes it exact finite difference scheme, but evidently, that scheme is not as exact finite difference scheme because of inexact boundary conditions. It should be noted that all numerical schemes required a fine mesh size in order to obtain the desired accuracy of the numerical solutions at high wave number.

Yau Shu Wong [2] presented a tremendous work in the field of computational wave propagation. He presented an exact finite difference scheme by making the exact finite difference scheme for derivative boundary condition, but just up to second order accuracy.

We extend the idea of Yau Shu wong [2] to construct novel finite difference schemes with accuracy of order four in one dimensions as well as in two dimensions. It performs better at high wave number with less computational cost. We also present the generalized expression for the coefficient of the middle node for any order accurate modified finite difference scheme.

2.1 Central finite difference scheme of any order

We start with the following expression given in the book of Cohen [8]

$$\lambda_i = \prod_{l=1, l\neq i}^{\frac{n}{2}} \frac{l^2}{l^2 - i^2}.$$

Where n is the order of the scheme.

Example for fourth order scheme:

$$\lambda_1 = \prod_{l\neq 1}^{2} \frac{4}{4-1} = \frac{4}{3}$$

$$\lambda_2 = \prod_{l\neq 2, l=1}^{2} \frac{1}{1-4} = \frac{-1}{3}$$

and for sixth order scheme:

$$\lambda_1 = \prod_{l\neq 1, l=2}^{3} \frac{l^2}{l^2-1} = \frac{3}{2}$$

$$\lambda_2 = \prod_{l\neq 2, l=1}^{3} \frac{l^2}{l^2-4} = \frac{-3}{5}$$

$$\lambda_3 = \prod_{l\neq 3, l=1}^{3} \frac{l^2}{l^2-9} = \frac{1}{10}$$

$$u''(x) = \sum_{i=1}^{\frac{n}{2}} \lambda_i \left\{ \frac{u(x-ih) - 2u(x) + u(x+ih)}{(ih)^2} \right\} + O\left(h^n\right). \tag{2.1}$$

This is the generalized expression for the **central finite difference scheme of any order**. Now we started with the idea presented in the paper [1]. In this paper, a scheme called *modified central finite difference scheme* for the Helmholtz equation is constructed by replacing the coefficient of the middle node in the traditional central finite difference schemes.

In [1] the solution of discretized equations satisfies the solution of Helmholtz equation exactly only at the interior nodes .

Now we replace $'2'$ by $'\alpha'$ in (2.1), gives

$$u''(x) = \sum_{i=1}^{\frac{n}{2}} \lambda_i \left\{ \frac{u(x-ih) - \alpha u(x) + u(x+ih)}{(ih)^2} \right\} + O\left(h^n\right)$$

$$u''(x) = \sum_{i=1}^{\frac{n}{2}} \lambda_i \left\{ \frac{(2\cos ikh - \alpha) u(x)}{(ih)^2} \right\} + O\left(h^n\right).$$

This is the *any order accurate central finite difference scheme of second order derivative.*

2.2 Generalization of the coefficient of the middle node

The time is to generalize the coefficient of the middle node α, which gives us the finite difference scheme so that the solution of discretized equations satisfies the solution of the Helmholtz equation exactly at the interior nodes. Consider the

Helmholtz equation and substitute the above expression in

$$u''(x) + k^2 u(x) = 0$$

gives

$$\left(\sum_{i=1}^{\frac{n}{2}} \lambda_i \frac{1}{(ih)^2}(2\cos ikh - \alpha)\right) u(x) + k^2 u(x) = 0$$

$$\sum_{i=1}^{\frac{n}{2}} \left(\lambda_i \frac{2\cos ikh}{i^2} - \lambda_i \frac{\alpha}{i^2}\right) + h^2 k^2 = 0$$

$$\sum_{i=1}^{\frac{n}{2}} \left(\lambda_i \frac{2\cos ikh}{i^2}\right) - \sum_{i=1}^{\frac{n}{2}} \left(\lambda_i \frac{\alpha}{i^2}\right) + h^2 k^2 = 0$$

$$\sum_{i=1}^{\frac{n}{2}} \left(\lambda_i \frac{2\cos ikh}{i^2}\right) + h^2 k^2 = \sum_{i=1}^{\frac{n}{2}} \left(\lambda_i \frac{\alpha}{i^2}\right)$$

$$\sum_{i=1}^{\frac{n}{2}} \left(\lambda_i \frac{2\cos ikh}{i^2}\right) + h^2 k^2 = \alpha \sum_{i=1}^{\frac{n}{2}} \left(\lambda_i \frac{1}{i^2}\right)$$

$$\alpha = \frac{\sum_{i=1}^{\frac{n}{2}} \left(\lambda_i \frac{2\cos ikh}{i^2}\right) + h^2 k^2}{\sum_{i=1}^{\frac{n}{2}} \left(\lambda_i \frac{1}{i^2}\right)}.$$

Which is the general form of the coefficient of the middle node in the modified central finite difference scheme.

2.3 The use of bloch wave property

In this section, we discuss an alternative and a simple way for the construction of exact central finite difference scheme for the Helmholtz equation.
Reconsider the Helmholtz equation,

$$u''(x) = -k^2 u(x).$$

On differentiating twice, gives

$$u^{(4)}(x) = -k^2 u''(x).$$

Inserting the value of $u''(x)$ in the above expression

$$u^{(4)}(x) = -k^2(-k^2u(x))$$

$$u^{(4)}(x) = k^4u(x).$$

$$\vdots$$

$$u^{(2n)}(x) = (-1)^n(k^2)^n u(x) \quad \textit{where } n = 1, 2, 3, \ldots .$$

This is the recursive relation for even order derivative.

Consider the non-reflecting boundary condition

$$u'(x) - iku(x) = 0.$$

Making twice differentiation, gives

$$u'''(x) = -ik^3u(x).$$

Similarly

$$u^{(5)}(x) = ik^5u(x).$$

$$\vdots$$

$$u^{(2n+1)}(x) = (-1)^n ik^{2n+1}u(x) \quad \textit{where } n = 1, 2, 3, \ldots .$$

This is the recursive relation for odd order derivative.

Consider the Taylor series approximation of $u(x - 2h)$

$$\begin{aligned} u(x-2h) &= u(x) - 2hu'(x) + 2h^2u''(x) - \frac{4h^3}{3}u'''(x) + \frac{2h^4}{3}u^{(4)}(x) \\ &\quad - \frac{4h^5}{15}u^{(5)}(x) + \frac{4h^6}{45}u^{(6)}(x) - \cdots . \end{aligned}$$

Inserting the even or odd order derivative in terms of $u(x)$

$$\begin{aligned}
u(x-2h) \quad &= u(x) - 2hiku(x) + 2h^2(-k^2)u(x) - \frac{4h^3}{3}(-ik^3)u(x) + \frac{2h^4}{3}k^4u(x) \\
&-\frac{4h^5}{15}ik^5u(x) + \frac{4h^6}{45}(-k^6)u(x) - \cdots \\
&= \left(1 - 2ikh - 2h^2k^2 + \frac{4ih^3k^3}{3} + \frac{2h^4k^4}{3} - \frac{4ih^5k^5}{15} - \frac{4h^6k^6}{45} + \cdots\right)u(x) \\
&= \left(1 - 2ikh + \frac{(2ikh)^2}{2!} - \frac{(2ikh)^3}{3!} + \frac{(2ikh)^4}{4!} - \frac{(2ikh)^5}{5!} + \frac{(2ikh)^6}{6!} - \cdots\right)u(x) \\
&= e^{-2ikh}u(x).
\end{aligned}$$

Similarly, we have

$$u(x+2h) = e^{2ikh}u(x)$$

$$u(x+h) = e^{ikh}u(x)$$

$$u(x-h) = e^{-ikh}u(x).$$

Reconsider

$$u''(x) + k^2u(x) = 0.$$

Replacing second order derivative by second order accurate standard central finite difference scheme, gives

$$\frac{u(x-h) - 2u(x) + u(x+h)}{h^2} + k^2u(x) = 0.$$

Rearranging

$$u(x-h) - 2u(x) + u(x+h) + h^2k^2u(x) = \left(e^{-ikh} - 2 + e^{ikh} + k^2h^2\right)u(x).$$

Finally

$$u(x-h) - 2\cos khu(x) + u(x+h) = 0.$$

This is the *Exact central finite difference scheme* obtained without introducing the α as introduced in [1] and [2].

2.4 Exact finite difference scheme for Helmholtz equation constructed from fourth order accurate standard C.F.D scheme.

Consider the Helmholtz equation (1.1)

$$u''(x) = -k^2 u(x).$$

Using the fourth order accurate central finite difference approximation of second order derivative, that is

$$u''(x) = \frac{-u(x-2h) + 16u(x-h) - 30u(x) + 16u(x+h) - u(x+2h)}{12h^2} + 4h^4 u^{(6)}(x) + \cdots .$$

On substituting the above approximation in (1.1), we have

$$u(x-2h) - 16u(x-h) + 30u(x) - 16u(x+h) + u(x+2h) - 12h^2k^2u(x) = 0.$$

We know that

$$u(x-2h) = e^{-2ikh}u(x),\ \ u(x-h) = e^{-ikh}u(x),\ \ u(x+h) = e^{ikh}u(x),\ \ u(x+2h) = e^{2ikh}u(x).$$

Inserting in the above scheme,

$$u(x-2h)-16u(x-h)+30u(x)-16u(x+h)+u(x+2h)-12h^2k^2u(x) = \Big\{e^{-2ikh}-16e^{-ikh}+ 30 - 16e^{ikh} + e^{2ikh} + 12h^2k^2\Big\}.$$

$$u(x-2h) - 16u(x-h) + \Big\{2\cos 2kh - 32\cos kh\Big\} - 16u(x+h) + u(x+2h) = 0.$$

This is the *Exact central finite difference scheme constructed from fourth order accurate central finite difference scheme* valid for the nodes x_i where $i =$

$2, 3, ..., N-2$.

For $i = 0$, we have

$$u(0) = 1.$$

For $i = 1$, we use the fourth order accurate seven point forward difference approximation presented in [3], we have

$$9u(x-h)-9u(x)-19u(x+h)+34u(x+2h)-21u(x+3h)+7u(x+4h)-u(x+5h)+12h^2k^2u(x) = 0$$

using the bloch wave property

$$9u(x-h)-9u(x)-19u(x+h)+34u(x+2h)-21u(x+3h)+7u(x+4h)-u(x+5h)+ 12h^2k^2u(x) = \left\{9e^{-ikh}-9-19e^{ikh}+34e^{2ikh}-21e^{3ikh}+7e^{4ikh}-e^{5ikh}+12h^2k^2\right\}u(x).$$

Finally, we obtain

$$9u(x-h)-\left\{9e^{-ikh}-19e^{ikh}+34e^{2ikh}-21e^{3ikh}+7e^{4ikh}-e^{5ikh}\right\}u(x)-19u(x+h)+ 34u(x+2h)-21u(x+3h)+7u(x+4h)-u(x+5h) = 0.$$

Hence, this is *Exact finite difference scheme constructed from fourth order accurate seven point forward difference approximation* for Helmholtz equation at the node x_1.

For the node x_{N-1}.

Here we use the fourth order accurate seven point backward difference approximation of second order derivative and replacing it in Helmholtz equation, we have

$$u(x-5h)-7u(x-4h)+21u(x-3h)-34u(x-2h)+19u(x-h)+9u(x)-9u(x+h)+ 12h^2k^2u(x) = \left\{e^{-5ikh}-7e^{-4ikh}+21e^{-3ikh}-34e^{-2ikh}+19e^{-ikh}+9-9e^{ikh}+12h^2k^2\right\}u(x)$$

rearranging

$$u(x-5h)-7u(x-4h)+21u(x-3h)-34u(x-2h)+19u(x-h)-\left\{e^{-5ikh}-7e^{-4ikh}+ 21e^{-3ikh}-34e^{-2ikh}+19e^{-ikh}-9e^{ikh}\right\}u(x)-9u(x+h) = 0$$

Hence, this is *Exact finite difference scheme constructed from fourth order accurate seven point backward difference approximation* for Helmholtz equation at the node x_{N-1}.

Now we construct the scheme for the node x_N, here we need the fourth order accurate finite difference approximation of first order derivative presented in [3], given by

$$u'(1) = \frac{1}{12h}(3u_{N-4} - 16u_{N-3} + 36u_{N-2} - 48u_{N-1} + 25u_N)$$

which on inserting into the following

$$u'(1) - iku(1) = 0$$

$$3u(x-4h)-16u(x-3h)+36u(x-2h)-48u(x-h)+25u(x)-2ikhu(x) = \Big\{3e^{-4ikh}- 16e^{-3ikh} + 36e^{-2ikh} - 48e^{-ikh} + 25 - 2ikh\Big\}u(x)$$

rearranging

$$3u(x-4h) - 16u(x-3h) + 36u(x-2h) - 48u(x-h) - \Big\{3e^{-4ikh} - 16e^{-3ikh}+ 36e^{-2ikh} - 48e^{-ikh}\Big\}u(x) = 0$$

Hence, this is *Exact finite difference scheme* for Helmholtz equation at the boundary node x_N.

Finally, we have the system

$$u(0) = 1 \quad \text{for } N = 0;$$

$$9u_{j-1} - \Big\{9e^{-ikh} - 19e^{ikh} + 34e^{2ikh} - 21e^{3ikh} + 7e^{4ikh} - e^{5ikh}\Big\}u_j - 19u_{j+1}+ 34u_{j+2} - 21u_{j+3} + 7u_{j+4} - u_{j+5} = 0 \quad \text{for } j = 1;$$

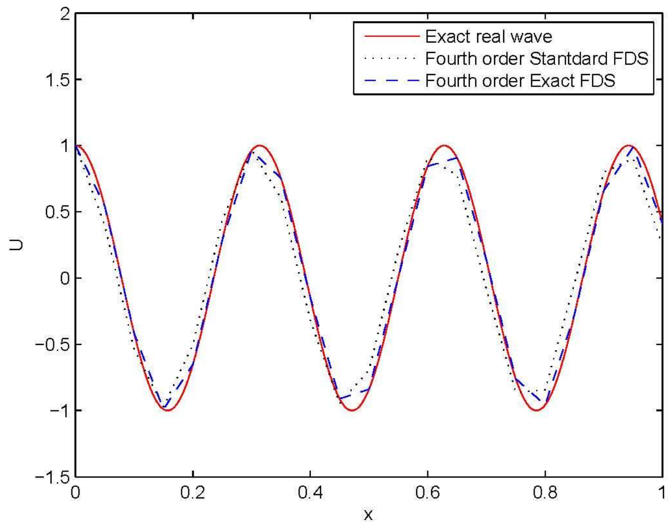


Figure 2.1: Comparison of Exact, fourth order accurate Standard C.F.D scheme and Fourth order Exact F.D scheme with wave number $k = 20$ at number of elements $N = 20$.

$$u_{j-2} - 16u_{j-1} + \left\{ e^{-2ikh} - 16e^{-ikh} - 16e^{ikh} + e^{2ikh} \right\} - 16u_{j+1} + u_{j+2} = 0 \quad \text{for } j = 2, 3, \ldots, N-2;$$

$$u_{j-5} - 7u_{j-4} + 21u_{j-3} - 34u_{j-2} + 19u_{j-1} - \left\{ e^{-5ikh} - 7e^{-4ikh} + \right.$$
$$\left. 21e^{-3ikh} - 34e^{-2ikh} + 19e^{-ikh} - 9e^{ikh} \right\} u_j - 9u_{j+1} = 0 \quad ; \text{for } j = N-1$$

$$3u_{j-4} - 16u_{j-3} + 36u_{j-2} - 48u_{j-1} - \left\{ 3e^{-4ikh} - 16e^{-3ikh} + \right.$$
$$\left. 36e^{-2ikh} - 48e^{-ikh} \right\} u_j = 0 \quad \text{for } j = N.$$

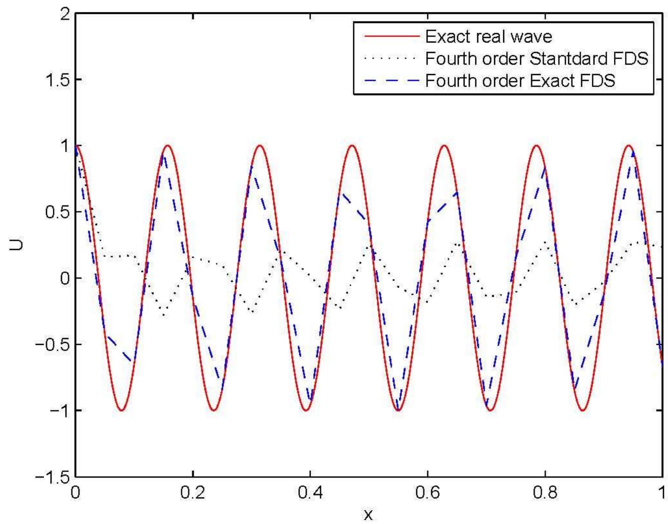


Figure 2.2: Fourth order Exact F.D scheme again exactly satisfies the solution of Helmholtz equation at high wave number $k = 40$ with number of elements $N = 20$.

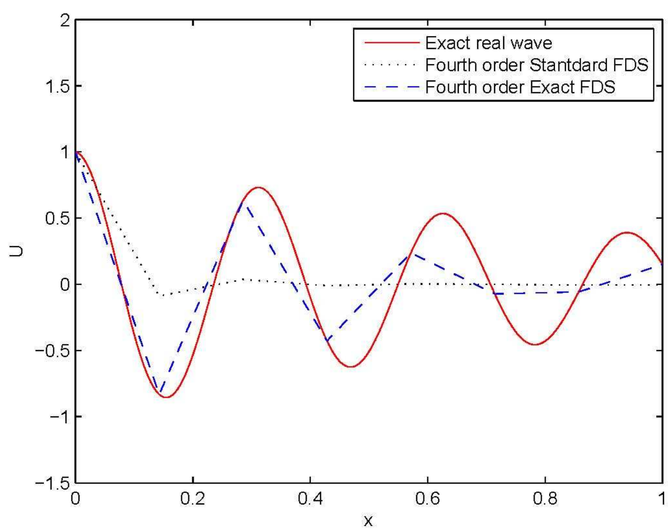


Figure 2.3: Comparison of Exact, fourth order accurate Standard C.F.D scheme and fourth order Exact F.D scheme with wave number $k = 20 + i$ at very less number of elements $N = 7$. Fourth order Exact F.D scheme satisfies the solution of Helmholtz equation exactly at very less number of elements with complex wave number.

2.5 Reduced modified central finite difference scheme in two dimensions

In this section we will study the two dimensional Helmholtz equation, it is defined as follows

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + k^2 u(x,y) = 0. \tag{2.2}$$

Here $k^2 = k_1^2 + k_2^2$, whereas $k_1 = k\cos\theta$ and $k_2 = k\sin\theta$.
Let us suppose that the exact solution of (2.2) is;

$$u(x,y) = e^{i(k_1 x + k_2 y)} \tag{2.3}$$

or

$$u(x,y) = e^{ik(x\cos\theta + y\sin\theta).}$$

Now let us see how the finite difference schemes have been developed before, that is;

$$\begin{aligned} u_{xx} &= \frac{1}{h^2}\Big\{u(x-h,y) - 2u(x,y) + u(x+h,y)\Big\} \\ u_{yy} &= \frac{1}{g^2}\Big\{u(x,y-g) - 2u(x,y) + u(x,y+g)\Big\}. \end{aligned}$$

For square mesh we have

$$u(x-h,y) + u(x,y-h) - (4 - h^2k^2)u(x,y) + u(x,y+h) + u(x+h,y) = 0.$$

By using the Nehrbass idea [1], this time we replaced u_{xx} and u_{yy} by the following,

$$\begin{aligned} u_{xx} &= \frac{1}{h^2}\Big\{u(x-h,y) - \alpha u(x,y) + u(x+h,y)\Big\} \\ u_{yy} &= \frac{1}{g^2}\Big\{u(x,y-g) - \alpha u(x,y) + u(x,y+g)\Big\}. \end{aligned}$$

Here we have taken the square mesh size. After substitution we have from equation (2.2), the form

$$u(x-h,y) + u(x,y-h) - \left(2\alpha - h^2k^2\right) u(x,y) + u(x,y+h) + u(x+h,y) = 0. \tag{2.4}$$

To find the value of α, we take the exact solution (2.3) and simplifying we have the form,

$$\Big\{2\cos(kh\cos\theta) - (2\alpha - h^2k^2) + 2\cos(kh\sin\theta)\Big\}u(x,y) = 0.$$

Since $u(x,y) \neq 0$ hence,

$$2\cos(kh\cos\theta) - (2\alpha - h^2k^2) + 2\cos(kh\sin\theta) = 0$$

$$\alpha = \frac{1}{2}h^2k^2 + \cos(kh\cos\theta) + \cos(kh\sin\theta).$$

Substituting the value of α in equation (2.4), we obtained the following exact scheme;

$$u(x-h,y)+u(x,y-h)-2(\cos(kh\cos\theta)+\cos(kh\sin\theta))u(x,y)+u(x,y+h)+u(x+h,y) = 0. \tag{2.5}$$

This is the *modified central finite difference scheme in two dimensions.*

2.6 The Dirchlet problem for the Helmholtz equation in two dimensions

We consider the Helmholtz equation in two dimensions

$$u_{xx} + u_{yy} + k^2u(x,y) = 0 \tag{2.6}$$

and solve it with the following Dirchlet boundary conditions

$$u = f_0(x) \quad \text{along } y = 0, \text{ and } 0 \leq x \leq 1$$

$$u = f_1(x) \quad \text{along } y = 1, \text{ and } 0 \leq x \leq 1$$

$$u = g_0(x) \quad \text{along } x = 0, \text{ and } 0 \leq y \leq 1$$

$$u = g_1(x) \quad \text{along } x = 1, \text{ and } 0 \leq y \leq 1.$$

The simplest region R is the unit square bounded by the lines $x = 0$, $x = 1$, $y = 0$ and $y = 1$ in (x,y) plane, these lines form ∂R on which the values of u is specified.

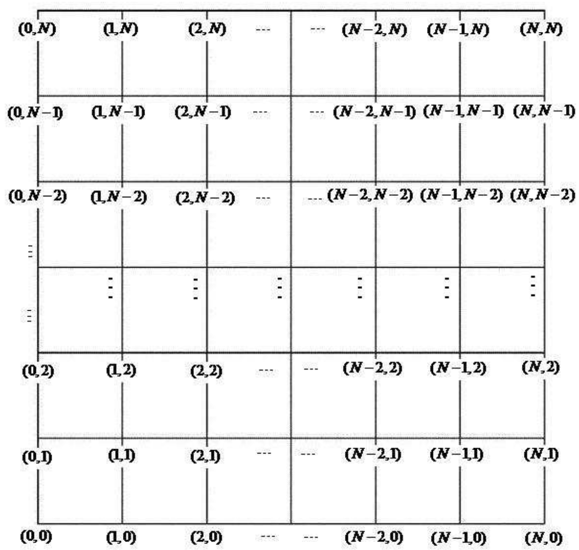


Figure 2.4: Grid

A square mesh is now superimposed on R and the value of u calculated at each point of the mesh. Usually, the mesh consists of equally spaced straight lines parallel to x, y-axis. Number of total points in a grid is $(M+2)^2$, number of interior points in a grid is M^2 and number of boundary points is $4M+4$. First we construct the schemes for the corner points $(1,1), (1, N-1), (N-1, 1)$ and $(N-1, N-1)$ of interior of the grid.

Implementation for the point $(1,1)$

In order to construct the scheme for the point $(1,1)$, replacing the second order partial derivatives by the fourth order accurate seven point forward difference approximation along x and y-axis in (2.6), which is presented in the thesis of S. A Taj [3], gives

$$\left\{9u_{i-1,j} - 9u_{i,j} - 19u_{i+1,j} + 34u_{i+2,j} - 21u_{i+3,j} + 7u_{i+4,j} - u_{i+5,j}\right\}+$$

$$\left\{9u_{i,j-1} - 9u_{i,j} - 19u_{i,j+1} + 34u_{i,j+2} - 21u_{i,j+3} + 7u_{i,j+4} - u_{i,j+5}\right\}+$$

$$12h^2k^2u_{i,j} = \left\{9e^{-ik_1h} - 9 - 19e^{ik_1h} + 34e^{2ik_1h} - 21e^{3ik_1h} + 7e^{4ik_1h} - e^{5ik_1h}\right\}+$$

$$\left\{9e^{-ik_2h} - 9 - 19e^{ik_2h} + 34e^{2ik_2h} - 21e^{3ik_2h} + 7e^{4ik_2h} - e^{5k_2h}\right\} + 12h^2k^2u_{i,j} = 0.$$

On rearranging

$$9u_{i-1,j} - \left\{9e^{-ik_1h} - 19e^{ik_1h} + 34e^{2ik_1h} - 21e^{3ik_1h} + 7e^{4ik_1h} - e^{5ik_1h} + 9e^{-ik_2h} - 19e^{ik_2h} +\right.$$

$$\left. 34e^{2ik_2h} - 21e^{3ik_2h} + 7e^{4ik_2h} - e^{5k_2h}\right\}u_{i,j} - 19u_{i+1,j} + 34u_{i+2,j} - 21u_{i+3,j} + 7u_{i+4,j} - u_{i+5,j} +$$

$$9u_{i,j-1} - 19u_{i,j+1} + 34u_{i,j+2} - 21u_{i,j+3} + 7u_{i,j+4} - u_{i,j+5} = 0.$$

This scheme is used for the corner node $u_{1,1}$ of interior of the grid.

Implementation for the point $(N-1, 1)$

In order to construct the scheme for the point $(N-1,1)$, we replace the second order partial derivative with respect to x by the fourth order accurate seven point backward difference approximation which is also presented in [3]and second order partial derivative with respect to y by fourth order accurate seven point forward difference approximation in (2.6), gives

$$\Big\{u_{i-5,j} - 7u_{i-4,j} + 21u_{i-3,j} - 34u_{i-2,j} + 19u_{i-1,j} + 9u_{i,j} - 9u_{i+1,j}\Big\}+$$
$$\Big\{9u_{i,j-1} - 9u_{i,j} - 19u_{i,j+1} + 34u_{i,j+2} - 21u_{i,j+3} + 7u_{i,j+4} - u_{i,j+5}\Big\}+$$
$$12h^2k^2u_{i,j} = \Big\{e^{-5ik_1h} - 7e^{-4ik_1h} + 21e^{-3ik_1h} - 34e^{-2ik_1h} + 19e^{-ik_1h} + 9 - 9e^{ik_1h}\Big\}+$$
$$\Big\{9e^{-ik_2h} - 9 - 19e^{ik_2h} + 34e^{2ik_2h} - 21e^{3ik_2h} + 7e^{4ik_2h} - e^{5ik_2h}\Big\} + 12h^2k^2u_{i,j} = 0.$$

On rearranging

$$u_{i-5,j} - 7u_{i-4,j} + 21u_{i-3,j} - 34u_{i-2,j} + 19u_{i-1,j} - \Big\{e^{-5ik_1h} - 7e^{-4ik_1h} + 21e^{-3ik_1h} -$$
$$34e^{-2ik_1h} + 19e^{-ik_1h} - 9e^{ik_1h} + 9e^{-ik_2h} - 19e^{ik_2h} + 34e^{2ik_2h} - 21e^{3ik_2h} + 7e^{4ik_2h} - e^{5k_2h}\Big\}u_{i,j} -$$
$$9u_{i+1,j} + 9u_{i,j-1} - 19u_{i,j+1} + 34u_{i,j+2} - 21u_{i,j+3} + 7u_{i,j+4} - u_{i,j+5} = 0.$$

This scheme is used for the corner node $u_{N-1,1}$ of interior of the grid.

Implementation for the point $(1, N-1)$

We replace the second order partial derivative with respect to x by the fourth order accurate seven point forward difference approximation and second order partial derivative with respect to y by fourth order accurate seven point backward difference approximation in (2.6) to construct the scheme for the point $(1, N-1)$, gives

$$\Big\{9u_{i-1,j} - 9u_{i,j} - 19u_{i+1,j} + 34u_{i+2,j} - 21u_{i+3,j} + 7u_{i+4,j} - u_{i+5,j}\Big\}+$$
$$\Big\{u_{i,j-5} - 7u_{i,j-4} + 21u_{i,j-3} - 34u_{i,j-2} + 19u_{i,j-1} + 9u_{i,j} - 9u_{i,j+1}\Big\}+$$
$$12h^2k^2u_{i,j} = \Big\{9e^{-ik_1h} - 9 - 19e^{ik_1h} + 34e^{2ik_1h} - 21e^{3ik_1h} + 7e^{4ik_1h} - e^{5ik_1h}\Big\}+$$
$$\Big\{e^{-5ik_2h} - 7e^{-4ik_2h} + 21e^{-3ik_2h} - 34e^{-2ik_2h} + 19e^{-ik_2h} + 9 - 9e^{ik_2h}\Big\} + 12h^2k^2u_{i,j} = 0.$$

On rearranging

$$9u_{i-1,j} - \Big\{9e^{-ik_1h} - 19e^{ik_1h} + 34e^{2ik_1h} - 21e^{3ik_1h} + 7e^{4ik_1h} - e^{5ik_1h} + e^{-5ik_2h} -$$
$$7e^{-4ik_2h} + 21e^{-3ik_2h} - 34e^{-2ik_2h} + 19e^{-ik_2h} - 9e^{ik_2h}\Big\} - 19u_{i+1,j} + 34u_{i+2,j} - 21u_{i+3,j} +$$
$$7u_{i+4,j} - u_{i+5,j} + u_{i,j-5} - 7u_{i,j-4} + 21u_{i,j-3} - 34u_{i,j-2} + 19u_{i,j-1} - 9u_{i,j+1} = 0.$$

This scheme is used for the corner node $u_{N-1,1}$ of interior of the grid.

Implementation for the point $(N-1, N-1)$

In order to construct the scheme for the point $(N-1, N-1)$, replacing the second order partial derivatives by the fourth order accurate seven point Backward difference approximation along x and y-axis in (2.6), gives

$$\Big\{u_{i-5,j} - 7u_{i-4,j} + 21u_{i-3,j} - 34u_{i-2,j} + 19u_{i-1,j} + 9u_{i,j} - 9u_{i+1,j}\Big\} +$$
$$\Big\{u_{i,j-5} - 7u_{i,j-4} + 21u_{i,j-3} - 34u_{i,j-2} + 19u_{i,j-1} + 9u_{i,j} - 9u_{i,j+1}\Big\} +$$
$$12h^2k^2u_{i,j} = \Big\{e^{-5ik_1h} - 7e^{-4ik_1h} + 21e^{-3ik_1h} - 34e^{-2ik_1h} + 19e^{-ik_1h} + 9 - 9e^{ik_1h}\Big\} +$$
$$\Big\{e^{-5ik_2h} - 7e^{-4ik_2h} + 21e^{-3ik_2h} - 34e^{-2ik_2h} + 19e^{-ik_2h} + 9 - 9e^{ik_2h}\Big\} + 12h^2k^2u_{i,j} = 0.$$

On rearranging

$$u_{i-5,j} - 7u_{i-4,j} + 21u_{i-3,j} - 34u_{i-2,j} + 19u_{i-1,j} - \Big\{e^{-5ik_1h} - 7e^{-4ik_1h} + 21e^{-3ik_1h} -$$
$$34e^{-2ik_1h} + 19e^{-ik_1h} - 9e^{ik_1h} + e^{-5ik_2h} - 7e^{-4ik_2h} + 21e^{-3ik_2h} - 34e^{-2ik_2h} + 19e^{-ik_2h} -$$
$$9e^{ik_2h}\Big\} - 9u_{i+1,j} + u_{i,j-5} - 7u_{i,j-4} + 21u_{i,j-3} - 34u_{i,j-2} + 19u_{i,j-1} - 9u_{i,j+1} = 0.$$

This scheme is used for the corner node $u_{N-1,N-1}$ of interior of the grid.

Left Boundary

We now construct the scheme for left boundary (excluding corner points) of interior of the grid by replacing second order partial derivative with respect to x by fourth order accurate seven point forward difference approximation and second order partial derivative with respect to y by fourth order accurate central finite difference approximation in (2.6).

Implementation for the points $(1, 2)$ and $(1, N-2)$

Firstly, we construct the schemes for the end points $(1, 2)$ and $(1, N-2)$ of the left boundary respectively.

$$\begin{aligned}
&\left\{9u_{i-1,j} - 9u_{i,j} - 19u_{i+1,j} + 34u_{i+2,j} - 21u_{i+3,j} + 7u_{i+4,j} - u_{i+5,j}\right\}+ \\
&\quad \left\{u_{i,j-2} - 16u_{i,j-1} + \theta_2 u_{i,j} - 16u_{i,j+1} + u_{i,j+2}\right\} + 12h^2k^2u_{i,j} = \\
&\quad \left\{9e^{-ik_1h} - 9 - 19e^{ik_1h} + 34e^{2ik_1h} - 21e^{3ik_1h} + 7e^{4ik_1h} - e^{5ik_1h}\right\}+ \\
&\qquad \left\{e^{-2ik_2h} - 16e^{-ik_2h} + \theta_2 - 16e^{ik_2h} + e^{2ik_2h}\right\} + 12h^2k^2u_{i,j} = 0.
\end{aligned}$$

where

$$\theta_2 = 2\cos 2k_2h - 32\cos k_2h.$$

On rearranging

$$\begin{aligned}
&-\left\{9e^{-ik_1h} - 19e^{ik_1h} + 34e^{2ik_1h} - 21e^{3ik_1h} + 7e^{4ik_1h} - e^{5ik_1h} + e^{-2ik_2h} - \right. \\
&\quad \left. 16e^{-ik_2h} - 16e^{ik_2h} + e^{2ik_2h}\right\} - 19u_{i+1,j} + 34u_{i+2,j} - 21u_{i+3,j} + 7u_{i+4,j} - \\
&\qquad u_{i+5,j} - 16u_{i,j-1} - 16u_{i,j+1} + u_{i,j+2} = -9e^{ik_2y_2} - e^{ik_1x_1}.
\end{aligned}$$

This scheme is used for the node $u_{1,2}$ of interior of the grid.

$$\left\{9u_{i-1,j} - 9u_{i,j} - 19u_{i+1,j} + 34u_{i+2,j} - 21u_{i+3,j} + 7u_{i+4,j} - u_{i+5,j}\right\} +$$
$$\left\{u_{i,j-2} - 16u_{i,j-1} + \theta_2 u_{i,j} - 16u_{i,j+1} + u_{i,j+2}\right\} + 12h^2k^2u_{i,j} =$$
$$\left\{9e^{-ik_1h} - 9 - 19e^{ik_1h} + 34e^{2ik_1h} - 21e^{3ik_1h} + 7e^{4ik_1h} - e^{5ik_1h}\right\} +$$
$$\left\{e^{-2ik_2h} - 16e^{-ik_2h} + \theta_2 - 16e^{ik_2h} + e^{2ik_2h}\right\} + 12h^2k^2u_{i,j} = 0.$$

where

$$\theta_2 = 2\cos 2k_2h - 32\cos k_2h.$$

On rearranging

$$-\left\{9e^{-ik_1h} - 19e^{ik_1h} + 34e^{2ik_1h} - 21e^{3ik_1h} + 7e^{4ik_1h} - e^{5ik_1h} + e^{-2ik_2h} -\right.$$
$$\left.16e^{-ik_2h} - 16e^{ik_2h} + e^{2ik_2h}\right\} - 19u_{i+1,j} + 34u_{i+2,j} - 21u_{i+3,j} + 7u_{i+4,j} -$$
$$u_{i+5,j} + u_{i,j-2} - 16u_{i,j-1} - 16u_{i,j+1} = -9e^{ik_2y_{N-2}} - e^{ik_1x_N + k_2}.$$

This scheme is used for the node $u_{1,N-2}$ of interior of the grid.

Implementation for the points $(1,3)$ to $(1, N-3)$

We now construct the scheme for the points $(1,3)$ to $(1, N-3)$. Again replacing second order partial derivative with respect to x by fourth order accurate seven point forward difference approximation and second order partial derivative with respect to y by fourth order accurate central finite difference approximation in (2.6).

$$
\begin{aligned}
&\Big\{9u_{i-1,j} - 9u_{i,j} - 19u_{i+1,j} + 34u_{i+2,j} - 21u_{i+3,j} + 7u_{i+4,j} - u_{i+5,j}\Big\} + \\
&\qquad \Big\{u_{i,j-2} - 16u_{i,j-1} + \theta_2 u_{i,j} - 16u_{i,j+1} + u_{i,j+2}\Big\} + 12h^2k^2u_{i,j} = \\
&\qquad \Big\{9e^{-ik_1h} - 9 - 19e^{ik_1h} + 34e^{2ik_1h} - 21e^{3ik_1h} + 7e^{4ik_1h} - e^{5ik_1h}\Big\} + \\
&\qquad\qquad \Big\{e^{-2ik_2h} - 16e^{-ik_2h} + \theta_2 - 16e^{ik_2h} + e^{2ik_2h}\Big\} + 12h^2k^2u_{i,j} = 0.
\end{aligned}
$$

On rearranging

$$
\begin{aligned}
&-\Big\{9e^{-ik_1h} - 19e^{ik_1h} + 34e^{2ik_1h} - 21e^{3ik_1h} + 7e^{4ik_1h} - e^{5ik_1h} + e^{-2ik_2h} - \\
&\quad 16e^{-ik_2h} - 16e^{ik_2h} + e^{2ik_2h}\Big\} - 19u_{i+1,j} + 34u_{i+2,j} - 21u_{i+3,j} + 7u_{i+4,j} - \\
&\qquad u_{i+5,j} + u_{i,j-2} - 16u_{i,j-1} - 16u_{i,j+1} + u_{i,j+2} = -9e^{ik_2y_m}
\end{aligned}
$$

where $m = 3, 4, 5, \dots, N-3$.

This scheme is used for the nodes of left boundary $u_{1,3}$ to $u_{1,N-3}$ of interior of the grid.

Right Boundary

Implementation for the points $(N-1, 2)$ and $(N-1, N-2)$

We now construct the schemes for the end points $(N-1, 2)$ and $(N-1, N-2)$ of the right boundary respectively. We use the fourth order accurate seven point backward difference approximation along x-axis and fourth order accurate central finite difference approximation along y-axis.

$$\Big\{u_{i-5,j} - 7u_{i-4,j} + 21u_{i-3,j} - 34u_{i-2,j} + 19u_{i-1,j} + 9u_{i,j} - 9u_{i+1,j}\Big\} +$$
$$\Big\{u_{i,j-2} - 16u_{i,j-1} + \theta_2 u_{i,j} - 16u_{i,j+1} + u_{i,j+2}\Big\} + 12h^2k^2u_{i,j} =$$
$$\Big\{e^{-5ik_1h} - 7e^{-4ik_1h} + 21e^{-3ik_1h} - 34e^{-2ik_1h} + 19e^{-ik_1h} + 9 - 9e^{ik_1h}\Big\} +$$
$$\Big\{e^{-2ik_2h} - 16e^{-ik_2h} + \theta_2 - 16e^{ik_2h} + e^{2ik_2h}\Big\} + 12h^2k^2u_{i,j} = 0.$$

where

$$\theta_2 = 2\cos 2k_2h - 32\cos k_2h.$$

On rearranging

$$u_{i-5,j} - 7u_{i-4,j} + 21u_{i-3,j} - 34u_{i-2,j} + 19u_{i-1,j} - \Big\{e^{-5ik_1h} - 7e^{-4ik_1h} + 21e^{-3ik_1h} -$$
$$34e^{-2ik_1h} + 19e^{-ik_1h} - 9e^{ik_1h} + e^{-2ik_2h} - 16e^{-ik_2h} - 16e^{ik_2h} + e^{2ik_2h}\Big\} -$$
$$16u_{i,j-1} - 16u_{i,j+1} + u_{i,j-2} = 9e^{i(k_1+k_2y_2)} - e^{ik_1x_{N-1}}.$$

This scheme is used for the node $u_{N-1,2}$ of interior of the grid.

$$\Big\{u_{i-5,j} - 7u_{i-4,j} + 21u_{i-3,j} - 34u_{i-2,j} + 19u_{i-1,j} + 9u_{i,j} - 9u_{i+1,j}\Big\} +$$
$$\Big\{u_{i,j-2} - 16u_{i,j-1} + \theta_2 u_{i,j} - 16u_{i,j+1} + u_{i,j+2}\Big\} + 12h^2k^2u_{i,j} =$$
$$\Big\{e^{-5ik_1h} - 7e^{-4ik_1h} + 21e^{-3ik_1h} - 34e^{-2ik_1h} + 19e^{-ik_1h} + 9 - 9e^{ik_1h}\Big\} +$$
$$\Big\{e^{-2ik_2h} - 16e^{-ik_2h} + \theta_2 - 16e^{ik_2h} + e^{2ik_2h}\Big\} + 12h^2k^2u_{i,j} = 0.$$

where

$$\theta_2 = 2\cos 2k_2h - 32\cos k_2h.$$

On rearranging

$$u_{i-5,j}-7u_{i-4,j}+21u_{i-3,j}-34u_{i-2,j}+19u_{i-1,j}-\Big\{e^{-5ik_1h}-7e^{-4ik_1h}+21e^{-3ik_1h}-34e^{-2ik_1h}+19e^{-ik_1h}-9e^{ik_1h}+e^{-2ik_2h}-16e^{-ik_2h}-16e^{ik_2h}+e^{2ik_2h}\Big\}+u_{i,j-2}-16u_{i,j-1}-16u_{i,j+1}=9e^{i(k_1+k_2y_{N-2})}-e^{i(k_1x_{N-1}+k_2)}.$$

This scheme is used for the node $u_{N-1,N-2}$ of interior of the grid.

Implementation for the points $(N-1,3)$ to $(N-1,N-3)$

Here we use fourth order accurate seven point backward difference approximation along x-axis and fourth order accurate central finite difference approximation along y-axis in (2.6).

$$\Big\{u_{i-5,j}-7u_{i-4,j}+21u_{i-3,j}-34u_{i-2,j}+19u_{i-1,j}+9u_{i,j}-9u_{i+1,j}\Big\}+\Big\{u_{i,j-2}-16u_{i,j-1}+\theta_2u_{i,j}-16u_{i,j+1}+u_{i,j+2}\Big\}+12h^2k^2u_{i,j}=\Big\{e^{-5ik_1h}-7e^{-4ik_1h}+21e^{-3ik_1h}-34e^{-2ik_1h}+19e^{-ik_1h}+9-9e^{ik_1h}\Big\}+\Big\{e^{-2ik_2h}-16e^{-ik_2h}+\theta_2-16e^{ik_2h}+e^{2ik_2h}\Big\}+12h^2k^2u_{i,j}=0.$$

where

$$\theta_2=2\cos 2k_2h-32\cos k_2h.$$

On rearranging

$$u_{i-5,j}-7u_{i-4,j}+21u_{i-3,j}-34u_{i-2,j}+19u_{i-1,j}-\Big\{e^{-5ik_1h}-7e^{-4ik_1h}+21e^{-3ik_1h}-34e^{-2ik_1h}+19e^{-ik_1h}-9e^{ik_1h}+e^{-2ik_2h}-16e^{-ik_2h}-16e^{ik_2h}+e^{2ik_2h}\Big\}+u_{i,j-2}-16u_{i,j-1}-16u_{i,j+1}+u_{i,j+2}=9e^{i(k_1+k_2y_m)}.$$

where $m = 3, 4, 5, \ldots, N-3$.
This scheme is used for the nodes $u_{N-1,3}$ to $u_{N-1,N-3}$ of right boundary of interior of the grid.

Bottom Boundary

Implementation for the points $(2,1)$ and $(N-2,1)$

We use fourth order accurate central finite difference approximation along x-axis and fourth order accurate seven point forward difference approximation along y-axis in (2.6) for the bottom boundary.

$$\Big\{u_{i-2,j} - 16u_{i-1,j} + \theta_1 u_{i,j} - 16u_{i+1,j} + u_{i+2,j}\Big\} + \Big\{9u_{i,j-1} - 9u_{i,j} - 19u_{i,j+1} + 34u_{i,j+2} - 21u_{i,j+3} + 7u_{i,j+4} - u_{i,j+5}\Big\} + 12h^2k^2u_{i,j} = \Big\{e^{-2ik_1h} - 16e^{-ik_1h} + \theta_1 - 16e^{ik_1h} + e^{2ik_1h}\Big\} + \Big\{9e^{-ik_1h} - 9 - 19e^{ik_1h} + 34e^{2ik_1h} - 21e^{3ik_1h} + 7e^{4ik_1h} - e^{5ik_1h}\Big\} + 12h^2k^2u_{i,j} = 0.$$

where

$$\theta_1 = 2\cos 2k_1h - 32\cos k_1h.$$

On rearranging

$$\begin{aligned} &-16u_{i-1,j} - \Big\{e^{-2ik_1h} - 16e^{-ik_1h} - 16e^{ik_1h} + e^{2ik_1h} + 9e^{-ik_2h} - 19e^{ik_2h} + \\ &\quad 34e^{2ik_2h} - 21e^{3ik_2h} + 7e^{4ik_2h} - e^{5ik_2h}\Big\} - 16u_{i+1,j} + u_{i+2,j} - 19u_{i,j+1} + \\ &\quad\quad 34u_{i,j+2} - 21u_{i,j+3} + 7u_{i,j+4} - u_{i,j+5} = -e^{ik_2y_1} - 9e^{ik_1x_2)}. \end{aligned}$$

This scheme is used for the node $u_{2,1}$ of interior of the grid.

Again we use fourth order accurate central finite difference approximation along x-axis and fourth order accurate seven point forward difference approximation along y-axis in (2.6)

$$\Big\{u_{i-2,j} - 16u_{i-1,j} + \theta_1 u_{i,j} - 16u_{i+1,j} + u_{i+2,j}\Big\} + \Big\{9u_{i,j-1} - 9u_{i,j} - 19u_{i,j+1} + 34u_{i,j+2} - 21u_{i,j+3} + 7u_{i,j+4} - u_{i,j+5}\Big\} + 12h^2k^2u_{i,j} = \Big\{e^{-2ik_1h} - 16e^{-ik_1h} + \theta_1 - 16e^{ik_1h} + e^{2ik_1h}\Big\} + \Big\{9e^{-ik_2h} - 9 - 19e^{ik_2h} + 34e^{2ik_2h} - 21e^{3ik_2h} + 7e^{4ik_2h} - e^{5ik_2h}\Big\} + 12h^2k^2u_{i,j} = 0.$$

where

$$\theta_1 = 2\cos 2k_1h - 32\cos k_1h.$$

On rearranging

$$u_{i-2,j} - 16u_{i-1,j} - \Big\{e^{-2ik_1h} - 16e^{-ik_1h} - 16e^{ik_1h} + e^{2ik_1h} + 9e^{-ik_2h} - 19e^{ik_2h} + 34e^{2ik_2h} - 21e^{3ik_2h} + 7e^{4ik_2h} - e^{5ik_2h}\Big\} - 16u_{i+1,j} - 19u_{i,j+1} + 34u_{i,j+2} - 21u_{i,j+3} + 7u_{i,j+4} - u_{i,j+5} = -e^{i(k_1+k_2y_1)} - 9e^{ik_1x_{N-2}}.$$

This scheme is used for the node $u_{N-2,1}$ of interior of the grid.

Implementation for the nodes $u_{3,1}$ to $u_{N-3,1}$

We now construct the scheme for the points $(3,1)$ to $(N-3,1)$ by using the fourth order accurate central finite difference approximation along x-axis and fourth order accurate seven point forward difference approximation along y-axis in (2.6)

$$\Big\{u_{i-2,j} - 16u_{i-1,j} + \theta_1 u_{i,j} - 16u_{i+1,j} + u_{i+2,j}\Big\} + \Big\{9u_{i,j-1} - 9u_{i,j} - 19u_{i,j+1} + 34u_{i,j+2} - 21u_{i,j+3} + 7u_{i,j+4} - u_{i,j+5}\Big\} + 12h^2k^2u_{i,j} = \Big\{e^{-2ik_1h} - 16e^{-ik_1h} + \theta_1 - 16e^{ik_1h} + e^{2ik_1h}\Big\} + \Big\{9e^{-ik_2h} - 9 - 19e^{ik_2h} + 34e^{2ik_2h} - 21e^{3ik_2h} + 7e^{4ik_2h} - e^{5ik_2h}\Big\} + 12h^2k^2u_{i,j} = 0.$$

where

$$\theta_1 = 2\cos 2k_1h - 32\cos k_1h.$$

On rearranging

$$u_{i-2,j} - 16u_{i-1,j} - \Big\{e^{-2ik_1h} - 16e^{-ik_1h} - 16e^{ik_1h} + e^{2ik_1h} + 9e^{-ik_2h} - 19e^{ik_2h} + 34e^{2ik_2h} - 21e^{3ik_2h} + 7e^{4ik_2h} - e^{5ik_2h}\Big\} - 16u_{i+1,j} + u_{i+2,j} - 19u_{i,j+1} + 34u_{i,j+2} - 21u_{i,j+3} + 7u_{i,j+4} - u_{i,j+5} = -9e^{ik_1x_m}.$$

where $m = 3, 4, 5, \ldots, N-3$.

This scheme is used for the nodes $u_{3,1}$ to $u_{N-3,1}$ of interior of the grid.

Top Boundary

Using the fourth order accurate central finite difference approximation along x-axis and fourth order accurate seven point backward difference approximation along y-axis in (2.6).

Implementation for the nodes $(2, N-1)$ and $(N-2, N-1)$

Firstly, we construct the schemes for the end points $(2, N-1)$ and $(N-2, N-1)$ of the top boundary respectively.

$$\Big\{u_{i-2,j} - 16u_{i-1,j} + \theta_1 u_{i,j} - 16u_{i+1,j} + u_{i+2,j}\Big\} + \Big\{u_{i,j-5} - 7u_{i,j-4} + 21u_{i,j-3} - 34u_{i,j-2} + 19u_{i,j-1} + 9u_{i,j} - 9u_{i,j+1}\Big\} + 12h^2k^2u_{i,j} = \Big\{e^{-2ik_1h} - 16e^{-ik_1h} + \theta_1 - 16e^{ik_1h} + e^{2ik_1h}\Big\} + \Big\{e^{-5ik_2h} - 7e^{-4ik_2h} + 21e^{-3ik_2h} - 34e^{-2ik_2h} + 19e^{-ik_2h} + 9 - 9e^{ik_2h}\Big\} + 12h^2k^2u_{i,j} = 0.$$

where

$$\theta_1 = 2\cos 2k_1h - 32\cos k_1h.$$

On rearranging

$$-16u_{i-1,j}-\Big\{e^{-2ik_1h}-16e^{-ik_1h}-16e^{ik_1h}+e^{2ik_1h}+e^{-5ik_2h}-7e^{-4ik_2h}+21e^{-3ik_2h}-34e^{-2ik_2h}+19e^{-ik_2h}-9e^{ik_2h}\Big\}-16u_{i+1,j}+u_{i+2,j}-7u_{i,j-4}+21u_{i,j-3}-34u_{i,j-2}+19u_{i,j-1}-9u_{i,j+1}=-e^{ik_2y_{N-1}}-e^{i(k_1x_2+k_2)}.$$

This scheme is used for the node $u_{2,N-1}$ of interior of the grid.

$$\Big\{u_{i-2,j}-16u_{i-1,j}+\theta_1u_{i,j}-16u_{i+1,j}+u_{i+2,j}\Big\}+\Big\{u_{i,j-5}-7u_{i,j-4}+21u_{i,j-3}-34u_{i,j-2}+19u_{i,j-1}+9u_{i,j}-9u_{i,j+1}\Big\}+12h^2k^2u_{i,j}=\Big\{e^{-2ik_1h}-16e^{-ik_1h}+\theta_1-16e^{ik_1h}+e^{2ik_1h}\Big\}+\Big\{e^{-5ik_2h}-7e^{-4ik_2h}+21e^{-3ik_2h}-34e^{-2ik_2h}+19e^{-ik_2h}+9-9e^{ik_2h}\Big\}+12h^2k^2u_{i,j}=0.$$

where

$$\theta_1=2\cos 2k_1h-32\cos k_1h.$$

On rearranging

$$u_{i-2,j}-16u_{i-1,j}-\Big\{e^{-2ik_1h}-16e^{-ik_1h}-16e^{ik_1h}+e^{2ik_1h}+e^{-5ik_2h}-7e^{-4ik_2h}+21e^{-3ik_2h}-34e^{-2ik_2h}+19e^{-ik_2h}-9e^{ik_2h}\Big\}-16u_{i+1,j}-7u_{i,j-4}+21u_{i,j-3}-34u_{i,j-2}+19u_{i,j-1}-9u_{i,j+1}=-e^{i(k_1+k_2y_{N-1}}-e^{i(k_1x_{N-2}+k_2)}.$$

This scheme is used for the node $u_{N-2,N-1}$ of interior of the grid.

Implementation for the points $(3, N-1)$ to $(N-3, N-1)$

We now construct the scheme for the points $(3, N-1)$ to $(N-3, N-1)$ by using the fourth order accurate central finite difference approximation along x-axis and fourth order accurate seven point backward difference approximation along y-axis in (2.6)

$$\left\{u_{i-2,j} - 16u_{i-1,j} + \theta_1 u_{i,j} - 16u_{i+1,j} + u_{i+2,j}\right\} + \left\{u_{i,j-5} - 7u_{i,j-4} + 21u_{i,j-3} - 34u_{i,j-2} + 19u_{i,j-1} + 9u_{i,j} - 9u_{i,j+1}\right\} + 12h^2k^2u_{i,j} = \left\{e^{-2ik_1h} - 16e^{-ik_1h} + \theta_1 - 16e^{ik_1h} + e^{2ik_1h}\right\} + \left\{e^{-5ik_2h} - 7e^{-4ik_2h} + 21e^{-3ik_2h} - 34e^{-2ik_2h} + 19e^{-ik_2h} + 9 - 9e^{ik_2h}\right\} + 12h^2k^2u_{i,j} = 0.$$

where

$$\theta_1 = 2\cos 2k_1h - 32\cos k_1h.$$

On rearranging

$$u_{i-2,j} - 16u_{i-1,j} - \left\{e^{-2ik_1h} - 16e^{-ik_1h} - 16e^{ik_1h} + e^{2ik_1h} + e^{-5ik_2h} - 7e^{-4ik_2h} + 21e^{-3ik_2h} - 34e^{-2ik_2h} + 19e^{-ik_2h} - 9e^{ik_2h}\right\} - 16u_{i+1,j} + u_{i+2,j} - 7u_{i,j-4} + 21u_{i,j-3} - 34u_{i,j-2} + 19u_{i,j-1} - 9u_{i,j+1} = -e^{i(k_1x_m + k_2)}.$$

Where $m = 3, 4, 5, \ldots, N-3$.

This scheme is used for the nodes of the top boundary from $u_{3,N-1}$ to $u_{N-3,N-1}$ of interior of the grid.

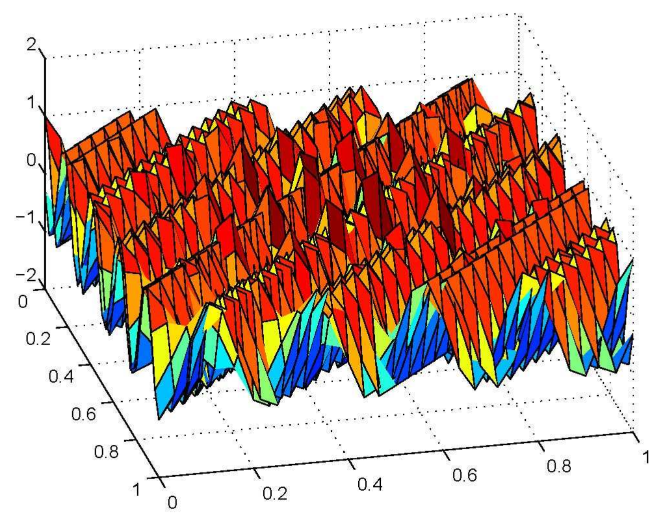


Figure 2.5: Comparison of two dimension exact wave, wave obtained by fourth order accurate Standard C.F.D scheme and wave obtained by fourth order Exact F.D scheme with wave number $k = 40$ at number of elements $N = 30$.

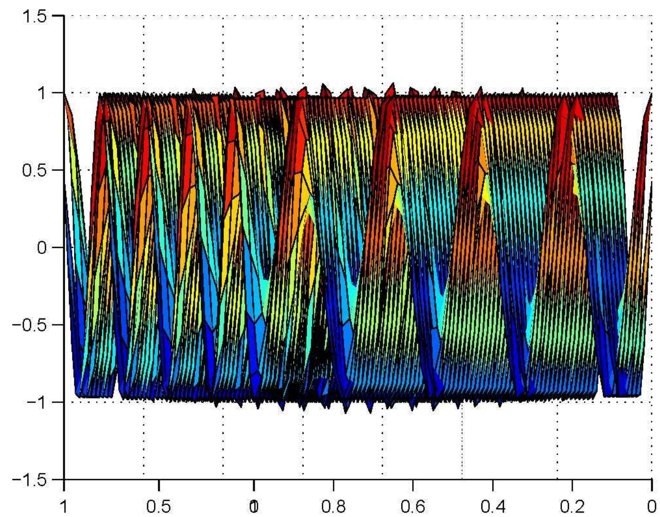


Figure 2.6: Comparison of two dimension exact wave, wave obtained by fourth order accurate Standard C.F.D scheme and wave obtained by fourth order Exact F.D scheme with wave number $k = 40$ at number of elements $N = 50$. Refining the mesh size in fourth order accurate S.F.D scheme reduces dispersion error but fourth order accurate exact F.D scheme gives an optimal results at high wave number with less computational cost.

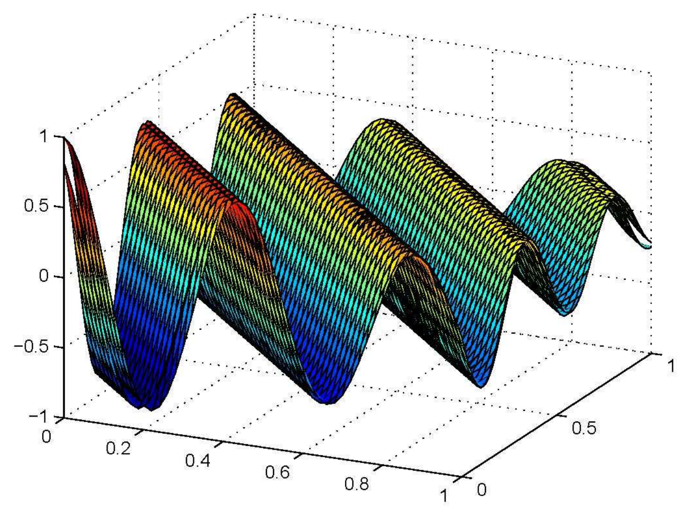


Figure 2.7: Comparison of two dimension exact wave, wave obtained by fourth order accurate Standard C.F.D scheme and wave obtained by fourth order Exact F.D scheme with pure complex wave number $k = 20 + i$ at number of elements $N = 50$. Complex

Chapter 3
Conclusions

Chapter 3

Conclusions

In this work, the novel central finite difference schemes are constructed by following the ideas of Nehrbass [1] and Yau shu [2] first by modifying the standard second order accurate central finite difference scheme. Later on relying only on the idea of Yau Shu in which he not only made differential equation exact but also boundary conditions exact. Novel central finite difference schemes were constructed by modifying standard central finite difference schemes accurate to arbitrary even order only for one dimensional Helmholtz equation. The generalized expression for the central node of scheme of novel scheme is given as well. Furthermore, the one dimensional exact operators were used for the approximation of Helmholtz equation in two dimensions. Detailed dispersive and dissipative properties of novel schemes are studied for Helmholtz equation and quantitative expressions are given. Finally accuracy of the novel schemes is shown to be superior compared with standard central finite difference scheme for the propagation of waves with any wave number with the help of numerical examples.

Chapter 4
References

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References

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