Fractional Integral Inequalities For (α-m) Convex Functions

By

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CIIT/SP14-BSMATH-007/LHR

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By

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Certificate

It is certified that Muhammad Irshad (**CIIT/SP14-BSMATH-007/LHR)** has carried out all the work related to this report under my supervision at the Department of Mathematics, COMSATS University Islamabad, Lahore Campus, and the work fulfills the requirement for award of BS degree.

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DEDICATION

To

My Family

ACKNOWLEDGEMENTS

Praise to be ALLAH, the Cherisher and Lordof the World, Most gracious and Most Merciful. Person is not perfect in all the contexts of this life. He has a limited mind and minor thinking approaches. It is the guidance from the almighty ALLAH that shows the man light in the darkness and the person finds his way in the light. Without this helping light, person is nothing but a helpless creature. Same is the case with us, as we experience all these phenomena during the completion of this project and have been successful in fulfilling this duty assigned to us only because of the help of ALLAH. The teaching of the Holy Prophet Muhammad (PBUH) were also the continuous source of guidance for us especially His order of getting knowledge and fulfilling one’s duty honestly was the key for motivating us. Credit of my humble efforts goes to my respectable teacher Dr. Saad Ihsaan Butt, under whose supervision I was able to complete this task.

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ABSTRACT

The aim of this project is to establish the Hermite Hadamard's inequality involving fractional integral for m-convex function. The obtained results have some relationship with [S.S Dragomire, R.P. Agrawal, two inequality for differentiable mapping and application to special means of real numbers and to trapezoidal formula, Appl.Math.Lett.,11(5)(1998), (91-95)].

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Introduction

1 Introduction

G.H Hardy Bohr reported ”all analysts spend half their time hunting through the literature for inequalities which they want to use but cannot prove”. The dynamic field of mathematical inequalities open many doors for analysts. Inequalities play an important role in many areas of mathematics. e.g number theory, geometry, functional analysis, probability, calculus, etc. In number theory for Diophantine approximations, in engineering and geometry for isoperimetric problems of convex bodies and in probability theory most of the laws put in terms of inequalities. Linear inequalities play a fundamental role in the theory of linear programming. Many inequalities are used in the theory of functions for the derivatives of the polynomials. Linear functional or linear operator in some spaces define or estimate by using classical inequalities. In functional analysis, the definition of norm in a function space requires it to meet the triangle inequality. Some problems in the theory of linear inequalities depends upon the theory of approximations, generated by P.L.Chebyshev. The Cauchy-Schwarz inequality is important because it connects the notion of an inner product with the notion of lenght. In multivariable calculus, inequality are used for the gradient vector of a function. Many inequality are very helpful to solve PDEs and integral equations. Jensen’s inequality shows the significant relation between mean and the mean of function values.

1.1 Integral inequalities

The integral inequalities of various types studied in most subjects involving mathematical analysis. They are particularly useful for approximation theory and numerical analysis in which estimates of approximation errors are involved. Integral inequalities provide explicit bounds on unknown functions. Integral inequalities are a necessary tool in the study of various classes of differential and integral equations. They are now used not only in mathematics but also in the areas of physics, technology and biological sciences. Many nonlinear dynamical systems are too complicated to be effectively analyzed by integral inequalities. In the study of qualitative behavior of solutions of certain nonlinear differential and integral equations some specific types of inequalities are needed in various situations. One of most important integral inequality is classical Hardy type inequality which describe the relation between L^p norms of the average value of function to L^p norm of function.
The classical Hardy inequality for $f \geq 0$ and $p > 1$ is given as:

$$\left(\int_0^\infty \left(\frac{1}{x}\int_0^x f(t)dt\right)^p dx\right)^{1/p} \leq \frac{p}{p-1}\left(\int_0^\infty f^p(x)dx\right)^{1/p}.$$

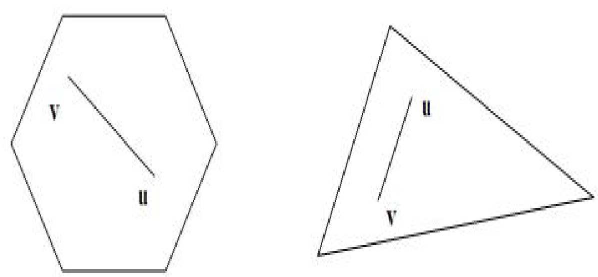
When $f \geq 0$ is a decreasing function, then the reversed Hardy inequality, given by P. F. Renaud (1986) in [10],

$$\int_0^\infty \left(\frac{1}{x}\int_0^x f(t)dt\right)^p dx \geq \frac{p}{p-1}\int_0^\infty f^p(x)dx$$

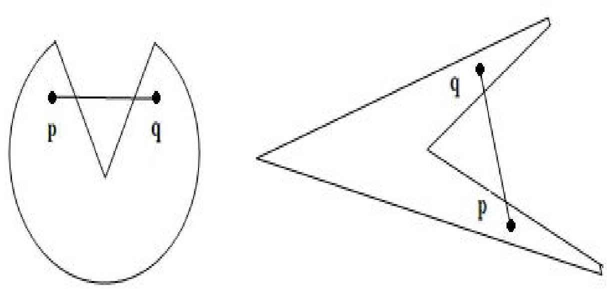
Convex functions: The modern view point on convex functions entails a powerful and elegant interaction between analysis and geometry, which makes the reader to share a sense of excitement. In a memorable paper dedicated to the Brunn-Minkowski inequality, R. J. Gardner [4],[5], described this reality in beautiful phrases: [convexity] appears like an octopus, tentacles reaching far and wide, its shape and color changing as it roams one area to the next. [And] it is quite clear that research opportunities abound. In recent years, the concept of convex functions has been generalized extensively. For this, their are several reasons. In fact, many areas in modern analysis directly or indirectly involve the applications of convex functions; further, convex functions are closely related to the theory of inequalities and many important inequalities are consequences of the applications of convex functions. Some definitions, characterizations, theorems and remarks about convex functions given below which are related to our research work.
The following definitions and results are given from [8].

1.1.1 Convex set

Definition 1. [7] A set $S \subset \Re^n$ is convex if the line segment joining any two point of the set also belongs to the set.
OR
A set $S \subset \Re^n$ is convex if $[u,v] \subset S$ for every $u,v \in S$ then $\lambda u + (1-\lambda)v$ must also belong to S for each $\lambda \in [0,1]$.



(both are convex)



(both are not convex)

1.1.2 Convex function

[7] A function $f : I \to \Re$ is called convex (i.e $I \subset \Re$) if for all $x_1, x_2 \in I$ and $\lambda \in [0,1]$,the inequality

$$f\Big(\lambda x_1 + (1-\lambda)x_2\Big) \leq \lambda f(x_1) + (1-\lambda) f(x_2)$$

holds.

1.1.3 Strictly convex function

[7] A function $f : I \to \Re$ is called convex (i.e $I \subset \Re$) if for all $x_1, x_2 \in I$ and $\lambda \in [0,1]$,the inequality

$$f\Big(\lambda x_1 + (1-\lambda)x_2\Big) < \lambda f(x_1) + (1-\lambda) f(x_2)$$

holds.

1.1.4 Concave function

[7] A function $f : I \to \Re$ is called convex (i.e $I \subset \Re$) if for all $x_1, x_2 \in I$ and $\lambda \in [0,1]$,the inequality

$$f\Big(\lambda x_1 + (1-\lambda)x_2\Big) \geq \lambda f(x_1) + (1-\lambda) f(x_2)$$

holds.
Note: A function $f : I \to \Re$ is called concave if "$-f$" is convex.

1.1.5 Strictly concave function

[7] A function $f : I \to \Re$ is called convex (i.e $I \subset \Re$) if for all $x_1, x_2 \in I$ and $\lambda \in [0,1]$,the inequality

$$f\Big(\lambda x_1 + (1-\lambda)x_2\Big) > \lambda f(x_1) + (1-\lambda) f(x_2)$$

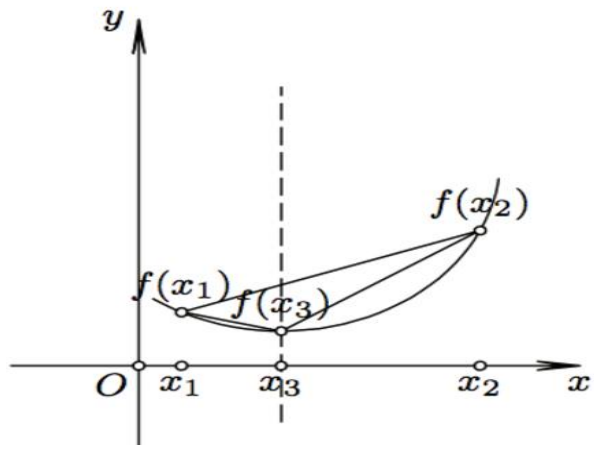
holds.

1.1.6 Geometric interpretation of convex function

[7] The geometrical interpretation of convexity or convex function

$$f\big(\lambda x_1 + (1-\lambda)x_2\big) \leq \lambda f(x_1) + \big(1-\lambda\big) f(x_2) \tag{1}$$

is that the graph of f lies below its chords. Take any $x_3 \in (x_1, x_2)$. There is $\lambda \in (0,1)$ such that $x_3 = \lambda x_1 + (1-\lambda)x_2$. Let dotted line passing through x_3 and parallel to the y axis. Let the chord(curved line) connecting the points $(x_1, f(x_1))$ and $(x_2, f(x_2))$. Assume that the dotted line and the chord(curved line) intersect at the $(x_3, f(x_3))$ point. The y coordinate (also called the height) of the $(x_3, f(x_3))$ point is: $\lambda f(x_1) + (1-\lambda) f(x_2)$.



The inequality 1 means exactly that the dotted line intersects the graph of a function below the chord(curved line). If f is strictly convex then the equality can hold in 1 if and only if $x_1 = x_2$.

Examples:

- $e^x, x^p(p \geq 1, x > 0), \frac{1}{x}(x \neq 0)$ are convex functions.
- $\log x(x > 0), sinx(0 \leq x \leq \pi), cosx(\frac{-\pi}{2} \leq x \leq \frac{\pi}{2})$ are concave functions.

Remarks:

1. For $x, y \in I$, $p, q \geq 0$ and $p + q > 0$, then
$$f\big(\lambda x + \big(1-\lambda\big)y\big) \leq \lambda f(x) + \big(1-\lambda\big)f(y) \tag{2}$$
is equivalent to
$$\frac{f(px+qy)}{p+q} \geq \frac{pf(x)+qf(y)}{p+q}$$
2. f is both convex and concave if and only if $f(x) = \lambda x + c$, for some $\lambda, c \in \Re$.
3. The geometrical interpretation of convexity or convex function is that the graph of f lies below its chords.
4. If $x_1, x_2.x_3$ are three points in I such that $x_1 < x_2 < x_3$, then 2 is equivalent to
$$\begin{vmatrix} x_1 & f(x_1) & 1 \\ x_2 & f(x_2) & 1 \\ x_3 & f(x_3) & 1 \end{vmatrix} = (x_3 - x_2)f(x_1) + (x_1 - x_3)f(x_2) + (x_2 - x_1)f(x_3) \geq 0$$
which is equivalent to
$$f(x_2) \leq \frac{x_2 - x_3}{x_1 - x_3}f(x_1) + \frac{x_1 - x_2}{x_1 - x_3}f(x_3)$$

or more symmetrically and without the condition of monotonicity on x_1, x_2 and x_3.

$$\frac{f(x_1)}{(x_1-x_2)(x_1-x_3)}+\frac{f(x_2)}{(x_2-x_3)(x_2-x_1)}+\frac{f(x_3)}{(x_3-x_1)(x_3-x_2)} \geq 0$$

5. If f is a convex function on I and if $x_1 \leq y_1, x_2 \leq y_2, x_1 \neq x_2, y_1 \neq y_2$, then the following inequality is

$$\frac{f(x_2)-f(x_1)}{x_2-x_1} \leq \frac{f(y_2)-f(y_1)}{y_2-y_1}$$

6. The are of the triangle whose vertices are $(x_1, f(x_1)), (x_2, f(x_2))$ and $(x_3, f(x_3))$ is given by

$$P = \frac{1}{2}\begin{vmatrix} x_1 & f(x_1) & 1 \\ x_2 & f(x_2) & 1 \\ x_3 & f(x_3) & 1 \end{vmatrix}$$

The area can be positive or negative. This will depends on weather the triangle is positively oriented(anticlockwise) or negatively oriented(clockwise). For a convex function, we have $P > 0$ and for concave function $P < 0$.

1.1.7 Properties of convex function

[7]

1. If f is convex and c is a constant, then the function cf is convex.

2. If a function $f(u)$ is convex and increasing and a function $u(x)$ is convex then the composition $f(u(x))$ is convex.

3. A function $f : [a,b] \to \Re$ is convex if and only if the set $\{[(x,y)|a \leq x \leq b, f(x) \leq y]\}$ is convex.

4. Sum of convex function is convex.e.g f_1, f_2 convex then $f_1 + f_2$ convex.

5. A function is convex if and only if it is convex on all lines. e.g f convex $\Leftrightarrow f(x_0+th)$ convex in t, $\forall x_0, h$

6. Positive multiple of convex function is convex.e.g f convex $\Rightarrow \alpha f$ convex, $\alpha > 0$.

7. If the is convex then its epigraph is convex set and vice versa.

8. Function of the form $Ax + b$ is convex(Affine).

9. Pointwise maximum.e.g f_1, f_2 convex $\Rightarrow max(f(x_1), f_2(x))$ convex.

10. Pointwise superimum.e.g f_α convex $\Rightarrow \sup_{\alpha \in A} f_\alpha$ convex.

1.1.8 J-convex function

[7] A function $f:[a,b]\rightarrow\Re$ is said to J-convex on $[a,b]$ if $\forall x,y\in[a,b]$ the inequality

$$f\left(\frac{x+y}{2}\right)\leq\frac{1}{2}(f(x)+f(y))$$

holds.

1.1.9 Strictly J-convex function

[7] A J-convex function f is said to be strictly J-convex function, if for all pairs of points (x,y), $x\neq y$ the inequality

$$f\left(\frac{x+y}{2}\right)<\frac{f(x)+f(y)}{2}$$

holds.

1.2 Riemann- Liouville Fractional Integrals

Definition 2. [11] Let $f\in L_1[a,b]$. The Riemann Liouville integrals $J^{\alpha}_{a+}f$ and $J^{\alpha}_{b-}f$ of order $\alpha>0$ with $a\geq 0$ are defined by

$$J^{\alpha}_{a+}f(x)=\tfrac{1}{\Gamma(\alpha)}\int_a^x(x-t)^{\alpha-1}f(t)dt,\quad x>a$$

and

$$J^{\alpha}_{b-}f(x)=\tfrac{1}{\Gamma(\alpha)}\int_x^b(t-x)^{\alpha-1}f(t)dt,\quad x<b$$

respectively. Here, $\Gamma(\alpha)$ is the Gamma function and $J^{0}_{a+}f(x)=J^{0}_{b-}f(x)=f(x)$.
For some recent results connected with fractional integral inequalities.

The aim of this project is to establish Hermite Hadamards inequalities for Riemann Liouville fractional integral and some other integral inequalities using the identity obtained for fractional integrals.

The concept of convexity and its various generalisations and extensions plays an important role in modern analysis. Numerous Hermite Hadamard type integral inequalities are presented for both convex functions as well as for log-convex, quasi-convex, s- convex, and other related classes of functions such as the Godunova-Levin class of functions. Since the logarithmic, and p-logarithmic means can be obtained as integral means of some particular convex functions, we provide here various applications of the H.-H. integral inequalities which complement, improve, and provide reverses of some classical inequalities for means. Many remarks and comments are given which reveal a rich interconnection between the above inequalities and create a natural platform for further research and applications.

Chapter 2
Hermite Hadamard Inequalities for fractional integral and related fractional Inequalities

The inequality discovered by Hermite and Hadamard for convex function are very important in the literature (see, e.g[1,p.137],[2]. These inequality state that if $f : I \to \mathbb{R}$ is a convex function on the interval I of real number and $a, b \in I$ with $a < b$, then

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2} \tag{3}$$

Both inequalities hold in the reversed direction if f is concave. We note that Hadamards inequality may be regarded as a refinement of the concept of convexity and it follows easily from Jensens inequality. Hadamards inequality for convex functions has received renewed attention in recent years and a remarkable variety of refinements and generalizations have been found; see, for example, [1] and the references cited therein.
The classical Hermite Hadamard inequality provides estimates of the mean value of a continuous convex function $f : [a,b] \to \mathbb{R}$.

Lemma 1.1. *[9] Let $f : I^\circ \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on $I^\circ, a, b \in I^\circ$ with $a < b$. If $f' \in L[a,b]$, then the following equality holds:*

$$\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx = \frac{b-a}{2}\int_0^1 (1-2t)f'(ta+(1-t)b)dt \tag{4}$$

Theorem 1.2. *[6] Let $f : I^\circ \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on $I^\circ, a, b \in I^\circ$ with $a < b$. If $|f'|$ is convex on $[a,b]$, then the following inequality holds:*

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx\right| \leq \frac{(b-a)}{8}(|f'(a)|+|f'(b)|). \tag{5}$$

In the following, we will give some necessary definitions and mathematical preliminaries of fractional calculus theory which are used further in this paper.

2 The Hermite-Hadamard Integral Inequality

We start this giving the simple proof of Hermite-Hadamard Integral Inequality.

Theorem 2.1. *[11] Let $f : [a,b] \to \mathbb{R}$ be a convex mapping on $[a,b]$. Then, equation* (3) *hold, which is known as the Hermite-Hadamard integral inequality.*

proof:

Since f is convex on I then for $t \in [0,1]$

$f(ta+(1-t)b) \leq tf(a)+(1-t)f(b)$

integrating with respect to t on $t \in [0,1]$

$\int_0^1 f(ta+(1-t)b)dt \leq \frac{f(a)+f(b)}{2}$

on the other hand since f is convex on I then for $t \in [0,1]$

$$f\left(\frac{a+b}{2}\right) = f\left(\frac{ta+(1-t)b)}{2} + \frac{(1-t)a+tb}{2}\right)$$

$$\leq \frac{1}{2}\left[f(ta+(1-t)b)+((1-t)a+tb)\right]$$

integrating with respect to t on $[0,1]$

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{2}\int_0^1 \left[f(ta+(1-t)b)+((1-t)a+tb)\right]dt$$

$$= \frac{1}{2}\left[\int_0^1 f(ta+(1-t)b)dt + \int_0^1 f((1-t)a+tb)dt\right]$$

by putting $s = 1-t$ in second integral on the right hand side

$$= \frac{1}{2}\left[\int_0^1 f(ta+(1-t)b)dt + \int_0^1 f(sa+(1-s)b)ds\right]$$

$$= \int_0^1 f(ta+(1-t)b)dt$$

$$f\left(\frac{a+b}{2}\right) \leq \int_0^1 f(ta+(1-t)b)dt \leq \frac{f(a)+f(b)}{2}$$

by putting $x = ta+(1-t)b, dx = (a-b)dt$
We get equation (3). $\square$

2.1 Hermite Hadamard's inequality for fractional integral

Hermite Hadamard's inequalities can be represented in fractional integral forms as follows.

Theorem 2.2. *[11] Let $f : [a,b] \to \mathbb{R}$ be a positive function with $0 \leq a < b$ and $f \in L_1[a,b]$. If f is a convex function on $[a,b]$, then the following inequalities for fractional integrals* *hold:*

$$f\left(\frac{a+b}{2}\right) \leq \frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}[J_{a+}^{\alpha} f(b) + J_{b-}^{\alpha} f(a)] \leq \frac{f(a)+f(b)}{2} \tag{6}$$

with $\alpha > 0$.

Proof: Since f is a convex function on $[a,b]$, we have for $x,y \in [a,b]$ with $\lambda = \frac{1}{2}$

$$f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda) f(y)$$

$$f\left(\frac{x+y}{2}\right) \leq \frac{f(x)+f(y)}{2} \tag{7}$$

i.e, with $x = ta + (1-t)b$, $y = (1-t)a + tb$

$$2f\left(\frac{a+b}{2}\right) \leq f(ta+(1-t)b) + f((1-t)a+tb). \tag{8}$$

Multiplying both sides of (8) by $t^{\alpha-1}$, then integrating the resulting inequality with respect to t over $[0,1]$, we obtain.

$$\frac{2}{\alpha} f\left(\frac{a+b}{2}\right) \leq \int_0^1 t^{\alpha-1} f(ta+(1-t)b)dt + \int_0^1 t^{\alpha-1} f((1-t)a+tb)dt$$

$$\leq \int_b^a \left(\frac{b-u}{b-a}\right)^{\alpha-1} f(u)\frac{du}{a-b} + \int_a^b \left(\frac{v-a}{b-a}\right)^{\alpha-1} f(v)\frac{dv}{b-a}$$

$$\leq \int_a^b \frac{(b-u)^{\alpha-1}}{(b-a)^{\alpha-1}} f(u)\frac{du}{b-a} + \int_a^b \frac{(v-a)^{\alpha-1}}{(b-a)^{\alpha-1}} f(v)\frac{dv}{b-a}$$

$$\leq \int_a^b \frac{(b-u)^{\alpha-1}}{(b-a)^{\alpha}} f(u)du + \int_a^b \frac{(v-a)^{\alpha-1}}{(b-a)^{\alpha}} f(v)dv$$

$$\leq \frac{\Gamma(a)}{\Gamma(a)} \frac{1}{(b-a)^{\alpha}} \left[\int_a^b (b-u)^{\alpha-1} f(u)du + \int_a^b (v-a)^{\alpha-1} f(v)dv \right]$$

$$\leq \frac{\Gamma(\alpha)}{(b-a)^{\alpha}}[J_{a+}^{\alpha} f(b) + J_{b-}^{\alpha} f(a)]$$

$$f\left(\frac{a+b}{2}\right) \leq \frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}[J_{a+}^{\alpha} f(b) + J_{b-}^{\alpha} f(a)]$$

and the first inequality is proved.

For the proof of the second inequality in (6) we first note that if f is a convex function, then, for $\lambda \in [0,1]$, it yields

$$f(ta+(1-t)b) \leq tf(a) + (1-t)f(b)$$

and

$$f((1-t)a+tb) \leq (1-t)f(a)+tf(b).$$

By adding these inequalities we have

$$f(ta+(1-t)b)+f((1-t)a+tb) \leq tf(a)+(1-t)f(b)+(1-t)f(a)+tf(b). \quad (9)$$

Then multiplying both sides of (9) by $t^{\alpha-1}$and integrating the resulting inequality with respect to t over [0, 1], we obtain
$\int_0^1 t^{\alpha-1} f(ta+(1-t)b)dt + \int_0^1 t^{\alpha-1} f((1-t)a+tb)dt \leq [f(a)+f(b)] \int_0^1 t^{\alpha-1}dt$

Hence,we get equation (6).□

Remark 2.3. *If in Theorem 2.2 we let $\alpha = 1$, then inequality (7) become inequality (5) of Theorem 1.2.*

2.2 Hermite Hadamard type inequalities for fractional integrals

We need the following lemma.

Lemma 2.4. *[11] Let $f : [a,b] \to \mathbb{R}$ be a differentiable mapping on (a,b) with $a < b$. If $|f'| \in [a,b]$, then the following equality for fractional integrals holds:*

$$\frac{f(a)+f(b)}{2} - \frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}[J^{\alpha}_{a+}f(b)+J^{\alpha}_{b-}f(a)] = \frac{b-a}{2}\int_0^1 [(1-t)^{\alpha}-t^{\alpha}]f'(ta+(1-t)b)dt. \quad (10)$$

***Proof**: It suffices to note that*
$I = \int_0^1 [(1-t)^{\alpha} - t^{\alpha}] f'(ta+(1-t)b)dt$

$$= \left[\int_0^1 [(1-t)^{\alpha} f'(ta+(1-t)b)dt\right] + \left[-\int_0^1 t^{\alpha} f'(ta+(1-t)b)dt\right]$$

$$= I_1 + I_2. \quad (11)$$

Integrating by parts

$$I_1 = \int_0^1 (1-t)^{\alpha} f'(ta+(1-t)b)dt$$

$$= (1-t)^{\alpha} \tfrac{f(ta+(1-t)b)}{a-b}\Big|_0^1 + \int_0^1 \alpha(1-t)^{\alpha} \tfrac{f(ta+(1-t)b)}{a-b}dt$$

$$= \tfrac{f(b)}{b-a} - \tfrac{\alpha}{b-a}\int_b^a \left(\tfrac{a-x}{a-b}\right)^{\alpha-1} \tfrac{f(x)}{a-b}dx$$

$$= \frac{f(b)}{b-a} - \frac{\Gamma(\alpha+1}{(b-a)^{\alpha+1}} J^{\alpha}_{b-} f(a) \tag{12}$$

and similarly we get,

$I_2 = -\int_0^1 t^{\alpha} f'(ta+(1-t)b)dt$

$= -\frac{t^{\alpha} f(ta+(1-t)b)}{a-b}|_0^1 + \alpha \int_0^1 t^{\alpha-1} \frac{f(ta+(1-t)b)}{a-b} dt$

$= \frac{f(a)}{b-a} - \frac{\alpha}{b-a} \int_b^a \left(\frac{b-x}{b-a}\right)^{\alpha-1} \frac{f(x)}{a-b} dx$

$$= \frac{f(a)}{b-a} - \frac{\Gamma(\alpha+1)}{(b-a)^{\alpha+1}} J^{\alpha}_{a+} f(b). \tag{13}$$

using (13) *and* (12) *in , it follows that*
$I = \frac{f(a)+f(b)}{b-a} - \frac{\Gamma(\alpha+1)}{(b-a)^{\alpha+1}} [J^{\alpha}_{a+} f(b) + J^{\alpha}_{b-} f(a)]$.
Thus, by multiplying both sides by $\frac{b-a}{2}$*, we have conclusion* (10). □

Remark 2.5. *If in Lemma 2.3, we let* $\alpha = 1$ *then equality* (10) *becomes equality* (3) *of Lemma 2.3. Using this Lemma, we can obtain the following fractional integral inequality.*

Theorem 2.6. *[3] Let* $f : [a,b] \to \mathbb{R}$ *be a differentiable mapping on (a, b) with* $a < b$*. If* $|f'|$ *is convex on* $[a,b]$*, then the following inequality for fractional integrals holds:*

$$\left| \frac{f(a)+f(b)}{2} - \frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}} [J^{\alpha}_{a+} f(b) + J^{\alpha}_{b-} f(a)] \right| \leq \frac{b-a}{2(\alpha+1)} \left(1 - \frac{1}{2^{\alpha}}\right) [f'(a) + f'(b)]. \tag{14}$$

Proof. *Using Lemma 2 and the convexity of* $|f'|$*, we find*

$\left| \frac{f(a)+f(b)}{2} - \frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}} [J^{\alpha}_{a+} f(b) + J^{\alpha}_{b-} f(a)] \right|$

$\leq \frac{b-a}{2} \int_0^1 |(1-t)^{\alpha} - t^{\alpha}| |f'(ta+(1-t)b)| dt$

$\leq \frac{b-a}{2} \int_0^1 |(1-t)^{\alpha} - t^{\alpha}| [t|f'(a)| + (1-t)|f'(b)|] dt$

$\leq \frac{b-a}{2} \Bigg\{ \int_0^{\frac{1}{2}} [(1-t)^{\alpha} - t^{\alpha}][t|f'(a)| + (1-t)|f'(b)|] dt$

$+ \int_{\frac{1}{2}}^1 [t^{\alpha} - (1-t)^{\alpha}][t|f'(a)| + (1-t)|f'(b)|] dt \Bigg\}$

$$= \frac{b-a}{2}(k_1+k_2) \tag{15}$$

Calculating k_1 and k_2,
we have.

$$k_1 = |f'(a)|\left[\int_0^{\frac{1}{2}} t(1-t)^{\alpha}dt - \int_0^{\frac{1}{2}} t^{\alpha+1}dt\right]$$

$$+|f'(b)|\left[\int_0^{\frac{1}{2}} (1-t)^{\alpha+1}dt - \int_0^{\frac{1}{2}} (1-t)t^{\alpha}dt\right]$$

$$= |f'(a)|\left[\frac{1}{(\alpha+1)(\alpha+2)} - \frac{\frac{1}{2}^{\alpha+1}}{\alpha+1}\right] + |f'(b)|\left[\frac{1}{(\alpha+2)} - \frac{\frac{1}{2}^{\alpha+1}}{\alpha+1}\right] \tag{16}$$

$$k_2 = |f'(a)|\left[\int_{\frac{1}{2}}^{1} t^{\alpha+1}dt - \int_{\frac{1}{2}}^{1} t(1-t)^{\alpha}dt\right] + |f'(b)|\left[\int_{\frac{1}{2}}^{1} (1-t)t^{\alpha}dt - \int_{\frac{1}{2}}^{1} (1-t)^{\alpha+1}dt\right]$$

$$= |f'(a)|\left[\frac{1}{(\alpha+2)} - \frac{\frac{1}{2}^{\alpha+1}}{\alpha+1}\right] + |f'(b)|\left[\frac{1}{(\alpha+1)(\alpha+2)} - \frac{\frac{1}{2}^{\alpha+1}}{(\alpha+1)}\right] \tag{17}$$

.

Thus if we use (17) *and* (16) *in* (15), *we obtain the inequality of* (13). *This completes the proof.* □

Remark 2.7. *If we take $\alpha = 1$ in Theorem 2.4, then inequality* (13) *becomes inequality* (3) *of Theorem 2.1.*

Chapter 3
HADAMARD TYPE INEQUALITIES FOR m-CONVEX FUNCTIONS

3 Generalized Hermite-Hadamard Integral Inequality

Definition 3. [1] The function $f : [0,b] \to \mathbb{R}$ is said to be m-convex, where $m \in [0,1]$,

if for every $x, y \in [0,b]$ and $t \in [0,1]$ we have:

$$f(tx + m(1-t)y) \le tf(x) + m(1-t)f(y)$$

Denote $K_m(b)$ by the set of the m-convex functions on $[0,b]$ for which $f(0) \le 0$.
Several papers have been written on m-convex functions and we refer the papers. In [2], the following inequality of Hermite-Hadamard type for m-convex functions hold:

Theorem 3.1. *Let $f : [0,\infty) \to \mathbb{R}$ be a m-convex function with $m \in (0,1]$. If $0 \le a < b < \infty$ and $f \in L1[a,b]$, then one has the inequality:*

$$\frac{1}{b-a}\int_a^b f(x)dx \le \min\left\{\frac{f(a)+mf(\frac{b}{m})}{2}, \frac{f(b)+mf(\frac{a}{m})}{2}\right\} \tag{18}$$

In [2], S.S. Dragomir proved the following theorem.

Theorem 3.2. *Let $f : [0,1) \to \mathbb{R}$ be a m-convex function with $m \in (0,1]$.if $f \in L1[am,b]$ where $0 \le a < b$, then one has the inequality:*

$$\frac{1}{m+1}\left[\frac{1}{mb-a}\int_a^{mb} f(x)dx + \frac{1}{b-ma}\int_{am}^b f(x)dx\right] \le \frac{f(a)+f(b)}{2} \tag{19}$$

In this chapter, we give the generalized of the result in previous chapter.

3.1 Fractional Inequalities of Hermite Hadamard's for m-Convex Functions

Theorem 3.3. *Let $f : [a,b] \to \mathbb{R}$ be a positive function with $0 \le a < b$ and $f \in L_1[a,b]$. If f is a m-convex function on $[a,b]$, then the following inequalities for fractional integrals hold:*

$$\begin{aligned} f\left(\tfrac{a+mb}{2}\right) &\le \tfrac{\Gamma(\alpha+1)}{2(mb-a)^\alpha}\left[J^\alpha_{a+}f(mb) + m^{\alpha+1}J^\alpha_{b-}f(\tfrac{a}{m})\right] \\ &\le \frac{m[f(b)+mf(\frac{a}{m^2})]}{2} + \frac{\alpha}{(\alpha+1)}\frac{[f(a)-m^2f\frac{a}{m^2}]}{2}, \qquad \text{with } \alpha > 0. \end{aligned} \tag{20}$$

Proof:
Since f is m-convex, so

$$f(tx + m(1-t)y) \le tf(x) + m(1-t)f(y)$$

Put $t = \frac{1}{2}$ $\qquad [x,y] \in [a,b]$,

$f\left(\frac{x+my}{2}\right) \leq \frac{f(x)+mf(y)}{2}$

$2f\left(\frac{x+my}{2}\right) \leq f(x)+mf(y)$

Put $x = ta+m(1-t)b, \qquad y = tb+\frac{(1-t)a}{m}$

$\frac{x+my}{2} = \frac{ta+m(1-t)b+mtb+(1-t)a}{2}$

$\frac{x+my}{2} = \frac{ta+mb-mtb+mtb+a-ta}{2} = \frac{a+mb}{2}$

$$2f\left(\frac{a+mb}{2}\right) \leq f(ta+m(1-t)b)+mf(tb+\frac{(1-t)a}{m}) \tag{21}$$

Multiplying both side of (21) by $t^{\alpha-1}$ and integrate with respect to t over $t \in [0,1]$.

$2f\left(\frac{a+mb}{2}\right)\int_0^1 t^{\alpha-1}dt \leq \int_0^1 t^{\alpha-1}f(ta+m(1-t)b)dt+m\int_0^1 t^{\alpha-1}f(tb+\frac{(1-t)a}{m})dt$

$\frac{2}{\alpha}f\left(\frac{a+mb}{2}\right) \leq \int_0^1 t^{\alpha-1}f(ta+m(1-t)b)dt+m\int_0^1 t^{\alpha-1}f(tb+\frac{(1-t)a}{m})dt$

$$\leq I_1+mI_2. \tag{22}$$

Now, we calculate the sperate integrate.

$I_1 = \int_0^1 t^{\alpha-1}f(ta+m(1-t)b)dt$

Put $u = ta+m(1-t)b$

$du = adt-mbdt = (a-mb)dt$

Also, $u = ta+mb-mtb$

$u-mb = t(a-mb)$

$\frac{mb-u}{mb-a} = t$

When $t = 0$ and $t = 1$ then $u = mb$ and $u = a$ respectively.

So, $I_1 = \int_{mb}^{a} \left(\frac{mb-u}{mb-a}\right)^{\alpha-1} f(u)\frac{du}{a-mb}$

$= \frac{\Gamma(\alpha)}{mb-a} \cdot \frac{1}{\Gamma(\alpha)} \int_{a}^{mb} \left(\frac{mb-u}{mb-a}\right)^{\alpha-1} f(u)du$

Now,

$J_{a+}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_{a}^{x} (x-t)^{\alpha-1} f(t)dt, \quad x > a$

$I_1 = \frac{\Gamma(\alpha)}{mb-a^{\alpha}} \frac{1}{\Gamma(\alpha)} \int_{a}^{mb} (mb-u)^{\alpha-1} f(u)du$

$I_1 = \frac{\Gamma(\alpha)}{(mb-a)^{\alpha}} J_{a+}^{\alpha} f(mb).$

In the similar way, we will calculate I_2.

$I_2 = \int_0^1 t^{\alpha-1} f(tb + \frac{a}{m}(1-t))dt$

Put $v = tb + \frac{a}{m}(1-t)$

$dv = bdt - \frac{a}{m}dt = \left(\frac{mb-a}{m}\right)dt$

and

$dt = \frac{mdv}{mb-a}$

Also, $v = tb + \frac{a}{m} - \frac{a}{m}t$

$v - \frac{a}{m} = t(b - \frac{a}{m})$

$\frac{mv-a}{m} = t(\frac{mb-a}{m})$

$\frac{mv-a}{mb-a} = t$

When $t = 0$ and $t = 1$ then $v = \frac{a}{m}$ and $v = b$ respectively.

So, $I_2 = \int_{a/m}^{b} \left(\frac{mv-a}{mb-a}\right)^{\alpha-1} f(v)\frac{mdv}{mb-a}$

$= \frac{m\Gamma(\alpha)}{(mb-a)^{\alpha}} \frac{1}{\Gamma(\alpha)} \int_{a/m}^{b} \left(mv-a\right)^{\alpha-1} f(v)dv$

Now,

$$J_{b-}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (t-x)^{\alpha-1} f(t)dt, \quad x < b$$

$$I_2 = \frac{m^{\alpha}\Gamma(\alpha)}{(mb-a)^{\alpha}} \cdot \frac{1}{\Gamma(\alpha)} \int_{a/m}^{b} \left(v - \frac{a}{m}\right)^{\alpha-1} f(v)dv$$

$$I_2 = \frac{m^{\alpha}\Gamma(\alpha)}{(mb-a)^{\alpha}} J_{b-}^{\alpha} f(\frac{a}{m})$$

$$\frac{2}{\alpha} f\left(\frac{a+mb}{2}\right) \leq \frac{\Gamma(\alpha)}{(mb-a)^{\alpha}} J_{a+}^{\alpha} f(mb) + \frac{m^{\alpha+1}\Gamma(\alpha)}{(mb-a)^{\alpha}} J_{b-}^{\alpha} f(\frac{a}{m})$$

$$\frac{2}{\alpha} f\left(\frac{a+mb}{2}\right) \leq \frac{\Gamma(\alpha)}{(mb-a)^{\alpha}} \left[J_{a+}^{\alpha} f(mb) + m^{\alpha+1} J_{b-}^{\alpha} f(\frac{a}{m})\right]$$

$$f\left(\frac{a+mb}{2}\right) \leq \frac{\Gamma(\alpha+1)}{2(mb-a)^{\alpha}} \left[J_{a+}^{\alpha} f(mb) + m^{\alpha+1} J_{b-}^{\alpha} f(\frac{a}{m})\right].$$

For the proof R.H.S of inequality (20)

$$f(ta + m(1-t)b) \leq tf(a) + m(1-t)f(b)$$

$$mf(tb + (1-t)\frac{a}{m}) = mf(tb + m(1-t)\frac{a}{m^2})$$

$$\leq mtf(b) + m^2(1-t)f(\frac{a}{m^2})$$

Now,

$$f(ta + m(1-t)b) + mf(tb + (1-t)\frac{a}{m})$$

$$\leq tf(a) + m(1-t)f(b) + mtf(b) + m^2(1-t)f(\frac{a}{m^2})$$

$$= tf(a) + mf(b) + m^2 f(\frac{a}{m^2}) - m^2 t f(\frac{a}{m^2})$$

$$= t\left[f(a) - m^2 f(\frac{a}{m^2})\right] + m\left[f(b) + mf(\frac{a}{m^2})\right]$$

$$f(ta + m(1-t)b) + mf(tb + (1-t)\frac{a}{m}) \leq m\left[f(b) + mf(\frac{a}{m^2})\right] + t\left[f(a) - m^2 f(\frac{a}{m^2})\right].$$

Multiplying both sides by $t^{\alpha-1}$ and integrate with respect to t over $[0,1]$, we obtain

$\int_0^1 t^{\alpha-1} f(ta+m(1-t)b)dt + \int_0^1 t^{\alpha-1} mf(tb+(1-t)\frac{a}{m})dt \leq \frac{m[f(b)+mf(\frac{a}{m^2})]}{\alpha} + \frac{[f(a)-m^2 f(\frac{a}{m^2})]}{\alpha+1}$.
So, we get

$\frac{\Gamma(\alpha+1)}{2(mb-a)^{\alpha}}\left[J^{\alpha}_{a+}f(mb)+m^{\alpha+1}J^{\alpha}_{b-}f(\frac{a}{m})\right]$

$\leq \frac{m[f(b)+mf(\frac{a}{m^2})]}{2} + \frac{[f(a)-m^2 f\frac{a}{m^2}]}{2(\alpha+1)}$.

Finally, we get inequality (20).

Remark 3.4. *For m=1, we get* (6) *of Theorem 2.1.*

Remark 3.5. *If in Theorem 3.3, we let* $\alpha = 1$ *and* $m = 1$*, then inequality* (20) *become inequality* (3) *of Theorem 2.2.*

3.2 Fractional Inequalities of Hermite Hadamard's type for m-Convex Functions

Lemma 3.6. *Let* $f : [a,b] \to \mathbb{R}$ *be a differential mapping on* (a,b) *with* $a < b$*. If* $f' \in L[a,b]$*, then the following equality for fractional integral hold:*

$$\frac{f(a)+f(mb)}{2} - \frac{\Gamma(\alpha+1)}{2(mb-a)^{\alpha}}\left[J^{\alpha}_{a+}f(mb)+J^{\alpha}_{-bm}f(a)\right]$$
$$= \frac{mb-a}{2}\int_0^1 [(1-t)^{\alpha} - t^{\alpha} f'(ta+m(1-t)b)]dt \tag{23}$$

Proof: *It suffices to note that*

$I = \int_0^1 [(1-t)^{\alpha} - t^{\alpha}] f'(ta+m(1-t)b)dt$

$I = \int_0^1 \left[(1-t)^{\alpha} f'(ta+m(1-t)b)\right]dt + \left[-\int_0^1 t^{\alpha} f'(ta+m(1-t)b)dt\right]$

$$I = I_1 + I_2 \tag{24}$$

$$I_1 = \int_0^1 [(1-t)^{\alpha} f'(ta+m(1-t)b)]dt \tag{25}$$

Integrating by parts,

$= (1-t)^{\alpha} \frac{f(ta+m(1-t)b)}{a-mb} \mid_0^1 + \int_0^1 \alpha(1-t)^{\alpha-1} \frac{f(ta+m(1-t)b)}{a-mb} dt$

$= \frac{-f(mb)}{(a-mb)} + \frac{\alpha}{(a-mb)} \int_0^1 f(ta+m(1-t)b)(1-t)^{\alpha-1} dt$
Put $x = ta+m(1-t)b$
$x - mb = (a-mb)t$
$\frac{x-mb}{a-mb} = t$
$t = \left(\frac{mb-x}{mb-a}\right)$ *and* $1-t = \frac{a-x}{a-mb}$.
When $t = 0$ *and* $t = 1$ *then* $x = mb$ *and* $x = a$ *respectively.*

$= \frac{f(mb)}{(mb-a)} - \frac{\alpha}{(mb-a)} \int_{mb}^{a} f(x) \left(\frac{a-x}{a-mb}\right)^{\alpha-1} \frac{-dx}{mb-a}$

$= \frac{f(mb)}{(mb-a)} - \frac{\alpha}{(mb-a)^{\alpha+1}} \int_a^{mb} f(x)(x-a)^{\alpha-1} dx$

$= \frac{f(mb)}{(mb-a)} - \frac{\alpha\Gamma(\alpha)}{(mb-a)^{\alpha+1}} \frac{1}{\Gamma(\alpha)} \int_a^{mb} (x-a)^{\alpha-1} f(x) dx$

$I_1 = \frac{f(mb)}{(mb-a)} - \frac{\Gamma(\alpha+1)}{(mb-a)^{\alpha+1}} J_{mb-}^{\alpha} f(a).$

$$I_2 = -\int_0^1 t^{\alpha} f'(ta+m(1-t)b)]dt \tag{26}$$

$= -\left[t^{\alpha} \frac{f(ta+m(1-t)b)}{(a-mb)} \mid_0^1 + \alpha \int_0^1 t^{\alpha-1} \frac{f(ta+m(1-t)b)}{(a-mb)} dt \right]$

$= \frac{f(a)}{(mb-a)} - \frac{\alpha}{(mb-a)} \int_0^1 t^{\alpha-1} f(ta+m(1-t)b) dt$

$= \frac{f(a)}{(mb-a)} - \frac{\alpha}{(mb-a)} \int_{mb}^{a} \left(\frac{mb-x}{mb-a}\right)^{\alpha-1} f(x) \frac{-dx}{mb-a}$

$= \frac{f(a)}{(mb-a)} - \frac{\alpha\Gamma\alpha}{(mb-a)^{\alpha+1}} \frac{1}{\Gamma(\alpha)} \int_a^{mb} (mb-x)^{\alpha-1} f(x) dx$

$I_2 = \frac{f(a)}{(mb-a)} - \frac{\Gamma(\alpha+1)}{(mb-a)^{\alpha+1}} J_{a+}^{\alpha} f(mb).$

Put the equations (25) *and* (26) *in equation* (24).

$I = \frac{f(a)+f(mb)}{(mb-a)} - \frac{\Gamma(\alpha+1)}{(mb-a)^{\alpha+1}}\left[J^{\alpha}_{mb-}f(a) + J^{\alpha}_{a+}f(mb)\right]$

Thus, by multiplying both sides by $\frac{mb-a}{2}$, we have conclusion (3.4).

Remark 3.7. *If in Lemma (3.4), we let $\alpha = 1$ and $m = 1$ then equality* (23) *becomes equality* (4) *of Lemma (2.1).*
Using this Lemma, we can obtain the following fractional integral inequality.

Theorem 3.8. *Let $f : [a,b] \to \mathbb{R}$ be a differential mapping on (a,b) with $a < b$. If $|f'|$ is a m-convex on $[a,b]$, then the following inequalities for fractional integrals holds:*

$$\left|\frac{f(a)+f(mb)}{2} - \frac{\Gamma(\alpha+1)}{2(mb-a)^{\alpha}}[J^{\alpha}_{b-}f(a) + J^{\alpha}_{a+}f(mb)]\right|$$
$$\leq \frac{m(b-a)}{2(\alpha+1)}\left(1-\frac{m}{2^{\alpha}}\right)[f'(a)+f'(b)]. \tag{27}$$

Proof: Using Lemma (2.4) and the convexity of $|f'|$, we find

$\int_0^1 \mid (1-t)^{\alpha} - t^{\alpha} \mid |f'(ta+m(1-t)b)|dt$

$= \int_0^1 \mid (1-t)^{\alpha} - t^{\alpha} \mid \left[t|f'(a)| + m(1-t)|f'(b)|\right]dt$

$= \int_0^{1/2}[(1-t)^{\alpha} - t^{\alpha}][t|f'(a)| + m(1-t)|f'(b)|]dt$

$+ \int_{1/2}^1 [t^{\alpha} - (1-t)^{\alpha}][t|f'(a)| + m(1-t)b|f'(b)|]dt$

$$K = K_1 + K_2 \tag{28}$$

$$K_1 = \int_0^{1/2}[(1-t)^{\alpha} - t^{\alpha}][t|f'(a)| + m(1-t)|f'(b)|]dt \tag{29}$$

$= \int_0^{1/2}[(1-t)^{\alpha} - t^{\alpha}][t|f'(a)|dt + \int_0^{1/2}[(1-t)^{\alpha} - t^{\alpha}][m(1-t)b|f'(b)|]dt$

$|f'(a)|[\int_0^{1/2} t(1-t)^{\alpha}dt - \int_0^{1/2} t^{\alpha+1}dt] + |f'(b)|[\int_0^{1/2} m(1-t)^{\alpha+1}dt - \int_0^{1/2} m(1-t)t^{\alpha}dt]$

$= |f'(a)|\left[-\int_1^{1/2}(1-u)u^{\alpha}du - \int_0^{1/2} t^{\alpha+1}dt\right]$

$$+|f'(b)|\Bigg[-\int_1^{1/2}m(u)^{\alpha+1}du-\int_0^{1/2}m(t)^{\alpha}dt+m\int_0^{1/2}t^{\alpha+1}dt\Bigg]$$

$$=|f'(a)|\Bigg[-\int_1^{1/2}(u)^{\alpha+1}du+\int_0^{1/2}(u)^{\alpha+1}du-\int_0^{1/2}t^{\alpha+1}dt\Bigg]$$

$$+|f'(b)|\Bigg[-\int_0^{1/2}m(u)^{\alpha+1}du-\int_0^{1/2}m(t)^{\alpha}dt+m\int_0^{1/2}(t)^{\alpha+1}\Bigg]dt$$

$$=|f'(a)|\Bigg[-\tfrac{(u)^{\alpha+1}}{\alpha+1}|_1^{1/2}+\tfrac{(u)^{\alpha+2}}{\alpha+2}|_1^{1/2}-\tfrac{(t)^{\alpha+2}}{\alpha+2}\Bigg]$$
$$+|f'(b)|\Bigg[-m\tfrac{(u)^{\alpha+2}}{\alpha+2}|_1^{1/2}-m\tfrac{(t)^{\alpha+1}}{\alpha+1}|_0^{1/2}+m\tfrac{(t)^{\alpha+1}}{\alpha+1}|_0^{1/2}+m\tfrac{(t)^{\alpha+2}}{\alpha+2}|_0^{1/2}\Bigg]$$

$$=|f'(a)|\Bigg[-\tfrac{(1/2)^{\alpha+1}}{\alpha+1}+\tfrac{(1)^{\alpha+1}}{\alpha+1}+\tfrac{(1/2)^{\alpha+2}}{\alpha+2}-\tfrac{(1)^{\alpha+2}}{\alpha+2}-\tfrac{(1/2)^{\alpha+2}}{\alpha+2}\Bigg]$$
$$+|f'(b)|\Bigg[-m\tfrac{(1/2)^{\alpha+2}}{\alpha+2}+m\tfrac{(1)^{\alpha+2}}{\alpha+2}-m\tfrac{(1/2)^{\alpha+1}}{\alpha+1}+m\tfrac{(1/2)^{\alpha+2}}{\alpha+2}\Bigg]$$

$$=|f'(a)|\Bigg[\{\tfrac{1}{\alpha+1}-\tfrac{1}{\alpha+2}\}-\tfrac{(1/2)^{\alpha+1}}{\alpha+1}\Bigg]+|f'(b)|\Bigg[m\tfrac{(1)^{\alpha+2}}{\alpha+2}-m\tfrac{(1/2)^{\alpha+1}}{\alpha+1}\Bigg]$$

$$K_1=|f'(a)|\Bigg[\tfrac{1}{(\alpha+1)(\alpha+2)}-\tfrac{(1/2)^{\alpha+1}}{\alpha+1}\Bigg]+|f'(b)|\Bigg[m\tfrac{(1)^{\alpha+2}}{\alpha+2}-m\tfrac{(1/2)^{\alpha+1}}{\alpha+1}\Bigg].$$

$$K_2=\int_{1/2}^{1}[t^{\alpha}-(1-t)^{\alpha}][t|f'(a)|+m(1-t)b|f'(b)|]dt \tag{30}$$

$$=|f'(a)|\Bigg[\int_{1/2}^{1}t^{\alpha+1}dt-\int_{1/2}^{1}t(1-t)^{\alpha}dt\Bigg]$$
$$+|f'(b)|\Bigg[\int_{1/2}^{1}m(1-t)t^{\alpha}dt-\int_{1/2}^{1}m(1-t)^{\alpha+1}dt\Bigg]$$

$$=|f'(a)|\Bigg[\int_{1/2}^{1}t^{\alpha+1}dt+\int_{1/2}^{0}(u)^{\alpha}(1-u)du\Bigg]$$
$$+|f'(b)|\Bigg[\int_{1/2}^{1}mt^{\alpha}-\int_{1/2}^{1}m(t)^{\alpha+1}dt+\int_{1/2}^{0}m(u)^(\alpha+1)du\Bigg]$$

$$=|f'(b)|\Bigg[\tfrac{((t)^{\alpha+1}}{\alpha+1}|_{1/2}^{1}+\tfrac{(u)^{\alpha+1}}{\alpha+1}|_{1/2}^{0}-\tfrac{(u)^{\alpha+2}}{\alpha+2}|_{1/2}^{0}\Bigg]$$
$$+|f'(b)|\Bigg[\tfrac{m(1)^{\alpha+1}}{\alpha+1}-m\tfrac{(1/2)^{\alpha+1}}{\alpha+1}-m\tfrac{(1)^{\alpha+2}}{\alpha+2}+\tfrac{(1/2)^{\alpha+2}}{\alpha+2}-m\tfrac{(1/2)^{\alpha+2}}{\alpha+2}\Bigg]$$

$$= |f'(a)|\left[\frac{1}{\alpha+2} - \frac{(1/2)^{\alpha+1}}{\alpha+1}\right] + |f'(b)|\left[\{m\frac{(1)^{\alpha+1}}{\alpha+1} - m\frac{(1)^{\alpha+2}}{\alpha+2}\} - m\frac{(1/2)^{\alpha+1}}{\alpha+1}\right]$$

$$K_2 = |f'(a)|\left[\frac{1}{\alpha+2} - \frac{(1/2)^{\alpha+1}}{\alpha+1} + |f'(b)|\left[\frac{m(\alpha+2)-m(\alpha+1)}{(\alpha+2)(\alpha+1)} - m\frac{(1/2)^{\alpha+1}}{\alpha+1}\right]\right.$$

$$K_2 = |f'(a)|\left[\frac{1}{\alpha+2} - \frac{(1/2)^{\alpha+1}}{\alpha+1}\right] + |f'(b)|\left[\frac{m}{(\alpha+2)(\alpha+1)} - m\frac{(1/2)^{\alpha+1}}{\alpha+1}\right]$$

Now put equations (29) and (30) in equation (28).

$$= |f'(a)|\left[\frac{1}{\alpha+2} - \frac{(1/2)^{\alpha+1}}{\alpha+1}\right] + |f'(b)|\left[\frac{m}{(\alpha+2)(\alpha+1)} - m\frac{(1/2)^{\alpha+1}}{\alpha+1}\right]$$
$$+ |f'(a)|\left[\frac{1}{(\alpha+1)(\alpha+2)} - \frac{(1/2)^{\alpha+1}}{\alpha+1}\right] + |f'(b)|\left[\frac{m}{\alpha+2} - m\frac{(1/2)^{\alpha+1}}{\alpha+1}\right]$$

$$= |f'(a)|\left[\frac{\alpha+2}{(\alpha+1)(\alpha+2)} - 2\frac{(1/2)^{\alpha+1}}{\alpha+1}\right] + m|f'(b)|\left[\frac{\alpha+2}{(\alpha+2)(\alpha+1)} - 2\frac{(1/2)^{\alpha+1}}{\alpha+1}\right]$$

$$= |f'(a)|\left[\frac{1}{(\alpha+1)} - 2\frac{(1/2)^{\alpha+1}}{\alpha+1}\right] + m|f'(b)|\left[\frac{\alpha+2}{(\alpha+2)(\alpha+1)} - 2\frac{(1/2)^{\alpha+1}}{\alpha+1}\right]$$

$$= \left[\frac{1}{(\alpha+1)} - 2\frac{(1/2)^{\alpha+1}}{\alpha+1}\right]\left[|f'(a)| + m|f'(b)|\right]$$

$$= \frac{1}{(\alpha+1)}\left[1 - 2(1/2)^{\alpha+1}\right]\left[|f'(a)| + m|f'(b)|\right]$$

$$= \left[|f'(a)| + m|f'(b)|\right]\frac{1}{(\alpha+1)}\left[1 - \frac{1}{2^{\alpha+1-1}}\right]$$

$$= \left[|f'(a)| + m|f'(b)|\right]\frac{1}{(\alpha+1)}\left[1 - \frac{1}{2^{\alpha}}\right]$$

Thus, by multiplying by $\frac{(mb-a)}{2}$, we have conclusion of (3.5).

Remark 3.9. *If we let* $\alpha = 1$ *and* $m = 1$ *then equality* (14) *becomes inequality* (5) *of Lemma (2.1).*

Chapter 4
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