Magic Labeling of Cycle Like Graphs

By

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It is certified that RABIA BIBI (CIIT/FA17-RMT-016/LHR) has carried out all the work related to this thesis under my supervision at the Department of Mathematics, COMSATS University Islamabad, Lahore Campus and the work fulfills the requirement for award of MS degree.

Date:______________________

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Dr. Sarfraz Ahmad
Associate Professor
Department of Mathematics

DEDICATION

To
My Parents

ACKNOWLEDGEMENT

First of all, I would like to thank Almighty Allah Who is most Gracious, Merciful, Omniscient and Omnipotent, and without the help of whom nothing could be completed. Peace and blessing of Allah be upon the Holy Prophet who exhort his followers to seek for knowledge from cradle to grave. It is great pleasure to acknowledge individuals and organizations help, which made this research possible.

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RABIA BIBI (CIIT/FA17-RMT-016)

ABSTRACT

Magic Labeling of Cycle Like Graphs

Graph labeling is play very vital role in graph theory applications like network addressing of communication, X-ray, circuit designing, crystallography, radar, missile guidance, astronomy and data base management.

A labeling is bijective mapping in which natural number are assigned to vertices or edges. If weights of all vertex have same sum, such labeling is called magic labeling. The notion of (a, d)-vertex-antimagic-total labeling was introduced by Baca et al. [5] in 2000. A labeling in which domain set consists of all vertices and edges is called total labeling. A labeling is called super (a, d)-vertex antimagic-total labeling (VAMTL) if we allot smallest number to vertices and then to edges. Hussain et al. [17, 18] formulated (a,d)-vertex antimagic total labeling on Harary graphs, however, many cases are still open.

This dissertation learning about the configuration of new results of vertex-magic total labeling and (a,d)-vertex antimagic total (VAT) labeling of Harary graphs. Also we construct same new results of super (a, d)-vertex antimagic total (VAT) labeling on Harary graphs.

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Chapter 1

Basic Concept of Graphs

1.1 Introduction

This chapter is based on basic concept of graphs also known basic results of graph labeling which are accessible. They are provided us sufficient basis for current study.

1.2 Basic Definitions

1.2.1 Preliminaries

Definition [23] " Consider graph G has set of some order pairs objects which are associated in relations, these inter connected items are known as vertices and denoted by $V(G)$, also some links between some pair of objects (i.e vertices) are known as edges and denoted by $E(G)$. These edges may be directed or undirected".

The figure 1.1 shows graphs A and B with
$V(A) = \{u, v, w, x\}$
$E(A) = \{uv, vw, wx, xu, xv\}$
$V(B) = \{a, b, c\}$,
$E(B) = \{d, e, f, g, h, i, j, k\}$;

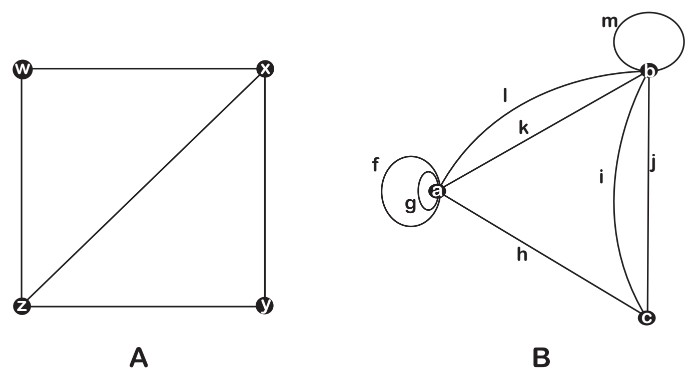


Figure 1.1: Graph G(V, E)

Definition [23] In figure B, "if an edge joins to it self is called self-loop".

Definition [23] In figure B, “if two or more edges having same endpoints is called multi-edge”.

Definition [23] In figure B , “a graph having multiple edges or self-loop is said to be general graph”.

Definition [23] Figure A represent simple graph, “if graph G is zero loop and zero multiple edge is said to be simple graph”.

Definition [23] “A graph is said to be *Null Graph* if it has vertices but no edges”.

Definition [23] “Graph G with one vertex is said to be *Trivial Graph*”.

1.2.2 Type of edge

Definition [23] “The notation used for order of graph G is $v = |V(G)|$, and is defined as the total number of vertices in graph $G(V,E)$”. For example figure (A)1.1 order is $|V(G)| = 4$.

Definition [23] “The notation used for size of graph G is $e = |E(G)|$, and is defined as the total number of edges in graph $G(V,E)$”. For example figure (A)1.1 size is $|E(G)| = 5$.

Definition [23] “Total number of edges on vertex v of graph G is said to be *degree of a vertex*, it is denoted by $deg(v)$”. In figure 1.1
d(b) = 6 and d(y) = 2.

In graph theory the role of degree has very important, Euler degree sum formula is very important to considered the relation between degree and size of graph G. Euler degree sum formula stated as sum of degree of graph G is equal to two times numbers of edge. i.e

$$\sum_{v \in V(G)} deg(v) = 2|E(G)| = 2e$$

1.2.3 Type of vertex

Definition [23] “If two vertices are joined through an edge is called Adjacent vertices”. In other word neighbor is Adjacent vertices. In figure 1.1 adjacent vertices of graph are x and y.

Definition [23] “If end point of e edge of v vertex, then we said v is incident on e, similarly e is incident on v”.

Definition [23] “If the vertex has degree zero such that it is not an endpoint of an edge of a graph, it is known as isolated vertex”.

Definition [23] “If an edge joins the endpoints, a vertex u which is joined through an edge to vertex v is called neighbor of v”.

Definition [23] “If G is a graph and open neighbor of v vertex is a set which consists all neighbors of v vertex, denoted as $N(v)$. Closed neighborhood of v vertex as under $N[v] = N(v) \bigcup \{v\}$”.

Definition [23] “Pendant vertex is a vertex which has degree one, for example end vertex or leaf”.

1.2.4 Subgraph

Definition [23] “If (i) If $V(H) \subseteq V(G)$.

(ii)If $E(H) \subseteq E(G)$.

after that we try to know that how to mark-down as $H \subseteq G$, hence H is called subgraph of $G(V,E)$”.

Definition [23] “H be proper subgraph $G(V,E)$, if $H \subset G(V,E)$, whichever $V(H) \subset V(G)$, or may be $E(H) \subset E(G)$”.

Definition [23] “H is said to be a spanning subgraph of graph G, if H is a subset of graph G, also $V(H) = V(G)$”.

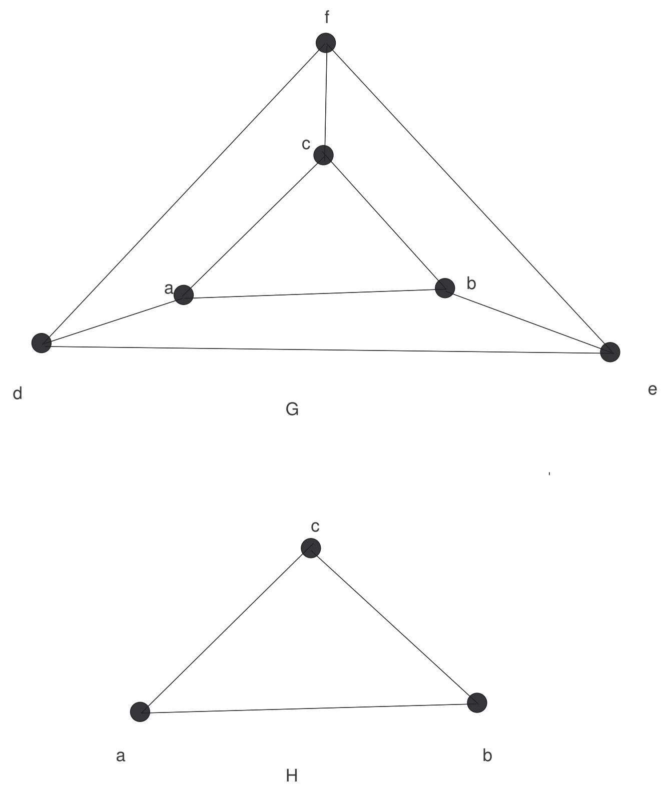


Figure 1.2: Subgraph H of Graph G

Definition [23] "$G(V,E)$ be a graph and H has *induced subgraph* of graph $G(V,E)$, if u, v belong to $V(H)$, uv belong to $E(G)$ then uv belong to $E(H)$".

Definition [23] "A nonempty vertices of set S in graph G, S is induce subgraph with vertex S. Its denoted as $< S >$ or $< S >_G$".

Definition [23] "For a non-empty set X of edges $(i,e)X \subseteq E(G)$, induce subgraph X which consists all vertices with one edge, subgraph $G(V,E)$ is known as edge induced subgraph".

1.2.5 Graphs related to Cycle, Walks, Path and Trail

Definition[23] "Consider a graph $G(V,E)$, v_0 vertex to v_n vertex walk having a alternating sequence having edges and vertices $W = \langle v_0, e_1, v_1, e_2, \cdots, v_(n-1), e_n, v_n \rangle$, such endpoints are $(e_i) = v_(i-1), v_i$, where $i = 1, \cdots, n$"

Definition [23] "The number of edge-steps of walk sequence, forms a length of walk".

Definition [23] "A trivial walk is a walk having 0 length, i.e., no edges and only one vertex".

Definition [23] "A nontrivial walk is closed walk that starting and ending point is same vertex".

Definition [23] "If no edges repeated in a walk that is called trail".

Definition [23] "If no vertex repeated in a path that is called a trail(excluding probably the first and last vertices)".

Definition [23] "If no edges and only one vertex in a walk, path or trail is known as trivial".

Definition [23] "Cycle is a nontrivial closed path".

Definition [23]"Graph G in which there is no cycle, then it is called acyclic graph".

Let u, v belong to $V(G)$. A vertices sequence in u to v *walk* starting from u and closing at v, adjacent vertices in walk W has defined by $W : \{v_0, v_1, \cdots, v_k\}$ for $k \geq 0$, v_i and $v_{(i+1)}$ has adjacent, each vertex V_i and edge $V_iV_{(i+1)}$ are lie on W, when $i = 0, 1, \cdots, (k-1)$ and $0 \leq i \leq k$.

$W_1 = uvxy$ is example of u to y walk and $W_2 = uvwyvu$ is example of u to u walk. If walk W is u to v and $u = v$ then W is said to be Closed Walk. W_2 is example of closed walk and if in a u to v walk W also $u \neq v$ then W is said to be Open. W_1 is example of open walk. Similarly, walk of length is the total edges meet in a walk (with multiples event of an edge). Also, if a length of walk is 0 then it is called trivial walk. Length of

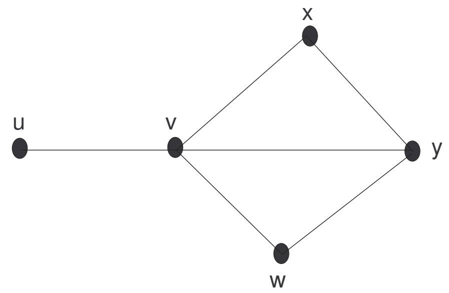


Figure 1.3: A Walk

W_1 is 3 and length of W_2 is 5.

Let u, v belong to $V(G)$, than u to v trail is a walk of graph G having no edges are traversed more then once. Trail examples are $W_6 = xvyw$ and $W_7 = vwyvx$, also a graph G, u to v walk having no vertices repeated is said path. W_6 is example of such path but W_7 is not a path. Every path is a trail.

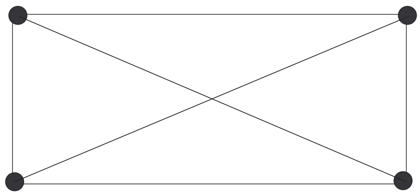
Theorem 1.2.1. *If graph G containing walk u to v with length l, then graph G having u to v path having length l.*

1.2.6 General Graph Families

Definition [23] "If vertex and edge sets of a graph are empty such graphs is called null graph".

Definition [23] "A graph having no edge and only one vertex v, then it is called trivial graph".

Definition [23] "If a graph having multi edges or self loops is called general graph". For example figure 1.1 is general graph.



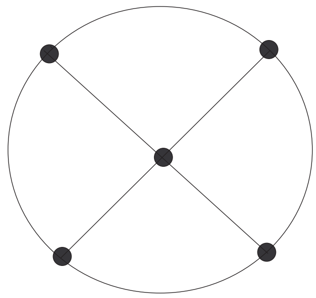


Figure 1.4: Simple Graph

Definition [23] “If a graph have not multi edges or self loops, then it is called simple graph”. For example figure 1.4 is simple graph:

Definition [23] “If V be vertex and E be edge sets of a graph G than $|V \bigcup E|$ be finite such graph is called finite graph”. For example figures 1.3 and 1.4 represent finite graphs.

Definition [23] “A G graph is said to be bipartite graph, if V is a vertex set of G graph divided into U, W subsets, in this connection every edge consists endpoint in U, other endpoint in W. A pair consist on U and W such graphs G is called bipartition of graph G. Bipartition subsets is U and W”.

Definition [23] “If each vertices is joined through an edge of a simple graph is known as complete graph. K_n represent a Complete graph”. Figure 1.6 is example of complete

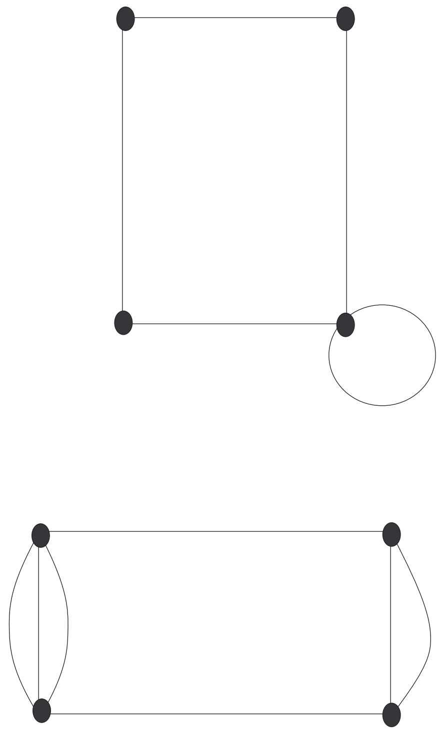


Figure 1.5: Not Simple Graph

graph.

Definition [23] " A simple bipartite graph is called complete bipartite graphs in which each vertex of first bipartition subset is connected with other vertices of second bipartition subset. If first bipartition subset consists m vertices and other bipartition subset consists n vertices then complete bipartite graph denoted as $K_{m;n}$". For example $K_{3,3}$ is shown in figure 1.7 below:

Definition [23] "If degree of all vertices of graph G is same such graphs is regular graphs. If a graph having K degree which is called $k-regular$ graph". In figure 1.6 all the graphs are regular.

Definition [48] "If we draw a graph without any crossings is called planar graph. These

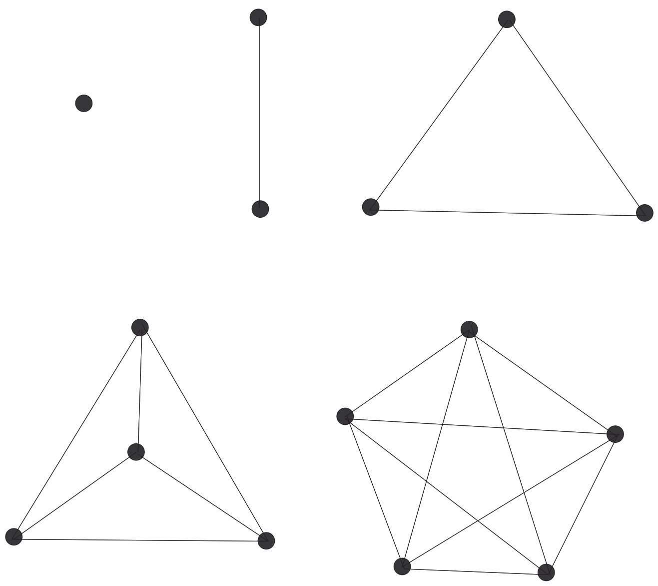


Figure 1.6: One to five vertices Complete Graph

drawing has planar embedding of graph G".

Definition [48] "G be a connected graph, if u, v be path of graph G and $u, v \varepsilon V(G)$ is called connected , otherwise graph G is called disconnected graph".

.

Definition [48] "A G graph is said to be digraph or directed graph, if triplet is consisting of edge set $E(G)$ of G, vertex set $V(G)$ of G and a function of every edge of vertices is ordered pair. Tail of edge is a ordered pair first vertex, head is second vertex, both are endpoints. We also asks an edge from tail to head".

1.2.7 Tree

It is said that a graph G that has no cycle is a-cyclic and a-cyclic connected graph is said to be a*tree*, also a graph have no cycles is called *forest*.

Definition [23] "Each tree of n vertices containing $n-1$ edges".

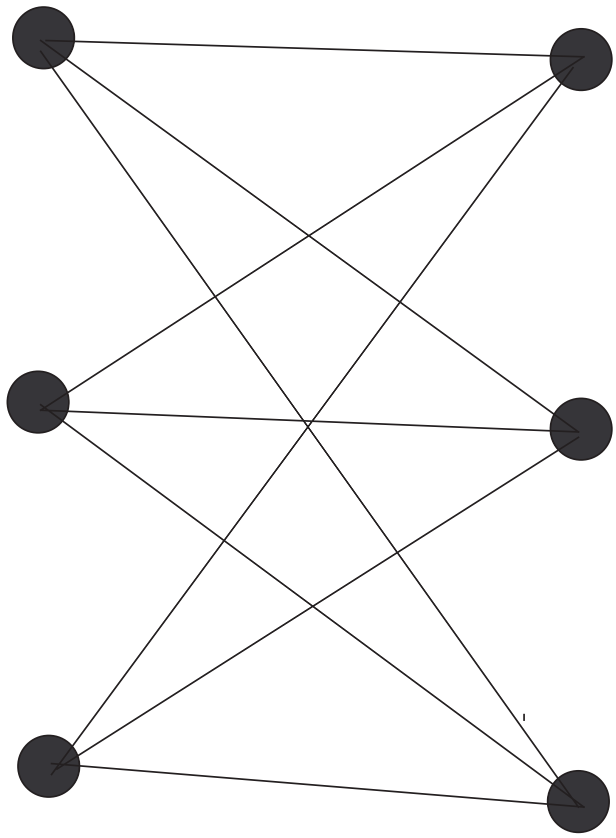


Figure 1.7: Complete Bipartite Graph $K_{3,3}$

Definition [48] “Connected a-cyclic graph is also a tree”.

Definition [48] “Spanning subgraph of a spanning tree is also a tree”.

Definition [48] “Forest is called a-cyclic graph”.

[48] “Tree be a associated forest, each part of a forest is also a tree”.

[48] “A graph is without cycles or odd cycles, such trees and forest must be bipartite”.

[48] “All paths always treated as a tree but tree is treated as a path iff it has maximum degree two”.

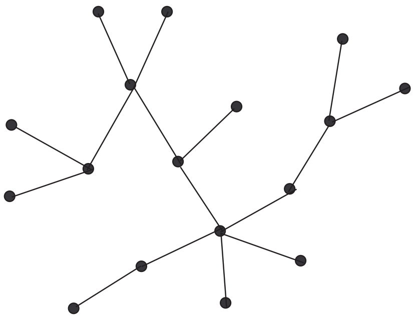


Figure 1.8: A tree

Definition [48] "If one vertex of Tree is adjacent to other vertices of tree, then it is called star. A star of n-vertex is also a bipartite graph $K_{1,n-1}$".

Definition [22] "Let G be a graph, we obtain $T_{(n,n,n,n)}$ graph by using four edges star $K_{1,4}$ graph of $n-1$ vertices, where $n \geq 1$. Hence $T_{(n,n,n,n)}$ is star graph $K_{1,4}$".

Definition [22] "If $n_i \geq 1,\ p \geq 5$ then we obtain a graph $G \cong T_{(n1,n2,\cdots,n_p)}$ by using vertices $n_i - 1$ to every $i-th$ edges of star graph $K_{1,p}$. Hence, $T_{(1,1,\cdots,1)}$ graph is star graph $K_{1,p}$".

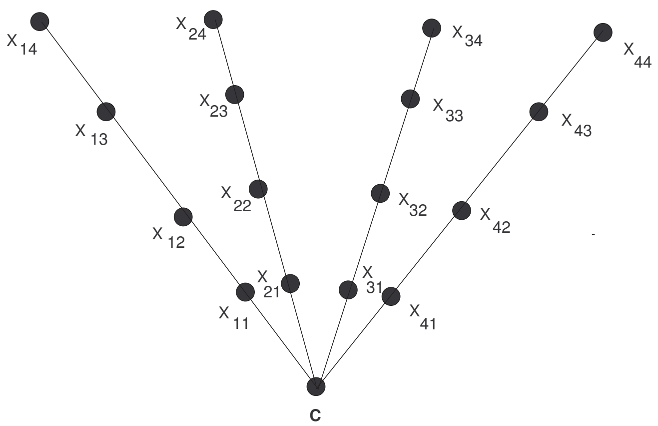


Figure 1.9: T(4,4,4,4)

Definition [48] "Let both G and H a simple graphs, an isomorphism from G to H be a bijection mapping $f : V(G) \to V(H)$ there exist $uv\varepsilon E(G)$ if and only if $f(u)f(v)\varepsilon E(H)$, such that G is isomorphic to H, which is denoted as $G \cong H$".

Definition [48] "A tree in which endpoints are removed which results in the formation of path graph, then it is said to be *caterpillar*".

Definition [48] "A tree having only two vertices are not end vertices is said to be double star. Caterpillar is a tree of order 3 or more, if we remove end vertices then produce a path. This path is called spine of caterpillar".

Theorem 1.2.2. *[48] A tree having only two vertices which are not end vertices are said to be Double star.*

Theorem 1.2.3. *[48] Following declaration have equivalent.*

1. If graph G is tree.

2. If unique path joined any two vertex of graph G.

3. If graph G must connected also $n = m + 1$.

4. If G must a-cyclic also $n = m + 1$

5. G have a-cyclic viceversa couple of nonadjacent joined with edge e.

Theorem 1.2.4. *Each connected graph having spanning tree.*

$W_n = C_n + K_1$ are a *wheel* graph. If any cycle in a graph which means a fixed hanging edges thats are adjointed to every vertex is called *Crown* graph. Suppose f_t mean union of one point t in 3-cycles, this is called a *friendship* graph. Prisms graphs appearance is $C_m \times P_2$. A graph in which C_n cycle have edge conclude at vertex having degree only one is said to be *sun* graph and is denoted as S_n.

1.2.8 Operation on graphs

Suppose $\overline{G}$ is *complement* of graph G, this *complement* has denoted by $\overline{G} = (V, V^{(2)} - E)$, two vertices of $G(V,E)$ adjacent vise versa vertices are not adjacent to $G(V,E)$, E_n

(empty graph) is the example, for details we use $\overline{K_n}$.

Suppose E_1, E_2 be sets of disjoint edges, V_1, V_2 be sets of vertex of graphs G_1, G_2 respectively and $G = G_1 \cup G_2$ are *union* of two graph having $E_1 \cup E_2$ and $V_1 \cup V_2$ are sets of edges and vertices respectively.

If G is connected graph then we can write its as nG, where n show the working of isomorphic graph $G(V,E)$.

Theorem 1.2.5. *G is disjointed, then $\overline{\mathrm{G}}$ must be joined.*

Chapter 2

Graph Labelings

Graph *labeling* is bijective mapping which carry graph element to number (usually all integers), if the domain set of all given vertices and edges, then such labeling is known as *total* labeling. If we use only set of vertices for labeling, then the labeling is known as *vertex labeling*, similarly if we use only set of edges for labeling, then the labeling is called *edge labeling* . So, different domains available for different types of labeling. Some types of graph labeling is anti-magic, graceful, harmonic and cordial labeling etc.

We try to discuss many types of labeling in this chapter, some types are as under EMT *labeling*, VMT *labeling*, VAT *labeling* and EMT *labeling*. We also discuss some known results about graph labeling, the existing results will very helpful for the construction of new results.

2.1 Magic labelings

In 1963 Sadláček "introduce concept of labeling by using magic squares for the first time in the field of number theory". Sadláček said that "a graph is magic, if the range consists of real numbers in an edge labeling. The sums around a vertex of all edge labels equal to a constant is called magic-labeling. Also a choice of independent vertex. In magic labeling if small labels are assigned to the edges, then the labeling of $G(V,E)$ is said to be *super-magic* labeling". This concept introduce by stewart in 1966. "The concept n^2 boxes of magic square demonstrate a complete bipartite graph $K_{n,n}$, such graphs are said to be super magic graph for all positive integer $n \neq 2$". Figure demonstrate a magic square of order 3 also Complete Bipartite Graph $K_{3,3}$ of super magic labeling.

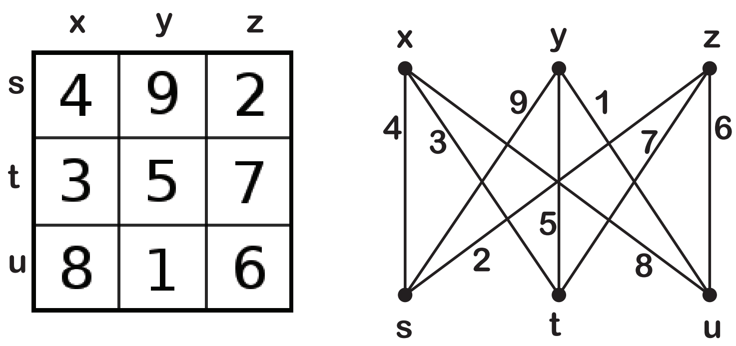


Figure 2.1: Order 3 Magic Square also $K_{3,3}$
Super magic labeling

2.2 Edge magic total labelings

2.2.1 Introduction

In 1970 Rosa and Kotzig [26]" clear the concept of *magic- labeling* in graph labeling G, a function δ: $V \cup E \rightarrow \{1,2,\cdots,|V|+|E|\}$, some integer k satisfying a condition $\delta(x)+\delta(y)+\delta(xy)=k$, $xy \in E$ where k is *magic constant*". Stewart differentiate the practice of graph labeling, Stewart said that the labeling is known as EMT *labeling*. Enomoto et al. [10] " said that if $\delta(V)=\{1,2,......,|V|\}$, such labeling δ is said to be SEMT *labeling* (as per Wallis [46] these labelings are called *strongly edge magic*)". Graph theory EMT *labeling* is also said to be EMT. Figure 3.2 shows EMT labeling of tree having 8 vertices.

2.2.2 Basic Counting

A vertices set of graph G is $\{x_1,x_2,\cdots,x_v\}$ along with edges e. We construct,

$$\sigma_i^j=(i+1)+(i+2)+\cdots+j=i(j-i)+\binom{j-i+1}{2}.$$

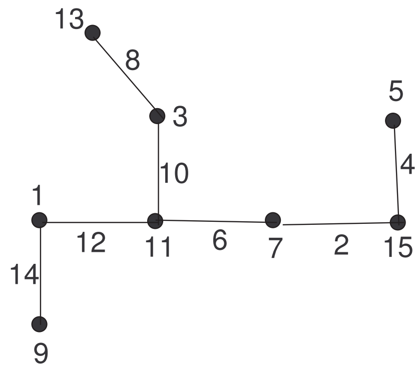


Figure 2.2: Edge-magic total Labeling when 8 vertices on tree having 24 magic constant

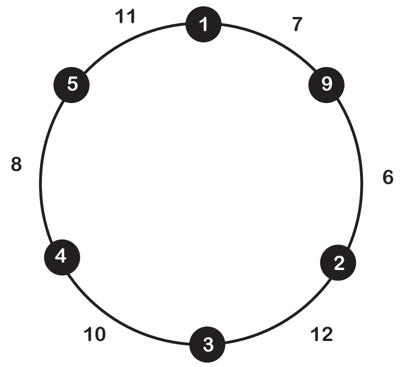


Figure 2.3: Cycle C_6 EMT Labeling

Consider δ edge magic total labeling having h magic-constant.

For adjacent vertices x, y we must labels xy as $h - \delta(x) + \delta(y)$. Labeling Sum of each and every one edge added to labeling Sum of each and every one vertex which is equal to sum of positive integers $v+e$. Vertex labels identify the complete labeling. If $S = \{a_i : 1 \leq i \leq v\}$ is the set of vertex labeling, s are sum of S elements. After that S be able to having least and biggest v, so

$$\sigma_0^v \leq s \leq \sigma_e^{v+e},$$

$$\binom{v+1}{2} \leq s \leq ve + \binom{v+1}{2}.$$

obviously, $\sum_{xy\in E}(\delta(xy)+\delta(x)+\delta(y)) = eh$. That sum enclose one time to each label and $d_i - 1$ times added to the lables of each vertex a_i. Hence

$$he = \sigma_0^{v+e} + \sum (d_i - 1)a_i.$$

If d_i is odd, edges e is even and $v+e \equiv 2 \pmod 4$, then quation is not satisfied .
so we contain:

Theorem 2.2.1. *[39] If $G(V,E)$ has even edges e, $v+e \equiv 2 \pmod 4$, each vertex of $G(V,E)$ contain odd number of degree, then there are no edge-magic-total-labeling in $G(V,E)$.*

2.2.3 Duality

A graph G with vertex p also a edge q on graph labeling δ, Dual labeling δ' is described as

$$\delta'(x) = (p+q+1) - \delta(x)$$

some edges xy;

$$\delta'(xy) = (p+q+1) - \delta(xy)$$

.

2.2.4 Some known results related to EMT labeling

Rosa and Kotzig "show that C_n have EMT labeling of every cases when $n \geq 3$ (We has see [8], [10], [16] and [38]), If n is odd vise versa n copies of disjoint-union of peterson graph P_2 has edge magic total labeling (EMTL)". They also added $n = 1,2,3,5$ or 6 if and only if K_n has EMT labeling (see [10], [9], [27]) , they also explain that all trees has edge magic total labelings (EMTL). Wallis [47] "specify all kinds of edge-magic total labelings (EMTL) of complete graphs". Wallis confirm that complete bipartite graphs, paths, cycles having single vertex connected to single edge and crowns are the examples of EMT labeling. Ringle, Enomoto, Nakamigana and Llado [10] "confirm that all the cases of complete bipartite graphs has edge magic total labeling (EMTL)". Ringle, Enomoto, Nakamigana and Llado [10] also explain that when $n \equiv 3 \pmod 4$, then wheels W_n have not edge-magic-total labeling (EMTL) while other all cases of wheels have edge magic total labeling (EMTL). This assumption has verified Philip, Rees, and Wallis [37] "its only possible when $n \equiv 0,1 \pmod 4$, similarly if, $n \equiv 6$ (mod 8) then this case has verified Simanjuntak, Slamin, Lin, Bača and Miller [41]. Assumption of all cases has confirmed by Fukuchi [12]".

2.2.5 Some known results related to super-EMT labeling

A spacial case of super-EMT labeling graph are EMT labeling δ, G having $\delta(V(G)) = \{1, 2, \cdots, |V|\}$.

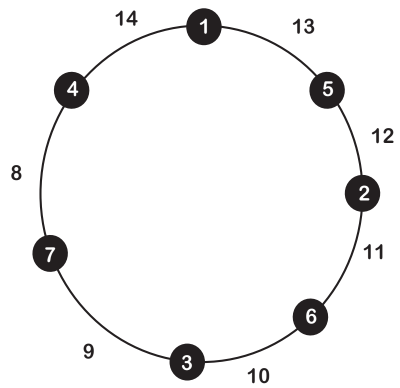


Figure 2.4: $C_6 Superedge-magic-totallabeling(SEMTL)$

Enomoto [10] expected and considered such a labeling. They verify the following: (Wallis [46] describe these *strongly edge magic* labeling), consider n is odd vise versa C_n is super-EM labeling; if, $n = 1$ or $m = 1$ then Caterpillar is SEML, similarly if, $n = 1, 2$ or 3 then K_n is SEML, they confirmed that graph G is SEML if graph having vertices p, edges q and $q \leq 2p - 3$. Enomoto [10] stated that each tree is SEML. [28] Shan and Lee verify the statement of Enomoto [10] with computer search of trees maximum to 17 vertices. Kotzig. [26] and Rosa. [27] "verify if n is odd then for nK_2

EMTL show that this labeling is a SEML". [41] "Slamin confirmed if $n = 3,4,5$ or 6 vise versa friendship graph is SEML. [11] Fukuchi proved that if n is odd at least 3 then generalized Petersen graph $(P(n,2))$ is SEML". [7] Ngurah and Baskoro explain if n is even also $n \geq 4$ then nP_3 is SEML.

2.3 Vertex-Magic-Total(VMT) Labeling

2.3.1 Introduction

In 1999 Slamin, Wallis, MacDougall and Miller [32] "introduced the concept of VMT*labeling*". For each vertex x, $\delta(x)+\sum\delta(xy) = h$, where h is constant, when sum are use for all adjacent vertices y to x, so k is called *magic constant* for the used of graph labeling δ. At vertex x sum of all labels under labeling of δ by $wt_{\delta}(x)$ is called *weight* of vertex x, we will want $wt_{\delta}(x) = h$ for all x. In this graph theory vertex-magic total labeling (VMTL) is also called VMT. Figure, 2.5 demonstrate an VMT labeling of cycle C_5. Each graph

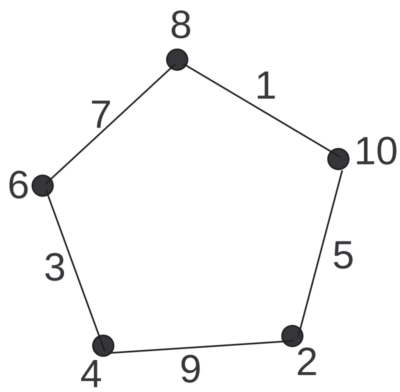


Figure 2.5: C_5 VMT labeling having 16 (magic constant)

is not vertex-magic-total labeling(VMTL), example is K_2, since $\delta(x) \neq \delta(y)$ therefore indicate $\delta(x)+\delta(xy) \neq \delta(y)+\delta(xy)$, thus in this case no labeling is possible. In the same way, any vertex x in graph $G(V,E)$ must has $\delta(x) = h$ so exclusion of repetitive label mean the single nearby vertex. Hence following Theorem holds.

Theorem 2.3.1. *[46] If $G(V,E)$ has nearby isolated-edge, or two nearby vertice isolated, then $G(V,E)$ is not (VMT).*

Theorem 2.3.2. *[46] "If G be graph and $\lambda : V \cup E \to \{0,1,\cdots,v+e-1\}$ are VMT labeling then 0 cannot be label of any vertex having degree 1".*

In graph theory, the history of graph labeling is very vast, most likely example in graph labeling is a Four Color Problem: find minimum numbers of 'colors' required for a surface of graph embedded to assign each vertex a 'color' and two adjacent vertices are not receive the same 'color'. In this example notable thing is that 'colors' used as a label but we have used letters and numbers without changing the necessary nature of the problem, in magic graph theory labels are always integers, modular integers, prime numbers or a group of elements.

The idea of vertex magic total labeling (VMTL) be somewhat recommended by following problem: the olympic emblem depending on five overlaps ring having 9 regions. If we hand over a pension money to self-effacing people, they asked deposit this money to every region. Strategy must allow the hand over money to take in any rings. Put the amounts $\$1, \$2, \cdots, \$9$ in the nine region, so amounts in every one ring must equal. Problem are understand for the labeling of 5 vertices path. For this one solution found figure is below.

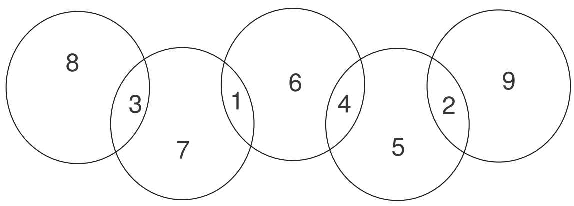


Figure 2.6: Solution found in olympic ring

2.3.2 Basic Counting of VMT Labeling

Consider the vertex labels sum is s_v and edge labels sum is s_e in VMT labeling δ of $G(V,E)$. The number of labels are started from $\{1,2,\cdots,v+e\}$ then sum of labeling is

$$s_v + s_e = \binom{v+e+1}{2}.$$

For all vertex x_i we contain $\delta(x_i) + \sum \delta(x_i y) = h$. Sum of all vertices x_i is equal to the addition of every vertex label one time and every edge label two times, hence

$$s_v + 2s_e = vh.$$

by adding two equations

$$s_e + \binom{v+e+1}{2} = vh.$$

2.3.3 Known results of VMT(vertex-magic-total) labeling

Above conversation show that all graphs have not a VMT labeling. Hence, study of graph categorization of vertex-magic total labeling (VMTL) is usual great affection. Slamin, Wallis, MacDougall and Miller, [32] obtains the following results recently:

Theorem 2.3.3. *C_n are n-cycle have a VMT labeling, for every $n \geq 3$.*

The every P_n path and $n \geq 3$ is VMT labeling, also every VMT labeling, of P_n is derive from the VMT labeling, of cycle C_n. Also they proved that complete bipartite graphs $K_{m,n}$ don't contain vertex-magic-total (VMT) labeling if $n > m+1$. The assumption to $K_{m,m+1}$ for m and K_n for all $n \geq 3$ declare VMT labeling. Phillips, Rees and Wallis [36] proved the following.

Theorem 2.3.4. *[36] "$K_{m,n}$ have vertex magic total (VMT) labeling vise versa m, n vary with by mostly 1".*

The concluding assumption was proved Lin and Miller that a case $n \equiv 0(mod\ 4)$ but the remaining cases be explain by Slamin, Wallis, MacDougall and Miller, [32]. McQuillan and Smith [34] "contain exposed that for odd n, K_n have vertex magic total labeling(VMTL) having h as magic constant vise versa $\frac{n(n^2+3)}{4} \leq h \leq \frac{n(n+1)^2}{4}$".

Miller, MacDougall and Bača [31] "has explained the generalized Petersen graphs $P(n,k)$ state VMT labeling, where $k \leq \frac{n}{2} - 1$ for only even n. The assumption for each $P(n,k)$ have VMT labeling, if $k \leq \frac{n-1}{2}$ also each prisms $C_n \times P_2$ has VMT". Slamin, Miller and Bača [6] demonstrate these assumptions. MacDougall et.al [32, 33, 30] "show that wheel W_n contain a VMT labeling vise versa $n \leq 11$, friendship graph are VMT labeling vise versa digit of triangle is 3 at most".

Wallis [45] "confirm that if regular graph has VMT labeling then we create a labeling of VMT labeling from a given labeling. A VMT labeling δ for a G, define a new function $\delta' : V \cup E \to \{1,2,....,v+e\}$ as

$$\delta'(x) = v+e+1-\delta(x)$$

at vertex x,

$$\delta'(xy) = v+e+1-\delta(xy)$$

for all edges xy.

If graph $G(V,E)$ is regular having degree r, where h is magic- constant for labeling δ then h' magic constant is known as

$$h' = (r+1)(v+e+1)-h$$

This new VMT labeling δ' is called *dual* of the labeling δ".

Froncek et al. [13] "show $K_5 \times C_{2n+1}$ and $C_n \times C_{2m+1}$ declared vertex magic total (VMT) labeling". [25] Kovář "show various results about certain products of VMT labeling of a regular graphs. If graph G, $2r+1$ regular VMT (vertex magic total) labeling graph is divided into two regular graphs, $r+1$ and r-regular, then for even n,

$G\times K_5$, $G\times C_n$ have VMT labeling".

Vertex magic total labeling of disconnected graph has not broadly discuss. Disconnected graphs are those graph having components has regular as well as isomorphic, Wallis [46] proved following result:

Theorem 2.3.5. *[46] "Consider a regular graph $G(V,E)$ having degree r, admits VMT labeling, then*
(i) When r is even, then nG has a VMT labeling, when n is odd positive integer.
(ii)When r is odd, then nG has vertex-magic, when n is positive integer".

The above theorems apprehension then VMT labeling of disconnected graphs, its components have regular graph and isomorphic graph. The components of disconnected graphs have not regular graphs, [15] "Gray show the VMT labeling of star on 3 vertices $tK_{1,2}$ for t copies".

2.4 Antimagic total labelings(ATL)

2.4.1 Vertex-anti-magic-total-labeling

In 2000 Baca [2] "introduced a concept of (a,d)-*VAT* labeling. A graph is called (a,d)-*VAT* labeling, if here exist a bijective mapping $\delta : V\bigcup E\rightarrow\{1,2,....,v+e\}$, wherever e is size and v is order of graph G, if $wt(x)=\delta(x)+\sum\limits_{e=xy}\delta(xy)$ where x is adjacent to y for every $x\in V(G)$ then $\{a,a+d,\cdots,a+(v-1)d\}$". If we start labeling from a set $\{1,2,\cdots,v\}$ such labeling is said to be super (a,d)-*VAT* labeling. In 2003 Bača [2] "established the idea of (a,d)-VAT labeling. Let order of G is v and size of G is e than (a,d)-VAT *labeling* only exist if bijective mapping $\delta : V\cup E\rightarrow\{1,2,\cdots,v+e\}$, hence $\{\ delta(x)+\sum\limits_{xy\in E(G)}\delta(xy)\}$, wherever sum of all vertices y is nearby x, for $x\in V(G)$ is $\{a,a+d,\cdots,a+(v-1)d\}$. If we start vertex labeling from a set $\{1,2,\cdots,v\}$, such labeling is called *super (a,d)-VAT labeling*". In 2005 Sugeng, Miller, Lin, and Bača "proved the C_n contain a super (a, d)-VAT labelings vise versa for odd n when $d=0$ or 1 or 2". In 2010 [44] M. Tariq Rahim, Kashif Ali, Imran Javaid, "provide us disjoint union of (a, d)-VAT labeling of cycle and sun graphs". In 2011 Sugeng and Bong "constructed super (a, d)-VAT labeling on circulant-graph $C_n(1,2,3)$, when $d = 0,1,2,3,4,8$".

Suppose G be graph, Harary graph has a graph create from cycle C_p along with joining of two vertices having distance t in C_p and is denoted by C_p^t where $t \geq 2$ and $n \geq 4$. In 2010 [44] M. Tariq Rahim, Kashif Ali, Imran Javaid, "provide us disjoint union of (a,d)-EAT- labeling of cycle and Harary graphs". In 2010 [18] M.Hussain, Kashif Ali, M. T. Rahim, Edy Tri Baskoro "proved many (a, d)-VAT labeling of Harary graphs also disjoint union for k isomorphic copies on Harary graphs when p is odd and $p \geq 5$". In [1] C. Balbuena, E. Barker, K. C. Das, Y. Lin, M. Miller, J. Ryan, Slamin, K. Sugeng and M. Tkac "provide us (a,d)-VAT-labeling on Harary graph". In 2012 [18] Muhammad Hussain, E.T. Baskoro, and Kashif Ali "provide us super-EAT labeling and super-VAT labeling of Harary graphs".

In [2] Bača demonstrate following results:

Theorem 2.4.1. *[2] "Each super-VAT graph has $(a,1)$-VAT-labeling".*

Theorem 2.4.2. *[2] "Each (a,d)-VAT-labeling of G is $(a+e+1,d+1)$ VAT-labeling, if $d > 1$ then each anti-magic graph $G(V,E)$ is $(a+v+e,d-1)$ VAT-labeling".*

Theorem 2.4.3. *For a, d, cycle and path has (a,d) VAT-labeling .*

Using the above result in [2, 3, 42] Bača get many results of (a,d)-VAT-labelings, for generalization of Petersen graph. Lin, Baca, Miller and Sugeng [42] demonstrate following results:

Theorem 2.4.4. *If $d = 0$ or $d = 1$ or $d = 2$, also n has odd vise versa C_n has super (a,d)-VAT-labeling.*

Theorem 2.4.5. *[3] "For $d = 2$ and $n \geq 3$ is odd, or $d = 3$ and $n \geq 3$ vise versa P_n has super-(a,d)-VAT labeling".*

Miller, Baca, Lin and Sugeng [42] demonstrate that there is no even order tree has super $(a,1)$-VAT labeling. Slamin [29] hold that when $n > 20$ then W_n is no (a,d)-VAT labeling. In [43] Tezer and Cahit demonstrate Following results:

Theorem 2.4.6. *If $a \geq 3$ and $d \geq 6$ then P_n and C_n have not (a,d)-VAT Labeling.*

Simanjuntak, Ngurah and Baskoro [35] "give us idea of generalized Petersen, $P(n,m)$ graph in (a,d)-VAT labeling, so all cases are: $(a,d) = (8n+3,2)$ when $n \geq 3$ and $1 \leq m \leq \lfloor \frac{n-1}{2} \rfloor$; $(a,d) = (\frac{15n+5}{2},1)$ when Odd $n \geq 5$ and $m = 2$; $(a,d) = (\frac{21n+5}{2},1)$ when Odd $n \geq 5$ and $m = 2$; $(a,d) = (\frac{15n+5}{2},1)$ when Odd $n \geq 7$ and $m = 3$; $(a,d) = (\frac{21n+5}{2},1)$ when Odd $n \geq 7$ and $m = 3$; $(a,d) = (\frac{15n+5}{2},1)$ and $(a,d) = (\frac{21n+5}{2},1)$ when Odd $n \geq 9$ and $m = 4$."

2.4.2 Known results on (a,d)-VAT-Labeling

Edy Tri Baskoro, Muhammad Hussain, M. T. Rahim and Kashif Ali [18, 17]"describe a *(a,d)-VAT-labeling* also *$Super-(a,d)$-VAT labeling*, there exist injection mapping $\delta : V \bigcup E \to \{1,2,\cdots,v+e\}$, for this $\{\delta(x)+\delta(xy)+\delta(y)|xy \in E\}$ are $\{a,a+d,\cdots,a+(v-1)d\}$. If start labeling from a set $\{1,2,\cdots,v\}$, such labeling λ is said to be *super-(a,d)-VAT(vertex-antimagic total) labeling*". Edy Tri Baskoro, Muhammad Hussain, M. T. Rahim and Kashif Ali [18, 17] demonstrate following results:

Theorem 2.4.7. *For $t \geq 2$ and $p \geq 5$, $G \cong C_p^t$ admit $(8p+3,1)$ super-VAT labeling, when $p \neq 2t$.*

Theorem 2.4.8. *Let $t \geq 2$, $n \geq 5$ and $k \geq 2$, $G \cong kC_p^t$ admits $(8p+3,1)$ super-VAT labeling, when $p \neq 2t$.*

Theorem 2.4.9. *If even $t \geq 2$, $p \geq 4$ then $G \cong C_p^t$ is not admits (a,d) super-VAT labeling, given d is even also $p \neq 2t$.*

Theorem 2.4.10. *For odd $t \geq 2$, $p \geq 5$ also $G \cong C_p^t$ admits $(\frac{13p+9}{2},3)$ super-VAT labeling, also given $p \neq 2t$.*

Theorem 2.4.11. *Let $p \geq 5$, $t \geq 2$ then $G \cong C_p^t$ admits $(7p+3,1)$ VAT labeling also given $p \neq 2t$.*

Theorem 2.4.12. *Let $n \geq 5$, $k \geq 2$ also $t \geq 2$ then $G \cong C_p^t$ admit $(7nk+3,1)$ VAT labeling, also given $p \neq 2t$.*

M. Tariq Rahim, Kashif Ali and Imran Javaid [44] "studied the relationship between some families of graphs". J.Joy Priscilla and K.Thirusangu [24] "studied the r-copies of Vertex-Antimagic Total Labelings. They also demonstrate some results with the help of results stated in [18, 17]".

Theorem 2.4.13. *[24] For odd $p \geq 5$, $t \geq 2$ also $r > 0$ then rC_p^t is antimagic labeling.*

Theorem 2.4.14. *[24] For odd $p \geq 5$, $t \geq 2$ also $r > 0$ then rC_p^t admit $(\frac{12r+1p}{2}, 3)$ VAT(vertex antimagic total) labeling.*

Theorem 2.4.15. *[24] Let $p \geq 5$, $t \geq 2$ also $r > 0$ then rC_p^t is antimagic-labeling given $p \neq 2t$ with vertex weight $W_k \doteq \{(9k+3r-5)p+3, (9k+3r-5)p+4, \cdots, (9k+3r-5)p+p+2\}$, for every Kth copy, here $k \doteq 1,2,3,\cdots,r$.*

Theorem 2.4.16. *[24] Let $p \geq 5$, $t \geq 2$ also $r > 0$ then rC_p^t is antimagic-labeling given $p \neq 2t$ with vertex weight $W_k \doteq \{(3k+6r-2)p+3, (3k+6r-2)p+4, \cdots, (3k+6r-2)p+p+2\}$, for every Kth Copy, here $k \doteq 1,2,3,\cdots,r$.*

Chapter 3

Total Labeling on Harary Graph

3.1 Introduction

Suppose G be graph, Harary graph has a graph create from cycle C_p along with joining of two vertices having distance t in C_p and is denoted by C_p^t where $t \geq 2$ and $n \geq 4$. Figure 3.1 shows C_7^2 Harary graph.

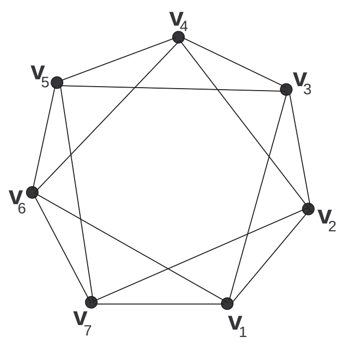


Figure 3.1: C_7^2.

In 1999 Slamin, Wallis, MacDougall and Miller [32] "introduce the idea of VAT*(vertex-magic-total) labeling*". In 2003 Bača [2] "establish the notion of (a,d)-VAT labeling. If we start vertex labeling from a set $\{1,2,\cdots,v\}$, such labeling is called *super* (a,d)-*VAT labeling*". Bača [2] demonstrate following results:

Theorem 3.1.1. *Each super VAT graph has* $(a,1)$*-VAT-labeling.*

Theorem 3.1.2. *For a, d, cycle and path has* (a,d) *VAT labeling .*

M.Hussain, Kashif Ali, M.T. Rahim, E. T. Baskoro [17, 18] describe a super and (a,d)*-VAT* labeling of following results:

Theorem 3.1.3. *[18] For* $t \geq 2$ *and* $p \geq 5$, $G \cong C_p^t$ *admit* $(8p+3,1)$ *super-VAT labeling, when* $p \neq 2t$.

Theorem 3.1.4. *[17] Let* $p \geq 5$, $t \geq 2$ *then* $G \cong C_p^t$ *admits* $(7p+3,1)$ *VAT labeling also given* $p \neq 2t$.

Theorem 3.1.5. *For $p \geq 5$ where p is odd, $t \geq 2$, also $G \cong C_p^t$ admit super $(\frac{17p+5}{2}, 0)$ VAT-labeling, when $p \neq 2t$.*

3.2 VAT(vertex-antimagic-total) and SVAT(super vertex anti-magic total) Labeling on Harary Graph:

In this section we construct (a,d)-VAT and Super-(a,d) SVAMT labeling also holds in Harary graph C_n^t as well as for disjoint union of K identical of Harary graphs copies.

Lemma 3.2.1. *[18]For each $m \geq 5$, $t \geq 2$, also compute that $G \cong C_m^t$ admit (a,d)-VAT then $d < 9$ where $m \neq 2t$ and $d < 6$ where $m = 2t$.*

Proof. Consider $m = |V(G)|$, $n = |E(G)|$. Suppose here exists bijective mapping

$$\delta : V(G) \cup E(G) \to \{1, 2, \cdots, |V| + |E|\}$$

has also (a,d)-VAT

$W = \{\delta(a) + \sum \delta(ab) : ab \in E(G)\}$

$= \{a, a+d, \cdots, a+(m-1)d\}$ be a weights of vertex set.

It is simple in the direction to observe that the smallest probable vertex weight within (a,d)-VAT labeling be

$$1 + (m+1) + (m+2) + (m+3) + (m+4) = 4m + 11.$$

If $m \neq 2t$, then the maximum weight of a vertex is no more than

$$m + 3m + (3m-1) + (3m-2) + (3m-3) = 13m - 6.$$

Thus, we have

$$a + (m-1)d \leq 13m - 6,$$

and

$$d \leq \frac{9m-17}{m-1} < 9.$$

If $m = 2t$,

$$1+(m+1)+(m+2)+(m+3)+(m+4) = 4m+11.$$

after that greatest vertex weight is no other than

$$m+\frac{5m}{2}+(\frac{5m}{2}-1)+(\frac{5m}{2}-2)-4) = \frac{17m}{2}-3$$

Hence, we have

$$d \leq \frac{11m-20}{2(m-1)} < 6.$$

Theorem 3.2.2. *For $n \geq 5$ where n is odd then $t \geq 2$, also $G \cong C_n^t$ admits a $(6n+5,3)$ VAT labeling, where $n \neq 2t$.*

Proof.

Consider that $a = |V(G)|$ also $b = |E(G)|$, then we denotes vertex set and edge set of G as follows:

$$V = \{v_l^m : 1 \leq l \leq n, 1 \leq m \leq k\},$$

$$E = \{v_l^m v_{l+1}^m : 1 \leq l \leq n, 1 \leq m \leq k\} \cup \{v_l^m v_{l+t}^m \ : \ 1 \leq l \leq n, 1 \leq m \leq k\}.$$

Where all indices has use in mod n.

We construct a labeling $\delta : V \cup E \rightarrow \{1, 2, \cdots, a+b\}$ as:

$$\lambda(v_l^m) = (3a+4-3l-m)k-m-1, \qquad 1 \leq l \leq n,\ 1 \leq m \leq k.$$

$$\lambda(v_l^m v_{l+t}^m) = (3l-4+m)k+m+1, \qquad 1 \leq l \leq n,\ 1 \leq m \leq k.$$

$$\lambda(v_l^m v_{l+1}^m) = \begin{cases} \frac{(a+b)}{2}+3k-3(m-1)+3(l-1), & \text{for } , l = t, t+2, \cdots, n-1. \\ \\ 3k-3(m-1)+3(l-1), & \text{for } , l = t+1, t+3, \cdots, n-2. \end{cases}$$

At the end we contain set of vertices, $\{v_l^m : 1 \leq l \leq n,\ 1 \leq m \leq k\}$, now that the vertex v_t have a weight $6n+5$ and the set of all vertices are depending on consecutive integers $6n+5, 6n+8, 6n+11, \cdots, 7n+4$ having $a = 6n+5$ and $d = 3$. Hence δ has a $(6n+5, 3)$ VAT(vertex antimagic total) labeling, when $n \neq 2t$.

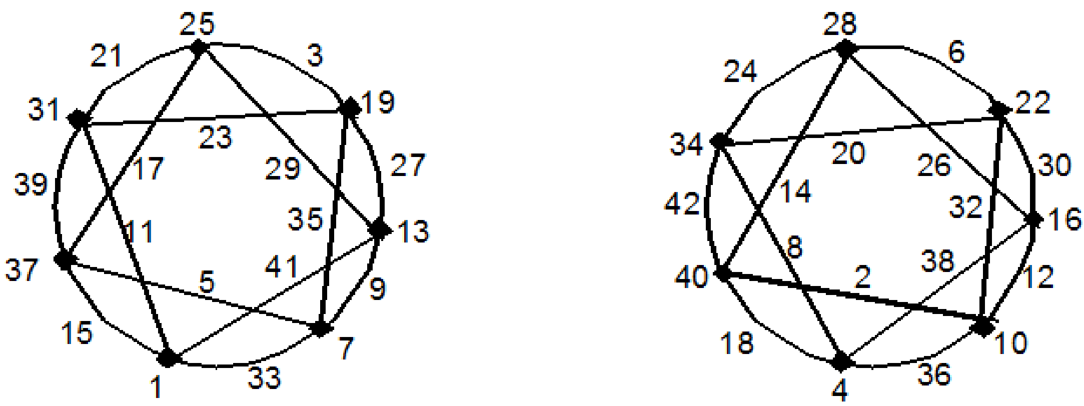


Figure 3.2: $2C_7^2$ of $(89, 3)$ of VATL.

Corollary 3.2.3. *For $n \geq 5$ where n is odd, $t \geq 2$, also $G \cong C_n^t$ admit $(6n+3, 3)$ VAT(vertex antimagic total) labeling, where $n \neq 2t$.*

Theorem 3.2.4. *For $n \geq 5$ where n is odd, $t \geq 2$, also $G \cong C_n^t$ admit $(6n+4, 3)$ VAT labeling, where $n \neq 2t$.*

Proof.

Consider that $a = |V(G)|$ also $b = |E(G)|$ and we denote vertex and edge set of G as follow:

$$V = \{v_l^m : 1 \leq l \leq n, 1 \leq m \leq k\},$$

$$E = \{v_l^m v_{l+1}^m : 1 \leq l \leq n, 1 \leq m \leq k\} \cup \{v_l^m v_{l+t}^m \ : \ 1 \leq l \leq n, 1 \leq m \leq k\}.$$

All indices has use in mod n.

We construct labeling $\delta : V \cup E \to \{1, 2, \cdots, a+b\}$ as:

$$\lambda(v_l^m) = (3a + 4 - 3l - m)k - m, \qquad 1 \leq l \leq n, \ 1 \leq m \leq k.$$

$$\lambda(v_l^m v_{l+t}^m) = (3l - 4 + m)k + m, \qquad 1 \leq l \leq n, \ 1 \leq m \leq k.$$

$$\lambda(v_l^m v_{l+1}^m) = \begin{cases} \frac{(a+b)}{2} + 3k - 3(m-1) + 3(l-1), & \text{for } , l = t, t+2, \cdots, n-1. \\ \\ 3k - 3(m-1) + 3(l-1), & \text{for } , l = t+1, t+3, \cdots, n-2. \end{cases}$$

At the end we contain set of vertices, $\{v_l^m : 1 \leq l \leq n, \ 1 \leq m \leq k\}$, now that the vertex v_t have a weight $6n+5$ and the set of all vertices are depending on consecutive integers $6n+5, 6n+8, 6n+11, \cdots, 7n+4$ having $a = 6n+5$ and $d = 3$. hence δ has a $(6n+5, 3)$ VAT(vertex antimagic total) labeling, where $n \neq 2t$.

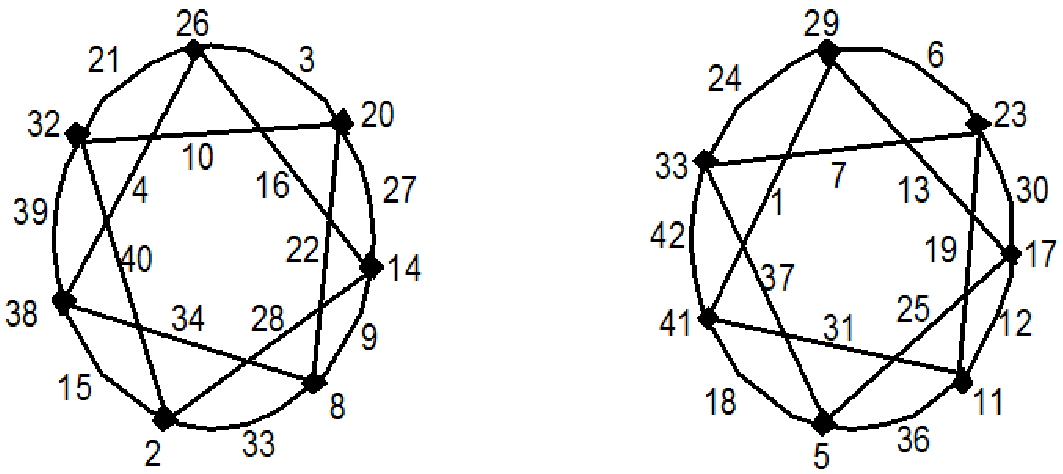


Figure 3.3: $2C_7^2$ of $(88, 3)$ of VATL.

Corollary 3.2.5. *For $n \geq 5$ where n is odd then $t \geq 2$, also $G \cong C_n^t$ admit $(6n+2,3)$ VAT(vertex antimagic total) labeling, where $n \neq 2t$.*

Theorem 3.2.6. *For $n \geq 5$ where n is odd and $t \geq 2$, also $G \cong C_n^t$ admit $(\frac{12n+6}{2},3)$ VAT labeling, where $n \neq 2t$.*

Proof.

Consider that $a = |V(G)|$ also $b = |E(G)|$ and we denote vertex and edge set of G as follow:

$$V = \{v_l^m : 1 \leq l \leq n, 1 \leq m \leq k\},$$

$$E = \{v_l^m v_{l+1}^m : 1 \leq l \leq n, 1 \leq m \leq k\} \cup \{v_l^m v_{l+t}^m \ : \ 1 \leq l \leq n, 1 \leq m \leq k\}.$$

All indices has use in mod n.

We construct labeling $\delta : V \cup E \to \{1,2,\cdots,a+b\}$ as:

$$\lambda(v_l^m) = (3a+4-3l-m)k-m+1, \qquad 1 \leq l \leq n,\ 1 \leq m \leq k.$$

$$\lambda(v_l^m v_{l+t}^m) = (3l-4+m)k+m+1, \qquad 1 \leq l \leq n,\ 1 \leq m \leq k.$$

$$\lambda(v_l^m v_{l+1}^m) = \begin{cases} \frac{(a+b)}{2}+2k-3(m-1)+3(l-1), & \text{for } ,l = t,t+2,\cdots,n-1. \\ \\ 3k-3(m-1)+3(l-1)-2, & \text{for } ,m = t+1,t+3,\cdots,n-2. \end{cases}$$

At the end we contain set of vertices, $\{v_l^m : 1 \leq l \leq n,\ 1 \leq m \leq k\}$, now that the vertex v_t have a weight $\frac{12n+6}{2}$ and the set of all vertices are depending on consecutive integers $\frac{12n+6}{2}, \frac{12n+9}{2}, \frac{12n+12}{2}, \cdots, \frac{13n+5}{2}$ having $a = \frac{12n+6}{2}$ and $d = 3$. hence δ has a $(\frac{12n+6}{2},3)$ VAT(vertex antimagic total) labeling, when $n \neq 2t$.

Theorem 3.2.7. *For each odd $n \geq 5$, $t \geq 2$, also $G \cong C_n^t$ admit $(\frac{12n+8}{2},3)$ VAT labeling, where $n \neq 2t$.*

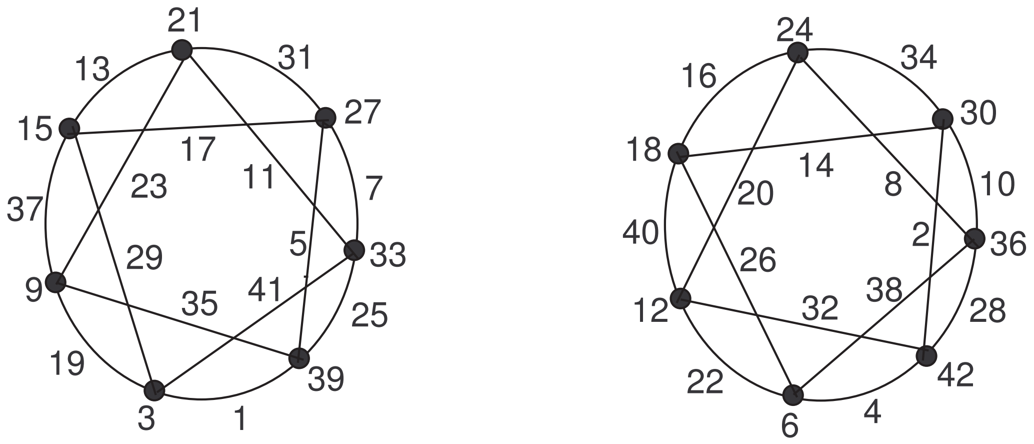


Figure 3.4: $2C_7^2$ of $(87,3)$ of VATL.

Proof.

Consider that $a = |V(G)|$ also $b = |E(G)|$ and we denote vertex and edge set of G as follow:

$$V = \{v_l^m : 1 \leq l \leq n, 1 \leq m \leq k\},$$

$$E = \{v_l^m v_{l+1}^m : 1 \leq l \leq n, 1 \leq m \leq k\} \cup \{v_l^m v_{l+t}^m \ : \ 1 \leq l \leq n, 1 \leq m \leq k\}.$$

All indices has use in mod n.

We construct labeling $\delta : V \cup E \rightarrow \{1, 2, \cdots, a+b\}$ as:

$$\lambda(v_l^m) = (3a + 4 - 3l - m)k - m, \qquad 1 \leq l \leq n,\ 1 \leq m \leq k.$$

$$\lambda(v_l^m v_{l+t}^m) = (3l - 4 + m)k + m, \qquad 1 \leq l \leq n,\ 1 \leq m \leq k.$$

$$\lambda(v_l^m v_{l+1}^m) = \begin{cases} \frac{(a+b)}{2} - 3k - 3(m-1) + 3(l-1), & \text{for } ,l = t, t+2, \cdots, n-1. \\ \\ a + b - 3k + 3(l-1) - 3(m-1), & \text{for } ,l = t+1, t+3, \cdots, n-2. \\ 3k - 3(m-1), & \text{for } ,l = n, m = 1, 2, \cdots, n. \end{cases}$$

At the end we contain set of vertices, $\{v_l^m : 1 \leq l \leq n,\ 1 \leq m \leq k\}$, now that the vertex v_t have a weight $\frac{12n+8}{2}$ and the set of all vertices are depending on consecutive integers $\frac{12n+8}{2}, \frac{12n+11}{2}, \frac{12n+14}{2}, \cdots, \frac{13n+7}{2}$ having $a = \frac{12n+8}{2}$ and $d = 3$. hence δ has a $(\frac{12n+8}{2}, 3)$-VAT(vertex-antimagic-total) labeling, where $n \neq 2t$.

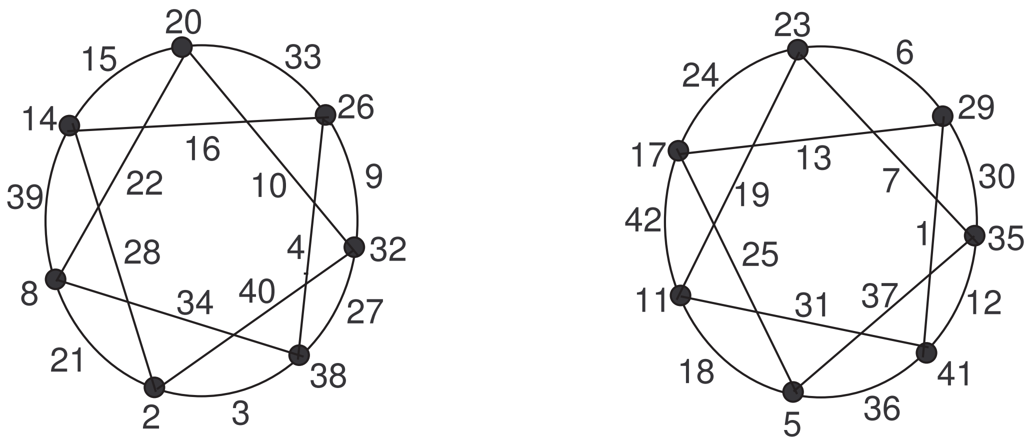


Figure 3.5: $2C_7^2$ of $(88,3)$ of VATL.

Theorem 3.2.8. *For odd $n \geq 5$, $t \geq 3$, also $G \cong C_n^t$ admit Super $(\frac{16n+6}{2}, 1)$(VAMT) labeling, where $n \neq 2t$.*

Proof.

Consider that $a = |V(G)|$ also $b = |E(G)|$ and we denote vertex and edge set of G as follow:

$$V = \{v_l^m : 1 \leq l \leq n, 1 \leq m \leq k\},$$

$$E = \{v_l^m v_{l+1}^m : 1 \leq l \leq n, 1 \leq m \leq k\} \cup \{v_l^m v_{l+t}^m\ :\ 1 \leq l \leq n, 1 \leq m \leq k\}.$$

All indices has use in mod n.

We construct labeling $\delta : V \cup E \to \{1, 2, \cdots, a+b\}$ as:

$$\lambda(v_l^m) = (l-1)k + m, \qquad 1 \leq l \leq n,\ 1 \leq m \leq k.$$

$$\lambda(v_l^m v_{l+t}^m) = b-(l-1)k-(m-1), \qquad 1 \le l \le n,\ 1 \le m \le k.$$

$$\lambda(v_l^m v_{l+1}^m) = \begin{cases} a+b+(m-1)-1, & \text{for } ,l=n. \\ \\ a+b-3k+(m-1)-(l-1), & \text{for } ,l=t,t+2,\cdots,n-1. \\ a+b-lk+(m-1)+l, & \text{for } ,l=t+1,t+3,\cdots,n-2. \end{cases}$$

At the end we contain set of vertices, $\{v_l^m : 1 \le l \le n,\ 1 \le m \le k\}$, now that the vertex v_t have a weight $\frac{16n+6}{2}$ and the set of all vertices are depending on consecutive integers $\frac{16n+6}{2}, \frac{16n+7}{2}, \frac{16n+8}{2}, \cdots, \frac{17n+5}{2}$ having $a = \frac{16n+7}{2}$ and $d = 1$. hence δ has a $(\frac{16n+6}{2}, 1)$ VAT(vertex antimagic total) labeling, when $n \neq 2t$.

Figure 3.8 shows the $(\frac{16n+6}{2}, 1)$-VAT labeling of C_9^3.

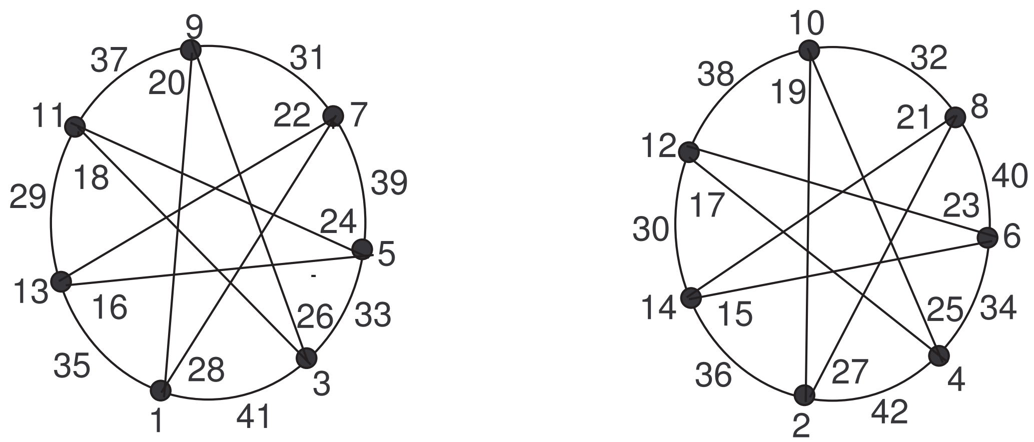


Figure 3.6: $2C_7^3$ of $(115, 1)$ of VATL.

Corollary 3.2.9. *For each odd $n \ge 5$, $t \ge 3$, also $G \cong C_n^t$ admit $(\frac{12n+6}{2}, 1)$ VAT(vertex antimagic total) labeling, when $n \neq 2t$.*

Chapter 4

Conclusions and Open Problems

4.1 Conclusions

In this thesis, we found some new results related to vertex-antimagic total (VAMT) labeling and Super (VAMT) labeling on Harary Graph when $d = 1,3$ We are incapable to create super and (a,d)-(VAMT) labeling of Harary graph for $d = 1,3$. Also, (a,d)-(VAMT) labeling as for as super (a,d)-(VAMTL) for non-isomorphic Harary graph copies have still unknown for $d = 1,3$.
Followings are the future direction in this regard:

4.2 Open problems

4.2.1. *For $n \geq 5$ where n is odd, $k,t \geq 2$, also $G \cong KC_n^t$ admit (a,d) (VAMT) and super (a,d) (VAMT) labeling for other values of d, where $n \neq 2t$.*

Chapter 6

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