Solution of Differential Equations by Lindstedt-Poincare Techniques

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CIIT/FA14-BSMATH-003/LHR

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By

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I **Shakeel Tariq** registration number **CIIT/FA14-BSMATH-003/LHR** hereby declare that I have produced the work presented in this report, during the scheduled period of study.

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Certificate

It is certified that **<u>Shakeel Tariq (CIIT/FA14-BSMATH-000/LHR)</u>** has carried out all the work related to this report under my supervision at the Department of Mathematics, COMSATS University Islamabad, Lahore Campus, and the work fulfills the requirement for award of BS degree.

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DEDICATION

To

Father and Mother

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Preface

Many of the problems facing physicists, engineers, and applied mathematicians involve such difficulties as nonlinear boundary conditions at complex known or unknown boundaries that preclude solving them exactly. Consequently, solutions are approximated using numerical techniques, analytic techniques, and combinations of both. Foremost among the analytic techniques are the systematic methods of perturbations (asymptotic expansions) in terms of a small or a large parameter or coordinate.

Although the techniques are described by means of examples that start with simple ordinary equations that can be solved exactly and progress toward complex partial-differential equations, the material is concise and advanced and therefore is intended for researchers and advanced graduate students only. The purpose of this techniques, however, is to present the material in an elementary way that makes it easily accessible to advanced undergraduates and first-year graduate students in a wide variety of scientific and engineering fields. As a result of teaching perturbation methods for eight years to first-year and advanced graduate students at Virginia Poly-techniques and amplified their description considerably. The techniques are described by means of simple examples that consist mainly of algebraic and ordinary-differential equations.

The material in Chapters 1 and 2 is about solution of differential equations by different perturbation Methods. Chapter 1 discusses “The Straightforward Expansions” method.

Contents

CHAPTER 1

The Straightforward Expansion

In this chapter, we find approximate solution of differential equations that depend on a small parameter. To find the approximate solution, we use the Straightforward expansion method which is based on asymptotic expansion. The solutions of differential equations are represented as an asymptotic expansion in terms of small parameter.

1.1 Example 1

Consider that

$$\ddot{u}+u+\varepsilon u^3=0 \tag{1.1.1}$$

initial conditions are

$$u(0)=x_0 \quad \dot{u}(0)=\dot{x}_0 \tag{1.1.2}$$

The solution u of our problem is a function of the independent variable t and the parameter ε. Hence, we write $u=u(t;\varepsilon)$, where the parameter ε is separated from the independent variable t by a semicolon.

When $\varepsilon=0$, (1.1.1) reduces to

$$\ddot{u}+u=0 \tag{1.1.3}$$

Whose general solution is

$$u_0=a_0\cos(t+\beta_0) \tag{1.1.4}$$

Where a_0 and β_0 are arbitrary constants. When ε is small but different from zero, the general solution of (1.1.1) is no longer given by (1.1.4), and a correction must be added to it. We try a correction in the form of a power series in ε, that is, we let [1, 2]

$$u(t;\varepsilon)=u_0(t)+\varepsilon u_1(t)+\varepsilon^2 u_2(t)+\varepsilon^3 u_3(t)+\dots \tag{1.1.5}$$

Here, we restrict our discussion to the first term in the correction series. Thus, we seek an approximate solution in the form

$$u(t;\varepsilon)=u_0(t)+\varepsilon u_1(t)+O(\varepsilon^2) \tag{1.1.6}$$

Since only one term is kept in the correction series, we call (6) a first-order expansion. We substitute (1.1.6) into (1.1.1) and obtain

$$\ddot{u}_0+\varepsilon\ddot{u}_1+O\left(\varepsilon^2\right)+u_0+\varepsilon u_1+O\left(\varepsilon^2\right)+\varepsilon\left[u_0+\varepsilon u_1+O\left(\varepsilon^2\right)\right]^3=0 \tag{1.1.7}$$

Using the binomial theorem to expand the last term, we have

$$\begin{aligned}\left[u_0+\varepsilon u_1+O\left(\varepsilon^2\right)\right]^3&=u_0^{\,3}+3u_0^{\,2}\left[\varepsilon u_1+O\left(\varepsilon^2\right)\right]+3u_0\left[\varepsilon u_1+O\left(\varepsilon^2\right)\right]^2\\ &+\left[\varepsilon u_1+O\left(\varepsilon^2\right)\right]^3=u_0^{\,3}+3\varepsilon u_0^{\,2}u_1+O\left(\varepsilon^2\right)\end{aligned} \tag{1.1.8}$$

Substituting (1.1.8) into (1.1.7) and collecting coefficients of equal powers of ε, we obtain

$$\ddot{u}_0+u_0+\varepsilon\left(\ddot{u}_1+u_1+u_0^{\,3}\right)+O\left(\varepsilon^2\right)=0 \tag{1.1.9}$$

We note that since we are only interested in terms up to $O(\varepsilon)$, we need only the terms that are independent of ε from the quantity inside the brackets in (1.1.7). This fact can be used to minimize the algebra. Setting $\varepsilon=0$ in (1.1.9) gives

$$\ddot{u}_0+u_0=0 \tag{1.1.10}$$

Then, (1.1.9) becomes

$$\varepsilon\left(\ddot{u}_1+u_1+u_0^{\,3}\right)+O\left(\varepsilon^2\right)=0 \tag{1.1.11}$$

Dividing (1.1.11) by ε yields

$$\ddot{u}_1+u_1+u_0^{\,3}+O(\varepsilon)=0 \tag{1.1.12}$$

Setting $\varepsilon=0$ in (1.1.12) gives

$$\ddot{u}_1+u_1+u_0^{\,3}=0 \tag{1.1.13}$$

Comparing (1.1.10) and (1.1.13) with (1.1.9), we note that the former equations can be obtained by simply setting each of the coefficients of ε equal to zero in (1.1.9). This is how one usually derives the equations governing u_0 and u_1. Moreover, we note that (1.1.10) and (1.1.13) have to be solved in succession. One first solves (1.1.10) for u_0, substitutes the result into (1.1.13), and then solves for u_1.

The general solution of (1.1.10) can be written as

$$u_0=a_0\cos\left(t+\beta_0\right) \tag{1.1.14}$$

Where a_0 and β_0 are arbitrary constants. Substituting for u_0 into (1.1.13) yields

$$\ddot{u}_1 + u_1 = -a_0{}^3 \cos^3\left(t + \beta_0\right) \tag{1.1.15}$$

Equation (1.1.15) is an inhomogeneous equation whose general solution consists of the sum of a homogeneous solution and a particular solution. To determine a particular solution, we find it convenient to express the inhomogeneous term in a Fourier series using the trigonometric identity given as

$$\cos^3\theta = \frac{1}{4}\cos 3\theta + \frac{3}{4}\cos\theta$$

And rewrite (1.1.15) as

$$\ddot{u}_1 + u_1 = -\frac{3}{4}a_0{}^3 \cos\left(t + \beta_0\right) - \frac{1}{4}a_0{}^3 \cos\left(3t + 3\beta_0\right) \tag{1.1.16}$$

The homogeneous solution of (1.1.16) can be expressed as

$$u_{1_h} = a_1 \cos\left(t + \beta_1\right) \tag{1.1.17}$$

Where a_1 and β_1 are arbitrary constants. Since (1.1.16) is linear, one can be use the principle of superposition and determine particular solutions as the sum of two particular solutions corresponding to the two inhomogeneous terms. That is, one determines a particular solution by adding two particular solutions of the following equations:

$$\ddot{u}_1 + u_1 = -\frac{3}{4}a_0{}^3 \cos\left(t + \beta_0\right) \tag{1.1.18}$$

$$\ddot{u}_1 + u_1 = -\frac{1}{4}a_0{}^3 \cos\left(3t + 3\beta_0\right) \tag{1.1.19}$$

A particular solution of (1.1.18) is

$$u_{1_p}{}^{(1)} = -\frac{3}{8}a_0{}^3 t \sin\left(t + \beta_0\right) \tag{1.1.20}$$

A particular solution of (1.1.19) is

$$u_{1_p}{}^{(2)} = \frac{1}{32}a_0{}^3 \cos\left(3t + 3\beta_0\right) \tag{1.1.21}$$

Hence, according to the principle of superposition, a particular solution of (1.1.16) is

$$u_{1_p} = -\frac{3}{8}a_0{}^3 t \sin\left(t + \beta_0\right) + \frac{1}{32}a_0{}^3 \cos\left(3t + 3\beta_0\right) \tag{1.1.22}$$

Therefore, the general solution of (1.1.16) is

$$u_1 = a_1 \cos(t + \beta_1) - \frac{3}{8} a_0^3 t \sin(t + \beta_0) + \frac{1}{32} a_0^3 \cos(3t + 3\beta_0) \tag{1.1.23}$$

Substituting for u_0 and u_1 from (14) and (1.1.23), respectively, into (1.1.6) yields the following first-order expansion for the general solution of (1.1.1):

$$u = a_0 \cos(t + \beta_0) + \varepsilon \left[a_1 \cos(t + \beta_1) - \frac{3}{8} a_0^3 t \sin(t + \beta_0) + \frac{1}{32} a_0^3 \cos(3t + 3\beta_0) \right] + \dots \tag{1.1.24}$$

Where $a_0, a_1, \beta_0, and\ \beta_1$ are arbitrary constants. We started a second-order equation that can satisfy two initial conditions but it appears that we ended up with four arbitrary constants. It turns out that the constants $a_0, a_1, \beta_0, and\ \beta_1$ are not arbitrary and that the two initial conditions (1.1.2) are enough to determine them. To see this, we substitute (1.1.24) into (1.1.2) and obtain

$$x_0 = a_0 \cos \beta_0 + \varepsilon \left(a_1 \cos \beta_1 + \frac{1}{32} a_0^3 \cos 3\beta_0 \right) + \dots \tag{1.1.25}$$

$$\dot{x}_0 = -a_0 \sin \beta_0 - \varepsilon \left(a_1 \sin \beta_1 + \frac{3}{8} a_0^3 \sin \beta_0 + \frac{3}{32} a_0^3 \sin 3\beta_0 \right) + \dots \tag{1.1.26}$$

Transposing all terms to one side and equating the coefficients of the powers of ε to zero in (1.1.25) and (1.1.26) is equivalent to equating the coefficients of like powers of ε on both sides of these equations. The results are

Order ε^0

$$x_0 = a_0 \cos \beta_0 \tag{1.1.27}$$

$$\dot{x}_0 = -a_0 \sin \beta_0 \tag{1.1.28}$$

Order ε

$$a_1 \cos \beta_1 = -\frac{1}{32} a_0^3 \cos 3\beta_0 \tag{1.1.29}$$

$$a_1 \sin \beta_1 = -\frac{3}{8} a_0^3 \sin \beta_0 - \frac{3}{32} a_0^3 \sin 3\beta_0 \tag{1.1.30}$$

Squaring and adding (1.1.27) and (1.1.28) yields

$$x_0^2 + \dot{x}_0^2 = a_0^2 \cos^2 \beta_0 + a_0^2 \sin^2 \beta_0 = a_0^2$$

Or

$$a_0 = \left(x_0{}^2 + \dot{x}_0{}^2\right)^{1/2} \tag{1.1.31}$$

Then, solving (1.1.27) and (1.1.28) for β_0 gives

$$\beta_0 = -\sin^{-1}\left[\frac{\dot{x}_0}{\left(x_0{}^2 + \dot{x}_0{}^2\right)^{1/2}}\right] = \cos^{-1}\left[\frac{x_0}{\left(x_0{}^2 + \dot{x}_0{}^2\right)^{1/2}}\right] \tag{1.1.32}$$

Similarly, solving (1.1.29) and (1.1.30) yields

$$a_1 = \frac{1}{32}a_0{}^3\left[\cos^2 3\beta_0 + 9\left(\sin 3\beta_0 + 4\sin\beta_0\right)^2\right]^{1/2} \tag{1.1.33}$$

$$\beta_1 = -\sin^{-1}\frac{3a_0{}^3\left(\sin 3\beta_0 + 4\sin\beta_0\right)}{32a_1} = \cos^{-1}\left[-\frac{a_0{}^3\cos 3\beta_0}{32a_1}\right] \tag{1.1.34}$$

Thus, once a_0 and β_0 are calculated from (1.1.31) and (1.1.32), a_1 and β_1 are calculated from (1.1.33) and (1.1.34).

Let us return to (1.1.24) and note that

$$\begin{aligned} a_0\cos\left(t+\beta_0\right) + \varepsilon a_1\cos\left(t+\beta_1\right) &= a_0\cos t\cos\beta_0 - a_0\sin t\sin\beta_0 + \varepsilon a_1\cos t\cos\beta_1 - \varepsilon a_1\sin t\sin\beta_1 \\ &= \left(a_0\cos\beta_0 + \varepsilon a_1\cos\beta_1\right)\cos t - \left(a_0\sin\beta_0 + \varepsilon a_1\sin\beta_1\right)\sin t \\ &= a\cos t\cos\beta - a\sin t\sin\beta \end{aligned}$$

$$a_0\cos\left(t+\beta_0\right) + \varepsilon a_1\cos\left(t+\beta_1\right) = a\cos\left(t+\beta\right) \tag{1.1.35}$$

Where

$$a\cos\beta = a_0\cos\beta_0 + \varepsilon a_1\cos\beta_1 \tag{1.1.36}$$

$$a\sin\beta = a_0\sin\beta_0 + \varepsilon a_1\sin\beta_1 \tag{1.1.37}$$

It follows from (1.1.36) and (1.1.37) that

$$a_0 = a + O(\varepsilon) \quad \beta_0 = \beta + O(\varepsilon) \tag{1.1.38}$$

Using (1.1.35) and (1.1.38), we rewrite (1.1.24) as

$$u = a\cos\left(t+\beta\right) + \varepsilon\left\{-\frac{3}{8}\left[a+O(\varepsilon)\right]^3 t\sin\left[t+\beta+O(\varepsilon)\right] + \frac{1}{32}\left[a+O(\varepsilon)\right]^3\cos\left[3t+3\beta+O(\varepsilon)\right]\right\} + \ldots$$

Hence,

$$u = a\cos(t+\beta) + \varepsilon a^3\left[-\frac{3}{8}t\sin(t+\beta) + \frac{1}{32}\cos(3t+3\beta)\right] + \dots \tag{1.1.39}$$

Note that above expansion is nonuniform or breakdown for long times and call it expansions pedestrian expansions. The reason for the breakdown of this expansion is the presence of the term $t\sin(t+\beta)$, a product of algebraic and circular terms. Such terms are called mixed-secular terms. Thus, for the expansion to be uniform, the corrections must be free of secular terms.

1.2 Example 2

Consider that

$$\begin{aligned} &\ddot{u} + u + \alpha_2 u^2 + \alpha_3 u^3 = 0 \\ &u(0) = x_0 \quad \dot{u}(0) = \dot{x}_0 \end{aligned} \tag{1.2.1}$$

Where α_2 and α_3 are constants.

We determine a second-order straightforward expansion to the solutions of (1.2.1) for small but finite amplitudes. To carry out a straightforward expansion for small but finite amplitudes for (1.2.1), we need to introduce a small parameter because none appear explicitly in this equation. To this end, we seek an expansion in the form [1. 2]

$$u = \varepsilon u_1(t) + \varepsilon^2 u_2(t) + \varepsilon^3 u_3(t) + \dots \tag{1.2.2}$$

Where ε is a small dimensionless parameter that is a measure of the amplitude of oscillation. It can be used as a bookkeeping or crutching device and set equal to unity if the amplitude is taken to be small as described below.

Substituting (1.2.2) into (1.2.1) gives

$$\begin{aligned} \varepsilon\ddot{u}_1 + \varepsilon^2\ddot{u}_2 + \varepsilon^3\ddot{u}_3 + \dots + \varepsilon u_1 + \varepsilon^2 u_2 + \varepsilon^3 u_3 + \dots + \alpha_2\left(\varepsilon u_1 + \varepsilon^2 u_2 + \varepsilon^3 u_3 + \dots\right)^2 \\ + \alpha_3\left(\varepsilon u_1 + \varepsilon^2 u_2 + \varepsilon^3 u_3 + \dots\right)^3 = 0 \end{aligned} \tag{1.2.3}$$

Using the Binomial theorem to expand the terms in parentheses in (1.2.3) and keeping terms to $O(\varepsilon^3)$, we obtain

$$\varepsilon(\ddot{u}_1 + u_1) + \varepsilon^2\left(\ddot{u}_2 + u_2 + \alpha_2 u_1^2\right) + \varepsilon^3\left(\ddot{u}_3 + u_3 + 2\alpha_2 u_1 u_2 + \alpha_3 u_1^3\right) + \dots = 0 \tag{1.2.4}$$

Equating each of the coefficients of ε to zero in (1.2.4) yields

$$\ddot{u}_1 + u_1 = 0 \tag{1.2.5}$$

$$\ddot{u}_2 + u_2 = -\alpha_2 u_1^2 \tag{1.2.6}$$

$$\ddot{u}_3 + u_3 = -2\alpha_2 u_1 u_2 - \alpha_3 u_1^3 \tag{1.2.7}$$

Which can be solved sequentially for u_1, u_2 and u_3.

The general solution of (1.2.5) can be expressed as

$$u_1 = a\cos(t+\beta) \tag{1.2.8}$$

Where a and β are constants. Then, (1.2.6) becomes

$$\ddot{u}_2 + u_2 = -\alpha_2 a^2 \cos^2(t+\beta) = -\frac{1}{2}\alpha_2 a^2 - \frac{1}{2}\alpha_2 a^2 \cos(2t+2\beta) \tag{1.2.9}$$

As before, we do not include the solution of the homogeneous problem for u_2. Moreover, a particular solution for (1.2.9) can be obtained as the sum of two particular solutions, one for each of the following equations:

$$\ddot{u}_2^{(1)} + u_2^{(1)} = -\frac{1}{2}\alpha_2 a^2 \tag{1.2.10}$$

$$\ddot{u}_2^{(2)} + u_2^{(2)} = -\frac{1}{2}\alpha_2 a^2 \cos(2t+2\beta) \tag{1.2.11}$$

A particular solution of (1.2.10) is

$$u_{2p}^{(1)} = -\frac{1}{2}\alpha_2 a^2 \tag{1.2.12}$$

Whereas a particular solution of (1.2.11) is

$$u_{2p}^{(2)} = \frac{1}{6}\alpha_2 a^2 \cos(2t+2\beta) \tag{1.2.13}$$

Hence,

$$u_2 = u_{2p}^{(1)} + u_{2p}^{(2)} = -\frac{1}{2}\alpha_2 a^2 + \frac{1}{6}\alpha_2 a^2 \cos(2t+2\beta) \tag{1.2.14}$$

Substituting (1.2.8) and (1.2.13) into (1.2.7) gives

$$\ddot{u}_3 + u_3 = -2\alpha_2 a\cos(t+\beta)\left[-\frac{1}{2}\alpha_2 a^2 + \frac{1}{6}\alpha_2 a^2 \cos(2t+2\beta)\right] - \alpha_3 a^3 \cos^2(t+\beta)$$

Or

$$\ddot{u}_3 + u_3 = \left(\frac{5}{6}\alpha_2^2 - \frac{3}{4}\alpha_3\right)a^3\cos(t+\beta) - \left(\frac{1}{4}\alpha_3 + \frac{1}{6}\alpha_2^2\right)a^3\cos(3t+3\beta) \tag{1.2.15}$$

Since (1.2.15) is linear,

$$u_3 = {u_{3p}}^{(1)} + {u_{3p}}^{(2)} \tag{1.2.16}$$

Where ${u_{3p}}^{(1)}$ and ${u_{3p}}^{(2)}$ are particular solutions of the following equations:

$$\ddot{u}_3^{(1)} + u_3^{(1)} = \left(\frac{5}{6}\alpha_2^{\,2} - \frac{3}{4}\alpha_3\right)a^3\cos(t+\beta) \tag{1.2.17}$$

$$\ddot{u}_3^{(2)} + u_3^{(2)} = -\left(\frac{1}{4}\alpha_3 + \frac{1}{6}\alpha_2^{\,2}\right)a^3\cos(3t+3\beta) \tag{1.2.18}$$

It follows that

$$u_{3_p}^{\;(1)} = \left(\frac{5}{12}\alpha_2^{\,2} - \frac{3}{8}\alpha_3\right)a^3 t\sin(t+\beta) \tag{1.2.19}$$

$$u_{3_p}^{\;(2)} = \frac{1}{8}\left(\frac{1}{4}\alpha_3 + \frac{1}{6}\alpha_2^{\,2}\right)a^3\cos(3t+3\beta) \tag{1.2.20}$$

Substituting (1.2.19) and (1.2.20) into (1.2.16) yields

$$u_3 = \left(\frac{5}{12}\alpha_2^{\,2} - \frac{3}{8}\alpha_3\right)a^3 t\sin(t+\beta) + \frac{1}{8}\left(\frac{1}{4}\alpha_3 + \frac{1}{6}\alpha_2^{\,2}\right)a^3\cos(3t+3\beta) \tag{1.2.21}$$

Substituting (1.2.8), (1.2.14), and (1.2.21) into (1.2.2) yields the following third-order [up to $O(\varepsilon^3)$] straightforward expansion:

$$u = \varepsilon a\cos(t+\beta) + \frac{1}{6}\alpha_2\varepsilon^2 a^2\left[\cos(2t+2\beta) - 3\right] + \varepsilon^3 a^3\begin{bmatrix}\left(\dfrac{5}{12}\alpha_2^{\,2} - \dfrac{3}{8}\alpha_3\right)t\sin(t+\beta) \\ +\dfrac{1}{8}\left(\dfrac{1}{4}\alpha_3 + \dfrac{1}{6}\alpha_2^{\,2}\right)\cos(3t+3\beta)\end{bmatrix} + \cdots \tag{1.2.22}$$

We note that the dependence of u on ε and a is in the combination εa. Thus, one can set $\varepsilon = 1$ and consider a as the perturbation parameter. The straightforward expansion (1.2.22) breaks down for $t \geq O(\varepsilon^{-1}a^{-1})$ because the second correction term is the same order or longer than the first correction term, owing to the presence of the mixed-secular term.

CHAPTER 2

The Lindstedt-Poincare Technique

As discussed in the preceding chapter, the breakdown in the straightforward expansion is due to its failure to account for the nonlinear dependence of the frequency on the nonlinearity. Thus, any expansion that does not account for a nonlinear frequency is doomed to failure. A number of techniques that yield uniformly valid expansions have been developed. In this chapter, we find approximate and uniform solution of chapter one's differential equations by Lindstedt-Poincare Technique.

2.1 Example 1

Consider that Eq. (1.1.1) subject to Eq. (1.1.2). To account for the nonlinear dependence of the frequency, we explicitly exhibit the frequency ω of the system in the differential equation. To this end, we introduce the transformation [1, 2]

$$\tau = \omega t \tag{2.1.1}$$

Where ω is a constant that depends on ε. Then, we need to change the independent variable from t to τ. Using the chain rule, we transform the derivatives according to

$$\frac{d}{dt} = \frac{d\tau}{dt}\frac{d}{d\tau} = \omega\frac{d}{dt}$$

$$\frac{d^2}{dt^2} = \omega\frac{d^2}{dtd\tau} = \omega\frac{d\tau}{dt}\frac{d^2}{d\tau^2} = \omega^2\frac{d^2}{d\tau^2}$$

Hence, (2.1.1) becomes

$$\omega^2 u'' + u + \varepsilon u^3 = 0 \tag{2.1.2}$$

Where the prime indicates the derivative with respect to τ. We note that the actual frequency of the system now appears explicitly in the equation. Until now, both u and ω are unknowns. We seek approximate solutions for them in the form of power series in terms of ε. That is, we let

$$u = u_0(\tau) + \varepsilon u_1(\tau) + \ldots \tag{2.1.3}$$

$$\omega = 1 + \varepsilon\omega_1 + \ldots \tag{2.1.4}$$

We note that the first term in the expansion of ω is the linear frequency; in this case, the linear frequency is unity. The corrections to the linear frequency are determined in the course of the analysis by requiring the expansion of u to be uniform for all τ.

Substituting (2.1.3) and (2.1.4) into (2.1.2) gives

$$\left(1+\varepsilon\omega_1+\ldots\right)^2\left(u_0''+\varepsilon u_1''+\ldots\right)+\left(u_0+\varepsilon u_1+\ldots\right)+\varepsilon\left(u_0+\varepsilon u_1+\ldots\right)^3=0$$

Which, upon expansion, becomes

$$u_0''+2\varepsilon\omega_1 u_0''+\varepsilon u_1''+\ldots+u_0+\varepsilon u_1+\ldots+\varepsilon u_0{}^3+\ldots=0$$

Or

$$u_0''+u_0+\varepsilon\left(u_1''+u_1+u_0{}^3+2\omega_1 u_0''\right)+\ldots=0 \tag{2.1.5}$$

Equating each of the coefficients of ε^0 and ε to zero yields

$$u_0''+u_0=0 \tag{2.1.6}$$

$$u_1''+u_1=-u_0{}^3+2\omega_1 u_0'' \tag{2.1.7}$$

The general solution of (2.1.6) is

$$u_0=a\cos\left(\tau+\beta\right) \tag{2.1.8}$$

Where a and β are constants. Then, (2.1.7) becomes

$$u_1''+u_1=-a^3\cos^3\left(\tau+\beta\right)+2\omega_1 a\cos\left(\tau+\beta\right)$$

Or

$$u_1''+u_1=\left(2\omega_1 a-\frac{3}{4}a^3\right)\cos\left(\tau+\beta\right)-\frac{1}{4}a^3\cos\left(3\tau+3\beta\right) \tag{2.1.9}$$

The particular solution of (2.1.9) is

$$u_1=\frac{1}{2}\left(2\omega_1 a-\frac{3}{4}a^3\right)\tau\sin\left(\tau+\beta\right)+\frac{1}{32}a^3\cos\left(3\tau+3\beta\right) \tag{2.1.10}$$

We note that the particular solution of u_1 contains a mixed-secular term. Which makes the expansion nonuniform. For a uniform expansion, we cannot permit such secular terms to appear in $u_1, u_2, u_3 \ldots$. In contrast with the straightforward expansion, where the secular term cannot be annihilated unless $a=0$ (i.e., trivial solution for u), in this case, we still have to choose the parameter ω_1. Thus, we choose it to annihilate the secular term. To this end, we set the coefficient of the secular term zero. That is,

$$2\omega_1 a - \frac{3}{4}a^3 = 0 \tag{2.1.11}$$

Then, (2.1.10) becomes

$$u_1 = \frac{1}{32}a^3\cos\left(3\tau + 3\beta\right) \tag{2.1.12}$$

Disregarding the trivial case $a = 0$, we find that (2.1.11) is satisfied if

$$\omega_1 = \frac{3}{8}a^2 \tag{2.1.13}$$

We note that, to determine the condition (2.1.11) for the elimination of the secular term from u_1, we do not need to determine the particular solution first as done above. Instead, we only need to inspect the inhomogeneous terms in (2.1.9) governing u_1 and choose the coefficient of $\cos\left(\tau + \beta\right)$ to be zero because this is the term that produces the secular term in u_1.

Substituting (2.1.8) and (2.1.12) into (2.1.3) yields

$$u = a\cos\left(\tau + \beta\right) + \frac{1}{32}a^3\cos\left(3\tau + 3\beta\right) + \dots \tag{2.1.14}$$

Substituting for ω_1 from (2.1.13) into (2.1.4) yields

$$\omega_1 = 1 + \frac{3}{8}\varepsilon a^2 + \dots \tag{2.1.15}$$

Since $\tau = \omega t$, we rewrite (2.1.14)

$$u = a\cos\left[\left(1 + \frac{3}{8}\varepsilon a^2\right)t + \beta\right] + \frac{1}{32}\varepsilon a^3\cos\left[3\left(1 + \frac{3}{8}\varepsilon a^2\right)t + 3\beta\right] + \dots \tag{2.1.16}$$

Thus, the expansion (2.1.16) is uniform to first-order because secular terms do not appear in it and the correction term (the term proportional to ε) is small compared with the first term.

Next, we compare the period obtained in this section by using the Lindstedt-Poincare technique with the approximate period obtained in the preceding section by expanding the integrand in the exact solution. Since the value $u = -x_0$ is the intersection of the closed trajectory with u axis, $\dot{u} = 0$ when $u = -x_0$. Using these values as initial conditions in (2.1.16), we have

$$-x_0 = a\cos\beta + \frac{1}{32}\varepsilon a^3\cos 3\beta + \ldots \tag{2.1.17}$$

It follows from (2.1.17) that $a^2 = x_0{}^2$ because $\beta = 0$ for $\dot{u}(0) = 0$. Hence, (2.1.15) becomes

$$\omega = 1 + \frac{3}{8}\varepsilon x_0{}^2 + \ldots$$

But $T_a = 2\pi/\omega$, therefore

$$T_a = 2\pi\left(1 + \frac{3}{8}\varepsilon x_0{}^2 + \ldots\right)^{-1} = 2\pi\left(1 - \frac{3}{8}\varepsilon x_0{}^2\right) + \ldots \tag{2.1.18}$$

Which is in full agreement with approximate period obtained in the preceding section in the exact solution.

2.2 Example2

Introducing the transformation $\tau = \omega t$ [1, 2],

$$\omega^2 u'' + u + \alpha_2 u^2 + \alpha_3 u^3 = 0 \tag{2.2.1}$$

Where the prime denotes the derivative with respect to τ. Next, we expand ω and u in power of ε as

$$u = \varepsilon u_1(\tau) + \varepsilon^2 u_2(\tau) + \varepsilon^3 u_3(\tau) + \cdots \tag{2.2.2}$$

$$\omega = 1 + \varepsilon\omega_1 + \varepsilon^2\omega_2 + \cdots \tag{2.2.3}$$

As discussed in the preceding chapters, the first term in the expansion of ω is the linear frequency, which is unity in this case. Substituting (2.2.2) and (2.2.3) into (2.2.1), we have

$$\left(1 + \varepsilon\omega_1 + \varepsilon^2\omega_2 + \cdots\right)^2\left(\varepsilon u_1'' + \varepsilon^2 u_2'' + \varepsilon^3 u_3'' + \cdots\right) + \varepsilon u_1 + \varepsilon^2 u_2 + \varepsilon^3 u_3 + \cdots$$
$$+ \alpha_2\left(\varepsilon u_1 + \varepsilon^2 u_2 + \varepsilon^3 u_3 + \cdots\right)^2 + \alpha_3\left(\varepsilon u_1 + \varepsilon^2 u_2 + \varepsilon^3 u_3 + \cdots\right)^3 = 0$$

Using binomial theorem to expand the exponentiated quantities and keeping terms to $O\left(\varepsilon^3\right)$, we obtain

$$\left(1 + 2\varepsilon\omega_1 + \varepsilon^2\omega_1^2 + 2\varepsilon^2\omega_2\right)\left(\varepsilon u_1'' + \varepsilon^2 u_2'' + \varepsilon^3 u_3''\right) + \varepsilon u_1 + \varepsilon^2 u_2 + \varepsilon^3 u_3 +$$
$$+ \alpha_2\left(\varepsilon^2 u_1{}^2 + 2\varepsilon^3 u_1 u_2\right) + \alpha_3\varepsilon^3 u_1{}^3 + \cdots = 0$$

Multiplying the first two terms and equating the coefficients of like powers of ε to zero yields

$$u_1'' + u_1 = 0 \tag{2.2.4}$$

$$u_2'' + u_2 = -2\omega_1 u_1'' - \alpha_2 u_1^{\,2} \tag{2.2.5}$$

$$u_3'' + u_3 = -2\omega u_2'' - \left(\omega_1^{\,2} + 2\omega_2\right)u_1'' - 2\alpha_2 u_1 u_2 - \alpha_3 u_1^{\,3} \tag{2.2.6}$$

The general solution of (2.2.4) can be expressed as

$$u_1 = a\cos\left(\tau + \beta\right) \tag{2.2.7}$$

Where a and β constants. Then, (2.2.5) becomes

$$u_2'' + u_2 = 2\omega_1 a\cos\left(\tau + \beta\right) - \alpha_2 a^2 \cos^2\left(\tau + \beta\right)$$

Or

$$u_2'' + u_2 = 2\omega_1 a\cos\left(\tau + \beta\right) - \frac{1}{2}\alpha_2 a^2 - \frac{1}{2}\alpha_2 a^2 \cos\left(2\tau + 2\beta\right) \tag{2.2.8}$$

Eliminating the secular terms from u_2 demands that $\omega_1 = 0$. Then, the solution of (2.2.8) can be obtained as

$$u_2 = -\frac{1}{2}\alpha_2 a^2 + \frac{1}{6}\alpha_2 a^2 \cos\left(2\tau + 2\beta\right) \tag{2.2.9}$$

Substituting (2.2.7) and (2.2.9) into (2.2.6) and using the fact that $\omega_1 = 0$, we obtained

$$u_3'' + u_3 = 2\omega_2 a\cos\left(\tau + \beta\right) - 2\alpha_2 a\cos\left(\tau + \beta\right)\begin{bmatrix} -\frac{1}{2}\alpha_2 a^2 \\ +\frac{1}{6}\alpha_2 a^2 \cos\left(2\tau + 2\beta\right) \end{bmatrix} - \alpha_3 a^3 \cos^3\left(\tau + \beta\right) \tag{2.2.10}$$

Using trigonometric identities, we rewrite (2.2.10) as

$$u_3'' + u_3 = \left(2\omega_2 a - \frac{3}{4}\alpha_3 a^3 + \frac{5}{6}\alpha_2^{\,2} a^3\right)\cos\left(\tau + \beta\right) - \left(\frac{1}{4}\alpha_3 + \frac{1}{6}\alpha_2^{\,2}\right)a^3 \cos\left(3\tau + 3\beta\right) \tag{2.2.11}$$

Eliminating the secular term from (2.2.11) demands that

$$2\omega_2 a - \frac{3}{4}\alpha_3 a^3 + \frac{5}{6}\alpha_2^{\,2} a^3 = 0$$

Or

$$\omega_2 = \frac{3}{8}\alpha_3 a^2 - \frac{5}{12}\alpha_2^{\,2} a^2 \tag{2.2.12}$$

Substituting (2.2.7) and (2.2.9) into (2.2.2) yields, whereas substituting (2.2.12) into (2.2.3), using $\tau = \omega t$, and recalling that $\omega_1 = 0$ yields (2.2.30). Thus, the Lindstedt-Poincare technique produced an expansion that is in full agreement with that obtained.

Reference

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