Some Integral Inequalities via Riemann-Livouille Fractional Integrals

By

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CIIT/FA16-RMT-001/LHR

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Declaration

I Muhammad Shuja Ur Rehman Babar, **CIIT/FA16-RMT-001/LHR** hereby declare that I have produced the work presented in this thesis, during the scheduled period of study. I also declare that I have not taken any material from any source except referred to wherever due, that amount of plagiarism is within acceptable range. If a violation of HEC rules on research has occurred in this thesis, I shall be liable to punishable action under the plagiarism rules of the HEC.

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It is certified that Muhammad Shuja Ur Rehman Babar, CIIT/FA16-RMT-001/LHR has carried out all the work related to this thesis under my supervision at the Department of Mathematics, COMSATS University Islamabad, Lahore campus and the work fulfills the requirement for award of MS degree

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DEDICATION

To

My Parents and family, specially to my elder brother Zia Babar.

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In the name of Allah, the most gracious and the most merciful.
The Prophet (S.A.W) said: “Seeking knowledge is obligatory for every muslim”.
First, I recognize the creator and finisher of my faith and in memorial of my life, I say thank ALLAH Almighty and Prophet Muhammad (S.A.W) for everything. I am grateful to Almighty ALLAH, without Whose endowments it was difficult to finish this proposal.
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Abstract

Mathematical inequalities contribute highly to error estimation. In the past few decades, many authors have done their studies about the error analysis of some quadrature rules of Newton-Cotes type. Moreover, Convexity is highly recognized for its vital and primary task in the expansion of numerous section of mathematics, for example, mathematical finance, economics, management sciences and optimization theory. Hence, in this dissertation we are focusing on the detailed studies of the function of convexity in the theory of inequalities and generalizations of new integral inequalities for differentiable mapping along with the Simpsons type as well as Hermite-Hadamards type. Here, we derive some explicit bounds for the mid-point, Trapezoid and Simpsons quadrature rule with the aid of modern theory of inequalities and general Peano-Kernel approach in coherent with the convexity $(s,m)-$convex at most first derivative.

In our monograph we shall generalize some integral inequalities for generalized convex function and, in parallel, developments shall be made on concavity.

With the above mentioned technique, we are able to focus on the larger classes of functions allowing us to conduct different quadrature rule that have restrictions on the behavior of the integrand. By applying the Holder inequality and power mean equality, different generalizations and melioration for a formal inequalities and differentiable mapping of the functions where the function under specific condition is convex are considered. The results presented here extend various inequalities of the Simpson’s and Hermite-Hadamard.

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Chapter 1
Preliminaries and Introduction

1 Preliminaries and Introduction

In the field of mathematics, the term 'inequality' indicates an inconsistency concerning two inequalities, which is applied to imitate the association between two substances. Therefore, an 'inequality' is merely two quantities that are equivalent with each other. As calculus started to surface in the $19th$ century, the position and function of the inequalities gradually became vital and important.

When referring to modern mathematics, inequalities practically plays an imperative part in all areas of mathematics. There are numerous applications of inequalities that are discovered in a variety of sciences fields such as, physical and engineering sciences.

Within the numerical investigation of a definite integral of a real function $f(t)$ over an interval $[a\ ,b]$, the estimation is incredibly a remarkable predicament. For that reason, numerous ways have emerged in literature to solve these predicaments there of, introducing us to one of the most common process used, which is the Newton-Cotes formulas (for example Simpson, trapezoidal, and midpoint, formulas).

Therefore bounds for these estimations deals with the involvement of higher order derivatives, such as, for the midpoint and the trapezoidal formulas, the error bound deals with the involvement of a second-order derivative.

Specifically, the error bound of Simpson's approach deals with the involvement of a fourth-order derivative, and for that reason this approach encompasses numerous difficulties. For instance, it expects countless differentiation (Let say our supposition the derivatives hold), with bounded derivatives, that builds and create this set of functions to be ineffective and rigid in solving such conundrum.

Latest developments show that the usage of modern theory of inequalities is exercised largely and that a lot of dedicated and committed work is being done in order to institute more than a few generalization of the Simpson's, trapezoid and midpoint, these are intended for mapping of bounded variation and for Lipschitzian, monotonic, and absolutely continuous mapping additionally for n-times differentiable through kernels to purify the error bounds of existing inequalities.

This dissertation is highly focused on investigating, numerous inequalities modification of Simpson's and Hermite-Hadamard's type using convexity, $(s,m)-$convex, functions and consequently, get exact bounds utilizing the Peano kernel approach and the modern theory of inequalities to attain overt and unequivocal bounds.

The involvement of these error bounds were derived simply up to first derivative.

With the application of Hölder and power mean inequalities, and overview of the acquired outcome were to be measured. Moreover, developments for the primal inequalities in the literature for functions under specific condition are convex are specified. Lastly, a number of error estimates and some special means for trapezoid, midpoint and Simpson's rules will be derived.

By all means, this will allow this thesis to impart on a study of various prominent and primary inequalities investigated by Hermite-Hadamard, through the different forms of convex functions. It will then be able to bring about in this area of research a number of recent discoveries, and in doing so, within a cohesive time frame.

1.1 Problem Statement

The focal statement of this dissertation is committed into instituting and conferring numerous inequalities of Hermite-Hadamard via Riemann Livouille fractional integral nature through the usage of the different types of convexities , like as $(s,m)-$convex function. We are then able to acquire inequalities referred to above by means of these convexities. Moreover, in places where the trapezoid , midpoint and Simpson's quadrature rules is not functional, it will be conferred to within this dissertation.

1.2 Aims and Objectives

The problem related to the expansion of integral inequalities that present error bounds on the physical magnitudes. That is, inequalities of Hermite- Hadamard's type are aimed to be overseen in this dissertation in such a way:

1) To expand the function of convexity within the theory of inequalities.

2) Deliberation of distinctive inequalities, thus, is creating within the theory of optimization the evident relevance of these inequalities.

3) Deliberation of special means with relation to its purpose in discovering unequivocal connections of these means.

4) The estimation of the optimal error bound for such inequalities under the assumptions under consideration.

5) Extending the variations of Hermite-Hadamard's type therefore, working on to achieve some more inequalities of Hermite-Hadamard's type.

6) To present some new inequalities of Hermite-Hadamard's utilizing different kind of convexity (such as $(s,m)-$convex).

This dissertation is attempt towards simplifying some findings on Hermite-Hadamard type inequalities via Riemann Livouille fractional integral taking the view the above narrative aims. The tracks in which the generalized versions of Hermite-Hadamard type inequalities via Riemann Livouille fractional integral are aimed to be applied are its uses in numerical integration and some special means will also be given.

1.3 Invesgative Technique

With the usage of two sets of convexities chartered collectively and the general Peano kernel and Montgomery distinctiveness, they are used in terms of second derivatives of a real functions to create alternative inequalities of Hermite-Hadamard type via Riemann Livouille fractional integral.
With the application of the Hölder and power-mean inequalities, numerous generalities, alterations and developments for the equivalent form of power of these inequalities are measured. As a result, a number of new error estimates for some quadrature rules and for some special means are derived.

1.4 Summary

This dissertation set out to introduce the generalization and application of inequalities in relation to the Hermite-Hadamard and Simpson's type. In order to acquire such generalizations, it has been taken into account to adjust and alter the Peano kernel concerned by bringing into being arbitrary parameters. The dissertation comprises four chapters.
This briefly describes as under:

Chapter 1 depicts an outline of where the incentive and aims are identified within the study.

Chapter 2 illustrates numerous basic concepts of convex functions as well as a number of its properties. Leading on to the brief discussion associated with several modifications and generalizations in relation to prior studies applicable to the Hermite-Hadamard and Simpson's type.
It is with the full intention of recapitulating a number of chorological contributions by several researchers that has fused integral inequalities with error estimations.

Chapter 3 presents the different modifications, improvement and new variations of the Hermite-Hadamard type inequalities via Riemann Livouille fractional integral through convex and $(s,m)-$convex function.

Chapter 4 initiates the some findings regarding to the current study with refrence to Simpson's type inequality. Moreover , several modifications, improvement and new variations of the Simpson's type type through $(s,m)-$convex function are considered.

Chapter 2
Literature Review

2 Literature Review

We start this section by narrating some important integral inequalities given in ([1], [2]) and their further developments. To start we would like to introduced some notions with some assumptions.

2.1 Function

In mathematics, a function is a relation between a set of inputs and a set of permissible outputs with the property that each input is related to exactly one output. An example is the function that relates each real number y to its square y^2.
In most fields of modern mathematics various types of functions are regarded as ”the central objects of investigation ”. A function can be represented or described in a number of ways . A formula or algorithm has also been used to define some functions by explaining that how to compute the output for a given input. As well as other functions are defined by a picture, called the graph of the function.

2.1.1 Definition

A function f from A to B is a subset of the Cartesian product $A \times B$ subject to the following condition: every element of A is the first component of one and only one ordered pair in the subset.

2.1.2 Examples

(1) $2x_1 + 3x_2 = 6$ is a function, because you can solve for x_1 :

$$2x_1 + 3x_2 = 6$$

$$2x_1 = -3x_2 + 6$$

and

$$x_1 = \frac{-3}{2}x_2 + 3$$

(2) On the other hand, $z^2 + 3t = 6$ is not a function , because you cannot solve for a unique z.

$$z^2 + 3t = 6$$

$$z^2 = -3t + 6$$

$$z = \pm\sqrt{}(-3t + 6)$$

(3) The equation $a = b^2$ is not a function because both $(1,1)$ and $(1,-1)$ satisfy the equation , and a function by definition cannot have two different b-values for the same a value.
(4) The domain is chosen to be the set of natural numbers$(1,2,3,4...)$, and the codomain is the set of integers $(...,-3,-2,-1,0,1,2,3...)$ .The function associates to any natural number n the number $4 - n$. For example, to 1 it associates 3 and to 10 it associates -6.

2.2 Continuity

A function which instinctively leads to small changes in output as result of small changes in input is defined as a continuous function([3]). If not a function is said to be a "discontinuous function. A continuous function along with a continuous inverse function is known as "bi-continuous ". functions continuity is one of the fundamental notions of topology.
A function from the set of real numbers to the real numbers can be represented by a graph in the Cartesian plane, such a function is continuous if, the graph is a single unbroken curve with no "holes" or "jumps".
When we plot function values generated in a laboratory or collected in the field , we often connect the plotted points with an unbroken curve to show what the functions values are likely to have been at the times we did not measure. In doing so, we are assuming that we are working with a continuous function, so its outputs vary continuously with the inputs and do not jump from one value to another without taking on the values in between.
The limit of a continuous function as x approaches c can be found simply by calculating the value of the function at c.
Any function whose graph without lifting the pencil, can be sketched in one continuous motion over its domain is an example of a continuous function. Now what it means for a function to be continuous we scrutinize it more accurately.
What is Continuity?

2.2.1 Definition

The function f is continuous at some point c of its domain if the limit of $f(y)$ as y approaches c through the domain of f exists and is equal to $f(c)$. In mathematical notation, this is written as

$$\lim_{y\to c} f(y) = f(c)$$

In detail this means three conditions . first, f has to be defined at c. Second, the limits on the left hand side and right hand side of that equation has to exist. Third, the value of this limit must equal $f(c)$.

2.2.2 Examples

(1) The function

$$f(t) = \frac{2t - 1}{t + 2}$$

is defined for all real numbers $t \neq -2$ and is continuous at every such point. The question of continuity at $t = -2$ does not arise, since $t = -2$ is not in the domain of f.
(2) The sine function $g(t) = \frac{sint}{t}$ defined for all $t \neq 0$ is continuous at these points.
(3) The function

$$f(t) = \begin{cases} t+1 & if\, t>2 \\ t-1 & if\, t<2 \end{cases}$$

Here.
When $t > 2$, $f(t) = t + 1$ is a polynomial, so it is continuous . When $t < 2$, $f(t) = t - 2$

is also a polynomial, so it is continuous. At $t = 2$, f is not defined, so it is not continuous. Thus, f is continuous everywhere , except at 2.

2.3 Differentiation

In calculus, a derivative helps to determine the rate at which a function changes, and it is regarded as one of the most important ideas in calculus. There are number of applications of derivatives as it is used to calculate velocity and acceleration, to set levels of production so as to maximize, efficiency to estimate the rate of spread of a disease, and for many other uses. With the help of derivative we can estimate complicated functions.

2.3.1 The Derivative as a function

The derivative as a function derived from f by considering the limit at each point of the domain of f.

Defination

The derivative of the function $f(x)$ with respect to the variable x is the function f' whose value at x is

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

provided the limit exists. The domain of f is the set of points in the domain of f for which the limit exists, and the domain may be the same or smaller than the domain of f. If exists at a particular x, we say that f is differentiable (has a derivative)at x. If exists at every point in the domain of f, we call f differentiable.

2.3.2 Differentiable on an interval; One-sided derivatives

A function $y = f(x)$ is differentiable on an open interval (finite or infinite) if it has a derivative at each point of the interval. It is differentiable on a closed interval $[a,b]$ if it is differentiable on the interior (a,b) and if the limits

$$\lim_{h \to 0^+} \frac{f(a+h) - f(a)}{h}$$

Right-hand derivative at a.

$$\lim_{h \to 0^-} \frac{f(b+h) - f(b)}{h}$$

left-hand derivative at b.

exist at the endpoints. Right-hand and left-hand derivatives may be defined at any point of a functions domain. A function has a derivative at a point if and only if it has left-hand and right-hand derivatives there, and these one-sided derivatives are equal.

2.3.3 When does a function not have a derivative at a point?

How about a function that is continuous all over but is not differentiable in all places?. These results pretty often with piecewise functions, although two intervals might be connected, the slope can change radically at their junction.

2.3.4 Can we differentiate any function anywhere?

The Functions that look like straight lines in the neighborhood of the point , (at which you want to differentiate), to those functions only the differentiation can be applied. Differentiating is finding the slope of the line. It looks like (the tangent line to the function we are considering). No tangent line means no derivative. Also when the tangent line is straight vertical the derivative would be infinite and that is no good either.

2.4 Integration

Integration is referred as an elegant technique or tool which is used to calculate the volumes and areas of general shapes like spheres, cones etc. As a tool its usability range is very wide because its use is not restricted up to calculation of areas and volumes but also have number of application in several other fields like sciences, engineering , and statistics. Basically the concept at the back of integration is the effective computation of several quantities by splitting them in small pieces and then summing the contributions from each small part.

2.4.1 Estimating with finite sums

Finite sums are effectively estimate the area average, values and the distance that an object covered over time. Finite sums play key role in order to define the integral. By taking sum of collection of rectangle areas, the area of a region with a curved boundary can be estimated. More rectangles can contribute towards the more accuracy of the approximation.

2.4.2 Indefinite integral

An anti-derivative of a function g is a function G, if $G' = g$. In order to find out function g anti-derivative, most often we can reverse the process of differentiation. The notation used to refer to anti-derivatives is the indefinite integral. $\int g(t)dt$ means the anti-derivative of g with respect to t . If G is an anti-derivative of g, we can write $\int g(t)dt = G + c$, where c is known as the constant of integration.

2.4.3 Definite integral

The definite integral that is used to represent the left-hand and right-hand approximations is a suitable notation. $\int_a^b g(x)dx$ means the area of the region bounded by g, the $y-axis$ and the lines $x = a$ and $x = b$. Writing $\int_a^b g(x)dx$ is equivalent to writing

$$\lim_{n\to\infty} \sum_{k=0}^{n-1} g(c_k)\Delta x_k$$

on the interval $[a,b]$.
Where Δx_k is the width of kth rectangle and c_k is the x-coordinate of the point where the kth rectangle touches $g(x)$.

2.4.4 Integrable and non-integrable functions

That functions which are continuous on the interval $[a, b]$ are integrable on that particular interval and the functions which are not continuous may or may not be integrable. Those discontinuous functions that are increasing on $[a \ , b]$ are included in integrable function as well as the piecewise-continuous functions are also integrable.
There are numerous function that are non-differentiable in nature for example those having discontinuities, sharp turns, and vertical slopes.
Discontinuous functions are also known as non-differentiable, though sharp turns and vertical slopes functions are called integrable.
Let's say, the function $s = |t|$ has a sharp point at $t = 0$, so the function is non-differentiable at this point. Though, for all values of t the same function is integrable. Its one of the extremely numerous illustrations of a function which is not differentiable but integrable. So, the set of differentiable functions is actually a subset of the set of integrable functions. However, practically the integral computing of most functions is more difficult than derivative computation. For integrability to fail, a function needs to be appropriately discontinuous so that the region between its graph and the x-axis cannot be approximated well.
Example A non-integrable function on the interval $[0, 1]$ is the function

$$g(t) = \begin{cases} 1 & if\ t\ is\ rattional \\ 0 & if\ t\ is\ irrational \end{cases}$$

2.4.5 The mean value theorem for integrals

The mean value theorem for integrals asserts as if g is a continuous function on $[a, b]$, there exists at least one c on $[a \ , b]$ such that

$$g\ (c) = \frac{1}{(b - \ a)} \int_a^b g(t)\ dt$$

That is to say the mean value theorem for integrals states that every continuous function, at least once on an interval attains its average value.

2.5 Integral inequalities

Various type of integral inequalities are utilized in most subjects involving mathematical analysis. They are particularly useful for approximation theory and numerical analysis in which estimates of approximation errors are involved. Integral inequalities provide explicit bounds on unknown functions. Integral inequalities are a necessary tool in the study of various classes of differential and integral equations. They are now used not only in mathematics but also in the areas of physics, computer and biological sciences. Many nonlinear dynamical systems are too complicated to be effectively analyzed by integral inequalities. In the study of qualitative behavior of solutions of certain nonlinear differential and integral equations, some specific types of inequalities are needed in various situations. ([4])

1. **Midpoint inequality:**
   The mid point inequality is known as

$$\left| \int_x^y g(t)dt - (y - x)g\left(\frac{x + y}{2}\right) \right| \leq \frac{(y - x)^3}{12} \left\| g''(t) \right\|_\infty$$

   Let $g : [x,y] \to R$ be a twice differentiable mapping such that $g''(t)$ exists on (x,y) and $\left\| g''(t) \right\|_\infty = \sup_{t\in(x,y)} \left| g''(t) \right| < \infty$.

2. **Trapezoid inequality:**

$$\left| \int_x^y g(t)dt - (y - x)\frac{g(x) + g(y)}{2} \right| \leq \frac{(y - x)^3}{12} \left\| g''(t) \right\|_\infty$$

3. **Hermite- Hadamard's inequality:**

$$g\left(\frac{x + y}{2}\right) \leq \frac{1}{y - x} \int_x^y g(t)dt \leq \frac{g(x) + g(y)}{2}.$$

4. **Simpson's type inequality:**
   Let $g : [x ,y] \to \mathbb{R}$ be a four times continuously differentiable mapping on (x,y) and $\left\| g^{(4)} \right\|_\infty = \sup_{t\in(x,y)} \left| g^{(4)}(t) \right| < \infty$, then we have the following inequality:

$$\left| \left[\frac{1}{6}g(x) + \frac{2}{3}g\left(\frac{x+y}{2}\right) + \frac{1}{6}g(y)\right] - \frac{1}{y-x}\int_x^y g(t)\,dt \right| \leq \frac{1}{2880} \left\| g^{(4)} \right\|_\infty (y-x)^4.$$

5. **Power means inequality:**
   Let r be a non-zero real number. We define the $r-mean$ or rth power mean of non-negative numbers $a_1, \ldots, a_n$ an as follows:

$$M^r(a_1, \ldots a_n) = \left(\frac{1}{n}\sum_{i=1}^{n}(a_i)^r\right)^{\frac{1}{r}}$$

   If $r < 0$, and $a_k = 0$ for some k, we define $(a_1, \ldots, a_n) = 0$.
   The ordinary arithmetic mean is M^1, M^2 is the quadratic mean, M^{-1} is the harmonic mean.
   Furthermore we define the 0-mean to be equal to the geometric mean:

$$M^0(a_1, \ldots a_n) = \left(\prod_{i=1}^{n}(a_i)\right)^{\frac{1}{n}}$$

   Then for any real numbers r, s such that $r < s$, the following inequality holds:

$$M^r(a_1, \ldots a_n) \leq M^s(a_1, \ldots a_n)$$

   Equality holds if and only if $a_1 = \ldots = a_n$, or $s \leq 0$ and $a_k = 0$ for some k.

6. **Holder's inequality:**
Let $[\xi_i]_{i=1}^n$ and $[\Phi_i]_{i=1}^n$ be positive n-tuples and p, q be two non-zero numbers such that $\frac{1}{p} + \frac{1}{q} = 1$ then

$$(\Sigma_{k=1}^n \xi_k \Phi_k) \leq (\Sigma_{k=1}^n \xi_k^p)^{\frac{1}{p}} (\Sigma_{k=1}^n \phi_k^q)^{\frac{1}{q}}$$

Equality holds if and only if $\frac{\xi_1^p}{\Phi_1^q} = \frac{\xi_2^p}{\Phi_2^q} \cdots \frac{\xi_n^p}{\Phi_n^q}$.

2.6 Convex function

The modern view point on convex functions implies a strong and clear interaction between analysis, engineering and geometry, making the reader to share a sense of excitement. In an article R. J. Gardner [5], described this reality in beautiful words " convexity like an octopus tentacles and changes its color and shape, cruise area to another and obviously, the research opportunities abound". In recent years, the concept of convex functions generalized on a large scale. Many fields of modern analysis directly or indirectly includes the applications of convex functions.
Some definitions, characterizations theorems and remarks about convex functions are given below which are related to our research work. (see [5]Ch:1)

2.6.1 Convex set

A set $C \subset \mathbb{R}^n$ is convex if $[\alpha\ , \beta] \subset C$ for every $\alpha,\ \beta \in C$ then $\lambda\alpha + \ (1- \ \lambda)\beta$ must also belong to C for each $\lambda \in [0, 1]$.
Example:
Let $v = (3, 2)$. show that the set $C = \{u \in \mathbb{R}^{\,2} \mid u.v \leq 9\}$ is a convex set
Let $C = \{u \in \mathbb{R}^{\,2} \mid u.v \leq 9\}$
The set C is convex if for any $u, w \in C\ ,\ \forall\, t \in [0, 1]$,
The point $tu\ +\ (1\ -\ t)\,w \in C\ , \forall\, t \in [0, 1]$,
Let $u, w \in C\ ,\ \forall\, t \in [0, 1]$,
$(tu\ +\ (1\ -\ t)\,w)\,.v$
$= (tu).v\ +\ [(1\ -\ t)\,w]\,.v$
$= t(u.v)\ +\ (1\ -\ t)\,(w.v)$
$\leq t \times 9\ +\ (1\ -\ t) \times 9$
$= 9.$

2.6.2 Convex function

A function $\Psi : I \to \mathbb{R}$ is called convex (i.e $I \subset \mathbb{R}$) if for all $\xi_1, \xi_2 \in I$ and $\lambda \in [0\ , 1]$, the inequality

$$\Psi\Big(\lambda\xi_1 + \big(1-\ \lambda\big)\xi_2\Big) \leq \lambda\Psi(\xi_1) + \big(1-\ \lambda\big)\Psi(\xi_2),$$

holds. If the strict inequality holds in above inequality then the function is said to be strictly convex function.

2.6.3 Concave function

If "$-\Psi$" is convex then Ψ is called concave function.

2.6.4 Geometric interpretation of convex function

The geometrical interpretation of convex function is that the graph of Ψ lies below its chords.

$$\Psi\big(\lambda\xi_1+\big(1-\lambda\big)\xi_2\big)\leq\lambda\Psi(\xi_1)+\big(1-\lambda\big)\Psi(\xi_2). \tag{1}$$

Examples:

- $e^x,\ x^p(p\geq 1, x>0), \frac{1}{x}(x\neq 0)$ are convex functions.
- $\log x(x>0),\ sinx(0\leq x\ \leq\pi), cosx(\frac{-\pi}{2}\leq x\ \leq\frac{\pi}{2})$ are concave functions.

Remarks:

1. For $\xi_1,\xi_2\in I, p,q\geq 0, p+q>0, then$
$$\Psi\big(\lambda\xi_1+\big(1-\lambda\big)\xi_2\big)\leq\lambda\Psi(\xi_1)+\big(1-\lambda\big)\Psi(\xi_2), \tag{2}$$
is equivalent to
$$\frac{\Psi(p\xi_1+q\xi_2)}{p+q}\geq\frac{p\Psi(\xi_1)+q\Psi(\xi_2)}{p+q}.$$
2. Ψ is both convex and concave if and only if $\Psi(\xi)\ =\lambda\xi\ +\gamma$, for some $\lambda,\gamma\in\mathbb{R}$.
3. If $\xi_1,\ \xi_2,\xi_3$ are three points in I such that $\xi_1<\xi_2<\xi_3$, then 2 is equivalent to
$$\begin{vmatrix}\xi_1 & \Psi(\xi_1) & 1\\ \xi_2 & \Psi(\xi_2) & 1\\ \xi_3 & \Psi(\xi_3) & 1\end{vmatrix}=(\xi_3-\ \xi_2)\Psi(\xi_1)+\ (\xi_1-\ \xi_3)\Psi(\xi_2)+(\xi_2-\ \xi_1)\Psi(\xi_3)\geq 0,$$
which is equivalent to
$$\Psi(\xi_2)\leq\frac{\xi_2-\ \xi_3}{\xi_1-\ \xi_3}\Psi(\xi_1)+\frac{\xi_1-\ \xi_2}{\xi_1-\ \xi_3}\Psi(\xi_3),$$
or more symmetrically and without the condition of monotonicity on ξ_1,ξ_2,ξ_3
$$\frac{\Psi(\xi_1)}{(\xi_1-\ \xi_2)(\xi_1-\xi_3)}\ +\ \frac{\Psi(\xi_2)}{(\xi_2-\xi_3)(\xi_2\ -\xi_1)}\ +\frac{\Psi(\xi_3)}{(\xi_3-\ \xi_1)(\xi_3-\ \xi_2)}\geq 0.$$

4. If Ψ is a convex function on I and if $\xi_1 \leq \varphi_1, \xi_2 \leq \varphi_2, \xi_1 \neq \xi_2, \varphi_1 \neq \varphi_2$, then the following inequality

$$\frac{\Psi(\xi_2)-\Psi(\xi_1)}{\xi_2-\xi_1} \leq \frac{\Psi(\varphi_2)-\Psi(\varphi_1)}{\varphi_2-\varphi_1},$$

holds.

5. The area of the triangle whose vertices are $(\xi_1,\Psi(\xi_1)),(\xi_2,\Psi(\xi_2))$ and $(\xi_3,\Psi(\xi_3))$ is given by

$$P=\frac{1}{2}\begin{vmatrix}\xi_1 & \Psi(\xi_1) & 1\\ \xi_2 & \Psi(\xi_2) & 1\\ \xi_3 & \Psi(\xi_3) & 1\end{vmatrix},$$

The area of the triangle depends on the rotation if the rotation is anticlockwise then area will be positive and if the rotation is clockwise then area will be negative. $P>0$ represent a convex function and $P<0$ express a concave function.

2.6.5 Properties of convex function

1. Composition of convex function is convex.i.e if Ψ_1 is convex and increasing and a function Ψ_2 is convex then the composition $\Psi_1(\Psi_2(\xi))$ is convex.
2. Addition of convex function is convex.i.e Ψ_1,Ψ_2 convex then $\Psi_1+\Psi_2$ convex.
3. Scalar multiplication of convex function is convex.i.e Ψ convex $\Rightarrow \alpha\Psi$ convex, $\alpha>0$.
4. Function of the form $Ax+b$ is convex(Affine).
5. Pointwise maximum.i.e Ψ_1,Ψ_2 convex $\Rightarrow max(\Psi(\xi_1),\Psi(\xi_2))$ convex.
6. Pointwise superimum.i.e Ψ_α convex $\Rightarrow \sup_{\alpha\in A}\Psi_\alpha$ convex([6],([7]).

In [8], Dragomir et. al proved the following recent developments on Simpson's inequality for which the remainder is expressed in terms of derivatives lower than the fourth.

Theorem 1. *Let $f:[a,b]\to\mathbb{R}$ is a differentiable mapping whose derivative is continuous on (a,b) and $f'\in L[a,b]$. Then we have the following inequality:*

$$\left|\left[\frac{1}{6}f(a)+\frac{2}{3}f\left(\frac{a+b}{2}\right)+\frac{1}{6}f(b)\right]-\frac{1}{b-a}\int_a^b f(x)\,dx\right| \leq \frac{(b-a)}{3}\left\|f'\right\|_1, \qquad (3)$$

where $\|f'\|_1 = \int_a^b |f'(x)|\,dx$.

The bound of (3) for L-Lipschitzian mapping was given in by $\frac{5}{36}(b-a)$.

In ([9]) Hwang et. al presented inequalities for differentiable convex functions which are linked with Simpson's inequality, and the main inequality in ([10]),([11]),([12]) pointed out, is as follows.

Theorem 2. *Let $I \subset \mathbb{R}$ be an open interval, $a,b \in I$ with $a < b$ and $f : [a,b] \to \mathbb{R}$ be a differentiable function such that f' is integrable and $0 < \alpha \leq 1$ on (a,b) with $a < b$. If $|f'|$ is convex on $[a,b]$, then the following integral inequality holds:*

$$
\begin{aligned}
&\left|\frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}\left[J_{a^+}^{\alpha} f(b)+J_{b^-}^{\alpha} f(a)\right]-\left[\frac{5^{\alpha}-1}{6^{\alpha}} f\left(\frac{a+b}{2}\right)+\frac{6^{\alpha}-5^{\alpha}+1}{6^{\alpha}} \frac{f(a)+f(b)}{2}\right]\right| \\
\leq\ & \left[\frac{1}{\alpha+1}\left(\frac{2^{\alpha}+1}{2^{\alpha+1}}-\frac{5^{\alpha+1}+1}{6^{\alpha+1}}\right)+\left(\frac{5^{\alpha}-1}{12.6^{\alpha}}\right)\right](b-a)\left(\left|f'(a)\right|+\left|f'(b)\right|\right) \qquad (4)
\end{aligned}
$$

Proposition 1. *Under the assumptions of Theorem 2 with $\alpha = 1$, we have the following inequality,*

$$
\left|\frac{1}{6}\left[f(a)+4 f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{1}{b-a} \int_a^b f(x)\, dx\right| \leq \frac{5(b-a)}{72}\left(\left|f'(a)\right|+\left|f'(b)\right|\right). \quad (5)
$$

Theorem 3. *Let $f : I \subset [0,\infty) \to \mathbb{R}$ be a differentiable mapping on I^0 such that $f' \in L_1[a,b]$ where $a,b \in I$ with $a < b$. If $|f'|^q$ is $(s,m)-$convex on $[a,b]$, for some fixed $s \in (0,1]$ and $q \geq 1$, then the following inequality holds:*

$$
\begin{aligned}
& \frac{1}{6}\left[f(ma)+4 f\left(\frac{ma+b}{2}\right)+f(b)\right]-\frac{1}{b-ma} \int_{ma}^b f(x)\, dx \\
\leq\ & \frac{(b-ma)}{2}\left(\frac{5}{72}\right)^{1-1/q} \\
\times\ & \left\{\left(u_1\left|f'(b)\right|^q+m u_2\left|f'(a)\right|^q\right)^{1/q}+\left(u_3\left|f'(b)\right|^q+u_4\left|f'(a)\right|^q\right)^{1/q}\right\}. \qquad (6)
\end{aligned}
$$

$$
u_1=\frac{\left(3^{-\alpha}\right)\left(2^{1-\alpha}\right)+3(\alpha)\left(2^{1-\alpha}\right)+3\left(2^{-\alpha}\right)}{6^3(\alpha+1)(\alpha+2)}, \quad u_2=\left(\frac{5}{72}-u_1\right),
$$

$$
\begin{aligned}
u_3 &= \frac{\left(5^{\alpha+2}\right)\left(3^{-\alpha}\right)\left(2^{1-\alpha}\right)-3(\alpha)\left(2^{1-\alpha}\right)-21\left(2^{-\alpha}\right)+6(\alpha)-24}{6^3(\alpha+1)(\alpha+2)}, \\
u_4 &= \left(\frac{5}{72}-u_3\right).
\end{aligned}
$$

Proposition 2. *Under the assumptions of Theorem 3 with $\alpha = 1$, if $|f'(x)|^q \leq Q \ \forall\, x \in I$, we have the following inequality,*

$$
\left|\frac{1}{6}\left[f(a)+4 f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{1}{b-a} \int_a^b f(x)\, dx\right| \leq \frac{5(b-a)}{36} . Q. \quad (7)
$$

In [13] Dragomir and Agarwal, obtained inequalities for differentiable convex mappings which are connected with the right-hand side of Hermite- Hadamard's (trapezoid) inequality and applied them to obtain some elementary inequalities for real numbers and in numerical integration which follows

Theorem 4. *Let $g : I \subset R \rightarrow R$ be a differentiable mapping on I^o where $x, y \in I$ with $x < y$. If $|g'|^q$ is convex on $[x, y]$, for some $q \geq 1$ then the following inequality holds:*

$$\left|\frac{g(x) + g(y)}{2} - \frac{1}{y-x}\int_x^y g(u)du\right| \leq \frac{y-x}{8}\left[\left|g'(x)\right| + \left|g'(y)\right|\right] \tag{8}$$

In [14], a variant of Hermite−Hadamard type inequalities was obtained, which follows as:

Theorem 5. *Let $f : I^\circ \subseteq \mathbb{R} \rightarrow \mathbb{R}$ be a differentiable function on I° and let $a, b \in I^\circ$ with $a < b$. If $|f'|$ is convex function on $[a, b]$, then the following inequality holds:*

$$\left|\frac{1}{y-x}\int_x^y g(u)du - g\left(\frac{x+y}{2}\right)\right| \leq \frac{y-x}{8}\left[\left|g'(x)\right| + \left|g'(y)\right|\right] \tag{9}$$

In [15] Yang, obtained Hermite-Hadamard's (trapezoid) inequalities for differentiable mapping for concave function.

Theorem 6. *Let $I \subset \mathbb{R}$ be an open interval , $l, m, n, P, Q \in I$ with $l \leq P \leq n \leq Q \leq m$ $(n \neq l, m)$ and $g : [x, y] \rightarrow \mathbb{R}$ be a differentiable function . If $|g'|^q$ is concave on $[x, y]$ and $1 \leq \theta \leq q$, then*

$$\left|(P-l)\,g(l) + (m-Q)\,g(m) + (Q-P)\,g(n) - \int_x^y g(u)du\right| \leq K(P, Q, n, \theta)\,.J(P, Q, n, \theta) \tag{10}$$

Where

$$K(P, Q, n, \theta) = \left(\frac{1}{2}\left[(P-l)^2 + (n-P)^2 + (Q-n)^2 + (m-Q)^2\right]\right)^{\frac{(\theta-1)}{\theta}},$$

and

$$J(P, Q, n, \theta)$$

$$= \left(\frac{1}{2}\left[(P-l)^2 + (n-P)^2\right]\left|g'\left(\frac{(P-l)^2 + (n-P)^2(2n-3l+P)}{3\left[(P-l)^2+(n-P)^2\right]} + l\right)\right|^{\theta}\right)^{\frac{(\theta-1)}{\theta}}$$

$$+ \left(\frac{1}{2}\left[(Q-n)^2+(m-Q)^2\right]\left|g'\left(m - \frac{(Q-n)^2(3m-2n-Q)+(m-Q)^3}{3\left[(Q-n)^2 + (m-Q)^2\right]}\right)\right|^{\theta}\right)^{\frac{(\theta-1)}{\theta}}$$

Proposition 3. *Under the assumptions of Theorem (6) with $P = Q = n = (l+m)/2$ and $\theta = 1$, then we get the following inequality,*

$$\left|\frac{g(x) + g(y)}{2} - \frac{1}{y-x}\int_x^y g(u)du\right| \leq \frac{y-x}{8}\left[\left|g'(\frac{5x+y}{6})\right| + \left|g'\left(\frac{x+5y}{6}\right)\right|\right] \tag{11}$$

Proposition 4. *Under the assumptions of Theorem* 6 *with* $P = x, Q = y, n = (l + m)/2$ *and* $\theta = 1$, *then we get the following inequality,*

$$\left| g\left(\frac{x + y}{2}\right) - \frac{1}{y-x}\int_x^y g(u)du \right| \leq \frac{y - x}{8}\left[\left| g'(\frac{2x + y}{3}) \right| + \left| g'\left(\frac{x + 2y}{3}\right) \right| \right] \tag{12}$$

Definition 1. Let $f \in L^1[a,b]$. The left$-$ sided and right$-$sided Riemann $-$Liouville fractional integrals of order $\alpha > 0$ with $a \geq 0$ are defined by

$$J_{a^+}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x - t)^{\alpha-1} f(t)dt, \quad a < x$$

and

$$J_{b^-}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (t - x)^{\alpha-1} f(t)dt, \quad x < b$$

respectively , where $\Gamma(.)$ is Gamma function and its definition is $\Gamma(\alpha) = \int_0^{\infty} e^{-u} u^{\alpha-1} du$. It is to be noted that $J_{a^+}^0 f(x) = J_{b^-}^0 f(x) = f(x)$. In the case of $\alpha = 1$, the fractional integral reduces to the classical integral.

Properties relating to this operator can be found in[16] and for useful details on Simpson's type inequalities connected with fractional integral inequalities.

the interested readers are directed to[17] ,[18],[19] ,[20].

Chapter 3
Main Results

3 Simpson's type inequalities

3.1 Introduction

Our aim in this section is to give some improvements and furture generalizations for Hwang and Shahid Qaiser et al [21] results, which are of trapezoid type via $(s,m)-$convex functions in second sense.
After that, we introduce some inequalities of midpoint type via $(s,m)-$convex function. Interested readers are directed to [22],[23],[24],[25],[26].
In [27] Hudzik and Maligranda considerd, among others, the class of functions which are $s-$convex in the first and second sense.

Definition 2. A function $f : [0,\infty) \to \mathbb{R}$ is said to be $s-$convex or f belongs to the class K_s^i if

$$f(ux+vy) \leq u^s f(x) + v^s f(y) \tag{13}$$

holds for all $x,y \in [0,\infty), u,v \in [0,1]$ and for some fixed $s \in (0,1]$

Note that, if $u^s + v^s = 1$, the above class of convex function is called $s-$convex in the first sense and represented by K_s^1 and if $u+v=1$,the above class is called $s-$convex in the second sense and represented by K_s^2.

In [28] V.G.Mihesan presented the class of $(s,m)-$convex fuction as reproduced below:

Definition 3. The function $f : [0,b] \to \mathbb{R}$ is said to be $(s,m)-$convex, where $(s,m) \in [0,1]^2$, if for every $x,y \in [0,b]$ and $t \in [0,1]$ we have if

$$f(tx+(1-t)y) \leq t^s f(x) + m(1-t^s) f(y) \tag{14}$$

Note that for $(s,m) \in \{(0,0),(s,0),(1,0),(1,m),(1,1),(s,1)\}$ one receive the following classes of functions respectively: increasing, $s-$starshaped, starshaped, m-convex and $s-$convex.

where $(\alpha,m) \in [0,1]^2$, for some fixed $s \in (0,1]$.

3.1.1 Inequalities of Simpson's type for $(s,m)-$convex

In order to prove our main results via $(s,m)-$convex function. we need the following lemma.

Lemma 1. *Let $I \subset \mathbb{R}$ be an open interval, $a, b \in I$ with $a < b$ and $f : [a,b] \to \mathbb{R}$ be a differentiable function such that f' is integrable and $0 < \alpha \leq 1$ on (a,b) with $a < b$. If $|f'|$ is convex on $[a,b]$, then the following integral inequality holds:*

$$\begin{aligned}
&\frac{1}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right] - \frac{\Gamma(\alpha+1)}{2(b-a)^\alpha}\left[J_{a^+}^\alpha f(b) + J_{b^-}^\alpha f(a)\right] \\
\leq\ & \frac{b-a}{2^{\alpha+1}}\left[L_1 + L_2 + (2^\alpha - 1)(L_3 + L_4)\right],
\end{aligned}$$

where

$$
\begin{aligned}
L_1 &= \int_0^1 \tfrac{1}{6}\left(1-3(1-t)^{\alpha}\right) f'(tb+(1-t)\tfrac{a+b}{2})dt,\\
L_2 &= \int_0^1 \tfrac{1}{6}\left(1-3(1-t)^{\alpha}\right) f'(ta+(1-t)\tfrac{a+b}{2})dt,\\
L_3 &= \int_0^1 \left(\tfrac{1}{2(2^{\alpha}-1)}(1+t)^{\alpha}-\tfrac{1}{2(2^{\alpha}-1)}-\tfrac{1}{3}\right) f'(tb+(1-t)\tfrac{a+b}{2})dt,\\
L_4 &= \int_0^1 \left(\tfrac{1}{2(2^{\alpha}-1)}-\tfrac{1}{2(2^{\alpha}-1)}(1+t)^{\alpha}+\tfrac{1}{3}\right) f'(ta+(1-t)\tfrac{a+b}{2})dt.
\end{aligned}
$$

Theorem 7. *Let $I \subset \mathbb{R}$ be an open interval, $a,b \in I$ with $a < b$ and $f : [a,b] \to \mathbb{R}$ be a differentiable function such that f' is integrable and $0 < \alpha \leq 1$ on (a,b) with $a < b$. If $|f'|$ is $(s,m)-$convex on $[a,b]$, then the following integral inequality holds:*

$$
\begin{aligned}
&\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}\left[J_{a^+}^{\alpha}f(b)+J_{b^-}^{\alpha}f(a)\right]\\
\leq\ &\left[\left\{M_1\left|f'(a)\right|+2m(M_2-M_1)\left|f'(\frac{a+b}{2m})\right|+M_1\left|f'(b)\right|\right\}\right.\\
&\left.+\left\{\frac{M_3}{2(2^{\alpha}-1)}\left|f'(a)\right|+\frac{m(M_4-M_3)}{2(2^{\alpha}-1)}\left|f'(\frac{a+b}{2m})\right|+\frac{M_3}{2(2^{\alpha}-1)}\left|f'(b)\right|\right\}\right]. \quad (15)
\end{aligned}
$$

$$
\begin{aligned}
M_1 &= \int_0^1 \left|\frac{1}{6}\left(1-3(1-t)^{\alpha}\right)dt\right| t^s dt\\
&= \frac{(1-\nu)^{s+1}}{(s+1)}-\frac{\alpha(1-\nu)^{s+2}}{(s+2)}-\frac{(1-\nu)^{s+1}}{3(s+1)}\\
&\quad -\frac{1}{2(s+1)}+\frac{\alpha}{2(s+2)}-\frac{1}{6(s+1)},
\end{aligned}
$$

$$
\begin{aligned}
M_2 &= \frac{1}{6}\left[\int_0^{1-\nu}\left(3(1-t)^{\alpha}-1\right)dt+\int_{1-\nu}^1\left(1-3(1-t)^{\alpha}\right)dt\right]\\
&= \frac{1}{6}-\frac{1}{3}(1-\nu)-\frac{1}{(\alpha+1)}\nu^{\alpha+1}+\frac{1}{2(\alpha+1)},
\end{aligned}
$$

$$
\begin{aligned}
M_3 &= \frac{1}{2(2^{\alpha}-1)}\int_0^1\left|1-(1+t)^{\alpha}+\frac{2(2^{\alpha}-1)}{3}\right| t^s dt\\
&= \frac{2(2^{\alpha}-1)}{3(s+1)}(\sigma-1)^{s+1}-\frac{\alpha(\sigma-1)^{s+2}}{(s+2)}+\frac{(\sigma-1)^{s+1}}{(s+1)}\\
&\quad -\frac{2(s+2)(2^{\alpha}-1)-3\alpha(s+1)}{6(s+1)(s+2)},
\end{aligned}
$$

$$
\begin{aligned}
M_4 &= \frac{1}{6}\left[\int_0^{\sigma-1}\left(1-(1+t)^{\alpha}+\frac{2(2^{\alpha}-1)}{3}\right)dt+\int_{\sigma-1}^1\left((1+t)^{\alpha}-1-\frac{2(2^{\alpha}-1)}{3}\right)dt\right]\\
&= \left[\frac{1}{3}+\frac{1}{2(2^{\alpha}-1)}\right](\sigma-1)-\frac{\sigma^{\alpha+1}}{(2^{\alpha}-1)(\alpha+1)}+\frac{1}{(2^{\alpha}-1)(\alpha+1)}\left(2^{\alpha}+\frac{1}{2}\right)\\
&\quad -\left[\frac{2(2^{\alpha}-1)+3}{6(2^{\alpha}-1)}\right],
\end{aligned}
$$

$$
\nu=\left(\frac{1}{3}\right)^{\frac{1}{\alpha}} \qquad \sigma^{\alpha}=\frac{2(2^{\alpha}-1)}{3}+1,
$$

Proof. Using $(s,m)-$ convexity of $|f'|$, for all $t\in[0,1]$, we obtain

$$\begin{aligned}
|J_1| &\leq \int_0^1 \frac{1}{6}\left(1-3(1-t)^{\alpha}\right)|f'(tb+(1-t)\frac{a+b}{2})|dt\\
&= \int_0^1 \frac{1}{6}\left(1-3(1-t)^{\alpha}\right)|t^s f'(b)+m(1-t^s)f'(\frac{a+b}{2m})|dt\\
&\leq \int_0^1 \frac{1}{6}\left(1-3(1-t)^{\alpha}\right)\left\{t^s\left|f'(b)\right| +m(1-t^s)\left|f'(\frac{a+b}{2m})\right|\right\}dt\\
&= M_1\left|f'(b)\right| +m(M_2-M_1)\left|f'(\frac{a+b}{2m})\right|.
\end{aligned}$$

$$\begin{aligned}
|J_2| &\leq \int_0^1 \frac{1}{6}\left(3(1-t)^{\alpha}-1\right)|f'(ta+(1-t)\frac{a+b}{2})|dt\\
&= \int_0^1 \frac{1}{6}\left(3(1-t)^{\alpha}-1\right)|t^s f'(a)+m(1-t^s)f'(\frac{a+b}{2m})|dt\\
&\leq \int_0^1 \frac{1}{6}\left(3(1-t)^{\alpha}-1\right)\left\{t^s\left|f'(a)\right| +m(1-t^s)\left|f'(\frac{a+b}{2m})\right|\right\}dt\\
&= M_1\left|f'(a)\right| +m(M_2-M_1)\left|f'(\frac{a+b}{2m})\right|.
\end{aligned}$$

$$\begin{aligned}
|J_3| &\leq \frac{1}{2(2^{\alpha}-1)}\int_0^1\left((1+t)^{\alpha}-1-\frac{2(2^{\alpha}-1)}{3}\right)\times\\
&\quad |f'(tb+(1-t)\frac{a+b}{2})|dt\\
&= \frac{1}{2(2^{\alpha}-1)}\int_0^1\left((1+t)^{\alpha}-1-\frac{2(2^{\alpha}-1)}{3}\right)\times\\
&\quad |t^s f'(b)+m(1-t^s)f'(\frac{a+b}{2m})|dt\\
&\leq \frac{1}{2(2^{\alpha}-1)}\int_0^1\left((1+t)^{\alpha}-1-\frac{2(2^{\alpha}-1)}{3}\right)\times\\
&\quad \left\{t^s\left|f'(b)\right| +m(1-t^s)\left|f'(\frac{a+b}{2m})\right|\right\}dt\\
&= \frac{M_3}{2(2^{\alpha}-1)}\left|f'(b)\right|+\frac{m(M_4-M_3)}{2(2^{\alpha}-1)}\left|f'(\frac{a+b}{2m})\right|.
\end{aligned}$$

$$
\begin{aligned}
|J_4| &\le \frac{1}{2(2^\alpha-1)}\int_0^1\left((1+t)^\alpha-1-\frac{2(2^\alpha-1)}{3}\right)\times \\
&\quad |f'(ta+(1-t)\frac{a+b}{2})|dt \\
&= \frac{1}{2(2^\alpha-1)}\int_0^1\left((1+t)^\alpha-1-\frac{2(2^\alpha-1)}{3}\right)\times \\
&\quad |t^s f'(a)+m(1-t^s)f'(\frac{a+b}{2m})|dt \\
&\le \frac{1}{2(2^\alpha-1)}\int_0^1\left((1+t)^\alpha-1-\frac{2(2^\alpha-1)}{3}\right)\times \\
&\quad \left\{t^s\left|f'(a)\right|+m(1-t^s)\left|f'(\frac{a+b}{2m})\right|\right\}dt \\
&= \frac{M_3}{2(2^\alpha-1)}\left|f'(a)\right|+\frac{m(M_4-M_3)}{2(2^\alpha-1)}\left|f'(\frac{a+b}{2m})\right|.
\end{aligned}
$$

Remark 1. *If we take $\alpha=s=m=1$ in Theorem 7 then inequality (15) reduces to inequality* (5).

The corresponding version for powers of the absolute value of the derivative is incorporated in the following theorem.

Theorem 8. *Let $I\subset\mathbb{R}$ be an open interval, $a,b\in I$ with $a<b$ and $f:[a,b]\to\mathbb{R}$ be a differentiable function such that f' is integrable and $0<\alpha\le 1$ on (a,b) with $a<b$ and $0\le\lambda\le 1$. if $|f'|^q$ is a convex on $[a,b]$, with $q\ge 1$, then the following inequality holds*:

$$
\begin{aligned}
&\left|\left[\left(1-\frac{2}{2^\alpha}\lambda\right)f\left(\frac{a+b}{2}\right)+\lambda\frac{f(a)+f(b)}{2^\alpha}\right]-\frac{\Gamma(\alpha+1)}{2(b-a)^\alpha}\left[J_{a^+}^\alpha f(b)+J_{b^-}^\alpha f(a)\right]\right|\le\frac{(b-a)}{2^\alpha} \\
&\left[(M_2)^{1-1/q}\left\{\left(M_1\left|f'(b)\right|^q+m(M_2-M_1)\left|f'\left(\frac{a+b}{2m}\right)\right|^q\right)^{1/q}+\left(M_1\left|f'(a)\right|^q+m(M_2-M_1)\left|f'\left(\frac{a+b}{2m}\right)\right|^q\right)^{1/q}\right\}\right. \\
&+\left.(M_4)^{1-1/q}\left\{\left(M_3\left|f'(b)\right|^q+m(M_4-M_3)\left|f'\left(\frac{a+b}{2m}\right)\right|^q\right)^{1/q}+\left(M_3\left|f'(b)\right|^q+m(M_4-M_3)\left|f'\left(\frac{a+b}{2m}\right)\right|^q\right)^{1/q}\right\}\right]
\end{aligned}
\tag{16}
$$

Proof. Using the well -known power -mean integral inequality for $q>1$, and convexity of $|f'|^q$ we have

$$
\begin{aligned}
|J_5| &\le \left(\int_0^1\left|\frac{1}{6}\left(1-3(1-t)^\alpha\right)\right|dt\right)^{1-1/q} \\
&\quad\times\left(\int_0^1\left|\frac{1}{6}\left(1-3(1-t)^\alpha\right)\right|\left|f'\left(tb+(1-t)\frac{a+b}{2}\right)\right|^q dt\right)^{1/q} \\
&\le \left(\int_0^1\left|\frac{1}{6}\left(1-3(1-t)^\alpha\right)\right|dt\right)^{1-1/q} \\
&\quad\times\left(\int_0^1\left|\frac{1}{6}\left(1-3(1-t)^\alpha\right)\right|\left\{t^s\left|f'(b)\right|^q+m(1-t^s)\left|f'(\frac{a+b}{2m})\right|^q\right\}dt\right)^{1/q} \\
&\le (M_2)^{1-1/q}\left(M_1\left|f'(b)\right|^q+m(M_2-M_1)\left|f'(\frac{a+b}{2m})\right|^q\right)^{1/q}
\end{aligned}
$$

$$
\begin{aligned}
|J_6| \leq & \left(\int_0^1 \left|\frac{1}{6}\left(1 - 3(1 - t)^{\alpha}\right)\right| dt\right)^{1-1/q} \\
& \times \left(\int_0^1 \left|\frac{1}{6}\left(1 - 3(1 - t)^{\alpha}\right)\right| \left|f'\left(ta + (1 - t)\frac{a + b}{2}\right)\right|^q dt\right)^{1/q} \\
\leq & \left(\int_0^1 \left|\frac{1}{6}\left(1 - 3(1 - t)^{\alpha}\right)\right| dt\right)^{1-1/q} \\
& \times \left(\int_0^1 \left|\frac{1}{6}\left(1 - 3(1-t)^{\alpha}\right)\right| \left\{t^s \left|f'(a)\right|^q + m(1 - t^s)\left|f'(\frac{a + b}{2m})\right|^q\right\} dt\right)^{1/q} \\
\leq & (M_2)^{1-1/q}\left(M_1 \left|f'(a)\right|^q + m(M_2 - M_1)\left|f'(\frac{a + b}{2m})\right|^q\right)^{1/q}
\end{aligned}
$$

$$
\begin{aligned}
|J_7| \leq & \left(\frac{1}{2(2^{\alpha}-1)}\int_0^1 \left|\left((1 + t)^{\alpha} - 1 - \frac{2(2^{\alpha} - 1)}{3}\right)\right| dt\right)^{1-1/q} \\
& \times \left(\frac{1}{2(2^{\alpha}-1)}\int_0^1 \left|\left((1 + t)^{\alpha} - 1 - \frac{2(2^{\alpha} - 1)}{3}\right)\right| \left|f'\left(tb + (1 - t)\frac{a + b}{2}\right)\right|^q dt\right)^{1/q} \\
\leq & \left(\frac{1}{2(2^{\alpha}-1)}\int_0^1 \left|\left((1 + t)^{\alpha} - 1 - \frac{2(2^{\alpha} - 1)}{3}\right)\right| dt\right)^{1-1/q} \\
& \times \left(\frac{1}{2(2^{\alpha}-1)}\int_0^1 \left|\left((1 + t)^{\alpha} - 1 - \frac{2(2^{\alpha} - 1)}{3}\right)\right| \left\{t^s \left|f'(b)\right|^q + m(1 - t^s)\left|f'(\frac{a + b}{2m})\right|^q\right\} dt\right)^{1/q} \\
\leq & (M_4)^{1-1/q}\left(M_3\left|f'(b)\right|^q + m(M_4 - M_3)\left|f'\left(\frac{a + b}{2m}\right)\right|^q\right)^{1/q}
\end{aligned}
$$

$$
\begin{aligned}
|J_8| \leq & \left(\frac{1}{2(2^{\alpha}-1)}\int_0^1 \left|\left((1 + t)^{\alpha} - 1 - \frac{2(2^{\alpha} - 1)}{3}\right)\right| dt\right)^{1-1/q} \\
& \times \left(\frac{1}{2(2^{\alpha}-1)}\int_0^1 \left|\left((1 + t)^{\alpha} - 1 - \frac{2(2^{\alpha} - 1)}{3}\right)\right| \left|f'\left(ta + (1 - t)\frac{a + b}{2}\right)\right|^q dt\right)^{1/q} \\
\leq & \left(\frac{1}{2(2^{\alpha}-1)}\int_0^1 \left|\left((1 + t)^{\alpha} - 1 - \frac{2(2^{\alpha} - 1)}{3}\right)\right| dt\right)^{1-1/q} \\
& \times \left(\frac{1}{2(2^{\alpha}-1)}\int_0^1 \left|\left((1 + t)^{\alpha} - 1 - \frac{2(2^{\alpha} - 1)}{3}\right)\right| \left\{t^s \left|f'(a)\right|^q + m(1 - t^s)\left|f'(\frac{a + b}{2m})\right|^q\right\} dt\right)^{1/q} \\
\leq & (M_4)^{1-1/q}\left(M_3\left|f'(a)\right|^q + m(M_4 - M_3)\left|f'\left(\frac{a + b}{2m}\right)\right|^q\right)^{1/q}
\end{aligned}
$$

This completes the proof.

Remark 2. *If we take $\alpha = s = m = 1$ in Theorem 8, then inequality (16) reduces to inequality as obtained in Proposition 2.(see inequality (7))*

In the following theorem , we obtain estimate of Simpson's inequality for concave functions.

Theorem 9. *Let $f : [a,b] \to \mathbb{R}$ be a differentiable function on (a,b) such that $f' \in L^1\,[a,b]$. If $|f'|^q$ is concave on $[a,b]$, for some fixed $q > 1$ then the following inequality for fractional integrals holds for $\alpha > 0$:*

$$\left|\left[\frac{1}{6}f(a) + \frac{2}{3}f\left(\frac{a+b}{2}\right) + \frac{1}{6}f(b)\right] - \frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}\left[J^{\alpha}_{a^+}f(b) + J^{\alpha}_{b^-}f(a)\right]\right|$$
$$\leq \frac{(b-a)}{2^{\alpha+1}} \times \left[M_2\left\{\left|f'\left(\frac{M_5 b + (M_2 - M_5)\left(\frac{a+b}{2}\right)}{M_2}\right)\right| + \left|f'\left(\frac{M_5 a + (M_2 - M_5)\left(\frac{a+b}{2}\right)}{M_2}\right)\right|\right\}\right.$$
$$\left. + M_4\left(2^{\alpha} - 1\right)\left|f'\left(\frac{M_6 a + (M_4 - M_6)\left(\frac{a+b}{2}\right)}{M_4}\right)\right| + \left|f'\left(\frac{M_6 b + (M_4 - M_6)\left(\frac{a+b}{2}\right)}{M_4}\right)\right|\right]. \tag{17}$$

$$\begin{aligned} M_5 &= \int_0^1 \left|\frac{1}{6}\left(1 - 3(1-t)^{\alpha}\right)\right| t\,dt \\ &= \frac{-6(1-\nu)(\nu)^{\alpha+1}}{(\alpha+1)} - \frac{6(\nu)^{\alpha+2}}{(\alpha+1)(\alpha+2)} - \frac{(1-\nu)^2}{2} \\ &\quad + \frac{3}{(\alpha+1)(\alpha+2)} - \frac{1}{2}, \end{aligned}$$

$$\begin{aligned} M_6 &= \frac{1}{2(2^{\alpha}-1)}\int_0^1 \left|1 - (1+t)^{\alpha} + \frac{2(2^{\alpha}-1)}{3}\right| t\,dt \\ &= \frac{(\sigma-1)^2}{3} - \frac{2(\sigma)^{\alpha+1}(\sigma-1)}{(\alpha+1)\ (2^{\alpha}-1)} + \frac{2(\sigma)^{\alpha+2}}{(\alpha+2)(\alpha+1)(2^{\alpha}-1)} \\ &\quad + \frac{(\sigma-1)^2}{2(2^{\alpha}-1)} - \frac{1}{2(\alpha+2)(\alpha+1)(2^{\alpha}-1)} \\ &\quad - \frac{(2^{\alpha}-1)}{6(2^{\alpha}-1)} + \frac{2^{\alpha+1}}{2(\alpha+1)(2^{\alpha}-1)} - \frac{2^{\alpha+2}}{2(2^{\alpha}-1)(\alpha+2)(\alpha+1)} - \frac{1}{4(2^{\alpha}-1)}, \end{aligned}$$

Proof. Using the concavity of $|f'|^q$ and the power -mean inequality, we obtain

$$\begin{aligned} |f'|^q &\leq t|f'|^q + (1-t)|f'|^q \\ &\leq t|f'|^q + (1-t)|f'|^q. \end{aligned}$$

Hence
$$|f'(tx + (1-t)y)| \leq t|f'(x)| + (1-t)|f'(y)|.$$

So $|f'|$ is also concave . By the Jensen integral inequality , we have

$$\begin{aligned} |J_8| &\leq \left(\int_0^1 \left|\frac{1}{6}\left(1 - 3(1-t)^{\alpha}\right)\right| dt\right)\left|f'\left(\frac{\int_0^1 \left|\frac{1}{6}\left(1 - 3(1-t)^{\alpha}\right)\right|(tb + (1-t)\frac{a+b}{2})dt}{\int_0^1 \left|\frac{1}{6}\left(1 - 3(1-t)^{\alpha}\right)\right| dt}\right)\right| \\ &= M_2\left|f'\left(\frac{M_5 b + (M_2 - M_5)\left(\frac{a+b}{2}\right)}{M_2}\right)\right|. \end{aligned}$$

$$
\begin{aligned}
|J_9| &\leq \left(\int_0^1 \left|\frac{1}{6}\left(1-3(1-t)^{\alpha}\right)\right| dt\right) \left| f'\left(\frac{\int_0^1 \left|\frac{1}{6}\left(1-3(1-t)^{\alpha}\right)\right| (ta+(1-t)\frac{a+b}{2}) dt}{\int_0^1 \left|\frac{1}{6}\left(1-3(1-t)^{\alpha}\right)\right| dt}\right)\right| \\
&= M_2 \left| f'\left(\frac{M_5 a + (M_2 - M_5)\left(\frac{a+b}{2}\right)}{M_2}\right)\right|.
\end{aligned}
$$

$$
\begin{aligned}
|J_{10}| \leq & \left(\frac{1}{2(2^{\alpha}-1)} \int_0^1 \left|\left((1+t)^{\alpha} - 1 - \frac{2(2^{\alpha}-1)}{3}\right)\right| dt\right) \\
& \times \left| f'\left(\frac{\frac{1}{2(2^{\alpha}-1)} \int_0^1 \left|\left((1+t)^{\alpha} - 1 - \frac{2(2^{\alpha}-1)}{3}\right)\right| (ta+(1-t)\frac{a+b}{2}) dt}{\frac{1}{2(2^{\alpha}-1)} \int_0^1 \left|\left((1+t)^{\alpha} - 1 - \frac{2(2^{\alpha}-1)}{3}\right)\right| dt}\right)\right| \\
& = M_4 \left| f'\left(\frac{M_6 a + (M_4 - M_6)\left(\frac{a+b}{2}\right)}{M_4}\right)\right|.
\end{aligned}
$$

$$
\begin{aligned}
|J_{11}| \leq & \left(\frac{1}{2(2^{\alpha}-1)} \int_0^1 \left|\left((1+t)^{\alpha} - 1 - \frac{2(2^{\alpha}-1)}{3}\right)\right| dt\right) \\
& \times \left| f'\left(\frac{\frac{1}{2(2^{\alpha}-1)} \int_0^1 \left|\left((1+t)^{\alpha} - 1 - \frac{2(2^{\alpha}-1)}{3}\right)\right| (tb+(1-t)\frac{a+b}{2}) dt}{\frac{1}{2(2^{\alpha}-1)} \int_0^1 \left|\left((1+t)^{\alpha} - 1 - \frac{2(2^{\alpha}-1)}{3}\right)\right| dt}\right)\right| \\
& = M_4 \left| f'\left(\frac{M_6 b + (M_4 - M_6)\left(\frac{a+b}{2}\right)}{M_4}\right)\right|.
\end{aligned}
$$

This completes the proof.

Corollary 1. *If we take* $\alpha = 1$ *in Theorem* (9)*, then inequality* (17) *becomes as:*

$$
\begin{aligned}
& \left|\left[\frac{1}{6} f(a) + \frac{2}{3} f\left(\frac{a+b}{2}\right) + \frac{1}{6} f(b)\right] - \frac{1}{b-a} \int_a^b f(x) dx\right| \\
& \qquad \leq \frac{5(b-a)}{72} \left[\left| f'\left(\frac{29a+61b}{90}\right)\right| + \left| f'\left(\frac{61a+29b}{90}\right)\right|\right]. \qquad (18)
\end{aligned}
$$

Discussion

We have presented some generalizations of the inequalities of Simpson's type for differentiable mappings in the sense of $(s-m)$-convex functions utilizing Riemann-Liouville fractional integrals, which refines the results of [21], Theorem 2 inequality (5), S. Qaiser [21] Theorem 3 inequality (7). From Corollory, it is clear that Theorem 8 for a specified value refines M.Alomari [$Theorem(8)$, [29]].

Chapter 4
Main Results

4 Hermite-Hadamard's type inequalities

4.1 Introduction

This work brings results together for inequalities of Hermite-Hadamard's type and thus giving explicit error bounds in trapezoidal and midpoint rules. Interested reader can read [30],[31],([32],[33]),([34]).
In this chapter we refine inequalities of Hermite-Hadamard's type via $(s,m)-$convex function [13]. Some error estimates for the trapezoid and midpoint are obtained in terms of firts derivative.
Let us introduce some notations, these are very helpful to express our results in compact form.

4.1.1 Inequalities of Hermite-Hadamard's type for $(s,m)-$convex

In order to prove our main results via $(s,m)-$convex function. we need the following lemma.

Lemma 2. *Let $I \subset \mathbb{R}$ be an open interval, $a,b \in I$ with $a < b$ and $f : [a\ ,b] \to \mathbb{R}$ be a differentiable function such that f' is integrable and $0 < \alpha \leq 1$ on (a,b) with $a < b$,then the following identity for Riemann$-$Liouville fractional integrals holds:*

$$\begin{aligned} &\left(1 - \frac{2}{2^{\alpha}}\lambda\right) f\left(\frac{a+b}{2}\right) + \lambda\frac{f(a)+f(b)}{2^{\alpha}} - \frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}\left[J^{\alpha}_{a^+}f(b) + J^{\alpha}_{b^-}f(a)\right] \\ \leq\ & \frac{b-a}{2^{\alpha+1}}\left[L_1 + L_2 + (L_3 + L_4)\right], \end{aligned}$$

where

$$\begin{aligned} L_1 &= \textstyle\int_0^1 \left[(1-t)^{\alpha} - \lambda\right] f'\left(ta + (1-t)\frac{a+b}{2}\right) dt, \\ L_2 &= \textstyle\int_0^1 \left[\lambda - (1-t)^{\alpha}\right] f'\left(tb + (1-t)\frac{a+b}{2}\right) dt, \\ L_3 &= \textstyle\int_0^1 \left[2^{\alpha} - \lambda - (2-t)^{\alpha}\right] f'\left(t\frac{a+b}{2} + (1-t)a\right) dt, \\ L_4 &= \textstyle\int_0^1 \left[\lambda - 2^{\alpha} + (2-t)^{\alpha}\right] f'\left(t\frac{a+b}{2} + (1-t)b\right) dt. \end{aligned}$$

Proof. *Integrating by parts and substituting $u = ta + (1-t)\frac{a+b}{2}$*

$$\begin{aligned} I_1 &= \int_0^1 \left[(1-t)^{\alpha} - \lambda\right] f'\left(ta + (1-t)\frac{a+b}{2}\right) dt \\ &= \left.\frac{2\left[(1-t)^{\alpha} - \lambda\right] f\left(ta + (1-t)\frac{a+b}{2}\right) dt}{a-b}\right|_0^1 \\ &\qquad + \frac{2\alpha}{a-b}\int_0^1 (1-t)^{\alpha} f\left(ta + (1-t)\frac{a+b}{2}\right) dt \\ &= \frac{2\lambda}{b-a} f(a) + \frac{2(1-\lambda)}{b-a} f\left(\frac{a+b}{2}\right) - \frac{2^{\alpha+1}\alpha}{(b-a)^{\alpha+1}}\int_a^{\frac{a+b}{2}} (u-a)^{\alpha-1} f(u)du \end{aligned}$$

Analogously

$$I_2 = \frac{2\lambda}{b-a} f(b) + \frac{2(1-\lambda)}{b-a} f\left(\frac{a+b}{2}\right) - \frac{2^{\alpha+1}\alpha}{(b-a)^{\alpha+1}}\int_{\frac{a+b}{2}}^{b} (b-u)^{\alpha-1} f(u)du$$

$$I_3 = \frac{2\lambda}{b-a} f(a) + \frac{2(2^{\alpha}-1-\lambda)}{b-a} f\left(\frac{a+b}{2}\right) - \frac{2^{\alpha+1}\alpha}{(b-a)^{\alpha+1}}\int_a^{\frac{a+b}{2}} (b-u)^{\alpha-1} f(u)du$$

$$I_4 = \frac{2\lambda}{b-a} f(b) + \frac{2(2^{\alpha}-1-\lambda)}{b-a} f\left(\frac{a+b}{2}\right) - \frac{2^{\alpha+1}\alpha}{(b-a)^{\alpha+1}}\int_{\frac{a+b}{2}}^{b} (b-u)^{\alpha-1} f(u)du$$

Multiplying above equalities by $\frac{b-a}{2^{\alpha+2}}$, then adding , to get required identity.

Theorem 10. *Let $I \subset \mathbb{R}$ be an open interval, $a, b \in I$ with $a < b$ and $f : [a, b] \to \mathbb{R}$ be a differentiable function such that f' is integrable and $0 < \alpha \leq 1$ on (a, b) with $a < b$ and $0 \leq \lambda \leq 1$. If $|f'|$ is a convex on $[a,b]$, then the following inequality for Riemann−Liouville fractional integrals holds:*

$$\begin{aligned} & \left| \left(1 - \frac{2}{2^\alpha}\lambda\right) f\left(\frac{a+b}{2}\right) + \lambda \frac{f(a)+f(b)}{2^\alpha} - \frac{\Gamma(\alpha+1)}{2(b-a)^\alpha}\left[J^\alpha_{a^+} f(b) + J^\alpha_{b^-} f(a)\right] \right| \\ \leq \quad & \left[\left\{ Q_1 \left| f'(a) \right| + 2m (Q_2 - Q_1) \left| f'(\frac{a+b}{2m}) \right| + Q_1 \left| f'(b) \right| \right\} \right. \\ & \left. + \left\{ Q_3 \left| f'(a) \right| + m(Q_4 - Q_3) \left| f'(\frac{a+b}{2m}) \right| + Q_3 \left| f'(b) \right| \right\} \right]. \end{aligned} \tag{19}$$

$$Q_1 = \int_0^1 |(1-t)^\alpha - \lambda| t^s dt = \frac{(s)!}{(\alpha + s + 1)!} - \frac{\lambda}{(s+1)}$$

$$Q_2 = \int_0^1 |(1-t)^\alpha - \lambda| dt = \frac{1 - 2(1-\zeta)^{\alpha+1}}{\alpha+1} + (1-2\zeta)\lambda,$$

$$Q_3 = \int_0^1 |2^\alpha - (2-t)^\alpha - \lambda| t \, dt = \frac{2^\alpha}{s+1} - 2^\alpha \left[\frac{1}{(s+1)} - \frac{\alpha}{2(s+1)}\right] - \frac{\lambda}{(s+1)}$$

$$Q_4 = \int_0^1 |2^\alpha - (2-t)^\alpha - \lambda| \, dt = \frac{1 + 2^{\alpha+1} - 2(2-\xi)^{\alpha+1}}{\alpha+1} + (2^\alpha - \lambda)(1-2\xi)$$

$$\text{where} \qquad \zeta = 1 - \lambda^{\frac{1}{\alpha}} \qquad \text{and} \qquad \xi = 2 - (2^\alpha - \lambda)^{\frac{1}{\alpha}}$$

Proof. Using (s,m) convexity of $|f'|$, for all $t \in [0,1]$, we obtain

$$\begin{aligned} |L_1| &\leq \int_0^1 ((1-t)^\alpha - \lambda)|f'(tb + (1-t)\frac{a+b}{2})|dt \\ &= \int_0^1 ((1-t)^\alpha - \lambda)|t^s f'(b) + m(1-t^s) f'(\frac{a+b}{2m})|dt \\ &\leq \int_0^1 ((1-t)^\alpha - \lambda) \left\{ t^s \left| f'(b) \right| + m(1-t^s) \left| f'(\frac{a+b}{2m}) \right| \right\} dt \\ &= Q_1 \left| f'(b) \right| + m(Q_2 - Q_1) \left| f'(\frac{a+b}{2m}) \right|. \end{aligned}$$

$$\begin{aligned} |L_2| &\leq \int_0^1 ((1-t)^\alpha - \lambda) |f'(ta + (1-t)\frac{a+b}{2})|dt \\ &= \int_0^1 ((1-t)^\alpha - \lambda)|t^s f'(a) + m(1-t^s) f'(\frac{a+b}{2m})|dt \\ &\leq \int_0^1 ((1-t)^\alpha - \lambda) \left\{ t^s \left| f'(a) \right| + m(1-t^s) \left| f'(\frac{a+b}{2m}) \right| \right\} dt \\ &= Q_1 \left| f'(a) \right| + m(Q_2 - Q_1) \left| f'(\frac{a+b}{2m}) \right|. \end{aligned}$$

$$\begin{aligned}
|L_3| &= \int_0^1 (2^\alpha - (2-t)^\alpha - \lambda)\left\{|f'(tb + (1-t)\frac{a+b}{2})|\right\}dt \\
&= \int_0^1 (2^\alpha - (2-t)^\alpha - \lambda)\left\{|t^s f'(b) + m(1-t^s)f'(\frac{a+b}{2m})|\right\}dt \\
&\leq \int_0^1 (2^\alpha - (2-t)^\alpha - \lambda)\left\{t^s\left|f'(b)\right| + m(1-t^s)\left|f'(\frac{a+b}{2m})\right|\right\}dt \\
&= Q_3\left|f'(b)\right| + m(Q_4 - Q_3)\left|f'(\frac{a+b}{2m})\right|.
\end{aligned}$$

$$\begin{aligned}
|L_4| &\leq \int_0^1 (2^\alpha - (2-t)^\alpha - \lambda)\left\{|f'(ta + (1-t)\frac{a+b}{2})|\right\}dt \\
&= \int_0^1 (2^\alpha - (2-t)^\alpha - \lambda)\left\{|t^s f'(a) + m(1-t^s)f'(\frac{a+b}{2m})|\right\}dt \\
&\leq \int_0^1 (2^\alpha - (2-t)^\alpha - \lambda)\left\{t^s\left|f'(a)\right| + m(1-t^s)\left|f'(\frac{a+b}{2m})\right|\right\}dt \\
&= Q_3\left|f'(a)\right| + m(Q_4 - Q_3)\left|f'(\frac{a+b}{2m})\right|.
\end{aligned}$$

Corollary 2.

$$\left| (1-\lambda)f(\frac{a+b}{2}) + \lambda\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx \right|$$

$$\leq \left(\frac{b-a}{8}[2\lambda^2 - 2\lambda + 1](|f'(a)| + |f'(b)|)\right). \tag{20}$$

Remark 3. *If we take* $\lambda = 1$ *in inequality* 20 *then inequality* (20) *reduces to inequality* (8).

Remark 4. *If we take* $\lambda = 0$ *in inequality* 20 *then inequality* (20) *reduces to inequality* (9).

The corresponding version for powers of the absolute value of the derivative is incorporated in the following theorem.

Theorem 11. *$I \subset \mathbb{R}$ be an open interval, $a,b \in I$ with $a < b$ and $f : [a,b] \to \mathbb{R}$ be a differentiable function such that f' is integrable and $0 < \alpha \leq 1$ on (a,b) with $a < b$ and $0 \leq \lambda \leq 1$. if $|f'|^q$ is a convex on $[a,b]$, with $q \geq 1$, then the following inequality holds*:

$$\left|\left[\left(1 - \frac{2}{2^\alpha}\lambda\right)f\left(\frac{a+b}{2}\right) + \lambda\frac{f(a)+f(b)}{2^\alpha}\right] - \frac{\Gamma(\alpha+1)}{2(b-a)^\alpha}[J^\alpha_{a^+}f(b) + J^\alpha_{b^-}f(a)]\right| \leq \frac{(b-a)}{2^\alpha}$$
$$\left[(Q_2)^{1-1/q}\left\{\begin{array}{c} \left(Q_1|f'(a)|^q + m(Q_2 - Q_1)\left|f'\left(\frac{a+b}{2m}\right)\right|^q\right)^{1/q} \\ + \left(Q_1|f'(b)|^q + m(Q_2 - Q_1)\left|f'\left(\frac{a+b}{2m}\right)\right|^q\right)^{1/q} \end{array}\right\} + \right.$$
$$\left.(Q_4)^{1-1/q}\left\{\begin{array}{c} \left(Q_3\left|f'\left(\frac{a+b}{2}\right)\right|^q + m(Q_4 - Q_3)|f'(a)|^q\right)^{1/q} \\ + \left(Q_3\left|f'\left(\frac{a+b}{2}\right)\right|^q + m(Q_4 - Q_3)|f'(b)|^q\right)^{1/q} \end{array}\right\}\right]. \tag{21}$$

Proof. Using the well -known power -mean integral inequality for $q > 1$, and convexity of $|f'|^q$ we have

$$\begin{aligned}
|L1| &\leq \left(\int_0^1 |((1-t)^\alpha - \lambda)|\,dt\right)^{1-1/q} \times \left(\int_0^1 |((1-t)^\alpha - \lambda)| \left|f'\left(tb + (1-t)\frac{a+b}{2}\right)\right|^q dt\right)^{1/q} \\
&\leq \left(\int_0^1 |((1-t)^\alpha - \lambda)|\,dt\right)^{1-1/q} \\
&\times \left(\int_0^1 |((1-t)^\alpha - \lambda)| \left\{t^s \left|f'(b)\right|^q + m(1-t^s)\left|f'(\frac{a+b}{2m})\right|^q\right\} dt\right)^{1/q} \\
&\leq (Q_2)^{1-1/q}\left(Q_1\left|f'(b)\right|^q + m(Q_2 - Q_1)\left|f'(\frac{a+b}{2m})\right|^q\right)^{1/q}
\end{aligned}$$

$$\begin{aligned}
|L_2| &\leq \left(\int_0^1 |((1-t)^\alpha - \lambda)|\,dt\right)^{1-1/q} \\
&\times \left(\int_0^1 |((1-t)^\alpha - \lambda)| \left|f'\left(ta + (1-t)\frac{a+b}{2}\right)\right|^q dt\right)^{1/q} \\
&\leq \left(\int_0^1 |((1-t)^\alpha - \lambda)|\,dt\right)^{1-1/q} \\
&\times \left(\int_0^1 |((1-t)^\alpha - \lambda)| \left\{t^s \left|f'(a)\right|^q + m(1-t^s)\left|f'(\frac{a+b}{2m})\right|^q\right\} dt\right)^{1/q} \\
&\leq (Q_2)^{1-1/q}\left(M_1\left|f'(a)\right|^q + m(Q_2 - Q_1)\left|f'(\frac{a+b}{2m})\right|^q\right)^{1/q}
\end{aligned}$$

$$\begin{aligned}
|L_3| &\leq \left(\int_0^1 |2^\alpha - (2-t)^\alpha - \lambda|\,dt\right)^{1-1/q} \\
&\times \left(\int_0^1 |2^\alpha - (2-t)^\alpha - \lambda| \left|f'\left(tb + (1-t)\frac{a+b}{2}\right)\right|^q dt\right)^{1/q} \\
&\leq \left(\int_0^1 |2^\alpha - (2-t)^\alpha - \lambda|\,dt\right)^{1-1/q} \\
&\times \left(\int_0^1 |2^\alpha - (2-t)^\alpha - \lambda| \left\{t^s \left|f'(b)\right|^q + m(1-t^s)\left|f'(\frac{a+b}{2m})\right|^q\right\} dt\right)^{1/q} \\
&\leq (Q_4)^{1-1/q}\left(Q_3\left|f'(b)\right|^q + m(Q_4 - Q_3)\left|f'\left(\frac{a+b}{2m}\right)\right|^q\right)^{1/q}
\end{aligned}$$

$$
\begin{aligned}
|L_4| \quad \leq \quad & \left(\int_0^1 |2^\alpha - (2-t)^\alpha \ - \ \lambda| \, dt\right)^{1 \ - \ 1/q} \\
& \times \left(\int_0^1 |2^\alpha - (2-t)^\alpha \ - \ \lambda| \left| f'\left(ta \ + \ (1 \ - \ t)\frac{a+\ b}{2}\right)\right|^q dt\right)^{1/q} \\
\leq \quad & \left(\int_0^1 |2^\alpha - (2-t)^\alpha \ - \ \lambda| \, dt\right)^{1 \ - \ 1/q} \\
& \times \left(\int_0^1 |2^\alpha \ - \ (2 \ - \ t)^\alpha \ - \ \lambda| \left\{t^s \left|f'(a)\right|^q \ + \ m(1 - \ t^s)\left|f'(\frac{a+\ b}{2m})\right|^q\right\} dt\right)^{1/q} \\
\leq \quad & (Q_4)^{1-1/q}\left(Q_3 \left|f'(a)\right|^q \ + \ m(Q_4 \ - \ Q_3)\left|f'\left(\frac{a \ + \ b}{2m}\right)\right|^q\right)^{1/q}
\end{aligned}
$$

This completes the proof.

Corollary 3. *Under the assumptions Theorem 11, with $\alpha = s = m = 1$, $\lambda = 0$ in inequality (21), then the following inequality holds,*

$$
\left|f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(x)dx\right| \leq \frac{b-a}{8}\left[\left(\frac{1}{3}|f'(a)| + \frac{2}{3}\left|f'\left(\frac{a+b}{2}\right)\right|^q\right)^{\frac{1}{q}} + \left(\frac{1}{3}||f'(b)| + \frac{2}{3}\left|f'\left(\frac{a+b}{2}\right)\right|^q\right)^{\frac{1}{q}}\right]
$$

Corollary 4. *Under the assumptions Theorem 11, with $\alpha = s = m = 1$, $\lambda = 1$ in inequality (21), then the following inequality holds,*

$$
\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx\right| \leq \frac{b-a}{8}\left[\left(\frac{2}{3}|f'(a)| + \frac{1}{3}\left|f'\left(\frac{a+b}{2}\right)\right|^q\right)^{\frac{1}{q}} + \left(\frac{2}{3}||f'(b)| + \frac{1}{3}\left|f'\left(\frac{a+b}{2}\right)\right|^q\right)^{\frac{1}{q}}\right]
$$

Corollary 5. *Under the assumptions Theorem 11, with $\alpha = s = m = 1$, $\lambda = \frac{1}{3}$ in inequality (21), then the following inequality holds,*

$$
\begin{aligned}
&\left|\frac{1}{3}\left\{2f\left(\frac{a+b}{2}\right) + \frac{f(a)+f(b)}{2}\right\} - \frac{1}{b-a}\int_a^b f(x)dx\right| \leq \frac{5(b-a)}{72} \times \\
&\qquad \left[\left(\frac{16}{45}|f'(a) + \frac{29}{45}\left|f'\left(\frac{a+b}{2}\right)\right|^q\right)^{\frac{1}{q}} + \left(\frac{16}{45}|f'(b) + \frac{29}{45}\left|f'\left(\frac{a+b}{2}\right)\right|^q\right)^{\frac{1}{q}}\right]
\end{aligned}
$$

Theorem 12. *Let $I \subset \mathbb{R}$ be an open interval, $a, b \in I$ with $a < b$ and $f : [a\ , b] \to \mathbb{R}$ be a differentiable function such that f' is integrable and $0 < \alpha \leq 1$ on (a, b) with $a < b$ and $0 \leq \lambda \leq 1$. If $|f'|^{\frac{1}{q}}$ for $q \geq 1$ is a convex on $[a, b]$, then the following inequality for Riemann−Liouville fractional integrals holds:*

$$
\begin{aligned}
&\left|\left(1 \ - \ \frac{2}{2^\alpha}\lambda\right) f\left(\frac{a \ + \ b}{2}\right) \ + \ \lambda\frac{f(a) \ + \ f(b)}{2^\alpha} \ - \ \frac{\Gamma(\alpha \ + \ 1)}{2(b \ - \ a)^\alpha}\left[J^\alpha_{a^+} f(b) \ + \ J^\alpha_{b^-} f(a)\right]\right| \\
&\leq \frac{(b \ - \ a)}{2^{\alpha \ + \ 1}} \times \left[Q_2\left\{\left|f'\left(\frac{Q_5 b \ + \ (Q_2 \ - \ Q_5)\left(\frac{a \ + \ b}{2}\right)}{Q_2}\right)\right| \ + \ \left|f'\left(\frac{Q_5 a \ + (Q_2 \ - \ Q_5)\left(\frac{a \ + \ b}{2}\right)}{Q_2}\right)\right|\right\}\right. \\
&\qquad \left. + \ Q_4\left|f'\left(\frac{Q_6 a \ + (Q_4 \ - \ Q_6)\left(\frac{a \ + \ b}{2}\right)}{Q_4}\right)\right| \ + \ \left|f'\left(\frac{Q_6 b \ + \ (Q_4 \ - \ Q_6)\left(\frac{a \ + \ b}{2}\right)}{Q_4}\right)\right|\right].
\end{aligned} \tag{22}
$$

$$Q_5 = \int_0^1 |(1-t)^\alpha - \lambda| t\,dt = \frac{1-2(1-\zeta)^{\alpha+2}}{(\alpha+1)(\alpha+2)} - \frac{2\zeta(1-\zeta)^{\alpha+1}}{\alpha+1} + \frac{\lambda}{2}(1-2\zeta^2)$$

$$\begin{aligned} Q_6 &= \int_0^1 |2^\alpha - (2-t)^\alpha - \lambda| t\,dt \\ &= \frac{1+2^{\alpha+2}-2(2-\xi)^{\alpha+2}}{(\alpha+1)(\alpha+2)} + \frac{1-2\xi(2-\xi)^{\alpha+1}}{\alpha+1} + (2^\alpha - \lambda)(\frac{1}{2} - \xi^2) \end{aligned}$$

Proof. Using the concavity of $|f'|^q$ and the power -mean inequality, we obtain

$$\begin{aligned} |f'|^q &\le t|f'|^q + (1-t)|f'|^q \\ &\le t|f'|^q + (1-t)|f'|^q. \end{aligned}$$

Hence

$$|f'(tx+(1-t)y)| \le t|f'(x)| + (1-t)|f'(y)|.$$

So $|f'|$ is also concave . By the Jensen integral inequality , we have

$$\begin{aligned} |J_8| &\le \left(\int_0^1 |((1-t)^\alpha - \lambda)|\,dt\right) \left| f'\left(\frac{\int_0^1 |((1-t)^\alpha - \lambda)|(tb + (1-t)\frac{a+b}{2})dt}{\int_0^1 |((1-t)^\alpha - \lambda)|\,dt} \right) \right| \\ &= Q_2 \left| f'\left(\frac{Q_5 b + (Q_2 - Q_5)\left(\frac{a+b}{2}\right)}{Q_2} \right) \right|. \end{aligned}$$

$$\begin{aligned} |J_9| &\le \left(\int_0^1 |((1-t)^\alpha - \lambda)|\,dt\right) \left| f'\left(\frac{\int_0^1 |((1-t)^\alpha - \lambda)|(t\,a + (1-t)\frac{a+b}{2})dt}{\int_0^1 |((1-t)^\alpha - \lambda)|\,dt} \right) \right| \\ &= Q_2 \left| f'\left(\frac{Q_5 a + (Q_2 - Q_5)\left(\frac{a+b}{2}\right)}{Q_2} \right) \right|. \end{aligned}$$

$$\begin{aligned} |J_{10}| &\le \left(\int_0^1 |(2^\alpha - (2-t)^\alpha - \lambda)|\,dt\right) \\ &\times \left| f'\left(\frac{\int_0^1 |(2^\alpha - (2-t)^\alpha - \lambda)|(ta + (1-t)\frac{a+b}{2})dt}{\int_0^1 |(2^\alpha - (2-t)^\alpha - \lambda)|\,dt} \right) \right| \\ &= Q_4 \left| f'\left(\frac{Q_6 a + (Q_4 - Q_6)\left(\frac{a+b}{2}\right)}{Q_4} \right) \right|. \end{aligned}$$

$$\begin{aligned} |J_{11}| &\le \left(\int_0^1 |(2^\alpha - (2-t)^\alpha - \lambda)|\,dt\right) \\ &\times \left| f'\left(\frac{\int_0^1 |(2^\alpha - (2-t)^\alpha - \lambda)|(tb + (1-t)\frac{a+b}{2})dt}{\int_0^1 |2^\alpha - (2-t)^\alpha - \lambda|\,dt} \right) \right| \\ &= Q_4 \left| f'\left(\frac{Q_6 b + (Q_4 - Q_6)\left(\frac{a+b}{2}\right)}{Q_4} \right) \right|. \end{aligned}$$

This completes the proof.

Remark 5. *Under the assumptions Theorem* 12*, with* $\alpha = s = 1$, $\lambda = 1$ *in inequality* (22)*, reduces to inequality* (11).

Remark 6. *Under the assumptions Theorem* 12*, with* $\alpha = s = 1$, $\lambda = 0$ *in inequality* (22)*, reduces to inequality* (12).

Corollary 6. *Under the assumptions Theorem* 12*, with* $\alpha = s = 1$, $\lambda = \frac{1}{3}$ *in inequality* (22)*, then the following inequality holds,:*

$$\left| \left[\frac{1}{6} f(a) + \frac{2}{3} f\left(\frac{a+b}{2} \right) + \frac{1}{6} f(b) \right] - \frac{1}{b-a} \int_a^b f(x)dx \right| \\ \leq \frac{5(b-a)}{72} \left[\left| f'\left(\frac{29a+61b}{90} \right) \right| + \left| f'\left(\frac{61a+29b}{90} \right) \right| \right]. \quad (23)$$

DISCUSSION.

In this chapter, we highlighted the function of convexity as well as convacity and established some inequalities for celebrated HermiteHadamard type integral for differentiable mapping, which are associated with $(s,m)-convex$ function utilizing Riemann-Liouville fractional integrals. It is clear that Theorem 4 under a specified value refines S. S. Dragomir [13], Theorem 5 presents new results see Corollory 3,4 and 5 and Theorem 6 refines G. S. Yang [15].

Chapter 5
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