On Metric Dimension - Edge Metric Dimension and Topological Properties of Connected Graphs

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CIIT/FA14-PMATH-001/LHR

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DEDICATION

I dedicate this work to my mother and father,
Ustad Jaffar Ali (Late), a great teacher
and very kind human being.

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ABSTRACT

On Metric Dimension - Edge Metric Dimension and Topological Properties of Connected Graphs

The distance between any two vertices of a connected graph G is the length of a shortest path between them. Any two vertices/edges are said to be resolved by a vertex if they have different distances w.r.t. that vertex. A set $S \subset G$ is a resolving set/edge resolving set if every vertex/edge of G is resolved by some vertices of G, respectively. A topological index is a real number associated to a graph that describes its topological properties.

The main objective of this thesis is to study edge metric dimension of certain families of wheel related graphs and generalized Petersen graphs $P(n,3)$. It has been proved that the gear graph, the fan graph, the friendship graph, the sunflower graph and certain convex polytopes have unbounded edge metric dimension whereas the generalized Petersen graph $P(n,3)$ has a constant edge metric dimension. An explicit Matlab program has also been given to generate the edge resolving set and edge coding for generalized Petersen graphs $P(n,3)$. A comparative study between the metric dimension and the edge metric dimension of the aforementioned families of graphs has also been conducted.

The second part of the thesis studies the Zagreb and co-Zagreb indices of k-iterated strong double graphs and some eccentricity-based indices of Sierpinski graphs and their regularizations. Moreover, some explicit algorithms have also been given to verify the validity of the results.

List of Publications from Thesis

1. Rafiullah, M., Siddiqui, H. M. A., & Ahmad, S. (2019). Resolvability of some convex polytopes. Utilitas Math. 111, 309-323.

2. Yang, B., Rafiullah, M., Siddiqui, H. M. A., & Ahmad, S. (2019). On Resolvability Parameters of Some Wheel-Related Graphs. J. Chem. 2019, 1-9.

List of Submissions from Thesis

1. Computing Zagreb indices and coindices of strong double graphs.

2. Eccentricity based topological properties of Sierpinski network and its regularization.

3. Computing edge resolvability of generalized Petersen graph *P(n,3)*.

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Chapter 1

Introduction

Chemical graph theory is an important topological field of mathematical chemistry that deals with mathematical modelling of chemical compound structures [18]. A molecular structure of a compound consists of many atoms. We consider a simple molecular graph, say G which consist of non-hydrogen atoms and covalent bonds. In graph theory the non-hydrogen atoms are represented by a set of vertices $V = V(G)$ and the covalent bonds with the set of edges $E = E(G)$. The number of atoms and bonds in a structure are represented by $n = |V|$ and $m = |E|$ respectively. The valency of an atom is represented by $deg(v_i)$ and it is known as degree of vertex $v_i \in V(G)$. Specially, the hydrocarbons is a chemical compound which consist of carbon and hydrogen atoms. The molecular graphs of hydrocarbons represent the carbon skeleton of the molecule [18].

This dissertation deals with two types of graph theoretic parameters, namely resolving sets and topological descriptors. These parameters are based on the distances and the degrees of the graph, respectively. This dissertation is designed to discuss and classify the nature of edge metric dimension, metric dimension and the relation between them. Secondly, the topological descriptors also known as topological indices have also been studied for different classes of graphs. There are eight chapters in this thesis whose details are given as follows:

Some basic notions, concepts, types of graphs and graph operations are given in the second chapter.

In the third chapter, we have studied the strong double graph and kth-iterated strong double graph. We have discussed the generalized explicit formulas to calculate the first Zagreb index, the second Zagreb index, the first Zagreb coindex and the second Zagreb coindex for the strong double graph and kth-iterated strong double graph. We have also given an algorithm based on adjacency matrix to verify these indices and coindices.

In the fourth chapter, we have studied the Sierpinski network and its regularizations. We have computed the generalized closed mathematical formulas to calculate some indices based on the eccentricity of vertices of Sierpinski networks $S(n, k)$ and their regularizations $S^{+}(n, k)$, $S^{++}(n, k)$, that is eccentric-connectivity index (ECI), total eccentric index (TEI) and Zagreb indices based on eccentricity. Moreover, some graphical representations have been provided to analyze the behaviour of the aforementioned indices.

The chapter 5 is about edge metric dimension (EMD) of some specific families of wheel related graphs, such as Jahangir graph J_{2n}, helm graph H_n, sunflower graph Sf_n and friendship graph F_n, have been studied and it has been proved that all of these families have unbounded EMD. We have also compared the results with their metric dimensions.

Chapter 6 includes the study of the EMD of some wheel related convex polytopes denoted by $\mathbb{B}_{2n}$, $\mathbb{D}_{2n}$, $\mathbb{Q}_n$ and their bounds as well.

An analysis of generalized Petersen graph $P(n, 3)$ regarding edge resolving set and EMD is provided in chapter 7. A Matlab code is also presented in this chapter to calculate and verify the distance codes based on the basis set.

A conclusion of all the results and some open problems are given in chapter 8.

Chapter 2

Preliminaries

2.1 Basic concepts

A *graph* G consists of vertex set, $W = \{w_1, w_2, w_3, \ldots, w_n\}$ and edge set, $E = \{e_1, e_2, e_3, \ldots, e_m\}$. The *order* of a graph G is the number of elements in W and shown by $n = |W(G)|$ and the number of elements in the set E referred as *size* of graph and it is denoted by $m = |E(G)|$. The Figure 2.1 represent, a graph K of order $n = 9$ and size $m = 11$. An edge is a connection between two vertices such as e_1 connecting v_1 and v_2, this connection is shown as $e_1 = v_1v_2$. Any two vertices of G are called *adjacent* vertices if they are joined by an edge, the vertex v_1 is adjacent to vertex v_2 (and vice versa). Any two edges of G are known as *incident* if they are sharing one vertex, the edge e_1 and e_2 are *incident edges* because they are sharing vertex v_2. The edge e_1 is also called incident to vertex v_1 (or v_2).

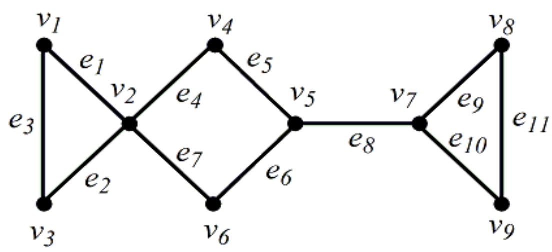


Figure 2.1: Graph K

Number of edges connected to w (edges incident to vertex w) referred as degree of w and it is represented by $deg(w)$. The vertex v_1 has degree 2 and the vertex v_2 has degree 4 in graph K. $\delta(G) = min\,\{deg(x) : \forall\ x \in V(G)\}$ is the *minimum degree* of graph G. Similarly, $\Delta(G) = max\,\{deg(x) : \forall\ x \in V(G)\}$ is the *maximum degree* of graph G. $d(G) = \frac{\sum_{i=1}^{n} deg(G)}{n}$ is the average degree of a graph of order n. The following relation exists between $\delta(G)$, $deg(v)$, $\Delta(G)$ and order of graph (n).

$$0 \leq \delta(G) \leq deg(x) \leq \Delta(G) \leq n - 1.$$

The $d = deg(v),\ v \in V(G)$ and the vertex v referred as an odd vertex if d is odd otherwise vertex v known as even vertex. An isolated vertex has zero degree and an ending (pendant) vertex of a graph has degree 1.

In a graph any adjacent vertices are called neighbours of each other and the set of neighbouring vertices of a vertex u is denoted by $N(u)$. So $deg(u) = |N(u)|$.

The set of neighbourhood of a vertex u is also known as open neighbourhood of vertex u. The closed neighbourhood is shown and defined by $N[u] = N[u] \cup \{u\}$. Any vertex u *dominates* its closed neighbourhood $N[u]$, so the vertex u dominates $deg(u) + 1$ number of vertices of graph G. The degree of the same vertex u in $\overline{G}$ is then given by $deg_{\overline{G}}(u) = n - 1 - deg(u)$.

A *walk* W is an arrangement of alternating vertices and edges such as W : $v_2e_4v_4e_5v_5\ e_6v_6$. An edge might be repeated in a walk. A Closed walk is a *walk* if the starting and ending vertices are alike, otherwise it is called an *open walk*. A *length of walk* is count of edges in a walk. A walk with zero length is known as a *trivial walk*. A walk known as a *trail* if there is no repetition of edges.

A walk without repetition of vertices is known as *path* and is shown by P_n where n represents the vertices in a path. The number of edges in a path P_n is known as the *length of path* and it is calculated by $k = n - 1$. There are two paths between v_1 and v_6 in graph K, one is P_3: $v_1e_1v_2e_7v_6$ and the second is $P_5 : v_1e_1v_2e_4v_4e_5v_5e_6v_6$. The length of P_3 and P_5 is 2 and 4, respectively. A *cycle* in a graph is a closed trail without repetition of vertices except the starting and ending vertex, denoted by C_n. The shortest length and the longest length of a cycle in a graph G is referred as the *girth* and *circumference*, respectively. In Figure 2.1 the graph K has girth 3 and circumference 4.

The *distance* is the length of a path with least number of edges between vertex g and vertex h of a graph, denoted by $d(g, h)$. The distance between v_1 and v_6 is $d(v_1, v_6) = 2$. The *geodesic distance* is defined as the shortest distance between two vertices of G [19]. The greatest distance between vertex w and another vertex u is referred as *eccentricity* of vertex w and represented by $\varepsilon_{cc}(w) = \max\limits_{u \in V}(d(w, u))$. The minimum eccentricity of any vertex of G is known as the *radius* and it can be denoted by $r = \min\limits_{w \in V}(\varepsilon_{cc}(w)) = \min\limits_{w \in V}\left(\max\limits_{u \in V}(d(w, u))\right)$. Maximum eccentricity of any vertex w of G is known as *diameter,* denoted by $d = \max\limits_{w \in V}(\varepsilon_{cc}(v)) = \max\limits_{w \in V}\left(\max\limits_{u \in V}(d(w, u))\right)$. A vertex w is named as *central vertex* of a graph G if $\varepsilon_{cc}(w) = r$ and it is known as *peripheral vertex* if $\varepsilon_{cc}(w) = d$.

A graph $\overline{G} = (V, \overline{E})$ is *complement* $G = (V, E)$, where $\overline{E}$ is the set of edges which do not belong to graph G. If a graph G is isomorphic to $\overline{G}$, then G is called as *self-complementary graph*. The size of a self-complementary graph is

$|E(G)| = \frac{|V|(|V|-1)}{4}$, where $|V| \ \equiv 0,\ 1\ (mod\ 4)$.

A matrix is a rectangular array consisting of rows and columns. A matrix denoted by a capital character of alphabet such as $A_{n\times m}$ where n is number of rows and m is number of columns. The size of a matrix is shown as $n \times m$. The elements of a matrix A are denoted by a small character of alphabets such as a_{kl} or $a(k,l)$, where $1 \leq k \leq n$ and $1 \leq l \leq m$.

Adjacency matrix A of a graph G of order n is a square matrix of order n, containing pairwise information of all vertices in the form of 0 and 1, where $n = |V(G)|$. If any two vertices w_k and w_l are adjacent, then the matrix element is $a(k,l) = 1$, otherwise the element is $a(k,l) = 0$, where $1 \leq k \leq n$ and $1 \leq l \leq n$. The adjacency matrix of graph K is shown below.

$$A = \begin{bmatrix} 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \end{bmatrix}$$

A rectangular matrix I of order $n \times m$ *is referred as incidence matrix* of G having order n and size m, which contains pairwise information of all vertices and edges in the form of 0 or 1. If vertex v_k and edge e_l are incident then the element of matrix is $i(k,\ l) = 1$, otherwise the element of matrix is $i(k,\ l) = 0$, where $1 \leq k \leq n$ and $1 \leq l \leq m$. The matrix I given below is the incidence matrix of graph K in Figure 2.1. The rows of matrix I are representing the vertices v_k of G and columns of matrix I are representing the edges e_l of G.

$$I = \begin{bmatrix} 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$

A rectangular matrix D of order $n \times m$ *is known as distance matrix* of G of order n and size m which contains pairwise information of distance between vertices and edges, denoted as $d(v_k, e_l)$ from vertex v_k to e_l, where $1 \leq k \leq n$ and $1 \leq l \leq m$. The rows of matrix D are representing the vertices v_k and columns of matrix D are representing the edges e_l.

$$D = \begin{bmatrix} 0 & 1 & 0 & 1 & 2 & 2 & 1 & 3 & 4 & 4 & 5 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 & 2 & 3 & 3 & 4 \\ 1 & 0 & 0 & 1 & 2 & 2 & 1 & 3 & 4 & 4 & 5 \\ 1 & 1 & 2 & 0 & 0 & 1 & 1 & 1 & 2 & 2 & 3 \\ 2 & 2 & 3 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 2 \\ 1 & 1 & 2 & 1 & 1 & 0 & 0 & 1 & 2 & 2 & 3 \\ 3 & 3 & 4 & 2 & 1 & 1 & 2 & 0 & 0 & 0 & 1 \\ 4 & 4 & 5 & 3 & 2 & 2 & 3 & 1 & 0 & 1 & 0 \\ 4 & 4 & 5 & 3 & 2 & 2 & 3 & 1 & 1 & 0 & 0 \end{bmatrix}$$

2.2 Graph operations

If $X_1 = (V_1(X_1), E_1(X_1))$ and $X_2 = (V_2(X_2), E_2(X_2))$ are two simple connected graphs, then;

1. The *join* of two graphs X_1 and X_2 is given by $X_1 + X_2$, with $V_1(X_1) \cup V_2(X_2)$ and $E_1(X_1) \cup E_2(X_2) \cup \{uw : \; u \in V_1(X_1), \; w \in V_2(X_2)\}$ as the vertex set

and the edge set, respectively.

2. The graph $X_1 \cup X_2$ is the *union* of the graph X_1 and the graph X_2 with edge set $E_1(X_1) \cup E_2(X_2)$ and vertex set $V_1(X_1) \cup V_2(X_2)$.

3. The *intersection* of graphs X_1 and X_2 is given by $X_1 \cap X_2$ with edge set $E_1(X_1) \cap E_2(X_2)$ and vertex set $V_1(X_1) \cap V_2(X_2)$.

4. The *Cartesian product* of X_1 and X_2 is represented by $X_1 \Box X_2$ with $V_1(X_1) \times V_2(X_2)$, and any two vertices (w_1, w_2) and (u_1, u_2) are neighbouring vertices in $X_1 \Box X_2$ if and only if either
   (a) $u_1 = w_1$ and u_2 connected to w_2 in X_2, or
   (b) $u_2 = w_2$ and u_1 connected to w_1 in X_1.
   A *ladder graph* L_n can be defined as $L_n = P_n \Box P_2$. A *grid graph* $X_{m,n}$ is defined as $X_{m,n} = P_m \Box P_n$. A *prism graph* Y_n is also defined by using Cartesian product as $Y_n = C_n \Box P_2$.
   Note: If both graphs X_1 and X_2 are connected, the Cartesian product is connected.

2.3 Types of graph

In this section, we will discuss some finite graphs, their notions and some basic properties. The finite word means countable, in graph theory it means the number of vertices and edges are countable in a graph G.

A graph of size zero is referred as *null graph.* Moreover it is a 0-regular graph, because the vertices have zero degree. The Null graph is also known as edgeless graph, represented by $\overline{K}_n$. The null graph is presented in Figure 2.2 (B). If all pair of vertices of G have path between them, then G is known as connected and otherwise it is a *disconnected graph.* The graphs shown in Figure 2.2 (A, C, D) are connected graphs and the Figure 2.4 (B) contains a disconnected graph.

A *directed graph* is a graph in which every edge is directed by an arrow indicating flow from a vertex v to another vertex u. Every edge is linked with ordered pair of vertices. The directed graph is also referred as digraph and it is shown in Figure 2.2 (D). A *multigraph* is a graph with multiple edges and loops. A loop is an edge which is incident to the same vertex as vertex v_2 contains a loop in Figure 2.2 (C).

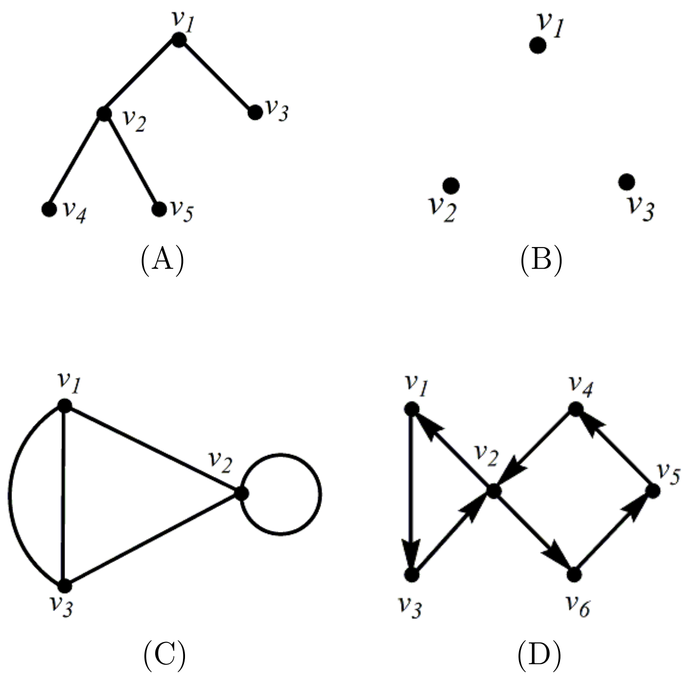


Figure 2.2: Some basic graphs

If all vertices of a graph G are connected to each other by a path of length one, then it is known as *complete graph,* denoted by K_n, where $n \geq 3$. The size of K_n is $C(n,2) = \frac{n!}{2!(n-2)!}$ and $deg(v) = n-1$. Graphs presented in Figure 2.3 (A) and (C) are complete graph K_3 and K_4, respectively. A *star graph* is a graph in which $n-1$ vertices are pendant to a single vertex w and denoted by S_n, where $deg(w) = n-1$. A star graph is presented in Figure 2.3 (D).

Acyclic graph is a graph in which no cycle exists, as presented in Figure 2.4 (A).

A *planar graph* is a graph in which no single edge crosses any other edge. This graph contains cycles which are known as faces of the graph. The planar graph is presented in Figure 2.5 (A) and a non-planar graph is also shown in Figure 2.5 (B).

If a vertex set $V(G) = U_1 \cup U_2$ such that U_1 and U_2 are disjoint then, the graph G is *bipartite if edges of G exist only between vertices of* U_1 and U_2. Moreover, if all vertices of U_1 are adjacent with vertices of U_2 only, then the graph is known as

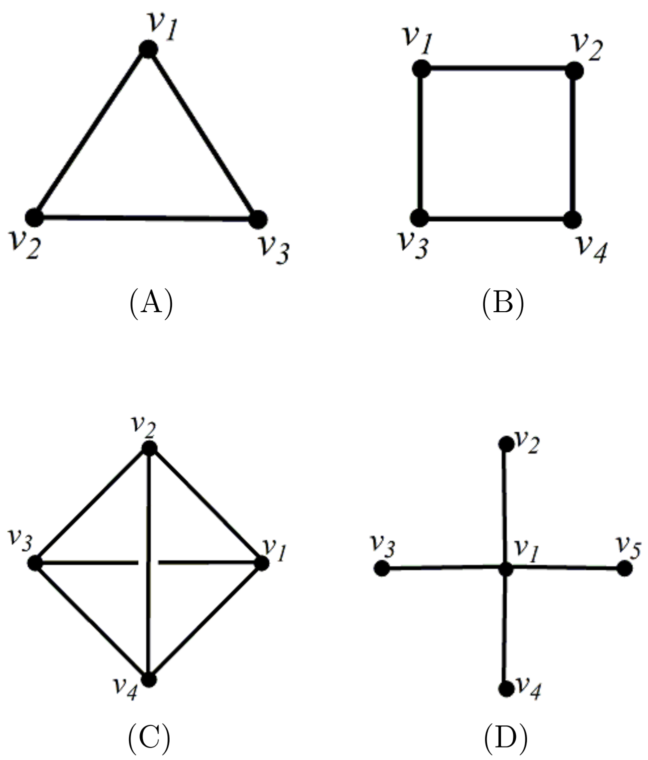


Figure 2.3: More basic graphs

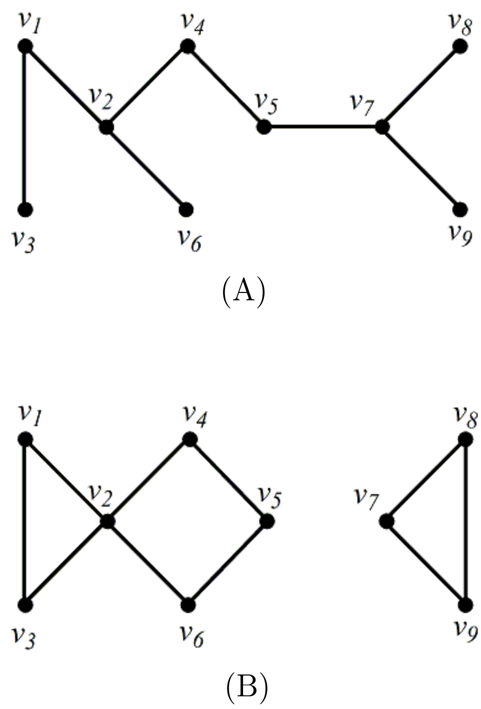
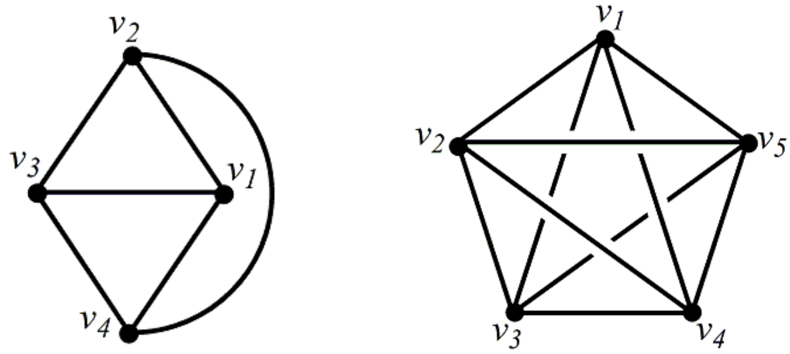


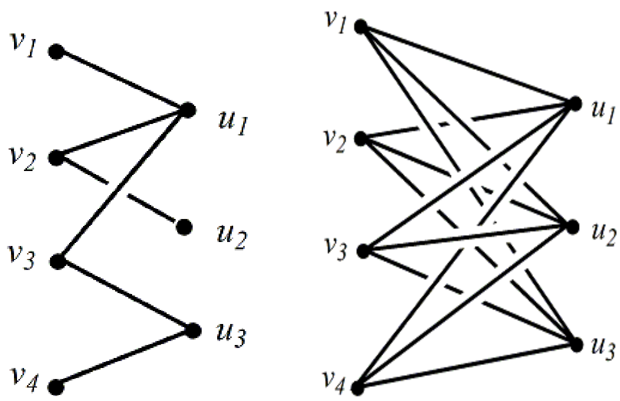
Figure 2.4: Acyclic and disconnected graphs

complete bipartite, represented by $K_{p,q}$, where $p = |U_1|$ and $q = |U_2|$. A bipartite graph and complete bipartite graph are presented in Figure 2.6 (A) and 2.6 (B), respectively.



(A) (B)

Figure 2.5: Planar and non-planar graphs



(A) (B)

Figure 2.6: Bipartite and complete bipartite graphs

2.4 Computational complexity

Graph theory is applied to model nature related problems in different fields, these models are based on objects and their relations. Some models can be solved by using algorithms. An algorithm in mathematics and computer science is a set of well defined instructions which are used to perform specific (or generalized) task.

An algorithm can be a simple process, such as addition of two real numbers, or a complex operation, such as monitoring real time data. The term algorithm was invented (used) by "Abdullah Muhammad bin Musa al-Khwarizmi" in 9th century and he is also known as "The father of Algebra". Algorithms are classified in three categories according to their structures such as sequence, branching and loops. Some popular algorithms used in graph theory are depth first search algorithm [105], breadth first search algorithm [7], Dijkstra's shortest path algorithm [31], Eulerian path algorithm [90], topological sort algorithm [75], Floyd-Warshall all pairs shortest path algorithm [39], Tarjan's strongly connected components algorithm [105], travelling Salesman problem algorithm [113], prim's algorithm [15], Kruskal's algorithm [2], and flood fill algorithm [3].

An algorithm is considered efficient if it takes short time in computation. The time taken by an algorithm to complete a task is known as time complexity (computational complexity). Time complexity is used to estimate the amount of basic operations which are performed by an algorithm and time taken by it. The time complexity is denoted by big O as $O(n)$, $O(2^n)$, $O(n^\alpha)$, and $O(log\ n)$, etc., where n is size of input. The argument appears in the big O notation helps to classify complexities such as n, 2^n, n^α and $log\ n$ represent linear time, exponential, polynomial time (if $\alpha > 1$) and logarithmic time, respectively.

A *polynomial time,* is a time taken by a computer to solve a problem and the time is upper bounded by a polynomial expression, i.e. $T(n) = O(n^k)$, where k is a constant and n is used for input size. If the resulting behaviour of an algorithm has one outcome or its outcome can be pre-determined then it is called *deterministic algorithm* otherwise it is called a *nondeterministic algorithm.* A task (or problem) which can be figure out in a polynomial time is known as a *P-problem* otherwise it is known as an *NP-problem.* If a problem A can be converted into a problem B (NP problem), it means that A is reducible to B, so problem B is at least as hard as A and it referred as *NP-hard problem.* If a problem is an NP and NP-hard than it is referred as an *NP-Complete problem.*

2.5 Topological indices

A topological index is a type of a molecular descriptor which provides the information of a molecular structure. In graph theory a molecular structure of a compound is considered as a graph. "The molecular descriptor is the final result of a logic and mathematical procedure which transforms chemical information encoded within a symbolic representation of a molecule into a useful number or the result of some standardized experiment" [106]. "Topological index is also known as connectivity index" [107]. Topological indices are largely applied in chemistry to develop the "quantitative structure-activity relationship" (QSAR) in which the characteristics of molecules can be correlated with their chemical structures [78].

Molecular descriptor is a numerical value (or values) of a molecular structure which contains the coded chemical information of a molecule. For many decades, the scientist have been developing such techniques to extract the information from a molecular structure by experiments and theoretical studies. The studies provide the quantitative relationship between the molecular structure, its chemical properties and its biological activity. Molecular descriptors are divided according to their structural dimensions such as

1. 0D-descriptors (example: constitutional descriptor of a compound),
2. 1D-descriptors (example: fingerprints descriptor),
3. 2D-descriptors (example: molecular graphs based descriptors),
4. 3D-descriptors (example: quantum-chemical descriptors)
5. 4D-descriptors (example: descriptors that are derived from "comparative molecular field analysis method" or GRID).

More than 1500 descriptors are reported in [43, 107] to measure the chemical properties of a compound in chemical graph theory, organic chemistry, pharmaceutical, biotechnology and nanotechnology. We consider 2D molecular descriptors in this thesis. The structure (geometry) of a chemical compound is based on atoms and bonds (covalent bonds). This molecular structure can be expressed through a graph by showing atoms as vertices and bonds as edges.

Molecular descriptors theoretically have been studied in the field of discrete mathematics, graph theory, algebraic topology, differential geometry, organic chemistry, physical chemistry, quantum chemistry and information theory. These molecular descriptors are handled by statistics, chemometrics and chemoinformatics., specially the regressions and correlations methods are largely used to study the molecular structures. Chemometrics is the study of extracting information from chemical structures (or material systems) by the compilation of data. Chemoinformatics, deals with the procedure and methods used to process information by using computational devices to solve problems related to chemistry. Molecular descriptors are applied in QSAR, quantitative structure–property relationships (QSPR), pharmacology, drug design, medical chemistry, analytic chemistry, genomics, proteomics, environmatrics, virtual screening and library searching.

Topological indices are classified into two main categories:

1. Topological indices based on distance.
2. Topological indices based on degree.

2.5.1 Topological indices based on distance

This category of indices is based on the distance between the pair of vertices in a connected graph. Some popular topological indices based on distance are given below:

- Wiener index
- Balaban index
- Harary index
- Szeged index, etc.

Wiener index

The Wiener index is a first topological index which was proposed by an American chemist, Harold Wiener in 1947 [112]. It is also known as Wiener number (or path number) [92] and is denoted by $W(G)$. He studied molecular structure of paraffin and established its correlation with a topological index. "The Wiener index is a useful topological index in a structure-property relationship because it is a measure of compactness of a molecule in terms of its structural characteristic, such as branching and cyclicity" [86]. Wiener is popular as a founder of chemical graph theory due to his suggested topological index which bridges between chemistry and graph theory.

Consider a simple and connected graph G having a vertex set $\{u_1, u_2, ...u_n\}$ and a edge set $\{e_1, e_2, ..., e_m\}$. The Wiener index of G can be computed as given below:

$$W(G) = \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} d(u_i, u_j).$$

where $d(u_i, u_j)$ is the distance between u_i and u_j.

Wiener index of some basic graphs [8, 92] are presented in the following list:

- Path graph P_n: $W(P_n) = \frac{1}{6}n(n^2 - 1)$,
- Cycle graph C_n, for $n \geq 3$: $W(C_n) = \begin{cases} \frac{1}{8}n^3, & \text{for } n \text{ even} \\ \frac{1}{8}(n-1)\ n\ (n+1), & \text{for } n \text{ odd} \end{cases}$,
- Complete graph K_n: $W(K_n) = 6n(n-1)$,
- Star graph S_n: $W(S_n) = (n-1)^2$,
- Gear graph (Jahangir graph) J_n: $W(J_n) = \frac{1}{6}n(n^2 - 1)$.

Balaban index

The Balaban index is a distance-based topological index which was proposed by Balaban over three decades ago [11, 12]. Babalan index of G is also referred as J index and is denoted by $J(G)$.

Consider an undirected graph G with $n = |V(G)|$ and $m = |E(G)|$. Then the J index can be defined as,

$$J(G) = \frac{m}{m-n+2}\sum_{uv\in E(G)} \frac{1}{\sqrt{w(u).w(v)}}$$

where $w(u) = \sum\limits_{v\in V(G)} d(u, v)$, $u \in V(G)$. The index J is also written as

$$J(G) = \frac{m}{\mu+1}\sum_{uv\in E(G)} \frac{1}{\sqrt{w(u).w(v)}}$$

where μ is cyclomatic number (or circuit rank) and it is defined as $\mu = m - n + 1$. If a graph contains c connected components, then the general formula of circuit rank is $\mu = m - n + c$ ($c = 1$, in a connected graph).

Harary index

The Harary index proposed by Plavšić *et al.* in [92], is reciprocal of Wiener index and is also known as RDSUM index (reciprocal distance sum index) [61]. Let G be a connected graph with a set of vertices $\{w_1, w_2, ...w_n\}$ and a set of edges $\{e_1, e_2, ..., e_m\}$. Then the Harary index is defined as

$$W(G) = \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{1}{d(w_i, w_j)}.$$

here, the notation $d(w_i, w_j)$ represent the distance between w_i and w_j.

Szeged index

The Szeged index was proposed by Gutman in 1994 [47]. This index is a modified form of Wiener index for those molecular structures which contain cycles [47, 48, 33].

Consider a graph G for $p, t, v, w \in V(G)$ and $(p, t) \in E(G)$, if v and w does not belong to some component of G, then $d(v, w)$ is not defined [49].

For $e = (v, u) \in E(G)$,

$$n_1(e \mid G) = |\{u \mid u \in V(G),\ d(u, p) < d(u, t)\}|$$

$$n_2(e \mid G) = |\{w \mid w \in V(G),\ d(w, t\ ) < d(w, p\ )\}|$$

where $n_1(e|G)$ denotes the number of vertices of graph closer to p than t, if $e = pt$. The concept of $n_2(e|G)$ is similar. The set $V(G)$ having same distances with p and t are not counted. Vertices belonging to components of G different from the component containing p and t are not counted. Since the Szeged index [47, 49] is based on $n_1(e|G)$ and $n_2(e|G)$. So it is defined as

$$Sz(G) = \sum_{e \in E(G)} n_1(e \mid G)\ n_2(e \mid G),\ \forall e \in E(G).$$

$Sz(G) = 0$, if G is an edgeless graph.

The chemical properties of Szeged index can be found in [49, 68, 121]. The useful chemical and pharmacological applications are discussed in [32, 67].

2.5.2 Topological indices based on degree

This section deals with an other class of topological indices which are based on the degree of vertices of a graph G. Due to their chemical significance, these indices are very important. These topological indices correlate with different physicochemical characteristic, like resonance energy, strain energy and boiling point in a more efficient way. Some examples for this topological class are given below.

- Zagreb index
- Randić index
- Atom-Bond Connectivity index
- Geometric-Arithmetic index, etc.

Zagreb index

In 1972, Gutman and Trinajstic [46] proposed a first degree based topological index referred as Zagreb index and after 3 decades in 2003 Nikolic *et al.* [85] elaborated this index. Consider graph G which is a simple connected finite graph with vertex set $V(G)$ and edge set $E(G)$. The first Zagreb index $M_1(G)$ and second Zagreb index $M_2(G)$ can be written as given below:

$$M_1(G) = \sum_{u_iv_j \in E(G)} [deg(u_i) + deg(v_j)]$$
$$M_2(G) = \sum_{u_iv_j \in E(G)} [deg(u_i)\ deg(v_j)]$$

The $M_1(G)$ also written as $\sum\limits_{v_i \in V(G)} [deg(v_i)]^2$.

Randic index

Randic index was suggested by Randić in 1976 [95] and this index is also referred as connectivity index. Randić index of a G can be written as:

$$R(G) = \sum_{vw \in E(G)} \frac{1}{\sqrt{deg(v)\ deg(w)}}$$

For further, reading we suggest the research papers [51, 53, 95, 96].

Atom-Bond Connectivity index (ABC index) In 1998, Estrada *et al.* [36] suggested ABC topological index. The ABC index is mostly applied to compute thermodynamic properties of a compounds [30]. This index is defined as:

$$ABC(G) = \sum_{vw \in E(G)} \frac{\sqrt{deg(v)\ + deg(w) - 2}}{deg(v)\ deg(w)}$$

Geometric-Arithmetic index The Geometric-Arithematic (GA) index is a topological index proposed by Vukičević and Furtula in 2009 [111]. The reason for proposing this new index is to improve the prediction of some properties of molecules. The Geometric-Arithematic index of a G is defined as given below:

$$GA(G) = \sum_{vw \in E(G)} \frac{2\sqrt{deg(v)\ deg(w)}}{deg(v) +\ deg(w)}$$

2.6 Metric dimension

In 1975 [104], Slater proposed the idea of a distance based graph theoretic parameter known as metric dimension (MD), and this concept was specially elaborated for graphs by Harary and Melter in 1976 [55]. After these two papers, lot of research papers related to its theoretical properties and applications have been published. The theoretical studies on metric dimension are highly contributed by several authors and they discussed new areas of graph theory and introduced application of metric dimensions such as, strong resolving sets [88], resolving dominating sets [20], local resolving sets [89], independent resolving sets [27], simultaneous metric generators [94], resolving partitions [24], mixed metric dimension [66, 97],

k-antiresolving sets [110], k-metric generators [37, 120] and strong resolving partitions [119]. The metric dimension has many applications in the digital world and in mathematical chemistry such as network verification [16], robot navigation, image processing and pattern recognition [81], mastermind game [23], chemistry [25, 26, 64] and pharmaceutical chemistry [64].

Definition 2.7 [24] *"Let $S = \{s_1, s_2, \cdots, s_k\}$ be an ordered subset of vertices of G and let w be a vertex of G. The representation $r(w|S)$ of w with respect to S is the ordered k-tuple $(d(w, s_1), d(w, s_2), \cdots, d(w, s_k))$. If $d(x, w) \neq d(y, w)$, then vertices x and y are distinguished by vertex w. If distinct vertices of G have distinct representations with respect to S, then S is called a resolving set for G".*

Definition 2.8 [21] *"A resolving set of minimum cardinality is called a basis for G and the cardinality is known as metric dimension of G and it is denoted as $dim(G)$".*

In 1979, Garey and Johnson [41] pointed out that the finding of "the metric dimension of a graph is an *NP*-hard problem". In 1994, Khuller *et al.* also showed by another construction that the "metric dimension of a graph is an *NP*-hard problem" [69].

2.8.1 Some results on metric dimensions

In this section, we present some results for metric dimension of well known graphs.

Theorem 2.8.1 [25] *"If G is a connected graph of order $n \geq 2$, the least positive integer k and diameter d, then $f(n, d) \leq dim(G) \leq n - d$".*

Theorem 2.8.2 [69] *"Let $G = (V, E)$ be a graph with metric dimension k and n number of vertices. Let D be the diameter of G. Then $n \leq D^k + k$".*

Theorem 2.8.3 [25, 69] *"A connected graph G of order n has dimension 1 if and only if $G = P_n$".*

Theorem 2.8.4 [25, 69] *"A connected graph G of order $n \geq 2$ has dimension $n - 1$ if and only if $G = K_n$".*

Theorem 2.8.5 [25] *"Let G be a connected graph of order $n \geq 4$. Then $dim(G) = n - 2$ if and only if $G = K_{s,t}$ $(s, t \geq 1)$, $G = K_s + \overline{K}_t$ $(s \geq 1,\ t \geq 2)$, or $G = K_s\ +\ (K_1 \cup K_t)$ $(s, t \geq 1)$".*

Theorem 2.8.6 [25] *"If T is a tree that is not a path, then $dim(T) = \sigma(T) - ex(T)$, where $\sigma(G)$ is the sum of the terminal degrees of the major vertices of G, and $ex(G)$ is the number of exterior major vertices of G"".*

Theorem 2.8.7 [69] *"The metric dimension of a d-dimensional grid ($d \geq 2$) is d".*

Theorem 2.8.8 [21] *"Let W_n be a wheel graph of order $n \geq 3$. Then $dim(W_n) = \lfloor \frac{2n+2}{5} \rfloor$, for $n \notin \{3, 6\}$".*

Theorem 2.8.9 [22] *"Let F_n be a fan graph of order $n \geq 1$. Then $dim(F_n) = \lfloor \frac{2n+2}{5} \rfloor$, for $n \notin \{1, 2, 3, 6\}$".*

Theorem 2.8.10 [109] *"Let J_n be a Jahangir (gear) graph of order $n \geq 2$. Then $dim(J_n) = \lfloor \frac{2n}{5} \rfloor$ for $n \geq 4$".*

Theorem 2.8.11 [63] *"Let $P(n, 2)$ be the generalized Petersen graph, then $dim(P(n, 2)) = 3 for\ n \geq 5$".*

Theorem 2.8.12 [58] *"Let $P(n, 3)$ be the generalized Petersen graph, then*

$$dim(P(n,3)) = \begin{cases} 3, & for\ n \equiv 1\ (mod\ 6)\ and\ n \geq 25. \\ 4, & for\ n \equiv 0\ (mod\ 6)\ and\ n \geq 24. \\ \leq 4, & for\ n \equiv 3,4,5\ (mod\ 6)\ and\ n \geq 17. \\ \leq 5, & for\ n \equiv 2\ (mod\ 6)\ and\ n \geq 8." \end{cases}$$

Theorem 2.8.12 for $P(n, 3)$ improved in 2015 [102] is given as

Theorem 2.8.13 [102] *"Let $P(n, 3)$ be the generalized Petersen graph, then*

$$dim(P(n,3)) = \begin{cases} 3, & for\ n \equiv 3\ (mod\ 6)\ and\ n \geq 25. \\ 4, & n \equiv 4\ (mod\ 6)\ and\ n \geq 24." \end{cases}$$

2.9 Edge metric dimension

Definition 2.10 [65] *"Let $G = (V, E)$ be a connected graph, let $v \in V$ be a vertex and let $e = uw \in E$ be an edge. The distance between the vertex v and the edge e is given by $d(e, v) = min\{d(u, v), d(w, v)\}$. A vertex $w \in V$ distinguishes two edges $e_1,\ e_2 \in E$ if $d(w, e_1) \neq d(w, e_2)$. A set S of vertices in a connected graph G is an edge metric generator for G if every two edges of G are distinguished by some vertex of S. The smallest cardinality of an edge metric generator for G is called the edge metric dimension and is denoted by $edim(G)$."*

Definition 2.11 [65] *"Let an ordered set of vertices $S = \{s_1,\ s_2,\ \cdots,\ s_k\}$ of a graph G, for any edge e in G, we refer to the k-vector (ordered k-tuple) $r(e|S) = (d(e, s_1),\ d(e, s_2),\ \cdots,\ d(e, s_k))$ as the edge metric representation of e with respect to S. In this sense, S is an edge metric generator for G if and only if for every pair of different edges $e_1,\ e_2$ of G, follows $r(e_1|S) \neq r(e_2|S)$."*

There is a controversy in the definition of edge resolving set and edge metric dimension. In literature, another version of edge metric dimension exists which is based on the distance between edges, denoted as $d(f, g)$. This version of edge metric dimension follows the following steps:

Step 1. Convert a graph G into a line graph $L(G)$.

Step 2. Find metric dimension, $dim(L(G))$ of line graph $L(G)$.

According to the above steps, the $dim(L(G)) = edim(G)$, for reference see [77].

But in our case the distance is based on vertex to edge, $d(v, e)$ as defined by Kelenc *et al.* in [65].

Let $\omega = (\eta_n)_{n\geq 1}$ be a family of connected graphs η_n of order $\varphi(n)$ for which $\lim\limits_{n\to\infty} \varphi(n) = \infty$. If for $a > 0$ such that $edim(\eta_n) \leq a$ then the family of ω has bounded edge metric dimension and if its not the case, then the $edim(\eta_n)$ is unbounded. Moreover if $edim(\eta_n) = a$, $\forall\ n \geq 1$, then ω has a constant edge metric dimension.

Recently, Kelenc *et al.* have proposed a notion $edim(G)$ for edge metric dimension (*EMD*) of graph and they have presented some results relating the notion of *MD* and *EMD* for some families of graph in [65] such as for path P_n:

$edim(P_n) = dim(P_n) = 1$ where $n \geq 2$, for cycle C_n: $edim(C_n) = dim(C_n) = 2$, for complete graph K_n: $edim(K_n) = dim(K_n) = n - 1$, where $n \geq 2$, for complete bipartite graph $K_{r,t}$: $edim(K_{r,t}) = dim(K_{r,t}) = r + t - 2$, where $r, t \geq 2$, for tree T_n which is not a path: $edim(T) = dim(T) = \sum\limits_{v \in V,\ L_v > 1} (L_v - 1)$, for grid graph $G = P_r \Box P_t$ (where $r \geq t \geq 2$): $edim(G) = dim(G) = 2$, for wheel graph ($G = S_{1,n}$): $edim(G) > dim(G)$ and for the grid graph $G = C_r \Box C_t$: $edim(C_r \Box C_t) < dim(C_r \Box C_t)$. They also proved that the $edim(G)$ is an NP-hard problem. Moreover, they have raised many open problems associated to edge metric dimension. Furthermore, in 2019, Zubrilina has classified some graphs on the same topic which has the $edim(G) = |V| - 1$ [124].

2.12 Applications

Metric dimension has many applications in the real world such as it can be used for optimization problems in complex networks, to analyze electrical networks and specially in coding and decoding (cryptography) and pharmaceutical chemistry, etc. We present here some of these applications.

2.12.1 Global positioning system

In this digital era, the Global Positioning System (GPS) is largely used for tracking a device or a vehicle. GPS is a technology which consists on the satellite-based navigation system which provides the information of time and location to a GPS device anywhere on the earth. GPS is an important tool for military, civil and commercial users such as tracking companies, digital devices and specially cell phones *etc.*

The GPS is based on the mathematical principal known as trilateration principle which is used to determine the location of a point by measurement of distances.

There are many GPS systems operated by different countries. Some of them are Global Positioning System (United States), Galileo (European Union), BeiDou (China), IRNSS-NAVIC (India), GLONASS (Russia) and QZSS (Japan). Every GPS system has at least 24 satellites around the earth. To locate the exact position of any object on the earth, the signals of four satellites are required. We can say every point on earth can be located by 4 satellites and the number of satellites are at least 24, ($n \geq 24$). So the metric dimension of GPS is 4.

2.12.2 Mastermind game

Mastermind is a popular logical game for two players. One player makes a secret code $s = [s_1, s_2, ..., s_n] \in \{1, 2, ..., k\}^n$ of n digits and hides it. This player is called the codemaker. The second player attempts to discover the hidden code $t = [t_1, t_2, ..., t_n] \in \{1, 2, ..., k\}^n$, who is called the codebreaker. There are some specific numbers of attempts (questions) and after every attempt, the codemaker uses pegs of two colors as a hint. The white color pegs mean that the one of guessed digits is correct and placed wrong. The black color pegs mean that one guessed digit is correct and placed right.

Mathematically these hints (answers) are represented with two integers, "first being the number of positions in which the secret vector and the question agree, denoted by $a(s,t) = |\{i : s_i = t_i, 1 \leq i \leq n\}|$. The second integer $b(s,t)$ is the maximum of $a(\overline{s}, t)$, where $\overline{s}$ ranges over all permutations of s. In the commercial version of this game, $n = 4$ and $k = 6$ are used. The secret code (vector) and each question is represented by four pegs each colored with one of six colors. Each answer is represented by $a(s,t)$ black pegs, and $b(s,t) - a(s,t)$ white pegs" [23]. Knuth showed in 1976 that four attempts (questions) are sufficient to find s in this case [74].

2.12.3 Social networks

Graph theory has an significant role in this era where the world has become a global village. Every one is connected with social networks through some devices. The connection between one device to another device (person to person) is known as relation and this relation is referred as an edge. The distance between vertex u (person) to an edge $e = xy$ (relation between person x and person y) is the shortest path between u and e. In a huge network, there are a lot of objects (and relations) and they are increasing day by day. It is very essential to know all properties (values) and constraints (limitations) of a network to analyze it. The graph theory presents all these networks in a graphical form and also provides some optimized solution to manage them. The edge resolvability and edge metric dimensions help to determine or identify unique relations of these objects in large networks and online communities.

2.12.4 Robot navigation

Robot is an electromechanical device which works automatically in different areas. Basically, the robot has no concept of direction and visibility. It can detect or sense the obstacles and landmarks by some sensors (or detectors) before any move. Robot motion needs to know a large map of relevant environment and solve the localization problem by taking minimum number of landmarks which exist in that environment.

With these implications, the robot is now able to understand the distance and use to optimize it. In graph theory the robot is a moving point and nodes are considered as landmarks or places with the edges representing the actions which connect the neighbouring landmarks. In this scenario, it is very important for a robot to identify its position uniquely by considering the minimum number of nodes and how far it is from every node. This problem largely dealt in graph theory to find resolving set or metric dimension.

Chapter 3

Zagreb indices and coindices of strong double graph

In this chapter, we present Zagreb indices and coindices of strong double graphs. We also study these indices for generalize kth-iterated strong double graphs. We use the concept of edge partition to reduce computation complexity and get computing formulae for these indices.

Definition 3.1 [84]*"The double graph of a simple graph G as the graph $D(G) = G \times T_2$. Since the direct product of a simple graph with any graph is always simple graph, it follows that the double of a simple graph is still a simple graph."*

Definition 3.2 [91]*"The strong double strong graph of the graph $G(n, m)$ with vertex set $V(G) = \{v_1, v_2, ..., v_n\}$ is the graph $SD(G)$ obtained by taking two copies of the graph G and joining each vertex v_i in one copy with the closed neighbourhood $N[v_i] = N(v_i) \cup \{v_i\}$ of the corresponding vertex in another copy."*

The Figure 3.1 represents the graph P_4 and its double graph $D(P_4)$. The strong double graphs $SD^{*1}(P_4)$ and the 2-iterated strong double graph $SD^{*2}(P_4)$ of P_4 are presented in Figure 3.2.

Figure 3.1

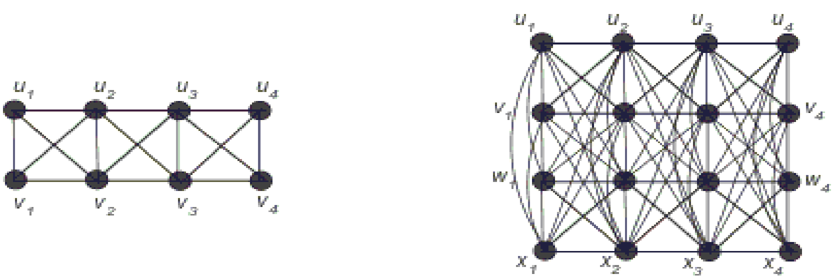


Figure 3.2

3.3 Introduction and preliminaries results

The first Zagreb index $M_1(G)$ and second Zagreb index $M_2(G)$ are defined as follow:

$$M_1(G) = \sum_{u_iv_j \in E(G)} [deg(u_i) + deg(v_j)]$$
$$M_2(G) = \sum_{u_iv_j \in E(G)} [deg(u_i)\ deg(v_j)] .$$

The Gutman and Trinajsti were suggested $M_1(G)$ and $M_2(G)$ in 1972 [46] and also explained by Nikolic *et al.* after 30 years in 2003 [85]. These indices reflect the extent of branching of the molecular structure which is based on carbon atoms and they are considered as molecular structure-descriptors [13, 106].

Recently, some unique forms of general Zagreb indices have been proposed, such as multiplicative Zagreb indices [50, 108, 115], multiplicative sum Zagreb index [35, 116], Zagreb coindices [6], and multiplicative Zagreb coindices [29].

The important variants of Zagreb index are first and second Zagreb coindices, which are defined as follow, respectively:

$$\overline{M}_1(G) = \sum_{u_i \neq v_j;\ u_iv_j \notin E(G)} [deg(u_i) + deg(v_j)]$$
$$\overline{M}_2(G) = \sum_{u_i \neq v_j;\ u_iv_j \notin E(G)} [deg(u_i)\ deg(v_j)]$$

Doslic [34] was introduced the $\overline{M}_1(G)$ and $\overline{M}_2(G)$ in 2008. In 2009 Ashrafi *et al.* [5] have find out "the extremal values of coindices for some particular types of graphs". Ashrafi *et al.* [6] has also explored their fundamental analytical properties and proposed explicit formulae of coindices under different graph operations as given below,

Proposition 3.3.1 [6] *"Let G be a simple graph on n vertices and m edges. Then $\overline{M}_1(G) = 2m(n-1) - M_1(G)$".*

Proposition 3.3.2 [6] *"Let G be a simple graph on n vertices and m edges. Then $\overline{M}_2(G) = 2m^2 - M_2(G) - \frac{1}{2}M_1(G)$".*

In 2016, Yang *et al.* [117] proposed the coindices of double graphs G^* and coindices for the kth-iterated double graphs G^{*k} as given below,

Theorem 3.3.3 [117] *"Let G be a nontrivial graph of order n and size m. Then $\overline{M}_1(G^*) = 8\overline{M}_1(G) + 8m$".*

Theorem 3.3.4 [117] *"Let G be a nontrivial graph of order n and size m. Then $\overline{M}_2(G^*) = 8\overline{M}_2(G) - 8M_1(G) + 16m^2$".*

Theorem 3.3.5 [117] *"Let G be a nontrivial graph of order n and size m, and let G^{k*} be its kth-iterated double graph. Then $\overline{M}_1(G^{*k}) = 8^k\overline{M}_1(G) + (2^{2k+1})(2^k - 1)m$".*

Theorem 3.3.6 [117] *"Let G be a nontrivial graph of order n and size m, and let G^{k*} be its kth-iterated double graph. Then $\overline{M}_2(G^{*k}) = 8^k\overline{M}_2(G) - \left[8^k\left(2^k-1\right)\right]M_2\left(G\right) + 2\left[8^k\left(2^k-1\right)\right]m^2$".*

In 2010, Ashrafi *et al.* [6] also computed the Zagreb coindices for graph operations such as join $G_1 + G_2$, Cartesian product $G_1 \square G_2$ and union $G_1 \cup G_2$ respectively as given under.

Proposition 3.3.7 [6] *"Let G_1 and G_2 be two simple graphs. Then $\overline{M}_1(G_1 \cup G_2) = \overline{M}_1(G_1) + \overline{M}_1(G_2) + 2(m_1n_2 + m_2n_1)$".*

Proposition 3.3.8 [6] *"Let G_1 and G_2 be two simple graphs. Then $\overline{M}_2(G_1 \cup G_2) = \overline{M}_2(G_1) + \overline{M}_2(G_2) + 2(m_1n_2 + m_2n_1)$".*

Proposition 3.3.9 [6] *"Let G_1 and G_2 be two simple graphs. Then $\overline{M}_1(G_1 + G_2) = \overline{M}_1(G_1) + \overline{M}_1(G_2) + 2(\overline{m}_1n_2 + \overline{m}_2n)$".*

Proposition 3.3.10 [6] *"Let G_1 and G_2 be two simple graphs. Then $\overline{M}_2(G_1 + G_2) = \overline{M}_2(G_1) + \overline{M}_2(G_2) + n_2\overline{M}_1(G_1) + n_1\overline{M}_1(G_2) + \overline{m}_1n_2^2 + \overline{m}_2n_1^2$".*

Proposition 3.3.11 [6] *"Let G_1 and G_2 be two simple graphs. Then $\overline{M}_1(G_1 \square G_2) = 2(n_1m_2 + n_2m_1)(n_1n_2 - 1) - 8m_1m_2 - n_2M_1(G_1) - n_1M_1(G_2)$".*

Proposition 3.3.12 [6] *"Let G_1 and G_2 be two simple graphs. Then $\overline{M}_2(G_1 \square G_2) = 2(n_1m_2 + n_2m_1)^2 - 4m_1m_2 - [n_1M_2(G_2) + n_2M_2(G_1)]$ $\left[\left(3m_2 + \frac{1}{2}n_2\right)M_1(G_1) + \left(3m_1 + \frac{1}{2}n_1\right)M_1(G_2)\right]$".*

In 2016, Gutman *et al.* established some relations between the first and second Zagreb index and coindex of G and its complements $\overline{G}$ in their research paper [52] as,

Theorem 3.3.13 [52] *"Let G be a graph with n vertices and m edges. Then $M_1(\overline{G}) = M_1(G) + n(n-1)^2 - 4m(n-1)$".*

Theorem 3.3.14 [52] *"Let G be a graph with n vertices and m edges. Then $\overline{M}_1(\overline{G}) = 2m(n-1) - M_1(G)$".*

Theorem 3.3.15 [52] *"Let G be a graph with n vertices and m edges. Then $M_2(\overline{G}) = \frac{1}{2}n(n-1)^3 - 3m(n-1)^2 + 2m^2 + \frac{2n-3}{2}M_1(G) - M_2(G)$".*

Theorem 3.3.16 [52] *"Let G be a graph with n vertices and m edges. Then $\overline{M}_2(\overline{G}) = m(n-1)^2 - (n-1)M_1(G) + M_2(G)$".*

Motivated by the work in [6, 117], we study the Zagreb indices and coindices of strong double graph of G.

3.4 Main results

For simplicity, we consider $SD(G) = G^{1*} = G^*$ and $G^{k*} = \left(G^{(k-1)*}\right)^*$ for $k \geq 2$. It is assumed $G^{0*} = G$ for the sake of consistency.

In the theorem given below, the first and second Zagreb indices of strong double graph discussed.

Theorem 3.4.1 *If G is a simple connected graph of order n and size m, then*
(i) $M_1(G^*) = 8M_1(G) + 16m + 2n$
(ii) $M_2(G^*) = 16M_2 + 12M_1(G) + 12m + n$

Proof. First we label every vertex of G as $\{v_1, v_2, ..., v_n\}$. Suppose that x_i and y_i are the corresponding clone vertices, in strong double graph G^*, of v_i for $i = 1, ..., n$.

There exists $deg_{G^*}(x_i) = deg_{G^*}(y_i) = 2deg_G(v_i) + 1$ for any given vertex v_i and its clone vertices x_i and y_i by the definition of strong double graph.

For $v_i, v_j \in V(G)$, if $v_iv_j \in E(G)$, then $x_ix_j \in E(G^*)$, $y_iy_j \in E(G^*)$, $x_iy_j \in E(G^*)$ and $y_ix_j \in E(G^*)$.

So we consider total contribution of adjacent vertex pairs for $M_1(G^*)$ and $M_2(G^*)$ in the following three types (cases):

Type 1: The adjacent vertex pairs $\{x_i, x_j\}$ and $\{y_i, y_j\}$, where $v_iv_j \in E(G)$.

Type 2: The adjacent vertex pairs $\{x_i, y_i\}$ for each $i = 1, ..., n$.

Type 3: The adjacent vertex pairs $\{x_i, y_j\}$ and $\{y_i, x_j\}$, where $v_iv_j \in E(G)$.

The total contribution of Type 1 adjacent vertex pairs in $M_1(G^*)$ is given by

$$\begin{aligned}
\sum_{x_ix_j \in E(G^*)} [deg_{G^*}(x_i) + deg_{G^*}(x_j)] &= \sum_{y_iy_j \in E(G^*)} [deg_{G^*}(y_i) + deg_{G^*}(y_j)] \\
&= \sum_{v_iv_j \in E(G)} [(2deg_G(v_i) + 1) + (2deg_G(v_j) + 1)] \\
&= 2 \sum_{v_iv_j \in E(G)} [deg_G(v_i) + deg_G(v_j)] \\
&+ 2\,|E(G)|_{v_iv_j \in E(G)} \\
&= 2M_1 + 2m. \qquad (3.1)
\end{aligned}$$

and $M_2(G^*)$ is given by

$$\begin{aligned}
\sum_{x_ix_j \in E(G^*)} [deg_{G^*}(x_i)\ deg_{G^*}(x_j)] &= \sum_{y_iy_j \in E(G^*)} [deg_{G^*}(y_i)\ deg_{G^*}(y_j)] \\
&= \sum_{v_iv_j \in E(G)} [(2deg_G(v_i) + 1)\ \ (2deg_G(v_j) + 1)] \\
&= 4 \sum_{v_iv_j \in E(G)} [deg_G(v_i)\ deg_G(v_j)] \\
&+ 2 \sum_{v_iv_j \in E(G)} [deg_G(v_i) + deg_G(v_j)]\ \ + |E(G)|_{v_iv_j \in E(G)} \\
&= 4M_2 + 2M_1 + m. \qquad (3.2)
\end{aligned}$$

The total contribution of Type 2 adjacent vertex pairs in $M_1(G^*)$ is given by

$$\begin{aligned}
&\sum_{x_i y_i \in E(G^*)} [deg_{G^*}(x_i) + deg_{G^*}(y_i)] \\
&= \sum_{v_i \in V(G)} [(2deg_G(v_i) + 1) + (2deg_G(v_i) + 1)] \\
&= 4 \sum_{v_i \in V(G)} deg_G(v_i) \ + 2\,|V(G)|_{v_i \in V(G)} \\
&= 8m + 2n \qquad (3.3)
\end{aligned}$$

and $M_2(G^*)$ is given by

$$\begin{aligned}
&\sum_{x_i y_i \in E(G^*)} [deg_{G^*}(x_i)\ deg_{G^*}(y_i)] \\
&= \sum_{v_i \in V(G)} [(2deg_G(v_i) + 1)\ \ (2deg_G(v_i) + 1)] \\
&= 4 \sum_{v_i \in V(G)} [deg_G(v_i)]^2 + 4 \sum_{v_i \in V(G)} deg_G(v_i) \ + |V(G)|_{v_i \in V(G)} \\
&= 4M_1(G) + 8m + n. \qquad (3.4)
\end{aligned}$$

The total contribution of Type 3 adjacent vertex pairs in $M_1(G^*)$ is given by

$$\begin{aligned}
\sum_{x_i y_j \in E(G^*)} [deg_{G^*}(x_i) + deg_{G^*}(y_j)] &= \sum_{y_i x_j \in E(G^*)} [deg_{G^*}(y_i) + deg_{G^*}(x_j)] \\
&= \sum_{v_i v_j \in E(G)} [(2deg_G(v_i) + 1) + (2deg_G(v_j) + 1)] \\
&= 2 \sum_{v_i v_j \in E(G)} [deg_G(v_i) + deg_G(v_j)] \\
&+ |E(G)|_{v_i v_j \in E(G)} \\
&= 2M_1(G) + 2m \qquad (3.5)
\end{aligned}$$

and $M_2(G^*)$ is given by

$$\sum_{x_i y_j \in E(G^*)} [deg_{G^*}(x_i)\ deg_{G^*}(y_j)] = \sum_{y_i x_j \in E(G^*)} [deg_{G^*}(y_i)\ deg_{G^*}(x_j)]$$

$$
\begin{aligned}
&= \sum_{v_iv_j \in E(G)} [(2deg_G(v_i)+1) \ \ (2deg_G(v_j)+1)] \\
&= 4 \sum_{v_iv_j \in E(G)} [deg_G(v_i) \ deg_G(v_j)] \\
&+ 2 \sum_{v_iv_j \in E(G)} [deg_G(v_i) + deg_G(v_j)] \ \ + |E(G)|_{v_iv_j \in E(G)} \\
&= 4M_2(G) + 2M_1(G) + m. \qquad (3.6)
\end{aligned}
$$

Therefore, by using Eq. (3.1), Eq. (3.2) and Eq. (3.5), we have

$$
\begin{aligned}
M_1(G^*) &= \sum_{x_ix_j \in E(G^*)} [deg_{G^*}(x_i) + deg_{G^*}(x_j)] + \sum_{y_iy_j \in E(G^*)} [deg_{G^*}(y_i) + deg_{G^*}(y_j)] \\
&+ \sum_{x_iy_i \in E(G^*)} [deg_{G^*}(x_i) + deg_{G^*}(y_i)] + \sum_{x_iy_j \in E(G^*)} [deg_{G^*}(x_i) + deg_{G^*}(y_j)] \\
&+ \sum_{y_ix_j \in E(G^*)} [deg_{G^*}(y_i) + deg_{G^*}(x_j)] \\
&= (2M_1(G) + 2m) + (2M_1(G) + 2m) + (8m + 2n) \\
&+ (2M_1(G) + 2m) + (2M_1(G) + 2m) \\
&= 8M_1(G) + 16m + 2n \qquad (3.7)
\end{aligned}
$$

and using Eq. (3.1), Eq. (3.4) and Eq. (3.6), we have

$$
\begin{aligned}
M_2(G^*) &= \sum_{x_ix_j \in E(G^*)} [deg_{G^*}(x_i) \ deg_{G^*}(x_j)] + \sum_{y_iy_j \in E(G^*)} [deg_{G^*}(y_i) \ deg_{G^*}(y_j)] \\
&+ \sum_{x_iy_i \in E(G^*)} [deg_{G^*}(x_i) \ deg_{G^*}(y_i)] + \sum_{x_iy_j \in E(G^*)} [deg_{G^*}(x_i) \ deg_{G^*}(y_j)] \\
&+ \sum_{y_ix_j \in E(G^*)} [deg_{G^*}(y_i) \ deg_{G^*}(x_j)] \\
&= (4M_2(G) + 2M_1(G) + m) + (4M_2(G) + 2M_1(G) + m) \\
&+ (4M_1(G) + 8m + n) + (4M_2(G) + 2M_1(G) + m) \\
&+ (4M_2(G) + 2M_1(G) + m) \\
&= 16M_2(G) + 12M_1(G) + 12m + n. \qquad (3.8)
\end{aligned}
$$

■

Theorem 3.4.2 *Let G be a simple connected graph of order n and size m, then*
(i) $\overline{M}_1(G^*) = 8\overline{M}_1(G) + 4n^2 - 4n - 8m$
(ii) $\overline{M}_2(G^*) = 16\overline{M}_2(G) + 8\overline{M}_1(G) + 2n^2 - 2n - 4m$

Proof. For the sake of convenience, we label all vertices of G as $\{v_1, v_2, ..., v_n\}$. Suppose that x_i and y_i are the corresponding clone vertices, in strong double graph G^*, of v_i for each $i = 1, ..., n$.
There exists $deg_{G^*}(x_i) = deg_{G^*}(yi) = 2deg_G(v_i) + 1$ for any given vertex v_i and its clone vertices x_i and y_i by the definition of strong double graph.
For $v_i, v_j \in V(G)$, if $v_iv_j \notin E(G)$, then $x_ix_j \notin E(G)$, $y_iy_j \notin E(G)$, $x_iy_j \notin E(G)$ and $y_ix_j \notin E(G)$.
So we consider total contributions of nonadjacent pairs of vertices for $\overline{M}_1(G^*)$ and $\overline{M}_2(G^*)$ in the following two types of cases:
Type 1: The pairs for nonadjacent vertex $\{x_i, x_j\}$ and $\{y_i, y_j\}$, where $v_iv_j \notin E(G)$.
Type 2: The pairs for nonadjacent vertex $\{x_i, y_j\}$ and $\{y_i, x_j\}$, where $v_iv_j \notin E(G)$.
The complete participation of Type 1 nonadjacent vertex pairs in $\overline{M}_1(G^*)$ is given by

$$\begin{aligned}\sum_{x_ix_j \notin E(G^*)} [deg_{G^*}(x_i) + deg_{G^*}(x_j)] &= \sum_{y_iy_j \notin E(G^*)} [deg_{G^*}(y_i) + deg_{G^*}(y_j)] \\ &= \sum_{v_iv_j \notin E(G)} [(2deg_G(v_i) + 1) + (2deg_G(v_j) + 1)] \\ &= 2 \sum_{v_iv_j \notin E(G)} [deg_G(v_i) + deg_G(v_j)] \;\; + 2\left|E(\overline{G})\right| \\ &= 2M_1(G) - 2m + n^2 - n. \end{aligned} \tag{3.9}$$

$\overline{M}_2(G^*)$ is given by

$$\begin{aligned}\sum_{x_ix_j \notin E(G^*)} [deg_{G^*}(x_i)\ deg_{G^*}(x_j)] &= \sum_{y_iy_j \notin E(G^*)} [deg_{G^*}(y_i)\ deg_{G^*}(y_j)] \\ &= \sum_{v_iv_j \notin E(G)} [(2deg_G(v_i) + 1)\ \ (2deg_G(v_j) + 1)]\end{aligned}$$

$$
\begin{aligned}
&= 4 \sum_{v_i v_j \notin E(G)} [deg_G(v_i)\ deg_G(v_j)] \\
&+ 2 \sum_{v_i v_j \notin E(G)} [deg_G(v_i) + deg_G(v_j)] \\
&+ \left|E(\overline{G})\right| \\
&= 4M_2(G) + 2M_1(G) + \frac{1}{2}\left(n^2 - n - 2m\right). \qquad (3.10)
\end{aligned}
$$

The complete participation of Type 2 nonadjacent vertex pairs in $\overline{M}_1(G^*)$ is given by

$$
\begin{aligned}
\sum_{x_i y_j \notin E(G^*)} [deg_{G^*}(x_i) + deg_{G^*}(y_j)] &= \sum_{y_i x_j \notin E(G^*)} [deg_{G^*}(y_i) + deg_{G^*}(x_j)] \\
&= \sum_{v_i v_j \notin E(G)} [(2deg_G(v_i) + 1) + (2deg_G(v_j) + 1)] \\
&= 2 \sum_{v_i v_j \notin E(G)} [deg_G(v_i) + deg_G(v_j)] \quad + 2\left|E(\overline{G})\right| \\
&= 2M_1(G) - 2m + n^2 - n \qquad (3.11)
\end{aligned}
$$

and $\overline{M}_2(G^*)$ is given by

$$
\sum_{x_i, y_j \notin E(G^*)} [deg_{G^*}(x_i) \cdot deg_{G^*}(y_j)] = \sum_{y_i, x_j \notin E(G^*)} [deg_{G^*}(x_i)\ \cdot deg_{G^*}(y_j)]
$$

$$
\begin{aligned}
&= \sum_{v_i, v_j \notin E(G)} [(2deg_G(v_i) + 1)\ \ (2deg_G(v_j) + 1)] \\
&= 4 \sum_{v_i, v_j \notin E(G)} [deg_G(v_i)\ \cdot deg_G(v_j)] \\
&+ 2 \sum_{v_i, v_j \notin E(G)} [deg_G(v_i) + deg_G(v_j)] \quad + \left|E(\overline{G})\right| \\
&= 4M_2(G) + 2M_1(G) + \frac{1}{2}\left(n^2 - n - 2m\right), \qquad (3.12)
\end{aligned}
$$

therefore, by using Eq. (3.9) and Eq. (3.11), we have

$$
\begin{aligned}
\overline{M}_1(G^*) =& \sum_{x_ix_j\notin E(G^*)} [deg_{G^*}(x_i) + deg_{G^*}(x_j)] + \sum_{y_iy_j\notin E(G^*)} [deg_{G^*}(y_i) + deg_{G^*}(y_j)] \\
&+ \sum_{x_iy_j\notin E(G^*)} [deg_{G^*}(x_i) + deg_{G^*}(y_j)] + \sum_{y_ix_j\notin E(G^*)} [deg_{G^*}(y_i) + deg_{G^*}(x_j)] \\
&= \left(2M_1(G) - 2m + n^2 - n\right) + \left(2M_1(G) - 2m + n^2 - n\right) \\
&+ \left(2M_1(G) - 2m + n^2 - n\right) + \left(2M_1(G) - 2m + n^2 - n\right) \\
&= 8M_1(G) - 8m + 4n^2 - 4n \qquad (3.13)
\end{aligned}
$$

and using Eq. (3.10) and (3.12), we have

$$
\begin{aligned}
\overline{M}_2(G^*) =& \sum_{x_i,x_j\notin E(G^*)} [deg_{G^*}(x_i)\cdot\ deg_{G^*}(x_j)] + \sum_{y_i,y_j\notin E(G^*)} [deg_{G^*}(y_i)\cdot\ deg_{G^*}(y_j)] \\
&+ \sum_{x_i,y_j\notin E(G^*)} [deg_{G^*}(x_i)\cdot\ deg_{G^*}(y_j)] + \sum_{y_i,x_j\notin E(G^*)} [deg_{G^*}(y_i)\ \cdot deg_{G^*}(x_j)] \\
&= \left(4M_2(G) + 2M_1(G) + \frac{1}{2}\left(n^2 - n - 2m\right)\right) + \\
&\left(4M_2(G) + 2M_1(G) + \frac{1}{2}\left(n^2 - n - 2m\right)\right) + \\
&\left(4M_2(G) + 2M_1(G) + \frac{1}{2}\left(n^2 - n - 2m\right)\right) + \\
&\left(4M_2(G) + 2M_1(G) + \frac{1}{2}\left(n^2 - n - 2m\right)\right) \\
&= 16M_2(G) + 8M_1(G) + 2\left(n^2 - n - 2m\right). \qquad (3.14)
\end{aligned}
$$

■

Now, we present the first and second indices and coindices of kth-iterated strong double graphs.

Theorem 3.4.3 *Let G be a nontrivial graph of order n and size m, and let G^{k*} be its kth-iterated strong double graph. Then*
(i) $M_1(G^{*k}) = 8^k M_1(G) + (2^k - 1)\ 2^{2(k+1)}m + (2^k - 1)^2\ 2^k n$
(ii) $M_2(G^{*k}) = 3\cdot 2^{3k-1}\left(2^k - 1\right) M_1(G) +\ 4^{2k}M_2(G) + 3(2^k - 1)^2 2^{2k}m + (2^k - 1)^3 2^{k-1}n$

(iii) $\overline{M}_1(G^{*k}) = 8^k\overline{M}_1(G) + (2^k-1)4^k\,(n^2-n-2m)$

(iv) $\overline{M}_2(G^{*k}) = 2^{4k}\overline{M}_2(G) + 2^{3k}\left(2^k-1\right)\overline{M}_1(G) + 2^{2k-1}(2^k-1)^2\,(n^2-n-2m)\,.$

Proof. For any nontrivial graph G with n vertices and m edges, the number of vertices in G^* is $2n$ and the number of edges in G^* is $2m$ plus those edges between the sets $\{x_1, x_2, ..., x_n\}$ and $\{y_1, y_2, ..., , y_n\}$, that is $4^k m + n$. Now we deduce that G^{k*} has $2^k m$ vertices and $4^k m + (2^{2k-1} - 2^{k-1})n$ edges.

(i) $M_1(G^{k*}) = 8^k M_1(G) + 2^{2(k+1)}(2^k-1)m + 2^k(2^k-1)^2 n.$

As we know $M_1(G^*) = 8M_1(G) + 16m + 2n.$

Using the size of strong double graph $m^* = 4^k m + (2^{2k-1} - 2^{k-1})n$, we have

$$M_1(G^*) = 8M_1(G) + 16\left\{4^k m + (2^{2k-1} - 2^{k-1})n\right\} + 2n$$
$$M_1(G^{k*}) = 8M_1(G^{(k-1)*}) + 4\cdot 2^{2k}m + (2^{2k-1} - 2^{k-1} + 2)n.$$

By Theorem 3.4.1 and the definition of kth-iterated strong double graph, for $k \geq 1$, we can write $M_1(G^{k*})$ as

$$M_1(G^{k*}) = 8M_1(G^{(k-1)*}) + 4\cdot 2^{2k}m + (8\cdot 2^{2k-2} - 8\cdot 2^{k-1} + 2^k)n$$

$$\begin{aligned}
M_1(G^{k*}) &= 8\left\{8M_1(G^{(k-2)*}) + 4\cdot 2^{2k-2}m + (8\cdot 2^{2k-4} - 8\cdot 2^{k-2} + 2^{k-1})n\right\} \\
&\quad +4\cdot 2^{2k}m + (8\cdot 2^{2k-2} - 8\cdot 2^{k-1} + 2^k)n \\
&= 8^2 M_1(G^{(k-2)*}) + 12\cdot 2^{2k}m + 2^{2k-1}\left(6\cdot 2^{2k} - 15\cdot 2^k\right)n \\
&= 8^2\left\{8M_1(G^{(k-3)*}) + 4\cdot 2^{2k-4}m + (8\cdot 2^{2k-6} - 8\cdot 2^{k-3} + 2^{k-2})n\right\} \\
&\quad +12\cdot 2^{2k}m \;+ 2^{2k-1}\left(6\cdot 2^{2k} - 15\cdot 2^k\right)n \\
&= 8^3 M_1(G^{(k-3)*}) + 28\cdot 2^{2k}m + \left(14\cdot 2^{2k} - 63\cdot 2^k\right)n \\
&\;\;\vdots \\
M_1(G^{*k}) &= 8^k M_1(G) + 2^{2(k+1)}(2^k-1)m + 2^k(2^k-1)^2 n.
\end{aligned}$$

(ii) $M_2(G^{*k}) = 3\cdot 2^{3k-1}\cdot\left(2^k-1\right)M_1(G) + 4^{2k}M_2(G) + 3\cdot 2^{2k}\cdot(2^k-1)^2 m + 2^{k-1}(2^k-1)^3 n.$

As we know

$$M_2(G^*) = 16M_2(G) + 12M_1(G) + 12m + n.$$

Using $M_1(G^*) = 8M_1(G) + 4 \cdot 2^{2k}m + (2^{2k-1} - 2^{k-1} + 2)n$ and the size of of strong double graph is $m^* = 4^k m + (2^{2k-1} - 2^{k-1})n$, we have

$$\begin{aligned} M_2(G^*) &= 16M_2(G) + 12\left\{8M_1(G) + 4 \cdot 2^{2k}m + (2^{2k-1} - 2^{k-1} + 2)n\right\} \\ &\quad + 12\left\{4^k m + (2^{2k-1} - 2^{k-1})n\right\} + n \\ M_2(G^*) &= 16M_2(G) + 96M_1(G) + \left(192 + \ 12 \cdot 2^{2k}\right) m \\ &\quad + (6 \cdot 2^{2k} - 6 \cdot 2^k + 25)n. \end{aligned}$$

By Theorem 3.4.1 and the definition of kth-iterated strong double graph, for $k \geq 1$, we can write $M_2(G^*)$ as,

$$\begin{aligned} M_2(G^*) \quad &= \quad 16M_2(G^{(k-1)*}) + 12 \cdot 2^{3(k-1)}M_1(G) + \left(6 \cdot 2^{3(k-1)} + 6 \cdot 2^{5(k-1)}\right) m \\ &\quad\quad +(6 \cdot 4^{k-1} - 6 \cdot 2^{k-1} + 5^{2(k-1)})n \end{aligned}$$

$$\begin{aligned} M_2(G^{k*}) &= 16\left\{16M_2(G^{(k-2)*}) + 12 \cdot 2^{3(k-2)}M_1(G) + \left(6 \cdot 2^{3(k-2)} + 6 \cdot 2^{5(k-2)}\right) m\right\} \\ &\quad + 16 \cdot (6 \cdot 4^{k-2} - 6 \cdot 2^{k-2} + 5^{2(k-2)})n + 12 \cdot 2^{3(k-1)}M_1(G) \\ &\quad + \left(6 \cdot 2^{3(k-1)} + 6 \cdot 2^{5(k-1)}\right) m + (6 \cdot 4^{k-1} - 6 \cdot 2^{k-1} + 5^{2(k-1)})n \\ &= 16^2 M_2(G^{(k-1)*}) + \frac{9 \cdot 2^{3(k)}}{2}M_1(G) + \left(\frac{9 \cdot 2^{3k}}{4} + \frac{9 \cdot 2^{5k}}{32}\right) m \\ &\quad + (\frac{15 \cdot 2^{2k}}{2} - \frac{41 \cdot 5^{2k}}{625} + 27 \cdot 2^k + 1)n \end{aligned}$$

$$\begin{aligned} &= 16^2\left\{16M_2(G^{(k-3)*}) + 12 \cdot 2^{3(k-3)}M_1(G) + \left(6 \cdot 2^{3(k-3)} + 6 \cdot 2^{5(k-3)}\right) m\right\} \\ &+ 16^2 \cdot (6 \cdot 4^{k-3} - 6 \cdot 2^{k-3} + 5^{2(k-3)})n + \frac{9 \cdot 2^{3(k)}}{2}M_1(G) \\ &+ \left(\frac{9 \cdot 2^{3k}}{4} + \frac{9 \cdot 2^{5k}}{32}\right) m + (\frac{15 \cdot 2^{2k}}{2} - \frac{41 \cdot 5^{2k}}{625} + 27 \cdot 2^k + 1)n \end{aligned}$$

$$= 16^3 M_2(G^{(k-3)*}) + \frac{21\cdot 2^{3k}}{2} M_1(G) + \left(\frac{21\cdot 2^{3k}}{4} + \frac{21\cdot 2^{5k}}{64}\right) m$$
$$+ (\frac{63\cdot 2^{2k}}{2} + \frac{1281\cdot 5^{2k}}{15625} + 219\cdot 2^k \ + 1)n$$
$$\vdots$$
$$M_2(G^{*k}) = 3\cdot 2^{3k-1}\cdot \left(2^k - 1\right) M_1(G) + \ 4^{2k} M_2(G) + 3\cdot 2^{2k}\cdot (2^k-1)^2 m +$$
$$2^{k-1}(2^k-1)^3 n.$$

(iii) $\overline{M}_1(G^{*k}) = 8^k \overline{M}_1(G) + (2^k - 1)4^k\left(n^2 - n - 2m\right).$

$$M_1(G^*) = 8\overline{M}_1(G^*) - 8m + 4n^2 + 4n.$$

By Theorem 3.4.2 and the definition of kth-iterated strong double graph, for $k \geq 1$, we write $M_1(G^{k*})$ as,

$$M_1(G^{k*}) = 8\overline{M}_1(G^{(k-1)*}) - 2^{2k+1}m + 2^{2k}n^2 + 2^{2k}n$$

$$M_1(G^{k*}) = 8\left\{8\overline{M}_1(G^{(k-2)*}) - 2^{2(k-1)+1}m + 2^{2(k-1)}n^2 + 2^{2(k-1)}n\right\}$$
$$- 2^{2k+1}m + 2^{2k}n^2 + 2^{2k}n$$
$$= 8^2 M_1(G^{(k-2)*}) - 6\cdot 2^{2k}m + 3\cdot 2^{2k}n^2 + 3\cdot 2^{2k}n$$
$$= 8^2\left\{8\overline{M}_1(G^{(k-3)*}) - 2^{2k-3}m + 2^{2k-4}n^2 + 2^{2k-4}n\right\} - 6\cdot 2^{2k}m$$
$$+ 3\cdot 2^{2k}n^2 + 3\cdot 2^{2k}n$$
$$= 8^3\overline{M}_1(G^{(k-3)*}) - 14\cdot 2^{2k}m + 7\cdot 2^{2k}n^2 + 7\cdot 2^{2k}n$$
$$\vdots$$
$$M_1(G^{*k}) = 8^k\overline{M}_1(G) + (2^k-1)4^k\left(n^2 - n - 2m\right).$$

(iv) $\overline{M}_2(G^{*k}) = 2^{4k}\overline{M}_2(G) + 2^{3k}\cdot\left(2^k - 1\right)\overline{M}_1(G) + \ 2^{2k-1}\cdot(2^k-1)^2\left(n^2 - n - 2m\right).$

$$\overline{M}_2(G^*) = 16\overline{M}_2(G) + 8\overline{M}_1(G) + \ 2\left(n^2 - n - 2m\right).$$

By Theorem 3.4.2 and the definition of kth-iterated strong double graph, for $k \geq 1$, we have

$$\overline{M}_2(G^{k*}) = 16\overline{M}_2(G^{(k-1)*}) + 8\overline{M}_1(G^{(k-1)*}) + \ 2\left(n^2 - n - 2m\right),$$

using the $\overline{M}_1(G^*) = 8\overline{M}_1(G) - 8m + 4n^2 + 4n$, we have

$$\begin{aligned}\overline{M}_2(G^{k*}) \ &= 16\overline{M}_2(G^{(k-1)*}) + 8\left\{8\overline{M}_1(G) - 8m + 4n^2 + 4n\right\} + \ 2\left(n^2 - n - 2m\right) \\ &= 16\overline{M}_2(G^{(k-1)*}) + 64\overline{M}_1(G) + \ 32\left(n^2 - n - 2m\right) + \ 2\left(n^2 - n - 2m\right)\end{aligned}$$

by rearranging the terms,we have

$$\begin{aligned}\overline{M}_2(G^{k*}) &= 16\overline{M}_2(G^{(k-1)*}) + 8\cdot 8^{(k-1)}\overline{M}_1(G) + \ 32^{(k-1)}\left(n^2 - n - 2m\right) \\ &\quad + \ 2^{(3k-3)}\left(n^2 - n - 2m\right)\end{aligned}$$

$$\begin{aligned}&= 16\left\{16\overline{M}_2(G^{(k-2)*}) + 8\cdot 8^{(k-2)}\overline{M}_1(G) + \ 32^{(k-2)}\left(n^2 - n - 2m\right)\right\} + \\ &+ \ 16\cdot 2^{(3k-6)}\left(n^2 - n - 2m\right) + 8\cdot 8^{(k-1)}\overline{M}_1(G) \\ &+ \ 32^{(k-1)}\left(n^2 - n - 2m\right) + \ 2^{(3k-3)}\left(n^2 - n - 2m\right)\end{aligned}$$

$$\begin{aligned}&= 16^2\overline{M}_2(G^{(k-2)*}) + 3\cdot 2^{(3k)}\overline{M}_1(G) + \frac{3\cdot 2^{(5k)}}{64}\left(n^2 - n - 2m\right) \\ &+ \ \frac{3\cdot 2^{(3k)}}{8}\left(n^2 - n - 2m\right) \\ &= 16^2\left\{16\overline{M}_2(G^{(k-3)*}) + 8\cdot 8^{(k-3)}\overline{M}_1(G) + \ 32^{(k-3)}\left(n^2 - n - 2m\right)\right\} \\ &+ 16^2\cdot \ 2^{(3k-9)}\left(n^2 - n - 2m\right) + 3\cdot 2^{(3k)}\overline{M}_1(G) + \frac{3\cdot 2^{(5k)}}{64}\left(n^2 - n - 2m\right) \\ &+ \ \frac{3\cdot 2^{(3k)}}{8}\left(n^2 - n - 2m\right)\end{aligned}$$

$$\begin{aligned}&= 16^3\overline{M}_2(G^{(k-3)*}) + 7\cdot 2^{(3k)}\overline{M}_1(G) + \ \frac{7\cdot 2^{(5k)}}{128}\left(n^2 - n - 2m\right) \\ &+ \frac{7\cdot 2^{(3k)}}{8}\left(n^2 - n - 2m\right) \\ &\vdots\end{aligned}$$

$$\overline{M}_2(G^{k*}) = 2^{4k}\overline{M}_2(G) + 2^{3k}\cdot\left(2^k - 1\right)\overline{M}_1(G) + \ 2^{2k-1}\cdot(2^k - 1)^2\left(n^2 - n - 2m\right).$$

■

Here we present an algorithm to verify first and second Zagreb indices, and first and second Zagreb coindices of finite simple connected graph. This algorithm is based on the adjacency matrix of the graph.

Algorithm 3.4.4 *Compute first Zagreb index IM_1, second Zagreb index IM_2, first Zagreb coindex CM_1 and second Zagreb coindex CM_2.*

Input: A adjacency matrix of a finite simple connected graph

Output: IM_1, IM_2, CM_1, CM_2

Variables used: nr = *number of rows in* A, nc = *number of column in* A,

```
IM_1 := 0, IM_2 := 0, CM_1 := 0, CM_2 := 0
for k = 1 to nr
    for l = k to nc
        s_k := count all values in k-th row of A,
        sl := count all values in l-th column of A.
        if A(k,l) = 1
            IM_1 := IM_1 + (S_k + S_l)
            IM_2 := IM_2 + (S_k * S_l)
        end if
        if A(k,j) = 0 and k ≠ l
            CM_1 := CM_1 + (S_k + S_l)
            CM_2 := CM_2 + (S_k * S_l)
        end if
    end for l
end for k
```

Example 3.4.5 *Let the following adjacency matrices A, B and C of path graph P_4 and strong double graph $SD^{1*}(P_4)$, $SD^{2*}(P_4)$, respectively. Apply suggested algorithm on these matrices to calculate first Zagreb index IM_1, second Zagreb index IM_2, first Zagreb coindex CM_1 and second Zagreb coindex CM_2. These adjacency matrices have been taken by drawing the respective graphs in Graph Tea software, this software easily available and it is free.*

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 \end{bmatrix}$$

$$C = \begin{Bmatrix} 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 \end{Bmatrix}$$

The result of algorithm is given below

$Graph$	$Matrix$	AM_1	AM_2	CM_1	CM_2
P_4	A	10	8	8	5
$SD^{1*}(P_4)$	B	136	288	88	156
$SD^{2*}(P_4)$	C	756	2652	864	2802

Now, we provide graphs of Zagreb indices and coindices. Such type of graphical

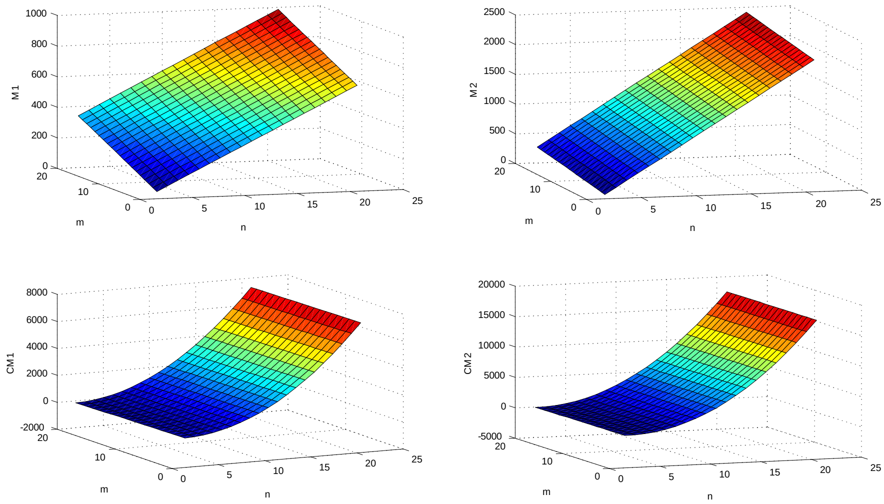


Figure 3.3: 3D surface plot of Zagreb indices and coindices of $SD(P_n)$

representation will be more helpful to study the dynamics of topological descriptors of the molecular graphs. Here we present the strong double graph of path graph, $SD(P_n)$, where $2 \leq n \leq 21$ and $m = n - 1$. In Figure 3.3, the first Zagreb index $M_1$and second Zagreb index M_2 are linear and the first Zagreb coindex $\overline{M}_1$ and second Zagreb coindex $\overline{M}_2$ are nonlinear by their behaviors. In Figure 3.4 we have drawn the curves of these indices and coindices for the $SD(P_n)$ to compare their relations, $M_1(G) \leq M_2(G) \leq \overline{M}_1(G) \leq \overline{M}_2(G)$.

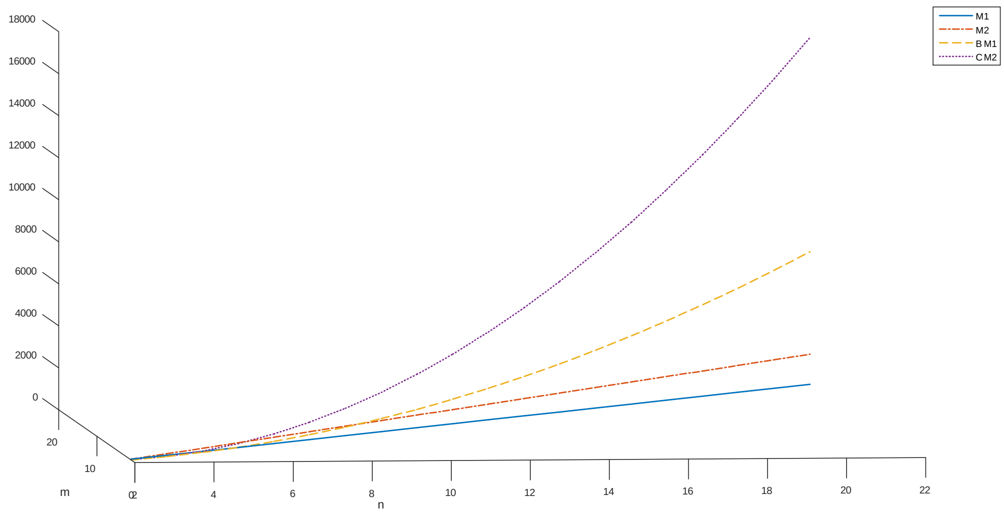


Figure 3.4: 3D curves plot of Zagreb indices and coindices of $SD(P_n)$

Chapter 4

Eccentricity based topological properties of Sierpinski network and its regularization

4.1 Introduction

A fractal is a scientific set that shows a rehashing design shown at each scale [17]. The fractals are a visual articulation of a rehashing example or equation that begins to be straightforward but gets logically more mind boggling [80]. It is otherwise called extending symmetry or advancing symmetry. If the replication is precisely the same at each scale, it is known as a self-comparable example [79]. One can undoubtedly watch that Sierpinski systems carry on like fractals.

Fractals to a great extent assist us with studying and comprehend the essential logical ideas, for example, the way microorganisms develop, designs in solidifying water, process of parting and combination, verdure of compound dissolving in water and symmetries of DNA, etc.

In 1997, Klavzar and Milutinovic [70] characterized Sierpinski graphs $S(n, k)$. These graphs are the speculation of the charts of Tower of Hanoi problem. Afterward, those diagrams have been known as Sierpinski graphs in [71], since their presentation was first spurred by topological investigations of Lipscomb's space [76, 44]. These diagrams have been very much concentrated since their presentation [72, 73]. The Sierpinski graphs $S(n, k)$ can be characterized recursively with the accompanying procedure: $S(1, k)$ is isomorphic to the entire graphs on k vertices, $S(n+1, k)$ is developed from $S(n, k)$ by duplicating n times graph $S(n, k)$ and including precisely one edge between each match of duplicates. At the point when $k = 3$, these graphs are precisely Tower of Hanoi diagrams and for bigger k, they can be viewed as diagram of a variety of Hanoi problem. Chemical graph theory is a sub discipline of mathematical chemistry which implements the tools from the graph theory to scientific demonstration of chemical occurrence [115].

A molecular graph G is a simple graph with $n = |V|$ vertices and $m = |E|$ edges. The vertices $v_i \in V(G)$ represent non-hydrogen atoms, the edges $(v_i, v_j) \in E(G)$ represent covalent bonds between the corresponding atoms and the degree of vertex $deg(v_i)$ show the valency of an atom. Specifically, hydrocarbons are formed just via carbon and hydrogen atoms and the molecular graphs of hydrocarbons represent the carbon skeleton of the molecule [18]. A topological index also called a topological descriptor is sort of a molecular descriptor that is computed based on the molecular graph of a chemical compound [107].

The eccentric-connectivity index (ECI), $\xi(G)$ of graph G is based on the combination of a vertex degree and its eccentricity, can be defined as

$$\xi(G) = \sum_{u \in V(G)} deg(u)\ \varepsilon_{cc}(u). \tag{4.1}$$

The total-eccentricity index represented by notation $\zeta(G)$ of G is defined as:

$$\zeta(G) = \sum_{u \in V(G)} \varepsilon_{cc}(u). \tag{4.2}$$

The relation exist $\xi(G) = k\zeta(G)$ if the graph G is k-consistent. The reader can consider the following research papers [1, 4, 38, 45, 56, 57, 122] for further studies about these indices. The Zagreb indices have been presented by Gutman and Trinajstic more than 30 years ago [46]. They have defined as:

$$M_1(G) = \sum_{v \in V(G)} deg(v)^2 \tag{4.3}$$

$$M_2(G) = \sum_{uv \in E(G)} deg(u)\ deg(v). \tag{4.4}$$

Ghorbani and Hosseinzadeh [42] proposed new forms of Zagreb indices as given below:

$$M_1^*(G) = \sum_{uv \in E(G)} [\varepsilon_{cc}(u)\ + \varepsilon_{cc}(v)] \tag{4.5}$$

$$M_1^{**}(G) = \sum_{u \in V(G)} \varepsilon_{cc}(u)^2 \tag{4.6}$$

$$M_2^*(G) = \sum_{uv \in E(G)} \varepsilon_{cc}(u)\ \varepsilon_{cc}(v). \tag{4.7}$$

Since in the sum of Eq. (4.5), the $\varepsilon_{cc}(u)$ occurs $deg(u)$ times for every $u \in V(G)$.

Therefore,

$$\begin{aligned} M_1^*(G) &= \sum_{uv\in E(G)} [\varepsilon_{cc}(u)\ + \varepsilon_{cc}(v)] \\ &= \sum_{u\in V(G)} deg(u)\ \varepsilon_{cc}(u) \\ &= \xi(G). \end{aligned}$$

4.1.1 Sierpinski graph

The Sierpinski graph $S(n,k)$ is defined in [70] as follows. For $n \geq 1$ and $k \geq 1$, the vertex set of $S(n,k)$ consists of all n-tuples of integers $1, 2, ..., k$, that is, $V(S(n,k)) = \{1, 2, , ..., k\}^n$. Any two different vertices $u = (u_1, u_2, ..., u_n)$ and $v = (v_1, v_2, ..., v_n)$ are adjacent if and only if there exists an $h \in [n]$ such that
(a) $u_t = v_t$ for $t \in [h-1]$;
(b) $u_h \neq v_h$;
(c) $u_t = v_h$ and $v_t = u_h$ for $t \in [h+1, n]$.

The graph $S^+(n,k)$ [73] is defined as follows. For $n \geq 1$ and $k \geq 1$, the graph $S^+(n,k)$ is obtained from Sierpinski graph $S(n,k)$ by adding a new vertex w and edges joining w with k extreme vertices of $S(n,k)$.

The graph $S^{++}(n,k)$ [73] is defined as follows. For $n = 1$, let $S^{++}(1,k) = K_{k+1}$. For $n \geq 2$ and $k \geq 1$, the graph $S^{++}(n,k)$ is obtained from the disjoint union of $k+1$ copies of $S(n-1,k)$ in which the corresponding extreme vertices in distinct copies of $S(n-1,k)$ are connected as the complete graph K_{k+1}.

Sierpinski graphs $S(n,k)$ for different values of n and k are represented in Figures 4.1 and Figure 4.2.

Regularizations of Sierpinski graph

For k, $n \in N$, all non-extreme vertices of the Sierpinski graph $S(n,k)$ have degree k and extreme vertices have degree $k-1$. So Sierpinski graphs are almost regular (obviously, $S(n,1)$, $S(0,k)$ and $S(1,k)$ are regular). This was the motivation to characterize two new families of Sierpinski-like graphs. As there are k vertices of

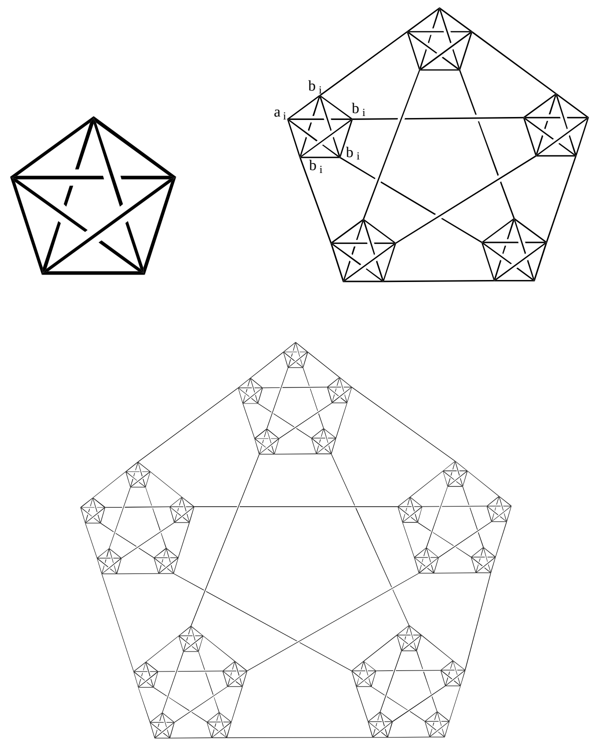


Figure 4.1: The Sierpinski graphs $S(1,5)$, $S(2,5)$ and $S(3,5)$

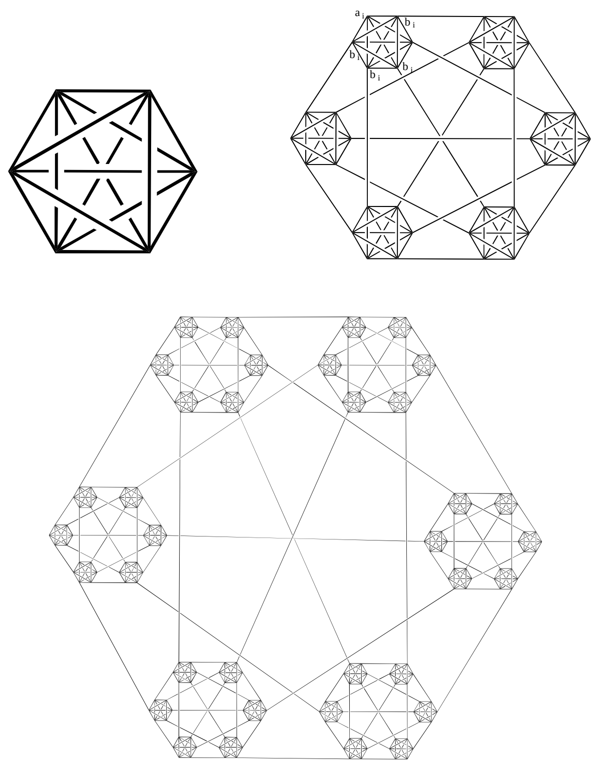


Figure 4.2: The Sierpinski graphs $S(1,6)$, $S(2,6)$ and $S(3,6)$

degree $k-1$ in $S(n,k)$, there are two natural approaches to regularize them: We either add another vertex to $S(n,k)$ and connect it with all the extreme vertices or we add another copy of $S(n-1,k)$ and connect the extreme vertices of $S(n,k)$ with the extreme vertices of that $S(n-1,k)$, respectively. To comprehend these two constructions better, see Figure 4.3 for the case $k = 4$ and $n = 2$. The first possibility gives us the graph $S^{+}(n,k)$, where the additional vertex w is known as the special vertex of $S^{+}(n,k)$.

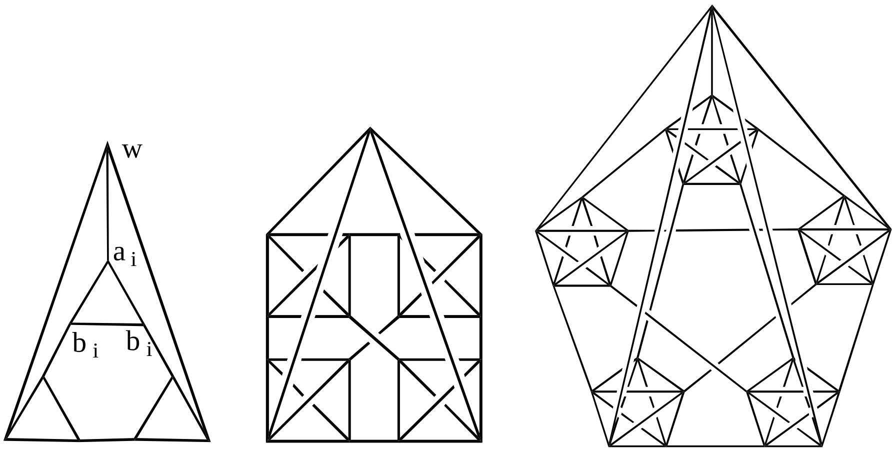


Figure 4.3: Regularization of Sierpinski graphs $S^{+}(2,3)$, $S^{+}(2,4)$ and $S^{+}(2,5)$.

The other regularization, i.e. adding another copy of $S(n-1,k)$ to $S(n,k)$, is denoted by $S^{++}(n,k)$. It can also be described as taking $k+1$ copies of $S(n-1,k)$ (when constructing a Sierpinski graph $S(n,k)$ we take just k such copies) and joining their extreme vertices in the sense of the complete graph K_{k+1}. We may think about the K_4 as the vertices of K_5. This construction is like the construction of a Sierpinski graph, but with complete graph of various orders. For more clarity about the construction of $S^{++}(n,k)$, see Figure 4.4.

4.2 Main results

In this section, we provide the generalized closed mathematical formulas for eccentric-connectivity index and zagreb-eccentricity indices of $S(n,k)$, $S^{+}(n,k)$ and $S^{++}(n,k)$.

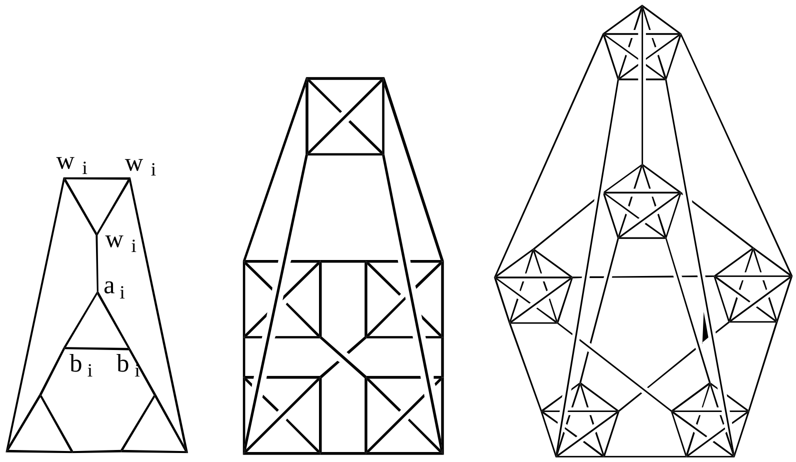


Figure 4.4: Regularization of Sierpinski graphs $S^{++}(2,3)$, $S^{++}(2,4)$ and $S^{++}(2,5)$.

4.2.1 The eccentric-connectivity index of $S(n,k)$, $S^{+}(n,k)$ and $S^{++}(n,k)$

In this section, we compute the eccentric-connectivity index of Sierpinski networks and their regularizations. The order of $S(n,k)$ is k^n and size is $\frac{k}{2}(k^n-1)$, the order of $S^{+}(n,k)$ is k^n+1 and the size is $\frac{k}{2}(k^n+1)$. The order of $S^{++}(n,k)$ is k^n+k and the size is $\frac{k^2}{2}(k+1)$.

Table 4.1: Vertex partition of $S(n,k)$.

representative	degree	eccentricity	frequency
a_i	$k-1$	2^n-1	k
b_i	k, $(n\geq 2)$	2^n-1	k^n-k

Theorem 4.2.1 *The eccentrici-connectivity index of $S(n,k)$ for $n\geq 1$, $k\geq 5$ is given by*

$$M_1^*(S(n,k)) = k(2^n-1)(k^n-1).$$

Proof. Using the symmetry of the Sierpinski graph $S(n,k)$ and one subgraph $S(1,k)$ of $S(2,k)$ as labeled in Figure 4.1 and Figure 4.2. We take one representative from a set of vertices which have same degree and eccentricity. These representatives are labeled by a_i for $1\leq i\leq k$ and b_i for $1\leq i\leq k^2-k$ as shown in Table 4.1. Here a_i considered as extreme vertices of Sierpinski graph $S(n,k)$.

Using Table 4.1 we can write eccentric-connectivity index of $S(1,5)$ as follows,

$$\begin{aligned}\xi(S(1,5)) &= \sum_{u\in V(G)} deg(u)\ \varepsilon(u)\\ &= 4\times(1)\times 5\\ &= 5(2^1-1)\times(5^1-1).\end{aligned}\tag{4.8}$$

The ECI of $S(n,5)$ for $n=2$ can be derived as follows

$$\begin{aligned}\xi(S(2,5)) &= \sum_{u\in V(G)} deg(u)\ \varepsilon(u)\\ &= \sum deg(a_i)\ \varepsilon(a_i)+\sum deg(b_i)\ \varepsilon(b_i)\\ &= 5\times(2^2-1)+(2^2-1)\times(5^2-1)\\ &= 5(2^2-1)\times(5^2-1)\end{aligned}\tag{4.9}$$

The ECI of $S(n,5)$ for $n=3$ calculated as

$$\begin{aligned}\xi(S(3,5)) &= \sum deg(a_i)\ \varepsilon(a_i)+\sum deg(b_i)\ \varepsilon(b_i)\\ &= 5\times(2^3-1)+(2^3-1)\times(5^3-1)\\ &= 5(2^3-1)\times(5^3-1)\end{aligned}\tag{4.10}$$

Similarly the ECI of $S(n,5)$ for $n\geq 4$, we have

$$\xi(S(n,5)) = 5(2^n-1)\times(5^n-1).\tag{4.11}$$

Now we can write ECI of $S(1,6)$ as follows

$$\begin{aligned}\xi(S(1,6)) &= \sum_{u\in V(G)} deg(u)\ \varepsilon(u)\\ &= 4\times(1)\times 6\\ &= 6(2^1-1)\times(6^1-1).\end{aligned}\tag{4.12}$$

The ECI of $S(n,6)$ for $n=2$ can be derived as follows

$$\begin{aligned}\xi(S(2,6)) &= \sum_{u\in V(G)} deg(u)\ \varepsilon(u)\\ &= \sum deg(a_i)\ \varepsilon(a_i) + \sum deg(b_i)\ \varepsilon(b_i)\\ &= 6\times(2^2-1)+(2^2-1)\times(6^2-1)\\ &= 6(2^2-1)\times(6^2-1) \qquad (4.13)\end{aligned}$$

The ECI of $S(n,6)$ for $n=3$ calculated as

$$\begin{aligned}\xi(S(3,6)) &= \sum deg(a_i)\ \varepsilon(a_i) + \sum deg(b_i)\ \varepsilon(b_i)\\ &= 6\times(2^3-1)+(2^3-1)\times(6^3-1)\\ &= 6(2^3-1)\times(6^3-1) \qquad (4.14)\end{aligned}$$

Similarly the ECI of $S(n,6)$ for $n\geq 4$, we have

$$\xi(S(n,6)) = 6(2^n-1)\times(6^n-1). \qquad (4.15)$$

By using Table 4.1 and observing the Eq. (4.11) and Eq. (4.15) we can complete our proof as follows

$$\xi(S(n,k)) = k(2^n-1)\times(k^n-1). \qquad (4.16)$$

■

Now we compute the closed-form expression for the eccentric-connectivity index of $S^+(2,k)$.

Table 4.2: Vertex partition of $S^+(n,k)$.

representative	degree	eccentricity	frequency
w	k	2	1
a_i	k	3	k
b_i	k	3	k^2-k

Theorem 4.2.2 *The eccentric-connectivity index of $S^+(2,k)$ for , $k\geq 3$ is given by*

$$\xi(S^+(2,k)) = 3 \times k^3 + 2 \times k.$$

Proof. Using the symmetry of the Sierpinski graph $S^+(2,k)$ and one subgraph $S(1,k)$ of $S^+(2,k)$ as labeled in Figure 4.3. We take one representative from a set of vertices which have same degree and eccentricity. These representatives are labeled by w, a_i for $1 \le i \le k$ and b_i for $1 \le i \le k^2 - k$ shown in Table 4.2. Here a_i considered as extreme vertices of regularization of Sierpinski graph $S^+(2,k)$. Using Table 4.2 we can write eccentric-connectivity index of $S^+(2,3)$ as follows,

$$\begin{aligned}
\xi(S^+(2,3)) &= \sum_{u \in V(G)} deg(u)\ \varepsilon(u) \\
&= deg(w)\ \varepsilon(w) + \sum deg(a_i)\ \varepsilon(a_i) + \sum deg(b_i)\ \varepsilon(b_i) \\
&= (3 \times 2) \times 1 + (3 \times 3) \times 3 + (3 \times 3) \times 6 \\
&= 3 \times 3^3 + 2 \times 3. \qquad (4.17)
\end{aligned}$$

The eccentric-connectivity index of $S^+(2,k)$ for $k = 4$ can be calculated as follows

$$\begin{aligned}
\xi(S^+(2,4)) &= \sum_{u \in V(G)} deg(u)\ \varepsilon(u) \\
&= deg(w)\ \varepsilon(w) + \sum deg(a_i)\ \varepsilon(a_i) + \sum deg(b_i)\ \varepsilon(b_i) \\
&= (4 \times 2) \times 1 + (4 \times 3) \times 4 + (4 \times 3) \times 12 \\
&= 3 \times 4^3 + 2 \times 4 \qquad (4.18)
\end{aligned}$$

Similarly the eccentric-connectivity index of $S^+(2,k)$ for $k \ge 5$ can be derived as

$$\xi(S^+(2,k)) = 3 \times k^3 + 2 \times k. \qquad (4.19)$$

■

Next we compute the closed-form expression for the eccentric-connectivity index of $S^{++}(2,k)$.

Theorem 4.2.3 *The eccentric-connectivity index of $S^{++}(2,k)$ for , $k \ge 3$ is given by*

Table 4.3: Vertex partition of $S^{++}(n,k)$.

representative	degree	eccentricity	frequency
w	k	3	k
a_i	k	3	k
b_i	k	3	k^2-k

$$\xi(S^{++}(2,k)) = 3 \times k^2 \times (k+1).$$

Proof. Using the symmetry of the Sierpinski graph $S^{++}(2,k)$ and one subgraph $S(1,k)$ of $S^{++}(2,k)$ as labeled in Figure 4.4. We consider one representative from a set of vertices which have same degree and eccentricity. These representatives are labeled by w_i for $1 \leq i \leq k$, a_i for $1 \leq i \leq k$ and b_i for $1 \leq i \leq k^2-k$ shown in Table 4.3. Here a_i vertices considered as extreme vertices of regularization of Sierpinski graph $S^{+}(2,k)$. Using Table 4.3 we can write eccentric-connectivity index of $S^{++}(2,3)$ as follows,

$$\begin{aligned}
\xi(S^{++}(2,3)) &= \sum_{u\in V(G)} deg(u)\ \varepsilon(u) \\
&= \sum deg(w_i)\ \varepsilon(w_i) + \sum deg(a_i)\ \varepsilon(a_i) + \sum deg(b_i)\ \varepsilon(b_i) \\
&= (3\times 3)\times 3 + (3\times 3)\times 3 + (3\times 3)\times 6 \\
&= 3\times 3^2 \times (3+1). \qquad (4.20)
\end{aligned}$$

The eccentric-connectivity index of $S^{++}(2,k)$ for $k=4$ can be calculated as follows

$$\begin{aligned}
\xi(S^{++}(2,4)) &= \sum_{u\in V(G)} deg(u)\ \varepsilon(u) \\
&= \sum deg(w_i)\ \varepsilon(w_i) + \sum deg(a_i)\ \varepsilon(a_i) + \sum deg(b_i)\ \varepsilon(b_i) \\
&= (4\times 3)\times 4 + (4\times 3)\times 4 + (4\times 3)\times 12 \\
&= 3\times 4^2 \times (4+1) \qquad (4.21)
\end{aligned}$$

The eccentric-connectivity index of $S^{++}(2,k)$ for $k \geq 5$ can be derived as follows

$$\xi(S^{++}(2,k)) = 3 \times k^2 \times (k+1). \qquad (4.22)$$

■

4.2.2 The Zagreb-eccentricity indices of $S(n,k)$, $S^{+}(2,k)$ and $S^{++}(2,k)$

In this section, we compute eccentricity based Zagreb indices M_1^{**} and M_2^{*} defined by Eq. (4.6) and Eq. (4.7) of the Sierpinski graph $S(n,k)$ and its regularizations $S^{+}(2,k)$ and $S^{++}(2,k)$.

Table 4.4: Edge partition of $S(n,k)$.

End vertices of edges	Eccentricities	Occurrence of edges
$[a_i,a_j]$, $i\neq j$, $n=1$	$[2^n-1,2^n-1]$	$\frac{k}{2}(k^n-1)$
$[a_i,b_j]$, $n\geq 2$	$[2^n-1,2^n-1]$	$k(k-1)$
$[b_i,b_j]$,$i\neq j$, $n\geq 2$	$[2^n-1,2^n-1]$	$\frac{k}{2}(k^n-2k+1)$

In the following theorem, we compute the closed-form expression for the second Zagreb-eccentricity of $S(n,k)$.

Theorem 4.2.4 *The second Zagreb-eccentricity index M_1^{**} of $S(n,k)$ for $n\geq 1$, $k\geq 5$ is given by*

$$M_1^{**}(S(n,k)) = k^n(2^n-1)^2.$$

Proof. Using the symmetry of the Sierpinski graph $S(n,k)$ and one subgraph $S(1,k)$ of $S(2,k)$ as labeled in Figure 4.1 and Figure 4.2. We take one representative from a set of edges whose end vertices have the same eccentricities occurrence. These representatives are labeled by a_i for $1\leq i\leq k$, $1\leq j\leq k$ and b_i for $1\leq i\leq k^2-k$ shown in Table 4.4. Here a_i vertices considered as extreme vertices of Sierpinski graph $S(n,k)$. Using Table 4.4 we can write second Zagreb-eccentricity index of $S(1,5)$ as follows,

$$\begin{aligned} M_1^{**}(S(1,5)) &= \sum_{u\in V(G)} \varepsilon(u)^2 \\ &= 5\times 1^2 \\ &= 5^1(2^1-1)^2, \end{aligned} \tag{4.23}$$

for $n = 2$ the second Zagreb-eccentricity index of $S(n,5)$ is given as

$$\begin{aligned} M_1^{**}(S(2,5)) &= \sum_{u \in V(G)} \varepsilon(u)^2 \\ &= \sum \varepsilon(a_i)^2 + \sum \varepsilon(b_i)^2 \\ &= 5 \times 3^2 + 20 \times 3^2 \\ &= 5^2(2^2 - 1)^2 \end{aligned} \tag{4.24}$$

and for $n \geq 3$ the second Zagreb-eccentricity index of $S(n,5)$ is given as

$$M_1^{**}(S(n,5)) = 5^n(2^n - 1)^2. \tag{4.25}$$

Now we can write second Zagreb-eccentricity index of $S(1,6)$ as follows

$$\begin{aligned} M_1^{**}(S(1,6)) &= \sum_{u \in V(G)} \varepsilon(u)^2 \\ &= 6 \times 1^2 \\ &= 6^1(2^1 - 1)^2, \end{aligned} \tag{4.26}$$

for $n = 2$ the second Zagreb-eccentricity index of $S(n,6)$ is given as

$$\begin{aligned} M_1^{**}(S(2,6)) &= \sum_{u \in V(G)} \varepsilon(u)^2 \\ &= \sum \varepsilon(a_i)^2 + \sum \varepsilon(b_i)^2 \\ &= 6 \times 3^2 + 30 \times 3^2 \\ &= 6^2(2^2 - 1)^2 \end{aligned} \tag{4.27}$$

and for $n \geq 3$ the second Zagreb-eccentricity index of $S(n,6)$ is given as

$$M_1^{**}(S(n,6)) = 6^n(2^n - 1)^2. \tag{4.28}$$

By using Table 4.4 and observing the Eq. 4.25 and Eq. 4.28 we can complete our proof as follows

$$M_1^{**}(S(n,k)) = k^n(2^n - 1)^2. \tag{4.29}$$

■

Next we compute the closed-form expression for the third Zagreb-eccentricity index of $S(n,k)$.

Theorem 4.2.5 *The third Zagreb-eccentricity index M_2^* of $S(n,k)$ for $n \geq 1$, $k \geq 5$ is given by*

$$M_2^*(S(n,k)) = \frac{k}{2}(k^n - 1) \times (2^n - 1)^2$$

Proof. Using the symmetry of the Sierpinski graph $S(n,k)$ and one subgraph $S(1,k)$ of $S(2,k)$ as labeled in Figure 4.1 and Figure 4.2. We take one representative from a set of edges whose end vertices have the same eccentricities. These representatives are labeled by a_i for $1 \leq i \leq k$, $1 \leq j \leq k$ and b_i for $1 \leq i \leq k^2 - k$ shown in Table 4.4. Here a_i vertices considered as extreme vertices of Sierpinski graph $S(n,k)$. Using Table 4.4 we can write third Zagreb-eccentricity index of $S(1,5)$ as follows,

$$\begin{aligned} M_2^*(S(1,5)) &= \sum_{uv \in E(G)} [\varepsilon(u)\varepsilon(v)] \\ &= 10 \times (1 \times 1) \\ &= \frac{5}{2}(5^1 - 1)(2^1 - 1)^2 \end{aligned} \tag{4.30}$$

the third Zagreb-eccentricity index of $S(n,5)$ for $n = 2$ can be written as follows

$$\begin{aligned} M_2^*(S(2,5)) &= \sum_{uv \in E(G)} [\varepsilon(u)\varepsilon(v)] \\ &= \sum [\varepsilon(a_i)\varepsilon(a_j)] + \sum [\varepsilon(b_i)\varepsilon(b_j)] \\ &= 20 \times (3 \times 3) + 40 \times (3 \times 3) \\ &= \frac{5}{2}(5^2 - 1)(2^2 - 1)^2 \end{aligned} \tag{4.31}$$

and the third Zagreb-eccentricity index of $S(n,5)$ for $n \geq 3$ written as follows

$$M_2^*(S(n,5)) = \frac{5}{2}(5^n - 1)(2^n - 1)^2. \tag{4.32}$$

Now we can write third Zagreb-eccentricity index of $S(1,6)$ as follows

$$\begin{aligned} M_2^*(S(1,6)) &= \sum_{uv\in E(G)} [\varepsilon(u)\varepsilon(v)] \\ &= 15 \times (1 \times 1) \\ &= \frac{6}{2}(6^1 - 1)(2^1 - 1)^2. \end{aligned} \tag{4.33}$$

The third Zagreb-eccentricity index of $S(n,6)$ for $n = 2$ can be written as follows

$$\begin{aligned} M_2^*(S(2,6)) &= \sum_{uv\in E(G)} [\varepsilon(u)\varepsilon(v)] \\ &= \sum[\varepsilon(a_i)\varepsilon(a_j)] + \sum[\varepsilon(b_i)\varepsilon(b_j)] \\ &= 30 \times (3 \times 3) + 75 \times (3 \times 3) \\ &= \frac{6}{2}(6^2 - 1)(2^2 - 1)^2 \end{aligned} \tag{4.34}$$

and the third Zagreb-eccentricity index of $S(n,6)$ for $n \geq 3$ can be defined as follows

$$M_2^*(S(n,6)) = \frac{6}{2}(6^n - 1)(2^n - 1)^2. \tag{4.35}$$

By using table 4.4 and observing the Eq. (4.32) and Eq. (4.35) we can complete our proof as follows

$$M_1^{**}(S(n,k)) = \frac{k}{2}(k^n - 1)(2^n - 1)^2. \tag{4.36}$$

■

Next we compute the closed-form expression for the second Zagreb-eccentricity index of $S^+(2,k)$.

Table 4.5: Edge partition of $S^+(n,k)$.

End vertices of edges	Eccentricities	Occurrence of edges
$[w, a_i]$	$[2, 3]$	k
$[a_i, b_j]$	$[3, 3]$	$k(k-1)$
$[b_i, b_j], i \neq j$	$[3, 3]$	$\frac{k}{2}(k^2 - 2k + 1)$

Theorem 4.2.6 *For $k \geq 3$, the second Zagreb-eccentricity index M_1^{**} of $S^+(2,k)$ is given by*

$$M_1^{**}(S^+(2,k)) = 9k^2 + 4).$$

Proof. Using the symmetry of the regularization $S^+(2,k)$ of the Sierpinski graph $S(n,k)$ as labeled in Figure 4.3. We consider one element from the set of edges whose end vertices have the same eccentricities occurrence. These elements are defined as w, a_i for $1 \leq i \leq k$, $1 \leq j \leq k(k-1)$ and b_i for $1 \leq i \leq \frac{k}{2}(k^2 - 2k + 1)$, $1 \leq j \leq \frac{k}{2}(k^2 - 2k + 1)$ as shown in Table 4.5. Here a_i vertices considered as extreme vertices. Using Table 4.5 we can write second Zagreb-eccentricity index of $S^+(2,3)$ as below,

$$\begin{aligned} M_1^{**}(S^+(2,3)) &= \sum_{u \in V(G)} \varepsilon(u)^2 \\ &= 2^2 \times 1 + 3^2 \times 3 \\ &= 9 \times 3^2 + 4 \end{aligned} \tag{4.37}$$

the second Zagreb-eccentricity index of $S^+(2,k)$ for $k = 4$ can be written as follows

$$\begin{aligned} M_1^{**}(S^+(2,4)) &= \sum_{u \in V(G)} \varepsilon(u)^2 \\ &= w + \sum \varepsilon(a_i)^2 + \sum \varepsilon(b_i)^2 \\ &= \ 2^2 \times 1 + 3^2 \times 4 + 3^2 \times 12 \\ &= 9 \times 4^2 + 4 \end{aligned} \tag{4.38}$$

and the second Zagreb-eccentricity index of $S^+(2,k)$ for $k \geq 5$ can be written as follows

$$M_1^{**}(S^+(2,k)) = 9 \times k^2 + 4. \tag{4.39}$$

■

Next we compute the closed-form expression for the third Zagreb-eccentricity index of $S^+(2,k)$.

Theorem 4.2.7 *The third Zagreb-eccentricity index M_2^* of $S^+(2,k)$ for $k \geq 3$ is* *given by*

$$M_2^*(S^+(2,k)) = \frac{3k}{2}(3k^2+1).$$

Proof. Using the symmetry of the regularization $S^+(2,k)$ of the Sierpinski graph $S(n,k)$ as labeled in Figure 4.3. We consider one element from the set of edges whose end vertices have the same eccentricities occurrence. These elements are defined as w, a_i for $1 \le i \le k$, $1 \le j \le k(k-1)$ and b_i for $1 \le i \le \frac{k}{2}(k^2-2k+1)$, $1 \le j \le \frac{k}{2}(k^2-2k+1)$ as shown in Table 4.5. Here a_i vertices considered as extreme vertices. Using Table 4.5. we can write third Zagreb-eccentricity index of $S^+(2,3)$ as follows,

$$\begin{aligned} M_2^*(S^+(2,3)) &= \sum_{uv\in E(G)} \varepsilon(u)\varepsilon(v) \\ &= (2\times 3)\times 3 + (3\times 3)\times 6 + (3\times 3)\times 6 \\ &= \frac{9}{2}(3\times 3^2+1) \end{aligned} \tag{4.40}$$

the third Zagreb-eccentricity index of $S^+(2,k)$ for $k=4$ can be written as follows

$$\begin{aligned} M_2^*(S^+(2,4)) &= \sum_{uv\in E(G)} \varepsilon(u)\varepsilon(v) \\ &= \sum \varepsilon(w)\varepsilon(a_i) + \sum \varepsilon(a_i)\varepsilon(b_j) + \sum \varepsilon(b_i)\varepsilon(b_j) \\ &= (2\times 3)\times 4 + (3\times 3)\times 12 + (3\times 3)\times 18 \\ &= \frac{12}{2}(3\times 4^2+1) \end{aligned} \tag{4.41}$$

and the third Zagreb-eccentricity index of $S^+(2,k)$ for $k \ge 5$ can be written as follows

$$M_2^*(S^+(2,k)) = \frac{3k}{2}(3k^2+1). \tag{4.42}$$

■

Now we compute the closed-form expression for the second Zagreb-eccentricity index of $S^{++}(2,k)$.

Theorem 4.2.8 *The second Zagreb-eccentricity index M_1^{**} of $S^{++}(2,k)$ for $k \ge 3$*

Table 4.6: Edge partition of $S^{++}(n,k)$.

End vertices of edges	Eccentricities	Occurrence of edges
$[w_i, w_j]$, $i \neq j$	$[3,3]$	k
$[w_i, a_i]$	$[3,3]$	k
$[a_i, b_j]$	$[3,3]$	$k(k-1)$
$[b_i, b_j], i \neq j$	$[3,3]$	$\frac{k}{2}(k^2-2k+1)$

is given by

$$M_1^{**}(S^{++}(2,k)) = 9k(k+4).$$

Proof. Using the symmetry of the regularization $S^{++}(2,k)$ of the Sierpinski graph $S(n,k)$ as labeled in Figure 4.4. We consider one element from the set of edges whose end vertices have the same eccentricities occurrence. These elements are defined as w_i for $1 \leq i \leq k$, a_i for $1 \leq i \leq k$, $1 \leq j \leq k(k-1)$ and b_i for $1 \leq i \leq \frac{k}{2}(k^2-2k+1)$, $1 \leq j \leq \frac{k}{2}(k^2-2k+1)$ as shown in Table 4.6. Here a_i vertices considered as extreme vertices. Using Table 4.6. we can write second Zagreb-eccentricity index of $S^{++}(2,3)$ as follows,

$$\begin{aligned} M_1^{**}(S^{++}(2,3)) &= \sum_{u \in V(G)} \varepsilon(u)^2 \\ &= 3^2 \times 3 + 3^2 \times 3 + 3^2 \times 6 \\ &= 27 \times (3+1) \end{aligned} \tag{4.43}$$

the second Zagreb-eccentricity index of $S^{++}(2,k)$ for $k = 4$ can be derived as follows

$$\begin{aligned} M_1^{**}(S^{++}(2,4)) &= \sum_{u \in V(G)} \varepsilon(u)^2 \\ &= \sum \varepsilon(w_i)^2 + \sum \varepsilon(a_i)^2 + \sum \varepsilon(b_i)^2 \\ &= \ 3^2 \times 4 + 3^2 \times 4 \times 4 + 3^2 \times 12 \\ &= 36 \times (4+1) \end{aligned} \tag{4.44}$$

and the second Zagreb-eccentricity index of $S^{++}(2,k)$ for $k \geq 5$ can be derived as follows

$$M_1^{**}(S^{++}(2,k)) = 9 \times k \times (k+1). \tag{4.45}$$

■

Next we calculate the closed-form expression for the third Zagreb-eccentricity index of $S^{++}(2,k)$.

Theorem 4.2.9 *The third Zagreb-eccentricity index M_2^* of $S^{++}(2,k)$ is given by $M_2^*(S^{++}(2,k)) = \frac{9k^2}{2}(k+1)$ for $k \geq 3$.*

Proof. Using the symmetry of the regularization $S^{++}(2,k)$ of the Sierpinski graph $S(n,k)$ as labeled in Figure 4.4. One can select an element from edge set whose end vertices have the same eccentricities occurrence. These elements are denoted by w, a_i for $1 \leq i \leq k$, $1 \leq j \leq k(k-1)$ and b_i for $1 \leq i \leq \frac{k}{2}(k^2-2k+1)$, $1 \leq j \leq \frac{k}{2}(k^2-2k+1)$ as shown in Table 4.6. Here a_i vertices considered as extreme vertices. Using Table 4.6 we can write third Zagreb-eccentricity index of $S^{++}(2,3)$ in following way,

$$\begin{aligned} M_2^*(S^{++}(2,3)) &= \sum_{uv\in E(G)} \varepsilon(u)\varepsilon(v) \\ &= (3\times 3)\times 3 + (3\times 3)\times 3 + (3\times 3)\times 6 + (3\times 3)\times 6 \\ &= \frac{81}{2}(3+1) \end{aligned} \tag{4.46}$$

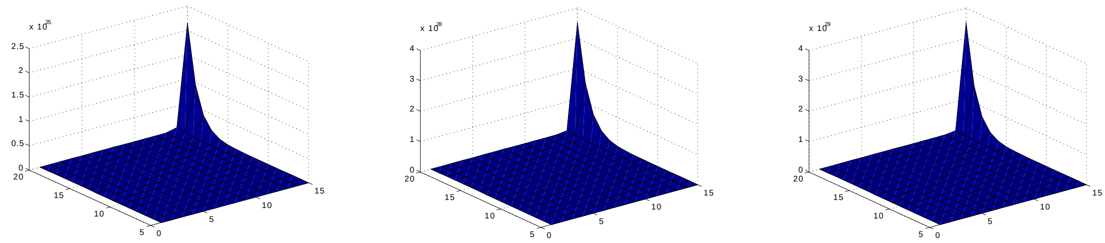
third Zagreb-eccentricity index of $S^{++}(2,k)$ for $k=4$ can be written as

$$\begin{aligned} M_2^*(S^{++}(2,4)) &= \sum_{uv\in E(G)} \varepsilon(u)\varepsilon(v) \\ &= \sum \varepsilon(w_i)\varepsilon(w_j) + \sum \varepsilon(w_i)\varepsilon(a_j) + \sum \varepsilon(a_i)\varepsilon(b_j) + \sum \varepsilon(b_i)\varepsilon(b_j) \\ &= (3\times 3)\times 6 + (3\times 3)\times 4 + (3\times 3)\times 12 + (3\times 3)\times 18 \\ &= \frac{144}{2}(4+1) \end{aligned} \tag{4.47}$$

and third Zagreb-eccentricity index of $S^{++}(2,k)$ for $k \geq 5$ can be written as

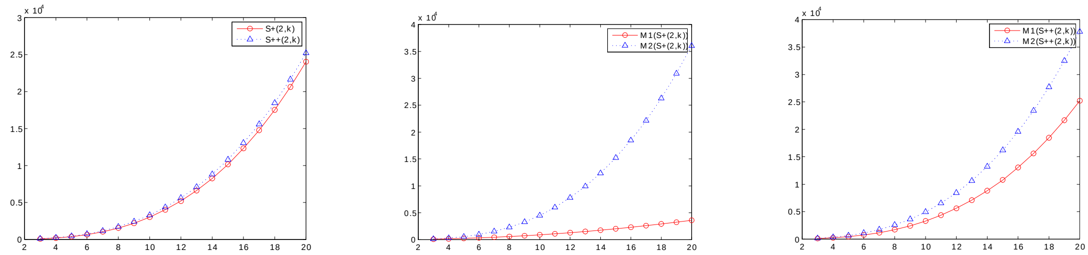
$$M_2^*(S^{++}(2,k)) = \frac{9k^2}{2}(k+1). \tag{4.48}$$

■



A B C

Figure 4.5



A B C

Figure 4.6

4.2.3 Graphical behaviours

In this section, the graphical behaviour of the aforementioned indices of Sierpinski graphs and their regularizations have been studied. The graphical representation will be more helpful to study the dynamics of topological descriptors of the molecular graphs. Here we present the eccentric-connectivity index of Sierpinski graph $\xi(S(n,k))$ where $n \geq 1$ and $k \geq 5$ in Figure 4.5 (A), the M_1^{**} of $\xi(S(n,k))$ in Figure 4.5 (B) and the M_2^{*} of $\xi(S(n,k))$ in Figure 4.5 (C), all are nonlinear by their behaviors.

In Figure 4.5 we have drawn the curves of the regularizations of the Sierpinski graphs $S^{+}(2,k)$ and $S^{++}(2,k)$ to understand their dynamics. The Figure 4.6 (A) shows the eccentric-connectivity index, Figure 4.6 (B) displays the second Zagreb-eccentricity index and Figure 4.6 (C) presents the third Zagreb-eccentricity index of $S^+(2,k)$ and $S^{++}(2,k)$, respectively.

Chapter 5

Resolvability of some wheel related graphs

5.1 Introduction

This chapter is based on the question raised by Kelenc *et al.* in [65] to characterize the families of graphs who have one of the relations $dim(G) = edim(G)$, $dim(G) < edim(G)$, $dim(G) > edim(G)$.
We consider Jahangir graph J_{2n}, helm graph H_n, sunflower graph Sf_n and friendship graph F_n for edge metric dimension (NP-hard problem). Our aim is to characterize these families of graphs by considering the nature of edge metric dimension. Moreover, we present some closed formulae to compute edge metric dimension of these graphs. We suggest some research papers on metric dimension of wheel related graphs for further reading, we refer [59, 102, 103].

The results of this chapter are published in the special issue *"Mathematical Tools for Solving Problems of Chemical Structure Generation"* of *"journal of chemistry"* in 2019 [118].

5.2 Main results

In this section, we discuss the edge metric dimensions and resolving sets of Jahangir graph J_{2n}, helm graph H_n, sunflower graph Sf_n and friendship graph F_n.

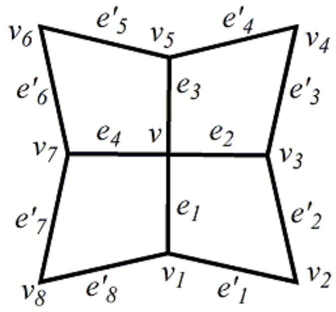
5.2.1 Jahangir graph (J_{2n})

The *Jahangir graph* J_{2n} generated from a wheel graph W_{2n} by removing n even (or odd) spokes. The order of J_{2n} is $2n + 1$ and size is $3n$. The gear graph G_n is used as an alternative name for the Jahangir graph [40]. Metric dimension of J_{2n} is computed in [109] and presented as

Theorem 5.2.1 [109] *"for $n \geq 4$ then the $dim(J_{2n}) = \lfloor \frac{2n}{3} \rfloor$".*

Now we present some observations for J_{2n} in the form of lemmas which will help us to compute its edge metric dimension.

Lemma 5.2.2 *The central vertex v is not the member of edge metric basis S of J_{2n} where $n \geq 6$.*



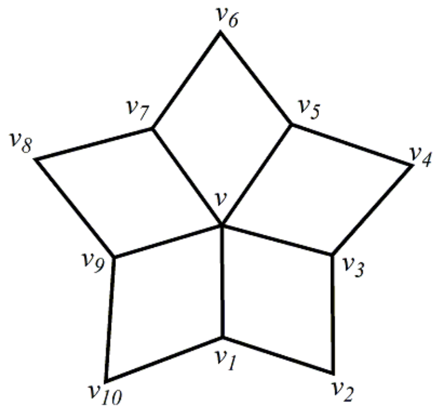


Figure 5.1: Jahangir graph J_{2n}

Proof. Let $J_{2n} = C_{2n} + K_1$ where the vertices of C_{2n} are $v_1, v_2, v_3, \cdots v_{2n}$. Let $e_j = vv_{2j-1}$ and $e'_i = v_iv_{i+1}$ are edges, where $j \in \{1, 2, ...n\}$ and $i \in \{1, 2, ..., 2n-1\}$ as shown in Figure 5.1. The vertices v_{2j-1} and v_{2j} are major and minor vertices, respectively. Since $d(e_j, v) = 0$ and $d(e'_i, v) = 1$, so vertex v is not member of any edge metric basis S of J_{2n}. ■

Let $S = \{v_{1_i}, v_{2_i}, ..., v_{r_i}\}$ is a basis of J_{2n} and $r = |S| \geq 2$ vertices on C_{2n} for $n \geq 6$. The pairs of vertices $\{v_{i_a}, v_{i+1_a}\}$ such that $1 \leq i \leq r-1$ and $\{v_{i_r}, v_{i_1}\}$ are said to be pairs of neighbouring vertices. We define the gap g_p such that $1 \leq p \leq r-1$ as the vertex set $\{v_j \mid i_p < j < i_{p+1}\}$ and $g_r = \{v_j \mid 1 \leq j < i$ or $i_r < j \leq 2n\}$. The number of gaps is r and some of which may be empty. If $|p-q| = 1$ or $r-1$, then the gaps g_p and g_q are known as the neighbouring gaps. Let $deg(u) = \alpha$ and $deg(v) = \beta$ or $deg(u) = \beta$ and $deg(v) = \alpha$ with $\alpha \leq \beta$, the gap between u and v is represented by $\alpha - \beta$ gap. The count of vertices belongs to a gap is known as the size of gap. There are three types of gaps observed on C_{2n} of J_{2n}, which are $2-2$, $2-3$ and $3-3$. We represent the major vertices by $\overset{*}{x}$ in the following results.

Lemma 5.2.3 *Let S is a basis set of Jahangir graph J_{2n} for $n \geq 6$. Then every $2-2$, $2-3$ and $3-3$ gap of set S consist of at most 5, 4 and 3 number of vertices, respectively.*

Proof. Let $e'_i = v_iv_{i+1}$ are edges on C_{2n} where $i \in \{1, 2, ..., 2n-1\}$ and $e_j = vv_{2j-1}$ are spokes edges of J_{2n} where $j \in \{1, 2, ..., n\}$. The S containing seven successive vertices $\overset{*}{x}_{i+1}$, x_{i+2}, ..., $\overset{*}{x}_{i+7}$of C_{2n} if there is a $2-2$ gap and it can be determined by x_i and x_{i+8} having $deg(\overset{*}{x}_{i+1}) = deg(\overset{*}{x}_{i+7}) = 3$, then we have $r(e_{i+2}|S) = r(e_{i+3}|S)$, a contradiction.

The existence of a $2-3$ gap containing six successive vertices of C_{2n}: $\overset{*}{x}_{i+1}$, x_{i+2}, ..., x_{i+6} determined by x_i and $\overset{*}{x}_{i+7}$with $deg(\overset{*}{x}_{i+1}) = 3$ and $deg(x_{i+6}) = 2$, it follows that $r(e'_{i+4}|S) = r(e'_{i+5}|S)$ and $r(e_i|S) = r(e_{i+1}|S)$.

If there is a $3-3$ gap of S containing five successive vertices of C_{2n}: x_{i+1}, $\overset{*}{x}_{i+2}$, ..., x_{i+5}, determined by $\overset{*}{x}_i$ and $\overset{*}{x}_{i+6}$ with $deg(x_{i+1}) = deg(x_{i+5}) = 2$ would imply $r(e_{i+1}|S) = r(e_{i+2}|S)$ and $r(e'_{i+2}|S) = r(e'_{i+3}|S)$, a contradiction.

Hence proved that every $2-2$, $2-3$ and $3-3$ gap of S consist of at most 5, 4 and 3 number of vertices, respectively. These gaps $2-2$, $2-3$ and $3-3$ are known as major gaps. ■

Lemma 5.2.4 *Let S be a basis of J_{2n} for $n \geq 6$. Then S has at most one major gap.*

Proof. Suppose that S contains two distinct major gaps g_p and g_q.

- $3-3$ and $3-3$: $x_{i+1}, \overset{*}{x}_{i+2},\ x_{i+3}$ and $y_{j+1},\ \overset{*}{y}_{j+2},\ y_{j+3}$, it follows $r(e_{i+1}|S) = r(e_{i+3}|S)$ and $r(e'_{i+3}|S) = r(e'_{i+4}|S)$;
- $3-3$ and $2-2$: $x_{i+1}, \overset{*}{x}_{i+2},\ x_{i+3}$ and $\overset{*}{y}_{j+1},\ y_{j+2},\ \overset{*}{y}_{j+3},\ y_{j+4},\ \overset{*}{y}_{j+5}$; in this case $r(e_{i+1}|S) = r(e_{i+4}|S)$;
- $3-3$ and $2-3$: $x_{i+1}, \overset{*}{x}_{i+2},\ x_{i+3}$ and $\overset{*}{y}_{j+1},\ y_{j+2},\ \overset{*}{y}_{j+3},\ y_{j+5}$; we have $r(e_{i+1}|S) = r(e_{i+3}|S)$ and $r(e'_{i+3}|S) = r(e'_{i+4}|S)$;
- $2-2$ and $2-2$: $\overset{*}{x}_{i+1},\ x_{i+2},\ \overset{*}{x}_{i+3},\ x_{i+4},\ x^*_{i+5}$ and $\overset{*}{y}_{j+1},\ y_{j+2},\ \overset{*}{y}_{j+3},\ y_{j+4},\ \overset{*}{y}_{j+5}$; in this case $r(e_{i+2}|S) = r(e_{i+5}|S)$, $r(e_{i+3}|S) = r(e_{i+4}|S)$ and $r(e'_{i+4}|S) = r(e'_{i+9}|S)$;
- $2-2$ and $2-3$: $\overset{*}{x}_{i+1},\ x_{i+2}, \overset{*}{x}_{i+3},\ x_{i+4},\ \overset{*}{x}_{i+5}$ and $\overset{*}{y}_{j+1},\ y_{j+2},\ \overset{*}{y}_{j+3},\ y_{j+4}$; in this case $r(e_{i+2}|S) = r(e_{i+5}|S)$, $r(e_{i+3}|S) = r(e_{i+4}|S)$ and $r(e'_{i+4}|S) = r(e'_{i+9}|S)$;
- $2-3$ and $2-3$: $\overset{*}{x}_{i+1},\ x_{i+2},\ \overset{*}{x}_{i+3},\ x_{i+4}$ and $\overset{*}{y}_{j+1},\ y_{j+2},\ \overset{*}{y}_{j+3},\ y_{j+4}$; in this case $r(e_i|S) = r(e_{i+1}|S)$, $r(e_{i+2}|S) = r(e_{i+5}|S)$ and $r(e'_i|S) = r(e'_{i+1}|S)$, which contradicts the hypothesis. ■

Lemma 5.2.5 *Let $n \geq 6$ and S be a basis set of Jahangir graph J_{2n}. Then their exist one major gap in any two neighbouring gaps of S and these gaps contains at* *most six number of vertices.*

Proof. If we take a $3-3$ gap which contain three vertices, then its neighbouring gap should contains three vertices that is only possible when we take $3-3$ as a neighbouring gap, which is not possible because Lemma 5.2.3 says that the basis only contain one major gap.

If we take major $2-2$ gap with five vertices then the neighbouring gap can not be a minor $2-2$ gap of three vertices nor minor $2-3$ gap of two vertices, we show below as case wise:

Case 1. If the vertices of gaps g_p and g_q are $\overset{*}{x}_{i+1},\ x_{i+2},\ \overset{*}{x}_{i+3},\ x_{i+4},\ \overset{*}{x}_{i+5}$ and $\overset{*}{x}_{i+7}, x_{i+8}, \overset{*}{x}_{i+9}$, respectively and determined by x_i, x_{i+6} and x_{i+10}, then we have $r(e_{i+3}|S) = r(e_{i+4}|S)$ and $r(e'_{i+6}|S) = r(e'_{i+8}|S)$, contradictions.

Case 2. If the vertices gaps g_p and g_q are $\overset{*}{x}_{i+1},\ x_{i+2},\ \overset{*}{x}_{i+3},\ x_{i+4},\ \overset{*}{x}_{i+5}$ and $\overset{*}{x}_{i+7},\ x_{i+8}$, respectively and determined by x_i, x_{i+6} and $\overset{*}{x}_{i+9}$, then we have $r(e_i|S) = r(e_{i+1}|S)$ and $r(e'_i|S) = r(e'_{i+1}|S)$.

If the gap is $2-3$ gap with four vertices, then the Lemma 5.2.3 is sufficient to prove that the neighbouring gap can not be minor $2-2$ gap with three vertices. In this case the vertices of gaps g_p and g_q are $\overset{*}{x}_{i+1},\ x_{i+2},\ \overset{*}{x}_{i+3},\ x_{i+4}$ and $\overset{*}{x}_{i-1},\ x_{i-2},\ \overset{*}{x}_{i-3}$, respectively, so $r(e_i|S) = r(e_{i+1}|S)$ and $r(e'_{i+2}|S) = r(e'_{i+3}|S)$.

■

Lemma 5.2.6 *Let S be a basis of J_{2n} with $n \geq 6$. Then any two minor neighbouring gaps consist on at most four number of vertices all together.*

Proof. Since by Lemma 5.2.3 any minor $2-2$, $2-3$ and $3-3$ gap contains 3, 2 and 1 vertex, respectively, it is sufficient to discuss the following cases for this Lemma:

Case 1. A $2-2$ gap with 3 vertices has a neighbouring $2-2$ gap with 3 vertices,

Case 2. A $2-2$ gap with 3 vertices has a neighbouring $2-3$ gap with 2 vertices cannot occur.

If Case 1. holds, then $r(e_i|S) = r(e_{i+1}|S)$ and $r(e'_i|S) = r(e'_{i+1}|S)$ and If Case 2. holds, then $r(e_i|S) = r(e_{i+1}|S)$ and $r(e'_i|S) = r(e'_{i+1}|S)$, so there are contradictions. ■

Theorem 5.2.7 *Let J_{2n} be a Jahangir graph for $n \geq 4$. Then $edim(J_{2n}) = \left\lfloor \frac{2n}{3} \right\rfloor$.*

Proof. We can easily see that $edim(J_8) = 2$, the possible resolving sets are $\{v_2, v_4\}$, $\{v_4, v_6\}$ and $\{v_6, v_8\}$. Similarly, we can also see that $edim(J_{10}) = 3$, the number of resolving sets are 15, few of them are $\{v_1, v_4, v_6\}$, $\{v_1, v_6, v_8\}$, $\{v_2, v_4, v_6\}$ and $\{v_2, v_4, v_7\}$.

First we show that $edim(J_{2n}) \leq \lfloor \frac{2n}{3} \rfloor$ for $n > 6$ by considering a resolving set in J_{2n} with $\lfloor \frac{2n}{3} \rfloor$ number of vertices. Here we suppose three cases as per to the residue class modulo 3 to which n belongs:

Case 1: $n \equiv 0\ (mod\ 3)$. Let $2n = 3t$ for t is even, $t \geq 4$, and $\lfloor \frac{2n}{3} \rfloor = t$. In this case $S = \{v_{6i-4}, v_{6i} : 1 \leq i \leq t/2\ - 1\} \cup \{v_1, v_{2n-4}\}$.

Case 2: $n \equiv 1\ (mod\ 3)$. Let $2n = 3t + 2$ for t is even, $t \geq 4$, and $\lfloor \frac{2n}{3} \rfloor = t$. We define $S = \{v_{6i-2}, v_{6i+2} : 1 \leq i \leq t/2\ - 1\} \cup \{v_2, v_{2n-4}\}$.

Case 3: $n \equiv 2\ (mod\ 3)$. Let $2n = 3t + 1$ for t is odd, $t \geq 5$, and $\lfloor \frac{2n}{3} \rfloor = t$ and $S = \{v_{6i-2}, v_{6i}\ : 1 \leq i \leq (t-1)/2\ \} \cup \{v_1\}$.

The set S contains only one major gap and remaining gaps are minor gaps which are $2-2$, $2-3$ and $3-3$ gaps containing 3, 2 and 1 vertex, respectively. Any two neighbouring gaps one of which is major have at most 6 number of vertices (Lemma 5.2.5) and the number of vertices of any two neighbouring gaps are at most 4 (Lemma 5.2.6).

Now we show that $edim(J_{2n}) \geq \lfloor \frac{2n}{3} \rfloor$. We consider S is a basis of J_{2n} having cardinality $r = |S|$. Then S generates r number of gaps on the rim of C_{2n}, denoted as $g_1, ..., g_r$ such that g_i and g_{i+1} for every $1 \leq i \leq r-1$. We also consider that g_1 and g_r are neighbouring gaps. According to Lemma 5.2.4 at most one of them, say g_1 , is a major gap. We can write by using Lemma 5.2.5 that $|g_1| + |g_2|\ \leq 6$, $|g_r|\ +\ |g_1|\ \leq\ 6$ and by Lemma 5.2.6, the relation for the remaining neighbouring gaps is $|g_i|\ +\ |g_{i+1}|\ \leq 4$ for every $i = 2, ..., r-1$.

So, we have

$$2(2n - r) = 2\sum_{i=1}^{r}|g_i|\ \leq 4(r-2) + 12$$

$$2(2n - r) \leq 4r + 4$$

$$r \geq \frac{2n-2}{3} \geq \lfloor \frac{2n}{3} \rfloor.$$

Since $r \in N$, for each $n \equiv 0, 1$ or $2(mod\ 3)$ we obtain $r \geq \lfloor \frac{2n}{3} \rfloor$. Hence the theorem is proved. ■

5.2.2 Helm graph (H_n)

A *helm graph* H_n can be generated from a wheel graph W_n by adding a single pendant edge with every vertex of C_n on its rim. The order and size of helm graph is $|V| = 2n + 1$ and $|E| = 3n$, respectively. We labeled the central vertex, cycle vertices and the pendant vertices as v, v_i and v_i', respectively, where $1 \leq i \leq n$. There are three types of edges and we labeled them in such a way $e_i = vv_i$, $e_i' = v_iv_{i+1}$ and $e_i'' = v_iv_i'$, see Figure 5.2.

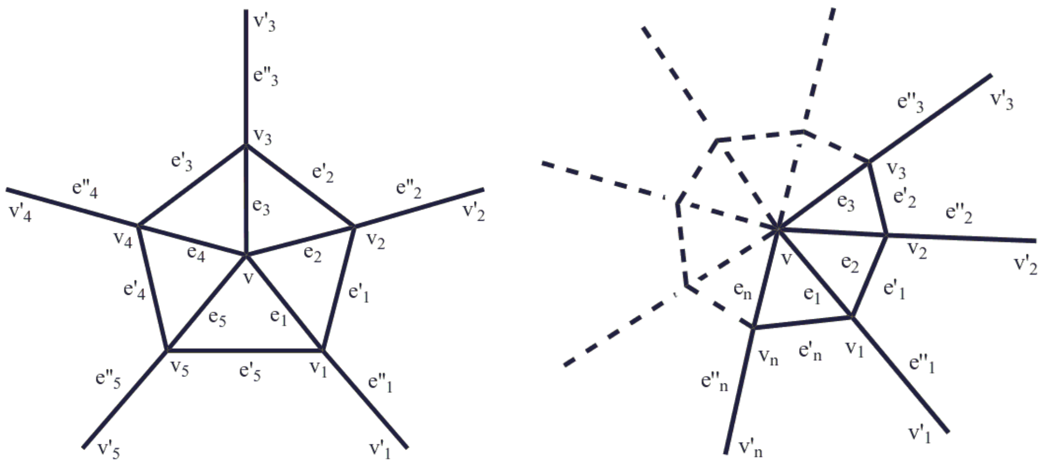


Figure 5.2: Helm graph H_n

The $dim(H_n)$ is proved in [62] as

Theorem 5.2.8 [62] *"If $n \notin \{3, 6\}$ then the $dim(H_n) = \lfloor \frac{2n+2}{5} \rfloor$".*

Theorem 5.2.9 *Let H_n be a helm graph. Then*

$$edim(H_n) = \begin{cases} n, & n = 3, 4 \\ n - 1, & n \geq 5 \end{cases}$$

Proof. for $n = 3$ or 4, then the proof is simple. Consider $n \geq 5$ and $V(H_n) = \{v, v_1, v_2, ..., v_n, v_1', v_2', ..., v_n'\}$, where $deg(v) = n$, vertices v_i are on C_n and they have degree 4 i.e., $deg(v_i) = 4$, and the v_i' are the pendant vertices having $deg(v_i) = 1$. To show that S is a minimum edge resolving set, we consider three cases as given below:

1. If $e_i = vv_i$, then e_i has distance 0 to v_i, distance 1 to v_j, where $j \in \{1, 2, ...n\}$ and $j \neq i$, distance 1 to v'_i, and distance 2 to v'_j, where $j \in \{1, 2, ...n\}$ and $j \neq i$.

2. If $e'_i = v_iv_{i+1}$, then e'_i has distance 0 to v_i and v_{i+1}, distance 1 to $\{v,\ v_{i-1},\ v_{i+2}\}$, distance 2 to all other vertices on C_n, distance 1 to $\{v'_i,\ v'_{i+1}\}$, distance 2 to $\{v'_{i-1},\ v'_{i+2}\}$ and distance 3 to all other pendant vertices.

3. If $e''_i = v_iv'_i$, then e''_i has distance 0 to v_i and v'_i, distance 1 to $\{v,\ v_{i-1},\ v_{i+1}\}$, distance 2 to $\{v'_{i-1},\ v'_{i+1}\}$ and distance 3 to all other pendant vertices.
   The above cases show that the edge metric representation with any $n-2$ vertices or less number of vertices of H_n is same for some edges. So S is edge metric generator with $n-1$ vertices and therefore

$$edim(H_n) \leq n - 1. \tag{5.1}$$

We can also show that the $edim(H_n) \nleq n-2$ by considering the S as a basis of H_n with $|S| = n-2$. For this we have the following cases:

- If we take $S = \{v_i, v_{i+1}, ..., v_{n-2}\}$, then $r(e_{i-1}|S) = r(e_{n-1}|S)$.
- If we suppose $S = \{v'_i, v'_{i+1}, ..., v'_{n-2}\}$, then $r(e_{i-1}|S) = r(e_{n-1}|S)$.
- If we consider $S = \{v_i, ..., v_{i+(n-4)}, v'_{i+(n-3)}\}$ and $S = \{v_i, v'_{i+1}, ..., v'_{i+(n-3)}\}$, then $r(e_i|S) = r(e'_i|S)$ and $r(e_{i+1}|S) = r(e_{i+2}|S)$, respectively.

So set S is not the basis of H_n. This implies that

$$edim(Hn) \geq n - 1. \tag{5.2}$$

Hence from Inequalities 5.1 and 5.2, we have $edim(H_n) = n - 1$. ■

Note: The total number of edge resolving set S for the helm graph (H_n) is $n + 1$.

5.2.3 Sunflower graph (Sf_n)

The *sunflower graph* Sf_n obtained from wheel graph W_n having a cycle C_n with vertex set $\{v_1, v_2, ..v_n\}$, by adding n new vertices $v'_1, v'_2, ..., v'_n$ with new edges $(v'_i v_i)$ and $(v'_i v_{i+1})$ for $1 \leq i \leq n$, as shown in Figure 5.3. The order and size of sunflower graph Sf_n can be determine as $|V| = 2n + 1$ and $|E| = 4n$, respectively. We labeled the central vertex, cycle vertices and the new added vertices by v, v_i and v'_i , respectively, where $1 \leq i \leq n$. We denote the cycle $v_1, v'_1, v_2, v'_2, ..., v'_n, v_1$ by C'_{2n}. There are three types of edges and we label them in the following manner, $e_i = vv_i$, $e'_i = v_i v_{i+1}$ and $e''_i = v_i v'_i$ or $e''_{i+1} = v_{i+1} v'_i$, consider Figure 5.3 for reference.

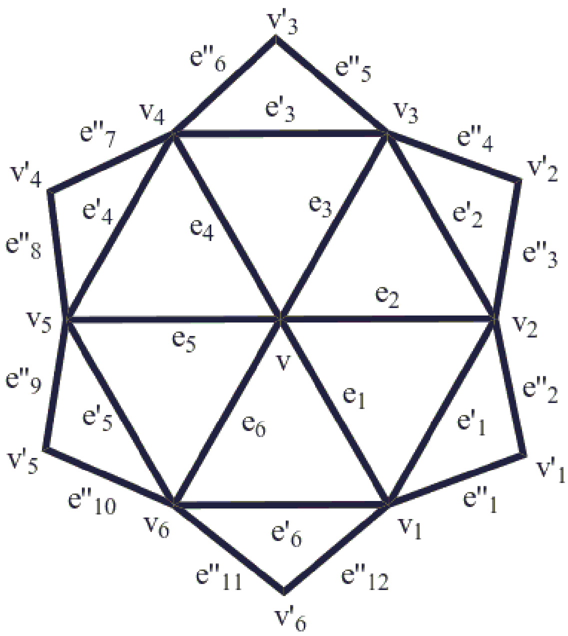


Figure 5.3: Sunflower graph Sf_6

The sunflower graph Sf_n for $n \geq 6$ has metric dimension

$$dim(Sf_n) = \begin{cases} \frac{n}{3},\ n \equiv 0\ (mod\ 3) \\ \left.\begin{matrix} \frac{n-1}{3}, \text{ if } n \text{ is odd} \\ \frac{n+2}{3}, \text{ if } n \text{ is even} \end{matrix}\right\},\ n \equiv 1\ (mod\ 3) \\ \frac{n+1}{3},\ n \equiv 2\ (mod\ 3) \end{cases}$$

Now we present some observations for Sf_n to prove its edge metric dimension.

i. There are two edge disjoint cycles: $C_n : v_1 e'_1 v_2\ e'_2 v_3 ...\ e'_{n-1} v_n e'_n v_1$ and C'_{2n} : $v_1 e''_1 v'_1\ e''_2 v_2 ...\ v_n e''_{2n-1} v'_n e''_{2n} v_1$.

ii. Every $2-2$ gap of S contains one vertex v_i or three vertices v_i, v'_i, v_{i+1} of C_n and C'_{2n}, respectively.

iii. There are three types of gaps, namely $2-2$, $2-5$ and $5-5$ with the maximum gap of size 3, 2 and 0, respectively.

iv. The resolving set S contains two $2-5$ gaps of size 2 and 0, when $n \equiv 2\ (mod\ 3)$.

v. The resolving set S mostly contain $2-2$ gaps.

vi. Any two, $2-2$ neighbouring gaps contain together at most 4 number of vertices.

vii. The neighbouring $2-2$ and $2-5$ gaps contain together at most 3 number of vertices.

viii. Only one vertex of degree 5 from C_n belongs to edge resolving set S of Sf_n, when $n \equiv 2\ (mod\ 3)$ and all remaining vertices belongs to C'_{2n}. If we consider any two consecutive vertices of the degree 5 in basis set S such as v_i and v_{i+1}, then $r(e'_{i+1}|S) = r(e'_{i+2}|S) = (1, 0, 2, 3, \cdots, 3)$ and $r(e'_i|S) = r(e_{i-1}|S) = (0, 1, 3, \cdots, 3, 2)$. So the $5-5$ gap is not possible in sunflower graph for edge metric dimension.

ix. Any two, $2-5$ neighbouring gaps contain together at most 2 vertices. If there are 4 vertices in $2-5$ neighbouring gaps for $v_i \in S$ where $1 \leq i \leq 2n$, then $r(e'_{i+1}|S) = r(e'_{i+3}|S) = (1, 1, 2, \cdots, 2)$, $r(e'_{i-1}|S) = r(e'_i|S) = (1, 2, \cdots, 1, 2)$ and $r(e''_{2n}|S) = r(e'_n|S) = (0, 2, 3, \cdots, 3)$.

x. The vertex v belongs to the center of Sf_n is not the member of any edge metric basis S of the Sf_n because of the $d(e_i, v) = 0$, $d(e'_i, v) = 1$ *and* $d(e''_j, v) = 1$ *where* $1 \leq i \leq n$ and $1 \leq j \leq 2n$.

We can also define the selection of basis S of Sf_n by the following claims.

Claim 5.2.10 *The resolving set S mostly contains $2-2$ gaps having one vertex.*

If a gap g of the size 1 with vertex v_{i+1} on C_n and $v'_i, v'_{i+1}, v'_{i+3},\ v'_{i+4} \in S$, then g contains two edges $e''_{i+1} = v'_iv_{i+1}$ and $e''_{i+2} = v_{i+1}v'_{i+1}$ on C'_{2n}. If $z \in \{e''_{i+1},\ e''_{i+2}\}$, then $r(z|S) = (1, 0, 1, 2, ..., 2,\ \overline{0, 1, 2, 3, ..., 3, 2})$ and $r(z|S) = (1, 0, 1, 2, ..., 2,\ \overline{1, 0, 2, 3, ..., 3, 2})$ where the normal part of the code and overlined part of the code represent the distance of z from v_i and v'_i type of vertices, respectively.

By observation v and Claim 5.2.10, there is no other edge in Sf_n having these type of codes.

Claim 5.2.11 *At most one* $2-2$ *gap of* S *has at most* 3 *number of vertices.*

If a gap g of the size 3 with vertices v_{i+3}, v'_{i+3}, v_{i+4} on C'_{2n} and v'_i, v'_{i+1}, v'_{i+2}, v'_{i+4}, $v'_{i+5} \in S$, then the g contains 5 number of edges, one of them $e'_{i+3} = v_{i+3}v_{i+4}$ on C_n and 4 of them $e''_{i+5} = v'_{i+2}v_{i+3}$, $e''_{i+6} = v_{i+3}v'_{i+3}$, $e''_{i+7} = v'_{i+3}v_{i+4}$, $e''_{i+8} = v_{i+4}v'_{i+4}$ on C'_{2n}. If $z \in \{\ e'_{i+3},\ e''_{i+5},\ e''_{i+6},\ e''_{i+7},\ e''_{i+8}\}$, then $r(z|S) = (2, 2, 1, 0, 0, 1, 2, ..., 2,\ \overline{3, 2, 1, 1, 1, 2, 3, ..., 3})$, $r(z|S) = (2, 2, 1, 0, 1, 2, ..., 2,\ \overline{3, 2, 0, 1, 2, 3, ..., 3})$, $r(z|S) = (2, 2, 1, 0, 1, 2, ..., 2, \overline{3, 2, 1, 0, 2, 3, ..., 3})$, $r(z|S) = (2, 2, 2, 1, 0, 1, 2, ..., 2, \overline{3, 3, 2, 0, 1, 3, ..., 3})$ and $r(z|S) = (2, 2, 2, 1, 0, 1, 2, ..., 2,\ \overline{3, 3, 2, 1, 0, 2, 3, ..., 3})$ where the normal part of the code and overlined part of the code represent the distance of z from v_i and v'_i type of vertices, respectively.

By observation iii and Claim 5.2.11, there is no other edge in Sf_n with these type of codes.

Claim 5.2.12 *At most one* $2-5$ *gap of* S *has* 2 *vertices.*

If a gap g of the size 2 with vertices v_n, v'_n and $v'_{n-7}, v'_{n-6}, v'_{n-4}, v'_{n-2},\ v'_{n-1}$, $v_{n+1} = v_i \in S$, then the g contains 4 number of edges, one of them $e'_n = v_nv_{n+1}$ on C_n and 3 of them $e''_{2n-2} = v'_{n-1}v_n$, $e''_{2n-1} = v_nv'_n$, $e''_{2n} = v'_nv_{n+1}$ on C'_{2n}. If $z \in \{e'_n,\ e''_{2n-2},\ e''_{2n-1},\ e''_{2n}\}$, then $r(z|S) = (0, 1, 2, ..., 2, 1, 0,\ \overline{1, 2, 3, ..., 3, 2, 1, 1})$, $r(z|S) = (1, 2, ..., 2, 1, 0, \overline{2, 3, ..., 3, 2, 0, 1})$, $r(z|S) = (1, 2, ..., 2, 1, 0, \overline{2, 3, ..., 3, 2, 1, 0})$ and $r(z|S) = (0, 1, 2, ..., 2, 0, 1,\ \overline{1, 2, 3, ..., 3, 2, 1})$ where the normal part of the code and overlined part of the code represent the distance of z from v_i and v'_i type of vertices, respectively.

By observations iii, iv and Claim 5.2.12, there is no other edge in Sf_n with these type of codes.

Claim 5.2.13 *Any two, $2-2$ neighbouring gaps contain together at most 4 number of vertices.*

Let two $2-2$ neighbouring gaps g_p and g_q have 1 and 3 number of vertices, respectively. The gaps g_p and g_q determined by v'_i, v'_{i+2} and v'_{i+3}. The gap g_p has one vertex v_{i+2} and two edges $e''_{i+3} = v'_{i+1}v_{i+2}$, $e''_{i+4} = v_{i+2}v'_{i+2}$ on C'_{2n}. The g_q has 3 vertices v_{i+3}, v'_{i+3}, v_{i+4} and 5 number of edges, one of them $e'_{i+3} = v_{i+3}v_{i+4}$ on C_n and 4 of them $e''_{i+5} = v'_{i+2}v_{i+3}$, $e''_{i+6} = v_{i+3}v'_{i+3}$, $e''_{i+7} = v'_{i+3}v_{i+4}$, $e''_{i+8} = v_{i+4}v'_{i+4}$ on C'_{2n}. If $v'_i, v'_{i+1}, v'_{i+2}, v'_{i+4}$, $v'_{i+5} \in S$ and $z \in \{e''_{i+3}$, e'_{i+3}, e''_{i+4}, e''_{i+5}, e''_{i+6}, e''_{i+7}, $e''_{i+8}\}$, then $r(z|S) = (2, 1, 0, 1, 2, ..., 2, \overline{2, 0, 1, 2, 3, ..., 3})$, $r(z|S) = (2, 2, 1, 0, 0, 1, 2, ..., 2, \overline{3, 2, 1, 1, 1, 2, 3, ..., 3})$, $r(z|S) = (2, 1, 0, 1, 2, ...2, \overline{2, 1, 0, 2, 3, ..., 3})$, $r(z|S) = (2, 2, 1, 0, 1, 2, ..., 2, \overline{3, 2, 0, 1, 2, 3, ..., 3})$, $r(z|S) = (2, 2, 1, 0, 1, 2, ..., 2, \overline{3, 2, 1, 0, 2, 3, ..., 3})$, $r(z|S) = (2, 2, 2, 1, 0, 1, 2, ..., 2, \overline{3, 3, 2, 0, 1, 3, ..., 3})$ and $r(z|S) = (2, 2, 2, 1, 0, 1, 2, ..., 2, \overline{3, 3, 2, 1, 0, 2, 3, ..., 3})$ where the normal part of the code and overlined part of the code represent the distance of z from v_i and v'_i type of vertices, respectively.

By observations iii, v, vi and Claim 5.2.13, there is no other edge in Sf_n having these type of codes.

Claim 5.2.14 *The neighbouring $2-2$ and $2-5$ gaps contain together at most 3 number of vertices.*

Let g_p and g_q be $2-2$ and $2-5$ neighbouring gaps having 1 and 2 vertices, respectively. These gaps can be determined by v'_{n-7}, v'_{n-1} and v_{n+1}. The gap g_p has one vertex v_{n-1} on C_n or two edges $e''_{2n-3} = v'_{n-1}v_{n-1}$, $e''_{2n-4} = v_{n-1}v'_n$ on C'_{2n}. The g_q has 2 vertices v_n, v'_n and 4 number of edges, one of them $e'_n = v_nv_{n+1}$ on C_n and 3 of them $e''_{2n-2} = v'_{n-1}v_n$, $e''_{2n-1} = v_nv'_n$, $e''_{2n} = v'_nv_{n+1}$ on C'_{2n}. If $v'_{n-7}, v'_{n-6}, v'_{n-4}, v'_{n-2}$, v'_{n-1}, $v_{n+1} = v_i \in S$ and $z \in \{e'_n$, e''_{2n-4}, e''_{2n-3}, e''_{2n-2}, e''_{2n-1}, $e''_{2n}\}$, then $r(z|S) = (0, 1, 2, ..., 2, 1, 0, \overline{1, 2, 3, ..., 3, 2, 1, 1})$, $r(z|S) = (2, ..., 2, 1, 0, 1, \overline{3, ..., 3, 2, 0, 1, 2})$, $r(z|S) = (2, ..., 2, 1, 0, 1, 1, 0, \overline{3, ..., 3, 2, 1, 0, 2})$, $r(z|S) = (1, 2, ..., 2, 1, 0, \overline{2, 3, ..., 3, 2, 0, 1})$, $r(z|S) = (1, 2, ..., 2, 1, 0, \overline{2, 3, ..., 3, 2, 1, 0})$ and $r(z|S) = (0, 1, 2, ..., 2, 0, 1, \overline{1, 2, 3, ..., 3, 2, 1})$ where the normal part of the code and overlined part of the code represent distance of z from v_i and v'_i type of vertices, respectively.

By observations iii, v, vii and Claim 5.2.14, there is no other edge in Sf_n with these type of codes.

Theorem 5.2.15 *Let Sf_n be a sunflower graph. Then then $edim(Sf_n) = \lceil \frac{2n}{3} \rceil$ If $n \geq 6$.*

Proof. We can easily see that $edim(Sf_5) = 5$. The resolving set for Sf_5 is $S = \{v_2, v_3, v_7, v_9, v_{10}\}$.

First we show that $edim(Sf_n) \leq \lfloor \frac{2n}{3} \rfloor$ for $n \geq 6$ by considering a resolving set S for Sf_n with $\lceil \frac{2n}{3} \rceil$ number of vertices. To show this theorem we discuss three different classes modulo 3 to which n belongs:

Case 1: $n \equiv 0 \ (mod\ 3)$. Let $2n = 3t$ for t is even, $t \geq 4$, and $\lceil \frac{2n}{3} \rceil = t$. In this case $S = \left\{v'_{3i-1}, v'_{3i} : 1 \leq i \leq t/2\ - 1\right\} \cup \left\{v'_{n-1}, v'_n\right\}$.

Case 2: $n \equiv 1 \ (mod\ 3)$. Let $2n = 3t + 2$ for t is even, $t \geq 4$, and $\lceil \frac{2n}{3} \rceil = t$. We define $S = \left\{v'_{3i}, v'_{3i+1} : 1 \leq i \leq t/2\right\} \cup \left\{v'_{n+2}\right\}$.

Case 3: $n \equiv 2 \ (mod\ 3)$. Let $2n = 3t + 1$ for t is odd, $t \geq 5$, and $\lceil \frac{2n}{3} \rceil = t$ and $S = \left\{v'_{3i+1}, v'_{3i+2}\ : 1 \leq i \leq (t-1)/2\ \right\} \cup \left\{v_2,\ v'_{n+2}\right\}$.

To show that $edim(Sf_n) \geq \lceil \frac{2n}{3} \rceil$, we consider the above aforementioned observations.

We suppose S is a basis of Sf_n of cardinality $r = |S|$. Then S generates r gaps which are denoted as $g_1, ..., g_r$ such that g_i and g_{i+1} for every $1 \leq i \leq r-1$ and also g_1 and g_r are neighbouring gaps. We can write $|g_1| + |g_2| \ \leq 3$, $|g_r| \ + \ |g_1| \ \leq \ 3$ and the relation for the remaining neighbouring gaps is $|g_i| \ + \ |g_{i+1}| \ \leq 4$ for every $i = 2, ..., r-1$.

So, we have

$$2(2n - r) = 2\sum_{i=1}^{r} |g_i| \ \leq 4(r-2) + 6$$

$$2(2n - r) \leq 4r - 2$$

$$r \geq \frac{4n+2}{6} = \frac{2n}{3} + \frac{1}{3} \geq \left\lceil \frac{2n}{3} \right\rceil$$

Since $r \in N$, for each $n \equiv 0, 1$ or $2(mod\ 3)$ we obtain $r \geq \lceil \frac{2n}{3} \rceil$ Hence the theorem is proved. ■

5.2.4 Friendship graph (F_n)

The *friendship graph* F_n can be generated from wheel graph by removing the alternative edges of the C_n cycle of W_n. The $|V| = 2n + 1$ and $|E| = 3n$ are the order and size of F_n, respectively. We label the central vertex, and remaining vertices as v and v_i, respectively, where $1 \leq i \leq 2n$. There are two types of edges and we label them in such a way $e_i = vv_i$ and $e'_i = v_iv_{i+1}$ for reference, see the Figure 5.4.

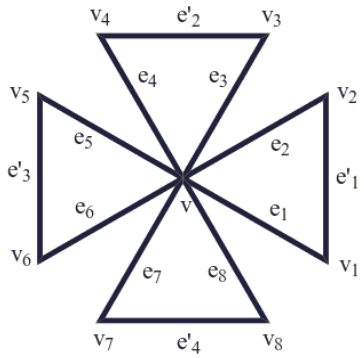


Figure 5.4: Friendship graph F_4

Mulyono and Wulandari have computed $dim(F_n)$ in [83] as

Theorem 5.2.16 [83] *"for $n \geq 2$ then the $dim(F_n) = n$".*

Now we show the $edim(F_n)$ in the next theorem.

Theorem 5.2.17 *Let F_n be a friendship graph for $n \geq 3$. Then*

$$edim(F_n) = 2n - 1.$$

Proof. Let $n \geq 3$ and $V(F_n) = \{v,\ v_1,\ v_2,\ ...,\ v_{2n}\}$, where $deg(v) = 2n$ and $deg(v_i) = 2$, where $i \in \{1,\ 2,\ \cdots,\ 2n\}$. To prove that S is a minimum edge resolving set, we consider the following cases:

1. If $e_i = vv_i$, then e_i has distance 0 to $\{v, v_i\}$ and distance 1 to all other vertices, where $i \in \{1, 2, ...2n\}$.

2. If $e_i' = v_i v_{i+1}$, then e_i' has distance 0 to v_i and v_{i+1} distance 1 to v and distance 2 to all other vertices.

The above cases show that the edge metric representation of any $2n-2$ vertices or less number of vertices of F_n is overlap (same) for some edges. So S can be edge metric generator if we take $2n-1$ number of vertices. Therefore $edim(F_n) = 2n-1$. We can also show that the $edim(F_n) \nleq 2n-2$ by considering the S is a basis of F_n and $|S| = 2n-2$, if we take $S = \{v_i, v_{i+1}, ..., v_{2n-2}\}$ then $r(e_{n-1}|S) = r(e_n|S)$ or $r(e_i|S) = r(e_n|S)$. So set S is not the basis of H_n. Hence proved that the $edim(F_n) = 2n-1$. ■

Note: The total number of edge resolving sets S for fan graph F_n is $2n$.

Chapter 6

Resolvability of some convex polytopes

6.1 Introduction

A polytope is a basic geometrical object bounded by regular sides. An object bounded by flat sides in two dimensions is called a polygon, and in three dimensions the object is known as a polyhedron. The two dimensional polygon and the three dimensional polyhedron are also known as 2-polytope and 3-polytope, respectively. Similarly, the n number of dimensions may contain n-polytope. There is a special type of polytope which is known as convex polytope, it has an additional property, in which it contains a convex set of points in the n-dimensional space $\mathbb{R}^n$. The convex polytopes are very useful in different areas of mathematics and applied sciences, especially in linear programming.

The results of this chapter are published in *"Utilitas Mathematica"* in 2019 [93].

6.2 Main results

In this section, we will study some wheel related convex polytopes and calculate their edge metric dimension (*EMD*).

6.2.1 Convex polytope graph $\mathbb{D}_{2n}$

The gear graph can be obtained from a cycle C_{2n} by introducing a vertex w adjacent to n number of alternate vertices w_1,w_3,$\cdots$,w_{2n-1} of C_{2n}. The gear graph is also referred as Jahangir graph and it is represented by J_{2n} [40, 109]. We consider first polytopes which is based on Jahangir graph and prism graph which is defined by Baca in [9] as,

Definition 6.3 [9] *"Let $l = \{1, 2, \cdots , 2n\}$ and $p = \{1, 2\}$ be index sets and D_{2n} be the graph of a prism. The prism graph D_{2n}, $n \geq 3$, is a trivalent graph which can be defined as the Cartesian product $P_2 \times C_{2n}$ of a path on two vertices with a cycle on n vertices, embedded in the plane. Let us denote the vertex set of D_{2n} can be denoted by $W(D_{2n}) = \{w_{j,k} : j \in p$ and $k \in l\}$ and edge set by $E(D_{2n}) = \{w_{j,k}w_{j,k+1} : j \in p$ and $k \in l\} \cup \{w_{1,k}w_{2,k} : k \in l\}$."*

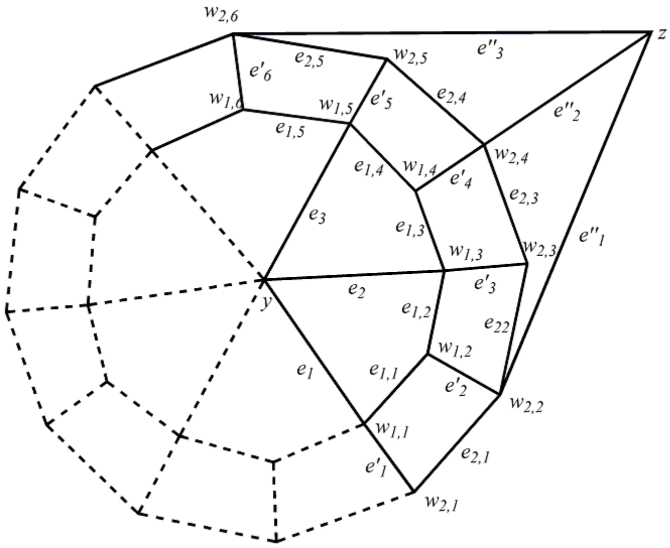


Figure 6.1: Polytope $\mathbb{D}_{2n}$

To show the different vertices by using index we consider some vertices as $w_{j,n+1} = w_{j,1}$, $w_{j,n+2} = w_{j,2}$ for $j \in p$. The prism graph D_{2n} has two types of face set, internal face set and external face set. An internal face set contains $2n$ 4-sided faces and the external face set contains two $2n$-sided faces [9, 59].

Baca defined a polytope based on prism graph as,

Definition 6.4 [9] *"Let y and z two new vertices and insert exactly one vertex y (z) into the internal (external) $2n$-sided faces of D_{2n}. Suppose that $n \geq 2$ and consider the graph $\mathbb{D}_{2n}$ with the vertex set $W(\mathbb{D}_{2n}) = W(D_{2n}) \cup \{y, z\}$ and the edge set $E(\mathbb{D}_{2n}) = E(D_{2n}) \cup \{y\ v_{1,2t-1} : t = 1, 2, \cdots, n\} \cup \{z\ v_{2,2t} : t = 1, 2, \cdots, n\}$. The convex polytope $\mathbb{D}_{2n}$, $n \geq 4$, is the plane graph on $|W(\mathbb{D}_{2n})| = 4n+2$ vertices, $|E(\mathbb{D}_{2n})| = 8n$ edges and consisting of $|F(\mathbb{D}_{2n})| = 4n$ faces."*

We labeled the vertices of polytope $\mathbb{D}_{2n}$ as shown in Figure 6.1. In this section, we study the *EMD* of $\mathbb{D}_{2n}$ for $n \geq 4$. We present that the family of $\mathbb{D}_{2n}$ polytope has unbounded *EMD*. The central vertex y or z does not belong to any basis because if y or z is the member of any edge metric basis S, then there lie two different vertices $w_{j,k}$ and $w_{j,t}$, for $j \in p$ and $1 \leq k = t \leq n$ such that $r(w_{j,k}|S) = r(w_{j,t}|S)$. We also know that the diameter of $\mathbb{D}_{2n}$ is 4. Therefore the basis consists on those vertices which belong to the rim of $\mathbb{D}_{2n}$.

A gap for neighbouring vertices $w_{j,k}$ and $w_{j,t}$ can be defined as $\alpha - \beta$ gap with $\alpha \leq \beta$ when $deg(w_{j,k}) = \alpha$ and $deg(w_{j,t}) = \beta$ or when $deg(w_{j,k}) = \beta$ and $deg(w_{j,t}) = \alpha$. Thus we have three different sort of gaps in $\mathbb{D}_{2n}$, that are, $3-3$, $4-3$ and $4-4$ gaps.

Lemma 6.4.1 *If S is basis set of $\mathbb{D}_{2n}$, then the $4-4$, $3-4$ and $3-3$ gaps of S consist of at most 1, 2 and 3 vertices, respectively.*

Proof. Let $e_{1,k} = w_{1,k}w_{1,k+1}$ be the edges on interior cycle $C_{1,2n}$, $e_{2,k} = w_{2,k}w_{2,k+1}$ be the edges on exterior cycle $C_{2,2n}$, $e'_k = w_{1,k}w_{2,k}$ are the edges which connect the vertices on the interior and exterior cycles, $e_j = y\ w_{1,j}$ are the spokes edges of interior cycle $C_{1,2n}$, and $e''_j = y\ w_{2,j+1}$ are the spokes edges of exterior cycle $C_{2,2n}$ of $\mathbb{D}_{2n}$ where $k \in \{1, 2, \cdots, 2n\}$ and $j \in \{1, 2, \cdots, n\}$ as shown in Figure 6.1.

Suppose $3-3$ gap of S containing five consecutive vertices $w_{1,k+1}$, $w_{1,k+2}$, $w_{1,k+3}$, $w_{1,k+4}$ and w_{k+5} of interior cycle $C_{1,2n}$ such that $deg(w_{1,k+1}) = deg(w_{1,k+5}) = 4$. In this case $r(e_{2,k+2} \mid S) = r(e_{2,k+3} \mid S)$, a contradiction.

Now, we choose a $3-4$ gap which containing four consecutive vertices $w_{1,k+1}$, $w_{1,k+2}$, $w_{1,k+3}$ and $w_{1,k+4}$ on interior cycle $C_{1,2n}$ such that $deg(w_{1,k+1}) = 4$, $deg(w_{1,k+5})$ $= 3$. It follows that $r(e_{k-1} \mid S) = r(e_{k+1} \mid S)$ and $r(e_{1,k-1} \mid S) = r(e_{1,k} \mid S)$, a contradiction.

Now let a $4-4$ gap containing three consecutive vertices $w_{1,k+1}$, $w_{1,k+2}$ and $w_{1,k+3}$ on interior cycle $C_{1,2n}$ such that $deg(w_{1,k+1}) = deg(w_{1,k+5}) = 3$. It follows that $r(e_{1,k+5} \mid S) = r(e_{1,k+6} \mid S)$, a contradiction. Hence proved that the $4-4$, $3-4$ and $3-3$ gaps of set S consist of at most 1, 2 and 3 vertices, respectively. ■

Lemma 6.4.2 *Any two neighbouring $3-3$ and $3-3$ gaps consist of at most 4 vertices all together.*

Proof. Let g_p and g_q be two $3-3$ neighbouring gaps with $w_{1,k+1}$, $w_{1,k+2}$, $w_{1,k+3}$ and $w_{1,k+5}$, $w_{1,k+6}$, $w_{1,k+7}$ vertices, respectively, determined by $w_{1,k}$, $w_{1,k+4}$ and $w_{1,k+8}$ with $deg(w_{1,k+2}) = deg(w_{1,k+6}) = 3$. Then we have $r(e'_{k+1} \mid S) = r(e_{2,k} \mid S)$, a contradiction. ■

Lemma 6.4.3 *Any two neighbouring $4-4$ and $4-4$ gaps consist of at most 2 vertices all together.*

Proof. Let g_p and g_q be two $4-4$ neighbouring gaps with $w_{1,k+1}$ and $w_{1,k+3}$, $w_{1,k+4}$, $w_{1,k+5}$ vertices, respectively, determined by $w_{1,k}$,$w_{1,k+2}$ and $w_{1,k+6}$ with $deg(w_{1,k+4}) = 4$. Then we have $r(e_{1,k+5} \mid S) = r(e_{1,k+6} \mid S)$ and $r(e_{2,k+5} \mid S) = r(e_{2,k+6} \mid S)$, a contradiction. ■

Lemma 6.4.4 *Any two neighbouring* $3-3$ *and* $3-4$ $(4-3$ *and* $3-3)$ *gaps consist of at most* 3 *vertices all together.*

Proof. Case 1: Let g_p and g_q be two $3-3$ and $3-4$ neighbouring gaps with $w_{1,k+1}$ and $w_{1,k+3}$, $w_{1,k+4}$, $w_{1,k+5}$, $w_{1,k+6}$ vertices, respectively, determined by $w_{1,k}$, $w_{1,k+2}$ and $w_{1,k+7}$ with $deg(w_{1,k+1}) = 3$ and $deg(w_{1,k+4}) = deg(w_{1,k+6}) = 4$. So it follows contradiction such as $r(e'_{k+5} \mid S) = r(e_{2,k+5} \mid S)$ and $r(e_{1,k+6} \mid S) = r(e_{1,k+7} \mid S)$.

Case 2: Consider g_p and g_q be two $4-3$ and $3-3$ neighbouring gaps with $w_{1,k+1}, w_{1,k+2}$ and $w_{1,k+4}$, $w_{1,k+5}$, $w_{1,k+6}$ vertices, respectively, determined by $w_{1,k}$, $w_{1,k+3}$ and $w_{1,k+7}$, with $deg(w_{1,k+2}) = 4$ and $deg(w_{1,k+5}) = 3$. Then we have $r(e_{2,k+2} \mid S) = r(e_{2,k+3} \mid S)$ and $r(e'_5 \mid S) = r(e'_7 \mid S)$, a contradiction.
■

First we discuss the edge metric dimension of $\mathbb{D}_{2n}$ for $n = 4, 5, 6$, which is easy to check. The $edim(\mathbb{D}_8) = 4$ with resolving set $S = \{w_1, w_2, w_4, w_6\}$. The edge metric dimension for $\mathbb{D}_{10}$ and $\mathbb{D}_{12}$ is 5 with resolving set $S = \{w_1, w_2, w_3, w_6, w_8\}$, $\{w_1, w_2, w_4, w_8, w_{10}\}$, respectively.

Theorem 6.4.5 *If* $\mathbb{D}_{2n}$ *be a convex polytope for* $n \geq 7$, *then* $edim(\mathbb{D}_{2n}) = \lceil \frac{4n-2}{5} \rceil$.

Proof. For $n \geq 7$, we have the following result. With the help of double inequality we present the prove of this theorem. So first we show that $edim(\mathbb{D}_{2n}) \leq \lceil \frac{4n-2}{5} \rceil$ by considering a resolving set for $n \geq 7$.

To prove $edim(\mathbb{D}_{2n}) \leq \lceil \frac{4n-2}{5} \rceil$ for $n \geq 7$, we consider five different cases based on modulus classes of 5 to which n belongs:

Case 1: Let $n \equiv 0 \ (mod\ 5)$. Then $n = 5t$, where $t \geq 2$, and $\lceil \frac{4n-2}{5} \rceil = 4t$. We define $S = \{w_{1,1}, w_{1,2}, w_{1,5}, w_{1,5}, w_{1,10}, w_{1,12}\} \cup \{w_{1,10k+5}, w_{1,10k+7}, w_{1,10(k+1)},$

$w_{1,10(k+1)+2} : 1 \leq k \leq t\ - 1\} \cup \{w_{1,2n-4},\ w_{1,2n-2}\}$ with $4t$ vertices satisfying Lemmas 6.4.1−6.4.4 and so S is a resolving set.

Case 2: Let $n \equiv 1\ (mod\ 5)$. Then $n = 5t + 1$, where $t \geq 2$, and $\lceil \frac{4n\ -\ 2}{5} \rceil = 4t+1$. We define $S = \{w_{1,1},\ w_{1,2},\ w_{1,4},\ w_{1,7},\ w_{1,9},\ w_{1,12},\ w_{1,14}\} \cup \{w_{1,10k+7},\ w_{1,10k+9},\ w_{1,10(k+1)+2},\ w_{1,10(k+1)+4} :\ 1 \leq k \leq t\ - 1\} \cup \{w_{1,2n-4},\ w_{1,2n-2}\}$ with $4t + 1$ vertices satisfying Lemmas 6.4.1−6.4.4 and so S is a resolving set.

Case 3: Let $n \equiv 2\ (mod\ 5)$. Then $n = 5t + 2$, where $t \geq 1$, and $\lceil \frac{4n\ -\ 2}{5} \rceil = 4t + 2$. In this case $S = \{w_{1,1},\ w_{1,2},\ w_{1,4},\ w_{1,6}\} \cup \{w_{1,10k-1},\ w_{1,10k+1},\ w_{1,10k+4},\ w_{1,10k+6} : 1 \leq k \leq t\ - 1\} \cup \{w_{1,2n-4},\ w_{1,2n-2}\}$ with $4t + 2$ vertices satisfying Lemmas 6.4.1−6.4.4 and so S is a resolving set.

Case 4: Let $n \equiv 3\ (mod\ 5)$. Then $n = 5t + 3$, where $t \geq 1$, and $\lceil \frac{4n\ -\ 2}{5} \rceil = 4t + 2$. We define $S = \{w_{1,1},\ w_{1,3},\ w_{1,6},\ w_{1,8}\} \cup \{w_{1,10k+1},\ w_{1,10k+3},\ w_{1,10k+6},\ w_{1,10k+8} : 1 \leq k \leq t\ - 1\} \cup \{w_{1,2n-4},\ w_{1,2n-2}\}$ with $4t + 2$ vertices satisfying Lemmas 6.4.1−6.4.4 and so S is a resolving set.

Case 5: Let $n \equiv 4\ (mod\ 5)$. Then $n = 5t + 4$, where $t \geq 1$, and $\lceil \frac{4n\ -\ 2}{5} \rceil = 4t + 3$. We define $S = \{w_{1,1},\ w_{1,3},\ w_{1,5},\ w_{1,8},\ w_{1,10}\} \cup \{w_{1,10k+3},\ w_{1,10k+5},\ w_{1,10k+8},\ w_{1,10(k+1)} : 1 \leq k \leq t\ - 1\} \cup \{w_{1,2n-4},\ w_{1,2n-2}\}$ with $4t + 3$ vertices satisfying Lemmas 6.4.1−6.4.4 and so S is a resolving set.

Now we prove that the $edim(\mathbb{D}_{2n}) \geq \lceil \frac{4n\ -\ 2}{5} \rceil$. Suppose S is a resolving set of $\mathbb{D}_{2n}$ and $|S| = r$. Then S includes r gaps on interior cycle, $C_{1,2n}$ of $\mathbb{D}_{2n}$, denoted by $g_1, \cdots, g_r$ such that g_k and g_{k+1} for every $1 \leq k \leq r - 1$ and also g_1 and g_r are neighbouring gaps. Four types of resolving sets can be observed on the basis of gaps. We will discuss them one by one:

Case 1: There are three neighbouring gaps, say g_{r-1}, g_r and g_1 which follow Lemma 6.4.2 such that $|g_{r-1}| + |g_r| \leq 4$, $|g_1| + |g_r| \leq 4$. The other three neighbouring gaps, say g_1, g_2 and g_3 have the relation as by Lemma 6.4.3,

$|g_1| + |g_2| \leq 2$, $|g_2| + |g_3| \leq 2$ and by Lemma 6.4.4, the relation for the remaining neighbouring gaps is $|g_k| + |g_{k+1}| \leq 3$ for every $k = 3, \cdots, r-2$.

$$\begin{aligned} 2(2n-r) &= 2\sum_{k=1}^{r}|g_k| \leq 3(r-4)+12 \\ r &\geq \frac{4n}{5} \geq \left\lceil \frac{4n - 2}{5} \right\rceil. \end{aligned}$$

Since $r \in \mathbb{N}$, for each $n \equiv 0$,1 $(mod\ 5)$ we obtain $r \geq \lceil \frac{4n-2}{5} \rceil$.

Case 2: There are three neighbouring gaps, say g_{r-1}, g_r and g_1 which follow Lemma 6.4.2 such that $|g_{r-1}| + |g_r| \leq 4$, $|g_1| + |g_r| \leq 4$. The other three neighbouring gaps, say g_1, g_2 and g_3 have the relation as by Lemma 6.4.3, $|g_1| + |g_2| \leq 2$, $|g_2| + |g_3| \leq 1$ and by Lemma 6.4.4, the relation for the remaining neighbouring gaps is $|g_k| + |g_{k+1}| \leq 3$ for every $k = 2, \cdots, r-2$.

$$\begin{aligned} 2(2n-r) &= 2\sum_{k=1}^{r}|g_k| \leq 3(r-5)+13 \\ r &\geq \frac{4n+2}{5} \geq \left\lceil \frac{4n - 2}{5} \right\rceil. \end{aligned}$$

Since $r \in \mathbb{N}$, for each $n \equiv 2$ $(mod\ 5)$ we obtain $r \geq \lceil \frac{4n-2}{5} \rceil$.

Case 3: There are three neighbouring gaps, say g_{r-1}, g_r and g_1 which follow Lemma 6.4.2 such that $|g_{r-1}| + |g_r| \leq 4$, $|g_1| + |g_r| \leq 4$ and by Lemma 6.4.4, the relation for the remaining neighbouring gaps is $|g_k| + |g_{k+1}| \leq 3$ for every $k = 1, \cdots, r-2$.

$$\begin{aligned} 2(2n-r) &= 2\sum_{k=1}^{r}|g_k| \leq 3(r-2)+8 \\ r &\geq \frac{4n-2}{5} \geq \left\lceil \frac{4n - 2}{5} \right\rceil. \end{aligned}$$

Since $r \in \mathbb{N}$, for each $n \equiv 3$ $(mod\ 5)$ we obtain $r \geq \lceil \frac{4n-2}{5} \rceil$.

Case 4: There are three neighbouring gaps, say g_{r-1}, g_r and g_1 which follow Lemma 6.4.2 such that $|g_{r-1}| + |g_r| \leq 4$, $|g_1| + |g_r| \leq 4$. The other two neighbouring gaps say g_1 and g_2 have the relation as by Lemma 6.4.3, $|g_1| + |g_2| \leq 2$ and by Lemma 6.4.4, the relation for the remaining neighbouring gaps is $|g_k| + |g_{k+1}| \leq 3$ for every $k = 2, \cdots, r-2$.

$$\begin{aligned} 2(2n-r) &= 2\sum_{k=1}^{r}|g_k| \leq 3(r-3)+10 \\ r &\geq \frac{4n-1}{5} \geq \left\lceil \frac{4n-2}{5} \right\rceil. \end{aligned}$$

Since $r \in \mathbb{N}$, for each $n \equiv 4 \ (mod\ 5)$ we obtain $r \geq \left\lceil \frac{4n-2}{5} \right\rceil$. ■

Imran and Siddiqui studied $dim(\mathbb{D}_n)$ and provided in the theorem [59] given below,

Theorem 6.4.6 [59] *"For every even integer $n \geq 8$, the $dim(\mathbb{D}_n) = \frac{n}{2}$".*

In the next section, we consider another convex polytope which is denoted by $\mathbb{B}_{2n}$ for edge metric dimension.

6.4.1 Convex polytope graph $\mathbb{B}_{2n}$

Definition 6.5 [10] *"Let $l = \{1, 2, \cdots, 2n\}$ and $J = \{1, 2, 3\}$ be index sets and B_{2n} be the graph of a biprism that can be defined as the Cartesian product $P_3 \square C_{2n}$ of a path on three vertices with a cycle on $2n$ vertices, embedded in the plane. Let us denote the vertex set of B_{2n} by $W(B_{2n}) = \{w_{j,k} : j \in p$ and $k \in l\}$ and the edge set by $E(B_{2n}) = \{w_{j,k}w_{j,k+1} : j \in J$ and $k \in l\} \cup \{w_{1,k}w_{2,k} : k \in l\} \cup \{w_{2,k}w_{3,k} : k \in l\}$. To make the convention let $w_{j,n+1} = w_{j,1}$, $w_{j,n+2} = w_{j,2}$ for $j \in p$."*

The convex polytope $\mathbb{B}_{2n}$ for $n \geq 4$ can be generated by B_{2n} as defined in [10] as

Definition 6.6 [10] *"The face set $F(B_{2n})$ contains $4n$ 4-sided faces and two $2n$-sided faces (internal and external). Insert exactly one vertex y into the internal $2n$-sided face of B_{2n} and exactly one vertex z into the external $2n$-sided face of B_{2n}. Suppose that $n \geq 4$, and consider the graph $\mathbb{B}_{2n}$ with vertex set $W(\mathbb{B}_{2n}) = W(B_{2n}) \cup \{y, z\}$ and the edge set $E(\mathbb{B}_{2n}) = E(B_{2n}) \cup \{y\ w_{1,t} : t = 1, 3, \cdots, 2n-1\} \cup \{z\ w_{3,t} : t = 1, 3, \cdots, 2n\}$."*

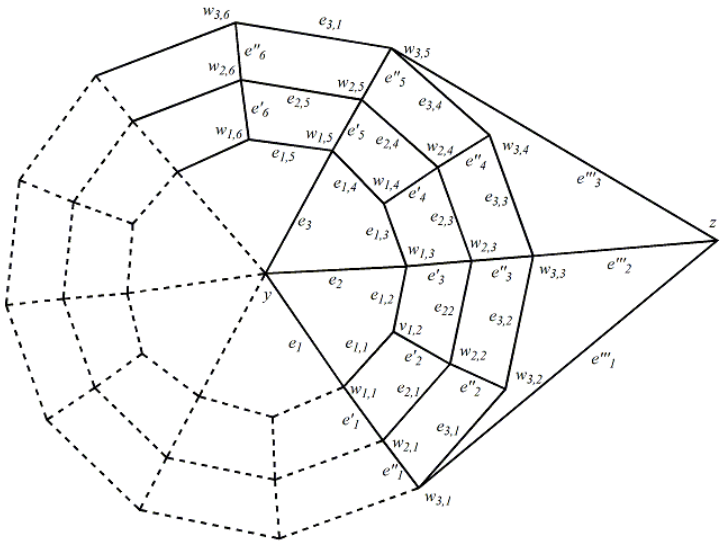


Figure 6.2: Polytope $\mathbb{B}_{2n}$

The graph of the convex polytope $\mathbb{B}_{2n}$ for $n \geq 4$ has an order $|W(\mathbb{B}_{2n})| = 6n+2$, size $|E(\mathbb{B}_{2n})| = 12n$ and the number of 4-sided faces is $F(\mathbb{B}_{2n}) = 6n$. We labelled the vertices of $\mathbb{B}_{2n}$ as shown in Figure 6.2.

There are two type of gaps in $\mathbb{B}_{2n}$, that are $3-3$ and $4-3$. Since there is only one vertex of degree 4 that can be part of resolving set, so there will be no gap of $4-4$.

Lemma 6.6.1 *If S is a basis of $\mathbb{B}_{2n}$, then the $3-3$ and $3-4$ gap of S consist of at most 3 and 2 number of vertices, respectively.*

Proof. Let $e_{1,k} = w_{1,k}w_{1,k+1}$ be the edges on inner cycle $C_{1,2n}$, $e_{2,k} = w_{2,k}w_{2,k+1}$ be the edges on middle cycle $C_{2,2n}$, $e_{3,k} = w_{3,k}w_{3,k+1}$ be the edges on outer cycle $C_{3,2n}$, $e'_k = w_{1,k}w_{2,k}$ are the edges which connect the vertices of $C_{1,2n}$ and $C_{2,2n}$, $e''_k = w_{2,k}w_{3,k}$ are the edges which connect the vertices of $C_{2,2n}$ and $C_{3,2n}$, $e_j = y\ w_{1,j}$ are the spokes edges of inner cycle $C_{1,2n}$, and $e'''_j = z\ w_{3,j+1}$ are the spokes edges of outer cycle $C_{3,2n}$, where $k \in \{1,3,\cdots,2n-1\}$ and $j \in \{1,3,\cdots,2n-1\}$.

Suppose the $3-3$ gap of S containing five consecutive vertices $w_{1,k+1}$, $w_{1,k+2}$, $w_{1,k+3}$, $w_{1,k+4}$ and $w_{1,k+5}$ of inner cycle $C_{1,2n}$ such that $deg(w_{1,k+1}) = deg(w_{1,k+5}) = 4$. In this case $r(e'_{k+2} \mid S) = r(e_{2,k+1} \mid S)$ and $r(e'_{k+4} \mid S) = r(e_{2,k+4} \mid S)$, a contradiction.

Now, we choose the $3-4$ gap which containing four consecutive vertices $w_{1,k+1}$, $w_{1,k+2}$, $w_{1,k+3}$, $w_{1,k+4}$ on inner cycle $C_{1,2n}$ such that $deg(w_{1,k+1}) = deg(w_{1,k+3}) = 4$, $deg(w_{1,k+2}) = deg(w_{1,k+4}) = 3$. It follows that $r(e'_{k+2} \mid S) = r(e_{2,k+1} \mid S)$, a contradiction.

Hence, it is proved that the $3-3$ and $3-4$ gap of S maximally consist of 3 and 2 vertices, respectively. ■

Lemma 6.6.2 *Any two neighbouring* $3-3$ *and* $3-3$ *gaps consist of at most* 4 *vertices all together.*

Proof. Let g_p and g_q be two $3-3$ neighbouring gaps containing $w_{1,k+1}$, $w_{1,k+2}$, $w_{1,k+3}$ and $w_{1,k+5}$, $w_{1,k+6}$, $w_{1,k+7}$ vertices, respectively determined by $w_{1,k}$, $w_{1,k+4}$ and $w_{1,k+8}$, where $deg(w_{1,k+2}) = deg(w_{1,k+6}) = 3$. Then we have $r(e_{1,k-2} \mid S) = r(e'_{k-1} \mid S)$ and $r(e_{2,k+3} \mid S) = r(e_{2,k+4} \mid S)$, a contradiction. ■

Lemma 6.6.3 *Any two neighbouring* $3-3$ *and* $3-4$ *(*$4-3$ *and* $3-3$*) gaps consist of at most* 3 *vertices all together.*

Proof. Case 1: Let g_p and g_q be two $3-3$ and $3-4$ neighbouring gaps containing $w_{1,k+1}$, $w_{1,k+2}$, $w_{1,k+3}$ and $w_{1,k+5}$, $w_{1,k+6}$ vertices, respectively determined by $w_{1,k}$, $w_{1,k+4}$ and $w_{1,k+7}$ with $deg(w_{1,k+2}) = 3$ and $deg(w_{1,k+5}) = 4$. Then we have $r(e_{2,k+3} \mid S) = r(e_{2,k+4} \mid S)$, a contradiction.

Case 2: Consider g_p and g_q be two $4-3$ and $3-3$ neighbouring gaps with $w_{1,k+1}$, $w_{1,k+2}$ and $w_{1,k+4}$, $w_{1,k+5}$, $w_{1,k+6}$ vertices, respectively determined by $w_{1,k}$, $w_{1,k+3}$ and $w_{1,k+7}$ with $deg(w_{1,k+2}) = 4$ and $deg(w_{1,k+5}) = 3$. Then there is a contradiction, such that $r(e_{2,k+2} \mid S) = r(e_{2,k+3} \mid S)$. ■

First we discuss the edge metric dimension of $\mathbb{B}_{2n}$ for $n = 4, 5, 6$, which is easy to check. The $edim(\mathbb{B}_8) = 3$ with the resolving set $S = \{w_1, w_4, w_6\}$. The edge metric dimension for $\mathbb{B}_{10}$ and $\mathbb{B}_{12}$ is 4 with resolving set $S = \{w_1, w_2, w_6, w_8\}$ and $\{w_2, w_4, w_8, w_{10}\}$, respectively.

Theorem 6.6.4 *If* $\mathbb{B}_{2n}$ *be a convex polytope for* $n \geq 7$*, then* $edim(\mathbb{B}_{2n}) = \lceil \frac{2n}{3} \rceil$.

Proof. For $n \geq 7$, we have the following result. We use double inequality to prove this theorem. First we show that $edim(\mathbb{B}_{2n}) \leq \lceil \frac{2n}{3} \rceil$ for $n \geq 7$ by constructing a resolving set. For this, we consider three cases as per modulus classes of 3 to which n belongs:

Case 1: Let $n \equiv 1\ (mod\ 3)$. Then $n = 3t+4$ for $t \geq 1$, and $\lceil \frac{2n}{3} \rceil = 2t+3$. In this case $S = \{w_{1,1}\} \cup \{w_{1,6k-2},\ w_{1,6k},: 0 \leq k \leq t\ \} \cup \{w_{1,2n-4},\ w_{1,2n-2}\}$, with $2t+3$ vertices satisfying Lemmas 6.6.1−6.6.3 and S is a resolving set.

Case 2: Let $n \equiv 2\ (mod\ 3)$. Then $n = 3t+5$ for $t \geq 1$, and $\lceil \frac{2n}{3} \rceil = 2t+4$. We define $S = \{w_{1,1},\ w_{1,2}\} \cup \{w_{1,6k},\ w_{1,6k+2} : 0 \leq k \leq t\ \} \cup \{w_{1,2n-4},\ w_{1,2n-2}\}$, with $2t+4$ vertices satisfying Lemmas 6.6.1−6.6.3 S is a resolving set.

Case 3: Let $n \equiv 0\ (mod\ 3)$. Then $n = 3t+6$ for $t \geq 1$, and $\lceil \frac{2n}{3} \rceil = 2t+4$. We define $S = \{w_{1,2},\ w_{1,4}\} \cup \{w_{1,6k+2},\ w_{1,6k+4} : 0 \leq k \leq t\ \} \cup \{w_{1,2n-4},\ w_{1,2n-2}\}$, with $2t+4$ vertices satisfying Lemmas 6.6.1−6.6.3 and S is a resolving set.

Now we show that the $edim(\mathbb{B}_{2n}) \geq \lceil \frac{4n}{3} \rceil$. Suppose S is a resolving set of $\mathbb{B}_{2n}$ and $|S| = r$. Then S includes r gaps on interior cycle, $C_{1,2n}$ of $\mathbb{B}_{2n}$, denoted by $g_1, \cdots, g_r$ such that g_k and g_{k+1} for every $1 \leq k \leq r-1$ and also g_1 and g_r are neighbouring gaps. We go through the following cases to show that $edim(\mathbb{B}_{2n}) \geq \lceil \frac{4n}{3} \rceil$:

Case 1: There are three neighbouring gaps, say g_{r-1}, g_r and g_1 which satisfy Lemma 6.6.3 such that $|g_{r-1}| + |g_r| \leq 3$, $|g_1| + |g_r| \leq 3$. By Lemma 6.6.2, the relation for the remaining neighbouring gaps is $|g_k| + |g_{k+1}| \leq 4$ for every $k = 1, \cdots, r-2$.

$$\begin{aligned} 2(2n-r) &= 2\sum_{k=1}^{r}|g_k| \leq 4(r-2)+6 \\ r &\geq \frac{4n+2}{6} \geq \frac{2n}{3} + \frac{1}{3} \geq \left\lceil \frac{2n}{3} \right\rceil. \end{aligned}$$

Since $r \in \mathbb{N}$, for each $n \equiv 0,\ 1\ (mod\ 3)$ we obtain $r \geq \lceil \frac{2n}{3} \rceil$.

Case 2: There are four neighbouring gaps, say g_{r-1}, g_r, g_1 and g_2 which satisfy Lemma 6.6.3 such that $|g_1| + |g_r| \leq 2$, $|g_1| + |g_2| \leq 3$ and $|g_{r-1}| + |g_r| \leq 3$ By Lemma 6.6.2, the relation for the remaining neighbouring gaps is $|g_k| + |g_{k+1}| \leq 4$

for every $k = 2, \cdots, r-2$.

$$\begin{aligned} 2(2n-r) &= 2\sum_{k=1}^{r}|g_k| \leq 4(r-3)+8 \\ r &\geq \frac{4n+4}{6} \geq \frac{2n}{3} + \frac{2}{3} \geq \left\lceil \frac{2n}{3} \right\rceil. \end{aligned}$$

Since $r \in \mathbb{N}$, for each $n \equiv 2$ $(mod\ 3)$ we obtain $r \geq \lceil \frac{2n}{3} \rceil$. ■

Imran and Siddiqui proved that the $dim(\mathbb{B}_n)$ is unbounded and they have provided the following theorem in [59].

Theorem 6.6.5 [59] *"For every even integer $n \geq 12$, the $dim(\mathbb{B}_n) = \lfloor \frac{n}{3} \rfloor + 2$".*

In the next section we find the edge metric dimension of convex polytope $\mathbb{Q}_n$.

6.6.1 Convex polytope $\mathbb{Q}_n$:

Definition 6.7 [102] *"Let $K = \{1, \cdots, n\}$ be an index set and Q_n be the graph of an antiprism. The antiprism Q_n, $n \geq 3$ is the planar graph. Let us denote the vertex set of Q_n by $W(Q_n) = \{w_{1,1}, w_{1,2}, \cdots, w_{1,n}, w_{2,1}, w_{2,2}, \cdots, w_{2,n}\}$ and the edge set $E(Q_n) = \{w_{1,k}w_{1,k+1} : k \in k\} \cup \{w_{2,k}w_{2,k+1} : k \in K\} \cup \{w_{1,k}w_{2,k-1}, w_{1,k}w_{2,k} : k \in K\}$. We make the convention that $w_{1,n+1} = w_{1,1}$ and $w_{2,n+1} = w_{2,1}$ to simplify later notations. The face set $F(Q_n)$ contains $2n$ 3-sided faces and 2 n-sided faces (internal and external). We insert exactly one vertex $y(z)$ into the internal (external) n-sided face of Q_n and consider the graph $\mathbb{Q}_n$ with the vertex set $W(\mathbb{Q}_n) = W(Q_n) \cup \{y, z\}$ and the edge set $E(\mathbb{Q}_n) = E(Q_n) \cup \{yv_{1,k} : k \in K\} \cup \{w_{2,k}z : k \in K\}$. Then $\mathbb{Q}_n$ is the plane graph consisting of 3-sided faces and constitutes an infinite class of convex polytopes."*

The $dim(\mathbb{Q}_n)$ proved in [102] as

Theorem 6.7.1 [102] *"If $n \geq 6$, then we have $dim(\mathbb{Q}_n) = \lfloor \frac{2n+4}{5} \rfloor$".*

Consider Figure 6.3 and let $w_{1,1}$, $w_{1,2}$, $\cdots$, $w_{1,n}$ and $w_{2,1}$, $w_{2,2}$, $\cdots$, $w_{2,n}$ represent the vertices of inner cycle $C_{1,n}$ and outer cycle $C_{2,n}$ of $\mathbb{Q}_n$, respectively. All these vertices have the same degree, $deg(w_{t,k}) = 5$. Let $e_k = y\, w_{1,k}$ are spokes edges

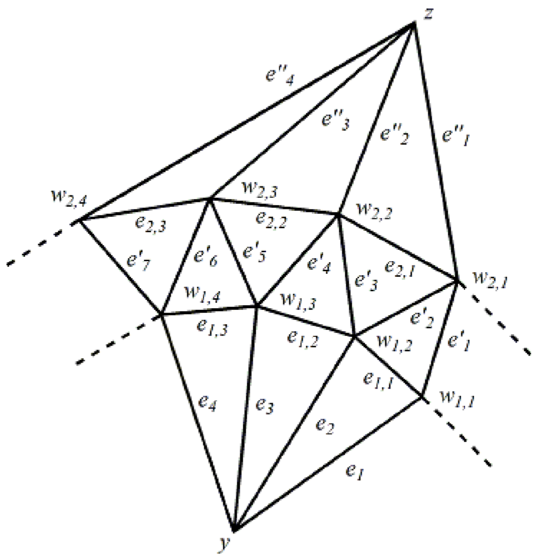


Figure 6.3: Polytope $\mathbb{Q}_n$

of inner cycle $C_{1,n}$, $e_{1,k} = w_{1,k}w_{1,k+1}$ be the edges of inner cycle, $e_{2,k} = w_{2,k}w_{2,k+1}$ be the edges of outer cycle $C_{2,n}$, $e'_{2k-1} = w_{1,k}w_{2,k}$ and $e'_{2k} = w_{1,k+1}w_{2,k}$ are the edges which connect the vertices of $C_{1,n}$ and $C_{2,n}$, $e''_k = z\ w_{2,k}$ are the spokes edges of outer cycle, where $k \in \{1, \cdots, n\}$.

We consider a convex polytope $\mathbb{Q}_n$ for $n \geq 3$ which is the member of an infinite class of convex polytopes. The vertices y and z belong to center do not be the member of basis S. The diameter of $\mathbb{Q}_n$ is 3, so if y or z the member of S, then there exist two different vertices $w_{t,k}$ and $w_{t,j}$, for $1 \leq k \neq j \leq n$ and $t = \{1, 2\}$ such that $r(w_{t,k}|S) = r(w_{t,j}|S)$. Therefore, the basis vertices only belong to the internal and (or) external cycles of $\mathbb{Q}_n$. The following different gap conditions are considered to select the basis vertices for $\mathbb{Q}_n$.

There is only one type of gap, namely $5-5$ gap.

Lemma 6.7.2 *If S is a basis of $\mathbb{Q}_n$, then $5-5$ gap contains at most 3 vertices.*

Proof. Suppose a $5-5$ gap containing four consecutive vertices $w_{t,k+1}$, $w_{t,k+2}$, $w_{t,k+3}$ and $w_{t,k+4}$ controlled by the vertices $w_{t,k}$ and $w_{t,k+5}$. Then $r(e_{k+1}|S) = r(e_{k+2}|S)$ if $t = 1$ and $r(e''_{k+1}|S) = r(e''_{k+2}|S)$ if $t = 2$. ■

Lemma 6.7.3 *Any two neighbouring gaps consist of at most 6 vertices all together.*

Proof. Let g_p and g_q be two neighbouring gaps containing vertices $w_{t,k+1}$, $w_{t,k+2}$, $w_{t,k+3}$ and $w_{t,k+5}$, $w_{t,k+6}$, $w_{t,k+7}$, $w_{1,k+8}$, respectively determined by the set of vertices $w_{t,k}$, $w_{t,k+4}$ and $w_{t,k+9}$. Then $r(e_{k+5}|S) = r(e_{k+6}|S)$ and $r(e'_{k+11}|S) = r(e'_{k+12}|S)$ if $t = 1$ and $r(e''_{k+5}|S) = r(e''_{k+6}|S)$ and $r(e'_{k+11}|S) = r(e'_{k+12}|S)$ if $t = 2$. ■

Every basis must contain vertices of the both cycles $C_{1,n}$ and $C_{2,n}$ with some particular formation which we shall define and prove in the following Lemmas.

Lemma 6.7.4 *If four consecutive gaps on $C_{1,n}$ containing $0, 0, 1$ and 0 vertices, respectively, then there are two consecutive gaps on $C_{2,n}$ containing 3 and 0 vertices, respectively.*

Proof. Let the gaps on $C_{1,n}$ determined by vertices $w_{1,k}$, $w_{1,k+1}$, $w_{1,k+2}$, $w_{1,k+4}$ and $w_{1,k+5}$ with gap size $0, 0, 1$ and 0, respectively and the gaps on $C_{2,n}$ determined by vertices $w_{2,k+2}$, $w_{2,k+7}$ and $w_{2,k+8}$ with gap size 4 and 0, respectively. Then $r(e'_{k+9}|S) = r(e'_{k+10}|S)$. If the gaps on $C_{2,n}$ determined by vertices $w_{2,k+2}$, $w_{2,k+6}$ and $w_{2,k+8}$ with gap size 3 and 1, respectively, then $r(e_{k+6}|S) = r(e_{k+7}|S)$. ■

Lemma 6.7.5 *If three consecutive gaps on $C_{1,n}$ containing $0, 1$ and 2 vertices, respectively, then there are three consecutive gaps on $C_{2,n}$ containing $2, 1$ and 0 vertices, respectively.*

Proof. Let the gaps on $C_{1,n}$ determined by vertices $w_{1,k}$, $w_{1,k+1}$, $w_{1,k+3}$ and $w_{1,k+6}$ with gap size $0, 1$ and 2, respectively and the gaps on $C_{2,n}$ determined by vertices $w_{2,k+2}$, $w_{2,k+6}$, $w_{2,k+8}$ and $w_{2,k+9}$ with gap size $3, 1$ and 0, respectively. Then $r(e_{k+4}|S) = r(e_{k+5}|S)$ and $r(e''_{k+4}|S) = r(e''_{k+7}|S)$. If the gaps on $C_{2,n}$ determined by vertices $w_{2,k+2}$, $w_{2,k+5}$, $w_{2,k+8}$ and $w_{2,k+9}$ with gap size $2, 2$ and 0, respectively, then $r(e_{k+4}|S) = r(e_{k+7}|S)$ and $r(e'_{k+9}|S) = r(e'_{k+13}|S)$. If the gaps on $C_{2,n}$ determined by vertices $w_{2,k+2}$, $w_{2,k+5}$, $w_{2,k+7}$ and $w_{2,k+9}$ with gap size $2, 1$ and 1, respectively, then $r(e_{k+7}|S) = r(e_{k+8}|S)$ and $r(e''_{k+4}|S) = r(e''_{k+6}|S)$. ■

Lemma 6.7.6 *If four consecutive gaps on $C_{1,n}$ containing $0, 1, 0$ and 2 vertices, respectively, then there are three consecutive gaps on $C_{2,n}$ containing $3, 1$ and 0 vertices, respectively.*

Proof. Let the gaps on $C_{1,n}$ determined by vertices $w_{1,k}$, $w_{1,k+1}$, $w_{1,k+3}$, $w_{1,k+4}$ and $w_{1,k+7}$ with gap size $0, 1, 0$ and 2, respectively and the gaps on $C_{2,n}$ determined by vertices $w_{2,k+2}$, $w_{2,k+7}$, $w_{2,k+9}$ and $w_{2,k+10}$ with gap size $4, 1$ and 0, respectively. Then $r(e_{k+5}|S) = r(e_{k+6}|S)$ and $r(e''_{k+5}|S) = r(e''_{k+8}|S)$. If the gaps on $C_{2,n}$ determined by vertices $w_{2,k+2}$, $w_{2,k+6}$, $w_{2,k+9}$ and $w_{2,k+10}$ with gap size $3, 2$ and 0, respectively, then $r(_{k+5}|S) = r(e_{k+8}|S)$ and $r(e''_{k+5}|S) = r(e''_{k+8}|S)$. If the gaps on $C_{2,n}$ determined by vertices $w_{2,k+2}$, $w_{2,k+6}$, $w_{2,k+8}$ and $w_{2,k+10}$ with gap size $3, 1$ and 1, respectively, then $r(e_{k+8}|S) = r(e_{k+9}|S)$ and $r(e''_{k+5}|S) = r(e''_{k+9}|S)$. ■

Lemma 6.7.7 *If four consecutive gaps on $C_{1,n}$ containing $0, 0, 2$ and 1 vertices, respectively, then there are three consecutive gaps on $C_{2,n}$ containing $2, 3$ and 0 vertices, respectively.*

Proof. Let the gaps on $C_{1,n}$ determined by vertices $w_{1,k}$, $w_{1,k+1}$, $w_{1,k+2}$, $w_{1,k+5}$ and $w_{1,k+7}$ with gap size $0, 0, 2$ and 1, respectively and the gaps on $C_{2,n}$ determined by vertices $w_{2,k+2}$, $w_{2,k+6}$, $w_{2,k+10}$ and $w_{2,k+11}$ with gap size $3, 3$ and 0, respectively. Then $r(e_{k+4}|S) = r(e_{k+9}|S)$ and $r(e''_{k+3}|S) = r(e''_{k+9}|S)$. If the gaps on $C_{2,n}$ determined by vertices $w_{2,k+2}$, $w_{2,k+5}$, $w_{2,k+10}$ and $w_{2,k+11}$ with gap size $2, 4$ and 0, respectively, then $r(_{k+4}|S) = r(e_{k+9}|S)$ and $r(e''_{k+3}|S) = r(e''_{k+9}|S)$. If the gaps on $C_{2,n}$ determined by vertices $w_{2,k+2}$, $w_{2,k+5}$, $w_{2,k+9}$ and $w_{2,k+11}$ with gap size $2, 3$ and 1, respectively, then $r(e_{k+9}|S) = r(e_{k+10}|S)$ and $r(e''_{k+3}|S) = r(e''_{k+10}|S)$. ■

Lemma 6.7.8 *If four consecutive gaps on $C_{1,n}$ containing $0, 0, 2$ and 0 vertices, respectively, then there are three consecutive gaps on $C_{2,n}$ containing $3, 3$ and 0 vertices, respectively.*

Proof. Let the gaps on $C_{1,n}$ determined by vertices $w_{1,k}$, $w_{1,k+1}$, $w_{1,k+3}$, $w_{1,k+5}$ and $w_{1,k+6}$ with gap size $0, 0, 2$ and 0, respectively and the gaps on $C_{2,n}$ determined by vertices $w_{2,k+2}$, $w_{2,k+7}$, $w_{2,k+11}$ and $w_{2,k+12}$ with gap size $4, 3$ and 0, respectively. Then $r(e_{k+4}|S) = r(e_{k+10}|S)$ and $r(e''_{k+3}|S) = r(e''_{k+10}|S)$. If the gaps on $C_{2,n}$ determined by vertices $w_{2,k+2}$, $w_{2,k+6}$, $w_{2,k+11}$ and $w_{2,k+12}$ with gap size $3, 4$ and 0, respectively, then $r(_{k+4}|S) = r(e_{k+10}|S)$ and $r(e''_{k+3}|S) = r(e''_{k+10}|S)$. If the gaps on $C_{2,n}$ determined by vertices $w_{2,k+2}$, $w_{2,k+6}$, $w_{2,k+10}$ and $w_{2,k+12}$ with gap size $3, 3$ and 1, respectively, then $r(e_{k+10}|S) = r(e_{k+11}|S)$ and $r(e''_{k+3}|S) = r(e''_{k+11}|S)$. ■

First we discuss the EMD of $\mathbb{Q}_n$ for $n = 5, .., 8$, which is easy to check. The $edim(\mathbb{Q}_n) = 6$ for $n = 5, 6, 7$ with resolving set $S = \{w_{1,1},\ w_{1,2},\ w_{1,3},\ w_{1,4},\ w_{2,1},\ w_{2,3}\}$, $S = \{w_{1,1},\ w_{1,2},\ w_{1,3},\ w_{1,5},\ w_{2,3},\ w_{2,5}\}$ and $S = \{w_{1,1},\ w_{1,2},\ w_{1,4},\ w_{1,6},\ w_{2,3},\ w_{2,7}\}$, respectively. The edge metric dimension for $\mathbb{Q}_8$ is 7 with resolving set $S = \{w_{1,1},\ w_{1,2},\ w_{1,4},\ w_{1,5},\ w_{1,7},\ w_{2,2},\ w_{2,6}\}$.

Theorem 6.7.9 *If $\mathbb{Q}_n$ be a convex polytope for $n \geq 9$, then $edim(\mathbb{Q}_n) = \lceil \frac{4n}{5} \rceil$.*

Proof. We consider double inequality condition to show this theorem. First we show that $edim(\mathbb{Q}_n) \leq \lceil \frac{4n}{5} \rceil$ by creating a resolving set. For this purpose, we consider five different cases as per modulus classes of 5 to which n belongs:

Case 1: Let $n \equiv 0\ (mod\ 5)$. Then $n = 5t$ for $t \geq 2$, and $\lceil \frac{4n}{5} \rceil = 4t$. We define $S = \{w_{1,1}, w_{1,2}, w_{1,4}, w_{1,7}\} \cup \{w_{1,5k+1}, w_{1,5k+2} : 2 \leq k \leq t-1\} \cup \{w_{2,3}, w_{2,6}\} \cup \{w_{2,5j-2}, w_{2,5j-1} : 2 \leq j \leq t\}$, with $4t$ vertices satisfying Lemma 6.7.2, Lemma 6.7.3 and Lemma 6.7.5. So S is a resolving set.

Case 2: Let $n \equiv 1\ (mod\ 5)$. Then $n = 5t + 1$ for $t \geq 2$, and $\lceil \frac{4n}{5} \rceil = 4t + 1$. We define $S = \{w_{1,1}, w_{1,2}, w_{1,4}, w_{1,5}, w_{1,8}\} \cup \{w_{1,5k+2}, w_{1,5k+3} : 2 \leq k \leq t-1\} \cup \{w_{2,3}, w_{2,7}\} \cup \{w_{2,5j-1}, w_{2,5j} : 2 \leq j \leq t\}$, with $4t + 1$ vertices satisfying Lemma 6.7.2, Lemma 6.7.3 and Lemma 6.7.6. So S is a resolving set.

Case 3: Let $n \equiv 2\ (mod\ 5)$. Then $n = 5t+2$ for $t \geq 2$, and $\lceil \frac{4n}{5} \rceil = 4t+2$. We define $S = \{w_{1,1}, w_{1,2}, w_{1,3}, w_{1,6}\} \cup \{w_{1,5k-2}, w_{1,5k-1} : 2 \leq k \leq t\} \cup \{w_{2,3}, w_{2,6}\} \cup \{w_{2,5j}, w_{2,5j+1} : 2 \leq j \leq t\}$, with $4t + 2$ vertices satisfying Lemma 6.7.2, Lemma 6.7.3 and Lemma 6.7.7. So S is a resolving set.

Case 4: Let $n \equiv 3\ (mod\ 5)$. Then $n = 5t+3$ for $t \geq 2$, and $\lceil \frac{4n}{5} \rceil = 4t+3$. We define $S = \{w_{1,1}, w_{1,2}, w_{1,3}, w_{1,6}, w_{1,7}\} \cup \{w_{1,5k-1}, w_{1,5k} : 2 \leq k \leq t\} \cup \{w_{2,3}, w_{2,7}\} \cup \{w_{2,5j+1}, w_{2,5j+2} : 2 \leq j \leq t\}$ and S contains $4t + 3$ vertices and satisfies Lemma 6.7.2, Lemma 6.7.3 and Lemma 6.7.8. So S is a resolving set.

Case 5: Let $n \equiv 4\ (mod\ 5)$. Then $n = 5t - 1$ for $t \geq 2$, and $\lceil \frac{4n}{5} \rceil = 4t$. We define $S = \{w_{1,1}, w_{1,2}, w_{1,3}\} \cup \{w_{1,5k}, w_{1,5k+1} : 1 \leq k \leq t-1\} \cup \{w_{2,3}\} \cup$

$\{w_{2,5j-3}, w_{2,5j-2} : 2 \leq j \leq t\}$, with $4t$ vertices satisfying Lemma Lemma 6.7.2, Lemma 6.7.3 and Lemma 6.7.4. So S is a resolving set.

Now we prove that the $edim(\mathbb{Q}_n) \geq \lceil \frac{4n}{5} \rceil$. Suppose S is a resolving set of $\mathbb{Q}_n$ of order $r = |S|$. Then S includes r gaps on $C_{1,n}$ and $C_{2,n}$ of $\mathbb{Q}_n$, denoted as $g_1, \cdots, g_r$ such that g_k and g_{k+1} for every $1 \leq k \leq r-1$ and also g_1 and g_r are neighbouring gaps. So we consider the five cases given below:

Case 1: There are five neighbouring gaps, say g_{r-1}, g_r, g_1, g_2 and g_3 which satisfy Lemma 6.7.5 such that $|g_{r-1}| + |g_r| \leq 1$, $|g_1| + |g_r| \leq 5$, $|g_1| + |g_2| \leq 5$ and $|g_2| + |g_3| \leq 1$. By Lemma 6.7.2 and Lemma 6.7.3, the relation for the remaining neighbouring gaps is $|g_k| + |g_{k+1}| \leq 3$ for every $k = 3, \cdots, r-2$.

$$\begin{aligned} 2(2n - r) &= 2\sum_{k=1}^{r} |g_k| \leq 3(r-4) + 12 \\ r &\geq \frac{4n}{5} \geq \left\lceil \frac{4n}{5} \right\rceil. \end{aligned}$$

Since $r \in \mathbb{N}$, for each $n \equiv 0\ (mod\ 5)$ we obtain $r \geq \lceil \frac{4n}{5} \rceil$.

Case 2: There are eight neighbouring gaps, say g_{r-1}, g_r and g_k, where $k \in \{1, \cdots, 6\}$ which satisfy Lemma 6.7.6 such that $|g_{r-1}| + |g_r| \leq 6$, $|g_1| + |g_r| \leq 5$, $|g_1| + |g_2| \leq 4$, $|g_2| + |g_3| \leq 2$ and three of them satisfy $|g_k| + |g_{k+1}| \leq 1$, where $k \in \{3, \cdots, 5\}$. By Lemma 6.7.2 and Lemma 6.7.3, the relation for the remaining neighbouring gaps is $|g_k| + |g_{k+1}| \leq 3$ for every $k = 6, \cdots, r-2$.

$$\begin{aligned} 2(2n - r) &= 2\sum_{k=1}^{r} |g_k| \leq 3(r-7) + 20 \\ r &\geq \frac{4n+1}{5} \geq \left\lceil \frac{4n}{5} \right\rceil. \end{aligned}$$

Since $r \in \mathbb{N}$, for each $n \equiv 1\ (mod\ 5)$ we obtain $r \geq \lceil \frac{4n}{5} \rceil$.

Case 3: There are six neighbouring gaps, say g_{r-1}, g_r and g_k, where $k \in \{1, \cdots, 4\}$ which satisfy Lemma 6.7.7 such that $|g_{r-1}| + |g_r| \leq 2$, $|g_1| + |g_r| \leq 1$,

$|g_1| + |g_2| = 0$, and two of them satisfy $|g_k| + |g_{k+1}| \leq 5$, where $k \in \{2,3\}$. By Lemma 6.7.2 and Lemma 6.7.3, the relation for the remaining neighbouring gaps is $|g_k| + |g_{k+1}| \leq 3$ for every $k = 4, \cdots, r-2$.

$$\begin{aligned} 2(2n-r) &= 2\sum_{k=1}^{r}|g_k| \leq 3(r-5)+13 \\ r &\geq \frac{4n+2}{5} \geq \left\lceil \frac{4n}{5} \right\rceil. \end{aligned}$$

Since $r \in \mathbb{N}$, for each $n \equiv 2$ $(mod\ 5)$ we obtain $r \geq \lceil \frac{4n}{5} \rceil$.

Case 4: There are eight neighbouring gaps, say g_{r-1}, g_r and g_k, where $k \in \{1, \cdots, 6\}$ which satisfy Lemma 6.7.8 such that $|g_{r-1}| + |g_r| = 0$, $|g_1| + |g_r| \leq 1$, $|g_1| + |g_2| \leq 1$, two of them satisfy $|g_k| + |g_{k+1}| \leq 2$, where $k \in \{2,3\}$ and two of them satisfy $|g_k| + |g_{k+1}| \leq 6$, where $k = \{4,5\}$. By Lemma 6.7.2 and Lemma 6.7.3, the relation for the remaining neighbouring gaps is $|g_k| + |g_{k+1}| \leq 3$ for every $k = 6, \cdots, r-2$.

$$\begin{aligned} 2(2n-r) &= 2\sum_{k=1}^{r}|g_k| \leq 3(r-7)+18 \\ r &\geq \frac{4n+3}{5} \geq \left\lceil \frac{4n}{5} \right\rceil. \end{aligned}$$

Since $r \in \mathbb{N}$, for each $n \equiv 3$ $(mod\ 5)$ we obtain $r \geq \lceil \frac{4n}{5} \rceil$.

Case 5: There are six neighbouring gaps, say g_{r-1}, g_r and g_k, where $k \in \{1, \cdots, 4\}$ which satisfy Lemma 6.7.4 such that $|g_{r-1}| + |g_r| \leq 0$, $|g_1| + |g_r| \leq 6$, $|g_1| + |g_2| \leq 6$, $|g_2| + |g_3| \leq 1$ and $|g_3| + |g_4| \leq 1$. By Lemma 6.7.2 and Lemma 6.7.3, the relation for the remaining neighbouring gaps is $|g_k| + |g_{k+1}| \leq 3$ for every $k = 4, \cdots, r-2$.

$$\begin{aligned} 2(2n-r) &= 2\sum_{k=1}^{r}|g_k| \leq 3(r-4)+8 \\ r &\geq \frac{4n+4}{5} \geq \left\lceil \frac{4n}{5} \right\rceil. \end{aligned}$$

Since $r \in \mathbb{N}$, for each $n \equiv 4$ $(mod\ 5)$ we obtain $r \geq \lceil \frac{4n}{5} \rceil$. ■

Chapter 7

Resolvability of generalized Petersen graph $P(n, 3)$

This chapter deals with the study of edge metric dimension (EMD) of generalized Petersen graph $P(n,3)$ for $n \geq 12$. The topological networks based on Petersen graph are considered cost-effective due to their regularity, high degree of symmetry, optimal connectivity (maximal fault-tolerance) and small diameter. In this chapter, the EDM of family of generalized Petersen graph $P(n,m)$ with $m = 3$ has been studied and shown that it is constant.

7.1 Introduction

The generalized Petersen graph is a simple connected graph with order $n = 10$ and size $m = 15$, named after Danish mathematician Julius Petersen. In 1969, the Watkins defined the generalized Petersen graph in [114]. There are many applications of Petersen graph in mathematics, medical science and engineering technologies, specially in networks. Some of them are hyper Petersen networks [28], folded Petersen networks [101], folded Petersen cube networks [87], Petersen-torus networks [98], three-dimensional Petersen-torus network [99], hierarchical Petersen networks [100] and chemical graph theory [14]. The $dim(P(n,3))$ computed and presented by Imran *et al.* in [58] as shown in the theorem below:

Theorem 7.1.1 [58] *"For the generalized Petersen graph $P(n,3)$ we have*

$$dim(P(n,3)) = \begin{cases} 3, & \textit{for } n \equiv 1 \pmod 6 \textit{ and } n \geq 13. \\ \leq 4, & \textit{for } n \equiv 0,3,4,5 \pmod 6 \textit{ and } n \geq 17. \\ \leq 5, & \textit{for } n \equiv 2 \pmod 6 \textit{ and } n \geq 8." \end{cases}$$

The result for the $dim(P(n,3))$ improved by Siddiqui in 2015 [103] as

Theorem 7.1.2 [103] *"For the generalized Petersen graph $P(n,m)$ for $m = 3$, we have* $dim(P(n,3)) = \begin{cases} 3, & \textit{for } n \equiv 1 \pmod 6 \textit{ and } n \geq 25. \\ 4, & \textit{otherwise } n \geq 24." \end{cases}$

The following theorem of Pigeonhole principle is very useful for our main results.

Theorem 7.1.3 [54] *"Pigeonhole principle: If m objects placed in n containers ($m > n$), then at least one container contains $\lceil \frac{m}{n} \rceil$ number of objects".*

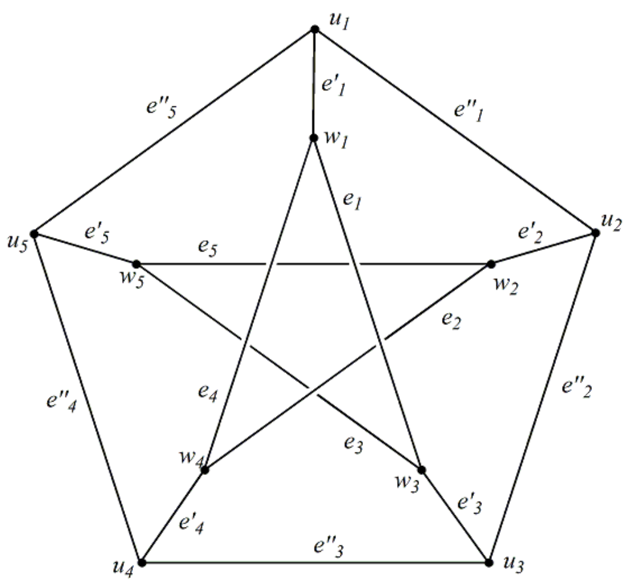


Figure 7.1: Petersen graph $P(5, 3)$

7.2 Main results

Generalized Petersen Graphs: A graph with vertex set $W = \{w_1, w_2, w_3, \ldots, w_n, u_1, u_2, u_3, \ldots, u_n\}$ and an edge set $E = \{e_k,\ e'_k,\ e''_k\}$, where $e_k = w_k w_{k+3}$, $e'_k = w_k u_k$ and $e''_k = u_k u_{k+1}$ for $1 \leq k \leq n$ as shown in Figure 7.1.We distinguished the inner and outer vertices with w_k and u_k, respectively. The $P(n, 3)$ is also know as 3-regular graph with order of $2n$ and size of $3n$. The indices (or subscripts) of vertices and edges considered as $w_{n+1} = w_1,\ w_{n+2} = w_2$ and $e_{n+1} = e_1,\ e_{n+2} = e_2$ respectively. Throughout this chapter we consider $c = \frac{n+6-r}{6}$ for $n \equiv r\ (mod\ 6)$.

A set of neighbouring edges of a vertex w at distance h, denoted by $N_h(w)$ and is defined as $N_h(w) = \{e \mid d(w, e) = h\}$, where $w \in W(G)$, $e \in E(G)$ and h is a positive integer.

We noted the following sets of neighbouring edges for a vertex w_k of $P(n, 3)$

where $1 \leq k \leq n$ and $n \geq 12$,

$$\begin{aligned}
N_1(w_k) &= \{e_{k+3}, e_{k+n-6}, e'_{k+3}, e'_{k+n-3}, e''_k, e''_{k+n-1}\}, && (7.1)\\
N_2(w_k) &= \{e'_{k+1},\ e'_{k+6}, e'_{k+n-1}, e''_{k+1}, e''_{k+2}, e''_{k+3}, e''_{k+n-4},\\
&\quad e''_{k+n-3}, e''_{k+n-2}\}, && (7.2)\\
N_3(w_k) &= \{e_{k+1},\ e_{k+n-4},\ e_{k+n-2},\ e_{k+n-1},\ e'_{k+2},\ e'_{k+4},\ e'_{k+n-4},\\
&\quad e'_{k+n-2},\ e''_{k+n-8},\ e''_{k+n-7},\ e''_{k+n-6},\ e''_{k+n-5}\}. && (7.3)
\end{aligned}$$

It is clear from Eq. (7.1) to Eq. (7.3) that, $|N_1(w)| \geq 6$, $|N_2(w)| \geq 9$ and $|N_3(w)| \geq 12$ for every $w \in W(P(n,3))$ where $n \geq 12$.

Lemma 7.2.1 *Let $G = P(n,3)$ be a graph and $|N_1(w)| \geq 6$, $|N_2(w)| \geq 9$ and $|N_3(w)| \geq 12$ for every $w \in W(P(n,3))$ where $n \geq 12$. Then $edkm(P(n,3)) \geq 4$.*

Proof. It is clear for every $w \in W(P(n,3))$, $e \in E(P(n,3))$ and $\bar{e} \in N_h(w)$ we have

$$d(w,e) - h \leq d(w,\bar{e}) \leq d(w,e) + h \qquad (7.4)$$

Suppose $edim(P(n,3)) = 3$ and the resolving set is $S = \{w_1, w_2, w_3\}$. As we know that $|N_1(w)| \geq 6$, $|N_2(w)| \geq 9$ and $|N_3(w)| \geq 12$, so we can deduce from Eq. (7.1) to Eq. (7.4) that there exist at least two edges $\bar{e}_1, \bar{e}_2 \in N_h(w_1)$ where $1 \leq h \leq 3$ such that $d(w_2,\bar{e}_1) = d(w_2,\bar{e}_2)$ or $d(w_3,\bar{e}_1) = d(w_3,\bar{e}_2)$, it is a contradiction. ■

The resolving sets of $P(n,3)$ follow congruence classes of modulo 6. So, we consider each class one by one to show that $edim(P(n,3)) = 4$ for $n \geq 12$.
For $n = 6$, one can easily verify the resolving set $S = \{w_k,\ w_{k+1},\ u_{k+2}\}$ and the *EMD* of $P(6,3)$ is 3.

Theorem 7.2.2 *For $n \geq 12$ and $n \equiv 0 \ (mod\ 6)$, the $edim(P(n,3)) = 4$.*

Proof. The following cases can be considered to prove $edim(P(n,3)) = 4$:

Case: 1. First we show for $n \geq 12$ the $edim(P(n,3)) \leq 4$ with resolving set $S = \{w_k,\ w_{k+1},\ w_{k+2},\ w_{k+3}\}$ for $1 \leq k \leq n$. The distance codes for all edges of $P(12,3)$ with respect to resolving set $S = \{w_1,\ w_2,\ w_3,\ w_4\}$ are shown in Table

7.1.

Table 7.1: Distance codes for $P(12,3)$

$d(.,.)$	w_1	w_2	w_3	w_4	$d(.,.)$	w_1	w_2	w_3	w_4	$d(.,.)$	w_1	w_2	w_3	w_4
e_1	0	3	3	0	e'_1	0	2	3	1	e''_1	1	1	2	2
e_2	3	0	3	3	e'_2	2	0	2	3	e''_2	2	1	1	2
e_3	4	3	0	3	e'_3	3	2	0	2	e''_3	2	2	1	1
e_4	1	4	3	0	e'_4	1	3	2	0	e''_4	2	2	2	1
e_5	4	1	4	3	e'_5	3	1	3	2	e''_5	3	2	2	2
e_6	4	4	1	4	e'_6	4	3	1	3	e''_6	3	3	2	2
e_7	1	4	4	1	e'_7	2	4	3	1	e''_7	3	3	3	2
e_8	4	1	4	4	e'_8	4	2	4	3	e''_8	3	3	3	3
e_9	3	4	1	4	e'_9	3	4	2	4	e''_9	2	3	3	3
e_{10}	0	3	4	1	e'_{10}	1	3	4	2	e''_{10}	2	2	3	3
e_{11}	3	0	3	4	e'_{11}	3	1	3	4	e''_{11}	2	2	2	3
e_{12}	3	3	0	3	e'_{12}	2	3	1	3	e''_{12}	1	2	2	2

It is clear from Table 7.1 that all edges e_k, e'_k and e''_k have different distance codes with respect to resolving set S. The code for e_1 w.r.t S is $C_S(e_1) = (0,3,3,0)$ and for e_4 w.r.t S is $C_S(e_4) = (1,4,3,0)$. So we can generalize the codes through some formula for $e_1$and e_4 as $C_S(e_{k+3l}) = \left(q,\ q+3,\ \left\lfloor\frac{(c-3)q}{c-2}\right\rfloor+3, \left\lfloor\frac{(c-3)q}{c-2}\right\rfloor\right)$, where $k=1$ and $0 \le q \le c-2$. We present this type of generalized formula for e_k, e'_k and e''_k in Table 7.2 to Table 7.4, respectively. We consider $b = \frac{n}{2}$ for this case.

Table 7.2: Distance code for e_i's w.r.t S of $P(n,3)$, where $n = 6t+6$, $t \in \mathbb{Z}^+$

$d(.,.)$	w_k	w_{k+1}	w_{k+2}	w_{k+3}
e_{k+3q} : $0 \le q \le c-2$	q	$q+3$	$\left\lfloor\frac{(c-3)q}{c-2}\right\rfloor+3$	$\left\lfloor\frac{(c-3)q}{c-2}\right\rfloor$
e_{k+3q-2} : $1 \le q \le c-2$	$q+2$	$q-1$	$q+2$	$\left\lfloor\frac{(c-3)q}{c-2}\right\rfloor+3$
e_{k+3q-1} : $1 \le q \le c-2$	$q+3$	$q+2$	$q-1$	$q+2$
$e_{k+n-3q-1}$: $1 \le q \le c-1$	$q+2$	$\left\lceil\frac{(c-3)(q-1)}{c-2}\right\rceil+4$	$\left\lceil\frac{(c-3)(q-1)}{c-2}\right\rceil+1$	$\left\lceil\frac{(c-3)(q-1)}{c-2}\right\rceil+4$
$e_{k+n-3q-2}$: $0 \le q \le c-2$	$q+3$	q	$q+3$	$\left\lceil\frac{(c-3)q}{c-2}\right\rceil+4$
e_{k+n-3q} : $1 \le q \le c-1$	$q-1$	$q+2$	$\left\lceil\frac{(c-3)(q-1)}{c-2}\right\rceil+4$	$\left\lceil\frac{(c-3)(q-1)}{c-2}\right\rceil+1$
e_{k+q-1}:$q=n$	3	3	0	3

The previous discussion helps us to conclude that S is a resolving set of $P(n,3)$

Table 7.3: Distance code for $e_i^{'}$'s w.r.t S of $P(n,3)$, where $n = 6t+6$, $t \in \mathbb{Z}^+$

$d(.,.)$	w_k	w_{k+1}	w_{k+2}	w_{k+3}
$e'_{k+3q} : 1 \le q \le c-2$	q	$q+2$	$\lvert q-1\rvert+2$	$\lvert q-1\rvert$
$e'_{k+3q-2} : 1 \le q \le c-1$	$q+1$	$q-1$	$q+1$	$\lvert q-2\rvert+2$
$e'_{k+3q-1} : 1 \le q \le c-1$	$q+2$	$q+1$	$q-1$	$q+1$
$e'_{k+n-3q-1} : 1 \le q \le c$	$q+2$	$q+3$	$q+1$	$q+3$
$e'_{k+n-3q-2} : 1 \le q \le c-3$	$q+3$	$q+1$	$q+3$	$q+4$
e'_{k+q+1}:$q=b$	$c+1$	$c-1$	$c+1$	c
$e'_{k+n-3q} : 1 \le q \le c-2$	q	$q+2$	$q+3$	$q+1$
e'_{k+q}:$q=b$	$c-1$	$c+1$	c	$c-2$

Table 7.4: Distance code for $e_i^{''}$'s w.r.t S of $P(n,3)$, where $n = 6t+6$, $t \in \mathbb{Z}^+$

$d(.,.)$	w_k	w_{k+1}	w_{k+2}	w_{k+3}
$e''_{k+3q} : 0 \le q \le c-1$	$q+1$	$q+1$	$\left\lfloor \frac{(c-2)q}{c-1} \right\rfloor + 2$	$\lvert q-1\rvert+1$
$e''_{k+3q} : q = c$	$q+1$	$q+1$	$\left\lfloor \frac{(c-2)q}{c-1} \right\rfloor + 3$	$\lvert q-1\rvert+1$
$e''_{k+3q-2} : 1 \le q \le c$	$q+1$	q	q	$\left\lfloor \frac{(c-2)(q-1)}{c-1} \right\rfloor + 2$
$e''_{k+3q-1} : 1 \le q \le c-1$	$q+1$	$q+1$	q	q
$e''_{n-3q+3} : 1 \le q \le c$	q	$q+1$	$q+1$	$q+1$
$e''_{n-3q+2} : 1 \le q \le c$	$q+1$	$q+1$	$q+1$	$\left\lceil \frac{(c-2)(q-1)}{c-1} \right\rceil + 3$
$e''_{n-3q+1} : 1 \le q \le c-1$	$q+1$	$q+1$	$q+2$	$q+2$

and the $edim(P(n,3)) \le 4$ for all $n \ge 12$ and $n \equiv 0 \ (mod\ 6)$.

Case: 2. The reverse inequality $edim(P(n,3)) \ge 4$ can be shown by using Theorem 7.1.3 and Lemma 7.2.1 that, there exist at least two edges $\bar{e}_1, \bar{e}_2 \in N_h(w_1)$ where $h \in \{1,2,3\}$ such that $d(w_2,\bar{e}_1) = d(w_2,\bar{e}_2)$ or $d(w_3,\bar{e}_1) = d(w_3,\bar{e}_2)$ so it is a contradiction. ■

Now we show that $edim(P(n,3)) = 4$ for $n \ge 12$ and $n \equiv 3 \ (mod\ 6)$. For $n = 9$, one can easily check the resolving set $S = \{w_k,\ w_{k+1},\ u_{k+4}\}$ and the *EMD* of $P(9,3)$ is 3.

Theorem 7.2.3 *For $n \ge 12$ and $n \equiv 3 \ (mod\ 6)$, the $edim(P(n,3)) = 4$.*

Proof. We use double inequality condition to prove $edim(P(n,3)) = 4$ in below two cases:

Case: 1. First we show that $edim(P(n,3)) \le 4$, for $n \ge 15$ with resolving set $S = \{w_k,\ w_{k+1},\ w_{k+2},\ w_{k+3}\}$, where $1 \le k \le n$. The distance codes for all edges of $P(15,3)$ with respect to resolving set $S = \{w_1,\ w_2,\ w_3,\ w_4\}$ are shown in Table 7.5.

Table 7.5: Distance codes for $P(15,3)$

$d(.,.)$	w_1	w_2	w_3	w_4	$d(.,.)$	w_1	w_2	w_3	w_4	$d(.,.)$	w_1	w_2	w_3	w_4
e_1	0	3	3	0	e'_1	0	2	3	1	e''_1	1	1	2	2
e_2	3	0	3	3	e'_2	2	0	2	3	e''_2	2	1	1	2
e_3	4	3	0	3	e'_3	3	2	0	2	e''_3	2	2	1	1
e_4	1	4	3	0	e'_4	1	3	2	0	e''_4	2	2	2	1
e_5	4	1	4	3	e'_5	3	1	3	2	e''_5	3	2	2	2
e_6	5	4	1	4	e'_6	4	3	1	3	e''_6	3	3	2	2
e_7	2	5	4	1	e'_7	2	4	3	1	e''_7	3	3	3	2
e_8	5	2	5	4	e'_8	4	2	4	3	e''_8	4	3	3	3
e_9	4	5	2	5	e'_9	4	4	2	4	e''_9	3	4	3	3
e_{10}	1	4	5	2	e'_{10}	2	4	4	2	e''_{10}	3	3	4	3
e_{11}	4	1	4	5	e'_{11}	4	2	4	4	e''_{11}	3	3	3	4
e_{12}	3	4	1	4	e'_{12}	3	4	2	4	e''_{12}	2	3	3	3
e_{13}	0	3	4	1	e'_{13}	1	3	4	2	e''_{13}	2	2	3	3
e_{14}	3	0	3	4	e'_{14}	3	1	3	4	e''_{14}	2	2	2	3
e_{15}	3	3	0	3	e'_{15}	2	3	1	3	e''_{15}	1	2	2	2

It is clear from Table 7.5 that all edges e_k, e'_k and e''_k have different distance codes with respect to resolving set S. The codes for e_1, e_4 and e_7 w.r.t S are $C_S(e_1) = (0,3,3,0)$, $C_S(e_4) = (1,4,3,0)$ and $C_S(e_7) = (2,5,4,1)$, respectively. We can generalize the codes as $C_S(e_{k+3q}) = \left(q, q+3, \lfloor \frac{3q}{4} \rfloor + 3, \lfloor \frac{3q}{4} \rfloor\right)$ where $k = 1$ and $0 \leq q \leq c-1$. We present such type of generalized formula for e_k, e'_k and e''_k in Table 7.6 to Table 7.8, respectively.

Table 7.6: Distance code for e_i's w.r.t S of $P(n,3)$, where $n = 6t+9$, $t \in \mathbb{Z}^+$

$d(.,.)$	w_k	w_{k+1}	w_{k+2}	w_{k+3}
$e_{k+3q} : 0 \leq q \leq c-1$	q	$q+3$	$\lfloor \frac{3q}{4} \rfloor + 3$	$\lfloor \frac{3q}{4} \rfloor$
$e_{k+3q-2} : 1 \leq q \leq c$	$q+2$	$q-1$	$q+2$	$\lfloor \frac{3q-3}{4} \rfloor + 3$
$e_{k+3q-1} : 1 \leq q \leq c-1$	$q+3$	$q+2$	$q-1$	$q+2$
$e_{k+n-3q-1} : 0 \leq q \leq c-1$	$\lfloor \frac{3q}{4} \rfloor$	$q+3$	q	$q+3$
$e_{k+n-3q-2} : 1 \leq q \leq c-2$	$q+3$	q	$q+3$	$q+4$
e_{k+n-3q} :$1 \leq q \leq c-1$	$q-1$	$q+2$	$q+3$	q
e_{k+q-1}: $q = n$	3	3	0	3

The preceding discussion show that S is a resolving set for $P(n,3)$ and hence the $edim(P(n,3)) \leq 4$ for all $n \geq 15$ and $n \equiv 3\ (mod\ 6)$.

Table 7.7: Distance code for $e_i^{'}$'s w.r.t S of $P(n,3)$, where $n = 6t+9$, $t \in \mathbb{Z}^+$

$d(.,.)$	w_k	w_{k+1}	w_{k+2}	w_{k+3}
$e'_{k+3q} : 0 \leq q \leq c-1$	q	$q+2$	$\lvert q-1\rvert+2$	$\lvert q-1\rvert$
$e'_{k+3q-2} : 1 \leq q \leq c$	$q+1$	$q-1$	$q+1$	$\lvert q-2\rvert+2$
$e'_{k+3q-1} : 1 \leq q \leq c-1$	$q+2$	$q+2$	$q-1$	$q+1$
$e'_{k+n-3q-1} : 0 \leq q \leq c-1$	$q+2$	$\left\lceil \frac{(c-2)q}{c-1} \right\rceil+3$	$\left\lceil \frac{(c-2)q}{c-1} \right\rceil+1$	$\left\lceil \frac{(c-2)q}{c-1} \right\rceil+3$
$e'_{k+n-3q-2} : 0 \leq q \leq c-3$	$q+3$	$q+1$	$q+3$	$q+4$
$e'_{k+n-3q-2} : q = c-2$	$q+3$	$q+1$	$q+3$	$q+3$
$e'_{k+n-3q} : 1 \leq q \leq c-2$	q	$q+2$	$q+3$	$q+1$
$e'_{k+n-3q} : q = c-1$	q	$q+2$	$c+1$	$c+2$

Table 7.8: Distance code for $e_i^{''}$'s w.r.t S of $P(n,3)$, where $n = 6t+9$, $t \in \mathbb{Z}^+$

$d(.,.)$	w_k	w_{k+1}	w_{k+2}	w_{k+3}
$e''_{k+3q} : 1 \leq q \leq c-1$	$q+1$	$q+1$	$\left\lfloor \frac{(c-2)q}{c-1} \right\rfloor+2$	$\lvert q-1\rvert+1$
$e''_{k+3q-2} : 1 \leq q \leq c$	$q+1$	q	q	$\left\lfloor \frac{(c-2)q}{c-1} \right\rfloor+2$
$e''_{k+3q-1} : 1 \leq q \leq c-1$	$q+1$	$q+1$	q	q
$e''_{k+n-3q+2} : 1 \leq q \leq c$	q	$q+1$	$\left\lceil \frac{(c-2)(q-1)}{c-1} \right\rceil+2$	$\left\lceil \frac{(c-2)(q-1)}{c-1} \right\rceil+2$
$e''_{k+n-3q+1} : 1 \leq q \leq c-1$	$q+1$	$q+1$	$q+1$	$\left\lfloor \frac{(c-2)(q-1)}{c-1} \right\rfloor+3$
$e''_{k+n-3q}:1 \leq q \leq c-2$	$q+1$	$q+1$	$q+2$	$q+2$
$e''_{k+n-3q} : q = c-1$	$q+1$	$q+1$	$q+2$	$q+1$

Case: 2. The reverse inequality $edim(P(n,3)) \geq 4$ can be shown by using Theorem 7.1.3 and Lemma 7.2.1 that there exist at least two edges $\bar{e}_1, \bar{e}_2 \in N_h(w_1)$ where $h \in \{1,2,3\}$ such that $d(w_2,\bar{e}_1) = d(w_2,\bar{e}_2)$ or $d(w_3,\bar{e}_1) = d(w_3,\bar{e}_2)$ so it is a contradiction. ■

Now we consider the case for $n \equiv 1 \ (mod\ 6)$ and $n \geq 12$ to prove that the $edim(P(n,3)) = 4$.

For $n = 7$, one can easily see that $S = \{w_k,\ w_{k+1}, w_{k+2},\ u_{k+4}\}$ is a resolving set of $P(7,3)$ with 4 elements, so the EMD of $P(7,3)$ is 4.

Theorem 7.2.4 *For $n \geq 12$ and $n \equiv 1 \ (mod\ 6)$, the $edim(P(n,3)) = 4$.*

Proof. We use double inequality condition in below two cases to prove $edim(P(n,3)) = 4$.

Case: 1. For $n \geq 13$, the resolving set is $S = \{w_k,\ w_{k+1},\ w_{k+2},\ u_{k+4}\}$, where $1 \leq k \leq n$ and the EMD of $P(n,3) \leq 4$. The set S consist of four vertices, three of them belong to inner cycle (star polygon) and one belongs to outer cycle. The distance codes for all edges of $P(13,3)$ with respect to resolving set $S = \{w_1,\ w_2,\ w_3,\ u_5\}$ are shown in Table 7.9.

Table 7.9: Distance codes for $P(13,3)$

$d(.,.)$	w_1	w_2	w_3	u_5	$d(.,.)$	w_1	w_2	w_3	u_5	$d(.,.)$	w_1	w_2	w_3	u_5
e_1	0	3	3	2	e'_1	0	2	3	3	e''_1	1	1	2	3
e_2	3	0	3	1	e'_2	2	0	2	2	e''_2	2	1	1	2
e_3	4	3	0	2	e'_3	3	2	0	2	e''_3	2	2	1	1
e_4	1	4	3	2	e'_4	1	3	2	1	e''_4	2	2	2	0
e_5	2	1	4	1	e'_5	3	1	3	0	e''_5	3	2	2	0
e_6	5	2	1	2	e'_6	4	3	1	1	e''_6	3	3	2	1
e_7	2	5	2	3	e'_7	2	4	3	2	e''_7	3	3	3	2
e_8	1	2	5	2	e'_8	2	2	4	2	e''_8	3	3	3	3
e_9	3	0	3	2	e'_9	4	2	2	3	e''_9	3	3	3	4
e_{10}	3	4	1	4	e'_{10}	3	4	2	4	e''_{10}	2	3	3	4
e_{11}	0	3	4	3	e'_{11}	1	3	4	3	e''_{11}	2	2	3	4
e_{12}	3	0	3	2	e'_{12}	3	1	3	3	e''_{12}	1	2	2	4
e_{13}	3	3	0	3	e'_{13}	2	3	1	4	e''_{13}	2	2	2	4

It is clear from Table 7.9 that all edges e_k, e'_k and e''_k have different distance codes with respect to S. We observe that the distance code of consecutive vertices of the inner cycle (star polygon) have a pattern, the code move diagonally downward in next row and next column. For example in Table 7.9. the distance code of e_1 w.r.t w_1 is $d(e_1,w_1) = 0$, the code move diagonally downward to next row and next column for e_2 w.r.t w_2 as $d(e_2,w_2) = 0$ and similarly, e_3 w.r.t w_3 is $d(e_3,w_3) = 0$. So, we generalize the distance codes for all edges w.r.t vertex w_1 and u_5 only. After that, the distance code for w_2 and w_3 can be generalized by using the scheme of diagonally downward to the next row and column.

The codes $C_S(e_1) = (0,3,3,2)$, $C_S(e_4) = (1,4,3,2)$, $C_S(e_7) = (2,5,2,3)$ and $C_S(e_{11}) = (0,3,4,3)$ can be generalize as

$$C_S(e_{k+3q-3}) = (d(e_{k+3q-3}, w_k),\ d(e_{k+3q-3}, w_{k+1}),\ d(e_{k+3q-3}, w_{k+2}),\ d(e_{k+3q-3}, u_{k+4})),$$

where $1 \leq k \leq n$. We present the generalized distance code for e_k, e'_k and e''_k with respect to vertex w_k and vertex u_{k+4} in Table 7.10 to Table 7.12, respectively. We consider $b = \frac{n-1}{2}$ for this case.

Table 7.10: Distance code for e_i's w.r.t v_i and u_{i+4} of $P(n,3)$, where $n = 6t + 7$, $t \in \mathbb{Z}^+$

$d(.,.)$	w_k	$d(.,.)$	u_{k+4}
$e_{k+3q-3} : 1 \le q \le \frac{b}{3} + 2$	$q-1$	$e_{k+3(q-1)} : 1 \le q \le \frac{b}{3} - 1$	$c-q$
$e_{k+3q-2} : 1 \le q \le \frac{b}{3} - 2$	$q+2$	$e_{k+3q-2} : 1 \le q \le \frac{b}{3} - 1$	$c-q-1$
$e_{k+3q-2} : \frac{b}{3} - 1 \le q \le \frac{2b}{3}$	$\frac{2b}{3} - q$	$e_{k+3q-1}: 1 \le q \le \frac{b}{3} - 1$	$c-q$
$e_{k+3q-1} : 1 \le q \le \frac{b}{3} - 1$	$q+3$	$e_{k+3q+3c-9}: 1 \le q \le \frac{b}{3} + 1$	$q+1$
$e_{k+3q-1} : \frac{b}{3} \le q \le \frac{2b}{3}$	$\frac{2b}{3} - q + 3$	$e_{k+3q+3c-8}: 1 \le q \le \frac{b}{3} + 1$	q
$e_{k+n-1} : 1 \le q \le c-q$	$c-q$	$e_{k+3q+3c-7} : 1 \le q \le \frac{b}{3}$	$q+1$
$ed_{k+n-1} : q = n$	3	$e_{k+n+q-3} : 1 \le q \le 2$	$c+q-2$

The Table 7.10 is used to show the code representation of e_k with respect to the vertices w_{k+1} and w_{k+2} as $d(e_k, w_{k+1}) = d(e_{k-1}, w_k)$, $d(e_k, w_{k+2}) = d(e_{k-2}, w_k)$, respectively. The Table 7.11 is used to present the code representation of e'_k with respect to w_{k+1} and w_{k+2} as $d(e'_k, w_{k+1}) = d(e'_{k-1}, w_k)$, $d(e'_k, w_{k+2}) = d(e'_{k-2}, w_k)$, respectively. The code representation of e''_k with respect to w_{k+1} and w_{k+2} presented as $d(e''_k, w_{k+1}) = d(e''_{k-1}, w_k)$, $d(e''_k, w_{k+2}) = d(e''_{k-2}, w_k)$, respectively in Table 7.12. It is proved that S is a resolving set of $P(n,3)$ and the $edim(P(n,3)) \le 4$ for all $n \ge 13$ and $n \equiv 1 \ (mod\ 6)$.

Case: 2. The reverse inequality $edim(P(n,3)) \ge 4$ can be shown by using Theorem 7.1.3 and Lemma 7.2.1 that there exist at least two edges $\bar{e}_1, \bar{e}_2 \in N_h(w_1)$ where $h \in \{1,2,3\}$ such that $d(w_2, \bar{e}_1) = d(w_2, \bar{e}_2)$ or $d(w_3, \bar{e}_1) = d(w_3, \bar{e}_2)$ so it is a contradiction. ■

Now we consider modulo class $n \equiv\ 4\ (mod\ \ 6)$ and $n \ge 12$ to show $edim(P(n,\ 3)) = 4$.

For $n = 10$, one can easily see that $S = \{w_k,\ w_{k+1},\ u_{k+2}\}$ is resolving set and the *EMD* of $P(10,3) = 3$.

Theorem 7.2.5 *For $n \ge 12$ and $n \equiv 4\ (mod\ 6)$, the $edim(P(n,3)) = 4$.*

Proof. We show this theorem by using double inequality condition in the below different cases:

Case: 1. For $n \ge 16$, the resolving set is $S = \{w_k,\ w_{k+1},\ w_{k+2},\ w_{k+3}\}$, where $1 \le k \le n$ and the *EMD* of $P(n,3) \le 4$. In this modulo class, the four consecutive vertices of S are on inner cycle of the $P(n,3)$. The distance codes for all edges of $P(16,3)$ with respect to resolving set $S = \{w_1,\ w_2,\ w_3,\ w_4\}$ are shown in Table 7.13. We prove this theorem as we have proved the Theorem 7.2.4.

Table 7.11: Distance code for $e_i^{'}$'s w.r.t v_i and u_{i+4} of $P(n,3)$, where $n = 6t+7$, $t \in \mathbb{Z}^+$

$d(.,.)$	w_k	$d(.,.)$	u_{k+4}
$e'_{k+3(q-1)}: 1 \le q \le \frac{b}{3}+2$	$q-1$	$e'_{k+3(q-1)}: 1 \le q \le \frac{b}{3}-1$	$c-q+1$
$e'_{k+3q-2}: 1 \le q \le \frac{b}{3}$	$q-1$	e'_{k+3q-2}: $1 \le q \le \frac{b}{3}-1$	$c-q$
$e'_{k+3q-1}: 1 \le q \le \frac{b}{3}$	$q+2$	e'_{k+3q-1}: $1 \le q \le \frac{b}{3}-2$	$c-q+1$
$e'_{k+3q+3c-4}: 1 \le q \le \frac{b}{3}$	$c-q+2$	$e'_{k+3q+3c-10}$: $1 \le q \le 2$	$3-q$
$e'_{k+3q+3c-5}: 1 \le q \le \frac{b}{3}-1$	$c-q$	$e'_{k+3q+3c-9}$: $1 \le q \le 2$	q
$e'_{k+3q+3c}: 1 \le q \le \frac{b}{3}-1$	$c-q$	e'_{k+q-2}: $q=1$	0
		$e'_{k+3q+3c-5}$: $1 \le q \le \frac{b}{3}$	$q+1$
		$e'_{k+3q+3c-4}$: $2 \le q \le \frac{b}{3}$	$q+2$
		$e'_{k+3q+3c-6}$: $2 \le q \le \frac{b}{3}$	$q+2$
		$e'_{k+n+q-3}$: $1 \le q \le 2$	$c+q-1$

It is clear from Table 7.13 that all edges e_k, e'_k and e''_k have different distance codes with respect to S. Due to the consecutive vertices we generalize code for all edges w.r.t vertex w_1 only, after that we define the relations for remaining vertices of the resolving set, w_2, w_3 and w_4. We present the generalized distance codes for e_k, e'_k and e''_k with respect to vertex w_k in Table 7.14 to Table 7.16, respectively. We consider $b = \frac{n}{2} - 2$ for this case.

Table 7.12: Distance code for $e_i^{''}$'s w.r.t v_i and u_{i+4} of $P(n,3)$, where $n = 6t+7$, $t \in \mathbb{Z}^+$

$d(.,.)$	w_k	$d(.,.)$	u_{k+4}
$e''_{k+3q-3}: 1 \le q \le \frac{b}{3}+1$	q	$e''_{k+q+3c-10}: 1 \le q \le 4$	$4-q$
$e''_{k+3q-2}: 1 \le q \le \frac{b}{3}$	$q+1$	$e''_{k+q+3c-6}: 1 \le q \le 4$	$q-1$
$e''_{k+3q-1}: 1 \le q \le \frac{b}{3}$	$q+1$	$e''_{k+3q+t-13}:$ $1 \le q \le c,\ 1 \le t \le 3$	$c-q+4$
$e''_{k+3q+3c-4}: 1 \le q \le \frac{b}{3}$	$c-q+1$	$e''_{k+3q+3c+t-14}:$ $1 \le q \le c+1,\ 1 \le t \le 3$	q
$e''_{k+3q+3c-5}: 1 \le q \le \frac{b}{3}$	$c-q+1$	$e''_{k+n+q-3}: 1 \le q \le 2$	$c+1$
$e''_{k+3q+3c}: 0 \le q \le \frac{b}{3}-1$	$c-q-1$		

The Table 7.14 is used to show the code representation of e_k with respect to the remaining vertices w_{k+1},w_{k+2} and w_{k+3} of S as $d(e_k, w_{k+1}) = d(e_{k-1}, w_k)$, $d(e_k, w_{k+2}) = d(e_{k-2}, w_k)$ and $d(e_k, w_{k+3}) = d(e_{k-3}, w_k)$, respectively. The Table

Table 7.13: Distance codes for $P(16,3)$

$d(.,.)$	w_1	w_2	w_3	w_4	$d(.,.)$	w_1	w_2	w_3	w_4	$d(.,.)$	w_1	w_2	w_3	w_4
e_1	0	3	3	0	e'_1	0	2	3	1	e''_1	1	1	2	2
e_2	3	0	3	3	e'_2	2	0	2	3	e''_2	2	1	1	2
e_3	4	3	0	3	e'_3	3	2	0	2	e''_3	2	2	1	1
e_4	1	4	3	0	e'_4	1	3	2	0	e''_4	2	2	2	1
e_5	3	1	4	3	e'_5	3	1	3	2	e''_5	3	2	2	2
e_6	5	3	1	4	e'_6	4	3	1	3	e''_6	3	3	2	2
e_7	2	5	3	1	e'_7	2	4	3	1	e''_7	3	3	3	2
e_8	2	2	5	3	e'_8	3	2	4	3	e''_8	4	3	3	3
e_9	5	2	2	5	e'_9	5	3	2	4	e''_9	4	4	3	3
e_{10}	3	5	2	2	e'_{10}	3	5	3	2	e''_{10}	3	4	4	3
e_{11}	1	3	5	2	e'_{11}	2	3	5	3	e''_{11}	3	3	4	4
e_{12}	4	1	3	5	e'_{12}	4	2	3	5	e''_{12}	3	3	3	4
e_{13}	3	4	1	3	e'_{13}	3	4	2	3	e''_{13}	2	3	3	3
e_{14}	0	3	4	1	e'_{14}	1	3	4	2	e''_{14}	2	2	3	3
e_{15}	3	0	3	4	e'_{15}	3	1	3	4	e''_{15}	2	2	2	3
e_{16}	3	3	0	3	e'_{16}	2	3	1	3	e''_{16}	1	2	2	2

Table 7.14: Distance code for e_i's w.r.t v_i of $P(n,3)$, where $n=6t+10$, $t\in\mathbb{Z}^+$

$d(.,.)$	w_k	$d(.,.)$	w_k
$e_{k+3(q-1)} : 1\le q\le \frac{b}{3}+1$	$q-1$	$e_{k+3q+3c-5} : 1\le q\le \frac{b}{3}+1$	$c-q$
$e_{k+3q-2} : 1\le q\le \frac{b}{3}-1$	$q+2$	$e_{k+q} : q=b+3$	c
$e_{k+3q-1} : 1\le q\le \frac{b}{3}$	c	$e_{k+3q+3c} : 1\le q\le \frac{b}{3}-1$	$c-q+1$
$e_{k+q} : q=b-2$	c	$e_{k+q-1} : q=n$	3
$e_{k+3q+3c-4} : 1\le q\le \frac{b}{3}+1$	$c-q+3$		

7.15 is used to present the code representation of e'_k with respect to w_{k+1},w_{k+2} and w_{k+3} of S as $d(e'_k,w_{k+1})=d(e'_{k-1},w_k)$, $d(e'_k,w_{k+2})=d(e'_{k-2},w_k)$ and $d(e'_k,w_{k+3})=d(e'_{k-3},w_k)$, respectively. The code representation of e''_k with respect to w_{k+1},w_{k+2} and w_{k+3} of S are $d(e''_k,w_{k+1})=d(e''_{k-1},w_k)$, $d(e''_k,\ w_{k+2})=d(e''_{k-2},\ w_k)$ and $d(e''_k,w_{k+3})\quad=d(e''_{k-3},w_k)$, respectively are presented in Table 7.16.
It is proved that S is a resolving set of $P(n,3)$ and the $edim(P(n,3))\le 4$ for all $n\ge 16$ and $n\equiv 4\ (mod\ 6)$.

Case: 2. The reverse inequality $edim(P(n,3))\ge 4$ can be proved by using Theorem 7.1.3 and Lemma 7.2.1 that there exist at least two edges $\bar{e}_1,\bar{e}_2\in N_h(w_1)$ where $h\in\{1,2,3\}$ such that $d(w_2,\bar{e}_1)=d(w_2,\bar{e}_2)$ or $d(w_3,\bar{e}_1)=d(w_3,\bar{e}_2)$ so it is a contradiction. ∎

Now we consider the modulo class $n\equiv 5\ (mod\ 6)$ and $n\ge 17$.

For $n = 11$, one can easily verify that the $S = \{w_k,\ w_{k+1},\ w_{k+2},\ u_{k+3}\}$ is resolving set and $edim(P(11,3)) = 4$.

Table 7.15: Distance code for $e_i^{'}$'s w.r.t v_i of $P(n,3)$, where $n = 6t+10$, $t \in \mathbb{Z}^+$

$d(.,.)$	w_k	$d(.,.)$	w_k
$e'_{k+3(q-1)} : 1 \le q \le \frac{b}{3}+2$	$q-1$	$e'_{k+3q+3c-4} : 1 \le q \le \frac{b}{3}+1$	$c-q+3$
$e'_{k+3q-2} : 1 \le q \le \frac{b}{3}$	$q+1$	$e'_{k+3q+3c-5} : 1 \le q \le \frac{b}{3}+1$	$c-q+1$
$e'_{k+3q-1} : 1 \le q \le \frac{b}{3}+1$	$q+2$	$e'_{k+3q+3c} : 1 \le q \le \frac{b}{3}$	$c-q+1$

Theorem 7.2.6 *For $n \ge 12$ and $n \equiv 5$ (mod 6), the $edim(P(n,3)) = 4$.*

Proof. The double inequality condition we use to prove that $edim(P(n,3)) \le 4$ and $edim(P(n,3)) \ge 4$ in the below different cases:

Case: 1. For $n \ge 17$, the resolving set is $S = \{w_k,\ w_{k+1},\ w_{k+2},\ u_{k+5}\}$, where $1 \le k \le n$ and the *EMD* of $P(n,3) \le 4$. In this modulo class, there are three consecutive vertices on inner cycle and one vertex belongs to outer cycle of $P(n,3)$. The distance codes for all edges of $P(17,3)$ with respect to resolving set $S = \{w_1,\ w_2,\ w_3,\ u_6\}$ are shown in Table7.17. We prove this theorem as we have proved the Theorem 7.2.4.

It is clear from Table 7.17 that all edges e_k, e'_k and e''_k have different distance codes with respect to S. We generalize distance codes for all edges to only vertex w_1 and u_6 then by using modulo class we define the relations for remaining vertices of the resolving set, w_2 and w_3. We present the generalized distance codes for e_k, e'_k and e''_k with respect to vertex w_k and vertex u_{k+5} in Table 7.18 to 7.20, respectively. We consider $b = \frac{n-1}{2}$ for this case.

Table 7.16: Distance code for $e_i^{''}$'s w.r.t v_i of $P(n,3)$, where $n = 6t+10$, $t \in \mathbb{Z}^+$

$d(.,.)$	w_k	$d(.,.)$	w_k
$e''_{k+3(q-1)} : 1 \le q \le \frac{b}{3}+1$	q	$e''_{k+3q+3c-4} : 1 \le q \le \frac{b}{3}+1$	$c-q+2$
$e''_{k+3q-2} : 1 \le q \le \frac{b}{3}+1$	$q+1$	$e''_{k+3q+3c-5} : 2 \le q \le \frac{b}{3}+1$	$c-q+2$
$e''_{k+3q-1} : 1 \le q \le \frac{b}{3}+1$	$q+1$	$e''_{k+3q+3c} : 0 \le q \le \frac{b}{3}$	$c-q$

The Table 7.18 is used to present the code representation of e_k with respect to the remaining vertices w_{k+1} and w_{k+2} of S as $d(e_k, w_{k+1}) = d(e_{k-1}, w_k)$, $d(e_k, w_{k+2}) = d(e_{k-2}, w_k)$, respectively. The Table 7.19 is used to show the code

Table 7.17: Distance codes for $P(17,3)$

$d(.,.)$	w_1	w_2	w_3	u_6	$d(.,.)$	w_1	w_2	w_3	u_6	$d(.,.)$	w_1	w_2	w_3	u_6
e_1	0	3	3	3	e'_1	0	2	3	4	e''_1	1	1	2	4
e_2	3	0	3	2	e'_2	2	0	2	3	e''_2	2	1	1	3
e_3	4	3	0	1	e'_3	3	2	0	2	e''_3	2	2	1	2
e_4	1	4	3	2	e'_4	1	3	2	2	e''_4	2	2	2	1
e_5	4	1	4	2	e'_5	3	1	3	1	e''_5	3	2	2	0
e_6	3	4	1	1	e'_6	4	3	1	0	e''_6	3	3	2	0
e_7	2	3	4	2	e'_7	2	4	3	1	e''_7	3	3	3	1
e_8	5	2	3	3	e'_8	4	2	4	2	e''_8	4	3	3	2
e_9	2	5	2	2	e'_9	3	4	2	2	e''_9	4	4	3	3
e_{10}	3	2	5	3	e'_{10}	3	3	4	3	e''_{10}	4	4	4	4
e_{11}	4	3	2	4	e'_{11}	4	3	3	4	e''_{11}	3	4	4	4
e_{12}	1	4	3	3	e'_{12}	2	4	3	3	e''_{12}	3	3	4	4
e_{13}	4	1	4	4	e'_{13}	4	2	4	4	e''_{13}	3	3	3	5
e_{14}	3	4	1	3	e'_{14}	3	4	2	4	e''_{14}	2	3	3	5
e_{15}	0	3	4	4	e'_{15}	1	3	4	4	e''_{15}	2	2	3	5
e_{16}	3	0	3	3	e'_{16}	3	1	3	4	e''_{16}	2	2	2	4
e_{17}	3	3	0	2	e'_{17}	2	3	1	3	e''_{17}	1	2	2	4

Table 7.18: Distance code for e_i's w.r.t v_i and u_{i+5} of $P(n,3)$, where $n = 6t+11$, $t \in \mathbb{Z}^+$

$d(.,.)$	w_k	$d(.,.)$	u_{k+5}
$e_{k+3q-3} : 1 \le q \le \frac{b}{3}+3$	$q-1$	$e_{k+q-1} : 1 \le q \le 3$	$4-q$
$e_{k+3q-2} : 1 \le q \le \frac{b}{3}+1$	$q+2$	$e_{k+q+2} : 1 \le q \le 2$	2
$e_{k+3q-1} : 1 \le q \le \frac{b}{3}-2$	$q+3$	$e_{k+3q+2} : 1 \le q \le c+1$	q
$e_{k+3q+3c-10} : 1 \le q \le c+1$	$c-q+2$	$e_{k+q+3} : 1 \le q \le c$	$q+1$
$e_{k+3q+3c-2} : 1 \le q \le c-1$	$c-q+2$	e_{k+3q+4}: $1 \le q \le c-1$	$q+2$
$e_{k+3q+3c+3} : 1 \le q \le \frac{b}{3}-2$	$c-q+1$	$e_{k+3q+3c+1}$: $1 \le q \le \frac{b}{3}$	$c-q+1$
$e_{k+n+q-3} : 1 \le q \le 2$	3	$e_{k+3q+3c+3}$: $1 \le q \le c-2$	$c-q+1$
		$e_{k+3q+3c+5}$: $1 \le q \le c-3$	$c-q+1$

representation of e'_k with respect to w_{k+1} and w_{k+2} as $d(e'_k, w_{k+1}) = d(e'_{k-1}, w_k)$, $d(e'_k, w_{k+2}) = d(e'_{k-2}, w_k)$, respectively. The code representation of e''_k with respect to w_{k+1} and w_{k+2} are $d(e''_k, w_{k+1}) = d(e''_{k-1}, w_k)$, $d(e''_k, w_{k+2}) = d(e''_{k-2}, w_k)$, respectively shown in Table 7.20.

We proved that the S is a resolving set of $P(n,3)$ and the $edim(P(n,3)) \le 4$ for all $n \ge 17$ and $n \equiv 5 \ (mod\ 6)$.

Case: 2. The reverse inequality $edim(P(n,3)) \ge 4$ is state forward from Theorem 7.1.3 and Lemma 7.2.1 that there exist at least two edges $\bar{e}_1, \bar{e}_2 \in N_h(w_1)$

where $h \in \{1, 2, 3\}$ such that $d(w_2, \bar{e}_1) = d(w_2, \bar{e}_2)$ or $d(w_3, \bar{e}_1) = d(w_3, \bar{e}_2)$ so it is a contradiction. ■

Now we consider $n \equiv 2\ (mod\ 6)$ to show that $edim(P(n, 3)) = 4$.

For $n = 8$ and 14 one can verify that resolving sets are $S = \{w_k,\ w_{k+1},\ w_{k+2},\ u_{k+3}\}$ and $S = \{w_k,\ w_{k+1},\ u_{k+2},\ u_{k+5}\}$, respectively; and the *EMD* of $P(n, 3)$ is 4.

Table 7.19: Distance code for $e_i^{'}$'s w.r.t v_i and u_{i+5} of $P(n, 3)$, where $n = 6t + 11$, $t \in \mathbb{Z}^+$

$d(.,.)$	w_k	$d(.,.)$	u_{k+5}
e'_{k+3q-3} : $1 \le q \le \frac{b}{3} + 3$	$q - 1$	$e'_{k+q+3t-1} : 0 \le t \le 1$, $1 \le q \le 3$	$5 - q - 2t$
e'_{k+3q-2} : $1 \le q \le \frac{b}{3} + 1$	$q + 1$	$e'_{k+q+6} : 1 \le q \le 2$	2
e'_{k+3q-1} : $1 \le q \le \frac{b}{3}$	$q + 2$	$e'_{k+q-1} : q = 7$	1
$e'_{k+3q+3c-4}$: $1 \le q \le c$	$c - q + 1$	$e'_{k+q+3t+7} : 1 \le t \le c - 2,$ $-1 \le q \le 1$	$3 + t - \lvert q \rvert$
$e'_{k+3q+3c-2}$: $1 \le q \le c$	$c - q + 2$	$e'_{k+q+3c+2} : 1 \le q \le 4$	$c + 1$
$e'_{k+3q+3c+3}$: $1 \le q \le c - 2$	$c - q + 1$	$e'_{k+q+3c+3t+5} : 1 \le t \le c - 3,$ $-1 \le q \le 1$	$c + 2 - t - \lvert q \rvert$
		$e'_{k+q-1} : q = n$	3

Theorem 7.2.7 *For $n \ge 12$ and $n \equiv 2\ (mod\ 6)$, the $edim(P(n, 3)) = 4$.*

Proof. By using double inequality condition we prove this theorem in the following cases:

Case: 1. For $n \ge 20$, the resolving set $S = \{w_k,\ w_{k+1},\ w_{k+2},\ u_{k+\frac{n}{2}-2}\}$, where $1 \le k \le n$ and the *EMD* of $P(n, 3) \le 4$. There are three consecutive vertices on inner cycle and one vertex belongs to outer cycle of $P(n, 3)$. The distance codes for all edges of $P(20, 3)$ with respect to resolving set $S = \{w_1,\ w_2,\ w_3,\ u_9\}$ are shown in Table 7.21. We prove this theorem in the same way as we have proved the Theorem 7.2.4. We consider $b = \frac{n}{2}$ for this case.

It is clear that all edges e_k, e'_k and e''_k have different distance codes with respect to S. We generalize distance codes for all edges to only vertex w_1 and u_6 then by using modulo class we define the relations for remaining vertices of the resolving set, w_2 and w_3. We present the generalized distance codes formula for e_k, e'_k and

e''_k with respect to vertex w_k and vertex $u_{k+\frac{n}{2}-2}$ of $P(n,3)$, where $n = 6t+14$, $t \in \mathbb{Z}^+$ in Table 7.22 to Table 7.24, respectively.

Table 7.20: Distance code for e''_i's w.r.t v_i and u_{i+5} of $P(n,3)$, where $n = 6t+11$, $t \in \mathbb{Z}^+$

$d(.,.)$	w_k	$d(.,.)$	u_{k+5}
$e''_{k+q-1} : q = 1$	$q+1$	$e''_{k+q-1} : 1 \leq q \leq 5$	$5-q$
$e''_{k+3q+t-3} : 1 \leq t \leq 3$, $1 \leq q \leq \frac{b}{3}+1$	$q+1$	$e''_{k+q+4} : 1 \leq q \leq 4$	$q-1$
$e''_{k+3q+3c+t-1} : 1 \leq t \leq 3$, $1 \leq q \leq \frac{b}{3}$	$c-q+1$	$e''_{k+3q+t+5} : 1 \leq t \leq 3$, $1 \leq q \leq c-1$	$q+3$
$e''_{k+q-1} : q = n$	1	$e''_{k+3q+3c+t+2} : 1 \leq t \leq 3$, $1 \leq q \leq c-3$	$c-q+2$
		$ed''_{k+n+q-3} : 1 \leq q \leq 2$	4

Table 7.21: Distance codes for $P(20,3)$

$d(.,.)$	w_1	w_2	w_3	u_9	$d(.,.)$	w_1	w_2	w_3	u_9	$d(.,.)$	w_1	w_2	w_3	u_9
e_1	0	3	3	4	e'_1	0	2	3	5	e''_1	1	1	2	5
e_2	3	0	3	3	e'_2	2	0	2	4	e''_2	2	1	1	4
e_3	4	3	0	2	e'_3	3	2	0	3	e''_3	2	2	1	4
e_4	1	4	3	3	e'_4	1	3	2	4	e''_4	2	2	2	4
e_5	4	1	4	2	e'_5	3	1	3	3	e''_5	3	2	2	3
e_6	4	4	1	1	e'_6	4	3	1	2	e''_6	3	3	2	2
e_7	2	4	4	2	e'_7	2	4	3	2	e''_7	3	3	3	1
e_8	5	2	4	2	e'_8	4	2	4	1	e''_8	4	3	3	0
e_9	3	5	2	1	e'_9	4	4	2	0	e''_9	4	4	3	0
e_{10}	3	3	5	2	e'_{10}	3	4	4	1	e''_{10}	4	4	4	1
e_{11}	5	3	3	3	e'_{11}	5	3	4	2	e''_{11}	4	4	4	2
e_{12}	2	5	3	2	e'_{12}	3	5	3	2	e''_{12}	4	4	4	3
e_{13}	4	2	5	3	e'_{13}	4	3	5	3	e''_{13}	4	4	4	4
e_{14}	4	4	2	4	e'_{14}	4	4	3	4	e''_{14}	3	4	4	4
e_{15}	1	4	4	3	e'_{15}	2	4	4	3	e''_{15}	3	3	4	4
e_{16}	4	1	4	4	e'_{16}	4	2	4	4	e''_{16}	3	3	3	5
e_{17}	3	4	1	4	e'_{17}	3	4	2	5	e''_{17}	2	3	3	5
e_{18}	0	3	4	4	e'_{18}	1	3	4	4	e''_{18}	2	2	3	5
e_{19}	3	0	3	4	e'_{19}	3	1	3	5	e''_{19}	2	2	2	5
e_{20}	3	3	0	3	e'_{20}	2	3	1	4	e''_{20}	1	2	2	5

The Table 7.22 contains the code representation of e_k with respect to the remaining vertices w_{k+1} and w_{k+2} of S as $d(e_k, w_{k+1}) = d(e_{k-1}, w_k)$, $d(e_k, w_{k+2}) = d(e_{k-2}, w_k)$, respectively. The Table 7.23 is used to show the code representation of e'_k with respect to w_{k+1} and w_{k+2} are $d(e'_k, w_{k+1}) = d(e'_{k-1}, w_k)$, $d(e'_k, w_{k+2}) =$

Table 7.22: Distance codes for e_i's w.r.t vertex v_i and vertex $u_{i+(n/2)-2}$ of $P(n,3)$, where $n = 6t + 14$

$d(.,.)$	w_k	$d(.,.)$	$u_{k+\frac{n}{2}-2}$
$e_{k+3q-3} : 1 \leq q \leq \frac{b}{3}+2$	$q-1$	$e_{k+3q+t-10}$: $1 \leq t \leq 3, 3 \leq q \leq c$	$c-q-t+4$
$e_{k+3q-2} : 1 \leq q \leq \frac{b}{3}-1$	$q+2$	$e_{k+q+3c-7} : 1 \leq q \leq 2$	2
$e_{k+3q-1} : 1 \leq q \leq c-3$	$q+3$	$e_{k+3q+3c+t-14}$: $1 \leq t \leq 3, 3 \leq q \leq c$	$q-2$
$e_{k+q+3c-8} : q = 1$	c	$e_{k+6q-4} : q = c-1$	$c-1$
$e_{k+3q+1} : 1 \leq q \leq \frac{b}{3}-1$	$q+3$	$e_{k+q+6c-10}$: $1 \leq q \leq 4$	c
$e_{k+3q+3c-5} : 1 \leq q \leq \frac{b}{3}$	$c-q+2$	e_{k+q-1}: $q = n$	$c-1$
$e_{k+3q+3c-7} : 1 \leq q \leq \frac{b}{3}+1$	$c-q$		
$e_{k+n+q-3} : 1 \leq q \leq 2$	3		

Table 7.23: Distance codes for $e_i^{'}$'s w.r.t vertex v_i and vertex $u_{i+(n/2)-2}$ of $P(n,3)$, where $n = 6t + 14$

$d(.,.)$	w_k	$d(.,.)$	$u_{k+\frac{n}{2}-2}$
e'_{k+3q-3} : $1 \leq q \leq \frac{b}{3}+2$	$q-1$	e'_{k+3q-3} : $1 \leq q \leq \frac{b}{3}-1$	$c-q+2$
e'_{k+3q-2} : $1 \leq q \leq \frac{b}{3}+1$	$q+1$	$e'_{k+3q-2} : 1 \leq q \leq \frac{b}{3}-1$	$c-q+1$
e'_{k+3q-1} : $1 \leq q \leq c-2$	$q+2$	$e'_{k+3q-1} : 1 \leq q \leq \frac{b}{3}-1$	$c-q$
$e'_{k+3q+3c-7}$: $1 \leq q \leq c$	$c-q+1$	$e'_{k+q+3c+3t-7} : t = 1, -2 \leq q \leq 2$	$\lvert q \rvert$
$e'_{k+3q+3c-2}$: $1 \leq q \leq c-1$	$c-q+1$	$e'_{k+3q+3c-4} : 1 \leq q \leq \frac{b}{3}$	$q+1$
$e'_{k+3q+3c}$: $1 \leq q \leq c-2$	$c-q+1$	$e'_{k+3q+3c-3}$: $-1 \leq q \leq \frac{b}{3}$	$q+2$
-	-	$e'_{k+3q+3c-2}$: $-1 \leq q \leq \frac{b}{3}$	$q+3$

Table 7.24: Distance codes for $e_i^{''}$'s w.r.t vertex v_i and vertex $u_{i+(n/2)-2}$ of $P(n,3)$, where $n = 6t + 14$

$d(.,.)$	w_k	$d(.,.)$	$u_{k+\frac{n}{2}-2}$
$e''_{k+q-1} : q = 1$	$q+1$	$e''_{k+3q-3} : q = 1$	q
$e''_{k+3q+t-3} : 1 \leq t \leq 3$, $1 \leq q \leq \frac{b}{3}+1$	$q+1$	$e''_{k+3q+t-3}$: $1 \leq t \leq 3, 1 \leq q \leq c-3$	$c-q+1$
$e''_{k+3q+3c+t-1} : 1 \leq t \leq 3$, $1 \leq q \leq \frac{b}{3}$	$c-q+1$	$e''_{k+q+3c-9} : 1 \leq q \leq 4$	$4-q$
$e''_{k+q-1} : q = n$	1	$e''_{k+q+3c-5} : 1 \leq q \leq 4$	$q-1$
-	-	$e''_{k+3q+3c+t-4}$: $1 \leq t \leq 3, 1 \leq q \leq c-2$	$q+3$
-	-	$e''_{k+n+q-3} : 1 \leq q \leq 2$	$c+1$

$d(e'_{k-2}, w_k)$, respectively. The code representation of e''_k with respect to w_{k+1} and w_{k+2} are $d(e''_k, w_{k+1}) = d(e''_{k-1}, w_k)$, $d(e''_k, w_{k+2}) = d(e''_{k-2}, w_k)$, respectively presented in Table 7.24.

Hence it is proved that S is a resolving set of $P(n, 3)$ and the $edim(P(n, 3)) \leq 4$ for all $n \geq 20$ and $n \equiv 2 \pmod 6$.

Case: 2. The reverse inequality $edim(P(n, 3)) \geq 4$ can be easily proved by using Theorem 7.1.3 and Lemma 7.2.1 that there exist at least two edges $\bar{e}_1, \bar{e}_2 \in N_h(w_1)$ where $\exists\ h \in \{1, 2, 3\}$ such that $d(w_2, \bar{e}_1) = d(w_2, \bar{e}_2)$ or $d(w_3, \bar{e}_1) = d(w_3, \bar{e}_2)$ so it is a contradiction. ■

In theorem 7.2.2 − 7.2.7 it is proved that the $edim(P(n, 3)) = 4$ for every $n \geq 12$ and the resolving set S presented as

$$S = \begin{cases} \{w_k, w_{k+1}, w_{k+2}, w_{k+3}\} & \text{for } n \equiv 0 \pmod 6 \text{ and } n \geq 12, \\ \{w_k, w_{k+1}, w_{k+2}, u_{k+4}\} & \text{for } n \equiv 1 \pmod 6 \text{ and } n \geq 13, \\ \{w_k, w_{k+1}, w_{k+2}, u_{k+n/2-2}\} & \text{for } n \equiv 2 \pmod 6 \text{ and } n \geq 20, \\ \{w_k, w_{k+1}, w_{k+2}, w_{k+3}\} & \text{for } n \equiv 3 \pmod 6 \text{ and } n \geq 15, \\ \{w_k, w_{k+1}, w_{k+2}, w_{k+3}\} & \text{for } n \equiv 4 \pmod 6 \text{ and } n \geq 16, \\ \{w_k, w_{k+1}, w_{k+2}, u_{k+3}\} & \text{for } n \equiv 5 \pmod 6 \text{ and } n \geq 17, \end{cases} \tag{7.5}$$

where $1 \leq k \leq n$.

7.3 Computer program

In this section, we present a computer program to generate distance codes of all edges e_k, e'_k and e''_k with respect to resolving set S, where $1 \leq k \leq n$. This program based on the generalized distance formulas which are presented in last section. This program is written in *Matlab lab software* and it has three parts or steps.

In first step the program receive the value of n as an input value, the number of vertices of Petersen graph $P(n, 3)$. In second step, program compute distance codes. The distance tables for edges e_k, e'_k and e''_k generated separately in third and last step.

1. function Petersen3N(n)
2. if n<12 || n==14 display(’n is less than 12 or n=14’); return end
3. %— code for Eq’s and Vn where n = 12, 18, 24,....
4. if n >=12 && mod(n, 6)==0
5. k=(n + 6 - mod(n,6))/6; l=n/2; q=1; e=zeros(n,4);
6. for z=0: k-2 e(3*z +q, :)=[z, 3+z, floor((k-3)*(z) / (k-2))+3, floor((k-3) * (z)/(-2+k))];   end
7. for z=1: k-1 e(3*(-2+z) +q, :)=[z+2, z-1, 2+z, floor((-3+k) * (-1+z) /(-2+k))+3]; end
8. for z=1: k-2 e(3*z-3 +q, :)=[3+z, 2+z, -1+z, 2+z]; end
9. for z=1: -1+k e(n- 3*z, :)=[-1+z+3, ceil((-3+k) * (z-1) / (-2+k))+4, ceil((-3+k) * (z-1) / (-2+k))+1 ,ceil((-3+k) * (-1+z) / (-2+k)) +4]; end
10. for z=0: k-2 e(n- 3*(-1+z), :)=[3+z, z, 3+z, ceil((-3+k)*(z) / (-2+k))+4]; end
11. for z=1: k-1 e(n- 3*(1+z),:)=[-1+z, 2+z, ceil((-3+k)* (-1+z)/ (-2+k))+4, ceil((-3+k) *(-1+z) / (-2+k))+1]; end
12. e(n,:)=[3,3,0,3];
13. mat={”};
14. for q=1:n mat{q,1}=[’e’, num2str(q)]; end
15. Table= array2table(e, ’VariableNames’, {’v1’, ’v2’, ’v3’, ’v4’}, ’RowNames’, mat);
16. disp(Table);
17. % ———- code for e’_q’s
18. k=(n+6- mod(n,6))/6; l=n/2; q=1; e=zeros(n,4);

19. for z=0:k-2 e(3*z +q, :)=[z, 2+ z, abs(z-1)+2, abs(z -1)]; end

20. for z=1:k-1 e(3*z-2 +q, :)=[z +1, z -1, z+1, abs(z -2)+ 2]; end

21. for z=1:k-1 e(3*z-1 +q, :)=[z +2, z+1, z-1, z+1]; end

22. for z=0:k-2 e(n-3*z, :)=[z+2, z +3, z +1, z+3]; end

23. for z=0:k-3 e(n- 3*z-1, :)=[z +3,z+1, z +3,z +4]; end

24. e(l+2,:)=[k+1, k-1, k+1, k];

25. for z=1:k -2 e(n -3*(1 +z),:)=[z, 2 +z, 3 +z, z +1]; end

26. e(l+1,:)=[k-1, k +1, k, k -2];

27. mat={”};

28. for q=1:n mat{q,1}=[’e”’, num2str(q)]; end

29. Table= array2table(e, ’VariableNames’, {’w1’, ’w2’, ’w3’, ’w4’}, ’RowNames’, mat);

30. disp(Table);

31. % ———————— code for e”_q’s

32. k=(n+6- mod(n,6))/6; l=n/2; q=1; e=zeros(n,4);

33. for z=0:k -2 e(3*z +q, :)=[1+z, 1+z, 2 + floor((-2 +k)*z / (-1 +k)), abs(-1 +z)+1]; end

34. z=k-1; e(3*z +q, :)=[1+z, 1+z, floor((-3 +k)*(-1 +z) / (-2 +k))+3, abs(-1 +z)+1];

35. for z=1:k -1 e(3*(-2+z) +q, :)=[1+z, z, z, floor((-2+k)*(-1+z) / (-1+k))+2]; end

36. for z=1:k -1 e(3*(-1+z) +q, :)=[1+z, 1+z, z, z]; end

37. for z=1:k -1 e(n -3*(2+z) +q, :)=[z, 1+z, 1+z, 1+z]; end

38. for z=1:k -1 e(n -3*(1+z) +q, :)=[1+z, 1+z, 1+z, ceil((-3+k)*(-1 +z) / (-2 +k)) +3]; end

39. for z=1:k -2 e(n- 3*z +q, :)=[1 +z, 1 +z, 2 +z, 2+z]; end

40. mat={”};

41. for q=1:n mat{q,1} =[’e””’, num2str(q)]; end

42. Table= array2table(e, ’VariableNames’, {’w1’, ’w2’, ’w3’, ’w4’}, ’RowNames’, mat);

43. disp(Table);

44. end

45. %— code for Eq and Vn where n = 13, 19, 25,.... and 1 <= q <= n

46. % v1, v2, v3, u(n-3)/2

47. if n>=13 && mod(n,6)==1

48. q=1; k=(n +6- mod(n,6)) /6; l=(n -1) /2; e= zeros(n ,1);

49. for z=1:l/3 +2 e(q + 3*(z-1),1) =z-1; end

50. for z=1:l/3 -2 e(q+ 3*z -2,1) =z +2; end

51. for z=l/3 -1:2*(l/3) e(q + 3*z-2,1) =2*(l/3) -z; end

52. for z=1:l/3 -1 e(q+ 3*z -1,1) =3 +z; end

53. for z=l/3:2*(l /3) e(q+ 3*z- 1,1) =2*(l /3) -z+ 3; end

54. for z=1:k -3 e(q +3*k+ 3*z,1) =k -z; end

55. for z=n e(q +n -1,1)=3; end

56. %—–for u_(n-3)/2—–

57. we=0; for z=1:l/3 -1 we(q + 3*(z -1),1) =k -z; end

58. for z=1:l /3 -1 we(q + 3*z -2,1) =k -z -1; end

59. for z=1:l /3 -1 we(q + 3*z-1,1) =k-z; end

60. for z=1:l /3 +1 we(q + 3*(k-2)+3*(z-1)) =z+1; end

61. for z=1:l /3 +1 we(q + 3*(k-2)+3*(z-1) +1,1) =z; end

62. for z=1:l /3 we(q + 3*(k-2)+3*(z-1) +2,1) =z+1; end

63. for z=1:2 we(q +n -3 +z, 1) =k +z -2; end

64. t=0;

65. for q=1: n

66. t(q, 1) = e(mod(q -1,n) +1, 1); t(q,2)= e(mod(q -2, n) +1, 1);

67. t(q ,3) = e(mod(q -3,n) +1, 1); t(q,4)= we(mod(q -1, n) +1, 1);

68. end

69. mat={”}; for q=1: n mat{q ,1} =[’e’,num2str(q)]; end

70. vt=[’u’,num2str((n-3)/2)];

71. Table= array2table(e, ’VariableNames’, {’w1’, ’w2’, ’w3’, ’vt’}, ’RowNames’, mat);

72. disp(Table);

73. % ——— code for e’_q’s

74. q=1; e = zeros(n, 1); k=(n + 6- mod(n,6)) /6; l=(n -1)/2;

75. for z=1:l /3+2 e(3*z -3 +q,1) = -1+z; end

76. for z=1:l /3 e(-6 +3*z +q) ,1) = 1+z; end

77. for z=1:l /3 e(3*z -1 +q,1) = z+2; end

78. for z=1:l /3 e(3*(k -1)+ 3*(-1+z) +q,1) = k +2-z; end

79. for z=1:l /3 e(3*(k -1)+ 3*(-2+z) +q,1) =k -z; end

80. for z=1:-1 + l /3 e(3*z + 3*k +q ,1) = -z +k; end

```
%——for w_(n-3)/2——
we=0; for z=1 :l/ 3-1 we( q+ 3*(-1+z),1) = - z+k+1; end
for z=1:l /3-1 we( q+ 3*z-2,1) = -z+k; end
for z=1:l/ 3-2 we( q+ 3*z-1,1) = k-z+1; end
for z=1:2 we( q+ 3*(-2+ k)+ 3*(-1 + z ) -1,1) = 3-z; end
for z=1:2 we( q+ 3*(-2 + k)+ 3*(-1+z) ) = z; end
z=l; we( z + q -2,1) = 0;
for z=1:l /3 we( q+ 3*(-2 +k) + 3*(z)+1, 1) = 1+ z; end
for z=1:l /3-1 we( q+ 3*(-2 +k) + 3*(z)+2, 1) = 2+ z; end
for z=2:l /3 we( q+ 3*(-2 +k )+ 3*z, 1) = 2+ z; end
for z=1: 2 we( q+ n- 3+ z, 1) = k + z -1; end
t=0;
for f=1:n
t(q, 1) = e( mod( q -1,n) +1, 1); t(q,2) = e(mod( q -2, n) +1, 1);
t(q ,3) = e( mod( q -3,n) +1, 1); t(q,4) = we(mod( q -1, n) +1, 1);
end
mat={”};
for q=1:n mat{q,1}=['e”', num2str(q)]; end
vt=['u', num2str(( n-3)/2)];
Table= array2table(e, 'VariableNames', {'w1', 'w2', 'w3', vt}, 'RowNames',
mat);
disp(Table);
% —— code for e”_i's
```

```
103. q=1; l = (n-1)/2; k=(n+6-mod( n ,6)) /6; e= zeros(n, 1);
104. for z=1:l /3+1 e( 3*z-3 + q ,1) =z; end
105. for z=1:l /3 e( -2+ 3*z + q ,1) =1 + z; end
106. for z=1:l /3 e( -1+ 3*z + q ,1) =1 + z; end
107. for z=1:l /3 e( 3*k-3 + 3*z-1 +q ,1) = k-z+1; end
108. for z=1:l /3 e( 3*k-5 + 3*z + q ,1) = k-z+1; end
109. for z=0:l /3-1 e( 3*z + 3*k + q ,1) = k-z-1; end
110. %——for w_(n-3)/2——
111. we = 0; for z =1:4 we(q + 3*(k -3) +z -1, 1) = 4 -z; end
112. for z=1:4 we(q + 3*(k -2) +z, 1) = z - 1; end
113. for z=4:k for t = 1:3 we(q + 3*(z-4) +t -1,1)  = k-z+4; end end
114. for z=4:k+1 for t=1:3 we(q + 3*(k) +3*(z -4) + t - 2 ,1) = 4 +z -4; end end
115. for z=1: 2 we(q+ n -3 +z,1)=k+1; end
116. t=0;
117. for f=1:n
118. t(q, 1) = e( mod( q -1,n) +1, 1); t(q,2) = e(mod( q -2, n) +1, 1);
119. t(q ,3) = e( mod( q -3,n) +1, 1); t(q,4) = we(mod( q -1, n) +1, 1);
120. end
121. mat={”};
122. for q=1:n mat{q,1}=['e””', num2str( q)]; end
123. vt=['u', num2str(( n-3)/2)];
124. Table= array2table(e, 'VariableNames', {'w1', 'w2', 'w3', vt}, 'RowNames',
     mat);
```

```
disp(Table);
end
%— code for Eq and Vn where n = 14, 20, 26,.... and 1 < = q  < = n
% v1, v2, v3, un/2 +1
if n>=14 && mod(n,6)==2
q=1; k=(n + 6 - mod(n , 6) ) /6; l=n/2; e= zeros( n ,1);
for z=1:l/3+2 e(q+ 3*( z-1), 1) = z - 1; end
for z=1:l/3-1 e(q+ 3*z -2, 1) = z +2; end
for z=1:k-3 e(q+ 3*z -1, 1) = 3 +z; end
for z=1 e(q + z + 3*k - 8, 1) = k; end
for z=1:l /3-1 e(q + 3*z + 1, 1) = z +3; end
for z=1:l /3 e(q + 3*z + 3*k - 5, 1) = k +2 -z; end
for z=1:l /3+1 e(q + 3*z + 3*k - 7, 1) = k -z; end
for z=1:k-3 e(q+3*k+ 3*z, 1) = k -z +1; end
for z=1:2 e(q + n- 3+z,1) =3; end
%——for w_(n-3)/2——
we=0;
for z=3: k for t=1:3 we(q + 3*z + t - 10, 1) =k +4 -z -t; end  end
for z=1: 2 we(q + z + 3*k - 7, 1) =2 ; end
for z=3: k
for t=1: 3 we(q + 3*z + 3*k + t - 14, 1) = t + (z -3); end  end
for z=k-1 we(q +  6*z - 4, 1) = k -1; end
for z=1:4 we(q + z +  6*k - 10, 1) = k; end
```

```
for z =1 we(q + n -z, 1) = k  -1; end
t=0;
for q=1:n
t(q, 1) = e( mod( q -1,n) +1, 1); t(q,2) = e(mod( q -2, n) +1, 1);
t(q ,3) = e( mod( q -3,n) +1, 1); t(q,4) = we(mod( q -1, n) +1, 1); end
mat={”};
for q=1:n mat{q ,1}=['e', num2str( q)]; end
vt=['u', num2str( n /2 -1)];
Table= array2table(e, 'VariableNames', {'w1', 'w2', 'w3', vt}, 'RowNames', mat);
disp(Table);
% —— code for e'_q's
q=1; k=(n +6 -mod(n, 6)) /6; l=n/2; e = zeros( n, 1);
for z=1:l /3+2 e(q+ 3*z - 3 ) ,1) = -1+ z; end
for z=1:l /3 +1 e(q-2 + 3*z ,1) = 1+ z; end
for z=1:-2 + k e(q+ 3*z-1 ,1) = 3 +z -1; end
for z=1: k e(q+ 3*z-7+ 3*k , 1) = k +1-z; end
for z=1:-1+ k e(q+ 3*z-2+ 3*k , 1) = k +1 -z; end
for z=1: - 2 +k e(q+ 3*z + 3*k, 1) = k+1 -z; end
%—for w_(n-3)/2—
we=0;
for z =1:l /3 -1 we(q+ 3*z -3, 1) = k + 2 -z; end
for z =1:l/ 3 -1 we(q+ 3*z -2, 1) = k +1- z; end
```

170. for z =1:l/ 3 -1 we(q+ 3*z -1, 1) = k - z; end

171. t=1;

172. for z=-2 :2 we(q+ 3*t+ 3*k -7+z, 1) = abs(z); end

173. for z=1 :l/ 3 we(q+ 3*z+ 3*k -4, 1) = 1 +z; end

174. for z=1 :l/ 3 we(q+ 3*z+ 3*k -3, 1) = 2 +z; end

175. for z=1 :l/ 3-1 we(q+ 3*z+ 3*k -2, 1) = 3 +z; end

176. for z=n we(q+ z- 1,1) = k; end

177. t=0;

178. for f=1:n

179. t(f,1) = e(mod(- 1 +f, n) + 1, 1); t(f, 2)= e(mod(-2 +f, n) +1,1);

180. t(f,3) = e(mod(- 3 +f, n) + 1, 1); t(f, 4)= we(mod(- 1 +f, n) +1, 1); end

181. mat={”};

182. for q=1:n mat{q,1} =['e”', num2str(q)]; end

183. vt=['u',num2str(n/2-1)];

184. Table= array2table(e, 'VariableNames', {'w1', 'w2', 'w3', vt}, 'RowNames', mat);

185. disp(Table);

186. % ——— code for e”_q's

187. q=1; k=(n+ 6 - mod(n ,6)) /6; l=n/ 2; e=zeros(n, 1);

188. for z =1 e(q +z -1, 1)=z; end

189. for z=1:l/ 3 for t=1:3 e(q + 3*z + t - 3, 1) = z +1; end end

190. for z=1:l/ 3 for t= 1:3

191. e(q + 3*z + 3*k + t - 6,1) = k + 1 -z; end end

192. for z=1 e(q +n -z ,1) = z; end
193. we=0; z=1; we(q+ 3*z -3, 1) = k + 1; end
194. for z =1:k-3 for t=1:3 we(q + 3*z + t - 3, 1) = k +1 -z; end end
195. for t=1:4 z =1 we(q+ 3*z + 3*k-12 +t, 1) = 4 -t; end
196. for t=1:4 z =1 we(q+ 3*z + 3*k -8 +t, 1) = t-1; end
197. for z =1: k -2 for t= 1:3 we(q +3*z + 3*k -4 +t, 1) = 3 +z; end end
198. for z =1: 2 we(q +n -3+z,1) = k +1; end
199. t=0;
200. for f=1:n
201. t(f,1) = e(mod(- 1 +f, n) + 1, 1); t(f, 2)= e(mod(-2 +f, n) +1,1);
202. t(f,3) = e(mod(- 3 +f, n) + 1, 1); t(f, 4)= we(mod(- 1 +f, n) +1, 1); end
203. mat={”};
204. for q=1:n mat{q,1} =[’e””’,num2str(q)]; end
205. vt=[’u’, num2str(n/2-1)];
206. Table= array2table(e, ’VariableNames’, {’w1’, ’w2’, ’w3’, vt}, ’RowNames’, mat);
207. disp(Table);
208. end
209. %— code for Eq and Vn where n = 15, 21, 27,.... and 1 < = q < = n
210. if n >=15 && mod(n,6) ==3
211. k=(n +6- mod(n , 6)) /6; l =(n -1) /2; q=1; e=zeros(n, 4);
212. for z=0: k-1 e(q +3*z , :) =[z, 3 +z, floor((3*z) /4)+3, floor((3*z) /4)]; end

213. for z=1: k e(q +3*z -2 , :) =[z +2, z -1, z+2, floor(3 *(z-1) /4)+3]; end

214. for z=1: k-1 e(q +3*z -1 , :) =[z +3, z +2, z -1, z +2]; end

215. for z=0: k-1 e(n -3*z , :) = [floor(3*(z) /4) +3, z +3, z, z +3]; end

216. for z=0: k-2 e(n -3*z -1 , :) = [z+3, z, z +3, z +4]; end

217. for z=1 :k-1 e(n -3*z +1 , :) = [z -1 z +2, z +3, z]; end

218. e(n, :) =[3, 3 ,0, 3];

219. mat={”};

220. for q=1:n mat{q,1} =[’e’, num2str(q)]; end

221. Table= array2table(e, ’VariableNames’, {’w1’, ’w2’, ’w3’, ’v4’}, ’RowNames’, mat);

222. disp(Table);

223. % —— code for e’_i’s

224. l = (n -1) /2; k= (n+ 6-mod(n,6)) /6; q = 1; e= zeros(n, 4);

225. for z = 0: k -1 e(3*z +q, :) = [z, z+2, 2+ abs(-1+z) , abs(-1+z)]; end

226. for z = 1: k e(3*z -2 +q, :) = [1+z, -1+z, 1+z, abs(-2+z) + 2]; end

227. for z = 1: k-1 e(3*z -1 +q, :) = [2+z, 1+z, -1+z, 1+z]; end

228. for z = 0: k-1 e(n -3*z, :) = [2+z, ceil((k-2)*z / (-1+k))+3, ceil((-2+k)*z / (-1+k)) +1, ceil((-2+ k) *z /(-1+ k)) +3]; end

229. for z = 0: k-3 e(n -3*z –1, :) = [3+z, 1+z, 3+z, 4+z]; end

230. z = k-2; e(n+ 3*z -1, :) = [3+z, 1+z, 3+z, 3+z];

231. for z =1 :k-2 e(n -3*z +1, :) = [z, 2+z, 3+z, 1+z]; end

232. z = k -1; e(n- 3*z +1, :)=[z, 2 +z, 1 +z, -1 +z];

233. mat={”};

234. for q=1:n mat{q,1} =['e”’,num2str(q)]; end

235. Table= array2table(e, 'VariableNames', {'w1', 'w2', 'w3', 'w4'}, 'RowNames', mat);

236. disp(Table);

237. % —— code for e”_q’s

238. k=(n+ 6 -mod(n, 6)) /6; l =(n -1) /2; q = 1; e = zeros(n, 4);

239. for z=0: k-1 e(1 +3*z ,:) =[1 +z, 1 +z, 2 +floor((-2+k)*z / (-1+k)), 1+ abs(-1+z)]; end

240. for z=1: k e(-1 +3*z, :) =[1 +z, z, z, 2+ floor((-2+k)*(-1+z) / (-1+k))]; end

241. for z=1: k-1 e(3*z,:) =[1+z, 1 +z,z ,z]; end

242. for z=1: k e(n +3-3*z, :) =[z, 1 +z, 2+ ceil((-2+k)*(-1+z) /(-1+k)), 2+ ceil((-2 +k)*(-1 +z) /(-1 +k))]; end

243. for z=1: k-1 e(n +2 -3*z, :) =[1 +z, 1 +z, 1+z, 3 + ceil((-2 +k)*(-1 +z) /(-1 +k))]’; end

244. for z=1: k-2 e(n +1 -3*z, :) =[1 +z, 1 +z, 2 +z, 2 +z]; end

245. z= k-1; e(1 +n -3*z, :) = [1 +z, 1+z, 2 +z, 1 +z];

246. mat={”};

247. for q=1:n mat{q,1} =['e””’,num2str(q)]; end

248. Table= array2table(e, 'VariableNames', {'w1', 'w2', 'w3', 'w4'}, 'RowNames', mat);

249. disp(Table);

250. end

251. %— code for Eq and Vn where n = 16, 22, 28,.... and 1 < = q < = n

```
if n> =16 & & mod(n, 6)==4
q=1; k =(n +2)/6; l =n/2 -2; e =zeros(n, 1);
for z =1:l/3 +1 e(q+ 3*(z -1), 1) = z -1; end
for z=1:l/3-1 e(q+ 3*z -2, 1) = z +2; end
for z=1:l/3 e(q+ 3*z -1, 1) = 3 +z; end
z=l-2; e(q +z,1 ) = k;
for z=1:l /3 +1 e(q+ 3*(k -1)+ 3*z -1, 1) = k + 3 -z; end
for z=1:l/ 3 +1 e(q+ 3*(k -1) + 3*z -2, 1) = k -z; end
z=l+3; e( q +z, 1) = k;
for z=1:l/3-1 e(q + 3*z + 3*k, 1) = k -z +1; end
z=n; e(q +z -1, 1) = 3;
t=0;
for q=1:n
t(f,1) = e( mod(- 1 +f, n) + 1, 1); t(f, 2)= e( mod( -2 +f, n) +1,1 );
t(f,3) = e( mod(- 3 +f, n) + 1, 1); t(f, 4)= we( mod(- 1 +f, n) +1, 1); end
mat={''};
for q=1:n mat{q,1} =['e',num2str( q)]; end
Table= array2table(e, 'VariableNames', {'w1', 'w2',' w3', 'w4'}, 'RowNames', mat);
disp(Table);
% —— code for e'_q's
q=1; k= (2 +n) /6; l= -2 +n /2; e =zeros( n, 1);
for z =1: 2 +l /3 e(q + 3*( -1 +z), 1) = -1 +z; end
```

274. for z=1: l/3 e(q - 2+ 3*z, 1) =1+z; end

275. for z=1: l/3+1 e(q -1+ 3*z, 1) =2+z; end

276. for z=2: l/3+1 e(q +3*(k-1)-1 + 3*z, 1) =-z +k +3; end

277. for z=1: l/3+1 e(q +3*(k-1)-2+ 3*z, 1) =-z +k +1; end

278. for z=1: l/3 e(q +3*z+ 3*k, 1) =-z +k +1; end

279. t=0;

280. for f=1:n

281. t(f, 1) = e(mod(- 1 +f, n) + 1, 1); t(f, 2)= e(mod(-2 +f, n) +1,1);

282. t(f, 3) = e(mod(- 3 +f, n) + 1, 1); t(f, 4)= we(mod(- 1 +f, n) +1, 1); end

283. mat={”};

284. for q=1:n mat{q,1} =['e”',num2str(q)]; end

285. Table= array2table(e, 'VariableNames', {'w1', 'w2', 'w3', 'w4'}, 'RowNames', mat);

286. disp(Table);

287. % —— code for e”_i's

288. q=1; k = (2 +n) /6; l = -2 +n /2; e = zeros(n, 1);

289. for z=1: l/3+1 e(q+ 3*(-1+z),1) = z; end

290. for z=1: l/3+1 e(q -2 + 3*z, 1) = 1 +z; end

291. for z=1: l/3+1 e(q -1 + 3*z, 1) = 1 +z; end

292. for z=2: l/3+1 e(q -1 -1+3*(k-1)+3*z, 1) = -z +k +2; end

293. for z=2: l/3+1 e(q -2 +3*(k -1) + 3*z, 1) = -z +k +2; end

294. for z=0: l/3 e(q+ 3*z+ 3*k, 1) = k -z; end

295. t=0;

```
for f=1: n
t(f, 1) = e( mod(- 1 +f, n) + 1, 1); t(f, 2)= e( mod( -2 +f, n) +1,1 );
t(f, 3) = e( mod(- 3 +f, n) + 1, 1); t(f, 4)= we( mod(- 1 +f, n) +1, 1); end
mat={”};
for q=1:n mat{q,1} =[’e””’,num2str( q)]; end
Table= array2table(e, ’VariableNames’, {’w1’, ’w2’, ’w3’, ’w4’}, ’RowNames’, mat);
disp(Table);
end
% n = 17, 23, 29,....
%—— code for Eq and Vn where 1<= q <=n
if n>=17 && mod(n,6)==5
q=1; k =(n+ 6-mod( n, 6)) /6; l =(n-1) /2; e = zeros(n, 1);
for z=1: l/3+3 e(q + 3*z-3, 1) = z -1; end
for z=1: l/3+1 e(q + 3*z-2, 1) = z +2; end
for z=1: l/3-2 e(q + 3*z-1, 1) = 3 +z; end
for z=1: k+2 e(q +3*z+ 3*k -10, 1) =k +2 -z; end
for z=1: k-1 e(q +3*z+ 3*k -2, 1) = k +2 -z; end
for z=1: l/3-2 e(q + 3*z +3*k +3, 1) = k +1 -z; end
for z=1: 2 e(q + n- 3+z, 1) =3; end
%—–for w_(n -3) /2—–
we=0; for z=1:3 we(q+ z-1, 1) = 4-z; end
for z=1: 2 we(q +2+ z, 1) =2; end
```

318. for z=1: k+1 we(q + 3*z +2, 1) = z; end

319. for z=1: k we(q+ 3*z +3, 1) =z +1; end

320. for z=1: k-1 we(q+ 3*z +4, 1) = z+2; end

321. for z=1: l/3 we(q+ 3*z+ 3*k +1, 1) = k+ 1-z; end

322. for z=1: k-2 we(q+ 3*z+ 3*k +3, 1) = k+ 1-z; end

323. for z=1: k-3 we(q+ 3*z+ 3*k +5, 1) = k+ 1-z; end

324. t=0;

325. for q=1:n

326. t(f, 1) = e(mod(- 1 +f, n) + 1, 1); t(f, 2)= e(mod(-2 +f, n) +1,1);

327. t(f, 3) = e(mod(- 3 +f, n) + 1, 1); t(f, 4)= we(mod(- 1 +f, n) +1, 1); end

328. mat={”};

329. for q=1:n mat{q, 1} =['e',num2str(q)]; end

330. Table= array2table(e, 'VariableNames', {w1', 'w2', 'w3', 'w4'}, 'RowNames', mat);

331. disp(Table);

332. % —— code for e'_q's

333. q=1; l=(n -1) /2; k=(n+ 6- mod(n,6)) /6; e= zeros(n, 1);

334. for z=1: l/3 +3 e(-3 + 3*z +q, 1) = -1 +z; end

335. for z=1: l/3 +1 e(-2 + 3*z +q, 1) = 1 +z; end

336. for z=1: l/3 e(3*z-1 +q, 1) = 2 +z; end

337. for z=1: k e(3*z+ 3*k -4 +q, 1)= k+ 1 -z; end

338. for z=1: k e(3*z+ 3*k -2 +q, 1)= k +2 -z; end

339. for z=1: k -2 e(3*z+ 3*k +3 +q, 1) = k +1 -z; end

340. %—–for w_(n -3) /2—–

341. we=0;

342. for t=0: 1 for z =1: 3 we(q +3*t +z-1, 1) =5-z-2*t; end end

343. for z=1: 2 we(q +6 +z, 1) = 2; end

344. z=7; we(q+ z-1, 1) = 1;

345. for t=1: k- 2 for z=-1: 1 we(q+ 3*t+ 7 +z, 1) = 3 +t -abs(z); end end

346. for z=1: 4 we(q+3 *k +z+ 2, 1) = k +1; end

347. for t=1: k-3 for z= -1: 1 we(q+ 3*k+3*t +5 +z, 1) = k +2 -t- abs(z); end end

348. z=n; we(q+z-1, 1) = 3;

349. t=0;

350. for f=1: n

351. t(f, 1) = e(mod(- 1 +f, n) + 1, 1); t(f, 2)= e(mod(-2 +f, n) +1,1);

352. t(f, 3) = e(mod(- 3 +f, n) + 1, 1); t(f, 4)= we(mod(- 1 +f, n) +1, 1); end

353. mat={”}; for q=1:n mat{q,1}= [’e”’,num2str(q)]; end

354. Table= array2table(e, ’VariableNames’, {’w1’, ’w2’, ’w3’, ’w4’}, ’RowNames’, mat);;

355. disp(Table);

356. % —- code for e”_q’s

357. q=1; k=(n +6 -mod(n,6)) /6; l=(-1+n) /2; e=zeros(n, 1);

358. for z=1 e(q-1 +z, 1) = z; end

359. for z=1: 1+l/3 for t=1: 3 e(q+ t+ 3*(z -1) , 1) =1 +z; end end

360. for z=1: l/3 for t=1: 3 e(q+ 3*k+ t +3*z -3 , 1) = k +1 -z; end end

```
for z=1 e(q + n -z, 1 ) = z; end
%——for w_(n-3)/2——
we=0; for z=1: 5 we(q+z- 1, 1) =5-z; end
for z=1: 4 we(q+z +4, 1) =-1+z; end
for z=1: k-1 for t=1:3 we(q+ 3*z +5+ t, 1) =3+z; end end
for z=1: k-3 for t=1:3 we(q +2 +3*z + 3*k +t,1) = k -z +2; end end
for z=1: 2 we(q+ n -3 +z, 1) = 4; end
t=0; for f=1: n
t(f, 1) = e( mod(- 1 +f, n) + 1, 1); t(f, 2)= e( mod( -2 +f, n) +1,1 );
t(f, 3) = e( mod(- 3 +f, n) + 1, 1); t(f, 4)= we( mod(- 1 +f, n) +1, 1); end
mat={”}; for q=1: n mat{q, 1}=[’e””’,num2str( q)]; end
Table= array2table(e, ’VariableNames’, {’w1’, ’w2’, ’w3’, ’w4’}, ’RowNames’, mat);
disp(Table );
end
end
```

Chapter 8

Conclusion and open problems

8.1 Conclusion

In this dissertation we have studied the topological indices and edge metric dimension of some families of simple connected graphs. The findings of this thesis can be summarized as follows:

- We have proposed generalized explicit formulas to calculate first Zagreb index $M_1(G)$, second Zagreb index $M_2(G)$, first Zagreb coindex $\overline{M}_1(G)$ and second Zagreb coindex $\overline{M}_2(G)$ for the strong double graph $SD(G)$ and k-iterated strong double graph $SD^{q*}(G)$. We have also presented an algorithm based on adjacency matrix to verify these indices and coindices by programming and numerically.

- We have computed generalized closed mathematical formulas to calculate the eccentric-connectivity index, total eccentric index and eccentricity based Zagreb indices of Sierpinski networks $S(n,q)$ and their regularizations $S^{+}(n,q)$, $S^{++}(n,q)$. Computer generated graphs are also presented.

- The edge metric dimension for certain families of wheel related graphs, like Jahangir graph J_{2n}, helm graph H_n, sunflower graph Sf_n and friendship graph F_n, have been studied and proved that all these families have unbounded edge metric dimension. We also compared the results with their metric dimensions and concluded that $dim(G) = edim(G)$ if $G \cong J_{2n}$, and $dim(G) < edim(G)$ if $G \cong Sf_n$, F_n, H_n.

- We have presented the edge metric dimension of some wheel related convex polytopes which are denoted by $\mathbb{B}_{2n}$, $\mathbb{D}_{2n}$ and $\mathbb{Q}_n$. We shown that these polytopes have unbounded edge metric dimension. Moreover, these families of polytopes admit the relation $edim(G) \geq dim(G)$.

- We have also investigated generalized Petersen graph $P(n,3)$ for edge resolving set and edge metric dimension. We have presented that the edge metric dimension of $P(n,3)$ is constant, that is $edim(P(n,3)) = 4$ for every $n \geq 12$. The inequality $edim(P(n,3)) \geq dim(P(n,3))$ holds for $P(n,3)$, where $n \geq 12$.

8.2 Open problems

There are many families of graph which are still open to characterize them on the basis of the nature of their metric dimension (MD), edge metric dimension (EMD) and mixed metric dimension. We suggest few of them here,

1. Determine the EMD of m-level gear graph $J_{2n,m}$ for $n \geq 6$ and $m \geq 2$.
2. Determine the EMD of m-level antiweb wheel graphs $AWW_{n,m}$ for $n \geq 8$ and $m \geq 2$.
3. Determine the EMD of m-level antiweb wheel graphs $AWJ_{2n,m}$ for $n \geq 15$ and $m \geq 2$.
4. Is there any class of convex polytope for which EMD is constant?
5. Is there any class of convex polytope for which $edim(G) \leq dim(G)$?
6. Determine the EMD of the generalized Petersen graph $P(n, m)$ for $m \geq 4$.
7. Characterize the families of graphs for $edim(G) = 2$ or 3.

Chapter 9

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