On Fuzzy Generalizations of Some Results in Finite Group Theory

By

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Certificate

It is certified that Ghous Ali **CIIT/FA14-MSMATH-025/LHR** has carried out all the work related to this thesis under my supervision at the Department of Mathematics, COMSATS Institute of Information Technology, Lahore and the work fulfills the requirement for award of MS degree.

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Abstract

In this thesis, we have investigated some basic results of fuzzy group theory by constructing various examples. In particular, we have studied the notions of fuzzy subgroup, normal fuzzy subgroup, fuzzy coset and fuzzy quotient group and have elaborated these concepts through examples.

We have proved that if ξ_1 and ξ_2 are fuzzy subgroups of a group $\mathfrak{G}$, then intersection of ξ_1 and ξ_2 is again a fuzzy subgroup of G. Also, we have shown that for two fuzzy subgroups ξ_1 and ξ_2 of a group G, if $\xi_1^{'}$ is a normal fuzzy subgroup of ξ_1, then $\xi_1^{'} \cap \xi_2$ is a normal fuzzy subgroup of $\xi_1 \cap \nu$.

Furthermore, we have defined the concept of a fuzzy subgroup of a fuzzy group.

We have formulated the notion of an m-polar fuzzy group as a generalization of a fuzzy group. Moreover, we have introduced m-polar fuzzy analogs of some results in fuzzy group theory.

Contents

Chapter 1

Introduction

Zadeh introduced the concept of fuzzy sets in [29]. His ideas provided tools and an approach to model imprecision and uncertainty present in phenomena that do not have sharp boundaries. The theory of fuzzy sets has developed in many directions due to its diverse applications in a vast variety of disciplines. Extensive literature on fuzzy subsets has been cited in [27].

Rosenfeld [20] used Zadeh's idea of a fuzzy set to define the concept of a fuzzy subgroup of a group and formulated a tool and a novel approach to establish the fuzzy analogs of many results in group theory. However, in 1979, Anthony and Sherwood [2] generalized the Rosenfeld's definition of fuzzy group. Sherwood [22] defined products of fuzzy subgroups using t-norms and gave some properties of these products. The notion of fuzzy subring and ideal was introduced by Liu in [16]. Since then, fuzzy versions of various basic properties and theorems in finite group theory have been developed.

Fuzzy version of Lagrange's Theorem has been studied in [18]. However, fuzzy quotient groups, characteristic fuzzy subgroups, conjugate fuzzy subgroups and more advanced topics like fuzzy solubility, fuzzy subgroups of Hall groups have been studied in [17]. Fuzzy isomorphism theorems have been studied in [17, 9]. Introductory fuzzy representations of fuzzy groups has been studied by Thampy Abraham in [3]. Recently, Marius Tarnauceanu has classified some cases of the fuzzy subgroups and fuzzy normal subgroups of finite groups [24, 25].

In a different direction however, Zhang [28] extended the idea of a fuzzy set and defined the notion of bipolar fuzzy set on a given set X as a mapping $A : X \to [-1, 1]$. In bipolar valued fuzzy sets the membership degree range of fuzzy sets is extended from the interval $[0, 1]$ to the interval $[-1, 1]$. This concept is used by researchers as a tool in bipolar fuzzy graphs and many other variety of fields, see, for examples, [4, 5, 6, 7, 8].

After the introduction of a bipolar fuzzy set by Zhang [28], several researchers generalized the concept of a bipolar fuzzy set. In 2014, Chen and his collaborators [11] introduced the notion of $\mathfrak{m}$-polar fuzzy sets as a generalization of bipolar fuzzy set and showed that bipolar fuzzy sets and 2-polar fuzzy sets are cryptomorphic mathematical notions and that we can obtain concisely one from the corresponding one in [11].

In [1], Akram and Adeel has applied the concept of $\mathfrak{m}$-polar fuzzy sets to trees.

Chapter 2

Literature Review

2.1 Fuzzy sets and fuzzy groups

In this chapter, we will elaborate some basic terminologies and theorems of fuzzy group theory. In particular, we are interested in extension principle of Zadeh, fuzzy cosets and fuzzy quotient groups. Moreover, we will examine basic terminologies of fuzzy group theory through detailed examples.

We have also given the definition of a fuzzy subgroup of a fuzzy group. Furthermore, we have proved the following theorems:

1. If ξ_1, ξ_2 are fuzzy subgroups of a group G, then their intersection is again a fuzzy subgroup of G.

2. If ξ_1, ξ_2 are fuzzy subgroups of a group G and $\xi_1^{'}$ is a normal fuzzy subgroup of ξ_1, then $\xi_1^{'} \cap \xi_2 \prec \xi_1 \cap \xi_2$.

From now on, we denote $[0,1]$ by $\mathcal{I}$.

Definition 2.1.1 (Fuzzy Subset)**.** *[17] Let $\mathfrak{X}$ be a non-empty set. A fuzzy subset ξ of $\mathfrak{X}$ is a function from $\mathfrak{X}$ to $\mathcal{I}$.*

The set of all fuzzy subsets of $\mathfrak{X}$ is dnoted by $FP(\mathfrak{X})$. It is also known as fuzzy power set of $\mathfrak{X}$.

Definition 2.1.2 (Support and Core of a fuzzy subset)**.** *[17] Let $\mathfrak{Y}$ be a non-empty set and $\xi \in FP(\mathfrak{Y})$,*
Then,

1. $\xi^* = \{\mathfrak{y} | \mathfrak{y} \in \mathfrak{Y}, \xi(\mathfrak{y}) > 0\}$

2. $\xi_* = \{\mathfrak{y} | \mathfrak{y} \in \mathfrak{Y}, \xi(\mathfrak{y}) = 1\}$

In the definition 2.1.2 ξ^* and ξ_* are known as ***support*** *and* ***core*** *of ξ respectively. If support of a fuzzy subset is finite then it is known as finite fuzzy subset. If support of a fuzzy subset is an infinite set then it is known as infinite fuzzy subset.*

Definition 2.1.3 (Fuzzy point)**.** *[17] Let $\mathfrak{Y}$ be a subset of $\mathfrak{X}$ and $p \in \mathcal{I}$.*
We define a fuzzy subset p_Y of $\mathfrak{X}$ as follows,

$$p_{\mathfrak{Y}}(x) = \left\{ \begin{array}{ll} p & if\ x \in \mathfrak{Y} \\ 0 & otherwise \end{array} \right\}$$

Particularly, if $\mathfrak{Y}$ is a singleton, then $p_{\mathfrak{Y}}$ is called a fuzzy point.

In group theory, we define some operations on groups to examine the properties of groups. A similar approach is used to examine the properties of fuzzy subgroups of a group. For the further discussion " $\mathfrak{G}$ " denotes the arbitrary group with identity e and a multiplicative binary operation. Moreover, minimum and maximum are denoted by "$\wedge$" and "$\vee$" respectively.

In [30], Zadeh proposed the so called extension principal. Zadeh's extension principal allows us to extend the domain of a mapping in a natural way. It is an important tool in the development of fuzzy set theory and other related areas.

Definition 2.1.4. *[17] Let $\mathfrak{f} : \mathfrak{X} \to \mathfrak{Y}$ be a function, and let ξ_1 and xi_2 are the fuzzy subsets of $\mathfrak{X}$ and $\mathfrak{Y}$ respectively. We characterize the fuzzy subsets $\mathfrak{f}(\xi_1)$ of $\mathfrak{Y}$ and $\mathfrak{f}^{-1}(\xi_2)$ of $\mathfrak{X}$ by $\forall\ \mathfrak{y} \in \mathfrak{Y}$,*

$$\mathfrak{f}(\xi_1)(\mathfrak{y}) = \left\{ \begin{array}{cc} \vee\{\xi_1(\mathfrak{x}) | \mathfrak{x} \in \mathfrak{X}, \mathfrak{f}(\mathfrak{x}) = \mathfrak{y}\} & if\ \mathfrak{f}^{-1}(\mathfrak{y}) \neq \phi \\ 0 & otherwise \end{array} \right\}$$

and $\forall \mathfrak{x} \in \mathfrak{X}$,

$$\mathfrak{f}^{-1}(\xi_2)(\mathfrak{x}) = \xi_2(\mathfrak{f}(\mathfrak{x})).$$

Then $\mathfrak{f}(\xi_1)$ and $\mathfrak{f}^{-1}(\xi_2)$ are called the image and pre-image(or inverse image) of ξ_1 and ξ_2 under $\mathfrak{f}$ respectively. It is the famous extension principle of Zadeh.

Definition 2.1.5. *[17] Let ξ_1 and ξ_2 be the fuzzy subsets of $\mathfrak{G}$. We define the fuzzy subsets $\xi_1 \cup \xi_2$ and $\xi_1 \cap \xi_2$ of $\mathfrak{G}$ as follows, for all $x \in \mathfrak{G}$,*

$$(\xi_1 \cup \xi_2)(x) = \xi_1(x) \vee \xi_2(x)$$

$$(\xi_1 \cap \xi_2)(x) = \xi_1(x) \wedge \xi_2(x).$$

Definition 2.1.6. *Let ξ_1 and ξ_2 be the fuzzy subsets of $\mathfrak{X}$. Then ξ_1 is said to be contained in ξ_2 if, for all $x \in \mathfrak{G}$,*

$$\xi_1(x) \leq \xi_2(x).$$

We now define fuzzy subgroupiod and fuzzy subgroup as Rosenfeld's paper published in 1971.

Definition 2.1.7 (Fuzzy subgroupoid)**.** *[20] Let ξ be a function from $\mathfrak{X}$ to $\mathcal{I}$, where $\mathfrak{X}$ is a groupoid, then ξ is called a fuzzy subgroupoid if, for all $x_1, x_2 \in \mathfrak{X}$,*

$$\xi(x_1x_2) \geq \xi(x_1) \wedge \xi(x_2).$$

Definition 2.1.8 (Fuzzy subgroup)**.** *[20] Let ξ be a fuzzy subset of a group $\mathfrak{G}$. Then ξ is called fuzzy subgroup of $\mathfrak{G}$ if for all $x_1, x_2 \in \mathfrak{G}$,*

1. $\xi(x_1x_2) \geq \xi(x_1) \wedge \xi(x_2)$
2. $\xi(x_1^{-1}) \geq \xi(x_1)$.

The set of all fuzzy subgroups of $\mathfrak{G}$ is denoted by "$F(\mathfrak{G})$". For the further discussion, we denote the fuzzy subset and fuzzy subgroup by $f.s$ and $F.S$ " respectively. Moreover, normal fuzzy subgroup and fuzzy coset are denoted by $N.F.S$ and $F.C$ respectively.

Lemma 2.1.9. *[17] Let $\mathfrak{G}$ be a finite group and ξ be a fuzzy subgroupiod of $\mathfrak{G}$. Then $\xi \in F(\mathfrak{G})$.*

Proof. From definition 2.1.7, it follows that $\xi(x_1x_2) \geq \xi(x_1) \wedge \xi(x_2)$. To show that $\xi(x_1^{-1}) \geq \xi(x_1)$, suppose that $e \neq x_1 \in \mathfrak{G}$. Then for $n > 1$,

Since $\mathfrak{G}$ is a finite group, we have $x_1^n = e$. This implies that inverse of x_1 is x_1^{n-1}.

Now

$$\begin{aligned}
\xi(x_1^{-1}) &= \xi(x_1^{n-1}) \\
&= \xi(x_1^{n-2}x) \\
&\geq \xi(x_1^{n-2}) \wedge \xi(x_1)(by\ using\ definition\ 2.1.7) \\
&\geq \xi(x_1) \wedge ... \wedge \xi(x_1) = \xi(x_1).
\end{aligned}$$

Hence ξ is a $F.S$ of $\mathfrak{G}$. □

Lemma 2.1.10. *[17] Let ξ be a $F.S$ of a group $\mathfrak{G}$. Then for all $g \in \mathfrak{G}$,*

1. $\xi(e) \geq \xi(g)$
2. $\xi(g) = \xi(g^{-1})$.

Proof. (1). Suppose that $g \in \mathfrak{G}$. We have

$$\begin{aligned}
\xi(e) &= \xi(gg^{-1}) \\
&\geq \xi(g) \wedge \xi(g^{-1})\ (since\ \xi \in F(\mathfrak{G})) \\
&\geq \xi(g) \wedge \xi(g) = \xi(g).
\end{aligned}$$

Hence $\xi(e) \geq \xi(g)$. □

Proof. (2). Assume that g belongs to $\mathfrak{G}$. Clearly, $\xi(g) = \xi((g^{-1})^{-1}) \geq \xi(g^{-1}) \geq \xi(g)$, since $\xi \in \mathfrak{G}$. Hence $\xi(g) = \xi(g^{-1})$. □

Remark 2.1.11. *[17] Let $\mathfrak{H} \lesssim \mathfrak{G}$. If ξ is a F.S of $\mathfrak{G}$, then $\xi|_{\mathfrak{H}} \in F(\mathfrak{H})$.*

In the remark 2.1.11 $\xi|_{\mathfrak{H}}$ is known as the restriction of ξ from the domain $\mathfrak{G}$ to $\mathfrak{H}$.

2.2 Examples of fuzzy subgroups

Example 2.2.1. *Let $K = \{e, a\} \times \{e, b\}$ be the non-cyclic group of order 4 and its elements can be written as $\{e, a, b, ab\}$. We define a f.s ξ of K by*

$$\xi(e) = \xi(ab) = s_\circ$$
$$\xi(a) = \xi(b) = s_1$$

$\forall\ s_\circ > s_1$ and $s_\circ, s_1 \in \mathcal{I}$.
Now we will verify that whether ξ is a F.S of K or not.

$$\xi(e) \geq \xi(e) \wedge \xi(e) \implies s_\circ \geq s_\circ$$

$$\xi(e.a) \geq \xi(e) \wedge \xi(a) \implies \xi(a) \geq s_\circ \wedge s_1 \implies s_1 \geq s_1$$

$$\xi(e.b) \geq \xi(e) \wedge \xi(b) \implies \xi(b) \geq s_\circ \wedge s_1 \implies s_1 \geq s_1$$

$$\xi(e.ab) \geq \xi(e) \wedge \xi(ab) \implies \xi(ab) \geq s_\circ \wedge s_\circ \implies s_\circ \geq s_\circ$$

$$\xi(a.a) \geq \xi(a) \wedge \xi(a) \implies \xi(e) \geq s_1 \wedge s_1 \implies s_\circ \geq s_1$$

$$\xi(a.b) \geq \xi(a) \wedge \xi(b) \implies \xi(ab) \geq s_1 \wedge s_1 \implies s_\circ \geq s_1$$

$$\xi(a.ab) \geq \xi(a) \wedge \xi(ab) \implies \xi(b) \geq s_1 \wedge s_\circ \implies s_1 \geq s_1$$

$$\xi(b.b) \geq \xi(b) \wedge \xi(b) \implies \xi(e) \geq s_1 \wedge s_1 \implies s_\circ \geq s_1$$

$$\xi(b.ab) \geq \xi(b) \wedge \xi(ab) \implies \xi(a) \geq s_1 \wedge s_\circ \implies s_1 \geq s_1$$

$$\xi(ab.ab) \geq \xi(ab) \wedge \xi(ab) \implies \xi(e) \geq s_\circ \wedge s_\circ \implies s_\circ \geq s_\circ$$

Hence $\xi \in F(K)$. Now we define a f.s $\xi_1 \in F(K)$ by

$$\xi_1(e) = s_\circ$$

$$\xi_1(a) = s_1$$

$$\xi_1(b) = \xi_1(ab) = s_2$$

We again check that whether ξ_1 is a F.S or not,

$$\xi_1(e.e) \geq \xi_1(e) \wedge \xi_1(e) \implies s_\circ \geq s_\circ$$

$$\xi_1(e.a) \geq \xi_1(e) \wedge \xi_1(a) \implies \xi_1(a) \geq s_\circ \wedge s_1 \implies s_1 \geq s_1$$

$$\xi_1(e.b) \geq \xi_1(e) \wedge \xi_1(b) \implies \xi_1(b) \geq s_\circ \wedge s_2 \implies s_2 \geq s_2$$

$$\xi_1(e.ab) \geq \xi_1(e) \wedge \xi_1(ab) \implies \xi_1(ab) \geq s_\circ \wedge s_2 \implies s_2 \geq s_2$$

$$\xi_1(a.a) \geq \xi_1(a) \wedge \xi_1(a) \implies \xi_1(e) \geq s_1 \wedge s_1 \implies s_\circ \geq s_1$$

$$\xi_1(a.b) \geq \xi_1(a) \wedge \xi_1(b) \implies \xi_1(ab) \geq s_1 \wedge s_2 \implies s_2 \geq s_2$$

$$\xi_1(a.ab) \geq \xi_1(a) \wedge \xi_1(ab) \implies \xi_1(b) \geq s_1 \wedge s_2 \implies s_2 \geq s_2$$

$$\xi_1(b.b) \geq \xi_1(b) \wedge \xi_1(b) \implies \xi_1(e) \geq s_2 \wedge s_2 \implies s_\circ \geq s_2$$

$$\xi_1(b.ab) \geq \xi_1(b) \wedge \xi_1(ab) \implies \xi_1(a) \geq s_2 \wedge s_2 \implies s_1 \geq s_2$$

$$\xi_1(ab.ab) \geq \xi_1(ab) \wedge \xi_1(ab) \implies \xi_1(e) \geq s_2 \wedge s_2 \implies s_\circ \geq s_2$$

Thus it is proved that ξ_1 is also a F.S of K.

Example 2.2.2. *Let $\mathfrak{G} = D_6 = \langle a, b : a^3 = e = b^2, bab = a^{-1}\rangle$. We define $\psi \in FP(\mathfrak{G})$ by*

$$\psi(e) = \psi(a^2b) = s_2$$

$$\psi(a) = \psi(a^2) = \psi(b) = \psi(ab) = s_1$$

$\forall\ s_\circ < s_1$ *and* $s_\circ, s_1 \in \mathcal{I}$. *Now we will verify that whether* ψ *is a subgroup or not,*

$$\psi(e.e) = \psi(e) = s_2 \implies \psi(e.e) \geq \psi(e) \wedge \psi(e) = s_2 \wedge s_2 = s_2$$

$$\psi(e.a) = \psi(a) = s_1 \implies \psi(e.a) \geq \psi(e) \wedge \psi(a) = s_2 \wedge s_1 = s_1$$

$$\psi(e.a^2) = \psi(a^2) = s_1 \implies \psi(e.a^2) \geq \psi(e) \wedge \psi(a^2) = s_2 \wedge s_1 = s_1$$

$$\psi(e.b) = \psi(b) = s_1 \implies \psi(e.b) \geq \psi(e) \wedge \psi(b) = s_2 \wedge s_1 = s_1$$

$$\psi(e.ab) = \psi(ab) = s_1 \implies \psi(e.ab) \geq \psi(e) \wedge \psi(ab) = s_2 \wedge s_1 = s_1$$

$$\psi(e.a^2b) = \psi(a^2b) = s_2 \implies \psi(e.a^2b) \geq \psi(e) \wedge \psi(a^2b) = s_2 \wedge s_2 = s_2$$

$$\psi(a.a) = \psi(a^2) = s_1 \implies \psi(a.a) \geq \psi(a) \wedge \psi(a) = s_1 \wedge s_1 = s_1$$

$$\psi(a.a^2) = \psi(e) = s_2 \implies \psi(a.a^2) \geq \psi(a) \wedge \psi(a^2) = s_1 \wedge s_1 = s_1$$

$$\psi(a.b) = \psi(ab) = s_1 \implies \psi(a.b) \geq \psi(a) \wedge \psi(b) = s_1 \wedge s_1 = s_1$$

$$\psi(a.ab) = \psi(a^2b) = s_2 \implies \psi(a.ab) \geq \psi(a) \wedge \psi(ab) = s_1 \wedge s_1 = s_1$$

$$\psi(a.a^2b) = \psi(b) = s_1 \implies \psi(a.a^2b) \geq \psi(a) \wedge \psi(a^2b) = s_1 \wedge s_2 = s_1$$

$$\psi(a^2.a) = \psi(e) = s_2 \implies \psi(a^2.a) \geq \psi(a^2) \wedge \psi(a) = s_1 \wedge s_1 = s_1$$

$$\psi(a^2.b) = \psi(a^2b) = s_2 \implies \psi(a^2.b) \geq \psi(a^2) \wedge \psi(b) = s_1 \wedge s_1 = s_1$$

$$\psi(a^2.ab) = \psi(b) = s_1 \implies \psi(a^2.ab) \geq \psi(a^2) \wedge \psi(ab) = s_1 \wedge s_1 = s_1$$

$$\psi(a^2.a^2b) = \psi(ab) = s_1 \implies \psi(a^2.a^2b) \geq \psi(a^2) \wedge \psi(a^2b) = s_1 \wedge s_2 = s_1$$

$$\psi(b.a) = \psi(a^2b) = s_2 \implies \psi(b.a) \geq \psi(b) \wedge \psi(a) = s_1 \wedge s_1 = s_1$$

$$\psi(b.a^2) = \psi(ab) = s_1 \implies \psi(b.a^2) \geq \psi(b) \wedge \psi(a^2) = s_1 \wedge s_1 = s_1$$

$$\psi(b.b) = \psi(e) = s_2 \implies \psi(b.b) \geq \psi(b) \wedge \psi(b) = s_1 \wedge s_1 = s_1$$

$$\psi(b.ab) = \psi(a^2) = s_1 \implies \psi(b.ab) \geq \psi(b) \wedge \psi(ab) = s_1 \wedge s_1 = s_1$$

$$\psi(b.a^2b) = \psi(a) = s_1 \implies \psi(b.a^2b) \geq \psi(b) \wedge \psi(a^2b) = s_1 \wedge s_2 = s_1$$

$$\psi(ab.a) = \psi(b) = s_1 \implies \psi(ab.a) \geq \psi(ab) \wedge \psi(a) = s_1 \wedge s_1 = s_1$$

$$\psi(ab.a^2) = \psi(ba^2.a^2) = s_2 \implies \psi(ab.a^2) \geq \psi(ab) \wedge \psi(a^2) = s_1 \wedge s_1 = s_1$$

$$\psi(ab.b) = \psi(a) = s_1 \implies \psi(ab.b) \geq \psi(ab) \wedge \psi(b) = s_1 \wedge s_1 = s_1$$

$$\psi(ab.ab) = \psi(e) = s_2 \implies \psi(ab.ab) \geq \psi(ab) \wedge \psi(ab) = s_1 \wedge s_1 = s_1$$

$$\psi(ab.a^2b) = \psi(a^2) = s_1 \implies \psi(ab.a^2b) \geq \psi(ab) \wedge \psi(a^2b) = s_1 \wedge s_2 = s_1$$

$$\psi(a^2b.a) = \psi(ab) = s_1 \implies \psi(a^2b.a) \geq \psi(a^2b) \wedge \psi(a) = s_2 \wedge s_1 = s_1$$

$$\psi(a^2b.a^2) = \psi(ba.a^2) = s_1 \implies \psi(a^2b.a^2) \geq \psi(a^2b) \wedge \psi(a^2) = s_2 \wedge s_1 = s_1$$

$$\psi(a^2b.b) = \psi(a^2) = s_1 \implies \psi(a^2b.b) \geq \psi(a^2b) \wedge \psi(b) = s_2 \wedge s_1 = s_1$$

$$\psi(a^2b.ab) = \psi(a) = s_1 \implies \psi(a^2b.ab) \geq \psi(a^2b) \wedge \psi(ab) = s_2 \wedge s_1 = s_1$$

$$\psi(a^2b.a^2b) = \psi(e) = s_2 \implies \psi(a^2b.a^2b) \geq \psi(a^2b) \wedge \psi(a^2b) = s_2 \wedge s_2 = s_2$$

Thus ψ is a F.S of $\mathfrak{G}$.

Now we define a f.s ψ_1 by

$$\psi_1(a^2b) = s_1$$

$$\psi_1(e) = s_2$$

$$\psi_1(a) = \psi_1(a^2) = \psi_1(b) = \psi_1(ab) = s_\circ$$

It can be easily seen that $\psi_1 \in F(\mathfrak{G})$.

Remark 2.2.3. *Let $\xi \in FP(\mathfrak{G})$ then ξ may not be a F.S of $\mathfrak{G}$.*

We give the following example to explain the remark 2.2.3.

Example 2.2.4. *Let $\mathfrak{G} = D_6 = \langle a, b : a^3 = e = b^2, bab = a^{-1}\rangle$.*

We define $\xi \in FP(D_6)$ by

$$\xi(a) = \xi(b) = s_\circ$$

$$\xi(e) = \xi(a^2b) = s_1$$

$$\xi(a^2) = \xi(ab) = s_2$$

$\forall\ s_\circ < s_2 < s_1$ and $s_\circ, s_1, s_2 \in \mathcal{I}$. In this example ξ is a f.s but it is not a F.S of D_6, because

$$\xi(a^2b.ab) = \xi(a^2b.ba^2) = \xi(a) = s_\circ$$

$$\xi(a^2b.ab) \geq \xi(a^2b) \wedge \xi(ab)$$
$$= s_1 \wedge s_2 = s_2$$

$\implies \xi(a^2b.a^3b) < \xi(a^2b) \wedge \xi(ab)$, *which shows that* ξ *is not a F.S of* D_6.

Example 2.2.5. *Let* D_8 *be a dihedral group of order* 8 *given by* $D_8 = \langle a, b : a^4 = e = b^2, bab = a^{-1}\rangle$. *We define* $\xi \in FP(D_8)$ *by*

$$\xi(e) = \xi(a^3b) = s_1$$

$$\xi(a) = \xi(a^2) = \xi(a^3) = \xi(b) = \xi(ab) = \xi(a^2b) = s_\circ$$

$\forall\ s_\circ < s_1$ *and* $s_\circ, s_1 \in \mathcal{I}$. *Now we will verify whether* ξ *ia a F.S or not.*

$$\xi(e.a^2) = \xi(a^2) = \xi(e) = s_1 \implies \xi(e.a^2) \geq \xi(e) \wedge \xi(a^2) = s_1 \wedge s_\circ = s_\circ$$
$$\xi(e.a^3) = \xi(a^3) = s_\circ \implies \xi(e.a^3) \geq \xi(e) \wedge \xi(a^3) = s_1 \wedge s_\circ = s_\circ$$
$$\xi(e.b) = \xi(b) = s_\circ \implies \xi(e.b) \geq \xi(e) \wedge \xi(b) = s_1 \wedge s_\circ = s_\circ$$
$$\xi(e.ab) = \xi(ab) = s_\circ \implies \xi(e.ab) \geq \xi(e) \wedge \xi(ab) = s_1 \wedge s_\circ = s_\circ$$
$$\xi(e.a^2b) = \xi(a^2b) = s_\circ \implies \xi(e.a^2b) \geq \xi(e) \wedge \xi(a^2b) = s_1 \wedge s_\circ = s_\circ$$
$$\xi(e.a^3b) = \xi(a^3b) = s_1 \implies \xi(e.a^3b) \geq \xi(e) \wedge \xi(a^3b) = s_1 \wedge s_1 = s_1$$
$$\xi(a^2.a^3) = \xi(a) = s_\circ \implies \xi(a^2.a^3) \geq \xi(a^2) \wedge \xi(a^3) = s_\circ \wedge s_\circ = s_\circ$$
$$\xi(a^2.b) = \xi(a^2b) = s_\circ \implies \xi(a^2.b) \geq \xi(a^2) \wedge \xi(b) = s_\circ \wedge s_\circ = s_\circ$$
$$\xi(a^2.ab) = \xi(a^3b) = s_1 \implies \xi(a^2.ab) \geq \xi(a^2) \wedge \xi(ab) = s_\circ \wedge s_\circ = s_\circ$$
$$\xi(a^2.a^2b) = \xi(b) = s_\circ \implies \xi(a^2.a^2b) \geq \xi(a^2) \wedge \xi(a^2b) = s_\circ \wedge s_\circ = s_\circ$$
$$\xi(a^2.a^3b) = \xi(ab) = s_\circ \implies \xi(a^2.a^3b) \geq \xi(a^2) \wedge \xi(a^3b) = s_\circ \wedge s_1 = s_\circ$$
$$\xi(a^3.a) = \xi(e) = s_1 \implies \xi(a^3.a) \geq \xi(a^3) \wedge \xi(a) = s_\circ \wedge s_\circ = s_\circ$$
$$\xi(a^3.a^2) = \xi(a) = s_\circ \implies \xi(a^3.a^2) \geq \xi(a^3) \wedge \xi(a^2) = s_\circ \wedge s_\circ = s_\circ$$
$$\xi(a^3.b) = \xi(a^3b) = s_1 \implies \xi(a^3.b) \geq \xi(a^3) \wedge \xi(b) = s_\circ \wedge s_\circ = s_\circ$$
$$\xi(a^3.ab) = \xi(b) = s_\circ \implies \xi(a^3.ab) \geq \xi(a^3) \wedge \xi(ab) = s_\circ \wedge s_\circ = s_\circ$$

$$\xi(a^3.a^2b) = \xi(ab) = s_\circ \implies \xi(a^3.a^2b) \geq \xi(a^3) \wedge \xi(a^2b) = s_\circ \wedge s_\circ = s_\circ$$

$$\xi(a^3.a^3b) = \xi(a^2b) = s_\circ \implies \xi(a^3.a^3b) \geq \xi(a^3) \wedge \xi(a^3b) = s_\circ \wedge s_1 = s_\circ$$

$$\xi(b.ab) = \xi(b.ba^3) = \xi(a^3) = s_\circ \implies \xi(b.ab) \geq \xi(b) \wedge \xi(ab) = s_\circ \wedge s_\circ = s_\circ$$

$$\xi(b.a^2b) = \xi(b.ba^2) = \xi(a^2) = s_\circ \implies \xi(b.a^2b) \geq \xi(b) \wedge \xi(a^2b) = s_\circ \wedge s_\circ = s_\circ$$

$$\xi(b.a^3b) = \xi(b.ba) = \xi(a) = s_\circ \implies \xi(b.a^3b) \geq \xi(b) \wedge \xi(a^3b) = s_\circ \wedge s_1 = s_\circ$$

$$\xi(ab.a^2b) = \xi(ab.ba^2) = \xi(a^3) = s_\circ \implies \xi(ab.a^2b) \geq \xi(ab) \wedge \xi(a^2b) = s_\circ \wedge s_\circ = s_\circ$$

$$\xi(ab.a^3b) = \xi(ab.ba) = \xi(a^2) = s_\circ \implies \xi(ab.a^3b) \geq \xi(ab) \wedge \xi(a^3b) = s_\circ \wedge s_1 = s_\circ$$

$$\xi(a^2b.a^2b) = \xi(a^2b.ba^2) = \xi(e) = s_1 \implies \xi(a^2b.a^2b) \geq \xi(a^2b) \wedge \xi(a^2b) = s_\circ \wedge s_\circ = s_\circ$$

$$\xi(a^2b.a^3b) = \xi(a^2b.ba) = \xi(a^3) = s_\circ \implies \xi(a^2b.a^3b) \geq \xi(a^2b) \wedge \xi(a^3b) = s_\circ \wedge s_1 = s_\circ$$

$$\xi(a^3b.a^3b) = \xi(a^3b.ba) = \xi(e) = s_1 \implies \xi(a^3b.a^3b) \geq \xi(a^3b) \wedge \xi(a^3b) = s_1 \wedge s_1 = s_1$$

All the above discussion shows that $\xi \in F(D_8)$.

Now we define $\xi_1 \in FP(D_8)$ *by*

$$\xi_1(e) = \xi_1(a) = \xi_1(a^3b) = s_1$$

$$\xi_1(a^2) = \xi_1(a^2b) = \xi_1(a^3) = \xi_1(b) = \xi_1(ab) = s_\circ$$

$\forall\ s_\circ < s_1$ *and* $s_\circ, s_1 \in \mathcal{I}$. *It is obvious that* $\xi_1 \notin F(D_8)$ *because*

$$\xi_1(a.a^3b) = \xi_1(b) = s_\circ$$

$$\xi_1(a.a^3b) \geq \xi_1(a) \wedge \xi_1(a^3b) = s_1 \wedge s_1 = s_1$$

$\implies \xi_1(a.a^3b) < \xi_1(a) \wedge \xi_1(a^3b)$, *which shows that* ξ_1 *is not a F.S of* D_8.

We define $\xi_2 \in FP(D_8)$ *by*

$$\xi_2(e) = \xi_2(a^3b) = \xi_2(a^3) = s_1$$

$$\xi_2(a) = \xi_2(a^2) = \xi_2(a^2b) = \xi_2(b) = \xi_2(ab) = s_\circ$$

$\forall\ s_\circ < s_1$ *and* $s_\circ, s_1 \in \mathcal{I}$. *Then* ξ_2 *is not a F.S of* D_8 *because*

$$\xi_2(a^3.a^3b) = \xi_2(a^2b) = s_\circ$$

$$\begin{aligned}\xi_2(a^3.a^3b) &\geq \xi_2(a^3) \wedge \xi_2(a^3b)\\ &= s_1 \wedge s_1 = s_1\end{aligned}$$

$\implies \xi_2(a^3.a^3b) < \xi_2(a^3) \wedge \xi_2(a^3b)$, *which shows that* ξ_2 *is not a F.S of* D_8.

One of the most important problem of fuzzy group theory is the classification of fuzzy subgroups of finite non-abelian groups. In [26] Marius Tarnauceanu developed a general method to enumerate fuzzy subgroups of certain finite non-abelian groups. In particular, explicit formulas have been obtained for dihedral groups.

Theorem 2.2.6. *[26] The number $h(D_{2p_1^{m_1}})$ of all distinct fuzzy subgroups of the dihedral group $D_{2p_1^{m_1}}$ is given by the following equality:*

$$h(D_{2p_1^{m_1}}) = \frac{2^{m_1}}{p_1 - 1}(p_1^{m_1-1} + p_1 - 2).$$

Theorem 2.2.7. *[26] The number $h(D_{2p_1^{m_1}p_2})$ of all distinct fuzzy subgroups of the dihedral group $h(D_{2p_1^{m_1}p_2})$ is given by the following equality,*

$$\begin{aligned} h(D_{2p_1^{m_1}p_2}) = \frac{2^{m_1}}{(p_1-1)^3}&[(m_1+2)p_1^{m_1+3}p_2 + 2p_1^{m_1+3} - (2m_1+5)p_1^{m_1+2}p_2 - \\ &- 3p_1^{m_1+2} + (m_1+3)p_1^{m_1+1}p_2 + p_1^{m_1+1} + (m_1+2)p_1^3 - p_1^2p_2 - \\ &- (4m_1+9)p_1^2 + 3p_1p_2 + (5m_1+11)p_1 - 2p_2 - (2m_1+4)]. \end{aligned}$$

Example 2.2.8. *Let $Q_8 = \{-1, 1, i, -i, j, -j, -k, k\}$ be a group of quaternions with eight elements.*
We define $\xi \in FP(Q_8)$ by

$$\xi(1) = s_\circ$$

$$\xi(-1) = \xi(\pm i) = s_1$$

$$\xi(\pm j) = \xi(\pm k) = s_2$$

Now we define $\xi_1 \in FP(Q_8)$ by

$$\xi_1(\pm 1) = s_\circ$$

$$\xi_1(\pm i) = s_1$$

$$\xi_1(\pm j) = \xi_1(\pm k) = s_2$$

We define a f.s ξ_2 of Q_8 by

$$\xi_2(\pm 1) = s_\circ$$

$$\xi_2(\pm i) = \xi_2(\pm j) = \xi_2(\pm k) = s_1$$

$\forall\ s_\circ > s_1 > s_2$ *and* $s_\circ, s_1, s_2 \in \mathcal{I}$. Then we can say ξ, ξ_1 and ξ_2 belongs to $F(Q_8)$, because it is easily observe as previous examples that ξ, ξ_1 and ξ_2 satisfy the axiom of $F.S$.

Lemma 2.2.9. *[17] If $\xi \in F(\mathfrak{G})$, then ξ^* and ξ_* are the subgroups of $\mathfrak{G}$.*

Theorem 2.2.10. *If ξ_1, ξ_2 are fuzzy subgroups of a group $\mathfrak{G}$, then their intersection is again a $F.S$ of $\mathfrak{G}$.*

Proof. We want to show that

$$(\xi_1 \cap \xi_2)(xy^{-1}) \geq (\xi_1 \cap \xi_2)(x) \wedge (\xi_1 \cap \xi_2)(y^{-1})$$

By definition 2.1.5, we have

$$\begin{aligned}
(\xi_1 \cap \xi_2)(xy^{-1}) &= \xi_1(xy^{-1}) \wedge \xi_2(xy^{-1}) \\
(\xi_1 \cap \xi_2)(xy^{-1}) &= \xi_1(x) \wedge \xi_1(y^{-1}) \wedge \xi_2(x) \wedge \xi_2(y^{-1})\ (Since\ \xi_1, \xi_2 \in F(\mathfrak{G})) \\
&\geq \xi_1(x) \wedge \xi_2(x) \wedge \xi_1(y^{-1}) \wedge \xi_2(y^{-1}) \\
&= (\xi_1 \cap \xi_2)(x) \wedge (\xi_1 \cap \xi_2)(y^{-1})
\end{aligned}$$

Hence it is proved that $\xi_1 \cap \xi_2 \in F(\mathfrak{G})$. □

Let $\mathfrak{G}$ be a group and $\mathfrak{H}$ be a subgroup of $\mathfrak{G}$, then the normalizer of $\mathfrak{H}$ in $\mathfrak{G}$ is defined as, $N_{\mathfrak{G}}(\mathfrak{H}) = \{x \in \mathfrak{G} | x\mathfrak{H} = \mathfrak{H}x\}$. We now discuss the normalizer of a $F.S$ ξ in $\mathfrak{G}$.

Definition 2.2.11. *[17] Suppose that ξ be a $F.S$ of a group $\mathfrak{G}$. Then normalizer of ξ in $\mathfrak{G}$ is defined as $N(\xi) = \{g_1 \mid g_1 \in \mathfrak{G}, \xi(g_1g_2) = \xi(g_2g_1)\ \forall\ g_2 \in \mathfrak{G}\}$.*

Theorem 2.2.12. *[17] Suppose $\xi \in F(\mathfrak{G})$. Let*

$$N(\xi) = \{g_1 | g_1 \in \mathfrak{G}, \xi(g_1g_2) = \xi(g_2g_1)\ \forall g_2 \in \mathfrak{G}\}.$$

Then $N(\xi) \lesssim \mathfrak{G}$.

Proof. It can be easily observed that, $e \in N(\xi)$. Suppose $g_1, g_2 \in N(\xi)$. For

$g_3 \in \mathfrak{G}$, we have

$$\begin{aligned}\xi(g_1g_2^{-1}.g_3) &= \xi(g_1.g_2^{-1}g_3)\\ &= \xi(g_2^{-1}g_3.g_1)\\ &= \xi(g_1^{-1}g_3^{-1}.g_2)\\ &= \xi(g_2.g_1^{-1}g_3^{-1})\\ &= \xi(g_3.g_1g_2^{-1}).\end{aligned}$$

Therefore, $g_1g_2^{-1} \in N(\xi)$. Hence $N(\xi) \lesssim \mathfrak{G}$. □

Proposition 2.2.13. *A F.S ξ of a group $\mathfrak{G}$ is self-normalizing F.S of $\mathfrak{G}$ iff ξ is a N.F.S of $\mathfrak{G}$.*

2.3 Level subsets and level subgroups

In this section we denote the level subset by $l.s$ and level subgroup by $L.S$.

Definition 2.3.1 (Level subset)**.** *[12] Let $\xi \in FP(X)$. For $t \in \mathcal{I}$, the set $\xi_t = \{x \in X|\ \xi(x) \geq t\}$ is called a l.s of the f.s ξ.*

Note that ξ_t is a subset of $\mathfrak{X}$ in the ordinary sense. The terminology "$l.s$" was introduced by Zadeh.[12]

Theorem 2.3.2. *[12] Suppose that $\xi \in F(\mathfrak{G})$, then the l.s ξ_t, for $t \in \mathcal{I},\ t \leq \xi(e)$, is a subgroup of $\mathfrak{G}$, where e is the identity of $\mathfrak{G}$.*

Proof. We know that $\xi_t = \{x \in \mathfrak{G} \mid \xi(x) \geq t\}$, then ξ_t is non-empty:

$$e \in \xi_t\ \forall\ t \in \mathcal{I}.$$

Let $g_1, g_2 \in \xi_t$, then $\xi(g_1) \geq t$ and $\xi(g_2) \geq t$ (by definition of $l.s$ defined in the definition 2.3.1). Since $\xi \in F(\mathfrak{G})$, $\xi(g_1g_2) \geq \xi(g_1) \wedge \xi(g_2)$.
This means $\xi(g_1g_2) \geq t$. Hence $g_1g_2 \in \xi_t$. Let $g_1 \in \xi_t$ implies $\xi(g_1) \geq t$. Since $\xi \in F(\mathfrak{G})$, implies that $\xi(g_1^{-1}) \geq \xi(g_1)$ and hence $\xi(g_1) \geq t$. This implies $g_1^{-1} \in \xi_t$. Therefore ξ_t is a subgroup of $\mathfrak{G}$. □

We now give an example to elaborate the theorem 2.3.2.

Example 2.3.3. *Suppose that $K = \{e, a, b, ab\}$ is a Klein four group with four elements. We define $\xi \in F(K)$ as follows,*

$$\xi(e) = t_\circ$$

$$\xi(a) = t_1$$

$$\xi(b) = \xi(ab) = t_2$$

$$\forall\ t_\circ > t_1 > t_2 \text{ and } t_\circ, t_1, t_2 \in \mathcal{I}.$$

Since $\xi \in F(K)$. Clearly (by 2.3.2), $\xi_{t_\circ} = \{e\}, \xi_{t_1} = \{e, a\}$ and $\xi_{t_2} = \{e, a, b, ab\}$ are the subgroups of K.

Following theorem provides a short way to determine whether a $f.s$ is a $F.S$ or not.

Theorem 2.3.4. *[12] Let $\mathfrak{G}$ be a finite group and $\xi \in FP(\mathfrak{G})$ such that $\xi_t \lesssim \mathfrak{G}$ for all $t \in \mathcal{I}, t \leq \xi(e)$, then $\xi \in F(\mathfrak{G})$.*

Proof. Suppose that $g_1, g_2 \in \xi_t$ and let $\xi(g_1) = t_1$ and $\xi(g_2) = t_2$. Then $g_1 \in \xi_{t_1}, g_2 \in \xi_{t_2}$. Let us assume $t_1 < t_2$. Then it implies $\xi_{t_2} \subset \xi_{t_1}$. So $g_2 \in \xi_{t_1}$. Thus $g_1, g_2 \in \xi_{t_1}$ and since $\xi_{t_1} \lesssim \mathfrak{G}$, by hypothesis, $g_1g_2 \in \xi_{t_1}$. Therefore $\xi(g_1g_2) \geq t_1 = \xi(g_1) \wedge \xi(g_2)$. Again, let $g_1 \in \mathfrak{G}$ and let $\xi(g_1) = t$. Then $g_1 \in \xi_t$. Since ξ_t is a subgroup, $g_1^{-1} \in \xi_t$. Therefore $\xi(g_1^{-1}) \geq t$ and hence $\xi(g_1^{-1}) \geq \xi(g_1)$. Thus $\xi \in F(\mathfrak{G})$. □

We give the following example to explain the remark 2.3.4

Example 2.3.5. *Let $S_3 = \{e, a, a^2, b, ab, a^2b\}$ be a symmetric group. We define a $f.s$ ξ of S_3 by*

$$\xi(a) = \xi(b) = t_\circ$$

$$\xi(e) = \xi(a^2b) = t_1$$

$$\xi(a^2) = \xi(ab) = t_2$$

From the theorem 2.3.4, we can say that ξ is not a $F.S$ of S_3 because $\xi_{t_1} = \{e, a, b, a^2b\}$ is not a subgroup of S_3.

Definition 2.3.6 (Level Subgroup)**.** *[12] Suppose that ξ be a $F.S$ of a group $\mathfrak{G}$. The subgroups $\xi_t = \{x \in \mathfrak{G} |\ \xi(x) \geq t\}$ for $t \in \mathcal{I}$ are called $L.Ss$ of ξ.*

The next theorem characterizes that not all the $L.S$s of a group are different.

Theorem 2.3.7. *[12] Let $\xi \in F(\mathfrak{G})$. Then ξ_{s_1}, ξ_{s_2}(with $s_1 < s_2$) are equal iff there does not exist $g \in \mathfrak{G}$ such that $s_1 < \xi(g) < s_2$.*

Proof. Suppose that $\xi_{s_1} = \xi_{s_2}$. Suppose on contrary that there exists $g \in \mathfrak{G}$ such that $s_1 < \xi(g) < s_2$, then $\xi_{s_2} \subsetneq \xi_{s_1}$, since $g \in \xi_{s_1}$ but not in ξ_{s_2}(From definition 2.3.1), which is contradiction. Now, suppose that there be no $g \in \mathfrak{G}$ such that $s_1 < \xi(g) < s_2$. Since $s_1 < s_2$ we have $\xi_{s_2} \subseteq \xi_{s_1}$. Let $g \in \xi_{s_1}$, then $\xi(x) \geq s_1$ and hence $\xi(g) \geq s_2$, since $\xi(g)$ does not lie between s_1 and s_2. Therefore $g \in \xi_{s_2}$. So $\xi_{s_1} \subseteq \xi_{s_2}$. Hence $\xi_{s_1} = \xi_{s_2}$. □

2.4 Fuzzy quotient subgroups

Definition 2.4.1 (Abelian Fuzzy Subset)**.** *[17] A $f.s$ ξ of a set $\mathfrak{X}$ is called an Abelian $f.s$ of X if and only if $\xi(xy) = \xi(yx)$ $\forall x, y \in X$.*

Theorem 2.4.2. *[17] Let $\mathfrak{G}$ be a group and $\xi \in FP(\mathfrak{G})$. Then for all $g_1, g_2 \in \mathfrak{G}$, the following conditions are equivalent:*

1. $\xi(g_1 g_2) = \xi(g_2 g_1)$.
2. $\xi(g_1 g_2 g_1^{-1}) = \xi(g_2)$.
3. $\xi(g_1 g_2 g_1^{-1}) \geq \xi(g_2)$.
4. $\xi(g_1 g_2 g_1^{-1}) \leq \xi(g_2)$.

Proof. (1) ⇒ (2): Suppose that $g_1, g_2 \in \mathfrak{G}$. Then

$$\begin{aligned}\xi(g_1 g_2 g_1^{-1}) &= \xi(g_1^{-1} g_1 g_2) \ (since\ \xi(g_1 g_2) = \xi(g_2 g_1)) \\ &= \xi(e.g_2) = \xi(g_2).\end{aligned}$$

(2) ⇒ (3): Obvious.
(3) ⇒ (4): Let $g_1, g_2 \in \mathfrak{G}$. Then $\xi(g_1 g_2 g_1^{-1}) \leq \xi(g_1^{-1}.g_1 g_2 g_1^{-1}.(g_1^{-1})^{-1}) = \xi(g_2)$.
(4) ⇒ (1): Suppose that $g_1, g_2 \in \mathfrak{G}$. Then

$$\begin{aligned}\xi(g_1 g_2) &= \xi(g_1.g_2 g_1..g_1^{-1}) \ (since\ \xi(g_1 g_2 g_1^{-1}) \leq \xi(g_2)) \\ &\leq \xi(g_2 g_1),\end{aligned}$$

and $\xi(g_2 g_1) = \xi(g_2.g_1 g_2.g_2^{-1}) \leq \xi(g_1 g_2)$. Hence $\xi(g_1 g_2) = \xi(g_2 g_1)$. □

Normal subgroups play a fundamental role in classical group theory. Because they are backbone in the structure of groups. Similarly, a $N.F.S$ plays an important role in the theory of fuzzy groups.

Definition 2.4.3. *[17] Let ξ be a f.s of a group $\mathfrak{G}$, then ξ is called $N.F.S$ of $\mathfrak{G}$ if it is an abelian f.s of $\mathfrak{G}$.*

We denote the set of all $N.F.S$s of $\mathfrak{G}$ by $NF(\mathfrak{G})$.

Example 2.4.4. *Let $Q_8 = \{\pm 1, \pm i, \pm j, \pm k\}$ be a group of quaternions with eight elements.*
We define $\psi_1 \in F(Q_8)$ by

$$\psi_1(1) = s_\circ$$

$$\psi_1(-1) = \psi_1(\pm i) = s_1$$

$$\psi_1(\pm j) = \psi_1(\pm k) = s_2$$

$\forall$ $s_\circ > s_1 > s_2$ *and* $s_\circ, s_1, s_2 \in \mathcal{I}$.

Now by using the definition 2.4.3, we will verify that $\psi_1 \lhd Q_8$.
We fix 1,

$$\psi_1(1.1) = \psi_1(1.1) \implies \psi_1(1) = \psi_1(1) \implies s_\circ = s_\circ$$

$$\psi_1(1.-1) = \psi_1(-1.1) \implies \psi_1(-1) = \psi_1(-1) \implies s_1 = s_1$$

$$\psi_1(1.i) = \psi_1(i.1) \implies \psi_1(i) = \psi_1(i) \implies s_1 = s_1$$

$$\psi_1(1.-i) = \psi_1(-i.1) \implies \psi_1(-i) = \psi_1(-i) \implies s_1 = s_1$$

$$\psi_1(1.j) = \psi_1(j.1) \implies \psi_1(j) = \psi_1(j) \implies s_2 = s_2$$

$$\psi_1(1.-j) = \psi_1(-j.1) \implies \psi_1(-j) = \psi_1(-j) \implies s_2 = s_2$$

$$\psi_1(1.k) = \psi_1(k.1) \implies \psi_1(k) = \psi_1(k) \implies s_2 = s_2$$

$$\psi_1(1.-k) = \psi_1(-k.1) \implies \psi_1(-k) = \psi_1(-k) \implies s_2 = s_2$$

Fix -1,

$$\psi_1(-1.1) = \psi_1(-1.1) \implies \psi_1(-1) = \psi_1(-1) \implies s_1 = s_1$$

$$\psi_1(-1.-1) = \psi_1(-1.-1) \implies \psi_1(1) = \psi_1(1) \implies s_\circ = s_\circ$$

$$\psi_1(-1.i) = \psi_1(i.-1) \implies \psi_1(-i) = \psi_1(-i) \implies s_1 = s_1$$

$$\psi_1(-1.-i) = \psi_1(-i.-1) \implies \psi_1(i) = \psi_1(i) \implies s_1 = s_1$$

$$\psi_1(-1.j) = \psi_1(j.-1) \implies \psi_1(-j) = \psi_1(-j) \implies s_2 = s_2$$

$$\psi_1(-1.-j) = \psi_1(-j.-1) \implies \psi_1(j) = \psi_1(j) \implies s_2 = s_2$$

$$\psi_1(-1.k) = \psi_1(k.-1) \implies \psi_1(-k) = \psi_1(-k) \implies s_2 = s_2$$

$$\psi_1(-1.-k) = \psi_1(-k.-1) \implies \psi_1(k) = \psi_1(k) \implies s_2 = s_2$$

Fix i,

$$\psi_1(i.1) = \psi_1(1.i) \implies \psi_1(i) = \psi_1(i) \implies s_1 = s_1$$

$$\psi_1(i.-1) = \psi_1(-1.i) \implies \psi_1(-i) = \psi_1(-i) \implies s_1 = s_1$$

$$\psi_1(i.i) = \psi_1(i.i) \implies \psi_1(-1) = \psi_1(-1) \implies s_1 = s_1$$

$$\psi_1(i.-i) = \psi_1(-i.i) \implies \psi_1(1) = \psi_1(1) \implies s_\circ = s_\circ$$

$$\psi_1(i.j) = \psi_1(j.i) \implies \psi_1(k) = \psi_1(-k) \implies s_2 = s_2$$

$$\psi_1(i.-j) = \psi_1(-j.i) \implies \psi_1(-k) = \psi_1(k) \implies s_2 = s_2$$

$$\psi_1(i.k) = \psi_1(k.i) \implies \psi_1(-j) = \psi_1(j) \implies s_2 = s_2$$

$$\psi_1(i.-k) = \psi_1(-k.i) \implies \psi_1(j) = \psi_1(-j) \implies s_2 = s_2$$

Fix $-i$,

$$\psi_1(-i.1) = \psi_1(1.-i) \implies \psi_1(-i) = \psi_1(-i) \implies s_1 = s_1$$

$$\psi_1(-i.-1) = \psi_1(-1.-i) \implies \psi_1(i) = \psi_1(i) \implies s_1 = s_1$$

$$\psi_1(-i.i) = \psi_1(i.-i) \implies \psi_1(1) = \psi_1(1) \implies s_\circ = s_\circ$$

$$\psi_1(-i.-i) = \psi_1(-i.-i) \implies \psi_1(-1) = \psi_1(-1) \implies s_1 = s_1$$

$$\psi_1(-i.j) = \psi_1(j.-i) \implies \psi_1(-k) = \psi_1(k) \implies s_2 = s_2$$

$$\psi_1(-i.-j) = \psi_1(-j.-i) \implies \psi_1(k) = \psi_1(k) \implies s_2 = s_2$$

$$\psi_1(-i.k) = \psi_1(k.-i) \implies \psi_1(j) = \psi_1(-j) \implies s_2 = s_2$$

$$\psi_1(-i.-k) = \psi_1(-k.-i) \implies \psi_1(-j) = \psi_1(j) \implies s_2 = s_2$$

Fix j,

$$\psi_1(j.1) = \psi_1(1.j) \implies \psi_1(j) = \psi_1(j) \implies s_2 = s_2$$

$$\psi_1(j.-1) = \psi_1(-1.j) \implies \psi_1(-j) = \psi_1(-j) \implies s_2 = s_2$$

$$\psi_1(j.i) = \psi_1(i.j) \implies \psi_1(-k) = \psi_1(k) \implies s_2 = s_2$$

$$\psi_1(j.-i) = \psi_1(-i.j) \implies \psi_1(k) = \psi_1(-k) \implies s_2 = s_2$$

$$\psi_1(j.j) = \psi_1(j.j) \implies \psi_1(1) = \psi_1(-1) \implies s_1 = s_1$$

$$\psi_1(j.-j) = \psi_1(-j.j) \implies \psi_1(1) = \psi_1(1) \implies s_\circ = s_\circ$$

$$\psi_1(j.k) = \psi_1(k.j) \implies \psi_1(i) = \psi_1(-i) \implies s_1 = s_1$$

$$\psi_1(j.-k) = \psi_1(-k.j) \implies \psi_1(-i) = \psi_1(i) \implies s_1 = s_1$$

Fix $-j$,

$$\psi_1(-j.1) = \psi_1(1.-j) \implies \psi_1(-j) = \psi_1(-j) \implies s_2 = s_2$$

$$\psi_1(-j.-1) = \psi_1(-1.-j) \implies \psi_1(j) = \psi_1(j) \implies s_2 = s_2$$

$$\psi_1(-j.i) = \psi_1(i.-j) \implies \psi_1(k) = \psi_1(-k) \implies s_2 = s_2$$

$$\psi_1(-j.-i) = \psi_1(-i.-j) \implies \psi_1(-k) = \psi_1(k) \implies s_2 = s_2$$

$$\psi_1(-j.j) = \psi_1(j.-j) \implies \psi_1(1) = \psi_1(1) \implies s_\circ = s_\circ$$

$$\psi_1(-j.-j) = \psi_1(-j.-j) \implies \psi_1(-1) = \psi_1(-1) \implies s_1 = s_1$$

$$\psi_1(-j.k) = \psi_1(k.-j) \implies \psi_1(-i) = \psi_1(i) \implies s_1 = s_1$$

$$\psi_1(-j.-k) = \psi_1(-k.-j) \implies \psi_1(i) = \psi_1(-i) \implies s_1 = s_1$$

Fix k,

$$\psi_1(k.1) = \psi_1(1.k) \implies \psi_1(k) = \psi_1(k) \implies s_2 = s_2$$

$$\psi_1(k.-1) = \psi_1(-1.k) \implies \psi_1(-k) = \psi_1(-k) \implies s_2 = s_2$$

$$\psi_1(k.i) = \psi_1(i.k) \implies \psi_1(j) = \psi_1(-j) \implies s_2 = s_2$$

$$\psi_1(k.-i) = \psi_1(-i.k) \implies \psi_1(-j) = \psi_1(j) \implies s_2 = s_2$$

$$\psi_1(k.j) = \psi_1(j.k) \implies \psi_1(-i) = \psi_1(i) \implies s_1 = s_1$$

$$\psi_1(k.-j) = \psi_1(-j.k) \implies \psi_1(i) = \psi_1(-i) \implies s_1 = s_1$$

$$\psi_1(k.k) = \psi_1(k.k) \implies \psi_1(-1) = \psi_1(-1) \implies s_1 = s_1$$

$$\psi_1(k.-k) = \psi_1(-k.k) \implies \psi_1(1) = \psi_1(1) \implies s_\circ = s_\circ$$

Fix $-k$,

$$\psi_1(-k.1) = \psi_1(1.-k) \implies \psi_1(-k) = \psi_1(-k) \implies s_2 = s_2$$

$$\psi_1(-k.-1) = \psi_1(-1.-k) \implies \psi_1(k) = \psi_1(k) \implies s_2 = s_2$$

$$\psi_1(-k.i) = \psi_1(i.-k) \implies \psi_1(-j) = \psi_1(j) \implies s_2 = s_2$$

$$\psi_1(-k.-i) = \psi_1(-i.-k) \implies \psi_1(j) = \psi_1(-j) \implies s_2 = s_2$$

$$\psi_1(-k.j) = \psi_1(j.-k) \implies \psi_1(i) = \psi_1(-i) \implies s_1 = s_1$$

$$\psi_1(-k.-j) = \psi_1(-j.-k) \implies \psi_1(-i) = \psi_1(i) \implies s_1 = s_1$$

$$\psi_1(-k.k) = \psi_1(k.-k) \implies \psi_1(1) = \psi_1(1) \implies s_\circ = s_\circ$$

$$\psi_1(-k.-k) = \psi_1(-k.-k) \implies \psi_1(-1) = \psi_1(-1) \implies s_1 = s_1$$

Thus, by definition 2.4.3, $\psi_1 \in NF(Q_8)$.

Remark 2.4.5. *Let $\mathfrak{G}$ be a finite abelian group then all of its fuzzy subgroups are N.F.Ss of $\mathfrak{G}$.*

Example 2.4.6. *Let $K = \{e, a\} \times \{e, b\}$ be the non-cyclic group of order 4 and its elements can be written as $\{e, a, b, ab\}$. We define $\xi \in F(K)$ by*

$$\xi(e) = \xi(ab) = t_\circ$$

$$\xi(a) = \xi(b) = t_1$$

$\forall\ t_\circ > t_1$ and $t_\circ, t_1 \in \mathcal{I}$. Since K is an abelian group so from the remark 2.4.5, $\xi \in NF(K)$.

Remark 2.4.7. *If ξ_1, ξ_2 are the fuzzy subgroups of $\mathfrak{G}$ and $\xi_1, \xi_2 \notin NF(\mathfrak{G})$, then $\xi_1 \cap \xi_2 \ntrianglelefteq \mathfrak{G}$.*

We give the following example to explain the above remark.

Example 2.4.8. *Let $\mathfrak{G} = D_6 = \{e, a, a^2, b, ab, a^2b\}$. Define $\xi_1, \xi_2 \in F(\mathfrak{G})$ as follows*

$$\forall\ t_\circ > t_1 > t_2 \text{ and } t_\circ, t_1, t_2 \in \mathcal{I}.$$

$$\xi_1(e) = \xi_1(a^2b) = t_\circ$$

$$\xi_1(a) = \xi_1(a^2) = \xi_1(b) = \xi_1(ab) = t_1$$

And

$$\xi_2(e) = \xi_2(a) = \xi_2(b) = \xi_2(a^2) = t_\circ$$

$$\xi_2(a^2b) = \xi_2(ab) = t_1$$

Clearly $\xi_1, \xi_2 \notin NF(\mathfrak{G})$ *because* $\xi_1(a^2b) \neq \xi_1(ba^2)$ *and* $\xi_2(ab) \neq \xi_2(a^2b)$*. From the definition of N.F.S it is immediate that*

$$(\xi_1 \cap \xi_2)(ab) \neq (\xi_1 \cap \xi_2)(ba).$$

Hence $\xi_1 \cap \xi_2 \ntriangleleft \mathfrak{G}$.

Definition 2.4.9. *Let* ξ_1 *and* $\xi_2 \in F(\mathfrak{G})$*. Then* ξ_1 *is called a F.S of a fuzzy group* ξ_2 *and is denoted by* $\xi_1 \lessdot \xi_2$*, if*

$$\xi_1(x) \leq \xi_2(x) \ \forall \ x \in \mathfrak{G}.$$

Definition 2.4.10. *[17] Let* ξ_1 *and* ξ_2 *be the fuzzy subgroups of* $\mathfrak{G}$ *and* ξ_1 *be a subset of* ξ_2*. Then* ξ_1 *is called a N.F.S of* ξ_2 *and is denoted by* $\xi_1 \prec \xi_2$*, if*

$$\xi_1(g_1 g_2 g_1^{-1}) \geq \xi_1(g_2) \wedge \xi_2(g_1) \ \forall \ g_1, g_2 \in \mathfrak{G}.$$

Remark 2.4.11. *[17]*

Let ξ_1 *be a F.S of* $\mathfrak{G}$*, then* ξ_1 *is a N.F.S of itself.*

Theorem 2.4.12. *[17] If* $\xi_1 \lhd \mathfrak{G}$ *and* $\xi_2 \in F(\mathfrak{G})$*, then* $\xi_1 \cap \xi_2 \prec \xi_2$*.*

Proof. Observe that $\xi_1 \cap \xi_2 \in F(\mathfrak{G})$ (by theorem 2.2.10) and $\xi_1 \cap \xi_2 \subseteq \xi_2$. Now

$$\begin{aligned}(\xi_1 \cap \xi_2)(g_1 g_2 g_1^{-1}) &= \xi_1(g_1 g_2 g_1^{-1}) \wedge \xi_2(g_1 g_2 g_1^{-1})(by\ definition\ 2.1.5)\\ &= \xi_1(g_2) \wedge \xi_2(g_1 g_2 g_1^{-1})\\ &\geq \xi_1(g_2) \wedge \xi_2(g_1) \wedge \xi_2(g_2) \wedge \xi_2(g_1^{-1})\\ &= (\xi_1 \cap \xi_2)(g_2) \wedge \xi_2(g_1)\end{aligned}$$

$\forall g_1, g_2 \in \mathfrak{G}$. Therefore, $\xi_1 \cap \xi_2 \prec \xi_2$. □

Theorem 2.4.13. *If* $\xi_1' \prec \xi_1$ *and* $\xi_1, \xi_2 \in F(\mathfrak{G})$*, then* $\xi_1' \cap \xi_2 \prec \xi_1 \cap \xi_2$*.*

Proof. It is easily observed that $\xi_1' \cap \xi_2 \in F(\mathfrak{G})$ and $\xi_1' \cap \xi_2 \subseteq \xi_1 \cap \xi_2$. Now

$$\begin{aligned}(\xi_1' \cap \xi_2)(xyx^{-1}) &= \xi_1'(xyx^{-1}) \wedge \xi_2(xyx^{-1})(by\ definition\ 2.1.5)\\ &\geq \xi_1'(y) \wedge \xi_1(x) \wedge \xi_2(x) \wedge \xi_2(y) \wedge \xi_2(x^{-1})\\ &= \xi_1'(y) \wedge \xi_2(y) \wedge \xi_1(x) \wedge \xi_2(x) \wedge \xi_2(x)\\ &= (\xi_1' \cap \xi_2)(y) \wedge (\xi_1 \cap \xi_2)(x)\end{aligned}$$

$\forall\ x, y \in \mathfrak{G}$. Therefore, $\xi_1' \cap \xi_2 \prec \xi_1 \cap \xi_2$. □

Theorem 2.4.14. *[17] Let $\xi \lhd \mathfrak{G}$. Then support and core of ξ are normal subgroups of $\mathfrak{G}$.*

Proof. Let $\xi \in NF(\mathfrak{G})$, then $\xi_*, \xi^* \lesssim \mathfrak{G}$ (by lemma 2.2.9). Now suppose that $g_1 \in \mathfrak{G}$ and $g_2 \in \xi_*$. Since $\xi \lhd \mathfrak{G}$, we have

$$\xi(g_1 g_2 g_1^{-1}) = \xi(g_2) = \xi(e)$$

and thus $g_1 g_2 g_1^{-1} \in \xi_*$. Therefore, $\xi_* \lhd \mathfrak{G}$. Now we want to show that $\xi^* \lhd \mathfrak{G}$. Suppose g_1 belongs to $\mathfrak{G}$ and g_2 belongs to ξ^*. Since $\xi \lhd \mathfrak{G}$, we have

$$\xi(g_1 g_2 g_1^{-1}) = \xi(g_2) > 0$$

and so $g_1 g_2 g_1^{-1} \in \xi^*$. Hence $\xi^* \lhd \mathfrak{G}$. □

Definition 2.4.15 (Fuzzy Coset)**.** *[18] Let $\xi \in F(\mathfrak{G})$. Let $\mathfrak{x} \in \mathfrak{G}$ and $\hat{\xi}$ be a map*

$$\hat{\xi}_{\mathfrak{x}} : \mathfrak{G} \longrightarrow \mathcal{I}$$

defined by

$$\hat{\xi}_{\mathfrak{x}}(g) = \xi(g\mathfrak{x}^{-1})\ \forall\ g \in \mathfrak{G} \tag{2.1}$$

$\hat{\xi}_{\mathfrak{x}}$ is known as the fuzzy right coset of ξ in $\mathfrak{G}$. $\hat{\xi}_{\mathfrak{x}}$ is also denoted by $\xi\mathfrak{x}$. Similarly, a f.s $\mathfrak{x}\xi$ is called a fuzzy left coset of ξ in $\mathfrak{G}$ if

$$\hat{\xi}_{\mathfrak{x}}(g) = \xi(\mathfrak{x}^{-1}g) \qquad \forall\ g \in \mathfrak{G}.$$

Example 2.4.16. *Let $K = \{e, a\} \times \{e, b\}$ be the non-cyclic group of order 4 and its elements can be written as $\{e, a, b, ab\}$. We define $\xi \in F(K)$ by*

$$\xi(e) = t_\circ$$

$$\xi(a) = t_1$$

$$\xi(b) = \xi(ab) = t_2$$

$\forall\ t_\circ > t_1 > t_2$ *and* $t_\circ, t_1, t_2 \in \mathcal{I}$.

The F.C corresponding to $e \in K$,

$$\hat{\xi}_e(e) = \xi(e) = t_\circ$$

$$\hat{\xi}_e(a) = \xi(a) = t_1$$

$$\hat{\xi}_e(b) = \xi(b) = t_2$$

$$\hat{\xi}_e(ab) = \xi(ab) = t_2$$

So F.C corresponding to e is $\{t_\circ, t_1, t_2, t_2\}$, *where* $\{t_\circ, t_1, t_2, t_2\}$ *is a* ***multi-set***, *which allows repetition.*

The F.C corresponding to $a \in K$,

$$\hat{\xi}_a(e) = \xi(a) = t_1$$

$$\hat{\xi}_a(a) = \xi(e) = t_\circ$$

$$\hat{\xi}_a(b) = \xi(ba) = t_2$$

$$\hat{\xi}_a(ab) = \xi(b) = t_2$$

So F.C corresponding to a is $\{t_1, t_\circ, t_2, t_2\}$.

The F.C corresponding to $b \in K$,

$$\hat{\xi}_b(e) = \xi(b) = t_2$$

$$\hat{\xi}_b(a) = \xi(ab) = t_2$$

$$\hat{\xi}_b(b) = \xi(e) = t_\circ$$

$$\hat{\xi}_b(ab) = \xi(a) = t_1$$

So F.C corresponding to b is $\{t_2, t_2, t_\circ, t_1\}$.

The F.C corresponding to $ab \in K$,

$$\hat{\xi}_{ab}(e) = \xi(ab) = t_2$$

$$\hat{\xi}_{ab}(a) = \xi(b) = t_2$$

$$\hat{\xi}_{ab}(b) = \xi(a) = t_1$$

$$\hat{\xi}_{ab}(ab) = \xi(e) = t_\circ$$

So F.C corresponding to ab is $\{t_2, t_2, t_1, t_\circ\}$.

Note that $\hat{\xi}_e, \hat{\xi}_a, \hat{\xi}_b$ and $\hat{\xi}_{ab}$ are the fuzzy cosets of $\mathfrak{G}$ with respect to ξ.

Example 2.4.17. *Let* $S_3 = \{e, a, a^2, b, ab, a^2b\}$ *be a symmetric group with six elements. We define* ξ *of* $F(S_3)$.

$$\xi(e) = \xi(a^2b) = t_\circ$$

$$\xi(a) = \xi(a^2) = \xi(b) = \xi(ab) = t_1$$

$$\forall\ t_\circ > t_1 \text{ and } t_\circ, t_1 \in \mathcal{I}.$$

The F.C corresponding to $e \in S_3$,

$$\hat{\xi}_e(e) = \xi(e) = t_\circ$$

$$\hat{\xi}_e(a) = \xi(a) = t_1$$

$$\hat{\xi}_e(a^2) = \xi(a^2) = t_1$$

$$\hat{\xi}_e(b) = \xi(b) = t_1$$

$$\hat{\xi}_e(ab) = \xi(ab) = t_1$$

$$\hat{\xi}_e(a^2b) = \xi(a^2b) = t_\circ$$

So F.C corresponding to e is $\{t_\circ, t_1, t_1, t_1, t_1, t_\circ\}$.
The F.C corresponding to $a \in S_3$,

$$\hat{\xi}_a(e) = \xi(a^2) = t_1$$

$$\hat{\xi}_a(a) = \xi(e) = t_\circ$$

$$\hat{\xi}_a(a^2) = \xi(a) = t_1$$

$$\hat{\xi}_a(b) = \xi(ab) = t_1$$

$$\hat{\xi}_a(ab) = \xi(a^2b) = t_\circ$$

$$\hat{\xi}_a(a^2b) = \xi(b) = t_1$$

So F.C corresponding to a is $\{t_1, t_\circ, t_1, t_1, t_\circ, t_1\}$.

The F.C corresponding to $a^2 \in S_3$,

$$\hat{\xi}_{a^2}(e) = \xi(a) = t_1$$

$$\hat{\xi}_{a^2}(a) = \xi(a^2) = t_1$$

$$\hat{\xi}_{a^2}(a^2) = \xi(e) = t_\circ$$

$$\hat{\xi}_{a^2}(b) = \xi(a^2b) = t_\circ$$

$$\hat{\xi}_{a^2}(ab) = \xi(b) = t_1$$

$$\hat{\xi}_{a^2}(a^2b) = \xi(ab) = t_1$$

So F.C corresponding to a^2 *is* $\{t_1, t_1, t_\circ, t_\circ, t_1, t_1\}$.

The F.C corresponding to $b \in S_3$,

$$\hat{\xi}_b(e) = \xi(b) = t_1$$

$$\hat{\xi}_b(a) = \xi(ab) = t_1$$

$$\hat{\xi}_b(a^2) = \xi(a^2b) = t_\circ$$

$$\hat{\xi}_b(b) = \xi(e) = t_\circ$$

$$\hat{\xi}_b(ab) = \xi(a) = t_1$$

$$\hat{\xi}_b(a^2b) = \xi(a^2) = t_1$$

So F.C corresponding to b is $\{t_1, t_1, t_\circ, t_\circ, t_1, t_1\}$.

The F.C corresponding to $ab \in S_3$,

$$\hat{\xi}_{ab}(e) = \xi(ab) = t_1$$

$$\hat{\xi}_{ab}(a) = \xi(a^2b) = t_\circ$$

$$\hat{\xi}_{ab}(a^2) = \xi(b) = t_1$$

$$\hat{\xi}_{ab}(b) = \xi(a^2) = t_1$$

$$\hat{\xi}_{ab}(ab) = \xi(e) = t_\circ$$

$$\hat{\xi}_{ab}(a^2b) = \xi(a) = t_1$$

So F.C corresponding to ab is $\{t_1, t_\circ, t_1, t_1, t_\circ, t_1\}$.
The F.C corresponding to $a^2b \in S_3$,

$$\hat{\xi}_{a^2b}(e) = \xi(a^2b) = t_\circ$$

$$\hat{\xi}_{a^2b}(a) = \xi(b) = t_1$$

$$\hat{\xi}_{a^2b}(a^2) = \xi(ab) = t_1$$

$$\hat{\xi}_{a^2b}(b) = \xi(a) = t_1$$

$$\hat{\xi}_{a^2b}(ab) = \xi(a^2) = t_1$$

$$\hat{\xi}_{a^2b}(a^2b) = \xi(e) = t_\circ$$

So F.C corresponding to e is $\{t_\circ, t_1, t_1, t_1, t_1, t_\circ\}$.

Then we can say that $\hat{\xi}_e, \hat{\xi}_a, \hat{\xi}_{a^2}, \hat{\xi}_b, \hat{\xi}_{ab}$ *and* $\hat{\xi}_{a^2b}$ *are the fuzzy cosets of* S_3 *with respect to* ξ.

Example 2.4.18. *Let* $Q_8 = \{\pm 1, \pm i, \pm j, \pm k\}$ *be a group of quaternions with eight elements.*
We define a F.S ϕ *of* Q_8 *by*

$$\phi(1) = t_\circ$$

$$\phi(-1) = \phi(\pm i) = t_1$$

$$\phi(\pm j) = \phi(\pm k) = t_2.$$

$\forall\ t_\circ > t_1 > t_2$ *and* $t_\circ, t_1, t_2 \in \mathcal{I}$.
The F.C corresponding to $1 \in Q_8$,

$$\hat{\phi}_1(1) = \phi(1) = t_\circ$$

$$\hat{\phi}_1(-1) = \phi(-1) = t_1$$

$$\hat{\phi}_1(i) = \phi(i) = t_1$$

$$\hat{\phi}_1(-i) = \phi(-i) = t_1$$

$$\hat{\phi}_1(j) = \phi(j) = t_2$$

$$\hat{\phi}_1(-j) = \phi(-j) = t_2$$

$$\hat{\phi}_1(k) = \phi(k) = t_2$$

$$\hat{\phi}_1(-k) = \phi(-k) = t_2$$

So, F.C corresponding to 1 *is* $\{t_\circ, t_1, t_1, t_1, t_2, t_2, t_2, t_2\}$.

The F.C corresponding to $-1 \in Q_8$,

$$\hat{\phi}_{-1}(1) = \phi(1. - 1) = \phi(-1) = t_1$$

$$\hat{\phi}_{-1}(-1) = \phi(-1. - 1) = \phi(1) = t_\circ$$

$$\hat{\phi}_{-1}(i) = \phi(i. - 1) = \phi(-i) = t_1$$

$$\hat{\phi}_{-1}(-i) = \phi(-i. - 1) = \phi(i) = t_1$$

$$\hat{\phi}_{-1}(j) = \phi(j. - 1) = \phi(-j) = t_2$$

$$\hat{\phi}_{-1}(-j) = \phi(-j. - 1) = \phi(j) = t_2$$

$$\hat{\phi}_{-1}(k) = \phi(k. - 1) = \phi(-k) = t_2$$

$$\hat{\phi}_{-1}(-k) = \phi(-k. - 1) = \phi(k) = t_2$$

So F.C corresponding to -1 *is* $\{t_1, t_\circ, t_1, t_1, t_2, t_2, t_2, t_2\}$.

The F.C corresponding to $i \in Q_8$,

$$\hat{\phi}_i(1) = \phi(1. - i) = \phi(-i) = t_1$$

$$\hat{\phi}_i(-1) = \phi(-1. - i) = \phi(i) = t_1$$

$$\hat{\phi}_i(i) = \phi(i. - i) = \phi(1) = t_\circ$$

$$\hat{\phi}_i(-i) = \phi(-i. - i) = \phi(-1) = t_1$$

$$\hat{\phi}_i(j) = \phi(j. - i) = \phi(-k) = t_2$$

$$\hat{\phi}_i(-j) = \phi(-j. - i) = \phi(-k) = t_2$$

$$\hat{\phi}_i(k) = \phi(k. - i) = \phi(-j) = t_2$$

$$\hat{\phi}_i(-k) = \phi(-k. - i) = \phi(j) = t_2$$

So F.C corresponding to i *is* $\{t_1, t_1, t_\circ, t_1, t_2, t_2, t_2, t_2\}$.

The F.C corresponding to $-i \in Q_8$,

$$\hat{\phi}_{-i}(1) = \phi(1.i) = \phi(i) = t_1$$

$$\hat{\phi}_{-i}(-1) = \phi(-1.i) = \phi(-i) = t_1$$

$$\hat{\phi}_{-i}(i) = \phi(i.i) = \phi(-1) = t_1$$

$$\hat{\phi}_{-i}(-i) = \phi(-i.i) = \phi(1) = t_\circ$$

$$\hat{\phi}_{-i}(j) = \phi(j.i) = \phi(-k) = t_2$$

$$\hat{\phi}_{-i}(-j) = \phi(-j.i) = \phi(k) = t_2$$

$$\hat{\phi}_{-i}(k) = \phi(k.i) = \phi(j) = t_2$$

$$\hat{\phi}_{-i}(-k) = \phi(-k.i) = \phi(-j) = t_2$$

So F.C corresponding to $-i$ *is* $\{t_1, t_1, t_1, t_\circ, t_2, t_2, t_2, t_2\}$.

The F.C corresponding to $j \in Q_8$,

$$\hat{\phi}_j(1) = \phi(1.-j) = \phi(-j) = t_2$$

$$\hat{\phi}_j(-1) = \phi(-1.-j) = \phi(j) = t_2$$

$$\hat{\phi}_j(i) = \phi(i.-j) = \phi(-k) = t_2$$

$$\hat{\phi}_j(-i) = \phi(-i.-j) = \phi(k) = t_2$$

$$\hat{\phi}_j(j) = \phi(j.-j) = \phi(1) = t_\circ$$

$$\hat{\phi}_j(-j) = \phi(-j.-j) = \phi(-1) = t_1$$

$$\hat{\phi}_j(k) = \phi(k.-j) = \phi(i) = t_1$$

$$\hat{\phi}_j(-k) = \phi(-k.-j) = \phi(-i) = t_1$$

So F.C corresponding to j *is* $\{t_2, t_2, t_2, t_2, t_\circ, t_1, t_1, t_1\}$.

The F.C corresponding to $-j \in Q_8$,

$$\hat{\phi}_{-j}(1) = \phi(1.j) = \phi(j) = t_2$$

$$\hat{\phi}_{-j}(-1) = \phi(-1.j) = \phi(-j) = t_2$$

$$\hat{\phi}_{-j}(i) = \phi(i.j) = \phi(k) = t_2$$

$$\hat{\phi}_{-j}(-i) = \phi(-i.j) = \phi(-k) = t_2$$

$$\hat{\phi}_{-j}(j) = \phi(j.j) = \phi(-1) = t_1$$

$$\hat{\phi}_{-j}(-j) = \phi(-j.j) = \phi(1) = t_\circ$$

$$\hat{\phi}_{-j}(k) = \phi(k.j) = \phi(-i) = t_1$$

$$\hat{\phi}_{-j}(-k) = \phi(-k.j) = \phi(i) = t_1$$

So F.C corresponding to $-j$ *is* $\{t_2, t_2, t_2, t_2, t_1, t_\circ, t_1, t_1\}$.

The F.C corresponding to $k \in Q_8$,

$$\hat{\phi}_k(1) = \phi(1. - k) = \phi(-k) = t_2$$

$$\hat{\phi}_k(-1) = \phi(-1. - k) = \phi(k) = t_2$$

$$\hat{\phi}_k(i) = \phi(i. - k) = \phi(-j) = t_2$$

$$\hat{\phi}_k(-i) = \phi(-i. - k) = \phi(j) = t_2$$

$$\hat{\phi}_k(j) = \phi(j. - k) = \phi(-i) = t_1$$

$$\hat{\phi}_k(-j) = \phi(-j. - k) = \phi(i) = t_1$$

$$\hat{\phi}_k(k) = \phi(k. - k) = \phi(1) = t_\circ$$

$$\hat{\phi}_k(-k) = \phi(-k. - k) = \phi(-1) = t_1$$

So F.C corresponding to k *is* $\{t_2, t_2, t_2, t_2, t_1, t_1, t_\circ, t_1\}$.

The F.C corresponding to $-k \in Q_8$,

$$\hat{\phi}_{-k}(1) = \phi(1.k) = \phi(k) = t_2$$

$$\hat{\phi}_{-k}(-1) = \phi(-1.k) = \phi(-k) = t_2$$

$$\hat{\phi}_{-k}(i) = \phi(i.k) = \phi(-j) = t_2$$

$$\hat{\phi}_{-k}(-i) = \phi(-i.k) = \phi(j) = t_2$$

$$\hat{\phi}_{-k}(j) = \phi(j.k) = \phi(i) = t_1$$

$$\hat{\phi}_{-k}(-j) = \phi(-j.k) = \phi(-i) = t_1$$

$$\hat{\phi}_{-k}(k) = \phi(k.k) = \phi(-1) = t_1$$

$$\hat{\phi}_{-k}(-k) = \phi(-k.k) = \phi(1) = t_\circ$$

So F.C corresponding to $-k$ *is* $\{t_2, t_2, t_2, t_2, t_1, t_1, t_1, t_\circ\}$.

Then we can say that $\hat{\phi}_1, \hat{\phi}_{-1}, \hat{\phi}_i, \hat{\phi}_{-i}, \hat{\phi}_j, \hat{\phi}_{-j}, \hat{\phi}_k, \hat{\phi}_{-k}$ *are the F.Cs of* Q_8 *with respect to* ϕ.

Example 2.4.19. *Let* $\mathfrak{G} = D_6 = \langle a, b : a^3 = e = b^2, bab = a^{-1}\rangle$. *We define a F.S* ν *of* S_3 *by*

$$\nu(e) = \nu(b) = 1$$

$$\nu(a) = \nu(a^2) = \nu(ab) = \nu(a^2b) = 0.5$$

Then the right F.C corresponding to a *in* S_3 *is given by*

$$\hat{\nu}_a(e) = \nu(a^2) = 0.5$$

$$\hat{\nu}_a(a) = \nu(e) = 1$$

$$\hat{\nu}_a(a^2) = \nu(a) = 0.5$$

$$\hat{\nu}_a(b) = \nu(ab) = 0.5$$

$$\hat{\nu}_a(ab) = \nu(a^2b) = 0.5$$

$$\hat{\nu}_a(a^2b) = \nu(b) = 1$$

So the right F.C $\nu a = (0.5, 1, 0.5, 0.5, 0.5, 1)$.

Now the left F.C corresponding to $a \in S_3$ *is given by*

$$\hat{\nu}_a(e) = \nu(a^2) = 0.5$$

$$\hat{\nu}_a(a) = \nu(e) = 1$$

$$\hat{\nu}_a(a^2) = \nu(a) = 0.5$$

$$\hat{\nu}_a(b) = \nu(a^2b) = 0.5$$

$$\hat{\nu}_a(ab) = \nu(b) = 1$$

$$\hat{\nu}_a(a^2b) = \nu(ab) = 0.5$$

So the left F.C $a\nu = (0.5, 1, 0.5, 0.5, 1, 0.5)$.

Hence $\nu a \neq a\nu$. *This happen because* $\nu \ntriangleleft S_3$.

Proposition 2.4.20. *[14] If* $\xi \lhd \mathfrak{G}$ *and* $x \in \mathfrak{G}$, *then the fuzzy right coset* ξx *is same as the fuzzy left coset* $x\xi$.

Proof. Let ξ be a $N.F.S$ of $\mathfrak{G}$ and $x \in \mathfrak{G}$. Then for any $g \in \mathfrak{G}$, $\xi(gx) = \xi(xg)$ and so

$$x\xi(g) = \xi(x^{-1}g) = \xi(gx^{-1}) = \xi x(g)$$

. Hence $x\xi = \xi x$. □

Theorem 2.4.21. *[18] Let $\xi \lhd \mathfrak{G}$ then it is constant on the conjugate classes of the group $\mathfrak{G}$. Moreover, the converse is also true.*

Proof. Let $\xi \lhd \mathfrak{G}$. Then

$$\xi(g_2^{-1}g_1g_2) = \xi(g_1g_2g_2^{-1}) = \xi(g_1)\ \forall g_1, g_2 \in \mathfrak{G}.$$

Conversely, suppose that ξ is constant on each conjugate class of $\mathfrak{G}$. Then

$$\xi(g_1g_2) = \xi(g_1g_2g_1g_1^{-1}) = \xi(g_1(g_2g_1)g_1^{-1}) = \xi(g_2g_1)\ \forall g_1, g_2 \in \mathfrak{G}.$$

Hence $\xi \lhd \mathfrak{G}$. □

We now give another formulation of a $N.F.S$ following the classical finite group theory. Let $\mathfrak{G}$ be a group and g_1, g_2 belongs to $\mathfrak{G}$, then the element $g_1^{-1}g_2^{-1}g_1g_2$, denoted by $[g_1, g_2]$, is called the commutator of g_1 and g_2.

We have $N \lhd \mathfrak{G}$ iff $[N, \mathfrak{G}] \leq N$.

Theorem 2.4.22. *[18] Let $\xi \in F(\mathfrak{G})$. Then $\xi \lhd \mathfrak{G}$ iff*

$$\xi([g_1, g_2]) \geq \xi(g_1)\ \forall\ g_1, g_2 \in \mathfrak{G}. \tag{2.2}$$

Proof. Suppose that $\xi \lhd \mathfrak{G}$ and $g_1, g_2 \in \mathfrak{G}$. Then

$$\begin{aligned}
\xi(g_1^{-1}g_2^{-1}g_1g_2) &\geq \xi(g_2^{-1}g_1g_2) \wedge \xi(g_1^{-1}) \\
&= \xi(g_1) \wedge \xi(g_1)\ (\because\ \xi \lhd \mathfrak{G}) \\
&= \xi(g_1).
\end{aligned}$$

Now suppose that $\xi([g_1, g_2]) \geq \xi(g_1)\ \forall\ g_1, g_2 \in \mathfrak{G}$. Then for $g_1, g_3 \in \mathfrak{G}$, we have

$$\begin{aligned}
\xi(g_1^{-1}g_3g_1) &= \xi(g_3g_3^{-1}g_1^{-1}g_3g_1) \\
&\geq \xi(g_3) \wedge \xi([g_3, g_1]) \\
&= \xi(g_3)\ by(2).
\end{aligned}$$

Thus

$$\xi(g_1^{-1}g_3g_1) \geq \xi(g_3)\ \forall g_3, g_1 \in \mathfrak{G}. \tag{2.3}$$

Again we get that

$$\xi(g_3) = \xi(g_1.g_1^{-1}g_3g_1g_1^{-1}) \geq \xi(g_1) \wedge \xi(g_1^{-1}g_3g_1). \tag{2.4}$$

Now if $\xi(g_1) \wedge \xi(g_1^{-1}g_3g_1) = \xi(g_1)$, then we get that $\xi(g_3) \geq \xi(g_1)$ $\forall g_1, g_3 \in \mathfrak{G}$, which, implies that ξ is a constant function, and there is nothing to prove. We are left with the possibility when $\xi(g_1) \wedge \xi(g_1^{-1}g_3g_1) = \xi(g_1^{-1}g_3g_1)$. Then 2.4 implies that $\xi(g_3) \geq \xi(g_1^{-1}g_3g_1)$ $\forall$ $g_1, g_3 \in \mathfrak{G}$. This implies that $\xi(g_1^{-1}g_3g_1) = \xi(g_3)$ $\forall$ $g_1, g_3 \in \mathfrak{G}$. Hence ξ is constant on the conjugacy classes of $\mathfrak{G}$. □

Theorem 2.4.23. *[17] Let $\xi \in F(\mathfrak{G})$. Then $\forall g_1, g_2 \in \mathfrak{G}$.*
$\hat{\xi}_{g_1} = \hat{\xi}_{g_2} \iff (\hat{\xi}_*)_{g_1} = (\hat{\xi}_*)_{g_2}$.

Proof. Suppose $\hat{\xi}_{g_1} = \hat{\xi}_{g_2}$. Then $\xi({g_1}^{-1}g_3) = \xi({g_2}^{-1}g_3)$ $\forall$ $g_3 \in \mathfrak{G}$. Now by replacing g_3 with g_2 yields $\xi({g_1}^{-1}g_2) = \xi({g_2}^{-1}g_2) = \xi(e)$ and thus ${g_1}^{-1}g_2 \in \xi_*$. Hence $(\hat{\xi}_*)_{g_1} = (\hat{\xi}_*)_{g_2}$.
Conversely, suppose that $(\hat{\xi}_*)_{g_1} = (\hat{\xi}_*)_{g_2}$. Then ${g_1}^{-1}g_2 \in \xi_*$ and ${g_2}^{-1}g_1 \in \xi_*$. Hence

$$\begin{aligned}
\xi({g_1}^{-1}g_3) &= \xi({g_1}^{-1}g_2.{g_2}^{-1}g_3) \\
&\geq \xi({g_1}^{-1}g_2) \wedge \xi({g_2}^{-1}g_3) \\
&= \xi(e) \wedge \xi({g_2}^{-1}g_3) \\
&= \xi({g_2}^{-1}g_3)
\end{aligned}$$

$\forall$ $g_3 \in \mathfrak{G}$. Similarly, $\xi({g_1}^{-1}g_3) = \xi({g_2}^{-1}g_3)$ $\forall$ $g_3 \in \mathfrak{G}$. Therefore, $\xi({g_1}^{-1}g_3) = \xi({g_2}^{-1}g_3)$ $\forall$ $g_3 \in \mathfrak{G}$ which shows that $\hat{\xi}_{g_1} = \hat{\xi}_{g_2}$. □

Theorem 2.4.24. *[17] Let $\xi \in NF(\mathfrak{G})$ and $g_1, g_2 \in \mathfrak{G}$. If $\hat{\xi}_{g_1} = \hat{\xi}_{g_2}$, then $\xi(g_1) = \xi(g_2)$.*

Proof. Let $\hat{\xi}_{g_1} = \hat{\xi}_{g_2}$. By theorem 2.4.23, ${g_1}^{-1}g_2 \in \xi_*$ and ${g_2}^{-1}g_1 \in \xi_*$. Since $\xi \in NF(\mathfrak{G})$ it follows that $\xi(g_1) = \xi({g_2}^{-1}g_1g_2) \geq \xi({g_2}^{-1}g_1) \wedge \xi(g_2) = \xi(e) \wedge \xi(g_2) = \xi(g_2)$.This implies, $\xi(g_1) \geq \xi(g_2)$.
Now we want to show that $\xi(g_2) \geq \xi(g_1)$. Similarly $\xi(g_2) = \xi({g_1}^{-1}g_2g_1) \geq \xi({g_1}^{-1}g_2) \wedge \xi(g_1) = \xi(e) \wedge \xi(g_1) = \xi(g_1)$. we get that, $\xi(g_2) \geq \xi(g_1)$. Hence $\xi(g_2) = \xi(g_1)$. □

Proposition 2.4.25. *[18] If $\xi \lhd \mathfrak{G}$. Let $x, y \in \mathfrak{G}$ then*

$$\hat{\xi}_x(gx) = \hat{\xi}_x(xg) = \xi(g) \qquad \forall g \in \mathfrak{G}.$$

Proof. We have

$$\begin{aligned}\hat{\xi}_x(gx) &= \hat{\xi}_x(xgx^{-1}x)\\ &= \xi(xg^{-1})\\ \hat{\xi}_x(gx) &= \xi(g).\end{aligned}$$

Since $\xi \lhd \mathfrak{G}$. Similarly,

$$\begin{aligned}\hat{\xi}_x(gx) &= \hat{\xi}(gxx^{-1})\\ &= \xi(g)\end{aligned}$$

□

Theorem 2.4.26. *[18] Let $\xi \lhd \mathfrak{G}$. Let $\mathcal{C}$ denote the set of all the F.Cs of ξ. Then $\mathcal{C}$ is a group under the composition*

$$\hat{\xi}_x \circ \hat{\xi}_y = \hat{\xi}_{xy} \qquad \forall\ x, y \in \mathfrak{G}. \tag{2.5}$$

Define a map $\bar{\xi} : \mathcal{C} \to \mathcal{I}$ by

$$\bar{\xi}(\hat{\xi}_x) = \xi(x) \qquad \forall\ x \in \mathfrak{G}. \tag{2.6}$$

Then $\bar{\xi} \in F(\mathcal{C})$.

Proof. We must show that the relation $\hat{\xi}_{\mathfrak{x}}(g) = \xi(g\mathfrak{x}^{-1})\ \forall\ g \in \mathfrak{G}$ is well defined. Let $x, y, x_0, y_0 \in \mathfrak{G}$ such that

$$\hat{\xi}_x = \hat{\xi}_{x_0} \quad \textit{and} \quad \hat{\xi}_y = \hat{\xi}_{y_0} \tag{2.7}$$

Then we must show that

$$\hat{\xi}_x \circ \hat{\xi}_y = \hat{\xi}_{x_0} \circ \hat{\xi}_{y_0}$$

that is, $\hat{\xi}_{xy} = \hat{\xi}_{x_0 y_0}$

We have, by definition 3.1,

$$\hat{\xi}_{xy}(g) = \xi(gy^{-1}x^{-1}) \qquad \forall\ g \in \mathfrak{G}$$

and

$$\hat{\xi}_{x_0 y_0}(g) = \xi(g{y_0}^{-1}{x_0}^{-1}) \qquad \forall\ g \in \mathfrak{G}.$$

Now

$$\xi(gy^{-1}x^{-1}) = \xi(gy_0{}^{-1}y_0y^{-1}x^{-1})$$
$$= \xi(gy_0{}^{-1}x_0{}^{-1}x_0y_0y^{-1}x^{-1})$$

$$\geq \xi(gy_0{}^{-1}x_0{}^{-1}) \wedge \xi(x_0y_0y^{-1}x^{-1}). \tag{2.8}$$

By 2.7 we have that

$$\xi(gx^{-1}) = \xi(gx_0^{-1}) \qquad \forall\ g \in \mathfrak{G} \tag{2.9}$$

and

$$\xi(gy^{-1}) = \xi(gy_0^{-1}) \qquad \forall\ g \in \mathfrak{G}. \tag{2.10}$$

Now in 2.9, substituting $x_0y_0y^{-1}$ for g, we have that

$$\xi(x_0y_0y^{-1}x^{-1}) = \xi(x_0y_0y^{-1}x_0^{-1})$$

Since $\xi \lhd \mathfrak{G}$,

$$= \xi(y_0y^{-1})$$

$$= \xi(e)$$

let $g = y_0$ in 2.10. Also $\xi(e) \geq \xi(gy_0{}^{-1}x_0{}^{-1})$, clearly $\xi(e) \geq \xi(u)\ \forall\ u \in \mathfrak{G}$, where $\xi \in F(\mathfrak{G})$. Hence 2.8 implies that

$$\xi(gy^{-1}x^{-1}) \geq \xi(gy_0{}^{-1}x_0{}^{-1})$$

Now

$$\xi(gy_0{}^{-1}x_0{}^{-1}) = \xi(gy^{-1}yy_0{}^{-1}x_0{}^{-1})$$
$$= \xi(gy^{-1}x^{-1}xyy_0{}^{-1}x_0{}^{-1})$$

$$\geq \xi((gy^{-1}x^{-1}) \wedge \xi(xyy_0{}^{-1}x_0{}^{-1}). \tag{2.11}$$

Now in 2.9, substituting $xyy_0{}^{-1}$ for g, we have that

$$\xi(xyy_0{}^{-1}x^{-1}) = \xi(xyy_0{}^{-1}x_0^{-1})$$

Since ξ is a $N.F.S$ of $\mathfrak{G}$,

$$\xi(y{y_0}^{-1}) = \xi(xy{y_0}^{-1}x_0^{-1})$$
$$\xi(e) = \xi(xy{y_0}^{-1}x_0^{-1})$$

let $g = y$ in 2.10. Also $\xi(e) \geq \xi(gy^{-1}x^{-1})$, clearly $\xi(e) \geq \xi(u)\ \forall\ u \in \mathfrak{G}$, where $\xi \in F(\mathfrak{G})$. Hence 2.11 implies that

$$\xi(g{y_0}^{-1}{x_0}^{-1}) \geq \xi(gy^{-1}x^{-1})$$

Hence we have that $\xi(g{y_0}^{-1}{x_0}^{-1}) = \xi(gy^{-1}x^{-1})$, so the composition refeq1 is well defined. Also, the composition is clearly associative. Also it is easy to see that $(\hat{\xi}_x)^{-1}$ is $\hat{\xi}_{x^{-1}}$ for $x \in \mathfrak{G}$. Hence it follows that F is a group.
Now, let $x, y \in \mathfrak{G}$. Then we have that

$$\begin{aligned}\bar{\xi}(\hat{\xi}_x \circ \hat{\xi}_y) &= \bar{\xi}(\hat{\xi}_{xy}) \\ &= \xi(xy) \\ &\geq \xi(x) \wedge \xi(y) \\ &= \bar{\xi}(\hat{\xi}_x) \wedge \bar{\xi}(\hat{\xi}_y)\end{aligned}$$

Further we have that

$$\bar{\xi}(\hat{\xi}_x) = \bar{\xi}(\hat{\xi}_{x^{-1}}) = \xi(x) = \xi(x^{-1}).$$

Hence, we have that $\bar{\xi} \lessdot F$. □

In the following example we see that if $\xi \in F(\mathfrak{G})$ and $\xi \ntrianglelefteq \mathfrak{G}$, then the operation "$\circ$" defined in (5) is not well defined.

Example 2.4.27. *Let D_8 be a dihedral group of order 8 given by*
$D_8 = \langle a, b : a^4 = e = b^2, bab = a^{-1}\rangle$.
Define $\xi : D_8 \to \mathcal{I}$ by setting

$$\xi(e) = \xi(a^3b) = 1$$

,

$$\xi(a) = \xi(a^2) = \xi(a^3) = \xi(b) = \xi(ab) = \xi(a^2b) = 0.6.$$

From the example 2.2.5, it follows that $\xi \lessdot D_8$ *and* $\xi \not\lhd D_8$ *because* $\xi(a^3.b) \neq \xi(b.a^3)$. *Now the cosets of* ξ *in* D_8 *are given by*

$$\xi e = (1, 0.6, 0.6, 0.6, 0.6, 0.6, 0.6, 1)$$

$$\xi a = (0.6, 1, 0.6, 0.6, 0.6, 0.6, 1, 0.6)$$

$$\xi a^2 = (0.6, 0.6, 1, 0.6, 0.6, 1, 0.6, 0.6)$$

$$\xi a^3 = (0.6, 0.6, 0.6, 1, 1, 0.6, 0.6, 0.6)$$

$$\xi b = (0.6, 0.6, 0.6, 1, 1, 0.6, 0.6, 0.6)$$

$$\xi ab = (0.6, 0.6, 1, 0.6, 0.6, 1, 0.6, 0.6)$$

$$\xi a^2 b = (0.6, 1, 0.6, 0.6, 0.6, 0.6, 1, 0.6)$$

$$\xi a^3 b = (1, 0.6, 0.6, 0.6, 0.6, 0.6, 0.6, 1)$$

Here $\xi b = \xi a^3, \xi a^2 = \xi ab$. *But*

$$\xi b \circ \xi a^2 = \xi a^2 b \neq \xi a^3 \circ \xi ab = \xi b.$$

Hence the operation $\circ$ *defined in 2.5 is well defined in case* ξ *is a N.F.S.*

Definition 2.4.28 (Fuzzy quotient group)**.** *[18] Let* ξ *be a N.F.S of* $\mathfrak{G}$. *Then the map* $\bar{\xi}$ *defined in 2.6 is called the fuzzy quotient group determined by* ξ. *Sometimes* $\bar{\xi}$ *is also denoted by* $\mathfrak{G}/\xi$.

It is easily observe that $\mathfrak{G}/\xi$ is a classical set and not a fuzzy set.

Example 2.4.29. *Let* $K = \{e, a\} \times \{e, b\}$ *be the non-cyclic group of order 4 and its elements can be written as* $\{e, a, b, ab\}$. *We define* $\xi \in NF(K)$ *by*

$$\xi(e) = t_\circ$$

$$\xi(a) = t_1$$

$$\xi(b) = \xi(ab) = t_2$$

$\forall\ t_\circ > t_1 > t_2$ *and* $t_\circ, t_1, t_2 \in \mathcal{I}$. *From the example 2.4.16 of F.Cs,* $K/\xi = \{\hat{\xi}_e, \hat{\xi}_a, \hat{\xi}_b, \hat{\xi}_{ab}\}$ *is a quotient group relative to N.F.S* ξ.

Example 2.4.30. *Let $Q_8 = \{\pm 1, \pm i, \pm j, \pm k\}$ be a group of quaternions with eight elements.*
we define a F.S ξ of Q_8 by

$$\xi(1) = t_\circ$$

$$\xi(-1) = \xi(\pm i) = t_1$$

$$\xi(\pm j) = \xi(\pm k) = t_2.$$

$\forall\ t_\circ > t_1 > t_2$ *and* $t_\circ, t_1, t_2 \in \mathcal{I}$.

By using example 2.4.18, we can say that $Q_8/\xi = \{\hat{\xi}_1, \hat{\xi}_{-1}, \hat{\xi}_i, \hat{\xi}_{-i}, \hat{\xi}_j, \hat{\xi}_{-j}, \hat{\xi}_k, \hat{\xi}_{-k}\}$ is a quotient group relative to $N.F.S\ \xi$.

Corollary 2.4.31. *[18] With the same notation as in theorem 2.4.26, we define a map*
$\gamma : \mathfrak{G} \to F$ *as follows:*

$$\gamma(g_1) = \hat{\xi}_{g_1}. \tag{2.12}$$

Then a homomorphism γ with kernel $\mathfrak{G}_\xi$ is given by

$$\mathfrak{G}_\xi = \{g_1 \in \mathfrak{G} : \xi(g_1) = \xi(e)\},$$

where identity of $\mathfrak{G}$ is e.

Proof. Let $g_1, g_2 \in \mathfrak{G}$. Then

$$\gamma(g_1 g_2) = \hat{\xi}_{g_1 g_2} = \hat{\xi}_{g_1} \circ \hat{\xi}_{g_2}$$

$$= \gamma(g_1)\gamma(g_2)$$

Thus, γ is a homomorphism. Also, the kernel of γ consists of all $g_1 \in \mathfrak{G}$ for which $\hat{\xi}_{g_1} = \hat{\xi}_e$, and this implies that $\xi(g_1) = \xi(e)$. □

Now we examine fuzzy version of the "fundamental theorem of homomorphism of groups."

Theorem 2.4.32. *[18] Let $\xi_1 \lhd \mathfrak{G}$, and F be the classical collection of all the F.Cs of ξ_1 with respect to $\mathfrak{G}$. Then every N.F.S of F corresponds to a N.F.S of $\mathfrak{G}$, in a natural way.*

Proof. Let $\xi_1^{'} \lhd F$. Define a map $\xi_2 : \mathfrak{G} \to \mathcal{I}$ as follows:

$$\xi_2(g_1) = \xi_1^{'}(\hat{\xi_1}_{g_1}) \qquad \forall g_1 \in \mathfrak{G}.$$

Since $\hat{\xi_1}_{g_1}$ is defined by 2.1, we give a proof that $\xi_2 \lessdot \mathfrak{G}$. Suppose that $g_1, g_2 \in \mathfrak{G}$. Then, we see that

$$\begin{aligned}
\xi_2(g_1g_2) &= \xi_1^{'}(\hat{\xi_1}_{g_1g_2}) \\
&= \xi_1^{'}(\hat{\xi_1}_{g_1}\hat{\xi_1}_{g_2}) \\
&\geq \xi_1^{'}(\hat{\xi_1}_{g_1}) \wedge \xi_1^{'}(\hat{\xi_1}_{g_2}) \\
&= \xi_2(g_1) \wedge \xi_2(g_2)
\end{aligned}$$

And we know that $\xi_2(g_1) = \xi_2({g_1}^{-1})$ $\forall g_1 \in \mathfrak{G}$. Hence $\xi_2 \in F(\mathfrak{G})$.
Further, if $\xi_1^{'} \lhd F$, then we have

$$\begin{aligned}
\xi_2(g_1g_2) &= \xi_1^{'}(\hat{\xi_1}_{g_1g_2}) \\
&= \xi_1^{'}(\hat{\xi_1}_{g_1} \circ \hat{\xi_1}_{g_2}) \\
&= \xi_1^{'}(\hat{\xi_1}_{g_2} \circ \hat{\xi_1}_{g_1}) \\
&= \xi_1^{'}(\hat{\xi_1}_{g_1g_2}) \\
&= \xi_2(g_2g_1)
\end{aligned}$$

Hence ξ_2 is $N.F.S$ of $\mathfrak{G}$. □

Chapter 3

m-Polar Fuzzy Analogs

In this chapter, we will elaborate m-polar fuzzy analog of some basic terminologies and theorems of fuzzy group theory.

Definition 3.0.1 (m-polar fuzzy set)**.** *[11] An m-polar fuzzy set (or a $\mathcal{I}^m$-set) on $\mathfrak{X}$ is a mapping $A : \mathfrak{X} \to \mathcal{I}^m$. The set of all m-polar fuzzy sets on $\mathfrak{X}$ is denoted by $m(\mathfrak{X})$.*

Note that $\mathcal{I}^m$(m-power of $\mathcal{I}$) is considered a poset with the point-wise order $\leq$, where m is an arbitrary ordinal number (we make an appointment that $m = \{n|n < m\}$ when $m > 0$), $\leq$ is defined by $x \leq y \Leftrightarrow p_i(x) \leq p_i(y)$for each $i \in m$ ($x, y \in \mathcal{I}^m$), and $p_i : \mathcal{I}^m \to \mathcal{I}$ is the ith projection mapping ($i \in m$). Also, $\mathbf{0} = (0, 0, \cdots, 0)$ is the smallest element in $\mathcal{I}^m$ and $\mathbf{1} = (1, 1, \cdots, 1)$ is the largest element in $\mathcal{I}^m$.

Definition 3.0.2. *Let $A \in m(\mathfrak{X})$,*
Then,

1. $A^* = \{x|x \in \mathfrak{X}, A(x) > 0\}$

2. $A_* = \{x|x \in \mathfrak{X}, A(x) = 1\}$

In the definition 3.0.2 A^* and A_* are known as ***support*** *and* ***core*** *of A respectively. If A^* is finite then A is called finite m-polar fuzzy subset. If A^* is infinite then A is called infinite m-polar fuzzy subset.*

3.1 m-polar fuzzy group

We now define m-polar fuzzy groupiod and m-polar fuzzy group.

Definition 3.1.1 (m-polar fuzzy groupoid)**.** *A function A_1 from a groupoid S to $\mathcal{I}^m$ is called an m-polar fuzzy groupoid of S if, for all $x, y \in S$,*

$$A_1(xy) \geq \inf(A_1(x), A_1(y)).$$

Definition 3.1.2. *An m-polar fuzzy set $A : \mathfrak{G} \to \mathcal{I}^m$ on a group $\mathfrak{G}$ is called an m-polar fuzzy group if the following conditions are satisfied:*

(1) $A(g_1g_2) \geq \inf(A(g_1), A(g_2))$,

(2) $A(g_1^{-1}) \geq A(g_1)$,

That is,

(1) $p_i \circ A(g_1 g_2) \geq \inf(p_i \circ A(g_1), p_i \circ A(g_2))$,

(2) $p_i \circ A({g_1}^{-1}) \geq p_i \circ A(g_1)$,

for all $g_1, g_2 \in \mathfrak{G}$, $i = 1, 2, 3, \cdots, m$.

The set of all m-polar fuzzy subgroups of a group $\mathfrak{G}$ is denoted by "$F_m(\mathfrak{G})$".

Lemma 3.1.3. *If A is an m-polar fuzzy subgroupoid of a finite group $\mathfrak{G}$, then A is an m-polar fuzzy subgroup of $\mathfrak{G}$.*

Proof. Let $e \neq g_1 \in \mathfrak{G}$ and $\mathfrak{G}$ be a finite group. If the order of g_1 is n, then ${g_1}^{-1} = {g_1}^{n-1}$. Now applying the definition 3.1.1 repeatedly, we have that

$$\begin{aligned} A({g_1}^{-1}) &= A({g_1}^{n-1}) = A({g_1}^{n-2} g_1) \\ &\geq \inf(A({g_1}^{n-2}), A(g_1)) \\ &\geq \inf(A(g_1), ..., A(g_1)) = A(g_1). \end{aligned}$$

Hence $A \in F_m(\mathfrak{G})$. □

From now on, we denote the m-polar fuzzy subset by $m.p.f.s$ and m-polar fuzzy subgroup by $m.P.F.S$.

3.2 Examples of m-polar fuzzy subgroups

Example 3.2.1. *Let $K = \{e, a, b, ab\}$ be a Klein four-group of order 4. We define an m-polar fuzzy subset $A : K \to \mathcal{I}^m$ by*

$$A(g) = \left\{ \begin{array}{ll} (0.5, 0.5, ..., 0.5) & if\ g = e \\ (0.1, 0.1, ..., 0.1) & otherwise \end{array} \right\}$$

Then A is an $m.P.F.S$ of K.

Example 3.2.2. *Let $\mathfrak{G} = D_6 = \langle a, b : a^3 = e = b^2, bab = a^{-1} \rangle$. We define an $m.p.f.s$ of $A : D_6 \to \mathcal{I}^m$ by*

$$A(e) = A(a^2 b) = (0.9, 0.9, ..., 0.9)$$

$$A(a) = A(a^2) = A(b) = A(ab) = (0.5, 0.5, ..., 0.5)$$

$$A(e.e) = A(e) = (0.9, 0.9, ..., 0.9)$$

$$A(e.e) \geq \inf(A(e), A(e)) = \inf((0.9, 0.9, ..., 0.9), (0.9, 0.9, ..., 0.9)) = (0.9, 0.9, ..., 0.9)$$

$$A(e.a) = A(a) = (0.5, 0.5, ..., 0.5)$$

$$A(e.a) \geq \inf(A(e), A(a)) = \inf((0.9, 0.9, ..., 0.9), (0.5, 0.5, ..., 0.5)) = (0.5, 0.5, ..., 0.5)$$

$$A(e.a^2) = A(a^2) = (0.5, 0.5, ..., 0.5)$$

$$A(e.a^2) \geq \inf(A(e), A(a^2)) = \inf((0.9, 0.9, ..., 0.9), (0.5, 0.5, ..., 0.5)) = (0.5, 0.5, ..., 0.5)$$

$$A(e.b) = A(b) = (0.5, 0.5, ..., 0.5)$$

$$A(e.b) \geq \inf(A(e), A(b)) = \inf((0.9, 0.9, ..., 0.9), (0.5, 0.5, ..., 0.5)) = (0.5, 0.5, ..., 0.5)$$

$$A(e.ab) = A(ab) = (0.5, 0.5, ..., 0.5)$$

$$A(e.ab) \geq \inf(A(e), A(ab)) = \inf((0.9, 0.9, ..., 0.9), (0.5, 0.5, ..., 0.5)) = (0.5, 0.5, ..., 0.5)$$

$$A(e.a^2b) = A(a^2b) = (0.9, 0.9, ..., 0.9)$$

$$A(e.a^2b) \geq \inf(A(e), A(a^2b)) = \inf((0.9, 0.9, ..., 0.9), (0.9, 0.9, ..., 0.9)) = (0.9, 0.9, ..., 0.9)$$

$$A(a.a) = A(a^2) = (0.5, 0.5, ..., 0.5)$$

$$A(a.a) \geq \inf(A(a), A(a)) = \inf((0.5, 0.5, ..., 0.5), (0.5, 0.5, ..., 0.5)) = (0.5, 0.5, ..., 0.5)$$

$$A(a.a^2) = A(e) = (0.9, 0.9, ..., 0.9)$$

$$A(a.a^2) \geq \inf(A(a), A(a^2)) = \inf((0.5, 0.5, ..., 0.5), (0.5, 0.5, ..., 0.5)) = (0.5, 0.5, ..., 0.5)$$

$$A(a.b) = A(ab) = (0.5, 0.5, ..., 0.5)$$

$$A(a.b) \geq \inf(A(a), A(b)) = \inf((0.5, 0.5, ..., 0.5), (0.5, 0.5, ..., 0.5)) = (0.5, 0.5, ..., 0.5)$$

$$A(a.ab) = A(a^2b) = (0.9, 0.9, ..., 0.9)$$

$$A(a.ab) \geq \inf(A(a), A(ab)) = \inf((0.5, 0.5, ..., 0.5), (0.5, 0.5, ..., 0.5)) = (0.5, 0.5, ..., 0.5)$$

$$A(a.a^2b) = A(b) = (0.5, 0.5, ..., 0.5)$$

$$A(a.a^2b) \geq \inf(A(a), A(a^2b)) = \inf((0.5, 0.5, ..., 0.5), (0.9, 0.9, ..., 0.9)) = (0.5, 0.5, ..., 0.5)$$

$$A(a^2.a) = A(e) = (0.9, 0.9, ..., 0.9)$$

$$A(a^2.a) \geq \inf(A(a^2), A(a)) = \inf((0.5, 0.5, ..., 0.5), (0.5, 0.5, ..., 0.5)) = (0.5, 0.5, ..., 0.5)$$

$$A(a^2.b) = A(a^2b) = (0.9, 0.9, ..., 0.9)$$

$$A(a^2.b) \geq \inf(A(a^2), A(b)) = \inf((0.5, 0.5, ..., 0.5), (0.5, 0.5, ..., 0.5)) = (0.5, 0.5, ..., 0.5)$$

$$A(a^2.ab) = A(b) = (0.5, 0.5, ..., 0.5)$$

$$A(a^2.ab) \geq \inf(A(a^2), A(ab)) = \inf((0.5, 0.5, ..., 0.5), (0.5, 0.5, ..., 0.5)) = (0.5, 0.5, ..., 0.5)$$

$$A(a^2.a^2b) = A(ab) = (0.5, 0.5, ..., 0.5)$$

$$A(a^2.a^2b) \geq \inf(A(a^2), A(a^2b)) = \inf((0.5, 0.5, ..., 0.5), (0.9, 0.9, ..., 0.9)) = (0.5, 0.5, ..., 0.5)$$

$$A(b.a) = A(a^2b) = (0.9, 0.9, ..., 0.9)$$

$$A(b.a) \geq \inf(A(b), A(a)) = \inf((0.5, 0.5, ..., 0.5), (0.5, 0.5, ..., 0.5)) = (0.5, 0.5, ..., 0.5)$$

$$A(b.a^2) = A(ab) = (0.5, 0.5, ..., 0.5)$$

$$A(b.a^2) \geq \inf(A(b), A(a^2)) = \inf((0.5, 0.5, ..., 0.5), (0.5, 0.5, ..., 0.5)) = (0.5, 0.5, ..., 0.5)$$

$$A(b.b) = A(e) = (0.9, 0.9, ..., 0.9)$$

$$A(b.b) \geq \inf(A(b), A(b)) = \inf((0.5, 0.5, ..., 0.5), (0.5, 0.5, ..., 0.5)) = (0.5, 0.5, ..., 0.5)$$

$$A(b.ab) = A(a^2) = (0.5, 0.5, ..., 0.5)$$

$$A(b.ab) \geq \inf(A(b), A(ab)) = \inf((0.5, 0.5, ..., 0.5), (0.5, 0.5, ..., 0.5)) = (0.5, 0.5, ..., 0.5)$$

$$A(b.a^2b) = A(a) = (0.5, 0.5, ..., 0.5)$$

$$A(b.a^2b) \geq \inf(A(b), A(a^2b)) = \inf((0.5, 0.5, ..., 0.5), (0.9, 0.9, ..., 0.9)) = (0.5, 0.5, ..., 0.5)$$

$$A(ab.a) = A(b) = (0.5, 0.5, ..., 0.5)$$

$$A(ab.a) \geq \inf(A(ab), A(a)) = \inf((0.5, 0.5, ..., 0.5), (0.5, 0.5, ..., 0.5)) = (0.5, 0.5, ..., 0.5)$$

$$A(ab.a^2) = A(ba^2.a^2) = A(a^2b) = (0.9, 0.9, ..., 0.9)$$

$$A(ab.a^2) \geq \inf(A(ab), A(a^2)) = \inf((0.5, 0.5, ..., 0.5), (0.5, 0.5, ..., 0.5)) = (0.5, 0.5, ..., 0.5)$$

$$A(ab.b) = A(a) = (0.5, 0.5, ..., 0.5)$$

$$A(ab.b) \geq \inf(A(ab), A(b)) = \inf((0.5, 0.5, ..., 0.5), (0.5, 0.5, ..., 0.5)) = (0.5, 0.5, ..., 0.5)$$

$$A(ab.ab) = A(e) = (0.9, 0.9, ..., 0.9)$$

$$A(ab.ab) \geq \inf(A(ab), A(ab)) = \inf((0.5, 0.5, ..., 0.5), (0.5, 0.5, ..., 0.5)) = (0.5, 0.5, ..., 0.5)$$

$$A(ab.a^2b) = A(a^2) = (0.5, 0.5, ..., 0.5)$$

$$A(ab.a^2b) \geq \inf(A(ab), A(a^2b)) = \inf((0.5, 0.5, ..., 0.5), (0.9, 0.9, ..., 0.9)) = (0.5, 0.5, ..., 0.5)$$

$$A(a^2b.a) = A(ab) = (0.5, 0.5, ..., 0.5)$$

$$A(a^2b.a) \geq \inf(A(a^2b), A(a)) = \inf((0.9, 0.9, ..., 0.9), (0.5, 0.5, ..., 0.5)) = (0.5, 0.5, ..., 0.5)$$

$$A(a^2b.a^2) = A(ba.a^2) = A(b) = (0.5, 0.5, ..., 0.5)$$

$$A(a^2b.a^2) \geq \inf(A(a^2b), A(a^2)) = \inf((0.9, 0.9, ..., 0.9), (0.5, 0.5, ..., 0.5)) = (0.5, 0.5, ..., 0.5)$$

$$A(a^2b.b) = A(a^2) = (0.5, 0.5, ..., 0.5)$$

$$A(a^2b.b) \geq \inf(A(a^2b), A(b)) = \inf((0.9, 0.9, ..., 0.9), (0.5, 0.5, ..., 0.5)) = (0.5, 0.5, ..., 0.5)$$

$$A(a^2b.ab) = A(a) = (0.5, 0.5, ..., 0.5)$$

$$A(a^2b.ab) \geq \inf(A(a^2b), A(ab)) = \inf((0.9, 0.9, ..., 0.9), (0.5, 0.5, ..., 0.5)) = (0.5, 0.5, ..., 0.5)$$

$$A(a^2b.a^2b) = A(e) = (0.9, 0.9, ..., 0.9)$$

$$A(a^2b.a^2b) \geq \inf(A(a^2b), A(a^2b)) = \inf((0.9, 0.9, ..., 0.9), (0.9, 0.9, ..., 0.9)) = (0.9, 0.9, ..., 0.9)$$

Thus A is an m.P.F.S of D_6. Now we define

$$A_1(a^2b) = (0.7, 0.7, ..., 0.7)$$

$$A_1(e) = (1, 1, ..., 1)$$

$$A_1(a) = A_1(a^2) = A_1(b) = A_1(ab) = (0.3, 0.3, ..., 0.3)$$

It is easily verified as above examples that A_1 is an m.P.F.S of D_6.

Remark 3.2.3. *Every m.p.f.s of a group $\mathfrak{G}$ is not always an m.P.F.S.*

We give the following examples to explain the remark 3.2.3.

Example 3.2.4. *Let $\mathfrak{G} = D_6 = \langle a, b : a^3 = e = b^2, bab = a^{-1}\rangle$. we define an m-polar fuzzy subset $A_1 : S_3 \to \mathcal{I}^m$ by*

$$A_1(e) = A_1(a^2b) = (1, 1, ..., 1)$$

$$A_1(a) = A_1(b) = (0.4, 0.4, ..., 0.4)$$

$$A_1(a^2) = A_1(ab) = (0.7, 0.7, ..., 0.7)$$

In this example A_1 is an m.p.f.s but it is not an m.P.F.S of D_6, because

$$A_1(a^2b.ab) = A_1(a^2b.ba^2) = A_1(a) = (0.4, 0.4, ..., 0.4)$$

$$A_1(a^2b.ab) \geq \inf(A_1(a^2b), A_1(ab))$$
$$= \inf((1, 1, ..., 1), (0.7, 0.7, ..., 0.7)) = (0.7, 0.7, ..., 0.7)$$

$\implies A_1(a^2b.a^3b) < \inf(A_1(a^2b), A_1(ab))$, *which shows that* A_1 *is not an m.P.F.S of* D_6.

Example 3.2.5. *Let* $D_8 = \{e, a, a^2, a^3, b, ab, a^2b, a^3b\}$ *be a dihedral group with eight elements. We define an m-polar fuzzy subset* $A : D_8 \to \mathcal{I}^m$ *by*

$$A(e) = A(a^3b) = (1, 1, ..., 1)$$

$$A(a) = A(a^2) = A(a^3) = A(b) = A(ab) = A(a^2b) = (0.8, 0.8, ..., 0.8)$$

Then A *is an m.P.F.S of* D_8. *Now we define* $A_1 \in m(D_8)$ *by*

$$A_1(e) = A_1(a) = A_1(a^3b) = (1, 1, ..., 1)$$

$$A_1(a^2) = A_1(a^2b) = A_1(a^3) = A_1(b) = A_1(ab) = (0.6, 0.6, ..., 0.6)$$

It is clear that A_1 *is not an m.P.F.S of* D_8 *because*

$$A_1(a.a^3b) = A_1(b) = (0.6, 0.6, ..., 0.6)$$

$$A_1(a.a^3b) \geq \inf(A_1(a), A_1(a^3b)) = \inf((1, 1, ..., 1), (1, 1, ..., 1)) = (1, 1, ..., 1)$$

$\implies A_1(a.a^3b) < \inf(A_1(a), A_1(a^3b))$, *which shows that* A_1 *is not an m.P.F.S of* D_8.

We define $A_2 \in m(D_8)$ *by*

$$A_2(e) = A_2(a^3b) = A_2(a^3) = (1, 1, ..., 1)$$

$$A_2(a) = A_2(a^2) = A_2(a^2b) = A_2(b) = A_2(ab) = (0.5, 0.5, ..., 0.5)$$

A_2 *is not an m.P.F.S of* D_8 *because*

$$A_2(a^3.a^3b) = A_2(a^2b) = (0.5, 0.5, ..., 0.5)$$

$$A_2(a^3.a^3b) \geq \inf(A_2(a^3), A_2(a^3b))$$
$$= \inf((1, 1, ..., 1), (1, 1, ..., 1)) = (1, 1, ..., 1)$$

$\implies A_2(a^3.a^3b) < \inf(A_2(a^3), A_2(a^3b))$, *which shows that* A_2 *is not an m.P.F.S of* D_8.

Example 3.2.6. *Let $Q_8 = \{\pm 1, \pm i, \pm j, \pm k\}$ be a group of quaternions. we define an m-polar fuzzy subset $A : Q_8 \to \mathcal{I}^m$ of Q_8 by*

$$A(1) = (1, 1, ..., 1)$$

$$A(-1) = A(\pm i) = (0.8, 0.8, ..., 0.8)$$

$$A(\pm j) = A(\pm k) = (0.5, 0.5, ..., 0.5)$$

Thus A is an $m.P.F.S$ of S_3.

Lemma 3.2.7. *Let $A \in F_m(\mathfrak{G})$. Then $\forall\ g_1 \in \mathfrak{G}$*

1. $A(e) \geq A(g_1)$
2. $A(g_1) = A({g_1}^{-1})$.

Proof. (1). Let $g_1 \in \mathfrak{G}$. We have

$$\begin{aligned} A(e) = A(g_1{g_1}^{-1}) &\geq \inf(A(g_1), A({g_1}^{-1}))\ (since\ A \in F_m(\mathfrak{G})) \\ &\geq \inf(A(g_1), A(g_1)) = A(g_1). \end{aligned}$$

□

Proof. (2). Let $g_1 \in \mathfrak{G}$. Clearly, $A(g_1) = A(({g_1}^{-1})^{-1}) \geq A({g_1}^{-1}) \geq A(g_1)$, since A is an $m.P.F.S$ of $\mathfrak{G}$. Hence $A(g_1) = A({g_1}^{-1})$. □

Lemma 3.2.8. *Let A be an $m.p.f.s$ of $\mathfrak{G}$. Then A_* and A^* are subgroups of $\mathfrak{G}$.*

Proof. We first define $A_* = \{g \mid g \in \mathfrak{G}, A(g) = 1\}$ and $A^* = \{g \mid g \in \mathfrak{G}, A(g) > 0\}$ and remembering that $g_1 \leq g_2$ iff $p_i(g_1) \leq p_i(g_2)$ ($g_1, g_2 \in \mathcal{I}^m$), and $p_i : \mathcal{I}^m \to \mathcal{I}$ is the ith projection mapping ($i \in m$).
Alearly, $A_* \neq \Phi$ since $e \in A_*$. Let $x, y \in A_*$ then $A(g_1) = 1$ and $A(g_2) = 1$. Thus $A(g_1g_2) \geq inf(A(g_1), A(g_2)$. Hence $g_1g_2 \in A_*$.
For $g_1 \in A_*$, $A(g_1^{-1}) \geq A(g_1)$ and hence $A(g_1^{-1}) = 1 \Rightarrow g_1 \in A_*$. Therefore, A_* is a subgroup of $\mathfrak{G}$. □

Theorem 3.2.9. *Let A_1 and A_2 be the $m.P.F.Ss$ of a group $\mathfrak{G}$. Then $A_1 \cap A_2$ is again an $m.P.F.S$ of $\mathfrak{G}$.*

Proof. We want to show that

$$(A_1 \cap A_2)(xy^{-1}) \geq \inf((A_1 \cap A_2)(x), (A_1 \cap A_2)(y^{-1}))$$

We have

$$\begin{aligned}
(A_1 \cap A_2)(xy^{-1}) &= \inf(A_1(xy^{-1}), A_2(xy^{-1})) \\
(A_1 \cap A_2)(xy^{-1}) &\geq \inf(\inf(A_1(x), A_1(y^{-1})), \inf(A_2(x), A_2(y^{-1}))) \ (\textit{Since } A_1, A_2 \in F_m(\mathfrak{G})) \\
&= \inf(\inf(A_1(x), A_2(x)), \inf(A_1(y^{-1}), A_2(y^{-1}))) \\
&= \inf((A_1 \cap A_2)(x), (A_1 \cap A_2)(y^{-1}))
\end{aligned}$$

Hence it is proved that $A_1 \cap A_2$ ia an $m.P.F.S$ of $\mathfrak{G}$. □

3.3 m-level subsets and m-level subgroups

Definition 3.3.1 (m-level subset)**.** *Let $A \in m(X)$. For $\boldsymbol{t} \in \mathcal{I}^m$, the set $A_t = \{x \in X \mid A(x) \geq t\}$ is called an m-level subset of an m-polar fuzzy subset A.*

Note that A_t is an ordinary subset of X.

Theorem 3.3.2. *Let $\mathfrak{G}$ be a group and A be an $m.P.F.S$ of $\mathfrak{G}$, then the m-level subset A_t, for $t \in \mathcal{I}^m,\ t \leq A(e)$, is a subgroup of $\mathfrak{G}$, where e is the identity of $\mathfrak{G}$.*

Proof. Since $A_t = \{g_1 \in \mathfrak{G} \mid A(g_1) \geq t\}$, then A_t is non-empty:

$$e \in A_t \ \forall\ t \in \mathcal{I}^m.$$

Let $g_1, g_2 \in A_t$, then $A(g_1) \geq t$ and $A(g_2) \geq t$ (by definition of m-level subset defined in the definition 3.3.1). Since $A \in F_m(\mathfrak{G})$, $A(g_1g_2) \geq \inf(A(g_1), A(g_2))$. This means $A(g_1g_2) \geq t$. Hence $g_1g_2 \in A_t$. Let $g_1 \in A_t$ implies $A(g_1) \geq t$. Since A is an $m.P.F.S$ $A({g_1}^{-1}) \geq A(g_1)$ and hence $A(g_1) \geq t$. This implies ${g_1}^{-1} \in A_t$. Therefore A_t is a subgroup of $\mathfrak{G}$. □

We now give an example to elaborate the theorem 3.3.2.

Example 3.3.3. *Let $K = \{e, a\} \times \{e, b\}$ be the non-cyclic group of order 4 and its elements can be written as $\{e, a, b, ab\}$. We define an $m.P.F.S$ $A : K \to \mathcal{I}^m$ of K by*

$$A(e) = t_\circ$$

$$A(a) = t_1$$

$$A(b) = A(ab) = t_2$$

$$\text{where } t_\circ > t_1 > t_2 \ \forall \ t_\circ, t_1, t_2 \in \mathcal{I}^m.$$

Since A is an m.P.F.S of K. Clearly (by 3.3.2), $A_{t_\circ} = \{e\}, A_{t_1} = \{e, a\}$ and $A_{t_2} = \{e, a, b, ab\}$ are the subgroups of K.

Following theorem provides a short way to determine whether an $m.p.f.s$ is an $m.P.F.S$ or not.

Theorem 3.3.4. *Let $\mathfrak{G}$ be a group and A be an m.p.f.s of $\mathfrak{G}$ such that A_t is a subgroup of $\mathfrak{G}$ for all $t \in \mathcal{I}^m, t \leq A(e)$, then A is an m.P.F.S of $\mathfrak{G}$.*

Proof. Suppose that $g_1, g_2 \in A_t$ and let $A(g_1) = t_1$ and $A(g_2) = t_2$. Then $g_1 \in A_{t_1}, g_2 \in A_{t_2}$. Let us assume $t_1 < t_2$. Then it implies $A_{t_2} \subset A_{t_1}$. So $g_2 \in A_{t_1}$. Thus $g_1, g_2 \in A_{t_1}$ and since A_{t_1} is a subgroup of $\mathfrak{G}$, by hypothesis, $g_1 g_2 \in A_{t_1}$. Therefore $A(g_1 g_2) \geq t_1 = \inf(A(g_1), A(g_2))$. Again, let $g_1 \in \mathfrak{G}$ and let $A(g_1) = t$. Then $g_1 \in A_t$. Since A_t is a subgroup of $\mathfrak{G}$, $g_1^{-1} \in A_t$. Therefore $A(g_1^{-1}) \geq t$ and hence $A(g_1^{-1}) \geq A(g_1)$. Thus A is an $m.P.F.S$ of $\mathfrak{G}$. □

Example 3.3.5. *Let $S_3 = \{e, a, a^2, b, ab, a^2b\}$ be a symmetric group with six elements. We define an m-polar fuzzy subset $A : S_3 \to \mathcal{I}^m$ by*

$$A(a) = A(b) = t_1$$

$$A(e) = A(a^2b) = t_\circ$$

$$A(a^2) = A(ab) = t_2$$

$\forall \ t_\circ, t_1, t_2 \in \mathcal{I}^m$, where $t_\circ > t_1 > t_2$. From the theorem 3.3.4 it is clear that A is not an m.P.F.S of S_3 because $A_{t_1} = \{e, a, b, a^2b\}$ is not a subgroup of S_3.

Definition 3.3.6 (m-level subgroup)**.** *Let $\mathfrak{G}$ be a group and A be an m.P.F.S of $\mathfrak{G}$. The subgroups $A_t = \{x \in \mathfrak{G} | \ A(x) \geq t\}$ for $t \in \mathcal{I}^m$ are called level subgroups of A.*

3.4 m-polar fuzzy quotient subgroups

Definition 3.4.1. *Let $A \in m(X)$, then A ia called m-polar commutative fuzzy subset of X if and only if $A(xy) = A(yx)$ $\forall x, y \in X$.*

Now we define m-polar normal fuzzy subgroup and for the further discussion, we denote it by $m.P.N.F.S$. Also, m-polar commutative fuzzy subset is denoted by $m.p.c.f.s$.

Definition 3.4.2. *Let $A \in F_m(\mathfrak{G})$, then A is called an $m.P.N.F.S$ of $\mathfrak{G}$ if it is an $m.p.c.f.s$ of $\mathfrak{G}$.*

Let $N_m(\mathfrak{G})$ denote the set of all $m.P.N.F.S$s of $\mathfrak{G}$.

Example 3.4.3. *Let $Q_8 = \{\pm 1, \pm i, \pm j, \pm k\}$ be a group of quaternions with eight elements.*
we define an $m.P.F.S$ $\mathfrak{V} : Q_8 \to \mathcal{I}^m$ of Q_8 by

$$\mathfrak{V}(1) = s_\circ$$

$$\mathfrak{V}(-1) = \mathfrak{V}(\pm i) = s_1$$

$$\mathfrak{V}(\pm j) = \mathfrak{V}(\pm k) = s_2$$

$\forall$ $s_\circ > s_1 > s_2$ *and* $s_\circ, s_1, s_2 \in \mathcal{I}^m$.

Now by using the definition 3.4.2 for an $m.P.N.F.S$ we will verify that $\mathfrak{V}$ is an $m.P.N.F.S$ of Q_8.
We fix 1,

$$\mathfrak{V}(1.1) = \mathfrak{V}(1.1) \implies \mathfrak{V}(1) = \mathfrak{V}(1) \implies s_\circ = s_\circ$$

$$\mathfrak{V}(1.-1) = \mathfrak{V}(-1.1) \implies \mathfrak{V}(-1) = \mathfrak{V}(-1) \implies s_1 = s_1$$

$$\mathfrak{V}(1.i) = \mathfrak{V}(i.1) \implies \mathfrak{V}(i) = \mathfrak{V}(i) \implies s_1 = s_1$$

$$\mathfrak{V}(1.-i) = \mathfrak{V}(-i.1) \implies \mathfrak{V}(-i) = \mathfrak{V}(-i) \implies s_1 = s_1$$

$$\mathfrak{V}(1.j) = \mathfrak{V}(j.1) \implies \mathfrak{V}(j) = \mathfrak{V}(j) \implies s_2 = s_2$$

$$\mathfrak{V}(1.-j) = \mathfrak{V}(-j.1) \implies \mathfrak{V}(-j) = \mathfrak{V}(-j) \implies s_2 = s_2$$

$$\mathfrak{V}(1.k) = \mathfrak{V}(k.1) \implies \mathfrak{V}(k) = \mathfrak{V}(k) \implies s_2 = s_2$$

$$\mathfrak{V}(1.-k) = \mathfrak{V}(-k.1) \implies \mathfrak{V}(-k) = \mathfrak{V}(-k) \implies s_2 = s_2$$

Fix -1,

$$\mathfrak{V}(-1.1) = \mathfrak{V}(-1.1) \implies \mathfrak{V}(-1) = \mathfrak{V}(-1) \implies s_1 = s_1$$

$$\mathfrak{V}(-1.-1) = \mathfrak{V}(-1.-1) \implies \mathfrak{V}(1) = \mathfrak{V}(1) \implies s_\circ = s_\circ$$

$$\mathfrak{V}(-1.i) = \mathfrak{V}(i.-1) \implies \mathfrak{V}(-i) = \mathfrak{V}(-i) \implies s_1 = s_1$$

$$\mathfrak{V}(-1.-i) = \mathfrak{V}(-i.-1) \implies \mathfrak{V}(i) = \mathfrak{V}(i) \implies s_1 = s_1$$

$$\mathfrak{V}(-1.j) = \mathfrak{V}(j.-1) \implies \mathfrak{V}(-j) = \mathfrak{V}(-j) \implies s_2 = s_2$$

$$\mathfrak{V}(-1.-j) = \mathfrak{V}(-j.-1) \implies \mathfrak{V}(j) = \mathfrak{V}(j) \implies s_2 = s_2$$

$$\mathfrak{V}(-1.k) = \mathfrak{V}(k.-1) \implies \mathfrak{V}(-k) = \mathfrak{V}(-k) \implies s_2 = s_2$$

$$\mathfrak{V}(-1.-k) = \mathfrak{V}(-k.-1) \implies \mathfrak{V}(k) = \mathfrak{V}(k) \implies s_2 = s_2$$

Fix i,

$$\mathfrak{V}(i.1) = \mathfrak{V}(1.i) \implies \mathfrak{V}(i) = \mathfrak{V}(i) \implies s_1 = s_1$$

$$\mathfrak{V}(i.-1) = \mathfrak{V}(-1.i) \implies \mathfrak{V}(-i) = \mathfrak{V}(-i) \implies s_1 = s_1$$

$$\mathfrak{V}(i.i) = \mathfrak{V}(i.i) \implies \mathfrak{V}(-1) = \mathfrak{V}(-1) \implies s_1 = s_1$$

$$\mathfrak{V}(i.-i) = \mathfrak{V}(-i.i) \implies \mathfrak{V}(1) = \mathfrak{V}(1) \implies s_\circ = s_\circ$$

$$\mathfrak{V}(i.j) = \mathfrak{V}(j.i) \implies \mathfrak{V}(k) = \mathfrak{V}(-k) \implies s_2 = s_2$$

$$\mathfrak{V}(i.-j) = \mathfrak{V}(-j.i) \implies \mathfrak{V}(-k) = \mathfrak{V}(k) \implies s_2 = s_2$$

$$\mathfrak{V}(i.k) = \mathfrak{V}(k.i) \implies \mathfrak{V}(-j) = \mathfrak{V}(j) \implies s_2 = s_2$$

$$\mathfrak{V}(i.-k) = \mathfrak{V}(-k.i) \implies \mathfrak{V}(j) = \mathfrak{V}(-j) \implies s_2 = s_2$$

Fix $-i$,

$$\mathfrak{V}(-i.1) = \mathfrak{V}(1.-i) \implies \mathfrak{V}(-i) = \mathfrak{V}(-i) \implies s_1 = s_1$$

$$\mathfrak{V}(-i.-1) = \mathfrak{V}(-1.-i) \implies \mathfrak{V}(i) = \mathfrak{V}(i) \implies s_1 = s_1$$

$$\mathfrak{V}(-i.i) = \mathfrak{V}(i.-i) \implies \mathfrak{V}(1) = \mathfrak{V}(1) \implies s_\circ = s_\circ$$

$$\mathfrak{V}(-i.-i) = \mathfrak{V}(-i.-i) \implies \mathfrak{V}(-1) = \mathfrak{V}(-1) \implies s_1 = s_1$$

$$\mathfrak{V}(-i.j) = \mathfrak{V}(j.-i) \implies \mathfrak{V}(-k) = \mathfrak{V}(k) \implies s_2 = s_2$$

$$\mathfrak{V}(-i.-j) = \mathfrak{V}(-j.-i) \implies \mathfrak{V}(k) = \mathfrak{V}(k) \implies s_2 = s_2$$

$$\mathfrak{V}(-i.k) = \mathfrak{V}(k.-i) \implies \mathfrak{V}(j) = \mathfrak{V}(-j) \implies s_2 = s_2$$

$$\mathfrak{V}(-i.-k) = \mathfrak{V}(-k.-i) \implies \mathfrak{V}(-j) = \mathfrak{V}(j) \implies s_2 = s_2$$

Fix j,

$$\mathfrak{V}(j.1) = \mathfrak{V}(1.j) \implies \mathfrak{V}(j) = \mathfrak{V}(j) \implies s_2 = s_2$$

$$\mathfrak{V}(j.-1) = \mathfrak{V}(-1.j) \implies \mathfrak{V}(-j) = \mathfrak{V}(-j) \implies s_2 = s_2$$

$$\mathfrak{V}(j.i) = \mathfrak{V}(i.j) \implies \mathfrak{V}(-k) = \mathfrak{V}(k) \implies s_2 = s_2$$

$$\mathfrak{V}(j.-i) = \mathfrak{V}(-i.j) \implies \mathfrak{V}(k) = \mathfrak{V}(-k) \implies s_2 = s_2$$

$$\mathfrak{V}(j.j) = \mathfrak{V}(j.j) \implies \mathfrak{V}(1) = \mathfrak{V}(-1) \implies s_1 = s_1$$

$$\mathfrak{V}(j.-j) = \mathfrak{V}(-j.j) \implies \mathfrak{V}(1) = \mathfrak{V}(1) \implies s_\circ = s_\circ$$

$$\mathfrak{V}(j.k) = \mathfrak{V}(k.j) \implies \mathfrak{V}(i) = \mathfrak{V}(-i) \implies s_1 = s_1$$

$$\mathfrak{V}(j.-k) = \mathfrak{V}(-k.j) \implies \mathfrak{V}(-i) = \mathfrak{V}(i) \implies s_1 = s_1$$

Fix $-j$,

$$\mathfrak{V}(-j.1) = \mathfrak{V}(1.-j) \implies \mathfrak{V}(-j) = \mathfrak{V}(-j) \implies s_2 = s_2$$

$$\mathfrak{V}(-j.-1) = \mathfrak{V}(-1.-j) \implies \mathfrak{V}(j) = \mathfrak{V}(j) \implies s_2 = s_2$$

$$\mathfrak{V}(-j.i) = \mathfrak{V}(i.-j) \implies \mathfrak{V}(k) = \mathfrak{V}(-k) \implies s_2 = s_2$$

$$\mathfrak{V}(-j.-i) = \mathfrak{V}(-i.-j) \implies \mathfrak{V}(-k) = \mathfrak{V}(k) \implies s_2 = s_2$$

$$\mathfrak{V}(-j.j) = \mathfrak{V}(j.-j) \implies \mathfrak{V}(1) = \mathfrak{V}(1) \implies s_\circ = s_\circ$$

$$\mathfrak{V}(-j.-j) = \mathfrak{V}(-j.-j) \implies \mathfrak{V}(-1) = \mathfrak{V}(-1) \implies s_1 = s_1$$

$$\mathfrak{V}(-j.k) = \mathfrak{V}(k.-j) \implies \mathfrak{V}(-i) = \mathfrak{V}(i) \implies s_1 = s_1$$

$$\mathfrak{V}(-j.-k) = \mathfrak{V}(-k.-j) \implies \mathfrak{V}(i) = \mathfrak{V}(-i) \implies s_1 = s_1$$

Fix k,

$$\mathfrak{V}(k.1) = \mathfrak{V}(1.k) \implies \mathfrak{V}(k) = \mathfrak{V}(k) \implies s_2 = s_2$$

$$\mathfrak{V}(k.-1) = \mathfrak{V}(-1.k) \implies \mathfrak{V}(-k) = \mathfrak{V}(-k) \implies s_2 = s_2$$

$$\mathfrak{V}(k.i) = \mathfrak{V}(i.k) \implies \mathfrak{V}(j) = \mathfrak{V}(-j) \implies s_2 = s_2$$

$$\mathfrak{V}(k.-i) = \mathfrak{V}(-i.k) \implies \mathfrak{V}(-j) = \mathfrak{V}(j) \implies s_2 = s_2$$

$$\mathfrak{V}(k.j) = \mathfrak{V}(j.k) \implies \mathfrak{V}(-i) = \mathfrak{V}(i) \implies s_1 = s_1$$

$$\mathfrak{V}(k.-j) = \mathfrak{V}(-j.k) \implies \mathfrak{V}(i) = \mathfrak{V}(-i) \implies s_1 = s_1$$

$$\mathfrak{V}(k.k) = \mathfrak{V}(k.k) \implies \mathfrak{V}(-1) = \mathfrak{V}(-1) \implies s_1 = s_1$$

$$\mathfrak{V}(k.-k) = \mathfrak{V}(-k.k) \implies \mathfrak{V}(1) = \mathfrak{V}(1) \implies s_\circ = s_\circ$$

Fix $-k$,

$$\mathfrak{V}(-k.1) = \mathfrak{V}(1.-k) \implies \mathfrak{V}(-k) = \mathfrak{V}(-k) \implies s_2 = s_2$$

$$\mathfrak{V}(-k.-1) = \mathfrak{V}(-1.-k) \implies \mathfrak{V}(k) = \mathfrak{V}(k) \implies s_2 = s_2$$

$$\mathfrak{V}(-k.i) = \mathfrak{V}(i.-k) \implies \mathfrak{V}(-j) = \mathfrak{V}(j) \implies s_2 = s_2$$

$$\mathfrak{V}(-k.-i) = \mathfrak{V}(-i.-k) \implies \mathfrak{V}(j) = \mathfrak{V}(-j) \implies s_2 = s_2$$

$$\mathfrak{V}(-k.j) = \mathfrak{V}(j.-k) \implies \mathfrak{V}(i) = \mathfrak{V}(-i) \implies s_1 = s_1$$

$$\mathfrak{V}(-k.-j) = \mathfrak{V}(-j.-k) \implies \mathfrak{V}(-i) = \mathfrak{V}(i) \implies s_1 = s_1$$

$$\mathfrak{V}(-k.k) = \mathfrak{V}(k.-k) \implies \mathfrak{V}(1) = \mathfrak{V}(1) \implies s_\circ = s_\circ$$

$$\mathfrak{V}(-k.-k) = \mathfrak{V}(-k.-k) \implies \mathfrak{V}(-1) = \mathfrak{V}(-1) \implies s_1 = s_1$$

Hence it shows that $\mathfrak{V}$ is an $m.p.c.f.s$ of Q_8. So, by definition of an $m.P.N.F.S$, $\mathfrak{V}$ is an $m.P.N.F.S$ of Q_8.

Remark 3.4.4. *Every $m.P.F.S$ of $\mathfrak{G}$ is normal if $\mathfrak{G}$ is an abelian group.*

Example 3.4.5. *Let $K = \{e, a, b, ab\}$ be a Klein four-group. We define an m-polar fuzzy subset A of K by*

$$A(e) = A(ab) = s_\circ$$

$$A(a) = A(b) = s_1$$

$\forall\ s_\circ > s_1$ and $s_\circ, s_1 \in \mathcal{I}^m$. Since K is an abelian group so from the remark 3.4.4 A is an $m.P.N.F.S$ of K.

Remark 3.4.6. *If $A_1, A_2 \in F_m(\mathfrak{G})$ and $A_1, A_2 \notin N_m(\mathfrak{G})$, then $A_1 \cap A_2$ is not an $m.P.N.F.S$ of $\mathfrak{G}$.*

We will prove it by using a counter example.

Example 3.4.7. *Let* $\mathfrak{G} = D_6 = \langle a, b : a^3 = e = b^2, bab = a^{-1}\rangle$. *We define* $A_1, A_2 \in F_m(D_6)$ *as follows*

$$A_1(e) = A_1(a^2b) = t_\circ$$

$$A_1(a) = A_1(a^2) = A_1(b) = A_1(ab) = t_1$$

And

$$A_2(e) = A_2(a) = A_2(b) = A_2(a^2) = t_\circ$$

$$A_2(a^2b) = A_2(ab) = t_1$$

$\forall\ t_\circ > t_1$ *and* $t_\circ, t_1 \in \mathcal{I}^m$. *Clearly* $A_1, A_2 \notin N_m(\mathfrak{G})$ *because* $A_1(a^2b) \neq A_1(ba^2)$ *and* $A_2(ab) \neq A_2(a^2b)$. *From the definition of m.P.N.F.S it is immediate that*

$$(A_1 \cap A_2)(ab) \neq (A_1 \cap A_2)(ba).$$

Hence $A_1 \cap A_2$ *is not an m.P.N.F.S of* D_6.

Definition 3.4.8. *Let* $A_1, A_2 \in F_m(\mathfrak{G})$ *and* $A_1 \subset A_2$. *Then* A_1 *is called an m.P.N.F.S of an m.P.F.S* A_2*,if*

$$A_1(xyx^{-1}) \geq \inf(A_1(y), A_2(x))\ \forall\ x, y \in \mathfrak{G}$$

From the above definition, we have

1. If $A_1 \in N_m(\mathfrak{G}), A_2 \in F_m(\mathfrak{G})$, and $A_1 \subset A_2$, then A_1 is an $m.P.N.F.S$ of A_2.

2. Every $m.P.F.S$ is an $m.P.N.F.S$ of itself.[17]

Theorem 3.4.9. *If* $A_1 \in N_m(\mathfrak{G})$ *and* $A_2 \in F_m(\mathfrak{G})$, *then* $A_1 \cap A_2$ *is an m.P.N.F.S of* A_2.

Proof. Observe that $A_1 \cap A_2 \in F_m(\mathfrak{G})$ and $A_1 \cap A_2 \subseteq A_2$. Now

$$\begin{aligned}(A_1 \cap A_2)(xyx^{-1}) &= \inf(A_1(xyx^{-1}), A_2(xyx^{-1})) \\ &= \inf(A_1(y), A_2(xyx^{-1})) \\ &\geq \inf(A_1(y), A_2(x), A_2(y), A_2(x^{-1})) \\ &= \inf((A_1 \cap A_2)(y), A_2(x))\end{aligned}$$

$\forall x, y \in \mathfrak{G}$. Therefore, $A_1 \cap A_2$ is an $m.P.N.F.S$ of $\mathfrak{G}$. □

Theorem 3.4.10. *Let $A_1, A_2, A_3 \in F_m(\mathfrak{G})$ be such that A_1 and A_2 are m-polar normal fuzzy subgroups of A_3. Then $A_1 \cap A_2$ is an $m.P.N.F.S$ of A_3.*

Proof. Since $A_1, A_2 \in F_m(\mathfrak{G})$, it follows that $A_1 \cap A_2 \in F_m(\mathfrak{G})$ and $A_1 \cap A_2 \subseteq A_3$. Now

$$\begin{aligned}(A_1 \cap A_2)(xyx^{-1}) &= \inf(A_1(xyx^{-1}), A_2(xyx^{-1})) \\ &\geq \inf(\inf(A_1(y), A_3(x)), \inf(A_2(y), A_3(x)))\ (\textit{since } A_1, A_2 \in N_m(A_3)) \\ &\geq \inf((A_1 \cap A_2)(y), A_3(x)).\end{aligned}$$

Hence $A_1 \cap A_2$ is an $m.P.N.F.S$ of A_3. □

Theorem 3.4.11. *If A'_1 is an $m.P.N.F.S$ of A_1 and $A_1, A_2 \in F_m(\mathfrak{G})$, then $A'_1 \cap A_2$ is an $m.P.N.F.S$ of $A_1 \cap A_2$.*

Proof. Clearly $A'_1 \cap A_2 \in F_m(\mathfrak{G})$ and $A'_1 \cap A_2 \subseteq A_1 \cap A_2$. Now

$$\begin{aligned}(A'_1 \cap A_2)(xyx^{-1}) &= \inf(A'_1(xyx^{-1}), A_2(xyx^{-1})) \\ &\geq \inf(A'_1(y), A_1(x), A_2(x), A_2(y), A_2(x^{-1})) \\ &= \inf(A'_1(y), A_2(y), A_1(x), A_2(x), A_2(x)) \\ &= \inf((A'_1 \cap A_2)(y), (A_1 \cap A_2)(x))\end{aligned}$$

$\forall x, y \in \mathfrak{G}$. Therefore, $A'_1 \cap A_2$ is an $m.P.N.F.S$ of $A_1 \cap A_2$. □

Theorem 3.4.12. *Let $A \in N_m(\mathfrak{G})$. Then A_* and A^* are normal subgroups of $\mathfrak{G}$.*

Proof. Clearly A_* and A^* are subgroups of $\mathfrak{G}$ because $A \in N_m(\mathfrak{G})$, (by lemma 3.2.8). Now we show that A_* and A^* are $m.P.N.F.S$ of $\mathfrak{G}$. Suppose that $g_1 \in \mathfrak{G}$ and $g_2 \in A_*$. Since A is an $m.P.N.F.S$ of $\mathfrak{G}$, we have

$$A(g_1 g_2 g_1^{-1}) = A(g_2) = A(e)$$

and thus $g_1 g_2 g_1^{-1} \in A_*$. Therefore, $A_* \lhd \mathfrak{G}$. Now let $g_1 \in \mathfrak{G}$ and $g_2 \in A^*$. Again by using condition (2) of theorem 2.4.2, we have

$$A(g_1 g_2 g_1^{-1}) = A(g_2) > 0$$

and so $g_1 g_2 g_1^{-1} \in A^*$. Hence $A^* \lhd \mathfrak{G}$. □

Now we define m-polar fuzzy coset and is denoted by $m.P.F.C$.

Definition 3.4.13 (m-polar fuzzy coset). *Let A be an $m.P.F.S$ of a group $\mathfrak{G}$. For any $a \in \mathfrak{G}$, we define a map*

$$\hat{A}_x : \mathfrak{G} \longrightarrow \mathcal{I}^m$$

by

$$\hat{A}_x(g) = A(gx^{-1}) \; \forall \; g \in \mathfrak{G} \tag{3.1}$$

$\hat{A}_x$ is called right $m.P.F.C$ of A in $\mathfrak{G}$. $\hat{A}_x$ is also denoted by xA.
Similarly, an $m.p.f.s$ $x\mu$ is called left $m.P.F.C$ of A in $\mathfrak{G}$, if

$$\hat{A}_x(g) = A(x^{-1}g) \; \forall \; g \in \mathfrak{G}.$$

Example 3.4.14. *Let $K = \{e, a\} \times \{e, b\}$ be the non-cyclic group of order 4 and its elements can be written as $\{e, a, b, ab\}$. We define an $m.P.F.S$ A of K by*

$$A(e) = t_\circ$$

$$A(a) = t_1$$

$$A(b) = A(ab) = t_2$$

$$\forall \; t_\circ > t_1 > t_2 \text{ and } t_\circ, t_1, t_2 \in \mathcal{I}^m.$$

The $m.P.F.C$ corresponding to $e \in K$,

$$\hat{A}_e(e) = A(e) = t_\circ$$

$$\hat{A}_e(a) = A(a) = t_1$$

$$\hat{A}_e(b) = A(b) = t_2$$

$$\hat{A}_e(ab) = A(ab) = t_2$$

So $m.P.F.C$ corresponding to e is $\{t_\circ, t_1, t_2, t_2\}$, where $\{t_\circ, t_1, t_2, t_2\}$ is a multi-set.
The $m.P.F.C$ corresponding to $a \in K$,

$$\hat{A}_a(e) = A(a) = t_1$$

$$\hat{A}_a(a) = A(e) = t_\circ$$

$$\hat{A}_a(b) = A(ba) = t_2$$

$$\hat{A}_a(ab) = A(b) = t_2$$

So m.P.F.C corresponding to a is $\{t_1, t_\circ, t_2, t_2\}$.

The m.P.F.C corresponding to $b \in K$,

$$\hat{A}_b(e) = A(b) = t_2$$

$$\hat{A}_b(a) = A(ab) = t_2$$

$$\hat{A}_b(b) = A(e) = t_\circ$$

$$\hat{A}_b(ab) = A(a) = t_1$$

So m.P.F.C corresponding to b is $\{t_2, t_2, t_\circ, t_1\}$.

The m.P.F.C corresponding to $ab \in K$,

$$\hat{A}_{ab}(e) = A(ab) = t_2$$

$$\hat{A}_{ab}(a) = A(b) = t_2$$

$$\hat{A}_{ab}(b) = A(a) = t_1$$

$$\hat{A}_{ab}(ab) = A(e) = t_\circ$$

So m.P.F.C corresponding to ab is $\{t_2, t_2, t_1, t_\circ\}$.

Note that $\hat{A}_e, \hat{A}_a, \hat{A}_b$ and $\hat{A}_{ab}$ are the m-polar fuzzy cosets of $\mathfrak{G}$ with respect to A.

Example 3.4.15. *Let* $\mathfrak{G} = D_6 = \langle a, b : a^3 = e = b^2, bab = a^{-1}\rangle$. *Define an m.P.F.S* A *of* S_3 *by*

$$A(e) = A(a^2b) = t_\circ$$

$$A(a) = A(a^2) = A(b) = A(ab) = t_1$$

$$\forall\ t_\circ > t_1 \text{ and } t_\circ, t_1 \in \mathcal{I}.$$

The m.P.F.C corresponding to $e \in S_3$,

$$\hat{A}_e(e) = A(e) = t_\circ$$

$$\hat{A}_e(a) = A(a) = t_1$$

$$\hat{A}_e(a^2) = A(a^2) = t_1$$

$$\hat{A}_e(b) = A(b) = t_1$$

$$\hat{A}_e(ab) = A(ab) = t_1$$

$$\hat{A}_e(a^2b) = A(a^2b) = t_\circ$$

So m.P.F.C corresponding to e is $\{t_\circ, t_1, t_1, t_1, t_1, t_\circ\}$.

The m.P.F.C corresponding to $a \in S_3$,

$$\hat{A}_a(e) = A(a^2) = t_1$$

$$\hat{A}_a(a) = A(e) = t_\circ$$

$$\hat{A}_a(a^2) = A(a) = t_1$$

$$\hat{A}_a(b) = A(ab) = t_1$$

$$\hat{A}_a(ab) = A(a^2b) = t_\circ$$

$$\hat{A}_a(a^2b) = A(b) = t_1$$

So m.P.F.C corresponding to a is $\{t_1, t_\circ, t_1, t_1, t_\circ, t_1\}$.

The m.P.F.C corresponding to $a^2 \in S_3$,

$$\hat{A}_{a^2}(e) = A(a) = t_1$$

$$\hat{A}_{a^2}(a) = A(a^2) = t_1$$

$$\hat{A}_{a^2}(a^2) = A(e) = t_\circ$$

$$\hat{A}_{a^2}(b) = A(a^2b) = t_\circ$$

$$\hat{A}_{a^2}(ab) = A(b) = t_1$$

$$\hat{A}_{a^2}(a^2b) = A(ab) = t_1$$

So m.P.F.C corresponding to a^2 *is* $\{t_1, t_1, t_\circ, t_\circ, t_1, t_1\}$.

The m.P.F.C corresponding to $b \in S_3$,

$$\hat{A}_b(e) = A(b) = t_1$$

$$\hat{A}_b(a) = A(ab) = t_1$$

$$\hat{A}_b(a^2) = A(a^2b) = t_\circ$$

$$\hat{A}_b(b) = A(e) = t_\circ$$

$$\hat{A}_b(ab) = A(a) = t_1$$

$$\hat{A}_b(a^2b) = A(a^2) = t_1$$

So m.P.F.C corresponding to b is $\{t_1, t_1, t_\circ, t_\circ, t_1, t_1\}$.

The m.P.F.C corresponding to $ab \in S_3$,

$$\hat{A}_{ab}(e) = A(ab) = t_1$$

$$\hat{A}_{ab}(a) = A(a^2b) = t_\circ$$

$$\hat{A}_{ab}(a^2) = A(b) = t_1$$

$$\hat{A}_{ab}(b) = A(a^2) = t_1$$

$$\hat{A}_{ab}(ab) = A(e) = t_\circ$$

$$\hat{A}_{ab}(a^2b) = A(a) = t_1$$

So m.P.F.C corresponding to ab is $\{t_1, t_\circ, t_1, t_1, t_\circ, t_1\}$.

The m.P.F.C corresponding to $a^2b \in S_3$,

$$\hat{A}_{a^2b}(e) = A(a^2b) = t_\circ$$

$$\hat{A}_{a^2b}(a) = A(b) = t_1$$

$$\hat{A}_{a^2b}(a^2) = A(ab) = t_1$$

$$\hat{A}_{a^2b}(b) = A(a) = t_1$$

$$\hat{A}_{a^2b}(ab) = A(a^2) = t_1$$

$$\hat{A}_{a^2b}(a^2b) = A(e) = t_\circ$$

So m.P.F.C corresponding to e is $\{t_\circ, t_1, t_1, t_1, t_1, t_\circ\}$.

Then we can say that $\hat{A}_e, \hat{A}_a, \hat{A}_{a^2}, \hat{A}_b, \hat{A}_{ab}$ *and* $\hat{A}_{a^2b}$ *are the m-polar fuzzy cosets of* S_3 *with respect to A.*

Example 3.4.16. *Let* $S_3 = \{e, a, a^2, b, ab, a^2b\}$ *be a symmetric group of order* 6, *where* $b^2 = e, a^3 = e$ *and* $ba^2 = ab$. *We define a fuzzy subgroup A of* S_3 *by*

$$A(e) = A(b) = 1$$

$$A(a) = A(a^2) = A(ab) = A(a^2b) = 0.5$$

Then the right fuzzy coset corresponding to a in S_3 is given by

$$\hat{A}_a(e) = A(a^2) = 0.5$$

$$\hat{A}_a(a) = A(e) = 1$$

$$\hat{A}_a(a^2) = A(a) = 0.5$$

$$\hat{A}_a(b) = A(ab) = 0.5$$

$$\hat{A}_a(ab) = A(a^2b) = 0.5$$

$$\hat{A}_a(a^2b) = A(b) = 1$$

So the right fuzzy coset $Aa = (0.5, 1, 0.5, 0.5, 0.5, 1)$.

Now the left fuzzy coset corresponding to $a \in S_3$ is given by

$$\hat{A}_a(e) = A(a^2) = 0.5$$

$$\hat{A}_a(a) = A(e) = 1$$

$$\hat{A}_a(a^2) = A(a) = 0.5$$

$$\hat{A}_a(b) = A(a^2b) = 0.5$$

$$\hat{A}_a(ab) = A(b) = 1$$

$$\hat{A}_a(a^2b) = A(ab) = 0.5$$

So the left fuzzy coset $aA = (0.5, 1, 0.5, 0.5, 1, 0.5)$.

Hence $Aa \neq aA$. This happen because $A \not\lhd S_3$.

Proposition 3.4.17. *[14] If A is an $m.P.N.F.S$ of $\mathfrak{G}$ and $x \in \mathfrak{G}$, then the m-polar right fuzzy coset Ax is same as the m-polar left fuzzy coset xA.*

Proof. Let A be an $m.P.N.F.S$ of $\mathfrak{G}$ and $x \in \mathfrak{G}$. Then for any $g \in \mathfrak{G}$, $A(gx) = A(xg)$ and so

$$xA(g) = A(x^{-1}g) = A(gx^{-1}) = Ax(g)$$

. Hence $xA = Ax$. □

Now we give another formulation of an $m.P.N.F.S$.

Theorem 3.4.18. *Let A be an $m.P.F.S$ of a group $\mathfrak{G}$. Then A is an $m.P.N.F.S$ of $\mathfrak{G}$ if and only if*

$$A([x,y]) \geq A(x)\ \forall\ x,y \in \mathfrak{G}. \tag{3.2}$$

Proof. Suppose that A is an $m.P.N.F.S$ of $\mathfrak{G}$. Let $x,y \in \mathfrak{G}$. Then

$$\begin{aligned} A(x^{-1}y^{-1}xy) &\geq \inf(A(y^{-1}xy), A(x^{-1})) \\ &= \inf(A(x), A(x))\ (since\ A\ is\ an\ m-polar\ normal\ fuzzy\ subgroup) \\ &= A(x). \end{aligned}$$

Conversely, assume that A satisfies the relation 3.2. Then for $x,z \in \mathfrak{G}$, we have

$$\begin{aligned} A(x^{-1}zx) &= A(zz^{-1}x^{-1}zx) \\ &\geq \inf(A(z), A([z,x])) \\ &= A(z)\ by (2). \end{aligned}$$

Thus

$$A(x^{-1}zx) \geq A(z)\ \forall z,x \in \mathfrak{G}. \tag{3.3}$$

Again we get that

$$A(z) = A(x.x^{-1}zxx^{-1}) \geq \inf(A(x), A(x^{-1}zx)). \tag{3.4}$$

Now if $\inf(A(x), A(x^{-1}zx)) = A(x)$, then we get that $A(z) \geq A(x)\ \forall x,z \in \mathfrak{G}$, implying that A is a constant function, and the result holds trivially. So we now consider the case when $\inf(A(x), A(x^{-1}zx)) = A(x^{-1}zx)$. Then 3.4 implies that $A(z) \geq A(x^{-1}zx)\ \forall\ x,z \in \mathfrak{G}$. Combining this inequality with 3.3 gives that $A(x^{-1}zx) = A(z)\ \forall\ x,z \in \mathfrak{G}$. Hence A is constant on the conjugate classes of $\mathfrak{G}$. □

Proposition 3.4.19. *If A is an $m.P.N.F.S$ of a group $\mathfrak{G}$. then for any $x,y \in \mathfrak{G}$ we have that*

$$\hat{A}_x(gx) = \hat{A}_x(xg) = A(g) \qquad \forall g \in \mathfrak{G}.$$

Proof. We have

$$\begin{aligned} \hat{A}_x(gx) &= \hat{A}_x(xgx^{-1}x) \\ &= A(xg^{-1}) \end{aligned}$$

$$\hat{A}_x(gx) = A(g).$$

Since A is an $m.P.N.F.S$ of $\mathfrak{G}$. Similarly,

$$\hat{A}_x(gx) = \hat{A}(gxx^{-1})$$

$$= A(g)$$

□

Theorem 3.4.20. *Let A be an $m.P.N.F.S$ of a group $\mathfrak{G}$. Let F be the set of all the m-polar fuzzy cosets of A. Then F is a group under the composition*

$$\hat{A}_x \circ \hat{A}_y = \hat{A}_{xy} \qquad \forall\ x, y \in \mathfrak{G}. \tag{3.5}$$

Define a map $\bar{A} : F \to \mathcal{I}^m$ by

$$\bar{A}(\hat{A}_x) = A(x) \qquad \forall\ x \in \mathfrak{G}. \tag{3.6}$$

Then $\bar{A}$ is an $m.P.F.S$ on F.

Proof. First, we show that the composition given by 3.1 is well defined. Let $x, y, x_0, y_0 \in \mathfrak{G}$ such that

$$\hat{A}_x = \hat{A}_{x_0} \quad \text{and} \quad \hat{A}_y = \hat{A}_{y_0} \tag{3.7}$$

Then we must show that

$$\hat{A}_x \circ \hat{A}_y = \hat{A}_{x_0} \circ \hat{A}_{y_0}$$

that is, $\hat{A}_{xy} = \hat{A}_{x_0y_0}$

We have, by definition 3.1,

$$\hat{A}_{xy}(g) = A(gy^{-1}x^{-1}) \qquad \forall\ g \in \mathfrak{G}$$

and

$$\hat{A}_{x_0y_0}(g) = A(g{y_0}^{-1}{x_0}^{-1}) \qquad \forall\ g \in \mathfrak{G}.$$

Now

$$A(gy^{-1}x^{-1}) = A(g{y_0}^{-1}y_0y^{-1}x^{-1})$$

$$= A(g{y_0}^{-1}{x_0}^{-1}x_0y_0y^{-1}x^{-1})$$

$$\geq \inf(A(gy_0{}^{-1}x_0{}^{-1}), A(x_0y_0y^{-1}x^{-1})). \tag{3.8}$$

By 3.7 we have that

$$A(gx^{-1}) = A(gx_0^{-1}) \qquad \forall\ g \in \mathfrak{G} \tag{3.9}$$

and

$$A(gy^{-1}) = A(gy_0^{-1}) \qquad \forall\ g \in \mathfrak{G}. \tag{3.10}$$

Now in 3.9, substituting $x_0y_0y^{-1}$ for g, we have that

$$A(x_0y_0y^{-1}x^{-1}) = A(x_0y_0y^{-1}x_0^{-1})$$

Since A is an $m.P.N.F.S$ of $\mathfrak{G}$,

$$= A(y_0y^{-1})$$

$$= A(e)$$

let $g = y_0$ in 3.10. Also $A(e) \geq A(gy_0{}^{-1}x_0{}^{-1})$, clearly $A(e) \geq A(u)\ \forall\ u \in \mathfrak{G}$, where $A \in N_m(G)$. Hence 3.8 implies that

$$A(gy^{-1}x^{-1}) \geq A(gy_0{}^{-1}x_0{}^{-1})$$

Now

$$\begin{aligned} A(gy_0{}^{-1}x_0{}^{-1}) &= A(gy^{-1}yy_0{}^{-1}x_0{}^{-1}) = A(gy^{-1}x^{-1}xyy_0{}^{-1}x_0{}^{-1}) \\ &\geq \inf(A((gy^{-1}x^{-1}), A(xyy_0{}^{-1}x_0{}^{-1})). \end{aligned} \tag{3.11}$$

Now in 3.9, substituting $xyy_0{}^{-1}$ for g, we have that

$$A(xyy_0{}^{-1}x^{-1}) = A(xyy_0{}^{-1}x_0^{-1})$$

Since A is an $m.P.N.F.S$ of $\mathfrak{G}$,

$$\begin{aligned} A(yy_0{}^{-1}) &= A(xyy_0{}^{-1}x_0^{-1}) \\ A(e) &= A(xyy_0{}^{-1}x_0^{-1}) \end{aligned}$$

let $g = y$ in 3.10. Also $A(e) \geq A(gy^{-1}x^{-1})$, since it is easy to see (for any $m.P.F.S$ A) that $A(e) \geq A(u)\ \forall\ u \in \mathfrak{G}$. Hence 3.11 implies that

$$A(gy_0{}^{-1}x_0{}^{-1}) \geq A(gy^{-1}x^{-1})$$

Hence we have that $A(gy_0{}^{-1}x_0{}^{-1}) = A(gy^{-1}x^{-1})$, so the composition 3.1 is well defined.

The composition defined in 3.1 is associative. Also $(\hat{A}_x)^{-1}$ is $\hat{A}_{x^{-1}}$ for $x \in \mathfrak{G}$. Hence F is a group.

Now, let $x, y \in \mathfrak{G}$. Then we have that

$$\begin{aligned}
\bar{A}(\hat{A}_x \circ \hat{A}_y) &= \bar{A}(\hat{A}_{xy}) \\
&= A(xy) \\
&\geq \inf(A(x), A(y)) \\
&= \inf(\bar{A}(\hat{A}_x), \bar{A}(\hat{A}_y))
\end{aligned}$$

Further we have that

$$\bar{A}(\hat{A}_x) = \bar{A}(\hat{A}_{x^{-1}}) = A(x) = A(x^{-1}).$$

Hence, we have that $\bar{A}$ is an $m.P.F.S$ on F. □

Example 3.4.21. *Let D_8 be a dihedral group of order 8 given by $D_8 = \{e, a, a^2, a^3, b, ab, a^2b, a^3b\}$, where we have $a^4 = b^2 = e$ and $a^3b = ba$. Define $A : D_8 \to \mathcal{I}$ by setting*

$$A(e) = A(a^3b) = 1$$

,

$$A(a) = A(a^2) = A(a^3) = A(b) = A(ab) = A(a^2b) = 0.6.$$

From the example 2.2.5, it follows that $A \in F(D_8)$ and $A \ntriangleleft D_8$ because $A(a^3.b) \neq A(b.a^3)$. Now the cosets of A in D_8 are given by

$$Ae = (1, 0.6, 0.6, 0.6, 0.6, 0.6, 0.6, 1)$$

$$Aa = (0.6, 1, 0.6, 0.6, 0.6, 0.6, 1, 0.6)$$

$$Aa^2 = (0.6, 0.6, 1, 0.6, 0.6, 1, 0.6, 0.6)$$

$$Aa^3 = (0.6, 0.6, 0.6, 1, 1, 0.6, 0.6, 0.6)$$

$$Ab = (0.6, 0.6, 0.6, 1, 1, 0.6, 0.6, 0.6)$$

$$Aab = (0.6, 0.6, 1, 0.6, 0.6, 1, 0.6, 0.6)$$

$$Aa^2b = (0.6, 1, 0.6, 0.6, 0.6, 0.6, 1, 0.6)$$

$$Aa^3b = (1, 0.6, 0.6, 0.6, 0.6, 0.6, 0.6, 1)$$

Here $Ab = Aa^3, Aa^2 = Aab$. *But*

$$Ab \circ Aa^2 = Aa^2b \neq Aa^3 \circ Aab = Ab.$$

Hence the operation " $\circ$ *" defined in* (5) *is well defined in case* A *is a normal fuzzy subgroup.*

Definition 3.4.22 (m-polar fuzzy quotient group)**.** *Let* A *be an m.P.N.F.S of a group* $\mathfrak{G}$. *Then the map* $\bar{A}$ *defined in 3.6 is called an* m*-polar fuzzy quotient group determined by* A. *Sometimes* $\bar{A}$ *is also denoted by* $\mathfrak{G}/A$.

It is easily observed that $\mathfrak{G}/A$ is a classical set and not an m-polar fuzzy set.

Example 3.4.23. *Let* $K = \{e, a\} \times \{e, b\}$ *be the non-cyclic group of order 4 and its elements can be written as* $\{e, a, b, ab\}$. *We define* $A \in N_m(K)$ *by*

$$A(e) = t_\circ$$

$$A(a) = t_1$$

$$A(b) = A(ab) = t_2$$

$$\forall\ t_\circ > t_1 > t_2 \text{ and } t_\circ, t_1, t_2 \in \mathcal{I}^m.$$

From the example 3.4.14 of m*-polar fuzzy cosets,* $K/A = \{\hat{A}_e, \hat{A}_a, \hat{A}_b, \hat{A}_{ab}\}$ *is an* m*-polar fuzzy quotient group.*

3.5 Conclusion

In this thesis, we have proved that if ξ_1 and ξ_2 are fuzzy subgroups of a group $\mathfrak{G}$, then their intersection is again a fuzzy subgroup of G. Also, we have shown that for two fuzzy subgroups ξ_1 and ξ_2 of a group G, if $\xi_1^{'}$ is a normal fuzzy subgroup of ξ_1, then $\xi_1^{'} \cap \xi_2$ is a normal fuzzy subgroup of $\xi_1 \cap \nu$. Furthermore, we have defined the concept of a fuzzy subgroup of a fuzzy group.

We have formulated the notion of an m-polar fuzzy group as a generalization of a fuzzy group. Moreover, we have introduced m-polar fuzzy analogs of some results in fuzzy group theory.

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