Generalizations Of Majorization, Favard and Berwald-Type Inequalities

and Related Results

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Declaration

I Ayesha Liaqat with registration number CIIT/FA16-RMT-019/LHR hereby declare that I have produced the work presented in this thesis, during the scheduled period of study. I also declare that I have not taken any material from any source except referred to wherever due that amount of plagiarism is within acceptable range. If a violation of HEC rules on research has occurred in this thesis, I shall be liable to punishable action under the plagiarism rules of the HEC

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It is certified that I Ayesha Liaqat & CIIT/FA16-RMT-019/LHR has carried out all the work related to this thesis under my supervision at the Department of Mathematics,COMSATS University Islamabad, Lahore campus and the work fulfills the requirement for award of MS degree

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DEDICATION TO

My beloved grandfather and my mother whom I still miss every day. I would like to dedicate my dissertation to my loving father, whose affection, understanding and most of all love make me able to get such success.

IN THE NAME OF

ALLAH

THE MOST GRACIOUS AND COMPASSIONATE

THE MOST MERCIFUL AND BENEIFICIENT

WHOSE HELP AND GUIDANCE ALWAYS

SOLICIT AT EVERY STEP AND MOMENT.

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Praise to be ALLAH, the Cherisher and Lord
of the World, Most gracious and Most Merciful

Person is not perfect in all the contexts of this life. He has a limited mind and minor thinking approaches. It is the guidance from the almighty ALLAH that shows the man light in the darkness and the person finds his way in the light. Without this helping light, person is nothing but a helpless creature. Same is the case with us, as we experience all these phenomena during the completion of this project and have been successful in fulfilling this duty assigned to us only because of the help of ALLAH. The teaching of the Holy Prophet Muhammad (PBUH) were also the continuous source of guidance for us especially His order of getting knowledge and fulfilling ones duty honestly was the key for motivating us. Credit of my humble efforts goes to my respectable teacher Dr., under whose supervision I was able to complete this task.
In the name of Allah Almighty, the most Beneficent, the most Merciful.

Chapter 1

Preliminaries and Introduction

1 Preliminaries and Introduction

Convex functions are closely related to the theory of inequalities and several well known inequalities are a result of the applications of convex functions [3] (*see also*[22]). The subject of convex functions has been discussed widely in [5].

1.1 Jensen convex functions and convex functions

The following definition is given in [22, p. 5].

Definition 1.1. "ψ is said to be convex in Jensen sense (J-sense) or ψ is said to be Jensen convex (J-convex) if for all $\tilde{u}, \tilde{v} \in I \subseteq \mathbb{R}$, the inequality

$$\psi\left(\frac{\tilde{u}+\tilde{v}}{2}\right) \leq \frac{\psi(\tilde{u})+\psi(\tilde{v})}{2}$$

holds."

The following definition is given in [22, p. 1].

Definition 1.2. " Let $I \subseteq \mathbb{R}$. Then $\psi : I \to \mathbb{R}$ is convex if for all $\tilde{u}, \tilde{v} \in I$ and for all $\kappa \in [0,1]$,

$$\psi[\kappa\tilde{u}+(1-\kappa)\tilde{v}] \leq \kappa\psi(\tilde{u})+(1-\kappa)\psi(\tilde{v}). \tag{1}$$

If reversed inequality holds in (1), then ψ is called concave."

Example 1.1. *Consider* $\psi : \mathbb{R} \to \mathbb{R}^+$ *defined by*

$$\psi(\tilde{u}) = |\tilde{u}|.$$

For any $\tilde{u}, \tilde{v} \in \mathbb{R}$ *and* $\kappa \in [0,1]$,

$$\psi[\kappa\tilde{u}+(1-\kappa)\tilde{v}] = |\kappa\tilde{u}+(1-\kappa)\tilde{v}| \leq |\kappa\tilde{u}|+|(1-\kappa)\tilde{v}|$$

implies

$$\psi[\kappa\tilde{u}+(1-\kappa)\tilde{v}] \leq \kappa|\tilde{u}|+(1-\kappa)|\tilde{v}| = \kappa\psi(\tilde{u})+(1-\kappa)\psi(\tilde{v})$$

and hence ψ *is convex.*

Example 1.2.

$$\psi(y) = y^n \begin{cases} convex\ on\ \mathbb{R}, & if\ n\ is\ even, \\ convex\ on\ \mathbb{R}^+, & if\ n\ is\ odd, \\ concave\ on\ \mathbb{R}^-, & if\ n\ is\ odd. \end{cases}$$

Example 1.3.

$$\Psi(y) = siny \begin{cases} convex\ on\ (\pi, 2\pi), \\ concave\ on\ (0, \pi). \end{cases}$$

Example 1.4.

$$\eta(y) = cosy \begin{cases} convex\ on\ \left(\frac{\pi}{2}, \pi\right), \\ concave\ on\ \left(0, \frac{\pi}{2}\right). \end{cases}$$

Remark 1.5. *If we put $\kappa = \frac{1}{2}$ in (1), then*

$$\psi\left(\frac{\tilde{u}+\tilde{v}}{2}\right) \leq \frac{\psi(\tilde{u})+\psi(\tilde{v})}{2}.$$

The next four results related to convex functions are given in [22, p. 2, p. 4-5].

Proposition 1.6. *"If ψ is a convex function defined on I and if $\tilde{u}_1 \leq \tilde{v}_1,\ \tilde{u}_2 \leq \tilde{v}_2,\ \tilde{u}_1 \neq \tilde{u}_2,\ \tilde{v}_1 \neq \tilde{v}_2$, then the following inequality is valid*

$$\frac{\psi(\tilde{u}_2) - \psi(\tilde{u}_1)}{\tilde{u}_2 - \tilde{u}_1} \leq \frac{\psi(\tilde{v}_2) - \psi(\tilde{v}_1)}{\tilde{v}_2 - \tilde{v}_1}.$$

If the function ψ is concave, then the inequality reverses."

Proposition 1.7. *"A function ψ be convex on an interval $I \subseteq \mathbb{R}$, if*

$$(\tilde{u}_3 - \tilde{u}_2)\psi(\tilde{u}_1) + (\tilde{u}_1 - \tilde{u}_3)\psi(\tilde{u}_2) + (\tilde{u}_2 - \tilde{u}_1)\psi(\tilde{u}_3) \geq 0$$

holds for every $\tilde{u}_1, \tilde{u}_2, \tilde{u}_3 \in I$ such that $\tilde{u}_1 < \tilde{u}_2 < \tilde{u}_3$."

Theorem 1.8. *"Let $\psi : I \to \mathbb{R}$ be convex. Then $\psi_+^{'}$ and $\psi_-^{'}$ exist and are increasing on I."*

Theorem 1.9. *"$\psi : (a,b) \to \mathbb{R}$ is convex if and only if there is at least one line of support for ψ at each $y \in (a,b)$,*

$$\psi(x) - \psi(y) \geq \chi(x-y) \quad for\ all\ x \in (a,b),$$

where χ depends on y and is given by $\chi = \psi'(y)$ when ψ' exists and $\chi \in [\psi'_-(y), \psi'_+(y)]$ when $\psi'_+(y) \neq \psi'_-(y)$."

Convex and J-convex functions are related as follows (see [19, p. 14]):

Theorem 1.10. *"Let $f : I \to \mathbb{R}$ be a continuous function. Then f is convex if and only if f is convex in the Jensen sense."*

The following is given in [22, p. 6].

Definition 1.3. "A finite sequence $\{a_m\}_{m=1}^{k}$ of real numbers is said to be a convex sequence if

$$2a_m \leq a_{m-1} + a_{m+1} \tag{2}$$

holds for all $m = 2, \ldots, k-1$. If (2) reverses, then $\{a_m\}_{m=1}^{k}$ is called concave."

1.2 Logarithmically convex functions in Jensen sense and logarithmically convex functions

The following definition is given in [11].

Definition 1.4. " A positive function ψ is called logarithmically convex or log-convex (l-convex) in J-sense on some interval $I \subseteq \mathbb{R}$ if

$$\psi(\tilde{u})\psi(\tilde{v}) \geq \psi^2\left(\frac{\tilde{u}+\tilde{v}}{2}\right)$$

holds for every $\tilde{u}, \tilde{v} \in I$."

Remark 1.11. *"It is easy to show that $\psi : I \to \mathbb{R}^+$ is l-convex in J-sense if and only if*

$$\zeta_1^2\psi(\tilde{u}) + 2\zeta_1\zeta_2\psi\left(\frac{\tilde{u}+\tilde{v}}{2}\right) + \zeta_2^2\psi(\tilde{v}) \geq 0$$

holds for every $\zeta_1, \zeta_2 \in \mathbb{R}$ and $\tilde{u}, \tilde{v} \in I$."

The following definition is given in [22, p. 7].

Definition 1.5. "A function $\psi : I \to (0,\infty)$ is called l-convex if $log\psi$ is convex. Equivalently, ψ is l-convex if for all $\tilde{u},\tilde{v} \in I$ and for all $\kappa \in [0,1]$, the inequality

$$\psi[\kappa\tilde{u} + (1-\kappa)\tilde{v}] \leq \psi^{\kappa}(\tilde{u})\psi^{1-\kappa}(\tilde{v})$$

holds."

Example 1.12. *The function $g(y) = \frac{1}{y}$, $y \in (0,\infty)$ is l-convex.*

Example 1.13. *The function $\eta(y) = e^{\frac{-y^2}{2}}$ is l-concave. Since $log\eta(y) = \frac{-y^2}{2}$ is a concave function of y.*

Example 1.14. *The function $y \mapsto e^{y^2}$ is called logarithmically convex.*

1.3 k-Convex functions

The following two definitions are given in [22, p. 14-15].

Definition 1.6. "The kth-order divided difference of a function $\psi : [\tilde{u},\tilde{v}] \to \mathbb{R}$ at mutually distinct points $y_0, y_1, ..., y_k \in [\tilde{u},\tilde{v}]$ is defined recursively by

$$[y_i;\psi] = \psi(y_i), \quad i = 0,\dots,k,$$
$$[y_0,\dots,y_k;\psi] = \frac{[y_1,\dots,y_k;\psi] - [y_0,\dots,y_{k-1};\psi]}{y_k - y_0}."$$

Definition 1.7. "A function ψ is k-convex $(k \geq 0)$ on $[\tilde{u},\tilde{v}]$ if and only if for all choices of $(k+1)$ distinct points $y_0, ..., y_k \in [\tilde{u},\tilde{v}]$,

$$[y_0, ..., y_k;\psi] \geq 0$$

holds."

The following definition is given in [22, p. 16].

Theorem 1.15. *"If $\psi^{(k)}$ exists, then ψ is k-convex if and only if $\psi^{(k)} \geq 0$."*

Example 1.16. *Let* $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ *be defined by*

$$\psi(y) = y^{\ell}.$$

We have

$$\begin{cases} \psi'(y) = \ell y^{\ell-1} \\ \text{and} \\ \psi''(y) = \ell(\ell-1) y^{\ell-2}, \end{cases}$$

$$\Longrightarrow \begin{cases} \psi''(y) > 0 \ for \quad \ell \in (1,\infty) \ or \ \ell \in (-\infty,0) \\ \text{and} \\ \psi''(y) < 0 \ for \quad \ell \in (0,1). \end{cases}$$

So ψ *is*

$$\begin{cases} strictly\ convex\ for \quad \ell \in (1,\infty) \ or \ \ell \in (-\infty,0) \\ \text{and} \\ strictly\ concave\ for \quad \ell \in (0,1). \end{cases}$$

Example 1.17. *Let* $\eta : \mathbb{R} \to \mathbb{R}^+$ *be defined by*

$$\eta(y) = e^{2y}.$$

Clearly

$$\begin{cases} \eta'(y) = 2e^{2y} \\ \text{and} \\ \eta''(y) = 4e^{2y}. \end{cases}$$

As $\eta''(y) \geq 0$ *for* $y \in \mathbb{R}$, η *is convex.*

Example 1.18. *Let the function* Φ *be defined by*

$$\Phi(y) = e^{-y}.$$

We have

$$\begin{cases} \Phi'(y) = -e^{-y} \\ and \\ \Phi''(y) = e^{-y}. \end{cases}$$

As $\Phi''(y) > 0$ for $y \in \mathbb{R}$, Φ is convex.

Example 1.19. *Consider $\Psi : \mathbb{R} \to \mathbb{R}$ defined by*

$$\Psi(y) = \ln(1 + e^{y}).$$

Obviously

$$\begin{cases} \Psi'(y) = \frac{e^{y}}{(1+e^{y})} \\ and \\ \Psi''(y) = \frac{e^{y}}{(1+e^{y})^{2}}. \end{cases}$$

As $\Psi''(y) > 0$ for $y \in \mathbb{R}$, Ψ is convex.

1.4 k-Exponentially convex functions

Let $I \subseteq \mathbb{R}$ be an open interval.

The definitions, in this section, are given in [21].

Definition 1.8. "A function ψ is k-exponentially convex (k-exp-convex) in J-sense on I if

$$\sum_{i,j=1}^{k} \zeta_i \zeta_j \psi\left(\frac{\tilde{u}_i + \tilde{u}_j}{2}\right) \geq 0 \tag{3}$$

holds for $\zeta_i, \zeta_j \in \mathbb{R}$ and every $\tilde{u}_i, \tilde{u}_j \in I$."

Definition 1.9. "A function $\psi : I \to \mathbb{R}$ is k-exp-convex if it is k-exp-convex in the J-sense and continuous on I."

Remark 1.20. "1*-exp-convex functions in the J-sense are non-negative functions. Also k-exp-convex functions in the J-sense are m-exp-convex functions in the J-sense for all $m \in \mathbb{N}$, $m \leq k$.*"

Proposition 1.21. *"If ψ is k-exp-convex in the J-sense on I, then the matrix* $\left[\psi\left(\frac{x_i+x_j}{2}\right)\right]_{i,j=1}^{m}$ *is positive semi-definite for all $m \in \mathbb{N}$, $m \leq k$. Particularly,*

$$\det\left[\psi\left(\frac{x_i+x_j}{2}\right)\right]_{i,j=1}^{m} \geq 0 \quad \textit{for every } m \in \mathbb{N},\ m \leq k,\ \ x_i \in I,\ \ i = 1,\ldots,k.$$

"

Definition 1.10. "A function $\psi : I \to \mathbb{R}$ is exp-convex in the J-sense if it is k-exp-convex in the J-sense for all $k \in \mathbb{N}$."

Definition 1.11. "A function $\psi : I \to \mathbb{R}$ is exp-convex if it is exp-convex in the J-sense and continuous."

This proposition is followed from [1].

Proposition 1.22. *"Let $\psi : I \to \mathbb{R}$. These statements are equivalent.*
(i) ψ is exponentially convex.
(ii) ψ is continuous and

$$\sum_{i,j=1}^{k} \zeta_i \zeta_j \psi\left(\frac{\tilde{u}_i+\tilde{u}_j}{2}\right) \geq 0$$

for $\zeta_i, \zeta_j \in \mathbb{R}$ and $\tilde{u}_i \in I$, $1 \leq i,\ j \leq k$."

Example 1.23. *$\psi(y) = c$ is exp-convex on $\mathbb{R}$ for any $c \in [0,\infty)$.*

Example 1.24. *$\psi(\tilde{u}) = ce^{\alpha\tilde{u}}$ is exp-convex on $c \in [0,\infty)$, $\alpha \in \mathbb{R}$.*

Example 1.25. *Let $\psi_p : [0,\infty) \to \mathbb{R}$ be defined by*

$$\psi_p(\tilde{u}) = \frac{1}{p^2} e^{p\tilde{u}^2}.$$

Then $p \mapsto \psi_p(\tilde{u})$, $p \mapsto \frac{d}{d\tilde{u}}\psi_p(\tilde{u})$ and $p \mapsto \frac{d^2}{d\tilde{u}^2}\psi_p(\tilde{u})$ are exp-convex defined on $(0,\infty)$ for all $y \in [0,\infty)$.

Example 1.26. *The function $\psi(\tilde{u}) = \tilde{u}^{-\beta}$ is exp-convex on $(0,\infty)$ for any $\beta \geq 0$.*

Remark 1.27. (i) *A positive real valued function* $\Psi : I \to \mathbb{R}$ *is l-convex in J-sense if and only if* Ψ *is 2-exp-convex in J-sense.*

(ii) *A positive real valued function* $\Phi : I \to \mathbb{R}$ *is l-convex if and only if* Φ *is 2-exp-convex.*

Remark 1.28. *From* (3) *we have these conclusions* :

(i) *If* ψ *is exp-convex defined on I then for all* $\tilde{u} \in I$, $\psi(\tilde{u}) \geq 0$. *Moreover for any* $c \in [0, \infty)$, $c\psi$ *is also exp-convex*;

(ii) *If* ψ_1 *and* ψ_2 *are exp-convex defined on I, then their sum is also exp-convex on I.*

Chapter 2

Literature Review

2 Literature Review

2.1 Majorization

The concept of majorization, which is closely related to the convexity, is discussed very nicely by Marshall and Olkin in [16] (*see also*[17]).
The following definition is given in [17].
" For fixed $k \geq 2$, let

$$\tilde{\mathbf{u}} = (\tilde{u}_1, ..., \tilde{u}_k)$$

and

$$\tilde{\mathbf{v}} = (\tilde{v}_1, ..., \tilde{v}_k)$$

denote two k-tuples and

$$\tilde{u}_{[1]} \geq \tilde{u}_{[2]} \geq ... \geq \tilde{u}_{[k]}$$

and

$$\tilde{v}_{[1]} \geq \tilde{v}_{[2]} \geq ... \geq \tilde{v}_{[k]}$$

be their ordered components.

$\tilde{\mathbf{u}}$ is said to be majorized by $\tilde{\mathbf{v}}$ or $\tilde{\mathbf{v}}$ is said to majorize $\tilde{\mathbf{u}}$ (denoted by $\tilde{\mathbf{u}} \prec \tilde{\mathbf{v}}$ *or* $\tilde{\mathbf{v}} \succ \tilde{\mathbf{u}}$) if

$$\sum_{i=1}^{m} \tilde{u}_{[i]} \leq \sum_{i=1}^{m} \tilde{v}_{[i]};\ m = 1, ..., k-1$$

and

$$\sum_{i=1}^{k} \tilde{u}_i = \sum_{i=1}^{k} \tilde{v}_i.$$

"

Example 2.1. *If* $\tilde{\mathbf{u}} = (2,1)$ *and* $\tilde{\mathbf{v}} = (3,0)$*, then*

$$(3,0) \succ (2,1).$$

Example 2.2. *If* $\tilde{\mathbf{u}} = (3,3,0)$ *and* $\tilde{\mathbf{v}} = (6,0,0)$*, then*

$$(6,0,0) \succ (3,3,0).$$

Example 2.3.

$$\left(\frac{1}{k},\ldots,\frac{1}{k}\right) \prec \left(\frac{1}{k-1},\ldots,\frac{1}{k-1},0\right) \prec \ldots \prec \left(\frac{1}{2},\frac{1}{2},0,\ldots,0\right) \prec (1,0,\ldots,0).$$

The well known majorization theorem states that (see [15, p. 11] and [18, p. 169]) :

Theorem 2.4. *"Let I be an interval in $\mathbb{R}$ and let $\tilde{\mathbf{u}}$, $\tilde{\mathbf{v}}$ be two k-tuples : $\tilde{u}_i, \tilde{v}_i \in I$ $(i = 1,\ldots,k)$. Then*

$$\sum_{i=1}^{k} \psi(\tilde{u}_i) \leq \sum_{i=1}^{k} \psi(\tilde{v}_i)$$

holds for every continuous convex function $\psi : I \to \mathbb{R}$ if and only if $\tilde{\mathbf{v}} \succ \tilde{\mathbf{u}}$ holds."

2.2 Favard and Berwald-type inequalities

Favard proved the following result in [22, p. 212].

Theorem 2.5. *"Let f be a non-negative continuous and concave defined on $[\tilde{u},\tilde{v}] \subset \mathbb{R}$, not identically zero and ψ be convex defined on $[0,2c] \subset \mathbb{R}$, where*

$$c = \frac{1}{\tilde{v} - \tilde{u}} \int_{\tilde{u}}^{\tilde{v}} f(x)dx. \tag{4}$$

Then

$$\frac{1}{2c} \int_{0}^{2c} \psi(y)dy \geq \frac{1}{\tilde{v} - \tilde{u}} \int_{\tilde{u}}^{\tilde{v}} \psi(f(x))dx."$$

Some results related to the generalizations of the Favard's inequality are given in [6, pp. 413-414].

Remark 2.6. *As $\psi(t) = t^s$ is convex for $s > 1$, we have*

$$\frac{1}{\tilde{v} - \tilde{u}} \int_{\tilde{u}}^{\tilde{v}} f^s(x)dx \leqslant \frac{2^s}{s+1} \left(\frac{1}{\tilde{v} - \tilde{u}} \int_{\tilde{u}}^{\tilde{v}} f(x)dx\right)^s. \tag{5}$$

If $s \in (0,1)$, then (5) reverses.

Karlin and Studden [6] presented a generalized result:

Theorem 2.7. *"Let f be a nonnegative continuous and concave defined on $[\tilde{u}, \tilde{v}]$, not identically zero; c is defined in Theorem 2.5 and let ψ be convex defined on $[d, 2c-d]$, where d satisfies $0 < d \leq f_{min}$ (where f_{min} is minimum of f). Then*

$$\frac{1}{2c-2d} \int_{d}^{2c-d} \psi(z)dz \geqslant \frac{1}{\tilde{v}-\tilde{u}} \int_{\tilde{u}}^{\tilde{v}} \psi(f(x))dx."$$

Remark 2.8. *For $\psi(y) = y^q$, $q > 1$, we have*

$$\frac{1}{(2c-2d)(q+1)} \left((2c-d)^{q+1} - d^{q+1}\right) \geq \frac{1}{\tilde{v}-\tilde{u}} \int_{\tilde{u}}^{\tilde{v}} \psi(f^q(x))dx.$$

Berwald gave a generalized form of Favard's inequality in [22, p. 214] (see also [6, pp. 413-414]):

Theorem 2.9. *"Let f is a non-negative, continuous concave function, not identically zero defined on $[\tilde{u}, \tilde{v}]$, and φ be a continuous and strictly monotonic function defined on $[0, z_0]$, where z_0 is sufficiently large. If α is the unique positive root of the equation*

$$\frac{1}{\alpha} \int_{0}^{\alpha} \varphi(y)dy = \frac{1}{\tilde{v}-\tilde{u}} \int_{\tilde{u}}^{\tilde{v}} \varphi(f(x))dx,$$

then for every function $\psi : [0, z_0] \to \mathbb{R}$ which is convex with respect to φ, we have

$$\frac{1}{\alpha} \int_{0}^{\alpha} \psi(y)dy \geqslant \frac{1}{\tilde{v}-\tilde{u}} \int_{\tilde{u}}^{\tilde{v}} \psi(f(x))dx."$$

Remark 2.10. *If we take $\varphi(x) = x^s$ and $\psi(x) = x^r$ and if $r > s > 0$, then*

$$\left(\frac{s+1}{\tilde{v}-\tilde{u}} \int_{\tilde{u}}^{\tilde{v}} f^s(x)dx\right)^{\frac{1}{s}} \geq \left(\frac{r+1}{\tilde{v}-\tilde{u}} \int_{\tilde{u}}^{\tilde{v}} f^r(x)dx\right)^{\frac{1}{r}}.$$

For monotone convex and concave functions f, some generalizations of Theorems 2.5 and 2.9 to the weighted version are given in [14].

Theorem 2.11. *(i) "Let f be a positive increasing concave function defined on $[\tilde{u}, \tilde{v}]$.*

Assume that ψ is a convex function defined on $[0, 2c_i]$, where

$$c_i = \frac{(\tilde{v} - \tilde{u}) \int_{\tilde{u}}^{\tilde{v}} f(t)\vartheta(t)dt}{2\int_{\tilde{u}}^{\tilde{v}} (t - \tilde{u})\vartheta(t)dt}.$$

Then

$$\frac{1}{\tilde{u} - \tilde{v}}\int_{\tilde{u}}^{\tilde{v}} \psi[f(t)]\vartheta(t)dt \leq \int_{0}^{1} \psi(2sc_i)\vartheta[\tilde{u}(1 - s)+\tilde{v}s]ds. \tag{6}$$

If f is decreasing convex function defined on $[\tilde{u},\tilde{v}]$ and $f(\tilde{u})=0$, then (6) *reverses.*

(ii) Let f be a positive decreasing concave function defined on $[\tilde{u},\tilde{v}]$. Assume that ψ is a convex function on $[0,2c_d]$, where

$$c_d = \frac{(\tilde{v} - \tilde{u})\int_{\tilde{u}}^{\tilde{v}} f(t)\vartheta(t)dt}{2\int_{\tilde{u}}^{\tilde{v}}(\tilde{v} - t)\vartheta(t)dt}.$$

Then

$$\frac{1}{\tilde{v} - \tilde{u}}\int_{\tilde{u}}^{\tilde{v}} \psi[f(t)]\vartheta(t)dt \leq \int_{0}^{1} \psi(2sc_d)\vartheta[\tilde{u}s+\tilde{v}(1 - s)]ds. \tag{7}$$

If f is decreasing convex function defined on $[\tilde{u},\tilde{v}]$ and $f(\tilde{v})=0$, then (7) *reverses."*

A weighted version of Berwald's inequality was presented in [14].

Theorem 2.12. *"Let ψ be convex w.r.t strictly increasing function φ defined on $[0,\infty)$ that is let $\psi\circ\varphi^{-1}$ is convex.*
(i) If f is a positive increasing concave function defined on $[\tilde{u},\tilde{v}]$ and if d_i is a positive root of the equation

$$\frac{1}{d_i}\int_{0}^{d_i} \varphi(y)\vartheta\left(\tilde{u}+\frac{\tilde{v}-\tilde{u}}{d_i}y\right)dy = \frac{1}{\tilde{v}-\tilde{u}}\int_{\tilde{u}}^{\tilde{v}} \varphi(f(t))\vartheta(t)dt, \tag{8}$$

then

$$\frac{1}{\tilde{v}-\tilde{u}}\int_{\tilde{u}}^{\tilde{v}} \psi(f(t))\vartheta(t)dt \leq \int_{0}^{1} \psi(sd_i)\vartheta[\tilde{u}(1-s)+\tilde{v}s]ds. \tag{9}$$

If convex function f is increasing defined on $[\tilde{u},\tilde{v}]$ with $f(\tilde{u})=0$, then (9) *reverses.*
(ii) If f is a positive decreasing concave function defined on $[\tilde{u},\tilde{v}]$ and if d_e is a positive root of the equation

$$\frac{1}{d_e}\int_{0}^{d_e} \varphi(y)\vartheta\left(\tilde{v}-\frac{\tilde{v}-\tilde{u}}{d_e}y\right)dy = \frac{1}{\tilde{v}-\tilde{u}}\int_{\tilde{u}}^{\tilde{v}} \varphi(f(t))\vartheta(t)dt,$$

then

$$\frac{1}{\tilde{v}-\tilde{u}}\int_{\tilde{u}}^{\tilde{v}}\psi(f(t))\vartheta(t)dt \leq \int_{0}^{1}\psi(sd_e)\vartheta[\tilde{u}s+\tilde{v}(1-s)]ds. \tag{10}$$

If f is decreasing convex function defined on $[\tilde{u},\tilde{v}]$ with $f(\tilde{v})=0$, then (10) reverses."

Remark 2.13. *If we put $\varphi(a)=a^p$ and $\psi(a)=a^q$ in Theorem 2.12. Let $0<p\leq q$.*

(i) If f (>0) is increasing and concave defined on $[\tilde{u},\tilde{v}]$, we have

$$\left(\frac{\int_{\tilde{u}}^{\tilde{v}} f(t)^q\vartheta(t)dt}{\int_{\tilde{u}}^{\tilde{v}}(t-\tilde{u})^q\vartheta(t)dt}\right)^{\frac{1}{q}} \leq \left(\frac{\int_{\tilde{u}}^{\tilde{v}} f(t)^p\vartheta(t)dt}{\int_{\tilde{u}}^{\tilde{v}}(t-\tilde{u})^p\vartheta(t)dt}\right)^{\frac{1}{p}}. \tag{11}$$

If f is an increasing and convex defined on $[\tilde{u},\tilde{v}]$ with $f(\tilde{u})=0$, then (11) reverses.

(ii) If f (>0) decreasing and concave defined on $[\tilde{u},\tilde{v}]$, we have

$$\left(\frac{\int_{\tilde{u}}^{\tilde{v}} f(t)^q\vartheta(t)dt}{\int_{\tilde{u}}^{\tilde{v}}(\tilde{v}-t)^q\vartheta(t)dt}\right)^{\frac{1}{q}} \leq \left(\frac{\int_{\tilde{u}}^{\tilde{v}} f(t)^p\vartheta(t)dt}{\int_{\tilde{u}}^{\tilde{v}}(\tilde{v}-t)^p\vartheta(t)dt}\right)^{\frac{1}{p}}. \tag{12}$$

If f is decreasing and convex defined on $[\tilde{u},\tilde{v}]$ with $f(\tilde{v})=0$, then (12) reverses.

Chapter 3

Main Results

3 Main Results

3.1 Generalizations of Majorization-type results

Let ψ be convex defined on I and suppose $\tilde{u}_1,\ldots,\tilde{u}_k,\tilde{v}_1,\ldots,\tilde{v}_k \in I$. Suppose that the sequence $\tilde{u}_1,\ldots,\tilde{u}_k$ majorizes $\tilde{v}_1,...,\tilde{v}_k$: that is the following hold :

$$\tilde{u}_1 \geq \ldots \geq \tilde{u}_k$$

$$\tilde{v}_1 \geq \ldots \geq \tilde{v}_k$$

$$\tilde{u}_1 \geq \tilde{v}_1$$

$$\tilde{u}_1+\tilde{u}_2 \geq \tilde{v}_1+\tilde{v}_2$$

$$\vdots$$

$$\tilde{u}_1+\tilde{u}_2+\ldots+\tilde{u}_{k-1} \geq \tilde{v}_1+\tilde{v}_2+\ldots+\tilde{v}_{k-1}$$

$$\tilde{u}_1+\tilde{u}_2+\ldots+\tilde{u}_{k-1}+\tilde{u}_k = \tilde{v}_1+\tilde{v}_2+\ldots+\tilde{v}_{k-1}+\tilde{v}_k.$$

Then

$$\psi(\tilde{u}_1)+\psi(\tilde{u}_2)+\ldots+\psi(\tilde{u}_k) \geq \psi(\tilde{v}_1)+\psi(\tilde{v}_2)+\ldots+\psi(\tilde{v}_k). \tag{13}$$

Inequality (13) is known as majorization inequality.

Fuchs proved a well known weighted version of Theorem 2.4 in [4] (see also [22, p. 323]), which is infact a generalization of Theorem 2.4.

Theorem 3.1. *Let $\vartheta=(\vartheta_1,\ldots,\vartheta_k)$ be a real k-tuple and $\tilde{\mathbf{u}}=(\tilde{u}_1,\ldots,\tilde{u}_k)$, $\tilde{\mathbf{v}}=(\tilde{v}_1,\ldots,\tilde{v}_k)$ be two decreasing real k-tuples* :

$$\sum_{i=1}^{m}\vartheta_i\tilde{u}_i \geq \sum_{i=1}^{m}\vartheta_i\tilde{v}_i, \quad m=1,\ldots,k-1,$$

and

$$\sum_{i=1}^{k}\vartheta_i\tilde{u}_i = \sum_{i=1}^{k}\vartheta_i\tilde{v}_i$$

hold. Let $\psi: I\to\mathbb{R}$ be convex and continuous. Then

$$\sum_{i=1}^{k}\vartheta_i\psi(\tilde{u}_i) \geq \sum_{i=1}^{k}\vartheta_i\psi(\tilde{v}_i) \tag{14}$$

holds. If ψ is concave, then (14) *reverses.*

Proof. Without loss of generality, let $\tilde{u}_i \neq \tilde{v}_i$ and define $e_i = (\psi(\tilde{u}_i) - \psi(\tilde{v}_i))/(\tilde{u}_i - \tilde{v}_i)$ $(i = 1,\ldots,k)$.Then from (13) we obtain $e_i \geq e_{i+1}$ for $i \leq k-1$ and the proof follows from

$$\sum_{i=1}^{k} \vartheta_i \psi(\tilde{u}_i) - \sum_{i=1}^{k} \vartheta_i \psi(\tilde{v}_i) = \sum_{i=1}^{k} \vartheta_i (\tilde{u}_i - \tilde{v}_i) e_i$$

$$= (\vartheta_k \tilde{u}_k - \vartheta_k \tilde{v}_k) e_k + \sum_{m=1}^{k-1} (\vartheta_m \tilde{u}_m - \vartheta_m \tilde{v}_m)(e_m - e_{m+1}).$$

□

Another interesting related result (see [2] and [22, p. 323]) states that :

Theorem 3.2. *Let $\tilde{\mathbf{u}}, \tilde{\mathbf{v}}$ be two decreasing k-tuples and ϑ be a real k-tuple. If*

$$\sum_{i=1}^{k} \vartheta_i \tilde{u}_i \leq \sum_{i=1}^{k} \vartheta_i \tilde{v}_i \quad for \quad m = 1,\ldots,k-1,k \tag{15}$$

holds, then (14) *holds for every continuous increasing convex function* $\psi : I \to \mathbb{R}$.
Let $\tilde{\mathbf{u}}, \tilde{\mathbf{v}}$ be two increasing k-tuples. If (15) *reverses, then* (14) *holds for every continuous decreasing convex function* $\psi : I \to \mathbb{R}$.

Proof. Analogous to the proof of Theorem 3.1. □

A majorization-type result (see [19, p. 35] and [12]) for the case when only one sequence is monotonic states that:

Theorem 3.3. *Let ψ be convex defined on $I \subseteq \mathbb{R}$, ϑ be k-tuple and $\tilde{\boldsymbol{u}}, \tilde{\mathbf{v}} \in I^k$ satisfying*

$$\sum_{i=1}^{m} \vartheta_i \tilde{v}_i \leq \sum_{i=1}^{m} \vartheta_i \tilde{u}_i, \quad m = 1,2,...,k-1, \tag{16}$$

$$\sum_{i=1}^{k} \vartheta_i \tilde{v}_i = \sum_{i=1}^{k} \vartheta_i \tilde{u}_i.$$

(i) If $\tilde{\mathbf{v}}$ is decreasing k-tuple, then

$$\sum_{i=1}^{k} \vartheta_i \psi(\tilde{v}_i) \leq \sum_{i=1}^{k} \vartheta_i \psi(\tilde{u}_i).$$

(ii) If $\tilde{\mathbf{u}}$ is an increasing k-tuple, then

$$\sum_{i=1}^{k} \vartheta_i \psi(\tilde{u}_i) \leq \sum_{i=1}^{k} \vartheta_i \psi(\tilde{v}_i).$$

Proof. (i) By using Theorem 1.9, the convexity of ψ implies

$$\psi(x) - \psi(y) \geq \psi'_+(y)(x-y). \tag{17}$$

Put $x = \tilde{u}_i$ and $y = \tilde{v}_i$ in (17), we have

$$\psi(\tilde{u}_i) - \psi(\tilde{v}_i) \geq \psi'_+(\tilde{v}_i)(\tilde{u}_i - \tilde{v}_i). \tag{18}$$

Since ϑ is a positive k-tuple, from (18), we have

$$\sum_{i=1}^{k} \vartheta_i [\psi(\tilde{u}_i) - \psi(\tilde{v}_i)] \geq \sum_{i=1}^{k} \vartheta_i \psi'_+(\tilde{v}_i)(\tilde{u}_i - \tilde{v}_i).$$

Simple calculation yields that

$$\sum_{i=1}^{k} \vartheta_i \psi'_+(\tilde{v}_i)(\tilde{u}_i - \tilde{v}_i) = \sum_{m=1}^{k-1} (\vartheta_m \tilde{u}_m - \vartheta_m \tilde{v}_m)[\psi'_+(\tilde{v}_m) - \psi'_+(\tilde{v}_{m+1})]. \tag{19}$$

Since $\tilde{v}$ is a decreasing k-tuple and ψ'_+ is an increasing, we obtain

$$\psi'_+(\tilde{v}_m) - \psi'_+(\tilde{v}_{m+1}) \geq 0, \tag{20}$$

and (16) can be written as

$$\sum_{i=1}^{m} \vartheta_i \tilde{u}_i - \sum_{i=1}^{m} \vartheta_i \tilde{v}_i \geq 0. \tag{21}$$

By using (20) and (21) in (19), the result is immediate.

(ii) Analogoous to the proof of (i).

□

Discrete case of Lemma 1 in [14] states that:

Lemma 3.4. *Let $\mathbf{y}$ be a positive k-tuple. If $\tilde{\mathbf{u}}$ is an increasing k-tuple, then*

$$\sum_{i=1}^{m} \tilde{u}_i y_i \sum_{i=1}^{k} y_i \leq \sum_{i=1}^{k} \tilde{u}_i y_i \sum_{i=1}^{m} y_i, \quad m = 1, 2, ..., k.$$

If $\tilde{\mathbf{u}}$ is a decreasing k-tuple, then the inequality reverses.

Proof. When $\tilde{\mathbf{u}}$ is an increasing real k-tuple, then

$$\begin{aligned}
\sum_{i=1}^{m}\tilde{u}_i y_i \sum_{i=1}^{k} y_i &= \sum_{i=1}^{m}\tilde{u}_i y_i \left[\sum_{i=1}^{m} y_i + \sum_{i=m+1}^{k} y_i\right] \\
&= \sum_{i=1}^{m}\tilde{u}_i y_i \sum_{i=1}^{m} y_i + \sum_{i=1}^{m}\tilde{u}_i y_i \sum_{i=m+1}^{k} y_i \\
&= \left[\sum_{i=1}^{k}\tilde{u}_i y_i - \sum_{i=m+1}^{k}\tilde{u}_i y_i\right]\sum_{i=1}^{m} y_i + \sum_{i=1}^{m}\tilde{u}_i y_i \sum_{i=m+1}^{k} y_i \\
&= \sum_{i=1}^{k}\tilde{u}_i y_i \sum_{i=1}^{m} y_i - \sum_{i=m+1}^{k}\tilde{u}_i y_i \sum_{i=1}^{m} y_i + \sum_{i=1}^{m}\tilde{u}_i y_i \sum_{i=m+1}^{k} y_i \\
&\leq \sum_{i=1}^{k}\tilde{u}_i y_i \sum_{i=1}^{m} y_i - \tilde{u}_m \sum_{i=m+1}^{k} y_i \sum_{i=1}^{m} y_i + \tilde{u}_m \sum_{i=1}^{m} y_i \sum_{i=m+1}^{k} y_i \\
&= \sum_{i=1}^{k}\tilde{u}_i y_i \sum_{i=1}^{m} y_i .
\end{aligned}$$

Analogously, the reverse inequality can be proved. □

3.2 Generalization of Favard's inequality and related results

If $\tilde{\mathbf{u}} = (\tilde{u}_1, \tilde{u}_2, \ldots, \tilde{u}_k)$ and $\tilde{\mathbf{v}} = (\tilde{v}_1, \tilde{v}_2, \ldots, \tilde{v}_k)$ are two k-tuples, then we define

$$\tilde{\mathbf{u}}\tilde{\mathbf{v}} = (\tilde{u}_1\tilde{v}_1, \tilde{u}_2\tilde{v}_2, \ldots, \tilde{u}_k\tilde{v}_k)$$

and

$$\frac{\tilde{\mathbf{u}}}{\tilde{\mathbf{v}}} = \left(\frac{\tilde{u}_1}{\tilde{v}_1}, \frac{\tilde{u}_2}{\tilde{v}_2}, \ldots, \frac{\tilde{u}_k}{\tilde{v}_k}\right)$$

with each $\tilde{v}_i \neq 0$ for $i = 1, \ldots, k$.

A generalization of discrete weighted Favard's inequality (see [12]) states that:

Theorem 3.5. *Let $\psi : (0,1) \to \mathbb{R}$ be convex and ϑ, $\tilde{\mathbf{u}}$ and $\tilde{\mathbf{v}}$ be positive k-tuples.*

(i) Let $\tilde{\mathbf{u}}/\tilde{\mathbf{v}}$ be decreasing k-tuple. If $\tilde{\mathbf{u}}$ is an increasing k-tuple, then

$$\sum_{i=1}^{k}\vartheta_i \psi\left(\frac{\tilde{u}_i}{\sum_{i=1}^{k}\tilde{u}_i\vartheta_i}\right) \leq \sum_{i=1}^{k}\vartheta_i \psi\left(\frac{\tilde{v}_i}{\sum_{i=1}^{k}\tilde{v}_i\vartheta_i}\right). \tag{22}$$

If $\tilde{\mathbf{v}}$ is a decreasing k-tuple, then (22) *reverses.*

(ii) Let $\tilde{\mathbf{u}}/\tilde{\mathbf{v}}$ be an increasing k-tuple. If $\tilde{\mathbf{v}}$ is an increasing k-tuple, then

$$\sum_{i=1}^{k} \vartheta_i \psi\left(\frac{\tilde{v}_i}{\sum_{i=1}^{k} \tilde{v}_i \vartheta_i}\right) \leq \sum_{i=1}^{k} \vartheta_i \psi\left(\frac{\tilde{u}_i}{\sum_{i=1}^{k} \tilde{u}_i \vartheta_i}\right). \tag{23}$$

If $\tilde{\mathbf{u}}$ is a decreasing k-tuple, then (23) *reverses.*

Proof. (i) If $\tilde{\mathbf{y}} = \tilde{\mathbf{v}}\vartheta$ and if $\tilde{\mathbf{w}} = \tilde{\mathbf{u}}/\tilde{\mathbf{v}}$, then by Lemma 3.4 we get

$$\sum_{i=1}^{k} \tilde{u}_i \vartheta_i \sum_{i=1}^{m} \tilde{v}_i \vartheta_i \leq \sum_{i=1}^{m} \tilde{u}_i \vartheta_i \sum_{i=1}^{k} \tilde{v}_i \vartheta_i, \qquad m = 1, \ldots, k,$$

which gives

$$\sum_{i=1}^{m} \vartheta_i \left(\frac{\tilde{v}_i}{\sum_{i=1}^{k} \tilde{v}_i \vartheta_i}\right) \leq \sum_{i=1}^{m} \vartheta_i \left(\frac{\tilde{u}_i}{\sum_{i=1}^{k} \tilde{u}_i \vartheta_i}\right).$$

As $\tilde{\mathbf{u}}$ is an increasing, by using Theorem 3.3, we have

$$\sum_{i=1}^{k} \vartheta_i \psi\left(\frac{\tilde{u}_i}{\sum_{i=1}^{k} \tilde{u}_i \vartheta_i}\right) \leq \sum_{i=1}^{k} \vartheta_i \psi\left(\frac{\tilde{v}_i}{\sum_{i=1}^{k} \tilde{v}_i \vartheta_i}\right).$$

Similarly, the remaining cases can be proved.

□

An application of Theorem 3.5 is states that:

Remark 3.6. *(a) Let $\psi(\tilde{w}) = \tilde{w}^t$, where $t > 1$ or $t < 0$.*

(i) Let $\tilde{\mathbf{u}}/\tilde{\mathbf{v}}$ be decreasing k-tuple. If $\tilde{\mathbf{u}}$ is an increasing k-tuple, then

$$\left(\frac{\sum_{i=1}^{k} \tilde{u}_i^t \vartheta_i}{\sum_{i=1}^{k} \tilde{v}_i^t \vartheta_i}\right) \leq \left(\frac{\sum_{i=1}^{k} \tilde{u}_i \vartheta_i}{\sum_{i=1}^{k} \tilde{v}_i \vartheta_i}\right)^t. \tag{24}$$

If $\tilde{\mathbf{v}}$ is a decreasing k-tuple, then (24) *reverses.*

(ii) Let $\tilde{\mathbf{u}}/\tilde{\mathbf{v}}$ be an increasing k-tuple. If $\tilde{\mathbf{v}}$ is an increasing k-tuple, then

$$\left(\frac{\sum_{i=1}^{k} \tilde{u}_i \vartheta_i}{\sum_{i=1}^{k} \tilde{v}_i \vartheta_i}\right)^t \leq \left(\frac{\sum_{i=1}^{k} \tilde{u}_i^t \vartheta_i}{\sum_{i=1}^{k} \tilde{v}_i^t \vartheta_i}\right). \tag{25}$$

If $\tilde{\mathbf{u}}$ is a decreasing k-tuple, then (25) *reverses.*

(b) If $\psi(\tilde{w}) = \tilde{w}^t$, $0 < t < 1$, then (24) reverses, (24), (25) and reverse of (25) hold.

The following corollary is given in [12] .

Corollary 3.7. *Let $\psi : [0,\infty) \to \mathbb{R}$ be convex, ϑ be a positive k-tuple.*

(i) If $\tilde{\mathbf{u}}$ is a positive increasing concave k-tuple, then

$$\sum_{i=1}^{k} \vartheta_i \psi\left(\frac{\tilde{u}_i}{\sum_{j=1}^{k} \vartheta_j \tilde{u}_j}\right) \leq \sum_{i=1}^{k} \vartheta_i \psi\left(\frac{i-1}{\sum_{j=1}^{k}(j-1)\vartheta_j}\right). \tag{26}$$

If $\tilde{\mathbf{u}}$ is an increasing convex real k-tuple and $\tilde{u}_1 = 0$, then (26) reverses.

(ii) If $\tilde{\mathbf{u}}$ is a positive decreasing concave k-tuple, then

$$\sum_{i=1}^{k} \vartheta_i \psi\left(\frac{\tilde{u}_i}{\sum_{j=1}^{k} \vartheta_j \tilde{u}_j}\right) \leq \sum_{i=1}^{k} \vartheta_i \psi\left(\frac{k-i}{\sum_{j=1}^{k}(k-j)\vartheta_j}\right). \tag{27}$$

If $\tilde{\mathbf{u}}$ is a decreasing convex real k-tuple and $\tilde{u}_k = 0$, then (27) reverses.

Proof. (i) If $\tilde{v}_1 = \varepsilon < \tilde{u}_1/\tilde{u}_2$, $\tilde{v}_i = i-1 (2 \leq i \leq k)$. Therefore, $\tilde{u}_i/\tilde{v}_i (1 \leq i \leq k)$ is a decreasing k-tuple. By using Theorem 3.5 (22), we obtain

$$\begin{aligned}\sum_{i=1}^{k} \vartheta_i \psi\left(\frac{\tilde{u}_i}{\sum_{j=1}^{k} \vartheta_j \tilde{u}_j}\right) &\leq \vartheta_1 \psi\left(\frac{\varepsilon}{\varepsilon\vartheta_1 + \sum_{j=2}^{k}(j-1)\vartheta_j}\right) \\ &+ \sum_{i=2}^{k} \vartheta_1 \psi\left(\frac{i-1}{\varepsilon\vartheta_1 + \sum_{j=2}^{k}(j-1)\vartheta_j}\right).\end{aligned}$$

By taking $\varepsilon \to 0$, we have

$$\begin{aligned}\sum_{i=1}^{k} \vartheta_i \psi\left(\frac{\tilde{u}_i}{\sum_{j=1}^{k} \vartheta_j \tilde{u}_j}\right) &\leq \vartheta_1 \psi\left(\frac{0}{(0)\vartheta_1 + \sum_{j=2}^{k}(j-1)\vartheta_j}\right) \\ &+ \sum_{i=2}^{k} \vartheta_i \psi\left(\frac{i-1}{(0)\vartheta_1 + \sum_{j=2}^{k}(j-1)\vartheta_j}\right), \\ &= \vartheta_1 \psi(0) + \sum_{i=2}^{k} \vartheta_i \psi\left(\frac{i-1}{\sum_{j=2}^{k}(j-1)\vartheta_j}\right), \\ &= \sum_{i=1}^{k} \vartheta_i \psi\left(\frac{i-1}{\sum_{j=2}^{k}(j-1)\vartheta_j}\right).\end{aligned}$$

Because $\tilde{\mathbf{u}}$ is an increasing convex k-tuple and $\tilde{u}_1 = 0$, further $\tilde{u}_i/(i-1)(2 \leq i \leq k)$ is an increasing k-tuple. By applying Theorem 3.5 (23), we get

$$\sum_{i=2}^{k} \vartheta_i \psi\left(\frac{i-1}{\sum_{j=2}^{k}(j-1)\vartheta_j}\right) \leq \sum_{i=2}^{k} \vartheta_i \psi\left(\frac{\tilde{u}_i}{\sum_{j=2}^{k} \vartheta_j \tilde{u}_j}\right),$$

equivalent to

$$\begin{aligned} \vartheta_1 \psi\left(\frac{0}{\sum\limits_{j=1}^{k}(j-1)\vartheta_j}\right) + \sum_{i=2}^{k} \vartheta_i \psi\left(\frac{i-1}{\sum\limits_{j=1}^{k}(j-1)\vartheta_j}\right) \quad &\leq \quad \vartheta_1 \psi\left(\frac{0}{\sum_{j=1}^{k} \vartheta_j \tilde{u}_j}\right) \\ &+ \quad \sum_{i=2}^{k} \vartheta_i \psi\left(\frac{\tilde{u}_i}{\sum_{j=1}^{k} \vartheta_j \tilde{u}_j}\right), \end{aligned}$$

which implies

$$\sum_{i=1}^{k} \vartheta_i \psi\left(\frac{i-1}{\sum_{j=1}^{k}(j-1)\vartheta_j}\right) \leq \sum_{i=1}^{k} \vartheta_i \psi\left(\frac{\tilde{u}_i}{\sum_{j=1}^{k} \vartheta_j \tilde{u}_j}\right).$$

The remaining cases can be proved similarly.

□

An important remark related to the application of Corollary 3.7 is the following:

Remark 3.8. *(a) Let $\psi(\tilde{w}) = \tilde{w}^t$, where $t > 1$.*

(i) If $\tilde{\mathbf{u}}$ is a positive increasing concave k-tuple, then

$$\left(\frac{\sum_{i=1}^{k} \tilde{u}_i^t \vartheta_i}{\sum_{i=1}^{k} (i-1)^t \vartheta_i}\right) \leq \left(\frac{\sum_{i=1}^{k} \tilde{u}_i \vartheta_i}{\sum_{i=1}^{k} (i-1) \vartheta_i}\right)^t. \tag{28}$$

If $\tilde{\mathbf{u}}$ is an increasing convex k-tuple and $\tilde{u}_1 = 0$, then (28) reverses.

(ii) If $\tilde{\mathbf{u}}$ is a positive decreasing concave k-tuple, then

$$\left(\frac{\sum_{i=1}^{k} \tilde{u}_i^t \vartheta_i}{\sum_{i=1}^{k} (k-i)^t \vartheta_i}\right) \leq \left(\frac{\sum_{i=1}^{k} \tilde{u}_i \vartheta_i}{\sum_{i=1}^{k} (k-i) \vartheta_i}\right)^t. \tag{29}$$

If $\tilde{\mathbf{u}}$ is decreasing convex k-tuple and $\tilde{u}_k = 0$, then (29) reverses.

(b) If $\psi(\tilde{w}) = \tilde{w}^t$, $0 < t < 1$, then (28) *reverses,* (28), (29) *and reverse of* (29) *hold.*

An extension of Theorem 1 in [20] is given as the following:

Theorem 3.9. *Let $\vartheta, \tilde{\boldsymbol{u}}$ and $\tilde{\boldsymbol{v}}$ be positive k-tuples. Let $\varphi, \psi : [0,\infty) \to \mathbb{R}$: φ be a strictly increasing function and ψ be convex with respect to φ i.e., $\psi \circ \varphi^{-1}$ be convex. Also suppose that*

$$\sum_{i=1}^{m} \vartheta_i \varphi(\tilde{v}_i) \leq \sum_{i=1}^{m} \vartheta_i \varphi(\tilde{u}_i), \qquad m = 1, \dots, k-1, \tag{30}$$

and

$$\sum_{i=1}^{k} \vartheta_i \varphi(\tilde{v}_i) = \sum_{i=1}^{k} \vartheta_i \varphi(\tilde{u}_i). \tag{31}$$

(i) If $\tilde{\boldsymbol{v}}$ is a decreasing k-tuple, then

$$\sum_{i=1}^{k} \vartheta_i \psi(\tilde{v}_i) \leq \sum_{i=1}^{k} \vartheta_i \psi(\tilde{u}_i).$$

(ii) If $\tilde{\boldsymbol{u}}$ is an increasing k-tuple, then

$$\sum_{i=1}^{k} \vartheta_i \psi(\tilde{u}_i) \leq \sum_{i=1}^{k} \vartheta_i \psi(\tilde{v}_i).$$

Proof. For the case when $\varphi(\tilde{w}) = \tilde{w}$, but proved in Theorem 3.3. □

3.3 Generalization of Berwald's inequality and related results

A generalization of discrete weighted Berwald's inequality (see [12]) states that:

Theorem 3.10. *Let $\vartheta, \tilde{\boldsymbol{u}}$ and $\tilde{\boldsymbol{v}}$ be positive k-tuples. Let $\varphi, \psi : [0,\infty) \to \mathbb{R}$: φ is a continuous and strictly increasing. Suppose that $\psi \circ \varphi^{-1}$ is convex. Let y_1 be*:

$$\sum_{i=1}^{k} \vartheta_i \varphi(y_1 \tilde{v}_i) = \sum_{i=1}^{k} \vartheta_i \varphi(\tilde{u}_i). \tag{32}$$

(i) Let $\tilde{\boldsymbol{u}}/\tilde{\boldsymbol{v}}$ be decreasing k-tuple. If $\tilde{\boldsymbol{u}}$ is an increasing k-tuple, we obtain

$$\sum_{i=1}^{k} \vartheta_i \varphi(\tilde{u}_i) \leq \sum_{i=1}^{k} \vartheta_i \varphi(y_1 \tilde{v}_i). \tag{33}$$

If $\tilde{\boldsymbol{v}}$ is a decreasing k-tuple, then (33) *reverses.*

(ii) Let $\tilde{\mathbf{u}}/\tilde{\mathbf{v}}$ is an increasing k-tuple. If $\tilde{\mathbf{v}}$ is an increasing k-tuple, then

$$\sum_{i=1}^{k} \vartheta_i \varphi(y_1 \tilde{v}_i) \leq \sum_{i=1}^{k} \vartheta_i \varphi(\tilde{u}_i). \tag{34}$$

If $\tilde{\mathbf{u}}$ is a decreasing k-tuple, then (34) *reverses.*

Proof. (i) Since φ is continuous, $G(y) = \sum_{i=1}^{k} \vartheta_i \varphi(y\tilde{v}_i)$ for $y \geq 0$ is continuous. If $\tilde{\mathbf{u}} > 0$ and φ is strictly increasing, we have $G(0) = \sum_{i=1}^{k} \vartheta_i \varphi(0) < \sum_{i=1}^{k} \vartheta_i \varphi(\tilde{u}_i)$. Because $\tilde{\mathbf{u}}/\tilde{\mathbf{v}}$ is bounded above in both cases, taking either $y_0 > \tilde{u}_i/\tilde{v}_i$ or $\tilde{u}_i < y_0 \tilde{v}_i$ for $i = 1, \ldots, k$. Therefore, $G(y_0) = \sum_{i=1}^{k} \vartheta_i \varphi(y_0 \tilde{v}_i) > \sum_{i=1}^{k} \vartheta_i \varphi(\tilde{u}_i)$. This shows that y_1 exist.

Since $\tilde{\mathbf{u}}/\tilde{\mathbf{v}}$ is a decreasing and φ is strictly increasing function, and because

$$\sum_{i=1}^{k} \vartheta_i \varphi(y_1 \tilde{v}_i) = \sum_{i=1}^{k} \vartheta_i \varphi(\tilde{u}_i),$$

there is an m :

$$\frac{\tilde{u}_i}{\tilde{v}_i} \geq y_1, \quad i = 1, \ldots, m \quad \textit{and} \quad \frac{\tilde{u}_i}{\tilde{v}_i} \leq y_1, \quad i = m+1, \ldots, k,$$

thus

$$\sum_{i=1}^{j} \vartheta_i \varphi(y_1 \tilde{v}_i) \leq \sum_{i=1}^{j} \vartheta_i \varphi(\tilde{u}_i), \quad j = 1, \ldots, k. \tag{35}$$

(35) can be proved as

$$\begin{aligned}
\sum_{i=1}^{j} \vartheta_i \varphi(y_1 \tilde{v}_i) &= \sum_{i=1}^{k} \vartheta_i \varphi(y_1 \tilde{v}_i) - \sum_{i=j+1}^{k} \vartheta_i \varphi(y_1 \tilde{v}_i) \\
&= \sum_{i=1}^{k} \vartheta_i \varphi(\tilde{u}_i) - \sum_{i=j+1}^{k} \vartheta_i \varphi(y_1 \tilde{v}_i) \\
&\leq \sum_{i=1}^{k} \vartheta_i \varphi(\tilde{u}_i) - \sum_{i=j+1}^{k} \vartheta_i \varphi(\tilde{u}_i) \\
&= \sum_{i=1}^{j} \vartheta_i \varphi(\tilde{u}_i).
\end{aligned}$$

From inequality (35), the equality (32), the supposition that $\psi \circ \varphi^{-1}$ be convex, $\tilde{\mathbf{u}}$ be an increasing and by Theorem 3.9, we get

$$\sum_{i=1}^{k} \vartheta_i \psi(\tilde{u}_i) \leq \sum_{i=1}^{k} \vartheta_i \psi(y_1 \tilde{v}_i).$$

From inequality (35), the equality (32), the supposition that $\psi \circ \varphi^{-1}$ be convex, $\tilde{\mathbf{v}}$ is decreasing and by Theorem 3.9, we get

$$\sum_{i=1}^{k} \vartheta_i \psi(y_1 \tilde{v}_i) \leq \sum_{i=1}^{k} \vartheta_i \psi(\tilde{u}_i).$$

Similarly the other cases can be proved .

□

An important remark related to the application of Theorem 3.10 is the following:

Remark 3.11. *Let* $\varphi(\tilde{w}) = \tilde{w}^s$ *and* $\phi(\tilde{w}) = \tilde{w}^t$ *be* : $0 < s \leq t$.

(i) Let $\tilde{\mathbf{u}}/\tilde{\mathbf{v}}$ *be a decreasing k-tuple. If* $\tilde{\mathbf{u}}$ *is an increasing k-tuple, then*

$$\left(\frac{\sum_{i=1}^{k} \vartheta_i \tilde{u}_i^t}{\sum_{i=1}^{k} \vartheta_i \tilde{v}_i^t}\right)^{\frac{1}{t}} \leq \left(\frac{\sum_{i=1}^{k} \vartheta_i \tilde{u}_i^s}{\sum_{i=1}^{k} \vartheta_i \tilde{v}_i^s}\right)^{\frac{1}{s}}. \tag{36}$$

If $\tilde{\mathbf{v}}$ *is a decreasing k-tuple, then* (36) *reverses.*

(ii) Let $\tilde{\mathbf{u}}/\tilde{\mathbf{v}}$ *be increasing k-tuple. If* $\tilde{\mathbf{v}}$ *is an increasing k-tuple, then*

$$\left(\frac{\sum_{i=1}^{k} \vartheta_i \tilde{u}_i^s}{\sum_{i=1}^{k} \vartheta_i \tilde{v}_i^s}\right)^{\frac{1}{s}} \leq \left(\frac{\sum_{i=1}^{k} \vartheta_i \tilde{u}_i^t}{\sum_{i=1}^{k} \vartheta_i \tilde{v}_i^t}\right)^{\frac{1}{t}}. \tag{37}$$

If $\tilde{\mathbf{u}}$ *is a decreasing k-tuple, then* (37) *reverses.*

The following corollary is given in [12].

Corollary 3.12. *Let* ϑ *be a positive k-tuple. Let* $\varphi, \psi : [0, \infty) \to \mathbb{R} : \varphi$ *be strictly increasing and continuous. Suppose* $\psi \circ \varphi^{-1}$ *is convex. Let* y_1 *and* y_2 *be* :

$$\sum_{i=1}^{k} \vartheta_i \varphi[(i-1)y_1] = \sum_{i=1}^{k} \vartheta_i \varphi(\tilde{u}_i), \tag{38}$$

and

$$\sum_{i=1}^{k} \vartheta_i \varphi[(n-i)y_2] = \sum_{i=1}^{k} \vartheta_i \varphi(\tilde{u}_i).$$

(i) If $\tilde{\mathbf{u}}$ is a positive increasing concave k-tuple, then

$$\sum_{i=1}^{k} \vartheta_i \psi(\tilde{u}_i) \leq \sum_{i=1}^{k} \vartheta_i \psi[(i-1)y_1]. \tag{39}$$

If $\tilde{\mathbf{u}}$ is an increasing convex k-tuple and $\tilde{u}_1 = 0$, then (39) *reverses.*

(ii) If $\tilde{\mathbf{u}}$ is a positive decreasing concave k-tuple, then

$$\sum_{i=1}^{k} \vartheta_i \psi(\tilde{u}_i) \leq \sum_{i=1}^{k} \vartheta_i \psi[(n-i)y_2]. \tag{40}$$

If $\tilde{\mathbf{u}}$ is a decreasing convex k-tuple and $\tilde{u}_n = 0$, then (40) *reverses.*

Proof. (i) Choose $\tilde{v}_1 = \upsilon < \tilde{u}_1/\tilde{u}_2$, $\tilde{v}_i = i-1$ $(2 \leq i \leq k)$. Therefore, $\tilde{u}_i/\tilde{v}_i$ $(1 \leq i \leq k)$ is a decreasing k-tuple. So, (38) becomes

$$\vartheta_1 \varphi(\upsilon y_1) + \sum_{i=2}^{k} \vartheta_i \varphi[(i-1)y_1] = \sum_{i=1}^{k} \vartheta_i \varphi(\tilde{u}_i).$$

By applying Theorem 3.10 (33), we get

$$\sum_{i=1}^{k} \vartheta_i \psi(\tilde{u}_i) \leq \vartheta_1 \psi(\upsilon y_1) + \sum_{i=2}^{k} \vartheta_i \psi[(i-1)y_1],$$

If $\upsilon \to 0$, then

$$\begin{aligned}
\sum_{i=1}^{k} \vartheta_i \psi(\tilde{u}_i) &\leq \vartheta_1 \psi(0) + \sum_{i=2}^{k} \vartheta_i \psi[(i-1)y_1], \\
&= 0 + \sum_{i=2}^{k} \vartheta_i \psi[(i-1)y_1], \\
&= \sum_{i=1}^{k} \vartheta_i \psi[(i-1)y_1].
\end{aligned}$$

Because $\tilde{\mathbf{u}}$ is an increasing convex k-tuple and $\tilde{u}_1 = 0$, then $\tilde{u}_i/(i-1)(2 \leq i \leq k)$ is an increasing k-tuple. So, (38) becomes

$$\sum_{i=2}^{k} \vartheta_i \varphi[(i-1)y_1] = \sum_{i=2}^{k} \vartheta_i \varphi(\tilde{u}_i).$$

By using Theorem 3.10 (34),

$$\sum_{i=2}^{k} \vartheta_i \psi[(i-1)y_1] \leq \sum_{i=2}^{k} \vartheta_i \psi(\tilde{u}_i).$$

$$\vartheta_1 \psi[(0)y_1] + \sum_{i=2}^{k} \vartheta_i \psi[(i-1)y_1] \leq \vartheta_1 \psi(0) + \sum_{i=1}^{k} \vartheta_i \psi(\tilde{u}_i).$$

$$\sum_{i=1}^{k} \vartheta_i \psi[(i-1)y_1] \leq \sum_{i=1}^{k} \vartheta_i \psi(\tilde{u}_i).$$

The remaining cases can be proved similarly. □

An important remark related to the application of Corollary 3.12 is the following:

Remark 3.13. *For $0 < s \leq t$, let $\varphi(\tilde{u}) = \tilde{u}^s$ and $\phi(\tilde{u}) = \tilde{u}^t$.*

(i) If $\tilde{\mathbf{u}}$ is an increasing positive concave k-tuple, then

$$\left(\frac{\sum_{i=1}^{k} \vartheta_i \tilde{u}_i^t}{\sum_{i=1}^{k} \vartheta_i (i-1)^t} \right)^{\frac{1}{t}} \leq \left(\frac{\sum_{i=1}^{k} \vartheta_i \tilde{u}_i^s}{\sum_{i=1}^{k} \vartheta_i (i-1)^s} \right)^{\frac{1}{s}}. \tag{41}$$

If $\tilde{\mathbf{u}}$ is an increasing convex k-tuple and $\tilde{u}_1 = 0$, then (41) reverses.

(ii) If $\tilde{\mathbf{u}}$ is a positive decreasing concave k-tuple, then

$$\left(\frac{\sum_{i=1}^{k} \vartheta_i \tilde{u}_i^t}{\sum_{i=1}^{k} \vartheta_i (n-i)^t} \right)^{\frac{1}{t}} \leq \left(\frac{\sum_{i=1}^{k} \vartheta_i \tilde{u}_i^s}{\sum_{i=1}^{k} \vartheta_i (n-i)^s} \right)^{\frac{1}{s}}. \tag{42}$$

If $\tilde{\mathbf{u}}$ is a decreasing convex k-tuple and $\tilde{u}_k = 0$, then (42) reverses.

3.4 Positive semi-definite matrix, exponential convexity, log-convexity and Lyapunov's inequality for the difference deduced from majorization-type result

We will use the following family of functions, convex with respect to $\varphi(x) = x^p (p > 0)$ defined on $(0, \infty)$:

$$\phi_s(x) := \begin{cases} \frac{p^2}{s(s-p)} x^s, & s \neq 0, p, \\ -p log x, & s = 0, \\ p x^p log x, & s = p. \end{cases}$$

Here, we present Lyapunov's inequality satisfying the following result (see [12]):

Theorem 3.14. *Let ϑ, $\tilde{u}$ and $\tilde{v}$ be positive k-tuples. Let $\tilde{v}$ be decreasing and*

$$\Delta_s := \sum_{i=1}^{k} \psi_s(\tilde{u}_i) - \sum_{i=1}^{k} \psi_s(\tilde{v}_i),$$

such that conditions (30) *and* (31) *are satisfied. Then the following statements are valid*:

(i) For $s_1, \ldots, s_k \in \mathbb{R}$ and for every $k \in \mathbb{R}$, $\left[\Delta_{\frac{s_i+s_j}{2}}\right]_{i,j=1}^{k}$ is positive semi-definite.

(ii) $s \to \Delta_s$ is exp-convex.

(iii) $s \to \Delta_s$ is l-convex.

Proof. (i) Consider

$$\Re(x) = \sum_{i,j}^{k} \zeta_i \zeta_j \psi_{q_{ij}}(x)$$

for $m = 1, \ldots, k$, $x > 0$, $\zeta \in \mathbb{R}$, $q_{ij} = \frac{q_i+q_j}{2}$. Let $\Re$ be convex with respect to $\varphi(x) = x^p \quad (p > 0)$.

$$\urcorner(x) = \Re(x^{\frac{1}{p}}) = \sum_{i,j}^{k} \zeta_i \zeta_j \psi_{q_{ij}}(x^{\frac{1}{p}}).$$

We have

$$\urcorner''(x) = \sum_{i,j}^{k} \zeta_i \zeta_j x^{\frac{q_{ij}}{p}-2}$$

$$= \left(\sum_{i}^{k} \zeta_i x^{\frac{q_i}{2p}} - 1\right)^2 > 0,\ x \in (0, \infty).$$

Therefore, $\Re$ is convex with respect to $\varphi(x) = x^p (p > 0)$ for $x \in (0, \infty)$.

By using Theorem 3.9 ,

$$\sum_{i=1}^{k} \vartheta_i \psi(\tilde{v}_i) \leq \sum_{i=1}^{k} \vartheta_i \psi(\tilde{u}_i),$$

we have

$$\sum_{i=1}^{k} \left(\sum_{i,j}^{m} \zeta_i \zeta_j \psi_{q_{ij}}(\tilde{u}_i)\right) - \sum_{i=1}^{k} \left(\sum_{i,j}^{m} \zeta_i \zeta_j \psi_{q_{ij}}(\tilde{v}_i)\right) \geq 0,$$

equivalent to

$$\sum_{i=1}^{m} \zeta_i \zeta_j \left[\sum_{i,j}^{k} \psi_{q_{ij}}(\tilde{u}_i) - \sum_{i=1}^{k} \psi_{q_{ij}}(\tilde{v}_i)\right] \geq 0,$$

which implies

$$\sum_{i=1}^{m} \zeta_i \zeta_j \Delta_{q_{ij}}(x) \geq 0. \tag{43}$$

From (43), it follows that $\left[\Delta_{\frac{q_i+q_j}{2}}\right]_{i,j=1}^{m}$ is positive semi-definite matrix.

(ii) Δ_s is continuous for $s \in \mathbb{R}$ since

$$\lim_{s\to 0} \Delta_s = \Delta_0 \quad \textit{and} \quad \lim_{s\to 1} \Delta_s = \Delta_1.$$

Then by using Proposition 1.22, we obtain exp-convexity of $s \to \Delta_s$.

(iii) If $s \to \Delta_s$ is exp-convex function, then $s \to \Delta_s$ is l-convex.

□

Here we present an important result (see [12]) satisfying Dresher-type inequality :

Theorem 3.15. *Let $s,t,x,y \in [0,\infty) : s \leq x$, $t \leq y$, $s \neq t$. Let Δ_s be same as in Theorem* 3.14. *Then*

$$\left(\frac{\Delta_t}{\Delta_s}\right)^{\frac{1}{t-s}} \leq \left(\frac{\Delta_y}{\Delta_x}\right)^{\frac{1}{y-x}}. \tag{44}$$

Proof. By using Proposition 1.6, we get

$$\frac{\psi(\tilde{u}_2) - \psi(\tilde{u}_1)}{\tilde{u}_2 - \tilde{u}_1} \leq \frac{\psi(\tilde{v}_2) - \psi(\tilde{v}_1)}{\tilde{v}_2 - \tilde{v}_1}, \tag{45}$$

where $\tilde{u}_1 \leq \tilde{v}_1$, $\tilde{u}_2 \leq \tilde{v}_2$, $\tilde{u}_1 \neq \tilde{u}_2$ and $\tilde{v}_1 \neq \tilde{v}_2$. From Theorem 3.14, Δ_s is l-convex, take $\psi(\tilde{u}) = log\Delta_{\tilde{u}}$, $\tilde{u}_1 = s$, $\tilde{u}_2 = t$, $\tilde{v}_1 = x$ and $\tilde{v}_2 = y$ in (45),
we have

$$\frac{log\Delta_t - log\Delta_s}{t-s} \leq \frac{log\Delta_y - log\Delta_x}{y-x}$$

and hence (44) is immediate. □

3.5 Exponential convexity, log-convexity and Lyapunov's inequality for the difference deduced from Favard-type inequalities

Let $\{\eta_s : (0,\infty) \to \mathbb{R} : s \in \mathbb{R}\}$ be a class of functions defined by

$$\eta_s(x) = \begin{cases} \frac{x^s}{s(s-1)}, & s \neq 0,1, \\ -logx, & s = 0, \\ xlogx, & s = 1. \end{cases} \tag{46}$$

As $\frac{d^2}{dx^2}\eta_s(x) = x^{s-2} = e^{(s-2)\ln x} > 0$, for $x > 0$, η_s is convex and $s \mapsto \frac{d^2}{dx^2}\eta_s(x)$ is exp-convex. Now we present an important result satisfying Lyapunov's inequality (*see* [11]):

Theorem 3.16. *Let f (>0) be continuous and concave defined on $[\tilde{u},\tilde{v}]$; c is defined in* (4) *and*

$$\Omega_s(f) := \begin{cases} \frac{1}{s(s-1)}\left[\frac{2^s}{s+1}\left(\frac{1}{\tilde{v}-\tilde{u}}\int\limits_{\tilde{u}}^{\tilde{v}} f(x)dx\right)^s - \frac{1}{\tilde{v}-\tilde{u}}\int\limits_{\tilde{u}}^{\tilde{v}} f^s(x)dx\right], & s \neq 0,1, \\ 1 - log2 - logc + \frac{1}{\tilde{v}-\tilde{u}}\int\limits_{\tilde{u}}^{\tilde{v}} logf(x)dx, & s = 0, \\ clog2 + clogc - \frac{1}{2}c - \frac{1}{\tilde{v}-\tilde{u}}\int\limits_{\tilde{u}}^{\tilde{v}} f(x)logf(x)dx, & s = 1. \end{cases} \tag{47}$$

Then for $s \in [0,\infty)$, $\Omega_s(f)$ is l-convex and for $r,s,t \in [0,+\infty)$,

$$\Omega_s^{t-r}(f) \leq \Omega_r^{t-s}(f)\Omega_t^{s-r}(f) \tag{48}$$

holds : $r < s < t$.

Proof. Consider ψ defined by

$$\psi(x) = a^2\eta_s(x) + 2a\tilde{w}\eta_r(x) + \tilde{w}^2\eta_t(x),$$

where $r = \frac{s+t}{2}$, η_s is the same as defined in (46) and $a,\tilde{w} \in \mathbb{R}$. For $s \neq 0,1$ and for $x > 0$, we have

$$\begin{aligned} \psi''(x) &= a^2x^{s-2} + 2a\tilde{w}x^{r-2} + \tilde{w}^2x^{t-2} \\ &= (ax^{s/2-1} + \tilde{w}x^{t/2-1})^2 \geq 0. \end{aligned}$$

As for $x>0$, $\psi(x)$ is convex, by applying Theorem 2.5, we have

$$\frac{1}{2c}\int_0^{2c}\left[a^2\eta_s(y)+2a\tilde{w}\eta_r(y)+\tilde{w}^2\eta_t(y)\right]dy$$

$$\geq \frac{1}{\tilde{v}-\tilde{u}}\int_{\tilde{u}}^{\tilde{v}}\left[\tilde{u}^2\eta_s(f(x))+2a\tilde{w}\eta_r(f(x))+\tilde{w}^2\eta_t(f(x))\right]dx,$$

which is equivalent to

$$a^2\left[\frac{1}{2c}\int_0^{2c}\eta_s(y)dy-\frac{1}{\tilde{v}-\tilde{u}}\eta_s(f(x))dx\right]+2a\tilde{w}\left[\frac{1}{2c}\int_0^{2c}\eta_r(y)dy-\frac{1}{\tilde{v}-\tilde{u}}\eta_r(f(x))dx\right]$$
$$+\tilde{w}^2\left[\frac{1}{2c}\int_0^{2c}\eta_t(y)dy-\frac{1}{\tilde{v}-\tilde{u}}\eta_t(f(x))dx\right]\geq 0. \tag{49}$$

Let

$$\Omega_s(f):=\frac{1}{2c}\int_0^{2c}\eta_s(y)dy-\frac{1}{\tilde{v}-\tilde{u}}\eta_s(f(x))dx. \tag{50}$$

By using η_s in (50), (47) is immediate. From (49), we have

$$a^2\Omega_s(f)+2a\tilde{w}\Omega_r(f)+\tilde{w}^2\Omega_t(f)\geq 0,$$

and by applying Remark 1.11, we get

$$\Omega_s(f)\Omega_t(f)\geq\Omega_r^2(f)=\Omega_{s+t/2}^2(f),$$

which implies that for $s\in[0,\infty)$, $\Omega_s(f)$ is l-convex in J-sense.

$\Omega_s(f)$ is continuous for $s\in[0,\infty)$ since

$$\lim_{s\to 0}\Omega_s(f)=\Omega_0(f)\quad and\quad \lim_{s\to 1}\Omega_s(f)=\Omega_1(f),$$

which shows that $\Omega_s(f)$ is continuous and hence $\Omega_s(f)$ is l-convex.

As $\Omega_s(f)$ is l-convex, $s\mapsto log\Omega_s(f)$ is convex. Take $\psi(s)=log\Omega_s(f)$, for $r,s,t\in[0,+\infty)$, applying Proposition 1.7

$$(t-r)log\Omega_s(f)\leq(t-s)log\Omega_r(f)+(s-r)log\Omega_t(f),$$

which is equivalent to (48). □

Here we present an important result (*see* [11]) satisfying Dresher-type inequality :

Theorem 3.17. *Let $s,t,x,y \in [0,\infty) : s \le x,\ t \le y,\ s \ne t$. Let f, $\Omega_s(f)$ be same as in Theorem* 3.16*. Then*

$$\left(\frac{\Omega_t(f)}{\Omega_s(f)}\right)^{\frac{1}{t-s}} \le \left(\frac{\Omega_y(f)}{\Omega_x(f)}\right)^{\frac{1}{y-x}}. \tag{51}$$

Proof. By using Proposition 1.6, we get

$$\frac{\psi(\tilde{u}_2)-\psi(\tilde{u}_1)}{\tilde{u}_2-\tilde{u}_1} \le \frac{\psi(\tilde{v}_2)-\psi(\tilde{v}_1)}{\tilde{v}_2-\tilde{v}_1}, \tag{52}$$

where $\tilde{u}_1 \le \tilde{v}_1$, $\tilde{u}_2 \le \tilde{v}_2$, $\tilde{u}_1 \ne \tilde{u}_2$ and $\tilde{v}_1 \ne \tilde{v}_2$. From Theorem 3.16, $\Omega_s(f)$ is l-convex, take $\psi(\tilde{u}) = log\Omega_{\tilde{u}}(f)$, $\tilde{u}_1 = s$, $\tilde{u}_2 = t$, $\tilde{v}_1 = x$ and $\tilde{v}_2 = y$ in (52),
we have

$$\frac{log\Omega_t(f)-log\Omega_s(f)}{t-s} \le \frac{log\Omega_y(f)-log\Omega_x(f)}{y-x}$$

and hence (51) is immediate. □

Theorem 3.18. ([*11*]) *Let f be continuous and concave defined on $[\tilde{u},\tilde{v}] : 0 < d < f_{min}$; c be defined in* (4)*, and*

$$\Phi_s(f) := \begin{cases} \frac{1}{s(s-1)}\left[\frac{(2c-d)^{s+1}}{(2c-2d)(s+1)} - \frac{d^{s+1}}{(2c-2d)(s+1)} - \frac{1}{\tilde{v}-\tilde{u}}\int_{\tilde{u}}^{\tilde{v}} f^s(x)dx\right], & s \ne 0,1, \\ \frac{1}{2c-2d}[2c + dlogd - 2d - (2c-d)log(2c-d)] + \int_{\tilde{u}}^{\tilde{v}} logf(x)dx, & s = 0, \\ \frac{1}{2(2c-2d)}[(2c-d)^2log(2c-d) - 2c^2 + 2dc - d^2logd + 2d] - \int_{\tilde{u}}^{\tilde{v}} f(x)logf(x)dx, & s = 1. \end{cases}$$

Then for $s \in [0,\infty)$, $\Phi_s(f)$ is l-convex and for $r,s,t \in [0,\infty)$

$$\Phi_s^{t-r}(f) \le \Phi_r^{t-s}(f)\Phi_t^{s-r}(f)$$

holds : $r < s < t$.

Proof. Analogous to Theorem 3.16 but here we apply Theorem 2.7 instead of Theorem 2.5. □

Theorem 3.19. ([*11*]) *Let* $s,t,x,y \in [0,\infty) : s \leq x,\ t \leq y,\ s \neq t,\ x \neq y$ *and let* $f,\ \Phi_s(f)$ *be same as in Theorem* 3.18*, then*

$$\left(\frac{\Phi_t(f)}{\Phi_s(f)}\right)^{\frac{1}{t-s}} \leq \left(\frac{\Phi_y(f)}{\Phi_x(f)}\right)^{\frac{1}{y-x}}.$$

Proof. Analogous to the proof of Theorem 3.17. □

Theorem 3.20. ([*11*])

(i) Let $f\ (>0)$ *be an increasing concave defined on* $[\tilde{u},\tilde{v}]$; c_i *as in Theorem* 2.11 *and*

$$\Phi_s(f) := \int_0^1 \psi_s(2rc_i)\tilde{w}[\tilde{u}(1-r)+\tilde{v}r]dr - \frac{1}{\tilde{v}-\tilde{u}}\int_{\tilde{u}}^{\tilde{v}} \psi_s(f(t))\tilde{w}(t)dt.$$

Then for $s \in [0,\infty)$, $\Phi_s(f)$ *is l-convex and for* $r,s,t \in [0,\infty)$,

$$\Phi_s^{t-r}(f) \leq \Phi_r^{t-s}(f) + \Phi_t^{s-r}(f)$$

holds : $r < s < t$.

(ii) Let f be an increasing convex defined on $[\tilde{u},\tilde{v}]$, $f(\tilde{u}) = 0$, $\widetilde{\Phi}_s(f) := -\Phi_s(f)$. *Moreover for* $s \in [0,\infty)$, $\widetilde{\Phi}_s(f)$ *is l-convex and for* $r,s,t \in [0,\infty)$,

$$\widetilde{\Phi}_s^{t-r}(f) \leq \widetilde{\Phi}_r^{t-s}(f)\,\widetilde{\Phi}_t^{s-r}(f)$$

holds : $r < s < t$.

Proof. Analogous to the proof of Theorem 3.16 but here we apply Theorem 2.11 instead of Theorem 2.5. □

Theorem 3.21. ([*11*])

(i) Let $s,t,x,y \in [0,\infty) : s \leq x,\ t \leq y,\ s \neq t,\ x \neq y$ *and let* $f,\ \Phi_s(f)$ *be same as in Theorem* 3.20(i). *Then*

$$\left(\frac{\Phi_t(f)}{\Phi_s(f)}\right)^{\frac{1}{t-s}} \leq \left(\frac{\Phi_y(f)}{\Phi_x(f)}\right)^{\frac{1}{y-x}}.$$

(ii) *Let $s,t,x,y \in [0,\infty) : s \le x,\ t \le y,\ s \ne t,\ x \ne y$ and let f, $\Phi_s(f)$ be same as in Theorem* 3.20(ii). *Then*

$$\left(\frac{\widetilde{\Phi}_t(f)}{\widetilde{\Phi}_s(f)}\right)^{\frac{1}{t-s}} \le \left(\frac{\widetilde{\Phi}_y(f)}{\widetilde{\Phi}_x(f)}\right)^{\frac{1}{y-x}}.$$

Proof. Analogous to the proof of Theorem 3.17. □

Theorem 3.22. ([*11*])

(i) *Let $f\ (>0)$ be decreasing and concave defined on $[\tilde{u},\tilde{v}]$; c_d is same as in Theorem* 2.11 *and*

$$\Upsilon_s(f) := \int_0^1 \phi_s(2rc_d)\tilde{w}[\tilde{u}(1-r)+\tilde{v}r]dr - \frac{1}{\tilde{v}-\tilde{u}}\int_{\tilde{u}}^{\tilde{v}} \phi_s(f(t))\tilde{w}(t)dt.$$

Then for $s \in [0,\infty)$, $\Upsilon_s(f)$ is l-convex and for $r,s,t \in [0,\infty)$,

$$\Upsilon_s^{t-r}(f) \le \Upsilon_r^{t-s}(f) + \Upsilon_t^{s-r}(f)$$

holds $: r < s < t$.

(ii) *Let f be a decreasing convex defined on $[\tilde{u},\tilde{v}]$, $f(\tilde{v}) = 0, \widetilde{\Upsilon}_s(f) := -\Upsilon_s(f)$. Then for $s \in [0,\infty)$, $\widetilde{\Upsilon}_s(f)$ is l-convex and for $r,s,t \in [0,\infty)$,*

$$\widetilde{\Upsilon}_s^{t-r}(f) \le \widetilde{\Upsilon}_r^{t-s}(f)\,\widetilde{\Upsilon}_t^{s-r}(f)$$

holds $: r < s < t$.

Proof. Similarly, as the proof of Theorem 3.16, but here we apply Theorem 2.11 instead of the Theorem 2.5. □

Theorem 3.23. ([*11*])

(i) *Let $s,t,x,y \in [0,\infty) : s \le x,\ t \le y,\ s \ne t,\ x \ne y$ and let f, $\Upsilon_s(f)$ be same as in Theorem* 3.22(i). *Then*

$$\left(\frac{\Upsilon_t(f)}{\Upsilon_s(f)}\right)^{1/(t-s)} \le \left(\frac{\Upsilon_y(f)}{\Upsilon_x(f)}\right)^{1/(y-x)}.$$

(ii) Let $s,t,x,y \in [0,\infty) : s \leq x,\ t \leq y,\ s \neq t,\ x \neq y$ and let f, $\widetilde{\Upsilon}_s(f)$ be same as in Theorem 3.22(ii). Then

$$\left(\frac{\widetilde{\Upsilon}_t(f)}{\widetilde{\Upsilon}_s(f)}\right)^{1/(t-s)} \leq \left(\frac{\widetilde{\Upsilon}_y(f)}{\widetilde{\Upsilon}_x(f)}\right)^{1/(y-x)}.$$

Proof. Similarly, as the proof of Theorem 3.17. □

Now, consider another family of convex functions (see [10]) as follows :

Let $\{\wp_s : (0,\infty) \to \mathbb{R} : s \in \mathbb{R}\}$ be a class of functions defined by

$$\wp_s(x) = \begin{cases} \frac{e^{sx}}{s^2}, & s \neq 0, \\ \frac{1}{2}x^2, & s = 0. \end{cases}$$

As $\frac{d^2}{dx^2}\Im_s(x) = e^{sx} > 0$, for $x > 0$, $\Im_s$ is convex and $s \mapsto \frac{d^2}{dx^2}\Im_s(x)$ is exp-convex.

Theorem 3.24. *Let f (>0) be concave and continuous defined on $[\tilde{u},\tilde{v}]$; c be same as defined in (4) and*

$$\Im_s(f) := \begin{cases} \frac{e^{2cs}}{2cs^3} - \frac{1}{2cs^3} - \frac{1}{s^2(\tilde{v}-\tilde{u})}\int\limits_{\tilde{u}}^{\tilde{v}} e^{sf(x)}, & s \neq 0; \\ \frac{2}{3}\left(\frac{1}{\tilde{v}-\tilde{u}}\int\limits_{\tilde{u}}^{\tilde{v}} f(x)dx\right)^2 - \frac{1}{4(\tilde{v}-\tilde{u})}\int\limits_{\tilde{u}}^{\tilde{v}} f^2(x)dx, & s = 0. \end{cases}$$

Then for $s \in [0,\infty)$, $\Im_s(f)$ is l-convex and for $r,s,t \in [0,\infty)$,

$$\Im_s^{t-r}(f) \leq \Im_r^{t-s}(f)\Im_t^{s-r}(f)$$

holds : $r < s < t$.

Proof. Analogous to the proof of Theorem 3.16. □

Remark 3.25. *Suppose that $\psi : (0,1) \to \mathbb{R}$ is convex and ϑ, $\tilde{\mathbf{u}}$ and $\tilde{\mathbf{v}}$ are positive k-tuples. Let $\tilde{\mathbf{u}}/\tilde{\mathbf{v}}$ be a decreasing k-tuple. If $\tilde{\mathbf{u}}$ is an increasing k-tuple, then the non-negative difference deduced from (22) is*

$$\sum_{i=1}^{k}\vartheta_i\psi\left(\frac{\tilde{v}_i}{\sum_{i=1}^{k}\tilde{v}_i\vartheta_i}\right) - \sum_{i=1}^{k}\vartheta_i\psi\left(\frac{\tilde{u}_i}{\sum_{i=1}^{k}\tilde{u}_i\vartheta_i}\right) \geq 0. \tag{53}$$

Exponential convexity for (53) can be proved similarly as in the previous section. Also for k-exponential convexity see [10].

3.6 Positive semi-definite matrix, exponential convexity, log-convexity and Lyapunov's inequality for the difference deduced from Berwald-type inequalities

Again we use the following family of functions, convex with respect to $\varphi(x) = x^p (p > 0)$ defined on $(0, \infty)$:

$$\phi_s(x) := \begin{cases} \frac{p^2}{s(s-p)} x^s, & s \neq 0, p, \\ -p log x, & s = 0, \\ p x^p log x, & s = p. \end{cases}$$

Theorem 3.26. ([*13*]) *Let f (> 0) be continuous and concave on $[\tilde{u}, \tilde{v}]$, $p > 0$ and*

$$\Gamma_s(f) := \begin{cases} \frac{p^2}{s(s-p)} \left[\left(\frac{p+1}{\tilde{v}-\tilde{u}} \int\limits_{\tilde{u}}^{\tilde{v}} f^p(x) dx \right)^{\frac{s}{p}} - \frac{s+1}{\tilde{v}-\tilde{u}} \int\limits_{\tilde{u}}^{\tilde{v}} f^s(x) dx \right], & s \neq 0, p, \\ p + \frac{p}{\tilde{v}-\tilde{u}} \int\limits_{\tilde{u}}^{\tilde{v}} log f(x) dx - log \left(\frac{p+1}{\tilde{v}-\tilde{u}} \int\limits_{\tilde{u}}^{\tilde{v}} f^p(x) dx \right), & s = 0, \\ \left(\frac{p+1}{\tilde{v}-\tilde{u}} \int\limits_{\tilde{u}}^{\tilde{v}} f^p(x) dx \right) log \left(\frac{p+1}{\tilde{v}-\tilde{u}} \int\limits_{\tilde{u}}^{\tilde{v}} f^p(x) dx \right) - & \\ \frac{p}{\tilde{v}} - \tilde{u} \int\limits_{\tilde{u}}^{\tilde{v}} f^p(x) dx - \frac{p(p+1)}{\tilde{v}-\tilde{u}} \int\limits_{\tilde{u}}^{\tilde{v}} f^p(x) log f(x) dx, & s = p. \end{cases} \tag{54}$$

Then for $s \in [0, \infty)$, $\Gamma_s(f)$ is l-convex and for $r, s, t \in [0, \infty)$,

$$\Gamma_s^{t-r}(f) \leq \Gamma_r^{t-s}(f) \Gamma_t^{s-r}(f)$$

holds : $r < s < t$.

Proof. Analogous to the proof of Theorem 3.16 but here we apply Theorem 2.9 instead of Theorem 2.5. □

Theorem 3.27. ([*13*]) *Let $s, t, x, y \in [0, \infty)$: $s \leq x$, $t \leq y$, $s \neq t$, $x \neq y$ and let f, $\Gamma_s(f)$ be same as in Theorem* 3.26.

Then

$$\left(\frac{\Gamma_t(f)}{\Gamma_s(f)} \right)^{\frac{1}{t-s}} \leq \left(\frac{\Gamma_y(f)}{\Gamma_x(f)} \right)^{\frac{1}{y-x}}$$

Proof. Similarly, as the proof of Theorem 3.17. □

Remark 3.28. ([*13*]) *Put* $p = 1$ *in* (54)*, we obtain*

$$\Gamma_s(f) := \begin{cases} \frac{1}{s(s-1)}\left[\left(\frac{2}{\tilde{v}-\tilde{u}}\int\limits_{\tilde{u}}^{\tilde{v}} f^p(x)dx\right)^{\frac{s}{p}} - \frac{s+1}{\tilde{v}-\tilde{u}}\int\limits_{\tilde{u}}^{\tilde{v}} f^s(x)dx\right], & s \neq 0,1, \\ 1 + \frac{1}{\tilde{v}-\tilde{u}}\int\limits_{\tilde{u}}^{\tilde{v}} log f(x)dx - log\left(\frac{2}{\tilde{v}-\tilde{u}}\int\limits_{\tilde{u}}^{\tilde{v}} f(x)dx\right), & s = 0, \\ \left(\frac{2}{\tilde{v}-\tilde{u}}\int\limits_{\tilde{u}}^{\tilde{v}} f(x)dx\right) log\left(\frac{2}{\tilde{v}-\tilde{u}}\int\limits_{\tilde{u}}^{\tilde{v}} (x)dx\right) - & \\ \frac{1}{\tilde{v}-\tilde{u}}\int\limits_{\tilde{u}}^{\tilde{v}} f(x)dx - \frac{2}{\tilde{v}-\tilde{u}}\int\limits_{\tilde{u}}^{\tilde{v}} f(x) log f(x)dx, & s = 1, \end{cases}$$

which is the same as $\Omega_s(f)$ *in* [*11*] *for Favard's inequality.*

Theorem 3.29. ([*13*])

(i) If d_i *is a positive root of the equation* (8) *for* $\varphi(x) = x^q$ $(q > 1)$ *and let* f (>0) *be an increasing and concave defined on* $[\tilde{u}, \tilde{v}]$

$$\Upsilon_s(f) := \int_0^1 \psi_s(rd_i)\tilde{w}[\tilde{u}(1-r) + \tilde{v}r]dr - \frac{1}{\tilde{v}-\tilde{u}}\int_{\tilde{u}}^{\tilde{v}} \psi(f(t))\tilde{w}(t)dt, \tag{55}$$

Then for $s \in [0,\infty)$, $\Upsilon_s(f)$ *is l-convex and for* $r,s,t \in [0,\infty)$,

$$\Upsilon_s^{t-r}(f) \leq \Upsilon_r^{t-s}(f)\Upsilon_t^{s-r}(f)$$

holds : $r < s < t$.

(ii) Let f *be convex defined on* $[\tilde{u}, \tilde{v}]$, $f(\tilde{u}) = 0$, $\overline{\Upsilon_s}(f) := -\Upsilon_s(f)$*, where* $\Upsilon_s(f)$ *be same as in* (55)*. Then for* $s \in [0,\infty)$, $\overline{\Upsilon_s}(f)$ *is l-convex and for* $r,s,t \in [0,\infty)$,

$$\overline{\Upsilon_s}^{t-r}(f) \leq \overline{\Upsilon_r}^{t-s}(f)\overline{\Upsilon_t}^{s-r}(f)$$

holds : $r < s < t$.

Proof. Same as Theorem 3.16 but here we apply Theorem 2.12(ii) instead of Theorem 2.9. □

Theorem 3.30. ([*13*]) *Let ϑ, $\mathbf{u}$ and $\mathbf{v}$ be positive k-tuples. Let $\mathbf{u}/\mathbf{v}$ be a decreasing k-tuple and y_1 is same as in* (32) *for $\Phi(x) = x^q$, $q > 0$:*

$$\sum_{i=1}^{k} \vartheta_i \psi(y_1 v_i) = \sum_{i=1}^{k} \vartheta_i \psi(u_i).$$

and also let

$$\Lambda_s = \sum_{i=1}^{k} \vartheta_i \psi(y_1 v_i) - \sum_{i=1}^{k} \vartheta_i \psi(u_i).$$

Then the following statements are valid :

(i) *For $s_1, \ldots, s_k \in \mathbb{R}$ and for every $k \in \mathbb{R}$, $\left[\Lambda_{\frac{s_i+s_j}{2}}\right]_{i,j=1}^{k}$ is positive semi-definite.*

(ii) *$s \to \Lambda_s$ is exp-convex.*

(iii) *$s \to \Lambda_s$ is l-convex.*

Proof. Same as the proof of Theorem 3.14 but here we use Theorem 3.10 instead of Theorem 3.9. □

Theorem 3.31. ([*13*]) *Let $s,t,x,y \in [0,\infty)$: $s \leq x$, $t \leq y$, $s \neq t$, $x \neq y$ and let f, $\Lambda_s(f)$ be same as in Theorem* 3.30*.*

Then

$$\left(\frac{\Lambda_t(f)}{\Lambda_s(f)}\right)^{\frac{1}{t-s}} \leq \left(\frac{\Lambda_y(f)}{\Lambda_x(f)}\right)^{\frac{1}{y-x}}.$$

Proof. Similarly, as the proof of Theorem 3.15. □

Chapter 4

Conclusion

4 Conclusion

At present many researchers are working on the refinements, extensions, improvements and generalizations of several well-known inequalities. The topics of our research work are topics of great interest (see for example [7], [8], [9]). In our work we present many significant results related to the weighted majorization inequality as well as the generalizations of Favard and Berwald-type inequalities with many interesting applications. Further, we consider the non-negative difference of the generalized Favard and Berwald inequalities and prove the log-convexity and exponential convexity. Finally, Lyapunov and Dresher-type inequalities are also presented.

Chapter 5

References

5 References

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