On Structure of Unitary Unit Group of $F_2^k \, 3^{1+2}$

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CIIT/FA16-RMT-020/LHR

MS

In

Mathematics

COMSATS University Islamabad

Lahore Campus

Spring, 2018

COMSATS University Islamabad, Lahore Campus

On Structure of Unitary Unit Group of $F_{2^k} 3^{1+2}$

A Thesis Presented to

COMSATS University Islamabad, Lahore Campus

In partial fulfillment

of the requirement for the degree of

MS (Mathematics)

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A Post Graduate Thesis is submitted to the Department of Mathematics as partial fulfillment of the requirement for the award of Degree of MS Mathematics.

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May , 2018

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It is certified that **<u>Muhammad Zafar (CIIT/FA16-RMT-020/LHR)</u>** has carried out all the work related to this thesis under my supervision at the Department of Mathematics, COMSATS University Islamabad, Lahore Campus, and the work fulfills the requirement for award of MS degree.

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DEDICATION

To

My family and teachers

ACKNOWLEDGEMENTS

All the praises be for ALLAH Almighty (the most beneficent and merciful) for bestowing infnite blessings upon me and giving me courage to achieve my goal. I love to pay my sincere thanks to my parents for their prayers and support. My special thanks and regards are for my advisor and teacher Dr. Adeel Farooq for his invaluable guidance, professional suggestions, tremendoussupport, and last but not least, patience. I admire his understanding, enthusiasm, commitment and sincerity with profession. I would like to give special thanks to Mr. Maqsood Ahmed who have given me his research work to proceed and have always been available for guidance and help. My thesis is completed because of his heartiest support. I would like to give a lot of thanks to my teachers Dr. Sarfraz, Dr. Kashif, Dr. Yousaf Bhatti, Dr. Imran, Dr. Yousaf Habib and others who teach, learn and encourage me during my course work. As a whole it wasgood learning experience and discussions on the subject would remain an asset for me in the future as well. I would also like to thank my friends and colleagues Javed Iqbal and Muhammad Imran who showed their support and encouragement to me for this endeavor.

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ABSTRACT

On Structure of Unitary Unit Group of $F_2{}^k\ 3^{1+2}$

In this thesis, we have constructed the group ring matrix of group algebra $F_2{}^k\ 3^{1+2}$. On the basis of this matrix, the center of unitary unit group have been established and then order of unitary unit group has also been computed.

By making use of conjugacy classes of the group 3^{1+2} a basis of center of the group algebra $F_2{}^k\ 3^{1+2}$ have been computed. Inside this center, we found the center of $V_*\ (F_2{}^k\ 3^{1+2})$ by using brute force. The basis of minimal ideals of $V_*\ (F_2{}^k\ 3^{1+2})$ have been computed by using GAP. The dimensions for augmentation ideal and Jacobson radical of $F_2{}^k\ 3^{1+2}$ has also been computed by GAP.

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Chapter 1

Introduction

The idea of the *group rings* was introduced by W.R.Hamilton in 1873, when he had an immense desire to construct the generalized form of complex numbers, but he could not achieve his goal, however he constructed a much important algebraic structure called quaternions [28]. In 1845, Sir A.Cayley introduced an extended form of quaternions called octonions, also known as Cayley Numbers [10]. After this Hamilton perceived that he could generalize this discovery and after soon he established *Hyper Complex Number System.* On this way T.Molien gave a detailed view on *group rings* in 1897. The algebraist who have given a lot of contributions to this field are I.N.Herstein, D.S.Passman, H.J.Zassenhaus, S.K.Sehgal and A.A.Bovdi.

An important isomorphic relationship that we will use in our main task is between *group rings* and *group ring matrices.* If R is an integral domain containing the identity then the representations give us a path to find the *units* of R through the properties of the matrices and if possible also through the determinant of the matrices. In [35] T.Hurley have established many useful results in this regards.

The product RG of a finite group G and a ring R (finite or infinite) contains the formal linear combinations of the form $\sum_{g\in G} a_g.g$, where $a_g \in R$ and $g \in G$. Let $p, q \in RG$ then addition and multiplication is defined as:

$$p + q = \sum_{g\in G}(a_g + b_g)g, \tag{1.1}$$

$$pq = \sum_{g,h\in G} a_g b_h gh. \tag{1.2}$$

with respect to the above addition and multiplication RG form a ring called *group ring.* The product RG is called *group algebra* if R is commutative or a field $\mathbb{F}$.

The homomorphism $A : RG \to RG$ is called the augmentation map and is defined as:

$$A\left(\sum_{g\in G}(a_g g)\right) = \sum_{g\in G} a_g. \tag{1.3}$$

The kernel of this homomorphism is called *augmentation ideal* and has a special notation $\Delta(RG)$. The set $U(RG)$ is called a *unit group* of RG if it contains all invertible elements of RG. If every element of $U(RG)$ has *augmentation* 1 then it is called *normalized unit group* and is denoted by $V(RG)$.

A matrix A is called *unitary* iff $A^* = \overline{A}^T = A^{-1}$. The set of *unitary matrices* $U_n(C)$ form a *unitary group.* If each $U \in U_n(C)$ has determinant 1 then it is called *special unitary* group $SU_n(C)$. Now we define an anti-automorphism $* : RG \to RG$ as

$$\left(\sum_{g\in G} a_g.g\right)^* = \sum_{g\in G} a_g.g^{-1}. \tag{1.4}$$

An element v in $V(RG)$ is said to be *unitary* iff $v^* = v^{-1}$. The elements in $V(RG)$ those satisfy $v^* = v^{-1}$ form a subgroup of $V(RG)$ that is called *unitary unit group* $V_*(RG)$. Furthermore $*$ has the following properties:

(i) $(v_1 + v_2)^* = v_1^* + v_2^*$.

(ii) $(v_1v_2)^* = v_2^*v_1^*$.

(iii) $(v^*)^* = v$.

Many researchers have given important results on *unit group* and *unitary unit group* for different *group algebras.* R.Sandling has established a basis for $V(\mathbb{F}_pG)$, where G is an abelian p-group and $\mathbb{F}_p$ is a Galois field [59]. $V(\mathbb{F}G)$ is a finite 2-group of order $|\mathbb{F}|^{|G|-1}$, if $\mathbb{F}$ is a finite field of characteristic 2 and G is a finite 2-group. Also R.Sandling gave the structure of $U(\mathbb{F}_{2^k}D_8)$ [60]. If G is an extra special group then $V_*(\mathbb{F}_{2^k}G)$ is normal in $V(\mathbb{F}_{2^k}G)$ (by Bovdi and Kovacs) [8]. L.Creedon and J.Gildea introduced the structure of $V_*(\mathbb{F}_{2^k}Q_8)$ and also Joe Gildea gave the structure of $V_*(\mathbb{F}_{2^k}D_8)$ [14, 24]. A.Bovdi and Erdei has established the structure of $V_*(\mathbb{F}_2G)$ for all groups of order 8 and 16 [5]. Bovdi and Szakacs introduced the basis for $V_*(\mathbb{F}G)$ where $\mathbb{F}$ has characteristic greater than 2. Also Bovdi and Rosa find the order of $V_*(\mathbb{F}_{2^k}G)$ for various casses of G. P.F.Pickel, J.Z.Goncaves, J.Gildea and M.Shirvani have given the various results on *unit group* [23, 25, 26, 27, 30].

The main goal of this thesis is to construct and understand the structure of *unitary unit group of group algebra* $\mathbb{F}_{2^k}3^{1+2}$ as well as *group algebra* $\mathbb{F}_{2^k}3^{1+2}$ itself. Also computational aspects about the structure of this group algebra forms the crucial part of this thesis.

1.1 Basic definitions and examples

This section contains some basic definitions from *group rings* as well as *group algebras*. Furthermore we will discuss some examples regarding to our work.

Definition 1.1.1.

Group Ring: For any finite group G having a group operation multiplicatively and a ring R with non-zero identity, we can define a product RG containing the formal linear combinations as:

$$RG = \left\{ p \in RG \mid \; p = \sum_{g \in G} a_g \; g, \; a_g \in R \right\}.$$

Let $p, q \in RG$, the addition and multiplication of p and q in RG is defined as:

$$p + q = \sum_{g \in G} (a_g + b_g) g, \tag{1.5}$$

$$pq = \sum_{g,h \in G} a_g b_h gh. \tag{1.6}$$

The addition and multiplication defined above make the product RG into a ring called *group ring*, where R is not necessarily commutative.

The product RG is commutative iff the group G is commutative. Also RG is defined as *group algebra* if R is a field $\mathbb{F}$.

Lemma 1.1.2.

Let G be a group with $|G| = m$ and R be a ring with $|R| = n$ then $|RG| = n^m$.

Example 1.1.3.

Let $G = D_4$, dihedral group of order 4 with generators r, s such that $r^2 = s^2 = 1$ and $rs = sr^{-1}$. Let $R = \mathbb{Z}$ than

$$\mathbb{Z}G = \{a_1 + a_2 r + a_3 s + a_4 rs \mid \; a_i \in \mathbb{Z}\}$$

is a *group algebra*. Let $p, q \in \mathbb{Z}G$ such that $p = r + 3s$ and $P = -2r + s$. The addition and multiplication of p and q is defined as:

$$p + q = (r + 3s) + (-2r + rs) = -r + 3s + rs$$

$$p.q = (r + 3s).(-2r + rs) = -2 + 3r + s - 6rs$$

Example 1.1.4.

Let $G = Q_8 = \langle x, y \mid x^4 = y^4 = 1,\ x^2 = y^2, xyx = y\rangle$ be a group of quaternions of order 8 and $R = \mathbb{F}_3 = \{0, 1, 2\}$ be field of three elements. The product:

$$\mathbb{F}_3Q_8 = \left\{a_0 + a_1x + a_2x^2 + a_3x^3 + a_4y + a_5xy + a_6x^2y + a_7x^3y \mid\ a_i \in \mathbb{F}_3\right\}$$

form a group algebra of order 3^8.

Example 1.1.5.

If $G = A_3 = \{a_1, a_2, a_3\}$ be an alternating group, where a_1, a_2 and a_3 are even permutations in S_3 and $R = \mathbb{F}_{3^k}$ be a field of characteristic 3. Than the product:

$$\mathbb{F}_{3^k}A_3 = \{\alpha_1a_1 + \alpha_2a_2 + \alpha_3a_3 \mid\ \alpha_i \in \mathbb{F}_{3^k}\}$$

generate a *group algebra* of order 3^{3k}.

Note: The generalized form of *group rings and group algebras* is *hyper complex number system* when $R = \mathbb{R}$ (set of reals).

Definition 1.1.6.

Unit group: In the product RG the elements having the inverses form a group:

$$U(RG) = \{p \in RG\ \mid p \textit{ is invertible }\}$$

called a unit group of RG.

Definition 1.1.7.

Zero-divisors: Let $p, q\ \in RG$ *such that* $pq = 0$ but $p \neq 0$ *and* $q \neq 0$ then p and q are called zero-divisors. The set of zero divisors is denoted by $ZD(RG)$.

Conjecture 1.1.8.

Zero-divisors conjecture: The zero-divisors conjecture states that unit group, set of zero divisors and $\{0\}$ partitioned the group ring i.e.,

$$RG = \{0\} \cup U(RG) \cup ZD(RG).$$

Definition 1.1.9.

Normalized unit group: Let $p \in U(RG)$ satisfying $A(p) = 1$, where A is an *augmentation map* then such p form a subgroup $V(RG)$ of $U(RG)$ called *normalized unit group* of RG i.e.,

$$V(RG) = \{q \in U(RG)\ \mid\ A(q) = 1\}.$$

Definition 1.1.10.

Unitary unit group: Let $p \in V(RG)$ such that $p^{-1} = p^*$ where $*$ is an $anti-automorphism$ of order 2 then p is called *unitary*. The set containing *unitary elements* is a subgroup of $V(RG)$ denoted by:

$$V_*(RG) = \left\{p \in V(RG) \ \mid \ p^* = p^{-1}\right\}$$

and is called *unitary unit group* of RG.

Example 1.1.11.

Let $G = K = \{1, a, b, c\}$ is Klein $4-group$ such that $a^2 = b^2 = c^2 = 1$ and $ab = c = ba$, $bc = a = cb$, $ac = b = ca$. Let $R = \mathbb{F}_2 = \mathbb{Z}_2$ is a field containing residue classes under modulo 2. The *group algebra* $\mathbb{Z}_2K$ is defined as:

$$\mathbb{Z}_2K = \{\alpha_1 + \alpha_2 a + \alpha_3 b + \alpha_4 c \mid \ \alpha_i \in \mathbb{Z}_2\}$$

or
$\mathbb{Z}_2K = \{0, 1, a, b, c, 1 + a, 1 + b, 1 + c, a + b, a + c, b + c, 1 + a + b, 1 + a + c, 1 + b + c, a + b + c, 1 + a + b + c\}$.
Let $p_0 = 0$, $p_1 = 1$, $p_2 = a$, $p_3 = b$, $p_4 = c$, $p_5 = 1 + a$, $p_6 = 1 + b$, $p_7 = 1 + c$, $p_8 = a+b$, $p_9 = a+c$, $p_{10} = b+c$, $p_{11} = 1+a+b$, $p_{12} = 1+a+c$, $p_{13} = 1+b+c$, $p_{14} = a + b + c$, $p_{15} = 1 + a + b + c$.
It can easily be checked that $p_ip_j = 1$ for $i = j$ and $i, j = 1, 2, 3, 4, 11, 12, 13, 14$, otherwise $p_ip_j \neq 1$ for $i, j = 0, 1, 2, ..., 15$. So, unit group of $\mathbb{Z}_2K$ is:

$$U(\mathbb{Z}_2K) = \{p_1, p_2, p_3, p_4, p_{11}, p_{12}, p_{13}, p_{14}\}.$$

Each element $p \in U(\mathbb{Z}_2K)$ has augmentation 1 i.e. $A(p) = 1$. This shows that $U(\mathbb{Z}_2K) \leq V(\mathbb{Z}_2K)$ and hence $U(\mathbb{Z}_2K) = V(\mathbb{Z}_2K)$. Also each $p \in V(\mathbb{Z}_2K)$ satisfy that $p^* = p^{-1}$, it implies that $V(\mathbb{Z}_2K) \leq V_*(\mathbb{Z}_2K)$, so we obtained the result $U(\mathbb{Z}_2K) = V(\mathbb{Z}_2K) = V_*(\mathbb{Z}_2K)$.

Definition 1.1.12.

Circulant matrices: Different fields of applied mathematics in which the linear algebra is used define the *circulant matrix* in different ways regarding to the structure of the problem. However *circulant matrix* is basically a square matrix which is generated by a *shift operator* $\tau : R^n \to R^n$, where R is a ring defined as:

$$\tau(a_0, a_1, a_2, ..., a_{n-1}) = (a_{n-1}, a_0, a_1, ..., a_{n-2})$$

for $n \geq 2$. A *circulant matrix* $M = circ(\mathbf{a})$ defined on a row or a column vector $\mathbf{a}$ associated to K^n is a square matrix whose rows or columns are generated by the iterations of *shift operator* defined on $\mathbf{a}$, its *ith* row is $\tau^{i-1}\mathbf{a}$ where $i = 1, 2, ..., n$:

$$M = circ(a_0, a_1, a_2, \ldots, a_{n-1}) = \begin{pmatrix} a_0 & a_1 & a_2 & \cdots & a_{n-1} \\ a_{n-1} & a_0 & a_1 & \cdots & a_{n-2} \\ a_{n-2} & a_{n-1} & a_0 & \cdots & a_{n-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_1 & a_2 & a_3 & \cdots & a_0 \end{pmatrix}.$$

The set $\mathbb{M}_n$ of *circulant matrices* is closed under addition and multiplication and also is commutative under multiplication by C.S.Cleghron, [11]. The set $\mathbb{M}_n$ form an $n-$*dimensional* vector space. For further details and background related to *circulant matrices*, see P.J.Davis, [17].

Definition 1.1.13.

Group matrix: Suppose we have a fixed listing of the elements of group G i.e.,

$$G = \{g_1, g_2, g_3, \ldots, g_n\}$$

Then the following matrix:

$$M(G) = \begin{pmatrix} g_1^{-1}g_1 & g_1^{-1}g_2 & g_1^{-1}g_3 & \cdots & g_1^{-1}g_n \\ g_2^{-1}g_1 & g_2^{-1}g_2 & g_2^{-1}g_3 & \cdots & g_2^{-1}g_n \\ g_3^{-1}g_1 & g_3^{-1}g_2 & g_3^{-1}g_3 & \cdots & g_3^{-1}g_n \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ g_n^{-1}g_1 & g_n^{-1}g_2 & g_n^{-1}g_3 & \cdots & g_n^{-1}g_n \end{pmatrix}$$

is called the matrix of G relative to this listing.

Definition 1.1.14.

Group ring matrix: Let $p = \sum_{j=1}^{n} \alpha_{g_j} g_j$ be an arbitrary element in RG. Then the matrix defined:

$$M(RG, p) = \begin{pmatrix} \alpha_{g_1^{-1}g_1} & \alpha_{g_1^{-1}g_2} & \alpha_{g_1^{-1}g_3} & \cdots & \alpha_{g_1^{-1}g_n} \\ \alpha_{g_2^{-1}g_1} & \alpha_{g_2^{-1}g_2} & \alpha_{g_2^{-1}g_3} & \cdots & \alpha_{g_2^{-1}g_n} \\ \alpha_{g_3^{-1}g_1} & \alpha_{g_3^{-1}g_2} & \alpha_{g_3^{-1}g_3} & \cdots & \alpha_{g_3^{-1}g_n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \alpha_{g_n^{-1}g_1} & \alpha_{g_n^{-1}g_2} & \alpha_{g_n^{-1}g_3} & \cdots & \alpha_{g_n^{-1}g_n} \end{pmatrix}$$

called the group ring matrix.

Example 1.1.15.

Let $RK = \{\alpha_1 + \alpha_2 a + \alpha_3 b + \alpha_4 c \mid \alpha_i \in R\}$ be a group ring, where K be Klein $4-group$ and $q = \sum_{j=1}^{4} \alpha_{g_j} g_j \in RK$ be an arbitrary element, then group ring matrix of q is defined as:

$$M(RK, q) = \begin{pmatrix} \alpha_{g_1^{-1}g_1} & \alpha_{g_1^{-1}g_2} & \alpha_{g_1^{-1}g_3} & \alpha_{g_1^{-1}g_4} \\ \alpha_{g_2^{-1}g_1} & \alpha_{g_2^{-1}g_2} & \alpha_{g_2^{-1}g_3} & \alpha_{g_2^{-1}g_4} \\ \alpha_{g_3^{-1}g_1} & \alpha_{g_3^{-1}g_2} & \alpha_{g_3^{-1}g_3} & \alpha_{g_3^{-1}g_4} \\ \alpha_{g_n^{-1}g_1} & \alpha_{g_n^{-1}g_2} & \alpha_{g_n^{-1}g_3} & \alpha_{g_n^{-1}g_4} \end{pmatrix}$$

or

$$M(RK, q) = \begin{pmatrix} \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 \\ \alpha_2 & \alpha_1 & \alpha_4 & \alpha_3 \\ \alpha_3 & \alpha_4 & \alpha_1 & \alpha_2 \\ \alpha_4 & \alpha_3 & \alpha_2 & \alpha_1 \end{pmatrix}.$$

1.2 Some important results

This section contains some important theorems which will be used as a powerful tool for next work. The following theorem plays a role of bridge between abstract algebra and linear algebra constructed by T.Hurley in [35].

Theorem 1.2.1.

Given a listing of the elements of a group G of order n there is a ring isomorphism between RG and the $n \times n$ $RG-matrices$ (group ring-matrices) over R. This ring isomorphism is given by $\sigma : w \mapsto M(RG, w)$. Suppose R has an identity. Then $w \in RG$ is a unit if and only if $\sigma(w)$ is unit in $M_n(R)$.

Following theorem is established in [23] by J. Gildea and is presented here to facilitate the reader.

Remark 1.2.2.

Let $C_n = \langle x \mid x^n = 1\rangle$ and $\alpha = \sum_{i=1}^{n} a_i x^{i-1} \in F_{p^k}C^n$ $where$ $a_i \in F_{p^k}$, p is a prime and F_{p^k} is the Galois field of p^k-elements. Then $\sigma(\alpha) = circ(a_1, a_2, a_3, \ldots, a_n)$, where σ is defined in theorem 1.2.1.

Theorem 1.2.3.

Let $A = circ(a_1, a_2, a_3, \ldots, a_{p^{\mathrm{m}}})$, where $a_i \in \mathbb{F}_{p^k}$, p is a prime and m $\in \mathbb{N}$. Then

$$A^{p^{\mathrm{m}}} = \sum_{i=1}^{p^{\mathrm{m}}} {a_i}^{p^{\mathrm{m}}} I_{p^{\mathrm{m}}}.$$

Proof: Let $\alpha = \sum_{i=1}^{p^{\mathrm{m}}} a_i x^{i-1} \in F_{p^k}C^{p^{\mathrm{m}}}$ *where* $a_i \in F_{p^k}$, p is a prime, F_{p^k} *and* $C^{p^{\mathrm{m}}}$ are defined in remark 3.1. Now

$$\alpha^{p^{\mathrm{m}}} = \left(\sum_{i=1}^{p^{\mathrm{m}}} a_i x^{i-1}\right)^{p^{\mathrm{m}}} = \sum_{i=1}^{p^m} a_i^{p^{\mathrm{m}}} \left(x^{i-1}\right)^{p^{\mathrm{m}}} = \sum_{i=1}^{p^m} a_i^{p^{\mathrm{m}}} \left(x^{p^{\mathrm{m}}}\right)^{i-1} = \sum_{i=1}^{p^{\mathrm{m}}} {a_i}^{p^{\mathrm{m}}}.$$

Let $A = \sigma(\alpha) = circ(a_1, a_2, a_3, \ldots, a_{p^{\mathrm{m}}})$, where σ is the isomorphism given in theorem 1.2.1.

$$\sigma(\alpha^{p^{\mathrm{m}}}) = [\sigma(\alpha)]^{p^{\mathrm{m}}} \Rightarrow A^{p^{\mathrm{m}}} = diag_{p^{\mathrm{m}}}\left({a_i}^{p^{\mathrm{m}}}\right) = {a_i}^{p^{\mathrm{m}}} I_{p^{\mathrm{m}}}.$$

The next theorem is presented in [9] by Bovdi and Rosa. The following proposition is presented in [15]

Proposition 1.2.4.

Let $\mathbb{F}$ be a finite field of *characteristic* 2. If

$$Q_{2^{n+1}} = \langle a, b \mid a^{2^n} = 1,\ a^{2^{n-1}} = b^2,\ a^b = a^{-1}\rangle$$

is the quaternion group of order 2^{n+1} then

$$|V_*(KQ_{2^{n+1}})| = 4.|\mathbb{F}|^{2^n}.$$

Chapter 2

The Unitary Unit Group of Group Algebra $\mathbb{F}_{2^m}Q_8$

This chapter contains the review of "On the structure of the unitary unit subgroup $V_*(\mathbb{F}_{2^m}Q_8)$ of the *group algebra* $\mathbb{F}_{2^m}Q_8$", where Q_8 is a group of quaternions of order 8 and $\mathbb{F}_{2^m}$ is any finite field of *characteristic* 2 having 2^m elements [15]. As Q_8 is an extra special group, so *unitary subgroup* $V_*(\mathbb{F}_{2^m}Q_8)$ is normal in $V(\mathbb{F}_{2^m}Q_8)$ by Bovdi and Kovacs [8].

2.1 The center of unitary subgroup $V_*(\mathbb{F}_{2^m}Q_8)$

Remark 2.1.1.

Let $Q_8 = \langle x, y \mid x^4 = 1, x^2 = y^2, xy = y^{-1}x\rangle$ be the group of quaternions of order 8.Let $p \in \mathbb{F}_{2^m}Q_8$ and is defined as:

$$p = \sum_{i=0}^{3} a_i x^i + \sum_{j=0}^{3} b_j x^j y,$$

where $a_i, b_j \in \mathbb{F}_{2^m}$. First we establish $\sigma(p)$, where σ is an isomorphism and has described in 1.2.1. Let

$$Q_8 = \{1, x, x^2, x^3, y, xy, x^2y, x^3y\}$$

be a fixed listing of group. According to this listing, the group matrix $M(Q_8)$ is defined as:

$$\begin{pmatrix} 1 & x & x^2 & x^3 & y & xy & x^2y & x^3y \\ x^3 & 1 & x & x^2 & x^3y & y & xy & x^2y \\ x^2 & x^3 & 1 & x & x^2y & x^3y & y & xy \\ x & x^2 & x^3 & 1 & xy & x^2y & x^3y & y \\ x^2y & xy & y & x^3y & 1 & x^3 & x^2 & x \\ x^3y & x^2y & xy & y & x & 1 & x^3 & x^2 \\ y & x^3y & x^2y & xy & x^2 & x & 1 & x^3 \\ xy & y & x^3y & x^2y & x^3 & x^2 & x & 1 \end{pmatrix}.$$

Now we form $\sigma(p)$ by replacing coefficients of these elements in $M(Q_8)$ therefore

$$\sigma(p) = \begin{pmatrix} a_0 & a_1 & a_2 & a_3 & b_0 & b_1 & b_2 & b_3 \\ a_3 & a_0 & a_1 & a_2 & b_3 & b_0 & b_1 & b_2 \\ a_2 & a_3 & a_0 & a_1 & b_2 & b_3 & b_0 & b_1 \\ a_1 & a_2 & a_3 & a_0 & b_1 & b_2 & b_3 & b_0 \\ b_2 & b_1 & b_0 & b_3 & a_0 & a_3 & a_2 & a_1 \\ b_3 & b_2 & b_1 & b_0 & a_1 & a_0 & a_3 & a_2 \\ b_0 & b_3 & b_2 & b_1 & a_2 & a_1 & a_0 & a_3 \\ b_1 & b_0 & b_3 & b_2 & a_3 & a_2 & a_1 & a_0 \end{pmatrix}. \tag{2.1}$$

The representation of $\sigma(q)$ in block-*circulant matrix* is $\sigma(p) = \begin{pmatrix} A & B \\ C & A^T \end{pmatrix}$,
where
$A = circ(a_0, a_1, a_2, a_3)$, $\qquad B = circ(b_0, b_1, b_2, b_3)$,
$C = circ(b_2, b_1, b_6, b_0, b_3)$.

If we apply $*$ (an anti-automorphism defined in equation 1.4) on p:

$$p^* = a_0 + a_1x^3 + a_2x^2 + a_3x + b_0^x 2y + b_1x^3y + b_2y + b_3xy,$$

according to this listing group matrix for p^* is:

$$\begin{pmatrix} 1 & x^3 & x^2 & x & x^2y & x^3y & y & xy \\ x & 1 & x^3 & x^2 & xy & x^2y & x^3y & y \\ x^2 & x & 1 & x^3 & y & xy & x^2y & x^3y \\ x^3 & x^2 & x & 1 & x^3y & y & xy & x^2y \\ y & x^3y & x^2y & xy & 1 & x & x^2 & x^3 \\ xy & y & x^3y & x^2y & x^3 & 1 & x & x^2 \\ x^2y & xy & y & x^3y & x^2 & x^3 & 1 & x \\ x^3y & x^2y & xy & y & x & x^2 & x^3 & 1 \end{pmatrix}.$$

We get $\sigma(p^*)$ after replacing coefficients of the relevant elements in above matrix:

$$\sigma(p^*) = \begin{pmatrix} a_0 & a_3 & a_2 & a_1 & b_2 & b_3 & b_0 & b_1 \\ a_1 & a_0 & a_3 & a_2 & b_1 & b_2 & b_3 & b_0 \\ a_2 & a_1 & a_0 & a_3 & b_0 & b_1 & b_2 & b_3 \\ a_3 & a_2 & a_1 & a_0 & b_3 & b_0 & b_1 & b_2 \\ b_0 & b_3 & b_2 & b_1 & a_0 & a_1 & a_2 & a_3 \\ b_1 & b_0 & b_3 & b_2 & a_3 & a_0 & a_1 & a_2 \\ b_2 & b_1 & b_0 & b_3 & a_2 & a_3 & a_0 & a_1 \\ b_3 & b_2 & b_1 & b_0 & a_1 & a_2 & a_3 & a_0 \end{pmatrix}.$$

It is evident from above matrices for $\sigma(q)$ and $\sigma(p^*)$ that

$$\sigma(p^*) = (\sigma(p))^T. \tag{2.2}$$

Proposition 2.1.2. *The center of unitary subgroup $V_*(\mathbb{F}_{2^m}Q_8)$ of normalized unit group $V(\mathbb{F}_{2^m}Q_8)$ is isomorphic to C_2^{4m} i.e.,*

$$Z\left(V_*(\mathbb{F}_{2^m}Q_8)\right) \cong C_2^{4m}.$$

Proof: Let $v = \sum_{i=0}^{3} a_i x^i + \sum_{j=0}^{3} b_j x^j y$ be an arbitrary element in $V(\mathbb{F}_{2^m}Q_8)$ where $\sum a_i = 1,\ a_i \in \mathbb{F}_{2^m}$ and let us denote for simplicity $V = V(\mathbb{F}_{2^m}Q_8)$. First we find the centralizer of the generator x in V. Consider the set

$$C_V(x) = \{v \in V \mid xv = vx\},$$

where

$$xv = \sum_{i=0}^{3} a_i x^{i+1} + \sum_{j=0}^{3} b_j x^{j+1} y$$

and

$$vx = \sum_{i=0}^{3} a_i x^{i+1} + \sum_{j=0}^{3} b_j x^{j+3} y.$$

Now,

$$vx - xv = \sum_{j=0}^{3} b_j (x^{j+3} - x^{j+1} y) = (b_1 - b_3)y + (b_2 - b_0)xy + (b_3 - b_1)x^2 y + (b_0 - b_2)x^3 y.$$

$vx - xv = 0$ iff $b_0 = b_2$ and $b_1 = b_3$. If $\alpha = \sum_{i=0}^{3} a_i x^i + c_1(y + x^2 y) + c_2(xy + x^3 y)$, where $\sum_{i=0}^{3} a_i = 0$ and $c_2 = b_1 = b_3$, $c_1 = b_0 = b_2$ then $\alpha x = x\alpha$. Thus

$$C_V(x) = \left\{ \sum_{i=0}^{3} a_i x^i + c_1(y + x^2 y) + c_2(xy + x^3 y) \mid \sum_{i=0}^{3} a_i = 0, c_1, c_2 \in \mathbb{F}_{2^m} \right\}.$$

Since the center of V is contained in $C_V(x)$, therefore we can write

$$Z(V) = \{\beta \in C_V(x) \mid \beta v = v\beta, for\ all\ v \in V\}.$$

The group ring matrix $\sigma(v)$ is same as $\sigma(p)$ which is already defined in 2.1. Group ring matrix of β is defined as:

$$\sigma(\beta) = \begin{pmatrix} a_0 & a_1 & a_2 & a_3 & c_1 & c_2 & c_1 & c_2 \\ a_3 & a_0 & a_1 & a_2 & c_2 & c_1 & c_2 & c_1 \\ a_2 & a_3 & a_0 & a_1 & c_1 & c_2 & c_1 & c_2 \\ a_1 & a_2 & a_3 & a_0 & c_2 & c_1 & c_2 & c_1 \\ c_1 & c_2 & c_1 & c_2 & a_0 & a_3 & a_2 & a_1 \\ c_2 & c_1 & c_2 & c_1 & a_1 & a_0 & a_3 & a_2 \\ c_1 & c_2 & c_1 & c_2 & a_2 & a_1 & a_0 & a_3 \\ c_2 & c_1 & c_2 & c_1 & a_3 & a_2 & a_1 & a_0 \end{pmatrix} \tag{2.3}$$

or

$$\sigma(\beta) = \begin{pmatrix} A & E \\ E & A^T \end{pmatrix},$$

where $E = circ(c_1, c_2, c_1, c_2)$.

$$\sigma(v)\sigma(\beta) - \sigma(\beta)\sigma(v) = \begin{pmatrix} A & B \\ C & A^T \end{pmatrix} \begin{pmatrix} A & E \\ E & A^T \end{pmatrix} - \begin{pmatrix} A & E \\ E & A^T \end{pmatrix} \begin{pmatrix} A & B \\ C & A^T \end{pmatrix}$$

or

$$\sigma(v)\sigma(\beta) - \sigma(\beta)\sigma(v) = \begin{pmatrix} E(C - B) & (B + E)(A - A^T) \\ (C + E)(A^T - A) & E(B - C) \end{pmatrix}.$$

$B - C = 0$ by using the relation $c_1 = b_0 = b_2$, $c_2 = b_1 = b_3$ and $A - A^T = 0$ iff $a_1 = a_3$, this implies that $a_0 = 1 + a_2$. Finally, we get the general form of each element of Z(V) is $1 + a_2 + a_1 x + a_2 x^2 + a_1 x^3 + b_0 y + b_1 xy + b_0 x^2 y + b_1 x^3 y$. This structure of Z(V) shows that, it has exponent 2.

Now let $\gamma = 1 + r + sx + rx^2 + sx^3 + ty + uxy + tx^2y + ux^3y \in Z(V)$, where r,s,t and u are in $\mathbb{F}_{2^m}$. We have to show that center of V is equal to the center of V_* (for simplicity we use $V_* = V_*(\mathbb{F}_{2^m}Q_8)$). For this let,

$$\gamma^* = \gamma^{-1}$$

$$\Leftrightarrow \sigma(\gamma^*) = \sigma(\gamma^{-1})$$

$$\Leftrightarrow (\sigma(\gamma))^T = (\sigma(\gamma))^{-1}$$

(by using 2.2 and as σ is an isomorphism)

$$\Leftrightarrow (\sigma(\gamma))^T(\sigma(\gamma)) = I. \tag{2.4}$$

Let

$$\sigma(\gamma) = \begin{pmatrix} F & G \\ G & F \end{pmatrix},$$

where $F = circ(1 + r, s, r, s)$ and $G = circ(t, u, t, u)$. Now equation 2.4 becomes

$$(\sigma(\gamma))^T(\sigma(\gamma)) = \begin{pmatrix} F & G \\ G & F \end{pmatrix}^T \begin{pmatrix} F & G \\ G & F \end{pmatrix}$$

or

$$(\sigma(\gamma))^T(\sigma(\gamma)) = \begin{pmatrix} F^2 + G^2 & 2FG \\ 2FG & F^2 + G^2 \end{pmatrix}.$$

As $F^2 + G^2 = (1+r)^2 + s^2 + r^2 + s^2 + t^2 + u^2 + t^2 + u^2 = 1$ (by using theorem 1.2.3) and $2FG = 0$ (characteristic of field is 2). Thus $Z(V)$ is contained in $Z(V_*)$ and

$$Z(V_*) = Z(V) \cong C_2^{4m}.$$

2.2 The structure of unitary subgroup $V_*(\mathbb{F}_{2^m}Q_8)$

Remark 2.2.1.

We can see that $Z(V_*)$ and Q_8 are normal subgroups in $V_*(\mathbb{F}_{2^m}Q_8)$ and some portion of $Z(V_*)$ and Q_8 are overlapping:

$$Z(V_*) \cap Q_8 = \left\{1, x^2\right\}.$$

The reason is that these elements are commute with all powers of xy.

Proposition 2.2.2.

The *unitary unit group* $V_*(\mathbb{F}_{2^m}Q_8)$ is equal to $Z(V_*(\mathbb{F}_{2^m}Q_8)).Q_8$.

Proof: Let $I = V_*(\mathbb{F}_{2^m}Q_8)$, $J = Z(V_*(\mathbb{F}_{2^m}Q_8))$ and $K = Q_8$, then by using the second isomorphism theorem

$$\frac{J.K}{J} \cong \frac{K}{J \cap K}. \tag{2.5}$$

As $|K| = 8$ and $|J \cap K| = 2$ implies that $|\frac{K}{J \cap K}| = 4$. Now equation 2.5 shows that

$$|J.K| = 4.|J| = 4.2^{4k} = 2^{4k+2}.$$

Therefore by using the proposition 1.2.4

$$Z(V_*(\mathbb{F}_{2^m}Q_8)).Q_8 = V_*(\mathbb{F}_{2^m}Q_8).$$

Theorem 2.2.3.

The *unitary unit group* $V_*(\mathbb{F}_{2^m}Q_8)$ is isomorphic to $C_2^{4k-1} \times Q_8$.

Proof: As we know that $Z(V_*(\mathbb{F}_{2^m}Q_8)) \cong C_2^{4m}$ forms a vector space over the field $\mathbb{F}_{2^m}$ having the dimension $4m$. The set of basis for this vector space is $\{x_1, x_2, ..., x_{4m}\}$ and let $x_{4m} = x^2$. Let us define a finitely generated group G such that $G = \langle x_1, x_2, ..., x_{4m-1}\rangle$. It implies that $G \cong C_2^{4k-1}$ and also $Z(V_*(\mathbb{F}_{2^m}Q_8)) \cong G \times \langle x_{4m}\rangle$ or $Z(V_*(\mathbb{F}_{2^m}Q_8)) \cong G \times \langle x^2\rangle$. Now we have $G \cap Q_8 = 1$ and $V_*(\mathbb{F}_{2^m}Q_8) = G.Q_8$. Since $G < Z(V_*(\mathbb{F}_{2^m}Q_8))$. Therefore $V_*(\mathbb{F}_{2^m}Q_8) \cong G \rtimes Q_8 \cong G \times Q_8$. Hence we can write:

$$V_*(\mathbb{F}_{2^m}Q_8) \cong C_2^{4m-1} \times Q_8.$$

Chapter 3

The Unitary Unit Group of The Group Algebra $\mathbb{F}_{2^k}3^{1+2}$

This chapter is about the discussion on the center $Z\left(V_*(\mathbb{F}_{2^k}3^{1+2})\right)$ of unitary unit subgroup $V_*(\mathbb{F}_{2^k}3^{1+2})$ of *group algebra* $\mathbb{F}_{2^k}3^{1+2}$ and its order, where

$$3^{1+2} = \left\langle x, y, z \mid x^3 = y^3 = 1 = z^3 = [x, z] = [y, z], z = [x, y]\right\rangle$$

is an extra special group of order 3^3. The permutation representation of the generators of 3^{1+2} is given below:
$x \mapsto (2, 6, 7)(3, 8, 9)$,
$y \mapsto (1, 2, 3)(4, 7, 8)(5, 6, 9)$,
$z \mapsto (1, 4, 5)(2, 7, 6)(3, 8, 9)$.
Further details about the generators and the other elements of this group is given in chapter 4.

3.1 The center of unitary unit group $V_*(\mathbb{F}_{2^k}3^{1+2})$

Remark 3.1.1.

Let $3^{1+2} = \{x_0, x_1, x_2, ..., x_{26}\}$ be a fixed listing of elements of 3^{1+2}. We label the elements of $3^{1+2})$ by the permutations in S_9. This labeling was done with the help of GAP.
$x_0 \mapsto 1$ (identity of group),
$x_1 \mapsto (2, 6, 7)(3, 8, 9) = x$,
$x_2 \mapsto (2, 7, 6)(3, 9, 8) = x^2$,
$x_3 \mapsto (1, 2, 3)(4, 7, 8)(5, 6, 9) = y$,
$x_4 \mapsto (1, 2, 9)(3, 4, 7)(5, 6, 8) = xy$,
$x_5 \mapsto (1, 2, 3)(4, 7, 8)(5, 6, 9) = x^2y$,
$x_6 \mapsto (1, 3, 2)(4, 8, 7)(5, 9, 6) = y^2$,
$x_7 \mapsto (1, 3, 7)(2, 5, 9)(4, 8, 6) = xy^2$,
$x_8 \mapsto (1, 3, 6)(2, 4, 8)(5, 9, 7) = x^2y^2$,
$x_9 \mapsto (1, 4, 5)(3, 9, 8) = y^2xy$,
$x_{10} \mapsto (1, 4, 5)(2, 6, 7) = x(y^2xy)$,
$x_{11} \mapsto (1, 4, 5)(2, 7, 6)(3, 8, 9) = x^2(y^2xy) = z$,
$x_{12} \mapsto (1, 5, 4)(3, 8, 9) = (xy)^2y$,
$x_{13} \mapsto (1, 5, 4)(2, 6, 7)(3, 9, 8) = x(xy)^2y$,
$x_{14} \mapsto (1, 5, 4)(2, 7, 6) = x^2(xy)^2y$,
$x_{15} \mapsto (1, 6, 3)(2, 8, 4)(5, 7, 9) = yx$,
$x_{16} \mapsto (1, 6, 9)(2, 3, 4)(5, 7, 8) = xyx$,
$x_{17} \mapsto (1, 6, 8)(2, 9, 4)(3, 5, 7) = x^2xy$,
$x_{18} \mapsto (1, 7, 3)(2, 9, 5)(4, 6, 8) = yx^2$,
$x_{19} \mapsto (1, 7, 9)(2, 8, 5)(3, 4, 6) = xyx^2$,

$x_{20} \mapsto (1,4,5)(2,7,6)(3,8,9) = x^2yx^2,$
$x_{21} \mapsto (1,8,2)(3,6,5)(4,9,7) = y^2x,$
$x_{22} \mapsto (1,8,7)(2,5,3)(4,9,6) = xy^2x,$
$x_{23} \mapsto (1,8,6)(2,4,9)(3,7,5) = x^2y^2x,$
$x_{24} \mapsto (1,9,2)(3,7,4)(5,8,6) = (xy)^2,$
$x_{25} \mapsto (1,9,7)(2,5,8)(3,6,4) = x^2(xy)^2,$
$x_{26} \mapsto (1,9,6)(2,4,3)(5,8,7) = x^2(xy)^2.$
Let $\mathbb{F}_{2^k}$ be a field having the characteristic 2 and order 2^k. We can form a *group algebra* $\mathbb{F}_{2^k}3^{1+2}$ which is defined as:

$$\mathbb{F}_{2^k}3^{1+2} = \left\{ \sum_{i=0}^{26} a_i x_i \mid \ a_i \in \mathbb{F}_{2^k}, x_i \in 3^{1+2} \right\},$$

containing 2^{27} elements.
Let if $p = \sum\limits_{i=0}^{26} a_i x_i \in \mathbb{F}_{2^k}3^{1+2}$ then group ring matrix of p is defined as:

$$\sigma(p) = \left(\begin{array}{ccc|ccc|ccc|ccc|ccc|ccc|ccc|ccc|ccc}
a_0 & a_1 & a_2 & a_3 & a_4 & a_5 & a_6 & a_7 & a_8 & a_9 & a_{10} & a_{11} & a_{12} & a_{13} & a_{14} & a_{15} & a_{16} & a_{17} & a_{18} & a_{19} & a_{20} & a_{21} & a_{22} & a_{23} & a_{24} & a_{25} & a_{26} \\
a_2 & a_0 & a_1 & a_5 & a_3 & a_4 & a_8 & a_6 & a_7 & a_{11} & a_9 & a_{10} & a_{14} & a_{12} & a_{13} & a_{17} & a_{15} & a_{16} & a_{20} & a_{18} & a_{19} & a_{23} & a_{21} & a_{22} & a_{26} & a_{24} & a_{25} \\
a_1 & a_2 & a_0 & a_4 & a_5 & a_3 & a_7 & a_8 & a_6 & a_{10} & a_{11} & a_9 & a_{13} & a_{14} & a_{12} & a_{16} & a_{17} & a_{15} & a_{19} & a_{20} & a_{18} & a_{22} & a_{23} & a_{21} & a_{25} & a_{26} & a_{24} \\
\hline
a_6 & a_{21} & a_{24} & a_0 & a_9 & a_{12} & a_3 & a_{20} & a_{16} & a_{26} & a_8 & a_{23} & a_{22} & a_{25} & a_7 & a_1 & a_{10} & a_{13} & a_2 & a_{11} & a_{14} & a_{15} & a_5 & a_{19} & a_{18} & a_{17} & a_4 \\
a_{24} & a_6 & a_{21} & a_{12} & a_0 & a_9 & a_{16} & a_3 & a_{20} & a_{23} & a_{26} & a_8 & a_7 & a_{22} & a_{25} & a_{13} & a_1 & a_{10} & a_{14} & a_2 & a_{11} & a_{19} & a_{15} & a_5 & a_4 & a_{18} & a_{17} \\
a_{21} & a_{24} & a_6 & a_9 & a_{12} & a_0 & a_{20} & a_{16} & a_3 & a_8 & a_{23} & a_{26} & a_{25} & a_7 & a_{22} & a_{10} & a_{13} & a_1 & a_{11} & a_{14} & a_2 & a_5 & a_{19} & a_{15} & a_{17} & a_4 & a_{18} \\
\hline
a_3 & a_{15} & a_{18} & a_6 & a_{26} & a_{22} & a_0 & a_{14} & a_{10} & a_4 & a_{16} & a_{19} & a_5 & a_{17} & a_{20} & a_{21} & a_8 & a_{25} & a_{24} & a_{23} & a_7 & a_1 & a_{12} & a_{11} & a_2 & a_{13} & a_9 \\
a_{18} & a_3 & a_{15} & a_{22} & a_6 & a_{26} & a_{10} & a_0 & a_{14} & a_{19} & a_4 & a_{16} & a_{20} & a_5 & a_{17} & a_{25} & a_{21} & a_8 & a_7 & a_{24} & a_{23} & a_{11} & a_1 & a_{12} & a_9 & a_2 & a_{13} \\
a_{15} & a_{18} & a_3 & a_{26} & a_{22} & a_6 & a_{14} & a_{10} & a_0 & a_{16} & a_{19} & a_4 & a_{17} & a_{20} & a_5 & a_8 & a_{25} & a_{21} & a_{23} & a_7 & a_{24} & a_{12} & a_{11} & a_1 & a_{13} & a_9 & a_2 \\
\hline
a_{12} & a_{13} & a_{14} & a_{16} & a_{17} & a_{15} & a_{24} & a_{25} & a_{26} & a_0 & a_1 & a_2 & a_9 & a_{10} & a_{11} & a_{19} & a_{20} & a_{18} & a_4 & a_5 & a_3 & a_6 & a_7 & a_8 & a_{21} & a_{22} & a_{23} \\
a_{14} & a_{12} & a_{13} & a_{15} & a_{16} & a_{17} & a_{26} & a_{24} & a_{25} & a_2 & a_0 & a_1 & a_{11} & a_9 & a_{10} & a_{18} & a_{19} & a_{20} & a_3 & a_4 & a_5 & a_8 & a_6 & a_7 & a_{23} & a_{21} & a_{22} \\
a_{13} & a_{14} & a_{12} & a_{17} & a_{15} & a_{16} & a_{25} & a_{26} & a_{24} & a_1 & a_2 & a_0 & a_{10} & a_{11} & a_9 & a_{20} & a_{18} & a_{19} & a_5 & a_3 & a_4 & a_7 & a_8 & a_6 & a_{22} & a_{23} & a_{21} \\
\hline
a_9 & a_{10} & a_{11} & a_{20} & a_{18} & a_{19} & a_{21} & a_{22} & a_{23} & a_{12} & a_{13} & a_{14} & a_0 & a_1 & a_2 & a_5 & a_3 & a_4 & a_{17} & a_{15} & a_{16} & a_{24} & a_{25} & a_{26} & a_6 & a_7 & a_8 \\
a_{11} & a_9 & a_{10} & a_{19} & a_{20} & a_{18} & a_{23} & a_{21} & a_{22} & a_{14} & a_{12} & a_{13} & a_2 & a_0 & a_1 & a_4 & a_5 & a_3 & a_{16} & a_{17} & a_{15} & a_{26} & a_{24} & a_{25} & a_8 & a_6 & a_7 \\
a_{10} & a_{11} & a_9 & a_{18} & a_{19} & a_{20} & a_{22} & a_{23} & a_{21} & a_{13} & a_{14} & a_{12} & a_1 & a_2 & a_0 & a_3 & a_4 & a_5 & a_{15} & a_{16} & a_{17} & a_{25} & a_{26} & a_{24} & a_7 & a_8 & a_6 \\
\hline
a_8 & a_{23} & a_{26} & a_2 & a_{11} & a_{14} & a_5 & a_{19} & a_{15} & a_{25} & a_7 & a_{22} & a_{21} & a_{24} & a_6 & a_0 & a_9 & a_{12} & a_1 & a_{10} & a_{13} & a_{17} & a_4 & a_{18} & a_{20} & a_{16} & a_3 \\
a_{26} & a_8 & a_{23} & a_{14} & a_2 & a_{11} & a_{15} & a_5 & a_{19} & a_{22} & a_{25} & a_7 & a_6 & a_{21} & a_{24} & a_{12} & a_0 & a_9 & a_{13} & a_1 & a_{10} & a_{18} & a_{17} & a_4 & a_3 & a_{20} & a_{16} \\
a_{23} & a_{26} & a_8 & a_{11} & a_{14} & a_2 & a_{19} & a_{15} & a_5 & a_7 & a_{22} & a_{25} & a_{24} & a_6 & a_{21} & a_9 & a_{12} & a_0 & a_{10} & a_{13} & a_1 & a_4 & a_{18} & a_{17} & a_{16} & a_3 & a_{20} \\
\hline
a_7 & a_{22} & a_{25} & a_1 & a_{10} & a_{13} & a_4 & a_{18} & a_{17} & a_{24} & a_6 & a_{21} & a_{23} & a_{26} & a_8 & a_2 & a_{11} & a_{14} & a_0 & a_9 & a_{12} & a_{16} & a_3 & a_{20} & a_{19} & a_{15} & a_5 \\
a_{25} & a_7 & a_{22} & a_{13} & a_1 & a_{10} & a_{17} & a_4 & a_{18} & a_{21} & a_{24} & a_6 & a_8 & a_{23} & a_{26} & a_{14} & a_2 & a_{11} & a_{12} & a_0 & a_9 & a_{20} & a_{16} & a_3 & a_5 & a_{19} & a_{15} \\
a_{22} & a_{25} & a_7 & a_{10} & a_{13} & a_1 & a_{18} & a_{17} & a_4 & a_6 & a_{21} & a_{24} & a_{26} & a_8 & a_{23} & a_{11} & a_{14} & a_2 & a_9 & a_{12} & a_0 & a_3 & a_{20} & a_{16} & a_{15} & a_5 & a_{19} \\
\hline
a_5 & a_{17} & a_{20} & a_8 & a_{25} & a_{21} & a_2 & a_{13} & a_9 & a_3 & a_{15} & a_{18} & a_4 & a_{16} & a_{19} & a_{23} & a_7 & a_{24} & a_{26} & a_{22} & a_6 & a_0 & a_{14} & a_{10} & a_1 & a_{12} & a_{11} \\
a_{20} & a_5 & a_{17} & a_{21} & a_8 & a_{25} & a_9 & a_2 & a_{13} & a_{18} & a_3 & a_{15} & a_{19} & a_4 & a_{16} & a_{24} & a_{23} & a_7 & a_6 & a_{26} & a_{22} & a_{10} & a_0 & a_{14} & a_{11} & a_1 & a_{12} \\
a_{17} & a_{20} & a_5 & a_{25} & a_{21} & a_8 & a_{13} & a_9 & a_2 & a_{15} & a_{18} & a_3 & a_{16} & a_{19} & a_4 & a_7 & a_{24} & a_{23} & a_{22} & a_6 & a_{26} & a_{14} & a_{10} & a_0 & a_{12} & a_{11} & a_1 \\
\hline
a_4 & a_{16} & a_{19} & a_7 & a_{24} & a_{23} & a_1 & a_{12} & a_{11} & a_5 & a_{17} & a_{20} & a_3 & a_{15} & a_{18} & a_{22} & a_6 & a_{26} & a_{25} & a_{21} & a_8 & a_2 & a_{13} & a_9 & a_0 & a_{14} & a_{10} \\
a_{19} & a_4 & a_{16} & a_{23} & a_7 & a_{24} & a_{11} & a_1 & a_{12} & a_{20} & a_5 & a_{17} & a_{18} & a_3 & a_{15} & a_{26} & a_{22} & a_6 & a_8 & a_{25} & a_{21} & a_9 & a_2 & a_{13} & a_{10} & a_0 & a_{14} \\
a_{16} & a_{19} & a_4 & a_{24} & a_{23} & a_7 & a_{12} & a_{11} & a_1 & a_{17} & a_{20} & a_5 & a_{15} & a_{18} & a_3 & a_6 & a_{26} & a_{22} & a_{21} & a_8 & a_{25} & a_{13} & a_9 & a_2 & a_{14} & a_{10} & a_0
\end{array}\right).$$

Where σ is defined in theorem 1.2.1. The representation of $\sigma(p)$ in block circulant matrix is given:

$$\sigma(p) = \begin{pmatrix} A_1 & A_2 & A_3 & \cdots & A_9 \\ A_{10} & A_{11} & A_{12} & \cdots & A_{18} \\ A_{19} & A_{20} & A_{21} & \cdots & A_{27} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ A_{73} & A_{74} & A_{75} & \cdots & A_{81} \end{pmatrix}.$$

Where

$$\begin{gathered}
A_1 = A_{31} = A_{41} = circ(a_0, a_1, a_2), \\
A_2 = circ(a_3, a_4, a_5), \\
A_3 = A_{35} = A_{45} = circ(a_6, a_7, a_8), \\
A_4 = A_{32} = A_{37} = circ(a_9, a_{10}, a_{11}), \\
A_5 = A_{28} = A_{40} = circ(a_{12}, a_{13}, a_{14}), \\
A_6 = circ(a_{15}, a_{16}, a_{17}), \\
A_7 = circ(a_{18}, a_{19}, a_{20}), \\
A_8 = A_{36} = A_{39} = circ(a_{21}, a_{22}, a_{23}), \\
A_9 = A_{30} = A_{44} = circ(a_{24}, a_{25}, a_{26}), \\
A_{10} = circ(a_6, a_{21}, a_{24}), \\
A_{11} = A_{51} = A_{61} = circ(a_0, a_9, a_{12}), \\
A_{12} = circ(a_3, a_{20}, a_{16}), \\
A_{13} = circ(a_{26}, a_8, a_{23}), \\
A_{14} = circ(a_{22}, a_{25}, a_7), \\
A_{15} = A_{52} = A_{56} = circ(a_1, a_{10}, a_{13}), \\
A_{16} = A_{47} = A_{60} = circ(a_2, a_{11}, a_{14}), \\
A_{17} = circ(a_{15}, a_5, a_{19}), \\
A_{18} = circ(a_{18}, a_{17}, a_4), \\
A_{19} = A_{67} = A_{77} = circ(a_3, a_{15}, a_{18}), \\
A_{20} = circ(a_6, a_{26}, a_{22}), \\
A_{21} = A_{71} = A_{81} = circ(a_0, a_{14}, a_{10}), \\
A_{22} = A_{68} = A_{73} = circ(a_4, a_{16}, a_{19}), \\
A_{23} = A_{64} = A_{76} = circ(a_5, a_{17}, a_{20}), \\
A_{24} = circ(a_{21}, a_8, a_{25}), \\
A_{25} = circ(a_{24}, a_{23}, a_7), \\
A_{26} = A_{72} = A_{75} = circ(a_1, a_{12}, a_{11}), \\
A_{27} = A_{66} = A_{80} = circ(a_2, a_{13}, a_9), \\
A_{29} = circ(a_{16}, a_{17}, a_{15}), \\
A_{33} = circ(a_{19}, a_{20}, a_{18}),
\end{gathered}$$

$$A_{34} = circ(a_4, a_5, a_3),$$
$$A_{38} = circ(a_{20}, a_{18}, a_{19}),$$
$$A_{42} = circ(a_5, a_3, a_4),$$
$$A_{43} = circ(a_{17}, a_{15}, a_{16}),$$
$$A_{46} = circ(a_8, a_{23}, a_{26}),$$
$$A_{48} = circ(a_5, a_{19}, a_{15}),$$
$$A_{49} = circ(a_{25}, a_7, a_{22}),$$
$$A_{50} = circ(a_{21}, a_{24}, a_6),$$
$$A_{53} = circ(a_{17}, a_4, a_{18}),$$
$$A_{54} = circ(a_{20}, a_{16}, a_3),$$
$$A_{55} = circ(a_7, a_{22}, a_{25}),$$
$$A_{57} = circ(a_4, a_{18}, a_{17}),$$
$$A_{58} = circ(a_{24}, a_6, a_{21}),$$
$$A_{59} = circ(a_{23}, a_{26}, a_8),$$
$$A_{62} = circ(a_{16}, a_3, a_{20}),$$
$$A_{63} = circ(a_{19}, a_{15}, a_5),$$
$$A_{65} = circ(a_8, a_{25}, a_{21}),$$
$$A_{69} = circ(a_{23}, a_7, a_{24}),$$
$$A_{70} = circ(a_{26}, a_{22}, a_6),$$
$$A_{74} = circ(a_7, a_{24}, a_{23}),$$
$$A_{78} = circ(a_{22}, a_6, a_{28}),$$
$$A_{79} = circ(a_{25}, a_{21}, a_8).$$

Let p^* be the unitary element for $p \in \mathbb{F}_{2^k}3^{1+2}$ is defined as:
$p^* = \sum\limits_{i=0}^{26} a_i x_i^{-1} = a_0x_0 + a_1x_2 + a_2x_1 + a_3x_6 + a_4x_{24} + a_5x_{21} + a_6x_3 + a_7x_{18} + a_8x_{15} + a_9x_{12} + a_{10}x_{14} + a_{11}x_{13} + a_{12}x_9 + a_{13}x_{11} + a_{14}x_{10} + a_{15}x_8 + a_{16}x_{26} + a_{17}x_{23} + a_{18}x_7 + a_{19}x_{25} + a_{20}x_{22} + a_{21}x_5 + a_{22}x_{20} + a_{23}x_{17} + a_{24}x_4 + a_{25}x_{19} + a_{26}x_{16}$.
Since all the circulant matrices in $\sigma(p)$ are also column wise circulant (by definition), so

$$\sigma(p^*) = (\sigma(p))^T. \tag{3.1}$$

Remark 3.1.2.

Let V be the *normalized unit group* in $\mathbb{F}_{2^k}3^{1+2}$, then we can write it as

$$V = V(\mathbb{F}_{2^k}3^{1+2}) = \left\{q \in \mathbb{F}_{2^k}3^{1+2}) \mid \ q \textit{ is invertible and } A(q)\textit{=1}\right\},$$

where $q = \sum\limits_{i=0}^{26} a_i x_i$ and A is *augmentation map.* Now we calculate *centralizers* of the generators $x = x_1$, $y = x_3$ and $z = x_{11}$. Let

$$C_V(x_1) = \{q \in V \mid \ x_1q = qx_1\}.$$

Where,
$x_1q = \sum_{i=0}^{26} a_ix_1x_i = a_0x_1 + a_1x_1^2 + a_2x_0 + a_3x_4 + a_4x_5 + a_5x_3 + a_6x_7 + a_7x_8 + a_8x_6 + a_9x_{10} + a_{10}x_{11} + a_{11}x_9 + a_{12}x_{13} + a_{13}x_{14} + a_{14}x_{12} + a_{15}x_{16} + a_{16}x_{17} + a_{17}x_{15} + a_{18}x_{19} + a_{19}x_{20} + a_{20}x_{18} + a_{21}x_{22} + a_{22}x_{23} + a_{23}x_{21} + a_{24}x_{25} + a_{25}x_{26} + a_{26}x_{24}$
and
$qx_1 = \sum_{i=0}^{26} a_ix_ix_1 = a_0x_1 + a_1x_1^2 + a_2x_0 + a_3x_{15} + a_4x_{16} + a_5x_{17} + a_6x_{21} + a_7x_{22} + a_8x_{23} + a_9x_{10} + a_{10}x_{11} + a_{11}x_9 + a_{12}x_{13} + a_{13}x_{14} + a_{14}x_{12} + a_{15}x_{18} + a_{16}x_{19} + a_{17}x_{20} + a_{18}x_3 + a_{19}x_4 + a_{20}x_5 + a_{21}x_{24} + a_{22}x_{25} + a_{23}x_{26} + a_{24}x_6 + a_{25}x_7 + a_{26}x_8$.
As $x_1q - qx_1 = (a_3 - a_{19})x_4 + (a_4 - a_{20})x_5 + (a_5 - a_{18})x_3 + (a_6 - a_{25})x_7 + (a_3 - a_{19})x_4 + (a_7 - a_{26})x_8 + (a_8 - a_{24})x_6 + (a_{15} - a_4)x_{16} + (a_{16} - a_5)x_{17} + (a_{17} - a_3)x_{15} + (a_{18} - a_{16})x_{19} + (a_{19} - a_{17})x_{20} + (a_{20} - a_{15})x_{18} + (a_{21} - a_7)x_{22} + (a_{22} - a_8)x_{23} + (a_{23} - a_6)x_{21} + (a_{24} - a_{22})x_{25} + (a_{25} - a_{23})x_{26} + (a_{26} - a_{21})x_{24}$.
So,
$x_1q - qx_1 = 0$ iff $a_3 = a_{17} = a_{19}$, $a_4 = a_{15} = a_{20}$, $a_5 = a_{16} = a_{18}$, $a_6 = a_{23} = a_{25}$, $a_7 = a_{21} = a_{26}$ $a_8 = a_{22} = a_{24}$.

If $k = \sum_{i=0}^{2} a_ix_i + \sum_{i=9}^{14} a_ix_i + d_3(x_3 + x_{17} + x_{19}) + d_4(x_4 + x_{15} + x_{20}) + d_5(x_5 + x_{16} + x_{18}) + d_6(x_6 + x_{23} + x_{25}) + d_7(x_7 + x_{21} + x_{26}) + d_8(x_8 + x_{22} + x_{24})$,
where $\sum_{i=0}^{2} a_i + \sum_{i=9}^{14} a_i + \sum_{j=3}^{8} d_j = 1$ and $d_3 = a_3 = a_{17} = a_{19}$, $d_4 = a_4 = a_{15} = a_{20}$, $d_5 = a_5 = a_{16} = a_{18}$, $d_6 = a_6 = a_{23} = a_{25}$, $d_7 = a_7 = a_{21} = a_{26}$, $d_8 = a_8 = a_{22} = a_{24}$ and $a_i, dj \in \mathbb{F}_{2^k}$ then $x_1k = kx_1$ so,

$$C_V(x_1) = \{k \in V \mid \ x_1k = kx_1\}.$$

Through the same steps we can check that for $y = x_3$, if
$l = a_0x_0 + a_3x_3 + a_6x_6 + a_{11}x_{11} + a_{13}x_{13} + a_{17}x_{17} + a_{19}x_{19} + a_{23}x_{23} + a_{25}x_{25} + d_1(x_1 + x_9 + x_{14}) + d_2(x_2 + x_{10} + x_{12}) + d_4(x_4 + x_{15} + x_{20}) + d_5(x_5 + x_{16} + x_{18}) + d_7(x_7 + x_{21} + x_{26}) + d_8(x_8 + x_{22} + x_{24})$,
where $a_0 + a_3 + a_6 + a_{11} + a_{13} + a_{17} + a_{19} + a_{23} + a_{25} + \sum_{j=1}^{8} d_j = 1$ and $d_1 = a_1 = a_9 = a_{14}$, $d_2 = a_2 = a_{10} = a_{12}$, $d_4 = a_4 = a_{15} = a_{20}$, $d_7 = a_7 = a_{21} = a_{26}$, $d_8 = a_8 = a_{22} = a_{24}$, $d_5 = a_5 = a_{16} = a_{18}$ and $a_i, j \in \mathbb{F}_{2^k}$, for *i=0,3,6,11,13,17,19,23,25 and j=1,2,4,5,7,8*, then $x_3l = lx_3$ so,

$$C_V(x_3) = \{l \in V \mid \ x_3l = lx_3\}.$$

Also it can be calculated easily for $z = x_{11}$, $x_{11}q - qx_{11} = 0$ for all $q \in V$, it implies that

$$C_V(x_{11}) = \{q \in V \mid \ zq = qz\} = V.$$

By using through the above calculations, we can write the relation:

$$Z(V) = \bigcap_{i=1,3,11} C_V(x_i).$$

Proposition 3.1.3.

The center of the *normalized unit group* $V(\mathbb{F}_{2^k}3^{1+2})$ is isomorphic to C_2^{10k}.

Proof: Since $Z(V)$ is contained in $C_V(x_1)$(using 3.1.2). So, it can be defined as:

$$Z(V) = \{k \in C_V(x_1) \mid qk = kq\},$$

where q and k defined in 3.1.2. Now, $Z(V)$ can be find by using through the representation theory. The representation of q in block circulant matrices is defined as:

$$\sigma(q) = \begin{pmatrix} A_1 & A_2 & A_3 & \cdots & A_9 \\ A_{10} & A_{11} & A_{12} & \cdots & A_{18} \\ A_{19} & A_{20} & A_{21} & \cdots & A_{27} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ A_{73} & A_{74} & A_{75} & \cdots & A_{81} \end{pmatrix},$$

where $A_i's$ are 3×3 circulant matrices already defined for $\sigma(p)$ with $i = 1, 2, 3, ..., 81$, but now for $q \in V$. Also the representation of k in block circulant matrices is defined as:

$$\sigma(k) = \begin{pmatrix} B_1 & B_2 & B_3 & \cdots & B_9 \\ B_{10} & B_{11} & B_{12} & \cdots & B_{18} \\ B_{19} & B_{20} & B_{21} & \cdots & B_{27} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ B_{73} & B_{74} & B_{75} & \cdots & B_{81} \end{pmatrix},$$

where
$A_1 = B_1$, $A_4 = B_4$, $A_5 = B_5$, $A_{11} = B_{11}$, $A_{15} = B_{15}$, $A_{16} = B_{16}$, $A_{21} = B_{21}$, $A_{26} = B_{26}$, $A_{27} = B_{27}$, $A_{28} = B_{28}$, $A_{31} = B_{31}$, $A_{32} = B_{32}$, $A_{37} = B_{37}$, $A_{40} = B_{40}$, $A_{41} = B_{41}$, $A_{47} = B_{47}$, $A_{51} = B_{51}$, $A_{52} = B_{52}$, $A_{56} = B_{56}$, $A_{60} = B_{60}$, $A_{61} = B_{61}$, $A_{66} = B_{66}$, $A_{71} = B_{71}$, $A_{72} = B_{72}$, $A_{75} = B_{75}$, $A_{80} = B_{80}$, $A_{81} = B_{81}$,
and
$B_2 = B_{12} = B_{19} = B_{33} = B_{43} = B_{53} = B_{63} = B_{67} = B_{77} = circ(d_3, d_4, d_5)$,
$B_3 = B_{10} = B_{20} = B_{35} = B_{45} = B_{49} = B_{59} = B_{69} = B_{79} = circ(d_6, d_7, d_8)$,
$B_6 = B_{17} = B_{32} = B_{34} = B_{38} = B_{54} = B_{57} = B_{68} = B_{73} = circ(d_4, d_5, d_3)$,
$B_7 = B_{18} = B_{23} = B_{29} = B_{42} = B_{48} = B_{62} = B_{64} = B_{76} = circ(d_5, d_3, d_4)$,

$B_8 = B_{13} = B_{24} = B_{36} = B_{39} = B_{50} = B_{55} = B_{70} = B_{74} = circ(d_7, d_8, d_6)$,
$B_9 = B_{14} = B_{25} = B_{30} = B_{44} = B_{46} = B_{58} = B_{65} = B_{78} = circ(d_8, d_6, d_7)$.

Now $qk = kq$ iff $\sigma(q)\sigma(k) = \sigma(k)\sigma(q)$.
Therefore:

$$\sigma(q)\sigma(k)-\sigma(k)\sigma(q) = \begin{pmatrix} A_1 & A_2 & A_3 & \cdots & A_9 \\ A_{10} & A_{11} & A_{12} & \cdots & A_{18} \\ A_{19} & A_{20} & A_{21} & \cdots & A_{27} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ A_{73} & A_{74} & A_{75} & \cdots & A_{81} \end{pmatrix} \begin{pmatrix} B_1 & B_2 & B_3 & \cdots & B_9 \\ B_{10} & B_{11} & B_{12} & \cdots & B_{18} \\ B_{19} & B_{20} & B_{21} & \cdots & B_{27} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ B_{73} & B_{74} & B_{75} & \cdots & B_{81} \end{pmatrix} -$$
$$\begin{pmatrix} B_1 & B_2 & B_3 & \cdots & B_9 \\ B_{10} & B_{11} & B_{12} & \cdots & B_{18} \\ B_{19} & B_{20} & B_{21} & \cdots & B_{27} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ B_{73} & B_{74} & B_{75} & \cdots & B_{81} \end{pmatrix} \begin{pmatrix} A_1 & A_2 & A_3 & \cdots & A_9 \\ A_{10} & A_{11} & A_{12} & \cdots & A_{18} \\ A_{19} & A_{20} & A_{21} & \cdots & A_{27} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ A_{73} & A_{74} & A_{75} & \cdots & A_{81} \end{pmatrix},$$

$\sigma(q)\sigma(k) - \sigma(k)\sigma(q) = 0$ iff $a_{11} = a_{13}$, $a_1 = a_9 = a_{14}$ and $a_2 = a_{10} = a_{12}$. Thus each element of $Z(V)$ takes the form $a_0x_0 + s(x_{11} + x_{13}) + t_1(x_1 + x_9 + x_{14}) + t_2(x_2 + x_{10} + x_{12}) + t_3(x_3 + x_{17} + x_{19}) + t_4(x_4 + x_{15} + x_{20}) + t_5(x_5 + x_{16} + x_{18}) + t_6(x_6 + x_{23} + x_{25}) + t_7(x_7 + x_{21} + x_{26}) + t_8(x_8 + x_{22} + x_{24})$,

where $a_0 + \sum_{l=1}^{8} t_l = 1$ or $a_0 = 1 + \acute{t}$, (where $\sum_{l=1}^{8} t_l = \acute{t}$) $s = a_{11} = a_{13}$ and $t_1 = a_1 = a_9 = a_{14}$, $t_2 = a_2 = a_{10} = a_{12}$, $t_3 = a_3 = a_{17} = a_{19}$, $t_4 = a_4 = a_{15} = a_{20}$, $t_5 = a_5 = a_{16} = a_{18}$, $t_6 = a_6 = a_{23} = a_{25}$, $t_7 = a_7 = a_{21} = a_{26}$, $t_8 = a_8 = a_{22} = a_{24}$.
The order of $Z(V)$ is 2^{10k}. So, we can write:

$$Z(V(\mathbb{F}_{2^k}3^{1+2})) \cong C_2^{10k}.$$

Proposition 3.1.4.

The center of the *unitary unit group* $V_*(\mathbb{F}_{2^k}3^{1+2})$ is isomorphic to C_2^{6k-1}.

Proof 1: First we prove this proposition without using representation theory. Let $\alpha = (1 + \acute{t})x_0 + s(x_{11} + x_{13}) + t_1(x_1 + x_9 + x_{14}) + t_2(x_2 + x_{10} + x_{12}) + t_3(x_3 + x_{17} + x_{19}) + t_4(x_4 + x_{15} + x_{20}) + t_5(x_5 + x_{16} + x_{18}) + t_6(x_6 + x_{23} + x_{25}) + t_7(x_7 + x_{21} + x_{26}) + t_8(x_8 + x_{22} + x_{24}) \in Z(V(\mathbb{F}_{2^k}3^{1+2})$.
where $\sum_{l=1}^{8} t_l = \acute{t}$, $s = a_{11} = a_{13}$ and $t_1 = a_1 = a_9 = a_{14}$, $t_2 = a_2 = a_{10} = a_{12}$, $t_3 = a_3 = a_{17} = a_{19}$, $t_4 = a_4 = a_{15} = a_{20}$, $t_5 = a_5 = a_{16} = a_{18}$, $t_6 = a_6 = a_{23} = a_{25}$, $t_7 = a_7 = a_{21} = a_{26}$, $t_8 = a_8 = a_{22} = a_{24}$.

We have to find $Z(V_*(\mathbb{F}_{2^k}3^{1+2})$ in $Z(V(\mathbb{F}_{2^k}3^{1+2})$, so $\alpha^* = \alpha^{-1}$ or $\alpha\alpha^* = 1$. Now,

$$\begin{aligned} \alpha\alpha^* = [(1+\acute{t})x_0 + s(x_{11}+x_{13}) + t_1(x_1+x_9+x_{14}) + t_2(x_2+x_{10}+x_{12}) \\ + t_3(x_3+x_{17}+x_{19}) + t_4(x_4+x_{15}+x_{20}) + t_5(x_5+x_{16}+x_{18}) \\ + t_6(x_6+x_{23}+x_{25}) + t_7(x_7+x_{21}+x_{26}) + t_8(x_8+x_{22}+x_{24})]\alpha^*. \end{aligned} \tag{3.2}$$

Where,
$\alpha^* = (1+\acute{t})x_0 + s(x_{11}+x_{13}) + t_2(x_1+x_9+x_{14}) + t_1(x_2+x_{10}+x_{12}) + t_6(x_3+x_{17}+x_{19}) + t_8(x_4+x_{15}+x_{20}) + t_7(x_5+x_{16}+x_{18}) + t_3(x_6+x_{23}+x_{25}) + t_5(x_7+x_{21}+x_{26}) + t_4(x_8+x_{22}+x_{24})$.

Let $\delta_1 = (1+\acute{t})x_0\alpha^* = (1+\acute{t})^2x_0 + (1+\acute{t})s(x_{11}+x_{13}) + (1+\acute{t})t_2(x_1+x_9+x_{14}) + (1+\acute{t})t_1(x_2+x_{10}+x_{12}) + (1+\acute{t})t_6(x_3+x_{17}+x_{19}) + (1+\acute{t})t_8(x_4+x_{15}+x_{20}) + (1+\acute{t})t_7(x_5+x_{16}+x_{18}) + (1+\acute{t})t_3(x_6+x_{23}+x_{25}) + (1+\acute{t})t_5(x_7+x_{21}+x_{26}) + (1+\acute{t})t_4(x_8+x_{22}+x_{24})$,

$\delta_2 = s(x_{11}+x_{13})\alpha^* = (1+\acute{t})s(x_{11}+x_{13}) + s^2(x_{11}+x_{13}) + st_2(0) + st_1(0) + st_6(0) + st_8(0) + t_7(x_5+x_{16}+x_{18}) + st_3(0) + st_5(0) + st_4(0)$,

$\delta_3 = t_1(x_1+x_9+x_{14})\alpha^* = (1+\acute{t})t_1(x_1+x_9+x_{14}) + st_1(0) + t_1t_2(x_2+x_{10}+x_{12}) + t_1^2(x_0+x_{11}+x_{13}) + t_1t_6(x_4+x_{15}+x_{20}) + t_1t_8(x_5+x_{16}+x_{18}) + t_1t_7(x_3+x_{17}+x_{19}) + t_1t_3(x_7+x_{21}+x_{26}) + t_1t_5(x_8+x_{22}+x_{24}) + t_1t_4(x_6+x_{23}+x_{25})$,

$\delta_4 = t_2(x_2+x_{10}+x_{12})\alpha^* = (1+\acute{t})t_2(x_2+x_{10}+x_{12}) + st_2(0) + t_2^2(x_0+x_{11}+x_{13}) + t_2t_1(x_1+x_9+x_{14}) + t_2t_6(x_5+x_{16}+x_{18}) + t_2t_8(x_3+x_{17}+x_{19}) + t_2t_7(x_4+x_{15}+x_{20}) + t_2t_3(x_8+x_{22}+x_{24}) + t_2t_5(x_6+x_{23}+x_{25}) + t_2t_4(x_7+x_{21}+x_{26})$,

$\delta_5 = t_3(x_3+x_{17}+x_{19})\alpha^* = (1+\acute{t})t_3(x_3+x_{17}+x_{19}) + st_3(0) + t_3t_2(x_4+x_{15}+x_{20}) + t_3t_1(x_5+x_{16}+x_{18}) + t_3t_6(x_6+x_{23}+x_{25}) + t_3t_8(x_7+x_{21}+x_{26}) + t_3t_7(x_8+x_{22}+x_{24}) + t_3^2(x_0+x_{11}+x_{13}) + t_3t_5(x_1+x_9+x_{14}) + t_3t_4(x_2+x_{10}+x_{12})$,

$\delta_6 = t_4(x_4+x_{15}+x_{20})\alpha^* = (1+\acute{t})t_4(x_4+x_{15}+x_{20}) + st_4(0) + t_4t_2(x_5+x_{16}+x_{18}) + t_4t_1(x_3+x_{17}+x_{19}) + t_4t_6(x_7+x_{21}+x_{26}) + t_4t_8(x_8+x_{22}+x_{24}) + t_4t_7(x_6+x_{23}+x_{25}) + t_4t_3(x_1+x_9+x_{14}) + t_4t_5(x_2+x_{10}+x_{12}) + t_4^2(x_0+x_{11}+x_{13})$,

$\delta_7 = t_5(x_5+x_{16}+x_{18})\alpha^* = (1+\acute{t})t_5(x_5+x_{16}+x_{18}) + st_5(0) + t_5t_2(x_3+x_{17}+x_{19}) + t_5t_1(x_4+x_{15}+x_{20}) + t_5t_6(x_8+x_{22}+x_{24}) + t_5t_8(x_6+x_{23}+x_{25}) + t_5t_7(x_7+x_{21}+x_{26}) + t_5t_3(x_2+x_{10}+x_{12}) + t_5^2(x_0+x_{11}+x_{13}) + t_5t_4(x_1+x_9+x_{14})$,

$\delta_8 = t_6(x_6 + x_{23} + x_{25})\alpha^* = (1 + \acute{t})t_6(x_6 + x_{23} + x_{25}) + st_6(0) + t_6t_2(x_7 + x_{21} + x_{26}) + t_6t_1(x_8 + x_{22} + x_{24}) + t_6^2(x_0 + x_{11} + x_{13}) + t_6t_8(x_1 + x_9 + x_{14}) + t_6t_7(x_2+x_{10}+x_{12})+t_6t_3(x_3+x_{17}+x_{19})+t_6t_5(x_4+x_{15}+x_{20})+t_6t_4(x_5+x_{16}+x_{18})$,

$\delta_9 = t_7(x_7 + x_{21} + x_{26})\alpha^* = (1 + \acute{t})t_7(x_7 + x_{21} + x_{26}) + st_7(0) + t_7t_2(x_8 + x_{22} + x_{24}) + t_7t_1(x_6 + x_{23} + x_{25}) + t_7t_6(x_1 + x_9 + x_{14}) + t_7t_8(x_2 + x_{10} + x_{12}) + t_7^2(x_0+x_{11}+x_{13})+t_7t_3(x_4+x_{15}+x_{20})+t_7t_5(x_5+x_{16}+x_{18})+t_7t_4(x_3+x_{17}+x_{19})$,

$\delta_{10} = t_8(x_8 + x_{22} + x_{24})\alpha^* = (1 + \acute{t})t_8(x_8 + x_{22} + x_{24}) + st_8(0) + t_8t_2(x_6 + x_{23} + x_{25}) + t_8t_1(x_7 + x_{21} + x_{26}) + t_8t_6(x_2 + x_{10} + x_{12}) + t_8^2(x_0 + x_{11} + x_{13}) + t_8t_7(x_1+x_9+x_{14})+t_8t_3(x_5+x_{16}+x_{18})+t_8t_5(x_3+x_{17}+x_{19})+t_8t_4(x_4+x_{15}+x_{20})$.

Now equation (3.2) becomes:

$$\begin{aligned}
\alpha\alpha^* &= \sum_{i=1}^{10} \delta_i \\
&= (1+\acute{t}^2)x_0 + s^2(x_{11}+x_{13}) \\
&\quad + (t_1^2+t_2^2+t_3^2+t_4^2+t_5^2+t_6^2+t_7^2+t_8^2)(x_0+x_{11}+x_{13}) + ((1+\acute{t})(t_1+t_2) \\
&\quad + t_1t_2+t_3t_4+t_3t_5+t_4t_5+t_3t_6+t_6t_8+t_7t_8)(x_1+x_9+x_{14}+x_2+x_{10}+x_{12}) \\
&\quad + ((1+\acute{t})(t_3+t_6)+t_3t_6+t_1t_4+t_1t_7+t_2t_5+t_2t_8+t_4t_7+t_5t_8)(x_3+x_{17} \\
&\quad + x_{19}+x_6+x_{23}+x_{25}) + ((1+\acute{t})(t_4+t_8)+t_4t_8+t_1t_5+t_2t_3+t_2t_7+t_3t_7 \\
&\quad + t_1t_6+t_5t_6)(x_4+x_{15}+x_{20}+x_8+x_{22}+x_{24}) + ((1+\acute{t})(t_5+t_7)+t_5t_7 \\
&\quad + t_1t_3+t_1t_8+t_2t_4+t_2t_6+t_3t_8+t_4t_6)(x_5+x_{16}+x_{18}+x_7+x_{21}+x_{26}).
\end{aligned} \tag{3.3}$$

$\alpha\alpha^* = 1$ iff $t_1 = t_2$, $t_3 = t_6$, $t_4 = t_8$, $t_5 = t_7$, and $\acute{t}^2 = s^2 = t_1^2 = t_3^2 = t_4^2 = t_5^2 \equiv 0$ (under the characteristic of field $\mathbb{F}_{2^k}$). It implies that $t_1, t_3, t_4, t_5, s, \acute{t} \in \{a \in \mathbb{F}_{2^k} \mid a \textit{ is divisible by } 2\}$.
Hence if $\gamma = (1+\acute{t}^2)x_0 + s^2(x_{11}+x_{13}) + u_1(x_1+x_9+x_{14}+x_2+x_{10}+x_{12}) + u_3(x_3+x_{17}+x_{19}+x_6+x_{23}+x_{25})+u_4(x_4+x_{15}+x_{20}+x_8+x_{22}+x_{24})+u_5(x_5+x_{16}+x_{18}+x_7+x_{21}+x_{26})$, where $u_1 = t_1 = t_2$, $u_3 = t_3 = t_6$, $u_4 = t_4 = t_8$, $u_5 = t_5 = t_7$, and $\acute{t}$, s, $u_j \in \{a \in \mathbb{F}_{2^k} \mid a \textit{ is divisible by } 2\}$ for $j = 1, 3, 4, 5$ then $\gamma \in Z(V_*(\mathbb{F}_{2^k}3^{1+2})$. The order of $Z(V_*(\mathbb{F}_{2^k}3^{1+2})$ is 2^{6k-1}. It shows that

$$Z(V_*(\mathbb{F}_{2^k}3^{1+2}) \cong C_2^{6k-1}.$$

Proof 2: Now we prove this proposition by using representation theory. Let $\alpha = (1+\acute{t})x_0 + s(x_{11}+x_{13}) + t_1(x_1+x_9+x_{14}) + t_2(x_2+x_{10}+x_{12}) + t_3(x_3+x_{17}+x_{19}) + t_4(x_4+x_{15}+x_{20}) + t_5(x_5+x_{16}+x_{18}) + t_6(x_6+x_{23}+x_{25}) + t_7(x_7+x_{21}+x_{26}) + t_8(x_8+x_{22}+x_{24}) \in Z(V(\mathbb{F}_{2^k}3^{1+2})$. We have to find $Z(V_*(\mathbb{F}_{2^k}3^{1+2})$

in $Z(V(\mathbb{F}_{2^k}3^{1+2})$, so $\alpha^* = \alpha^{-1}$ iff $\sigma(\alpha^*) = \sigma(\alpha^{-1})$ iff $(\sigma(\alpha))^T = (\sigma(\alpha))^{-1}$ iff $(\sigma(\alpha))(\sigma(\alpha))^T = I$. To prove this identity we have to first define $\sigma(\alpha)$, so

$$\sigma(\alpha) = \begin{pmatrix} C_1 & C_2 & C_3 & C_4 & C_5 & C_6 & C_7 & C_8 & C_9 \\ C_3 & C_1 & C_2 & C_8 & C_9 & C_4 & C_5 & C_6 & C_7 \\ C_2 & C_3 & C_1 & C_6 & C_7 & C_8 & C_9 & C_4 & C_5 \\ C_5 & C_7 & C_9 & C_1 & C_4 & C_2 & C_6 & C_3 & C_8 \\ C_4 & C_6 & C_8 & C_5 & C_1 & C_7 & C_2 & C_9 & C_3 \\ C_9 & C_5 & C_7 & C_3 & C_8 & C_1 & C_4 & C_2 & C_6 \\ C_8 & C_4 & C_6 & C_9 & C_3 & C_5 & C_1 & C_7 & C_2 \\ C_7 & C_9 & C_5 & C_2 & C_6 & C_3 & C_8 & C_1 & C_4 \\ C_6 & C_8 & C_4 & C_7 & C_2 & C_9 & C_3 & C_5 & C_1 \end{pmatrix}.$$

Where,
$C_1 = circ(a_0, t_1, t_2)$, $C_2 = circ(t_3, t_4, t_5)$, $C_3 = circ(t_6, t_7, t_8)$, $C_4 = circ(t_1, t_2, s)$
$C_5 = circ(t_2, s, t_1)$, $C_6 = circ(t_4, t_5, t_3)$ $C_7 = circ(t_5, t_3, t_4)$, $C_8 = circ(t_7, t_8, t_6)$
$C_9 = circ(t_8, t_6, t_7)$

Now,

$$(\sigma(\alpha))(\sigma(\alpha))^T = \begin{pmatrix} C_1 & C_2 & C_3 & C_4 & C_5 & C_6 & C_7 & C_8 & C_9 \\ C_3 & C_1 & C_2 & C_8 & C_9 & C_4 & C_5 & C_6 & C_7 \\ C_2 & C_3 & C_1 & C_6 & C_7 & C_8 & C_9 & C_4 & C_5 \\ C_5 & C_7 & C_9 & C_1 & C_4 & C_2 & C_6 & C_3 & C_8 \\ C_4 & C_6 & C_8 & C_5 & C_1 & C_7 & C_2 & C_9 & C_3 \\ C_9 & C_5 & C_7 & C_3 & C_8 & C_1 & C_4 & C_2 & C_6 \\ C_8 & C_4 & C_6 & C_9 & C_3 & C_5 & C_1 & C_7 & C_2 \\ C_7 & C_9 & C_5 & C_2 & C_6 & C_3 & C_8 & C_1 & C_4 \\ C_6 & C_8 & C_4 & C_7 & C_2 & C_9 & C_3 & C_5 & C_1 \end{pmatrix} \begin{pmatrix} C_1 & C_2 & C_3 & C_4 & C_5 & C_6 & C_7 & C_8 & C_9 \\ C_3 & C_1 & C_2 & C_8 & C_9 & C_4 & C_5 & C_6 & C_7 \\ C_2 & C_3 & C_1 & C_6 & C_7 & C_8 & C_9 & C_4 & C_5 \\ C_5 & C_7 & C_9 & C_1 & C_4 & C_2 & C_6 & C_3 & C_8 \\ C_4 & C_6 & C_8 & C_5 & C_1 & C_7 & C_2 & C_9 & C_3 \\ C_9 & C_5 & C_7 & C_3 & C_8 & C_1 & C_4 & C_2 & C_6 \\ C_8 & C_4 & C_6 & C_9 & C_3 & C_5 & C_1 & C_7 & C_2 \\ C_7 & C_9 & C_5 & C_2 & C_6 & C_3 & C_8 & C_1 & C_4 \\ C_6 & C_8 & C_4 & C_7 & C_2 & C_9 & C_3 & C_5 & C_1 \end{pmatrix}^T$$

$$= \begin{pmatrix} I_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & I_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & I_3 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & I_3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & I_3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & I_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & I_3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & I_3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & I_3 \end{pmatrix}.$$

In above matrix non-diagonal entries $\sum C_i C_j = 0$ for $i \neq j$ and $i, j = 1, 2, 3, ..., 9$ in each row and the diagonal entries $\sum_{i=1}^{9} C_i^2 = I_3$ iff same conditions

hold which are obtained in Proof 1, i.e. $t_1 = t_2$, $t_3 = t_6$, $t_4 = t_8$ and $\acute{t}^2 = s^2 = t_1^2 = t_3^2 = t_4^2 = t_5^2 \equiv 0$ (under the characteristic of field $\mathbb{F}_{2^k}$). It implies that $t_1, t_3, t_4, t_5, s, \acute{t} \in \{a \in \mathbb{F}_{2^k} \mid a \textit{ is divisible by } 2\}$. Thus the structure of α reduces to
$\gamma = (1+\acute{t})x_0 + s(x_{11}+x_{13}) + u_1(x_1+x_9+x_{14}+x_2+x_{10}+x_{12}) + u_3(x_3 + x_{17}+x_{19}+x_6+x_{23}+x_{15}) + u_4(x_4+x_{15}+x_{20}+x_8+x_{22}+x_{24}) + u_5(x_5 + x_{16}+x_{18}+x_7+x_{21}+x_{26})$, where $u_1 = t_1 = t_2$, $u_3 = t_3 = t_6$, $u_4 = t_4 = t_8$, $u_5 = t_5 = t_7$. Now $\alpha \in Z(V_*(\mathbb{F}_{2^k}3^{1+2}))$ and $|Z(V_*(\mathbb{F}_{2^k}3^{1+2}))| = 2^{6k-1}$. Hence,

$$Z(V_*(\mathbb{F}_{2^k}3^{1+2})) \cong C_2^{6k-1}.$$

3.2 Structure of $V_*(\mathbb{F}_{2^k}3^{1+2})$

Proposition 3.2.1.

The order of $V_*(\mathbb{F}_{2^k}3^{1+2})$ is 2^{13k}. On the basis of this computation it is easy to see that:

$$V_*(\mathbb{F}_{2^k}3^{1+2}) \cong C_2^{13k}.$$

Remark 3: The above computation was performed by using the GAP package Laguna.

Chapter 4

The Computational Aspects of Structure of The Group Algebra $\mathbb{F}_{2^k}3^{1+2}$

In this chapter some computational aspects of the structure of the group algebra $\mathbb{F}_{2^k}3^{1+2}$ are introduced by using through GAP. These computations are very useful in understanding the structure and proving some important results of $\mathbb{F}_{2^k}3^{1+2}$.

The permutation representation of the generators of 3^{1+2} can be obtained by first computing the permutation representation ρ of $G = \langle x, y, z : x^3 = y^3 = 1 = z^3 = [x,z] = [y,z],\ z = [x,y]\rangle$ and then evaluating the pre-images of the homomorphism between the two groups. The GAP commands `AsPermutation` and `GroupHomomorphismByImages` were used for this task. The details are given below:

$$x \mapsto (2,6,7)(3,8,9),$$

$$y \mapsto (1,2,3)(4,7,8)(5,6,9),$$
$$z \mapsto (1,4,5)(2,7,6)(3,8,9).$$

4.1 Center of the group algebra $\mathbb{F}_{2^k}3^{1+2}$

To find the center of $\mathbb{F}_{2^k}3^{1+2}$, we need the character table of 3^{1+2}. The GAP out put of the Character table of 3^{1+2} is given below:

```
           3  3  2  2  2  2  2  2  2  2  3  3

             1a 3a 3b 3c 3d 3e 3f 3g 3h 3i 3j
          2P 1a 3b 3a 3f 3h 3g 3c 3e 3d 3j 3i
          3P 1a 1a 1a 1a 1a 1a 1a 1a 1a 1a 1a

X.1           1  1  1  1  1  1  1  1  1  1  1
X.2           1  1  1  A  A  A /A /A /A  1  1
X.3           1  1  1 /A /A /A  A  A  A  1  1
X.4           1  A /A  1 /A  A  1 /A  A  1  1
X.5           1 /A  A  1  A /A  1  A /A  1  1
X.6           1  A /A  A  1 /A /A  A  1  1  1
X.7           1 /A  A /A  1  A  A /A  1  1  1
X.8           1  A /A /A  A  1  A  1 /A  1  1
X.9           1 /A  A  A /A  1 /A  1  A  1  1
X.10          3  .  .  .  .  .  .  .  .  B /B
X.11          3  .  .  .  .  .  .  .  . /B  B

A = E(3)^2
  = (-1-Sqrt(-3))/2 = -1-b3
B = 3*E(3)^2
  = (-3-3*Sqrt(-3))/2 = -3-3b3
```

The conjugacy classes of 3^{1+2} are:

- $\mathcal{C}_0 = \{1\}$.
- $\mathcal{C}_1 = \{(2,6,7)(3,8,9),(1,4,5)(3,9,8),(1,5,4)(2,7,6)\}$.
- $\mathcal{C}_2 = \{(2,7,6)(3,9,8),(1,4,5)(2,6,7),(1,5,4)(3,8,9)\}$.
- $\mathcal{C}_3 = \{(1,2,3)(4,7,8)(5,6,9),(1,6,8)(2,9,4)(3,5,7),(1,7,9)(2,8,5)(3,4,6)\}$.
- $\mathcal{C}_4 = \{(1,4,5)(2,7,6)(3,8,9)\}$.
- $\mathcal{C}_5 = \{(1,2,8)(3,5,6)(4,7,9),(1,6,9)(2,3,4)(5,7,8),(1,7,3)(2,9,5)(4,6,8)\}$.
- $\mathcal{C}_6 = \{(1,2,9)(3,4,7)(5,6,8),(1,6,3)(2,8,4)(5,7,9),(1,7,8)(2,3,5)(4,6,9)\}$.
- $\mathcal{C}_7 = \{(1,3,2)(4,8,7)(5,9,6),(1,8,6)(2,4,9)(3,7,5),(1,9,7)(2,5,8)(3,6,4)\}$.
- $\mathcal{C}_8 = \{(1,3,6)(2,4,8)(5,9,7),(1,8,7)(2,5,3)(4,9,6),(1,9,2)(3,7,4)(5,8,6)\}$.
- $\mathcal{C}_9 = \{(1,3,7)(2,5,9)(4,8,6),(1,8,2)(3,6,5)(4,9,7),(1,9,6)(2,4,3)(5,8,7)\}$.
- $\mathcal{C}_{10} = \{(1,5,4)(2,6,7)(3,9,8)\}$.

Now it is well known that class sums $\bar{C}_i = \sum\limits_{g \in C_i} g \in \mathbb{F}G$ form the basis of $Z(\mathbb{F}G)$.

Thus the basis for $Z(\mathbb{F}_{2^k}3^{1+2})$ is:

$\mathcal{B} = \{1, (2,6,7)(3,8,9) + (1,4,5)(3,9,8) + (1,5,4)(2,7,6), (2,7,6)(3,9,8) +(1,4,5)(2,6,7)+(1,5,4)(3,8,9), (1,2,3)(4,7,8)(5,6,9)+(1,6,8)(2,9,4)(3,5,7) +(1,7,9)(2,8,5)(3,4,6), (1,4,5)(2,7,6)(3,8,9), (1,2,8)(3,5,6)(4,7,9)+(1,6,9)(2,3,4)(5,7,8)+(1,7,3)(2,9,5)(4,6,8), (1,2,9)(3,4,7)(5,6,8)+(1,6,3)(2,8,4)(5,7,9) + (1,7,8)(2,3,5)(4,6,9), (1,3,2)(4,8,7)(5,9,6) + (1,8,6)(2,4,9)(3,7,5)+(1,9,7)(2,5,8)(3,6,4), (1,3,6)(2,4,8)(5,9,7)+(1,8,7)(2,5,3)(4,9,6) +(1,9,2)(3,7,4)(5,8,6), (1,3,7)(2,5,9)(4,8,6)+(1,8,2)(3,6,5)(4,9,7)+(1,9,6)(2,4,3)(5,8,7), (1,5,4)(2,6,7)(3,9,8)\}$.

Clearly, $|Z(\mathbb{F}_{2^k}3^{1+2})| = 2^{11k}$. We can now search inside $Z(\mathbb{F}_{2^k}3^{1+2})$ to find $Z(V_*(\mathbb{F}_{2^k}3^{1+2}))$ by making use of the GAP package Laguna.
Our main goal is to find Unitary units of $\mathbb{F}_{2^k}3^{1+2}$ but we can also obtain some information about the Unit group of $\mathbb{F}_{2^k}3^{1+2}$ by looking into the sub algebra of $\mathbb{F}_{2^k}3^{1+2}$. To accomplish this, we need composition series of 3^{1+2}.

$$1 = \mathcal{H}_0 \lhd \mathcal{H}_1 \lhd \mathcal{H}_2 \lhd \mathcal{H}_3 = 3^{1+2},$$

where $\mathcal{H}_1 = \langle (2,6,7)(3,8,9)\rangle$ and $\mathcal{H}_2 = \langle (1,4,5)(3,9,8), (2,6,7)(3,8,9)\rangle = 3^2$. We note that $\mathcal{H}_2$ is an elementary abelian group of order 9.
It is clear that the unit group of the sub algebra $\mathbb{F}_{2^k}3^2$ is a subgroup of the group algebra $\mathbb{F}_{2^k}3^{1+2}$.
We use GAP to find out that the Unit group of $\mathbb{F}_{2^k}3^2$ is an elementary abelian group of order 3^4. The minimal generating set for this 3^4 is:
$\{(1,5,4)(2,7,6), (1,5,4)(2,6,7)(3,9,8),$
$(1,4,5)(2,6,7)+(1,4,5)(2,7,6)(3,8,9)+(1,5,4)(3,8,9)+(1,5,4)(2,6,7)(3,9,8) + (1,5,4)(2,7,6), (2,7,6)(3,9,8) + (1,4,5)(2,6,7) + (1,4,5)(2,7,6)(3,8,9) + (1,5,4)(2,6,7)(3,9,8) + (1,5,4)(2,7,6)\}$.
The generating set of unit group of $\mathbb{F}_{2^k}3^2$ is contained in the generating set of unit group of $\mathbb{F}_{2^k}3^{1+2}$.

4.2 Minimal Ideals of $\mathbb{F}_{2^k}3^{1+2}$

An ideal M of group algebra FG is said to be minimal if every ideal I contain in M coincides with M. The minimal ideals are the simple Wedderburn components of the group algebra, corresponding to the absolutely irreducible characters of the group, and their dimensions are the squares of the character degrees.
A linear character is a homomorphism is defined on group G over field F to the multiplicative group of a field. Since there are 6 linear characters of the

underlying group 3^{1+2}, it follows that here are 6 minimal ideals of the group algebra $\mathbb{F}_{2^k}3^{1+2}$. We now give the basis of these ideals.

4.2.1 Ideal of dimension 1 of $\mathbb{F}_{2^k}3^{1+2}$

Basis:
$Id(G)+$
$(1,19,10)(2,20,11)(3,21,12)(4,22,13)(5,23,14)(6,24,15)(7,25,16)(8,26,17)(9,27,18)$
$+(1,7,4)(2,8,5)(3,9,6)(10,18,14)(11,16,15)(12,17,13)(19,26,24)(20,27,22)(21,25,23)$
$+(1,3,2)(4,6,5)(7,9,8)(10,12,11)(13,15,14)(16,18,17)(19,21,20)(22,24,23)(25,27,26)$
$+(1,10,19)(2,11,20)(3,12,21)(4,13,22)(5,14,23)(6,15,24)(7,16,25)(8,17,26)(9,18,27)$
$+(1,4,7)(2,5,8)(3,6,9)(10,14,18)(11,15,16)(12,13,17)(19,24,26)(20,22,27)(21,23,25)$
$+(1,2,3)(4,5,6)(7,8,9)(10,11,12)(13,14,15)(16,17,18)(19,20,21)(22,23,24)(25,26,27)$
$+(1,26,13)(2,27,14)(3,25,15)(4,20,16)(5,21,17)(6,19,18)(7,23,10)(8,24,11)(9,22,12)$
$+(1,21,11)(2,19,12)(3,20,10)(4,24,14)(5,22,15)(6,23,13)(7,27,17)(8,25,18)(9,26,16)$
$+(1,24,16)(2,22,17)(3,23,18)(4,27,10)(5,25,11)(6,26,12)(7,21,13)(8,19,14)(9,20,15)$
$+(1,20,12)(2,21,10)(3,19,11)(4,23,15)(5,24,13)(6,22,14)(7,26,18)(8,27,16)(9,25,17)$
$+(1,25,14)(2,26,15)(3,27,13)(4,19,17)(5,20,18)(6,21,16)(7,22,11)(8,23,12)(9,24,10)$
$+(1,9,5)(2,7,6)(3,8,4)(10,17,15)(11,18,13)(12,16,14)(19,25,22)(20,26,23)(21,27,24)$
$+(1,16,24)(2,17,22)(3,18,23)(4,10,27)(5,11,25)(6,12,26)(7,13,21)(8,14,19)(9,15,20)$
$+(1,8,6)(2,9,4)(3,7,5)(10,16,13)(11,17,14)(12,18,15)(19,27,23)(20,25,24)(21,26,22)$
$+(1,12,20)(2,10,21)(3,11,19)(4,15,23)(5,13,24)(6,14,22)(7,18,26)(8,16,27)(9,17,25)$
$+(1,6,8)(2,4,9)(3,5,7)(10,13,16)(11,14,17)(12,15,18)(19,23,27)(20,24,25)(21,22,26)$
$+(1,18,22)(2,16,23)(3,17,24)(4,12,25)(5,10,26)(6,11,27)(7,15,19)(8,13,20)(9,14,21)$
$+(1,14,25)(2,15,26)(3,13,27)(4,17,19)(5,18,20)(6,16,21)(7,11,22)(8,12,23)(9,10,24)$
$+(1,11,21)(2,12,19)(3,10,20)(4,14,24)(5,15,22)(6,13,23)(7,17,27)(8,18,25)(9,16,26)$
$+(1,22,18)(2,23,16)(3,24,17)(4,25,12)(5,26,10)(6,27,11)(7,19,15)(8,20,13)(9,21,14)$
$+(1,13,26)(2,14,27)(3,15,25)(4,16,20)(5,17,21)(6,18,19)(7,10,23)(8,11,24)(9,12,22)$
$+(1,5,9)(2,6,7)(3,4,8)(10,15,17)(11,13,18)(12,14,16)(19,22,25)(20,23,26)(21,24,27)$
$+(1,17,23)(2,18,24)(3,16,22)(4,11,26)(5,12,27)(6,10,25)(7,14,20)(8,15,21)(9,13,19)$
$+(1,27,15)(2,25,13)(3,26,14)(4,21,18)(5,19,16)(6,20,17)(7,24,12)(8,22,10)(9,23,11)$
$+(1,23,17)(2,24,18)(3,22,16)(4,26,11)(5,27,12)(6,25,10)(7,20,14)(8,21,15)(9,19,13)$
$+(1,15,27)(2,13,25)(3,14,26)(4,18,21)(5,16,19)(6,17,20)(7,12,24)(8,10,22)(9,11,23)$

4.2.2 Ideal of dimension 2 of $\mathbb{F}_{2^k}3^{1+2}$

Basis:
$Id(G)+$
$(1,7,4)(2,8,5)(3,9,6)(10,18,14)(11,16,15)(12,17,13)(19,26,24)(20,27,22)(21,25,23)$
$+(1,3,2)(4,6,5)(7,9,8)(10,12,11)(13,15,14)(16,18,17)(19,21,20)(22,24,23)(25,27,26)$

$+(1,10,19)(2,11,20)(3,12,21)(4,13,22)(5,14,23)(6,15,24)(7,16,25)(8,17,26)(9,18,27)$
$+(1,4,7)(2,5,8)(3,6,9)(10,14,18)(11,15,16)(12,13,17)(19,24,26)(20,22,27)(21,23,25)$
$+(1,2,3)(4,5,6)(7,8,9)(10,11,12)(13,14,15)(16,17,18)(19,20,21)(22,23,24)(25,26,27)$
$+(1,9,5)(2,7,6)(3,8,4)(10,17,15)(11,18,13)(12,16,14)(19,25,22)(20,26,23)(21,27,24)$
$+(1,16,24)(2,17,22)(3,18,23)(4,10,27)(5,11,25)(6,12,26)(7,13,21)(8,14,19)(9,15,20)$
$+(1,8,6)(2,9,4)(3,7,5)(10,16,13)(11,17,14)(12,18,15)(19,27,23)(20,25,24)(21,26,22)$
$+(1,12,20)(2,10,21)(3,11,19)(4,15,23)(5,13,24)(6,14,22)(7,18,26)(8,16,27)(9,17,25)$
$+(1,6,8)(2,4,9)(3,5,7)(10,13,16)(11,14,17)(12,15,18)(19,23,27)(20,24,25)(21,22,26)$
$+(1,18,22)(2,16,23)(3,17,24)(4,12,25)(5,10,26)(6,11,27)(7,15,19)(8,13,20)(9,14,21)$
$+(1,14,25)(2,15,26)(3,13,27)(4,17,19)(5,18,20)(6,16,21)(7,11,22)(8,12,23)(9,10,24)$
$+(1,11,21)(2,12,19)(3,10,20)(4,14,24)(5,15,22)(6,13,23)(7,17,27)(8,18,25)(9,16,26)$
$+(1,13,26)(2,14,27)(3,15,25)(4,16,20)(5,17,21)(6,18,19)(7,10,23)(8,11,24)(9,12,22)$
$+(1,5,9)(2,6,7)(3,4,8)(10,15,17)(11,13,18)(12,14,16)(19,22,25)(20,23,26)(21,24,27)$
$+(1,17,23)(2,18,24)(3,16,22)(4,11,26)(5,12,27)(6,10,25)(7,14,20)(8,15,21)(9,13,19)$
$+(1,15,27)(2,13,25)(3,14,26)(4,18,21)(5,16,19)(6,17,20)(7,12,24)(8,10,22)(9,11,23)$
$(1,19,10)(2,20,11)(3,21,12)(4,22,13)(5,23,14)(6,24,15)(7,25,16)(8,26,17)(9,27,18)$
$+(1,10,19)(2,11,20)(3,12,21)(4,13,22)(5,14,23)(6,15,24)(7,16,25)(8,17,26)(9,18,27)$
$+(1,26,13)(2,27,14)(3,25,15)(4,20,16)(5,21,17)(6,19,18)(7,23,10)(8,24,11)(9,22,12)$
$+(1,21,11)(2,19,12)(3,20,10)(4,24,14)(5,22,15)(6,23,13)(7,27,17)(8,25,18)(9,26,16)$
$+(1,24,16)(2,22,17)(3,23,18)(4,27,10)(5,25,11)(6,26,12)(7,21,13)(8,19,14)(9,20,15)$
$+(1,20,12)(2,21,10)(3,19,11)(4,23,15)(5,24,13)(6,22,14)(7,26,18)(8,27,16)(9,25,17)$
$+(1,25,14)(2,26,15)(3,27,13)(4,19,17)(5,20,18)(6,21,16)(7,22,11)(8,23,12)(9,24,10)$
$+(1,16,24)(2,17,22)(3,18,23)(4,10,27)(5,11,25)(6,12,26)(7,13,21)(8,14,19)(9,15,20)$
$+(1,12,20)(2,10,21)(3,11,19)(4,15,23)(5,13,24)(6,14,22)(7,18,26)(8,16,27)(9,17,25)$
$+(1,18,22)(2,16,23)(3,17,24)(4,12,25)(5,10,26)(6,11,27)(7,15,19)(8,13,20)(9,14,21)$
$+(1,14,25)(2,15,26)(3,13,27)(4,17,19)(5,18,20)(6,16,21)(7,11,22)(8,12,23)(9,10,24)$
$+(1,11,21)(2,12,19)(3,10,20)(4,14,24)(5,15,22)(6,13,23)(7,17,27)(8,18,25)(9,16,26)$
$+(1,22,18)(2,23,16)(3,24,17)(4,25,12)(5,26,10)(6,27,11)(7,19,15)(8,20,13)(9,21,14)$
$+(1,13,26)(2,14,27)(3,15,25)(4,16,20)(5,17,21)(6,18,19)(7,10,23)(8,11,24)(9,12,22)$
$+(1,17,23)(2,18,24)(3,16,22)(4,11,26)(5,12,27)(6,10,25)(7,14,20)(8,15,21)(9,13,19)$
$+(1,27,15)(2,25,13)(3,26,14)(4,21,18)(5,19,16)(6,20,17)(7,24,12)(8,22,10)(9,23,11)$
$+(1,23,17)(2,24,18)(3,22,16)(4,26,11)(5,27,12)(6,25,10)(7,20,14)(8,21,15)(9,19,13)$
$+(1,15,27)(2,13,25)(3,14,26)(4,18,21)(5,16,19)(6,17,20)(7,12,24)(8,10,22)(9,11,23)$

4.2.3 Ideal of dimension 2 of $\mathbb{F}_{2^k}3^{1+2}$

Basis:
$Id(G)+(1,3,2)(4,6,5)(7,9,8)(10,12,11)(13,15,14)(16,18,17)(19,21,20)$
$(22,24,23)(25,27,26)+(1,10,19)(2,11,20)(3,12,21)(4,13,22)(5,14,23)(6,15,24)$
$(7,16,25)(8,17,26)(9,18,27)+(1,4,7)(2,5,8)(3,6,9)(10,14,18)(11,15,16)(12,13,17)$

$(19,24,26)(20,22,27)(21,23,25)+(1,2,3)(4,5,6)(7,8,9)(10,11,12)(13,14,15)(16,17,18)$
$(19,20,21)(22,23,24)(25,26,27)+(1,26,13)(2,27,14)(3,25,15)(4,20,16)(5,21,17)$
$(6,19,18)(7,23,10)(8,24,11)(9,22,12)+(1,24,16)(2,22,17)(3,23,18)(4,27,10)(5,25,11)$
$(6,26,12)(7,21,13)(8,19,14)(9,20,15)+(1,25,14)(2,26,15)(3,27,13)(4,19,17)(5,20,18)$
$(6,21,16)(7,22,11)(8,23,12)(9,24,10)+(1,16,24)(2,17,22)(3,18,23)(4,10,27)(5,11,25)$
$(6,12,26)(7,13,21)(8,14,19)(9,15,20)+(1,12,20)(2,10,21)(3,11,19)(4,15,23)(5,13,24)$
$(6,14,22)(7,18,26)(8,16,27)(9,17,25)+(1,6,8)(2,4,9)(3,5,7)(10,13,16)(11,14,17)$
$(12,15,18)(19,23,27)(20,24,25)(21,22,26)+(1,18,22)(2,16,23)(3,17,24)(4,12,25)$
$(5,10,26)(6,11,27)(7,15,19)(8,13,20)(9,14,21)+(1,11,21)(2,12,19)(3,10,20)(4,14,24)$
$(5,15,22)(6,13,23)(7,17,27)(8,18,25)(9,16,26)+(1,22,18)(2,23,16)(3,24,17)(4,25,12)$
$(5,26,10)(6,27,11)(7,19,15)(8,20,13)(9,21,14)+(1,5,9)(2,6,7)(3,4,8)(10,15,17)$
$(11,13,18)(12,14,16)(19,22,25)(20,23,26)(21,24,27)+(1,17,23)(2,18,24)(3,16,22)$
$(4,11,26)(5,12,27)(6,10,25)(7,14,20)(8,15,21)(9,13,19)+(1,27,15)(2,25,13)(3,26,14)$
$(4,21,18)(5,19,16)(6,20,17)(7,24,12)(8,22,10)(9,23,11)+(1,23,17)(2,24,18)(3,22,16)$
$(4,26,11)(5,27,12)(6,25,10)(7,20,14)(8,21,15)(9,19,13)(1,19,10)(2,20,11)(3,21,12)$
$(4,22,13)(5,23,14)(6,24,15)(7,25,16)(8,26,17)(9,27,18)+(1,7,4)(2,8,5)(3,9,6)$
$(10,18,14)(11,16,15)(12,17,13)(19,26,24)(20,27,22)(21,25,23)+(1,10,19)(2,11,20)$
$(3,12,21)(4,13,22)(5,14,23)(6,15,24)(7,16,25)(8,17,26)(9,18,27)+(1,4,7)(2,5,8)$
$(3,6,9)(10,14,18)(11,15,16)(12,13,17)(19,24,26)(20,22,27)(21,23,25)+(1,26,13)$
$(2,27,14)(3,25,15)(4,20,16)(5,21,17)(6,19,18)(7,23,10)(8,24,11)(9,22,12)+$
$(1,21,11)$
$(2,19,12)(3,20,10)(4,24,14)(5,22,15)(6,23,13)(7,27,17)(8,25,18)(9,26,16)+$
$(1,20,12)$
$(2,21,10)(3,19,11)(4,23,15)(5,24,13)(6,22,14)(7,26,18)(8,27,16)(9,25,17)+$
$(1,25,14)$
$(2,26,15)(3,27,13)(4,19,17)(5,20,18)(6,21,16)(7,22,11)(8,23,12)(9,24,10)+$
$(1,9,5)$
$(2,7,6)(3,8,4)(10,17,15)(11,18,13)(12,16,14)(19,25,22)(20,26,23)(21,27,24)$
$+(1,8,6)(2,9,4)(3,7,5)(10,16,13)(11,17,14)(12,18,15)(19,27,23)(20,25,24)(21,26,22)$
$+(1,12,20)(2,10,21)(3,11,19)(4,15,23)(5,13,24)(6,14,22)(7,18,26)(8,16,27)(9,17,25)$
$+(1,6,8)(2,4,9)(3,5,7)(10,13,16)(11,14,17)(12,15,18)(19,23,27)(20,24,25)(21,22,26)$
$+(1,14,25)(2,15,26)(3,13,27)(4,17,19)(5,18,20)(6,16,21)(7,11,22)(8,12,23)(9,10,24)$
$+(1,11,21)(2,12,19)(3,10,20)(4,14,24)(5,15,22)(6,13,23)(7,17,27)(8,18,25)(9,16,26)$
$+(1,13,26)(2,14,27)(3,15,25)(4,16,20)(5,17,21)(6,18,19)(7,10,23)(8,11,24)(9,12,22)$
$+(1,5,9)(2,6,7)(3,4,8)(10,15,17)(11,13,18)(12,14,16)(19,22,25)(20,23,26)(21,24,27)$
$+(1,27,15)(2,25,13)(3,26,14)(4,21,18)(5,19,16)(6,20,17)(7,24,12)(8,22,10)(9,23,11)$
$+(1,15,27)(2,13,25)(3,14,26)(4,18,21)(5,16,19)(6,17,20)(7,12,24)(8,10,22)(9,11,23)$

4.2.4 Ideal of dimension 2 of $\mathbb{F}_{2^k}3^{1+2}$

Basis:

$Id(G)+$
$(1,19,10)(2,20,11)(3,21,12)(4,22,13)(5,23,14)(6,24,15)(7,25,16)(8,26,17)(9,27,18)+$
$(1,3,2)(4,6,5)(7,9,8)(10,12,11)(13,15,14)(16,18,17)(19,21,20)(22,24,23)(25,27,26)+$
$(1,10,19)(2,11,20)(3,12,21)(4,13,22)(5,14,23)(6,15,24)(7,16,25)(8,17,26)(9,18,27)+$
$(1,4,7)(2,5,8)(3,6,9)(10,14,18)(11,15,16)(12,13,17)(19,24,26)(20,22,27)(21,23,25)+$
$(1,2,3)(4,5,6)(7,8,9)(10,11,12)(13,14,15)(16,17,18)(19,20,21)(22,23,24)(25,26,27)+$
$(1,21,11)(2,19,12)(3,20,10)(4,24,14)(5,22,15)(6,23,13)(7,27,17)(8,25,18)(9,26,16)+$
$(1,24,16)(2,22,17)(3,23,18)(4,27,10)(5,25,11)(6,26,12)(7,21,13)(8,19,14)(9,20,15)+$
$(1,20,12)(2,21,10)(3,19,11)(4,23,15)(5,24,13)(6,22,14)(7,26,18)(8,27,16)(9,25,17)+$
$(1,12,20)(2,10,21)(3,11,19)(4,15,23)(5,13,24)(6,14,22)(7,18,26)(8,16,27)(9,17,25)+$
$(1,6,8)(2,4,9)(3,5,7)(10,13,16)(11,14,17)(12,15,18)(19,23,27)(20,24,25)(21,22,26)+$
$(1,14,25)(2,15,26)(3,13,27)(4,17,19)(5,18,20)(6,16,21)(7,11,22)(8,12,23)(9,10,24)+$
$(1,11,21)(2,12,19)(3,10,20)(4,14,24)(5,15,22)(6,13,23)(7,17,27)(8,18,25)(9,16,26)+$
$(1,22,18)(2,23,16)(3,24,17)(4,25,12)(5,26,10)(6,27,11)(7,19,15)(8,20,13)(9,21,14)+$
$(1,13,26)(2,14,27)(3,15,25)(4,16,20)(5,17,21)(6,18,19)(7,10,23)(8,11,24)(9,12,22)+$
$(1,5,9)(2,6,7)(3,4,8)(10,15,17)(11,13,18)(12,14,16)(19,22,25)(20,23,26)(21,24,27)+$
$(1,23,17)(2,24,18)(3,22,16)(4,26,11)(5,27,12)(6,25,10)(7,20,14)(8,21,15)(9,19,13)+$
$(1,15,27)(2,13,25)(3,14,26)(4,18,21)(5,16,19)(6,17,20)(7,12,24)(8,10,22)(9,11,23)$
$(1,7,4)(2,8,5)(3,9,6)(10,18,14)(11,16,15)(12,17,13)(19,26,24)(20,27,22)(21,25,23)+$
$(1,4,7)(2,5,8)(3,6,9)(10,14,18)(11,15,16)(12,13,17)(19,24,26)(20,22,27)(21,23,25)+$
$(1,26,13)(2,27,14)(3,25,15)(4,20,16)(5,21,17)(6,19,18)(7,23,10)(8,24,11)(9,22,12)+$
$(1,24,16)(2,22,17)(3,23,18)(4,27,10)(5,25,11)(6,26,12)(7,21,13)(8,19,14)(9,20,15)+$
$(1,25,14)(2,26,15)(3,27,13)(4,19,17)(5,20,18)(6,21,16)(7,22,11)(8,23,12)(9,24,10)+$
$(1,9,5)(2,7,6)(3,8,4)(10,17,15)(11,18,13)(12,16,14)(19,25,22)(20,26,23)(21,27,24)+$
$(1,16,24)(2,17,22)(3,18,23)(4,10,27)(5,11,25)(6,12,26)(7,13,21)(8,14,19)(9,15,20)+$
$(1,8,6)(2,9,4)(3,7,5)(10,16,13)(11,17,14)(12,18,15)(19,27,23)(20,25,24)(21,26,22)+$
$(1,6,8)(2,4,9)(3,5,7)(10,13,16)(11,14,17)(12,15,18)(19,23,27)(20,24,25)(21,22,26)+$
$(1,18,22)(2,16,23)(3,17,24)(4,12,25)(5,10,26)(6,11,27)(7,15,19)(8,13,20)(9,14,21)+$
$(1,14,25)(2,15,26)(3,13,27)(4,17,19)(5,18,20)(6,16,21)(7,11,22)(8,12,23)(9,10,24)+$
$(1,22,18)(2,23,16)(3,24,17)(4,25,12)(5,26,10)(6,27,11)(7,19,15)(8,20,13)(9,21,14)+$
$(1,13,26)(2,14,27)(3,15,25)(4,16,20)(5,17,21)(6,18,19)(7,10,23)(8,11,24)(9,12,22)+$
$(1,5,9)(2,6,7)(3,4,8)(10,15,17)(11,13,18)(12,14,16)(19,22,25)(20,23,26)(21,24,27)+$
$(1,17,23)(2,18,24)(3,16,22)(4,11,26)(5,12,27)(6,10,25)(7,14,20)(8,15,21)(9,13,19)+$
$(1,27,15)(2,25,13)(3,26,14)(4,21,18)(5,19,16)(6,20,17)(7,24,12)(8,22,10)(9,23,11)+$
$(1,23,17)(2,24,18)(3,22,16)(4,26,11)(5,27,12)(6,25,10)(7,20,14)(8,21,15)(9,19,13)+$
$(1,15,27)(2,13,25)(3,14,26)(4,18,21)(5,16,19)(6,17,20)(7,12,24)(8,10,22)(9,11,23),$

4.2.5 Ideal of dimension 18 of $\mathbb{F}_{2^k}3^{1+2}$

Basis:
$Id(G)+$
$(1,2,3)(4,5,6)(7,8,9)(10,11,12)(13,14,15)(16,17,18)(19,20,21)(22,23,24)(25,26,27)$
$(1,19,10)(2,20,11)(3,21,12)(4,22,13)(5,23,14)(6,24,15)(7,25,16)(8,26,17)(9,27,18)$
$+(1,20,12)(2,21,10)(3,19,11)(4,23,15)(5,24,13)(6,22,14)(7,26,18)(8,27,16)(9,25,17)$
$(1,7,4)(2,8,5)(3,9,6)(10,18,14)(11,16,15)(12,17,13)(19,26,24)(20,27,22)(21,25,23)$
$+(1,8,6)(2,9,4)(3,7,5)(10,16,13)(11,17,14)(12,18,15)(19,27,23)(20,25,24)(21,26,22)$
$(1,3,2)(4,6,5)(7,9,8)(10,12,11)(13,15,14)(16,18,17)(19,21,20)(22,24,23)(25,27,26)$
$+(1,2,3)(4,5,6)(7,8,9)(10,11,12)(13,14,15)(16,17,18)(19,20,21)(22,23,24)(25,26,27)$
$(1,10,19)(2,11,20)(3,12,21)(4,13,22)(5,14,23)(6,15,24)(7,16,25)(8,17,26)(9,18,27)$
$+(1,11,21)(2,12,19)(3,10,20)(4,14,24)(5,15,22)(6,13,23)(7,17,27)(8,18,25)(9,16,26)$
$(1,4,7)(2,5,8)(3,6,9)(10,14,18)(11,15,16)(12,13,17)(19,24,26)(20,22,27)(21,23,25)$
$+(1,5,9)(2,6,7)(3,4,8)(10,15,17)(11,13,18)(12,14,16)(19,22,25)(20,23,26)(21,24,27)$
$(1,26,13)(2,27,14)(3,25,15)(4,20,16)(5,21,17)(6,19,18)(7,23,10)(8,24,11)(9,22,12)$
$+(1,27,15)(2,25,13)(3,26,14)(4,21,18)(5,19,16)(6,20,17)(7,24,12)(8,22,10)(9,23,11)$
$(1,21,11)(2,19,12)(3,20,10)(4,24,14)(5,22,15)(6,23,13)(7,27,17)(8,25,18)(9,26,16)$
$+(1,20,12)(2,21,10)(3,19,11)(4,23,15)(5,24,13)(6,22,14)(7,26,18)(8,27,16)(9,25,17)$
$(1,24,16)(2,22,17)(3,23,18)(4,27,10)(5,25,11)(6,26,12)(7,21,13)(8,19,14)(9,20,15)$
$+(1,23,17)(2,24,18)(3,22,16)(4,26,11)(5,27,12)(6,25,10)(7,20,14)(8,21,15)(9,19,13)$
$(1,25,14)(2,26,15)(3,27,13)(4,19,17)(5,20,18)(6,21,16)(7,22,11)(8,23,12)(9,24,10)$
$+(1,27,15)(2,25,13)(3,26,14)(4,21,18)(5,19,16)(6,20,17)(7,24,12)(8,22,10)(9,23,11)$
$(1,9,5)(2,7,6)(3,8,4)(10,17,15)(11,18,13)(12,16,14)(19,25,22)(20,26,23)(21,27,24)$
$+(1,8,6)(2,9,4)(3,7,5)(10,16,13)(11,17,14)(12,18,15)(19,27,23)(20,25,24)(21,26,22)$
$(1,16,24)(2,17,22)(3,18,23)(4,10,27)(5,11,25)(6,12,26)(7,13,21)(8,14,19)(9,15,20)$
$+(1,17,23)(2,18,24)(3,16,22)(4,11,26)(5,12,27)(6,10,25)(7,14,20)(8,15,21)(9,13,19)$
$(1,12,20)(2,10,21)(3,11,19)(4,15,23)(5,13,24)(6,14,22)(7,18,26)(8,16,27)(9,17,25)$
$+(1,11,21)(2,12,19)(3,10,20)(4,14,24)(5,15,22)(6,13,23)(7,17,27)(8,18,25)(9,16,26)$
$(1,6,8)(2,4,9)(3,5,7)(10,13,16)(11,14,17)(12,15,18)(19,23,27)(20,24,25)(21,22,26)$
$+(1,5,9)(2,6,7)(3,4,8)(10,15,17)(11,13,18)(12,14,16)(19,22,25)(20,23,26)(21,24,27)$
$(1,18,22)(2,16,23)(3,17,24)(4,12,25)(5,10,26)(6,11,27)(7,15,19)(8,13,20)(9,14,21)$
$+(1,17,23)(2,18,24)(3,16,22)(4,11,26)(5,12,27)(6,10,25)(7,14,20)(8,15,21)(9,13,19)$
$(1,14,25)(2,15,26)(3,13,27)(4,17,19)(5,18,20)(6,16,21)(7,11,22)(8,12,23)(9,10,24)$
$+(1,15,27)(2,13,25)(3,14,26)(4,18,21)(5,16,19)(6,17,20)(7,12,24)(8,10,22)(9,11,23)$
$(1,22,18)(2,23,16)(3,24,17)(4,25,12)(5,26,10)(6,27,11)(7,19,15)(8,20,13)(9,21,14)$
$+(1,23,17)(2,24,18)(3,22,16)(4,26,11)(5,27,12)(6,25,10)(7,20,14)(8,21,15)(9,19,13)$
$(1,13,26)(2,14,27)(3,15,25)(4,16,20)(5,17,21)(6,18,19)(7,10,23)(8,11,24)(9,12,22)$
$+(1,15,27)(2,13,25)(3,14,26)(4,18,21)(5,16,19)(6,17,20)(7,12,24)(8,10,22)(9,11,23)$

Remark: We might have concluded immediately from the fact that the derived quotient of 3^{1+2} has order 6.

4.3 Augmentation Ideal of $\mathbb{F}_{2^k}3^{1+2}$

Let RG be a group ring and $\phi : RG \to R$ be an *augmentation map*, then the $Ker(\phi)$ is called an *augmentation ideal* and have a special notation $ker(\phi) = \Delta(RG)$.

Augmentation Ideal of ϕ defined on $\mathbb{F}_{2^k}3^{1+2}$ has dimension 26 and this was computed by using GAP.

4.4 Jacobson Radical of $\mathbb{F}_{2^k}3^{1+2}$

"Let E be an algebra over the field K. An ideal P of E is said to be primitive if E/P is primitive ring, that is if E/P has a faithful irreducible right module V. The intersection of all primitive ideals of E is JE The Jacobson Radical of E and E is semi simple iff $JE = 0$."

The Jacobson radical turns out to be of dimension zero. Thus $\mathbb{F}_{2^k}3^{1+2}$ is semi simple.

By using GAP composition sereis of $\mathbb{F}_{2^k}3^{1+2}$ consists of sub, algebra's of dimensions 1, 3, 21, 23 and 25.

Appendix

#Gap Code for generating elements of unitary unit group $V_*(F_{2^k}3^{1+2})$:

```
G:=Group((2,6,7)*(3,8,9), (1,2,3)*(4,7,8)*(5,6,9),
(1,4,5)*(2,7,6)*(3,8,9));

K:=GF(2);
KG:=GroupRing(K,G);

L:=[];
T:=Set(G);


 i1:=0;
while (i1<=1)    do


ii:=i1*T[1]^Embedding( G, KG );;
if( (Augmentation(ii)=Z(2)^0) and IsUnitary(ii) and
 not(ii in L)) then
       Add(L,ii);;
   fi;
                                 od;
                                 i1:=i1+1;

                            od;
                         Print("\n",L,"\n",Length(L));

                           for k in [1..12] do
                             S:=L;;
                             i2:=0;
                             j:=Length(L);;
```

```
while(i2<=1)
         do


   for i in [1..j] do

  ii:=S[i]+i2*T[k]^Embedding( G, KG );;

   if( (Augmentation(ii)=Z(2)^0) and IsUnitary(ii) and
 not(ii in L)) then
       Add(L,ii);;
   fi;
   od;
  i2:=i2+1;

 od;
 Print("\n",Length(L));
 od;
```

#Gap Code for generating group ring matrix of $F_{2^k}3^{1+2}$:

```
  #f:=FreeGroup("a","b","c");
# x:=f.1;y:=f.2;z:=f.3;
#rels:=[x^3,y^3,z^3,Comm(x,z),Comm(y,z),Comm(x,y)*z^-1];
 G:=f/rels;
#H:=Group((2,6,7)*(3,8,9), (1,2,3)*(4,7,8)*(5,6,9), (1,4,5)*(2,7,6)*(3,8,9));
#L:=[];
#K:=[];
#x:=(2,6,7)*(3,8,9);
#y:=(1,2,3)*(4,7,8)*(5,6,9);
#z:=(1,4,5)*(2,7,6)*(3,8,9);

       for i1 in [1..3]do
         for i2 in [1..3]do
           for i3 in [1..3]do
             for i4 in [1..3]do
               for i5 in [1..3]do
                 for i6 in [1..3]do
                   for i7 in [1..3]do
```

```
              ii:=(x^i1*y^i2*(z)^i3*(x)^i4*y^i5*z^i6*(x)^i7);;
 jj:=(f.1^i1*f.2^i2*(f.3)^i3*(f.1)^i4*f.2^i5*f.3^i6*(f.1)^i7);;
if (not(ii in L)) then
  Add(L,ii);;Add(K,jj);;#Print("\n",jj);
fi;


od;
od;
od;
od;
od;
od;
od;
for i in [1..27] do
Print("\n",L[i],"\t",K[i],"\n");
#Print("\n",K[i],"\n");
od;
#11111111232
#(x*y*x*y^2*x*y*x*y^2)^2;



 S:=[];
 #K:=Set(H);

  #for j in [1..27]do
   #  Print( "\n",L[j]^-1*L,"\t",K[j]^-1*K,"n");
#od;
a:=(2,6,7)*(3,8,9); b:=(1,2,3)*(4,7,8)*(5,6,9); c:=(1,4,5)*(2,7,6)*(3,8,9);
#T:=[(), a, a^2, b, a^2*b, a*b, b^2, a^2*b^2, a*b^2, b^2*a*b,
 a*b^2*a*b, a^2*b^2*a*b, (a*b)^2*b, a*(a*b)^2*b, b*a*b^2,
 a*b*a, b*a,   a^2*b*a, a^2*b*a^2, a*b*a^2, b*a^2, b^2*a,
 a^2*b^2*a, a*b^2*a, (a*b)^2, b*a*b, a*(a*b)^2];
#K:=[T[1],T[2],T[3],T[5],T[4],T[6],T[8],T[7],T[9].T[10],
T[11],T[12],T[13],T[14],T[16],T[17],T[20],T[19],T[18],
T[21],T[23],T[22],T[24],T[26],T[25],[27]];

#T:=[(), a, a^2, b, a*b,a^2*b,  b^2, a*b^2,a^2*b^2,
  b^2*a*b, a*b^2*a*b, a^2*b^2*a*b, (a*b)^2*b,
 a*(a*b)^2*b, b*a*b^2, b*a,a*b*a,    a^2*b*a,
```

```
 b*a^2, a*b*a^2,  a^2*b*a^2, b^2*a, a*b^2*a,
  a^2*b^2*a,  (a*b)^2,  a*(a*b)^2, b*a*b];
for i in [1..27] do

#Print("\n","Row", i,S[i]^-1*S,"\t",T[i]^-1*T,"\n");
for j in [1..27] do
Print("\t","a_{",Position(T,T[i]^-1*T[j])-1,"}","&");
if (j=27) then Print("\n", "\n");fi;
od;
od;
```

#Gap Code for one to one correspondence between two representations of $F_{2^k}3^{1+2}$:

```
f:=FreeGroup("a","b","c");
x:=f.1;y:=f.2;z:=f.3;
rels:=[x^3,y^3,z^3,Comm(x,z),Comm(y,z),Comm(x,y)*z^-1];
 G:=f/rels;
S:=Set(G);
H:=Group((2,6,7)*(3,8,9), (1,2,3)*(4,7,8)*(5,6,9), (1,4,5)*(2,7,6)*(3,8,9));
T:=Set(H);
u:=(2,6,7)*(3,8,9);
v:=(1,2,3)*(4,7,8)*(5,6,9);
w:=(1,4,5)*(2,7,6)*(3,8,9);

for i in [1..27] do
Print("\n",S[i],"\t",T[i],"\t","a_",i-1,"\n");
od;


for i in [1..27] do

#Print("\n","Row", i,S[i]^-1*S,"\t",T[i]^-1*T,"\n");
for j in [1..27] do
Print("\t","a_{",Position(T,T[i]^-1*T[j])-1,"}","&");
 if (j=27) then Print("\n", "\n");fi;
od;
od;
```

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