On Group Distance Graph Evaluation

MS Thesis
by
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COMSATS University Islamabad
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In partial fulfillment
of the requirement for the degree of

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in
Mathematics

by
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Dedication

To My Parents and All Family

Acknowledgements

Praise to be ALLAH, the Cherisher and Lord
of the World, Most gracious and Most Merciful

First and foremost, I would like to thank ALLAH Almighty (the most beneficent and most merciful) for giving me the strength, knowledge, ability and opportunity to undertake this research study and to preserve and complete it satisfactorily. Without countless blessing of ALLAH Almighty, this achievement would not have been possible. May His peace and blessings be upon His messenger Hazrat Muhammad (PBUH), upon his family, companions and whoever follows him. My insightful gratitude to Hazrat Muhammad (PBUH) Who is forever a track of guidance and knowledge for humanity as a whole. In my journey towards this degree, I have found a teacher, an inspiration, a role model and a pillar of support in my life, my kind.

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Abstract

On Group Distance Graph Evaluation

By

Attlah Abbas

Graph help us for building the interconnection network that are required by systems. The unique graph labeling help us to ensure the functionality of these networks.In this research we have discuss the labeling of certain graphs and how we develope the labeling pattern for these graphs.We have used modulo group combinatorial techniques to find out the labeling schemes.By using elements from modulo group we have define group distance and distance magic labeling.

We have calculate the weight of vertices and magic constant of graphs by finding the labeling of direct product of prism graph and cycle graph $\mathbb{P}_n \times C_4$ under modulo group $\mathbb{Z}_{8n}$ and $\mathbb{P}_n \times C_m$ under modulo group $\mathbb{Z}_m \times \mathbb{Z}_{2n}$ $\forall$ $n = 3$ and $m =\geq 4$

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Chapter 1
Introduction

Graph Theory is the study of connections that provide a helpful tool to calculate and simplify the moving parts of a dynamic system. Using networks of nodes and links that might extract every detail from computer data to city plans. It aids academics in analysing the optimal paths.It is used in social network connections,search engine ranking of hyperlinks,gps maps to determine the fastest route and other applications.

The father of graph theory, famous Swiss mathematician Leonhard Euler has been linked with developing the field with his work on "Königsberg bridge" problem in 18th century.

Konigsberg which was a city of Germany now a part of Russia and it is on the Pregel river. In this city there are seven bridges and the two islands are connected with these seven bridges. So the people of that city thought that how could one walk through konigsberg without crossing the every bridge more than once. So the mayor of the city wrote this problem to famous mathematician Leonhard Euler that is it possible to cross the konigsberg all bridges without crossing anyone of its bridges twice? At first he thought this problem is minor but it was not and this problem helped paved the way towards new mathematical branch which is **Graph Theory**.

He proved that its impossible to cross the bridges without repeating it. He used dots and lines which we called vertices and edges to solve this problem.He explained the vertices and edges and this thing leads towards a new graph which he called Eulerian Grap.A little over two centuries had passed since Euler's lecture on the Königsberg bridges, and Cayley was moved by an interest in particular analytical forms derived to investigate a certain class of graphs using differential calculus, the trees, during the time that listing was developing the idea of topology. A particular discipline of chemistry is theoretical chemistry. The consequences of this study has had on me is substantial. His main methods were methods for counting graphs with specific properties. The foundational resulting results made by P¨olya in 1935–1937 led to the further advancement of the theory of enumerative graphs.

De Bruijn generalised them in 1959. Cayley juxtapose his findings from the tree with a recent chemical composition scrutiny. The initial development of what is currently regarded as standard nomenclature in the discipline involved the fusion of mathematical and chemical principles. Specifically, Sylvester first used the term ”graph” in an article of research that was published in Nature in 1878. In that paper, he compares ”quantic invariants” and ”Co-variants,” two concepts in algebra. Molecular diagrams are a kind of imagery aid that represents a Denes Knig penned the first graph theory textbook,which was initially published in 1936. Frank Harary’s second book, which was published in and is regarded by many people as the ”classic textbook on the topic,” allowed professionals in chemistry, electrical engineering, and mathematics to work on it.It’s imperative that engineers and social scientists communicate with one another. Among those that are most well-known and captivating problems in graph theory is the four-color problem: Is it accurate to say that there are exactly the same amount of colours on every map created using the colours red, green, and blue? The regions of an airplanes can be painted in four different ways, so that any two portions that share a boundary can have distinct colours.The issue in this was first raised by Francis Guthrie in 1852 and was the first to be overcome. De Morgan’s letter to Hamilton from the same year contains a recorded account and a documented record. A selection of the numerous erroneous proofs that have been put out is provided below. Authors Cayley and Kempe are among the others.

After uncovering and generalising this problem some of the researchers examined the colours of graphs implanted on multiple surfaces. Issues with factorization are a new class of problems that Tait’s improved version gave rise to, and Petersen and Knig have studied them extensively. Turan’s 1941 findings, which drew from Ramsey’s Colorations work, led to the development of extremal graph theory, a novel subfield of graph theory. The four-color dilemma hasn’t been solved in more than a century.

Heinrich Heesch proposed a computer-aided approach to the problem in 1969. 1 Heesch’s proposal of ”discharging” is an essential factor of the electronic evidence that Kenneth Appel and Wolfgang Haken developed in 1976. The manifestation, which included using a computer to evaluate the characteristics of 1, 936 couples, was not widely acknowledged at the time because to its complexity. The self-governing expansion of topology between

1860 and 1930 nourished the theory of graphs, leading to the innovations of Kuratowski, Whitney, and Jordan. Collaboration is a key component in the growth of graph theory overall.

On the other side, topology was the use of novel algebraic techniques. Voltage and current in electric circuits is the earliest example of such an application, as determined by physicist Gustav Kirchhoff in his 1845 publication of Kirchhoff's Circuit Rules. Asymptotic probability estimation of graph theory, which has yielded a multitude of graph-theoretic discoveries, gave rise to the second branch of graph theory, the random graph, and the use of probabilistic techniques in it, notably in the work of Erdos and Renyi.

1.0.1 Applications

Graph theory is a subject which is being applied now a days in many fields.Realtionships and processes in range of domains, such as physics, bio,social and informational systems can be appropriately displayed using graphs. Graphs are particularly advantageous in computer science to depict the communication networks, data structures, computing equipment and computational activites. Mostly ther are reffered to as networks when nodes or edges incorporate properties.Directed graphs, for instance, can depict a website's hierarchy, with edges standing in for connections between web pages and vertices that represent individual web pages. Challenges in computer chip design, biology, booking a getaway, and social networking are all approached similarly. Computer science has made managing graphs a prominent topic, leading to the development of graph databases to store and retrieve long-lasting graph-structured data, as well as graph rewriting systems to define and visualise graph modifications.Linguistics also makes extensive utilisation of graph-theoretic methods. While directed acyclic graphs are used in contemporary syntactic theories like head-driven phrase structure grammar, tree-based graphs are still commonly used in the hierarchical framework of syntax and compositional semantics. Strategies like WordNet and VerbNet also show how crucial semantic networks are for modelling word meanings in context in machine linguistics. Graph theory has been applied in chemistry and physics for learning about condensed matter and molecular structures. In chemistry, atoms are delineated by vertices, while bonds that lies within molecules are visualized by edges. This helps to make it possible to process molecular structures with computers and do database searches. Graphs are used in computational neuroscience to explain functional interconnections between brain areas, which aid in the understanding of cognitive function, while they are utilised in statistical physics to analyse system components and physical process dynamics. Graphs can also depict the channels found in porous materials, which link the pores with smaller channels.In sociology, graph theory is becomes especially helpful for social network analysis tasks like determining actor status or monitoring the spread of rumours. Social network graphs include collaboration graphs that show teamwork in tasks like filmmaking, impact graphs that show individual influence, and friendship graphs that show personal contacts. Researchers can might investigate breeding habits, disease transmission, and the consequences of movement changes on species by using graph theory, which is frequently utilised in biological and environmental conservation. In graph theory, vertices represent habitats and edges show migration patterns.

1.0.2 Graph Labeling

A successful approach to giving numbers to the components of graphs so that particular values equals one another is achieved through graph labeling.Around the second half of 1960 graph labeling concept was first intoduced.Vertex,edge,distance and group distance magic are the few examples of various labelings which are inspired from A.Rosa's β valuation.Distance magic labeling leads to the exploration of group distance magic which was first presented as Σ labeling. Later on with the passage of time its name changes into one-vertex magic labeling and then in 2009 it named as distance magic labeling. In this method vertices of graph are labeled depending on their distances from one another.

1.0.3 Research Objective

Group distance magic labeling helps to seek out the structural and mathematical aspects of direct product of unlabeled regular graphs and uncover the characteristics they possess.The implementation of GDML attempts to create the framework for analysing regular graph products and highlighting their essential features. The ultimated objective of this research is to find the particular groups in which direct product of regular graphs reveals the features of group distance magic labeling.This research work helps us to have a good understanding between group theory,graph theory and the graph labeling.

Chapter 2
Basics of Graph Theory

2.1 Graph

A figurative representation of data is called graph which is made up of using lines, arrows and some different shapes to describe the connection across the variables.In other words graph is a mathematical structure used in mathematics and computer science to represent a collection of items (referred to as vertices or nodes) and the connections (referred to as edges or arcs) between them.Graphs have been utilised in enormous number of fields to solve several problems like finding the quikest pathways between two cities,improving traffic flow,assigning multiple channels to tv stations,maximizing useful resources in construction projects etc.We have discussed many types of graphs in this chapter and each graph have different properties upon which they are different from each other.

In mathematics graph is usually defined as ordered pair $G = (V, E)$ where V is the set of vertices and E is the set of edges. A graph is a system made up of two basic parts.

Vertices

Another term used of vertices is Nodes and they are the discrete components or entities that are shown in the graph. For convenience, a unique identifier or label is typically given to each vertex.

Edges

The links or connections between vertices are known as edges(arcs). The relationship, interdependence, or association between two vertices is represented by an edge. Both directed and undirected edges exsit.The directed graph is also named as digraphs. The edges of a **directed graph** have a distinct direction, signifying a one-way relationship.The direction is may b in or out. If the direction is towards the vertex it is counted as in degree and if the the direction of arrow is opposite to the vertex than it is counted as out degree.The degree indicates the quantity of edges attached to the vertices.The edges of an **undirected graph**, which shows a symmetric connection or a two-way relationship, have no direction

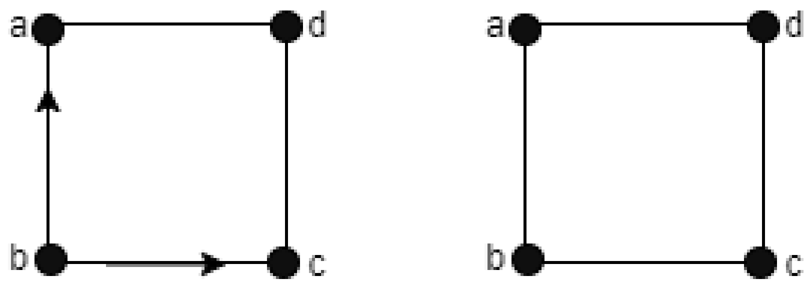

Figure 2.1: Undirected and Directed Graph

Hand Shaking Lemma

In a graph theory we have a lemma through which we can find the number of edges just by knowing the sum of the degree of the vertices. According to that lemma sum of degree of vertices is equal to two times of edges.

$$\sum_{i=1}^{n} d(v_i) = 2|E(G)|$$

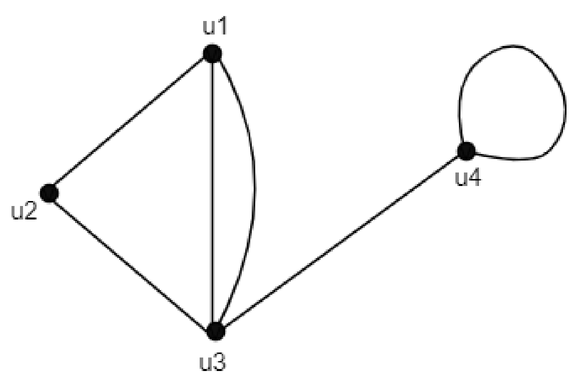

Figure 2.2: Hand Shaking Lemma

From the above example we have:

$$\Sigma 3+2+4+3 = 2(6)$$

$$12 = 12$$

Hand Shaking Di Lemma

We also have a lemma for directed graphs to find the edges of a graph.The lemma states that the sum of the degress coming towards the vertices is equal to the sum of degress going away from the vertices.

$$Indeg \sum_{i=1}^{n} d(v_i) = outdeg \sum_{i=1}^{n} d(v_i)$$

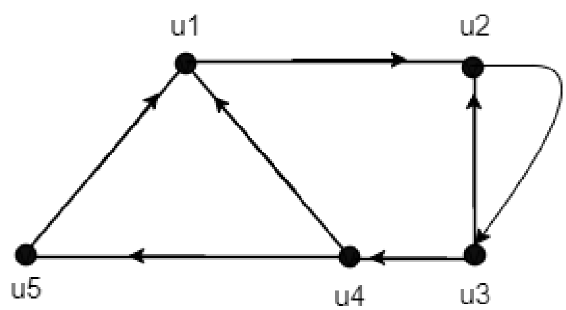

Figure 2.3: Hand Shaking Di-Lemma

From this example we have:

$$Indeg\Sigma 2+2+1+1+1 = outdeg\Sigma 1+1+1+2+2$$

$$7 = 7$$

One example of graph is **Online gaming network.** The players or the gaming accounts could represent the vertices. Each vertex is the players account and its connection with its friends or other team members could be the edges. In an **undirected graph**, the relationship between players are balanced. If we have three players A,B,C and player A is connected to player B and C it means player B and C are also connected to player A. In **directed graph** the relation between players are not balanced. If Player A is connected to Player B and C it dosen't mean that player B and C are also connected to player A.

Graphs are widely used in modelling and analysis of many real world scenarios covering social networks, transportation networks,gaming network and others. They served as a groundwork for a number of graph algorithms,analysis methods and offer an effective structure for figuring out the connections and relationships between entities

Graph are visually represented using diagrams, where vertices are represented with points/circles and the edges are represented with lines/arrows which are connected with vertices. Mathematically graphs are usually represented by adjacency matrices or adjacency lists which offers the compact way of storing the connections between vertices.

2.2 Different terminologies in graph

A graph without any multiple edges (if two vertices are connected there is only edge that indicates the connection between them) or loops (that there are no any edges that connect a vertex to itself) is known as **Simple Graph**.

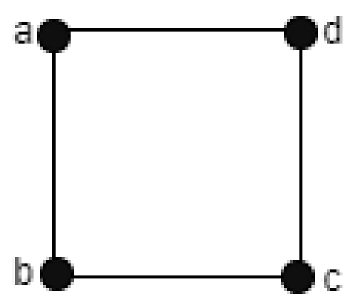

Figure 2.4: Simple Graph

A graph having multiple edges or loops is the **multigraph**.In multiples edges each edge is treated as unique edge despite they are connected to same pair of vertices.Multi graphs help us in transportation network like the multiple edges represent the different routes/mode of transportaion between same location.

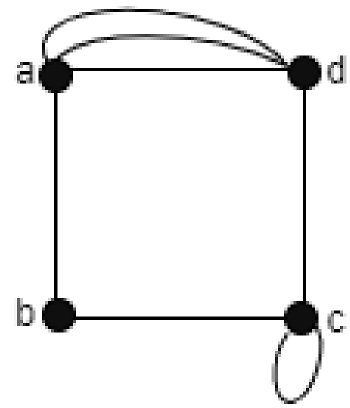

Figure 2.5: Multi Graph

The graph in which each edge is allocated a numerical value is called **weighted graphs**.In real world scenairo like in transportaion network they help us to modelling travel time, distance and the costs between locations.

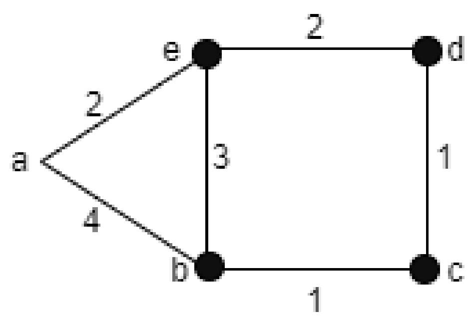

Figure 2.6: Weighted Graph

If a graph is part of another graph it is known as **subgraph**. For example a graph representing a transportation network having locations as vertices and roads as edges. A subgraph is formed by selecting a specific area between the location(vertices). In the example given below the first one is the graph and the second is the subgraph of first graph.

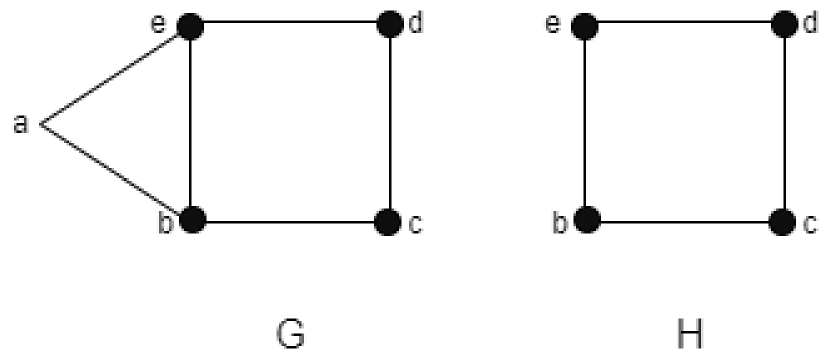

Figure 2.7: Subgraph

A kind of graph within which the edges connect each vertex is called **path graph**. The degree of path graphs are 1 or 2 with the exception of end points whose degree is 1. *Pn* is used to represent the path graph. In path graph if we have n vertices then we have $n-1$ edges

Figure 2.8: Path Graph

A vertex in graph that has no edges incident to it are known as **isolated vertex**. These vertices are disconnected from the rest of graph.

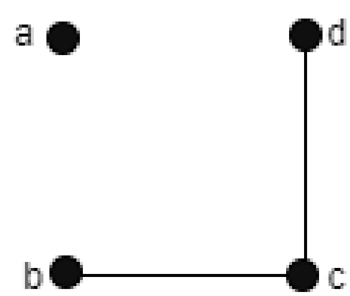

Figure 2.9: Isolated Vertex

Everytime when we are having a path between any pair of vertices in a graph is known as **connected graphs**. There is no isolated vertex in this graph.

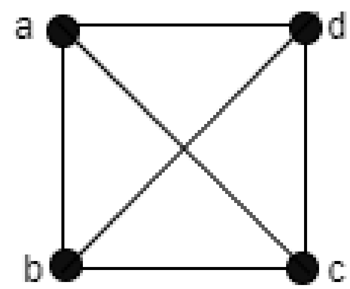

Figure 2.10: Connected Graph

Two vertices are said to be adjacent if they share one edge.Consider a graph.Two vertices a and c adjacent to each other in the example given below as there is an edge between them but the vertices b and c are not adjacent to each other.

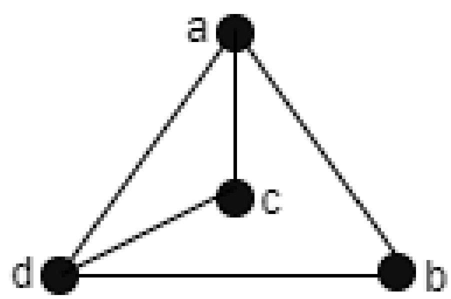

Figure 2.11: Adjacent Vertices

Complete graph K_n in which every 2 distinct vertices are connected by an edge. Given below K_2 and K_3 are the examples of complete graph.

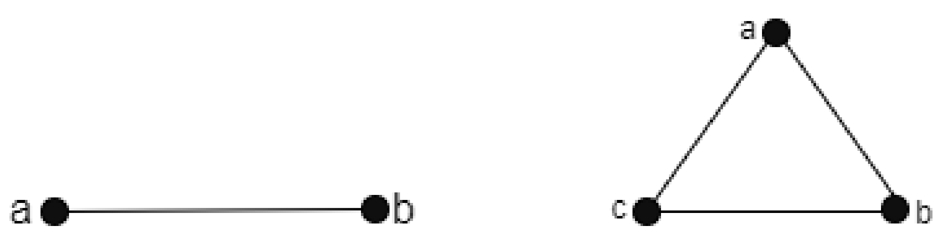

Figure 2.12: Complete Graph

Two or more edges incident with same vertices are termed as **parallel edges**.In example given below $x3$ and $x4$ are parallel to each other and $x5$ and $x6$ are parallel to each other.

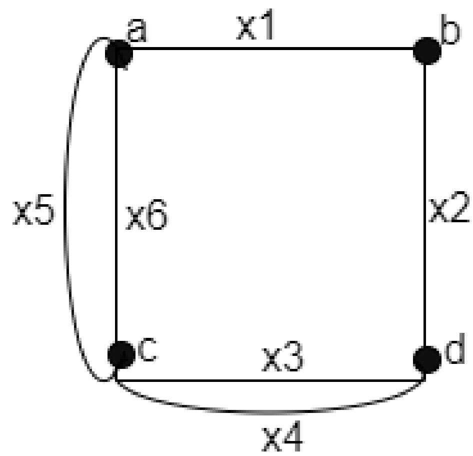

Figure 2.13: Parallel Edges

If we are having a graph and we will draw it in plane and it will not overlap/cross the edge then that graph is a **planner graph**. Like in the example given below graph $K4$ is embedded in plane.

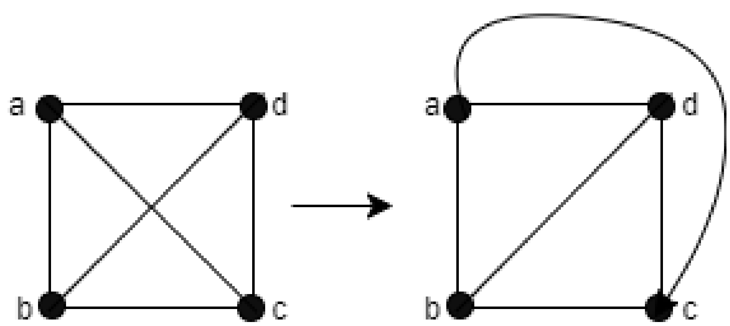

Figure 2.14: Planner Graph

The graphs that cannot be embedded in plane means the graph cannot be drawn in plane without edge crossing is known as **non-planner graph**. For example K_5 graph.

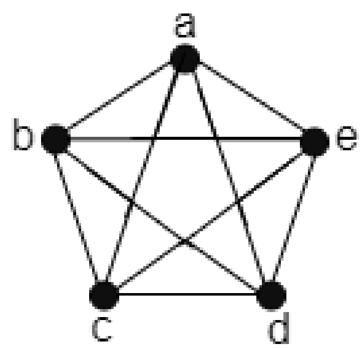

Figure 2.15: Non-Planner Graph

A graph in which every nodes have the same degree meaning each node has the same number of edges incident to it is known as **regular graphs**. The following graph is 2-regular.

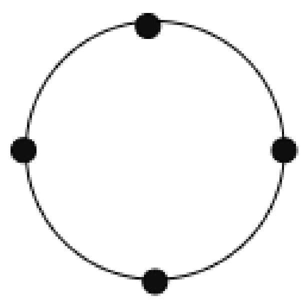

Figure 2.16: Regular Graph

The graph in which degree of every vertex is not same, the edges attach to each vertex are different in number is called **irregular graphs**. In example given below degree of inner vertex is 3 but the degree of other vertices are not 3.

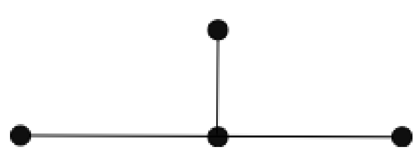

Figure 2.17: Irregular Graph

A graph having zero size but *n* order is known as **null graph**.To put it anotherway, a graph only having vertices with no connection between them.

Figure 2.18: Null Graph

If we start from any vertex that we want and then we travel across an edge to get another vertex is called **walk**. We can repeat the edges and vertices in walk.The length of walk is equal to its number of edges.

A walk in which there is no repetition of edges is termed as **trail**.

In graph **path** is also a walk but it does not visit any vertex more than once.

Circuit when we have closed trail having length 3 or more.

A walk is if we move from d towards b then e,b,d and then a.

In the example given below trail is moving around c,d,b,e,d,a ,

c,d,b,e is a path and

e,b,d,e is a circuit in the graph given below.

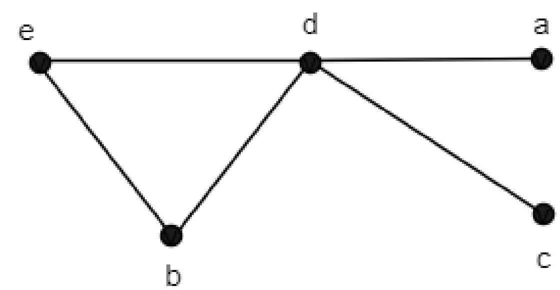

Figure 2.19: Walk,Trail, Path and Circuit in Graph

A path is called **eulerian path** in a graph if it traverse each edge in a graph once and only once.

$ue1ve2we3xe4ye5w$ is eulerian path in given graph

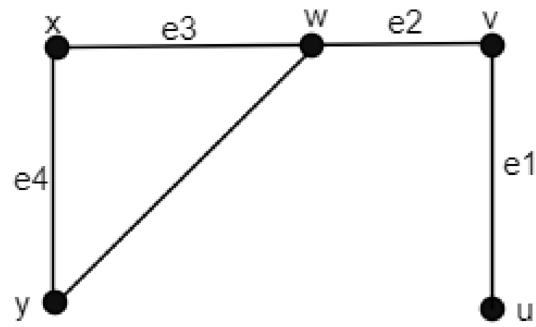

Figure 2.20: Eulerian Path

An **euler circuit** is like euler path but it starts and end at same vertex. If the vertex in a graph having odd degree then no euler circuit is possible.

y, x, w, v, u, w, y is the euler circuit in given graph.

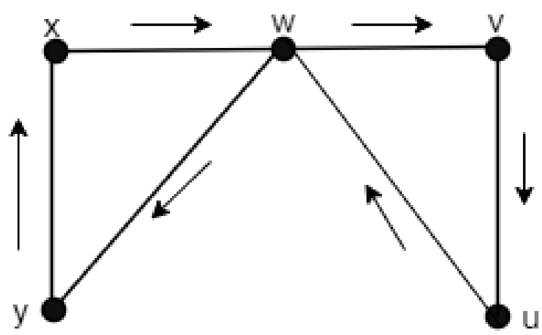

Figure 2.21: Euler Circuit

A graph that satisfy both the properties of euler path and euler circuit is known as **eulerian graph**.

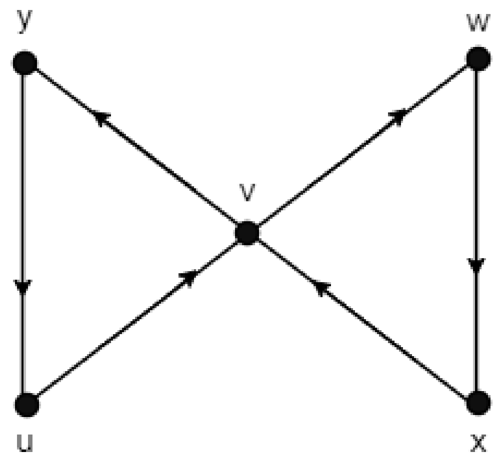

Figure 2.22: Eulerian Graph u, v, w, x, y, u

A simple path that goes through all vertices of graph but the ending point distinct, is known as **hamiltonian path**

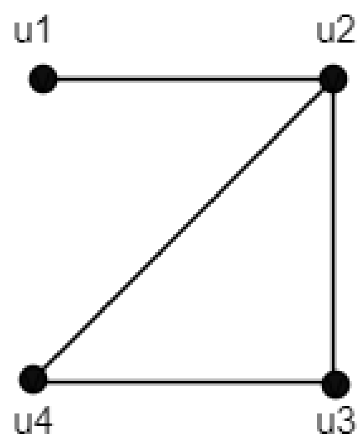

Figure 2.23: Hamiltonian Path u_1, u_2, u_3, u_4

Except the repition of starting and ending vertex, if a circuit in graph passes through each vertex only once is known as **hamiltonian circuit.**

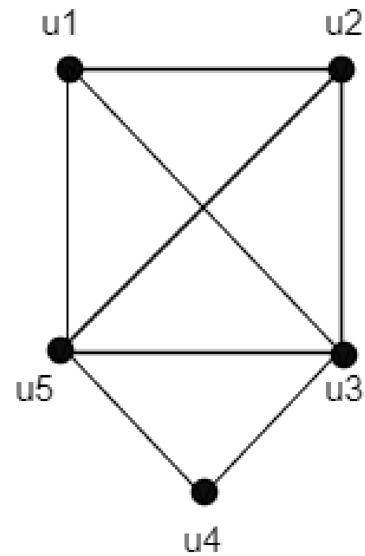

Figure 2.24: Hamiltonian Circuit u_1, u_2, u_3, u_4, u_5

A graph possessing hamiltonian circuit is named as **hamiltonian graph**.

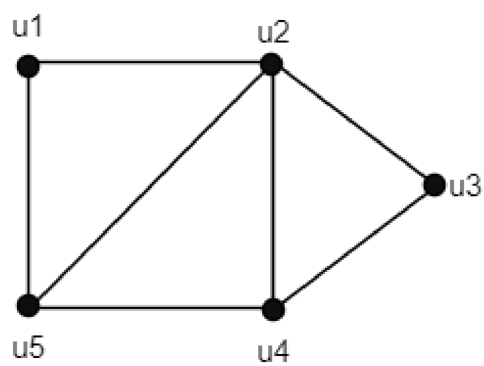

Figure 2.25: Hamiltonian Graph $u_1, u_2, u_3, u_4, u_5, u_1$

If the begining and the ending vertex in graph are same then it is known as **cycle graph** in graph theory.It is denoted as C_n A **simple cycle** in a graph pasess through distinct vertices and edges in a graph.

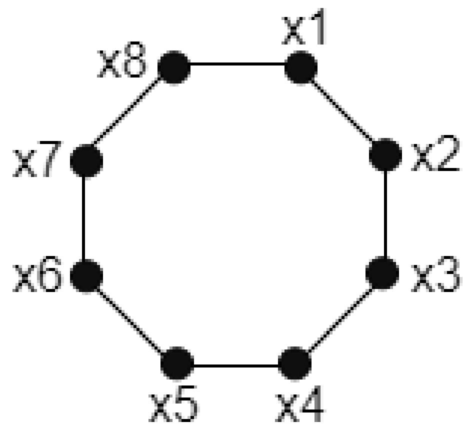

Figure 2.26: Cycle Graph

A graph having atleast one cycle which is path and that starts and end at a same point and doesn't go through any other point twice is known as **Cyclic Graph**

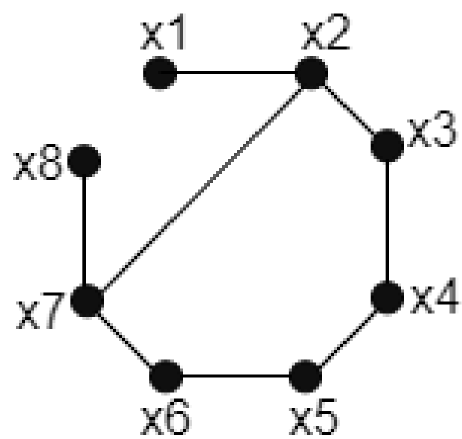

Figure 2.27: Cyclic Graph

A special sort of graph known as **wheel graph** is graph which is develops by joining one vertex to each vertex in cycle graph.W_n is a representation of wheel graph.

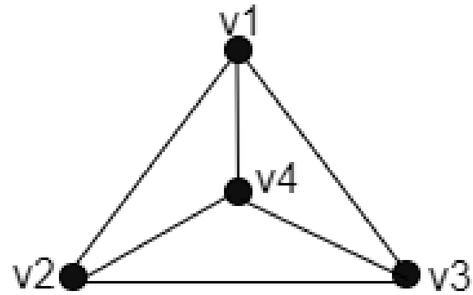

Figure 2.28: Wheel Graph W_4

If we are having two graphs and each and every vertex from both graphs are connected with each other then we can say that this is as **complete bipartite graph**.It is denoted as $K_{m,n}$

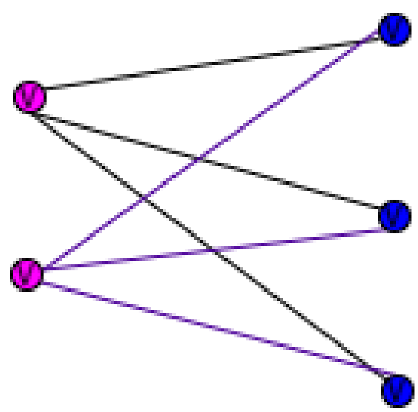

Figure 2.29: Complete Bipartite Graph $K_{2,3}$

When we have the cartesian product of a path graph P_2 with cycle C_n then we have a new graph which we named **prism graph**.The notation we used for prism is $\mathbb{P}_n$.

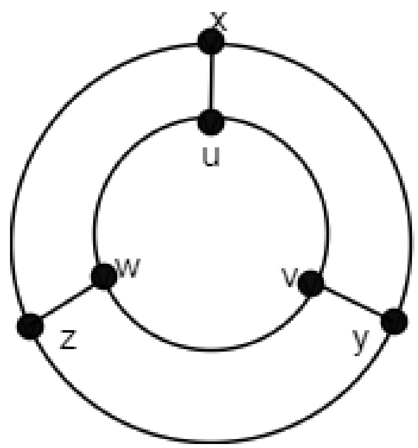

Figure 2.30: Prism Graph $\mathbb{P}_3$

A type of graph that is undirected and acyclic meaning there are no cycles init is known as **tree graph**. A tree with n vertices has exactly $n-1$ edges.

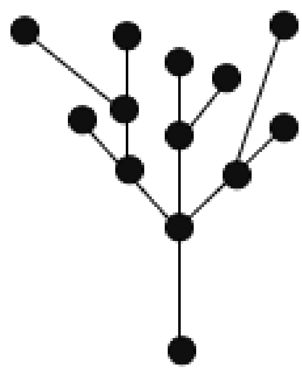
Figure 2.31: Tree Graph

If we want the **union of two graphs** then we will merge the vertex and edge set of both graphs.Mathematically we can write as $X = (v_x, e_x)$ $Y = (v_y, e_y)$ then the union is represented as $G = X \cup Y$, where vertex and edge set are $V = v_x \cup v_y$, $E = e_x \cup e_y$. $G =$

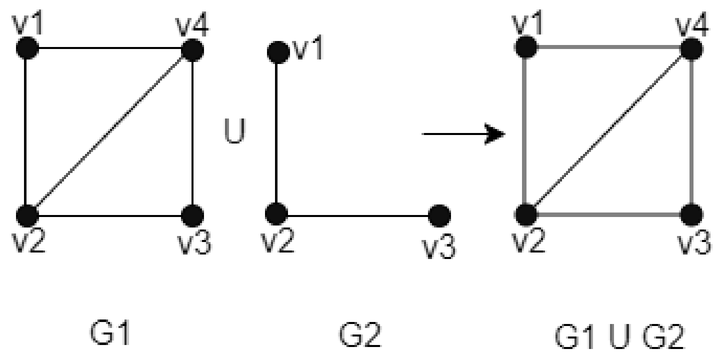

Figure 2.32: Union of Graph

Consider graphs $G1 = (V1, E1)$ and $G2 = (V2, E2)$. The **intersection of graphs** $G1, G2$ is shown as $G = G1 \cap G2$ where vertex set $V = V1 \cap V2$ and edge set $E = E1 \cap E2$.

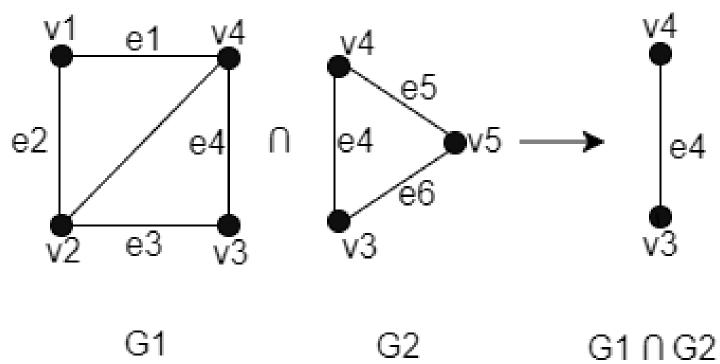

Figure 2.33: Intersection of Graph

A **complement of graph** G is a graph G' having all the vertices of graph G and there will be edge between vertices v' and e' of graph G' if and only if there is no edge between vertices v and e of graph G.

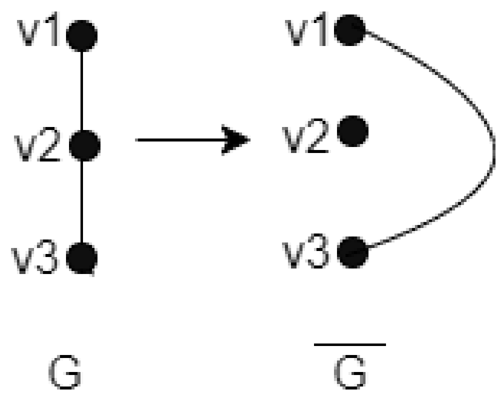

Figure 2.34: Complement of Graph

If the intersection of vertices of graphs are empty then the **sum of two graphs** $G+H$ is defined as the graph whose edge set is comprised of edges that are in both G and H along with the edges which are formed through the connection of each vertex in G to each vertex in H. The graph's vertex set is $V(G)+V(H)$.

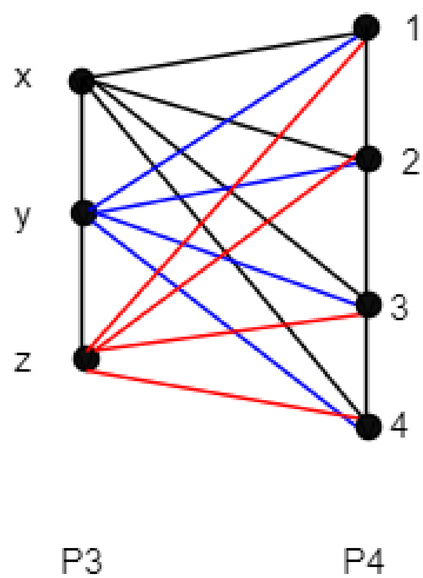

Figure 2.35: Sum of Two Graphs P_3 and P_4

Consider we have graph X having three vertices and two edges and graph Y with two vertices and one edge. Now if we take three copies of graph Y and then connect all vertices of each copies to corresponding vertex in X is known as **corona of two graphs**.

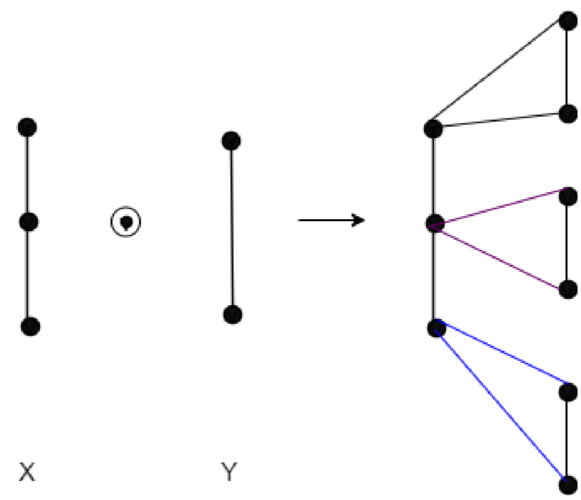

Figure 2.36: Corona of Graphs

If the edge sets of graphs $G1$ and $G2$ have a $1:1$ correspondence, meaning that if edge e_1 in $G1$ has end vertices u_1 and v_1, followed by the end point vertices u_2 and v_2 in $G2$ on the matching edge e_2 in $G2$ that corelate with u_1 and v_1, then $G1$ and $G2$ are said to be **isomorphic**. The two graphs, $G1$ and $G2$, in the example below are isomorphic.

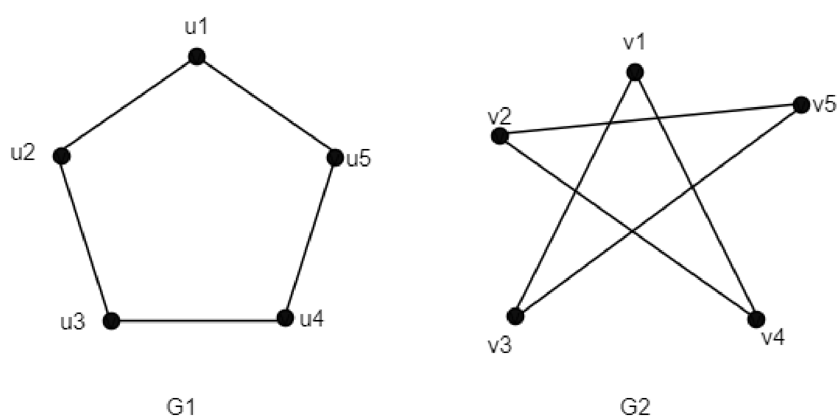

Figure 2.37: Isomorphism of Graphs

Line graph is represented by $L(G)$ and is defined as:

$L(G) = E(G)$

$E(L(G)) = \{uv : u, v \in L(G)$ and u, v are next to eachother in G.

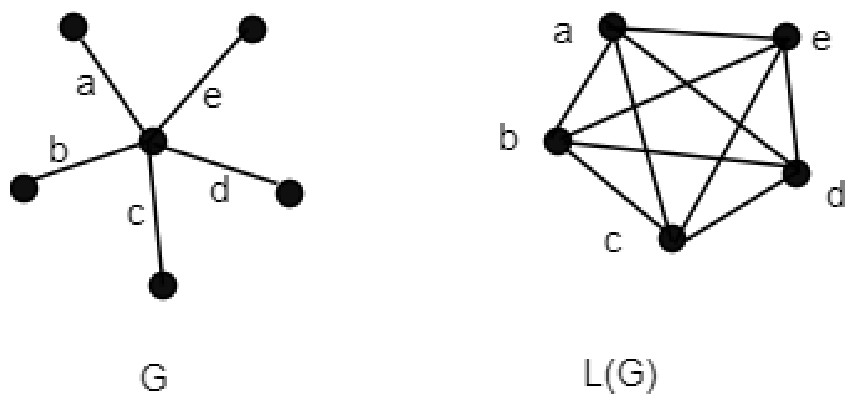

Figure 2.38: Line Graph

2.1.4 Graph Products

The process of combining two graphs G and H into a new graph is known as **Cartesian product** and we used to represent cartesian product of two graphs by $G\square H$.This product exists if and only if an edge exists between g and g' in G and $h = h'$ in H or an edge exists between h and h' in H and $g = g'$ in G. [4]

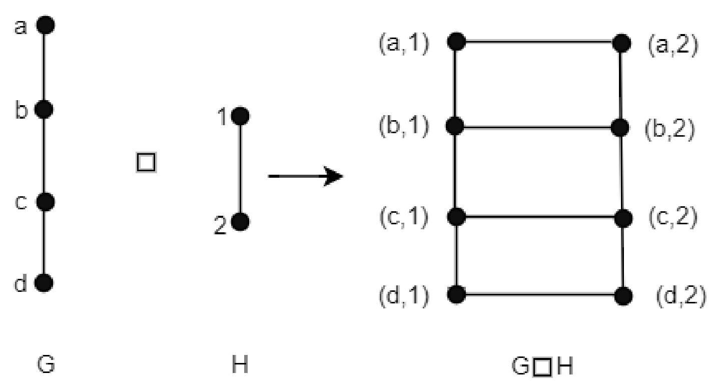

Figure 2.39: Cartesian Product of Graphs

Two graphs P and Q are said to be having **Direct Product** if and only if there is an edge between p and p' in P and q and q' in Q. Its is denoted by $P \times Q$. [4]

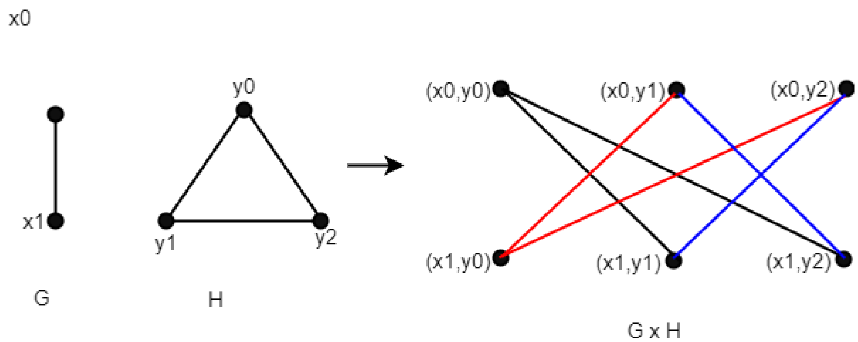

Figure 2.40: Direct Product of Graphs

Let an arbitrary regular graph X. Then the direct product of X with cycle graph of order 4 is a distance magic graph.

Composition of graph also known as **Lexicographic product of graph** G and H exists if in graph we have an edge between g and g' in G or $g = g'$ and an edge between h and h' in H. Mathematical notation for this product is $G \circ H$ or $G[H]$. [4]

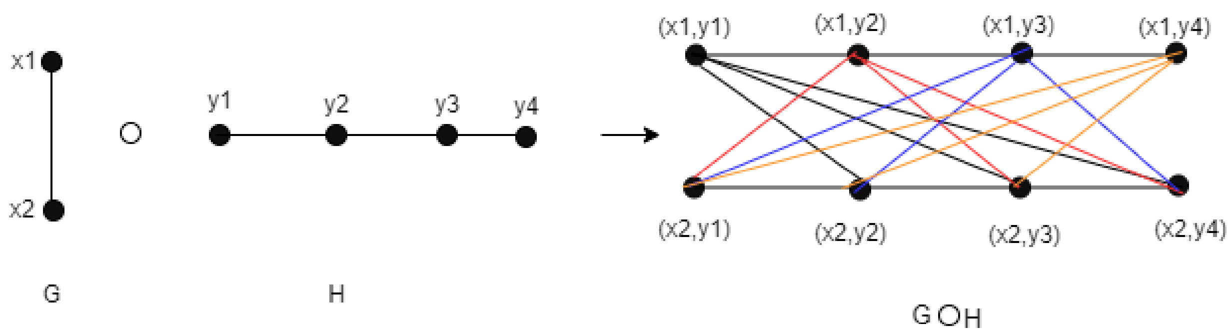

Figure 2.41: Lexicographic product of Graphs

Here we have some results of lexicographoc product which are:

Consider an r-regular graph and a cycle graph where r ranges greater than or equal to 1 and the range of n is greater or equal to 3.Then lexicographic product of graph admits a distance magic labeling.

Suppose we have intgers i, j, k and i, k are greater than equal to $1, 3$ and j is greater than 1. Then ${}_iC_k \circ K_j$ has DML iff j is even or i, j, k is odd and p is congurent to mod4.

$G \boxtimes H$ is basically representaion of **strong product**.A graph having vertex set $V(G) \times V(H)$ in strong product has adjacent vertices (g, h) and (g', h') in $G \boxtimes H$ iff $g = g'$ and h is next to h' in H $h = h'$ and g is adjacent to g' in G,or g is adjacent to g' in G and h is adjacent to h' in H. [13]

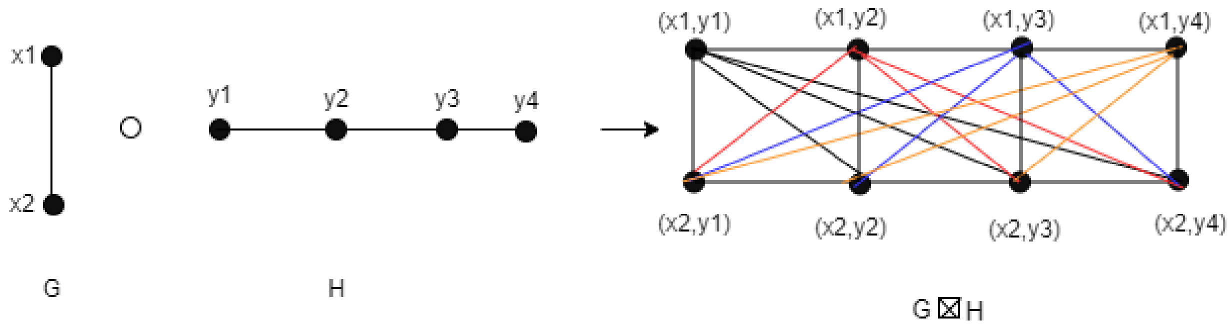

Figure 2.42: Strong Product of Graphs

If n is not congurent to 0 mod4 and is greater than 1 then the strong product of C_n and C_4 is not a DM graph.

In this chapter we discuss the main components of graphs that are vertices and edges.We aslo discuss the different types of graphs and how these different types of graphs help us in modelling and understanding different relationship and structures in real world scenario.Furthermore we discuss the the different type of graph products which are cartesian product, direct product, strong product and lexicographic product.

Chapter 3
Introduction of Graph Labelling

3.1 Overview of Graph Labeling

An important part of graph theory which includes a broad range of methods and techniques for providing labels to the elements of graph which are vertices and edges is known as Graph Labelling.To find a label that meets the appropriate requirement for the task at hand was the primary objective.Labeled graphs are useful models in the fields of database management, circuit design, coding theory, communication network addressing, and secret sharing schemes.The Conception of graph labeling was first put forward in mid-1960 and most techniques were inspired by Rosa's (1967) beta-valuation graphs [1].An injective map f from a graph $G's$ vertices to the set $\{0, 1, , q\}$ is the beta-valuation of a graph. This means that induced map g on graph's edges given as $g(xy) = |f(x) - f(y)|$ is aswell injective.For an enormous number of applications in coding theory,programme design,database administration,communicating entangalment adressing and private sharing schemes, labelled graphs are helpful. [1]

Definition 3.1.1. Graph Labeling is simply the act of giving labels (usually non-negative integers) to the graph's elements which can be vertices, edges or any combination of these.Formally we can say a labeling of graph $G(V, E)$ is a realtion that, given particular conditions maps graph components to a set of labels.

Tournament sheduling is one specific application of graph labelling. Like Round Robin Tournament, which is good for less number of teams.Each team palys aganist everyother team exactly once.In this we have equalized incomplete tournament and fair incomplete tuornament.

There are n teams and r rounds in an equalised incomplete tournament EIT(n, r). Each team in this matches up with particularly r opponents, while the overall number of opponents for every team remains fixed.A novel kind of labelling known as distance magic labelling of r-regular networks with n vertices has been generated by solving EIT(n, r).

A fair incomplete competition containing p rounds and n teams. In this each team plays every other team precisely so that the combined strength of the opponents that each team fails to score is equal..By finding solution of FIT we get a distance d-antimagic labelling in which by assigning labels to vertices,such that the sum of their labels is distinct for any pair of vertices at distance d.

3.2 Types of Graph Labeling

There are some important types of graph labelling depends on some properties of labels.

- Graceful Labelings
- Harmonious Labelings
- Magic Type Labeling

3.2.1 **Graceful Labeling**

The approach of labeling edges involves computing the absolute difference between the labels of its end vertices where m is the number of edges and vertices are labelled with different numbers which are selected from 0 to m. At first it was known as β-labeling but later on S.W.Golomb named it Graceful Labeling.[17]

- The path graph P_n is graceful for all $n \geq 1$
- K_n graphs are graceful if and only if n is less than equal to 4.

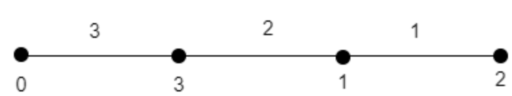

Figure 3.1: Graceful Labeling of P_3

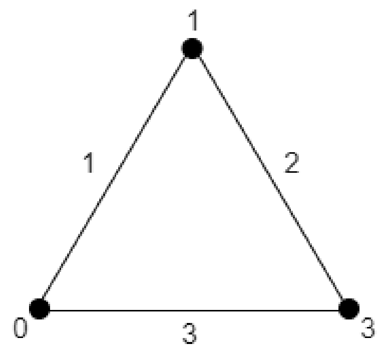

Figure 3.2: Graceful Labeling of K_3

3.2.2 Harmonious Labeling

A function f which is injective from $V(G) \to \mathbb{Z}_q$ and $f(xy) = (f(x) + f(y))(mod q)$ $\forall x, y \in E$, is an induced function which is bijection, then the injective function f is said to have a harmonious labeling of a graph G. [10] Given below is the example of harmonious labeling of two graphs

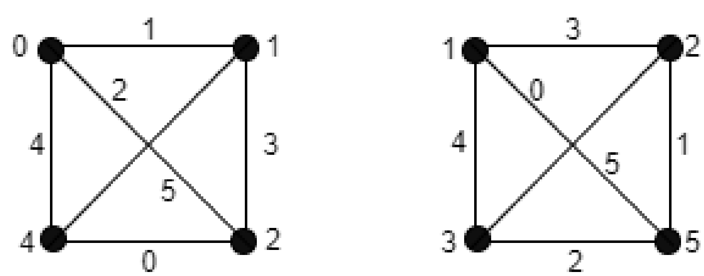

Figure 3.3: Harmonious Labeling of Graphs

3.2.3 Magic Labeling

Extension of magic squares and magic rectangles are magic squares.Extending this idea to graphs we get a sum of labels related to the vertices or edges taht are constant over the graph.in 1963 the first magic labeling was introduced by Sedlack. The graph's edges are labelled with real values, and each edge's total label sum is incident to a vertex has to remain constant.

Following are the different types of magic labeling.

Edge Magic Total Labeling:

A graph H (V, E) having a bijection from the union vertex and edge sets to $\{1,2,..., V \cup E\}$ such that for any edges xy,$\phi(x) + \phi(y) + \phi(xy)$ is constant, such as that constant is named as magic constant and this type of labeling is called magic labeling.This labeling was defined back in 1970 by Kotzig and Rosa. After years, Ringel and Llad rediscovered this idea in 1996, referring to it as EML. In 2001 Wallis named this valuation of the graph as edge magic total labeling just to differentiate it from other kinds of labelings.

Definition 3.2.1. Edge Magic Total Labeling is a mapping μ :V $\cup$ E $\rightarrow \{1,2,..., v+e\}$,there exist a positive number k for every edge $xy \in E(G)$, such that

$$\mu(x) + \mu(xy) + \mu(y) = K$$

where,

$$W_{\mu}(xy) = \mu(x) + \mu(xy) + \mu(y)$$

is called weight of edge xy.

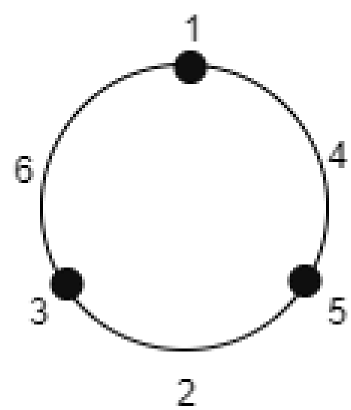

Figure 3.4: EMTL of C_3 with $K = 10$

Where K is magic constant.

- Path graph and cycle graph holds edge magic total labeling for n less than equal to 3 whereas n is odd and having magic constant $k = 3n + 1$.

-

Super Edge Magic Total Labeling :

To put it simplyv SEMTL is an EMTL if the vertices are labelled with samllest numbers.

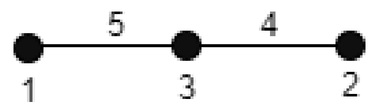

Figure 3.5: SEMTL of P_3 with $K = 9$

Vertex Magic Total Labeling:

Years back in 1999 the proposal of vertex magic total labeling was given by MacDougall, Miller, Slamin, and Wallis [3]. A mapping which is one to one from the vertices and edges onto the integers $\{1, 2, 3, ..., v+e\}$ so that the labels of incident edge and the total of the labels on the vertex are always be equal.Formally,if there is constant k.An injective mapping μ on a graph G (V, E),from V ∪ E to the set $\{1, 2, ..., |V| + |E|\}$ is a $VMTL$ such that ,

$$W_\mu(v) = \mu(v) + \sum \mu(vu), \forall v \in V.$$

where the sum is overall vertices u adjacent to v.[9]

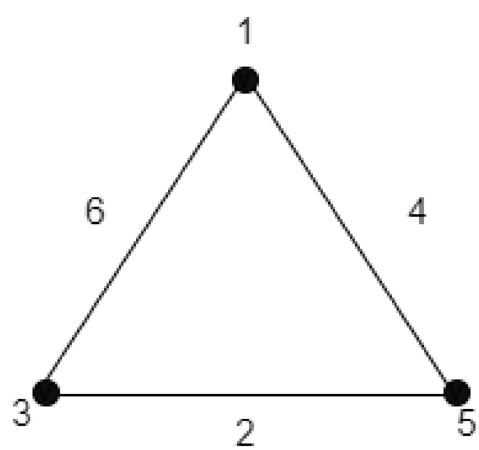

Figure 3.6: VMTL of Graph with $k = 10$

Where k is a magic constant.

Super Vertex Magic Total Labeling

VMTLμ is called super vertex magic total labeling if

$$\mu(V(G)) = \{1, 2, ..., |V| + |E|\}$$

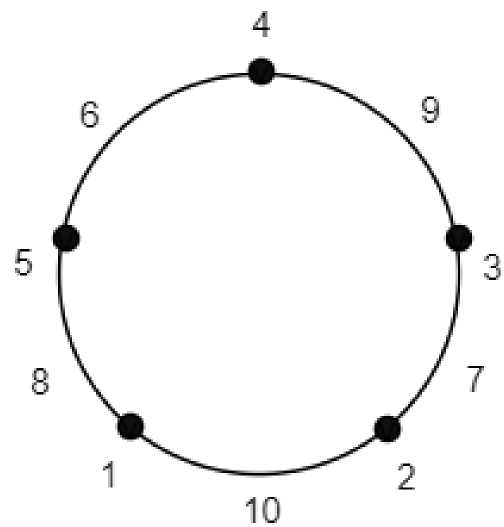

Figure 3.7: SVMTL of Graph with $k = 19$

3.2.4 Antimagic-type Labelings

Anti-magic graphs were initially suggested by Hatfield and Rigel in 1990.If the labels of the graph's q edges can be written as $\{1, 2, ..., q\}$ without doubling and the total labels of edges that are incident to each vertex differ that graph is termed as anti-magic.

Definition 3.2.2. A mapping $\phi : V \cup E \to \{$1,2,..., v+e $\}$ to form arithmetic sequence exist for a $>$ 0 with common difference d $\geq$ 0 is termed as edge antimagic total labeling . [9]

(a,d)**-Edge Antimagic Total Labeling**

For all graphs with $G=(V,E)$ a one to one mapping $\mu : V(G) \cup E(G) \to \{1,2,....,|V(G)|+|E(G)|\}$ is described as (a,d)-edge antimagic total labeling.Thus for two numbers $a>0$ and $d \geq 0$ the set of edge weight corresponds to $\{a, a+d, a+2d, ...a+(|E(G)|-1)d$.

In edge antimagic total labeling

Min. possible edge weight $= a \geq 6$

Max. possible edge weight $= (v+e)+(v+e-1)+(v+e-2)$.

Formula of d for edge anti magic total labeling.

If we have a bijection $\sigma : V \cup E = \{1,2,.......|V|+|E|\}$ admits EAMTL then:

The weight of any edge $\leq$ maximum weight

$a+(e-1)d \leq (v+e)+(v+e-1)+(v+e-2)$

$d \leq \frac{3v+3e-9}{e-1}$

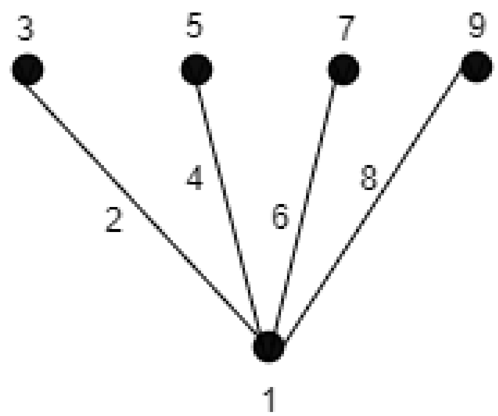

Figure 3.8: $(6,4)$ EAMTL of a Graph S_5

Super (a,d)-Edge Antimagic Total Labeling

An (a,d)-edge antimagic labeling (EAT) μ of a graph G is called super (a,d)-edge antimagic total labeling if $\mu(V) = \{1,2,.....|V|\}$.

In super edge antimagic total labeling

Min. possible edge weight $= a \geq 1+2+v+1$

Max. possible edge weight $= 3v+e-1$.

Formula of d for SEAMTL

The weight of any edge $\leq$ maximum weight

$a+(e-1)d \leq 3v+e-1$

$d \leq \frac{2v+e-5}{e-1}$

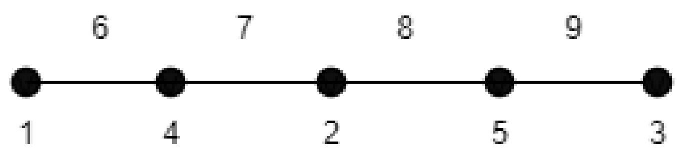

Figure 3.9: Super (a,d)-EAT Labeling of a Graph

(a,d)-Vertex-Antimagic Total Labelings

Definition 3.2.3. A linked graph $G(V,E)$ is said to have (a,d)-vertex-antimagic total labeling if it contains non-negative integers (a,d) and one-one and onto mapping.The mapping $g_f : V \to N$, which is induced is such that $g_f : E \to \{1,2,...,|E|\}$ defined as $g_f(V) = \{a, a+d, ..., a+(|V|-1)d|\}$;$g_f(v) = \sum\{f(uv)|uv\exists E(G)\}$ is injective.

In vertex antimagic total labeling

Min. possible vertex weight $= a \geq 6$

Max. possible vertex weight $= (v+e)+(v+e-1)+(v+e-2)$.

Formula of d for vertex anti magic total labeling.

The weight of any vertex $\leq$ maximum vertex weight

$a+(v-1)d \leq (v+e)+(v+e-1)+(v+e-2)$

$d \leq \frac{3v+3e-9}{v-1}$

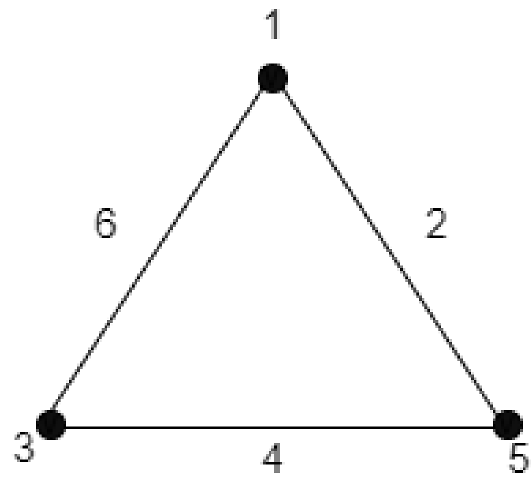

Figure 3.10: VAMTL of a Graph

Super (a,d)-Vertex-Antimagic Total Labelings

(a,d)-vertex anti magic labeling g_f is reffered as super (a,d)-vertex antimagic labeling if $g_f(V) = \{1,2,...,v\}$ and $g_f(E) = \{v+1, v+2, ...v+e\}$.

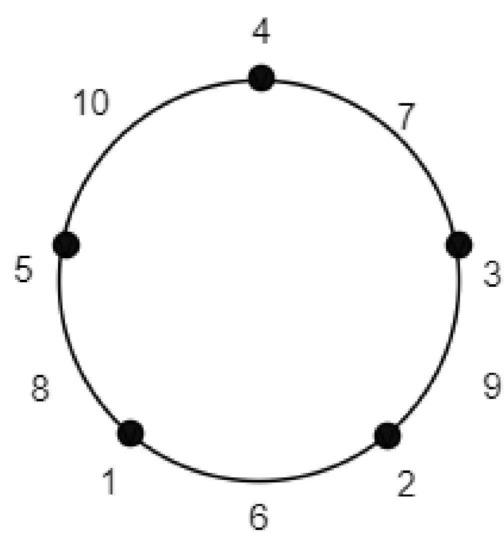

Figure 3.11: Super (a,d)-VATL of A graph

This chapter comprises many types of labeling, i.e.Vertex and edge magic total labeling,super-edge and vertex magic total labeling, edge and vertex antimagic total labeling,super edge and vertex antimagic total labeling and their correspondance graphs.We will discuss the group based magic and antimagic labelings and their outcomes in the upcoming chapter.

Chapter 4
Group Based Labelling

Assigning labels to vertices and edges of the graph based on properties/characteristics of group is known as **group based labelling**. Group based labelling provide a structured approach to represent and analyse different element of graph structure. In this, we select labels from group and apply them on vertices or edges which are the elements of graph in accordance with specific criteria. In group-based labelling we categorized them as following

- **colour**:It is used in graph colouring problems in which elements of graph are given colours in such a pattern that neighbouring elements have opposite colours.
- **weight**: In this edges are labelled with weight which represent different numerical values.
- **direction**: Edges are labelled for directions.

4.0.1 Distance Magic Labelling

A one-one and onto mapping from vertex set to {1,2,....,n} such that for each vertex x

$$\sum_{y \in N_G(x)} l(y) = k$$

,where $N_G(x)$ is the set of vertices of G adjacent to x, is DML of a graph G with magic constant k. A graph is referred as distance magic graph if it has a DML.[14]

It is technique in which vertices/edges are labelled in such a way that they have specific properties at a certain distances between labelled elements.The sum of labells whith in the graph has a uniform value known as **magic constant**. Magic constant is specific value associated with labelling of a graph. If we have a achieved a magic constant in graph it means that labelling of graph is satisfied and it will help in different application domains like network optimization, coding theory and cryptography. The notion of distance magic labeling emerged through the building of magic squares. A distance magic labelling of a graph $G = (V, E)$ is a bijection $\sigma : V \to \{1, 2, ..., |V|\}$ such that the total of labels in neighborhood $N(x)$ is not dependent on x is distance magic labeling of graph $G = (v, E)$. This total is known as magic constant.

Few results of group distance magic labeling are listed below.

- A necessary requirment for a graph G of order n to have magic labeling is

  $$k_n = \Sigma_{x \varepsilon V} deg(x) l(x)$$

  where k is magic constant. [12]

- A path graph P_n of n vertices holds distance magic labeling only for n equals 1 or 3. [12]

- Cycle graph C_n having length n is said to be DM graph if n is equal to 4 [12]

- If $n = 1$ then the complete graph having n vertices holds DML. [12]

- Tree graph T is distance magic if it is equal to path graph of length 1 or 3. [12]

- A graph will not be considered as distance magic graph if it has order n and two vertices has degree $n - 1$.[11]

- Assuming x, y, z are all from V and if the graph G is distance magic with y lying next to x and z and degree of x and y equalling two and that graph G either has a component that is isomorphic to C_4 or is itself is isomorphic to C_4. [3]

Balanced Distance Magic Labelling

A subclass of DML has been found which is called balanced distance labelling and it behaves well among products. A DM graph G with even number of vertices is said to be balanced if there exist a one-one and on-to mapping $l : V(G) \to \{1, ..., |V(G)|\}$ such that for every $p \in V(G)$ the following holds: if $q \in X(p)$ with $Y(q) = i$ then $\exists$ r $\in X(p)$, where $r \neq p$, with $l(r) = |V(G)| + 1 - i$. The r and p are the twin vertices where Y is called a balanced distance magic labelling. A balanced distance magic graph's twin vertices can not be next to one an other and that $N_G(r) = N_G(p)$.[2]

Some results on balance distance magic labeling.

Results

- From two graphs G and H, if one of them is balanced distance magic and the other graph is regular graph then the direct product of G and H is a balanced distance magic graph. [15]
- If n or m is qual to 4 then the direct product of cycle graph C_m and C_n is balanced distance magic graph. [15]

4.0.2 Group Distance Magic Labelling

Overview

The concept of Group Distance Magic Labelling (GDML) was initiated by Froncek. It is extension of distance magic labelling. In GDML instead of taking labels for the elements of graph from integers we take labels from the mathematical groups and these labels exhibits some specific properties related to distances between them in graph. Uptill now many researchers have worked on this topic and they explore that which graph holds such labelling and their properties under different groups.[6]

If there is an abelian group Y' of order n and a one-one map from the vertex set of$Y$ to the group elements such that$\Sigma_{y \in N_{G(x)} l(y)} = \sigma$ for all $x \in V$ where σ is the magic constant, then the graph Y is said to have GDML. This type of graph is known as Y' distance magic graph.

Uptill know much work have been done on distance magic labelling of groups such as tetravalent circulant graphs (the graphs included in this are finite, simple and undirected), Group distance magic and antimagic hypercubes,GD magic graphs $G[C4]$, Group Distance Magic Labeling of Graphs and their Direct Product of antiprism graphs and many more.

Some of the results from different researches are as follows:

- $C_n \square$ C_n is a group distance magic , for n greater than equal to 3 and is even. [2]

- For even n the n-dimensional hypercube Q_n is $\mathbb{Z}_n^2$ -distance magic. [2]

- The $G[\overline{K}_n]$ is distance magic for any even n where G be an arbitrary regular graph.

- Let G and H be isomorphic to A_3 and A_n where A_3 and A_n be anti-prism graphs such that n is equal to $3m$,where m is greater than equal to 1 and m not equal to $3k, k \geq 1$. $\mathbb{Z}_3 \times \mathbb{Z}_4 n$ be the module group of order $12n$. Then the graph $G \times H$ admits a $\mathbb{Z}_3 \times \mathbb{Z}_4 n$ - DM labeling.[2]

For order-n graph G and an order n abelian group H. $l : V(G) \to \mathscr{H}$ so that for each $x \in V$, $w(x) = \Sigma_{y \in N_G(x) \lambda(y) = \mu}$, where $\mu \in \mathscr{H}$ a one-one map is used for group distance magic labelling.Actually labels applied to graph elements are from abelian groups because they provide labelling which is mostly very close to the group distnace magic labelling.We have a proved result which states that if graph is a balanced distance magic graph than that graph is also a group distance magic graph. Another proved fact is that every DM graph is GDM graph under mudulo group $\mathbb{Z}_n$. bht the converse is not true. [2]

Here we have some proved results on magic graphs of $\mathbb{Z}_n$ distance.

- It is not $\mathbb{Z}_n$-distance magic in graph G if it is an r-regular distance magic on n vertices and r is not even. [16]

- $\mathbb{Z}_n$-distance magic graph is the direct product of cycle graphs C_m and C_n if m or n is $4, 8$ or m, n is equivalent to $0(mod 4)$.[2]

- If G and H are regular graphs and G and H both have $\mathbb{Z}_n$ balanced labeling then $G \boxtimes H$ has a $\mathbb{Z}_n$ -balanced labeling. [16]

4.0.3 Group Distance Antimagic Labelling

A bijection f from $V(G)$ to A in order that the weights of all vertices of G are different pairwise is termed as distance anti-magic labeling for groups. We claim that each element of A occurs as the weight of exactly one vertex of G since G and A have same cardinality. The results for group distance antimagic are as follows:

- A graph G r-regular with n even vertices. Therefore unless r is odd the graph G cannot be A-distance antimagic for every abelian group A of order n with exactly one involution.

- If n and r are both even in r regular graph of n vertices then G is not $\mathbb{Z}_n$ -distance antimagic.

- If we have r-regular graph G having n vertices and $r = 2k+1$ then the graph is not a $\mathbb{Z}_n$-distance magic.

4.0.4 Applications

Group distance and anti magic labels are used in tournament scheduling.An unbaised or equitable method of organising a competition in which each team plays each other once.In network designing several types of networks are designed and optimised.

4.0.5 Orientable Group Distance Magic Labelling

Overview

From the idea of group distance magic labelling Brayn Freyberg intoduced orientable group distance magic labelling.In this the vertices and edges of graph are labelled with the elements of group in a way that the group operations and distances related properties are satisfied and labelling respect the orientation of graph. In OGDML each edge is given a direction or orientation.For every edge there is a specific starting and ending vertex.

Just like group distance magic labelling alot of work have been done on orientable group distance magic labelling also like Orientable Group Distance Magic Labeling of Anti-Prism Graphs, OGDML of Direct Product of Anti-Prism Graphs with cycle, Orientable $\mathbb{Z}_n$-distance magic regular graphs, Orientable $\mathbb{Z}_n$-distance magic labeling of the cartesian product of two cycles. [16]

The proved results from above are as follows:

- Complete graph K_n when n is odd is orientable $\mathbb{Z}_n$-distance magic graph. [16]

- Consider a directed anti-prism graph A_n and $\mathbb{Z}_{2n}$ be the modulo group, then A_n admits orientable $\mathbb{Z}_{2n}$-DM labeling.

The only distinction between the methods used to define GDML and OGDML is how the weights of the vertices are determined.If we have an abelian group $\mathscr{H}$ and one-one map l from the vertex set of G to the group elements then a directed graph G is orientable group distance magic labeling iff $\Sigma_{y\in N^+_G(x)} \overrightarrow{l}(y) - \Sigma_{y\in N^-_G(x)} \overrightarrow{l}(y)$.

4.0.6 Applications

Just like GDML the applications of OGDML are mathematical modelling, cryptography,algoarthim designs, combinatorial designs.

In this chapter we allocate positive integers to vertices, edges and both in every labeling so that the weight of graph's vertices could be easily computed.We link positive integers to modulo groups with respect to addition in next chapter creating a labeling technique called group distance magic labeling. we will discuss the GDML of direct product of graphs which is our main work and will prove some results.

Chapter 5
Group Distance Magic Labelling of Graphs

In this chapter we provide the group distance magic labelling of direct product of prism graph with cycle in the following theorems.In the theorem given below,we have the direct product of prism graph with cycle $\mathbb{P}_n \times C_4$ under mudulo group $\mathbb{Z}_{2mn}$ for having a magic constant.

Theorem 5.0.1. *Let $G \cong \mathbb{P}_n$ and $H \cong C_4$, where $\mathbb{P}_n$ is prism graph of order n and C_4 is cycle graph of order 4 and $\mathbb{Z}_{8n}$ be the mudulo group of order 8n.Then the direct product of graph $G \times H$ admits a $\mathbb{Z}_{8n}$ distance magic labelling $\forall\ n \geq 3$.*

Proof

The vertex and edge set of $\mathbb{P}_n$ and C_4 are as follows:

$V(\mathbb{P}_n) = \{ x_t, y_t | 0 \leq t \leq n-1 \}$,

$E(\mathbb{P}n) = \{ x_t x_{t+1}, y_t y_{t+1}, x_t y_| 0 \leq t \leq n-2 \} \cup \{ \mathrm{x}_0 x_n - 1, y_0 y_n - 1, x_n - 1 y_{n-1} \}$,

$V\ (C_4) = \{\mathrm{x}'_t, 0 \leq t \leq 4-1 \}$,

$E\ (C_4) = \{\mathrm{x}'_t x'_t{+}_1, 0 \leq t \leq 4-2 \} \cup \{\mathrm{x}'_0 x'_4{-}_1\}$,

By using concept of direct product we get the vertex set below representing $\mathbb{P}_n \times C_4$,

$V\ (\mathbb{P}n \times C_4) = \{(\mathrm{x}'_t x'_r), (y_t, x'_r) | 0 \leq t \leq n-1, 0 \leq \mathrm{r} \leq \text{4-1} \}$.

Define: $l : V\ (\mathbb{P}_n \times C_4) \rightarrow \mathbb{Z}_{8n}$ under following way

$l\ (x_t, x'_r) = 2t + 4nr,\ 0 \leq r \leq 1$

$l\ (x_t, x'_r) = (4n_r - 1) mod(8n) - 2t,\ 2 \leq r \leq 3$

$l\ (y_t, x'_r) = 2t + 2n(1 + 2r),\ 0 \leq r \leq 1$

$l\ (y_t, x'_r) = (4nr - 12n + 1) mod(8n) - 2t,\ 2 \leq r \leq 3$

In the above graph $\mathbb{P}_n \times C_4$ we have two types of vertices which are representing the vertices and that are (x_t, x'_r) and (y_t, x'_r). Now we will use the labeling l to find the weight of each vertex.

The vertex type (x_t, x'_r) is adjacent to following vertices:

$$(x_{t+1}, x'_r), (x_{t+(n-1)}, x'_{r+1}), (y_t, x'_{r+1}), (x_{t+1}, x'_{r+2}), (x_{t+(n-1)}, x'_{r+2}), (y_t, x'_r + 2)$$

and the vertex type (y_t, x'_r) is adjacent to following vertices:

$$(x_t, x'_{r+1}), (x_{t+(n-1)}, x'_{r+1}), (y_{t+1}, x'_{r+1}), (x_t, x'_{r+2}), (x_{t+(n-1)}, x'_{r+2}), (y_{t+1}, x'_r + 2)$$

Weight of vertices type (x_t, x'_r) and (y_t, x'_r) can be calculated in such a way,

for $r = 0 \leq r \leq 1$

$$w(x_t, x'_r) = (x_{t+1}, x'_r), (x_{t+(n-1)}, x'_{r+1}), (y_t, x'_{r+1}), (x_{t+1}, x'_{r+2}), (x_{t+(n-1)}, x'_{r+2}), (y_t, x'_r + 2)$$

$w(x_t, x'_r) = l(2(t+1) + 4n(r+1)) + l(2(t+n-1) + 4n(r+1)) + l(2t + 2n(1 + 2(r+1))) + l(2(t+1) + 4n(r+2) + 2) + l(2(t+1) + (n-1) + 4n(r+2)) + l(2t + 2n(1 + 2(r+1)))$

for $r = 2 \leq r \leq 3$

$$w(x_t, x'_r) = (x_{t+1}, x'_r), (x_{t+(n-1)}, x'_{r+1}), (y_t, x'_{r+1}), (x_{t+1}, x'_{r+2}), (x_{t+(n-1)}, x'_{r+2}), (y_t, x'_r + 2)$$

$w(x_t, x'_r) = l((4n(r+1) - 1) mod 8n - 2(t+1)) + l((4n(r+1) - 1) mod 8n - 2(t+n-1)) + ((4n(r+1) - 12n + 1) mod 8n - 2t) + ((4n(r+2) - 1) mod 8n - 2(t+1)) + ((4n(r+2) - 1) mod 8n - 2(t+n-1)) + ((4n(r+2) - 12n + 1) mod 8n - 2t)$

for $r = 0 \leq r \leq 1$

$$w(y_t, x'_r) = (x_t, x'_{r+1}), (x_{t+(n-1)}, x'_{r+1}), (y_{t+1}, x'_{r+1}), (x_t, x'_{r+2}), (x_{t+(n-1)}, x'_{r+2}), (y_{t+1}, x'_r + 2)$$

$w(y_t, x'_r) = l(2t + 4n(r+1)) + l(2(t+n-1) + 4n(r+1)) + l(2(t+1) + 2n(1 + 2(r+1))) + l(2t + 4n(r+2)) + l(2(t+n-1) + 4n(r+2)) + l(2(t+1) + 2n(1 + 2(r+1)))$

for $r = 2 \leq r \leq 3$

$$w(y_t, x'_r) = (x_t, x'_{r+1}), (x_{t+(n-1)}, x'_{r+1}), (y_{t+1}, x'_{r+1}), (x_t, x'_{r+2}), (x_{t+(n-1)}, x'_{r+2}), (y_{t+1}, x'_r + 2)$$

$w(y_t, x'_r) = l((4nr+4n-1)mod8n-2t) + l((4nr+4n-1)mod8n-2t-2n+2) + ((4nr-8n+1)mod8n-2t-2) + ((4nr+8n-1)mod8n-2t) + ((4nr+8n-1)mod8n-2t-2n+2) + ((4nr-4n+1)mod8n-2t-2)$

The magic constant for $\mathbb{P}_n \times C_4$ under modulo $\mathbb{Z}_{8n}$ is:

$$k = (2mn-3)$$

Following are the examples of above theorem.

Example.5.0.1. Direct product of graph $\mathbb{P}_3 \times C_4$ under mudulo group $\mathbb{Z}_{24}$ having magic constant 21.

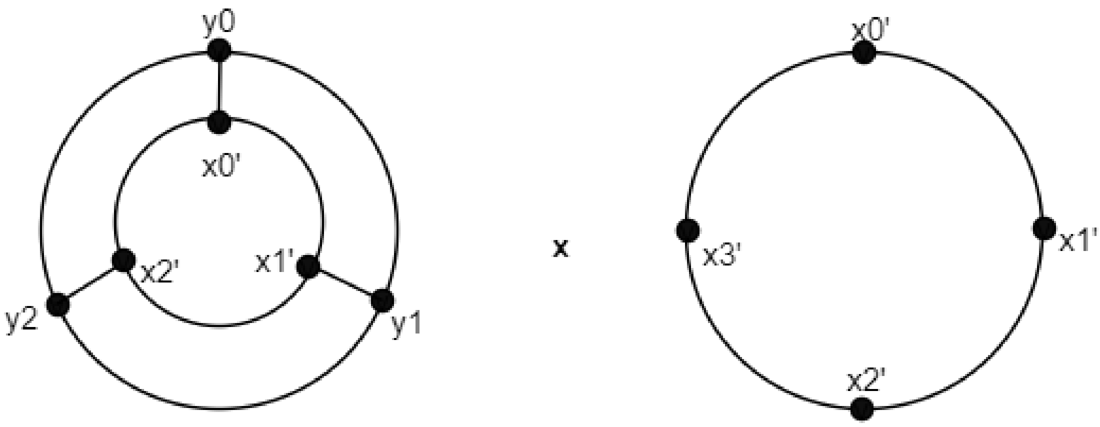

Figure 5.1: $\mathbb{P}_3 \times C_4$

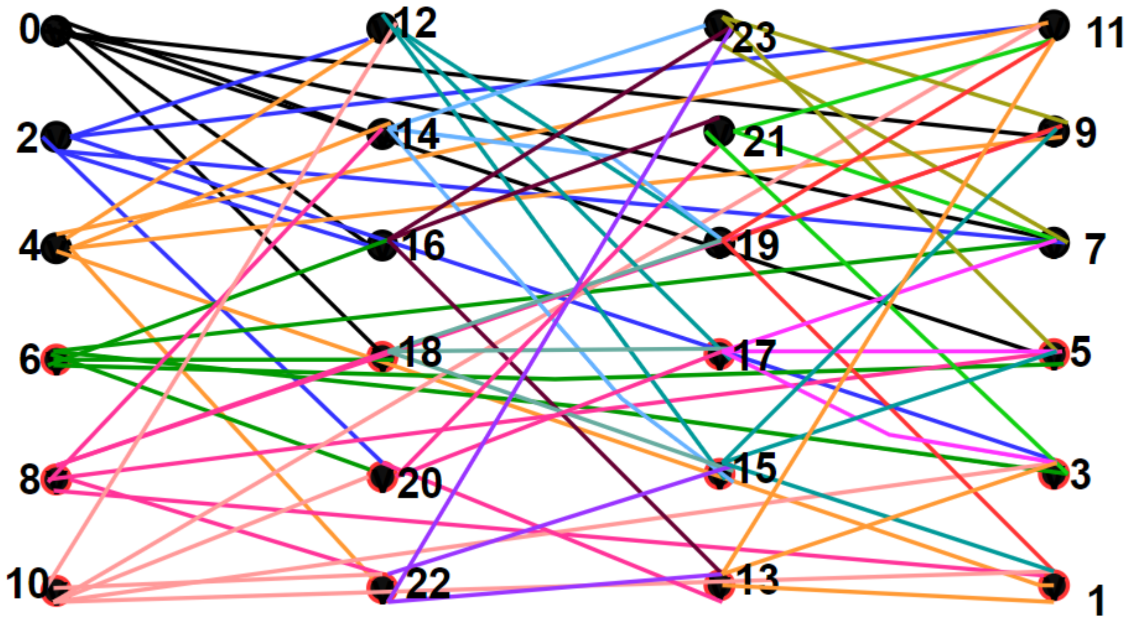

Figure 5.2: Labelling of Direct Product of graph $\mathbb{P}_3 \times C_4$ under $\mathbb{Z}_{24}$

Example.5.0.2. Direct product of graph $\mathbb{P}_4 \times C_4$ under mudulo group $\mathbb{Z}_{32}$ having magic constant 29.

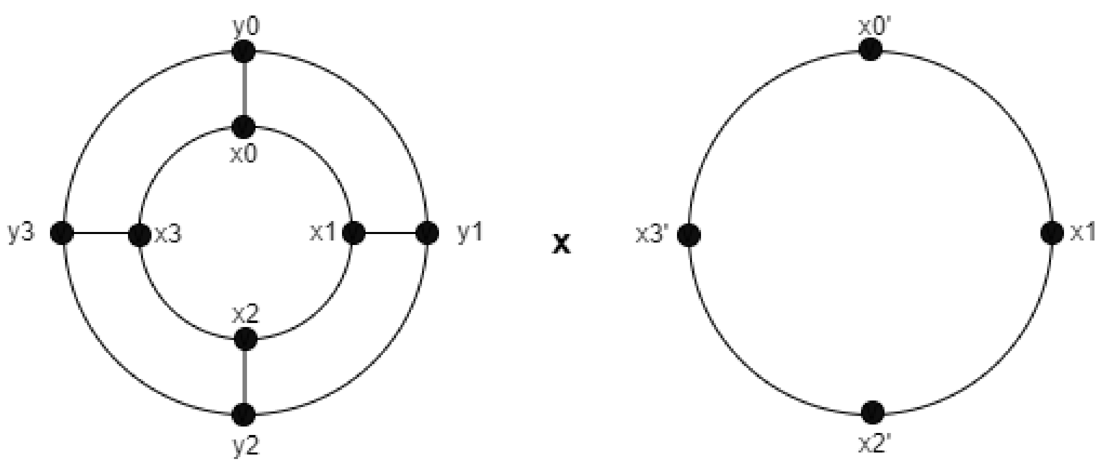

Figure 5.3: $\mathbb{P}_4 \times C_4$

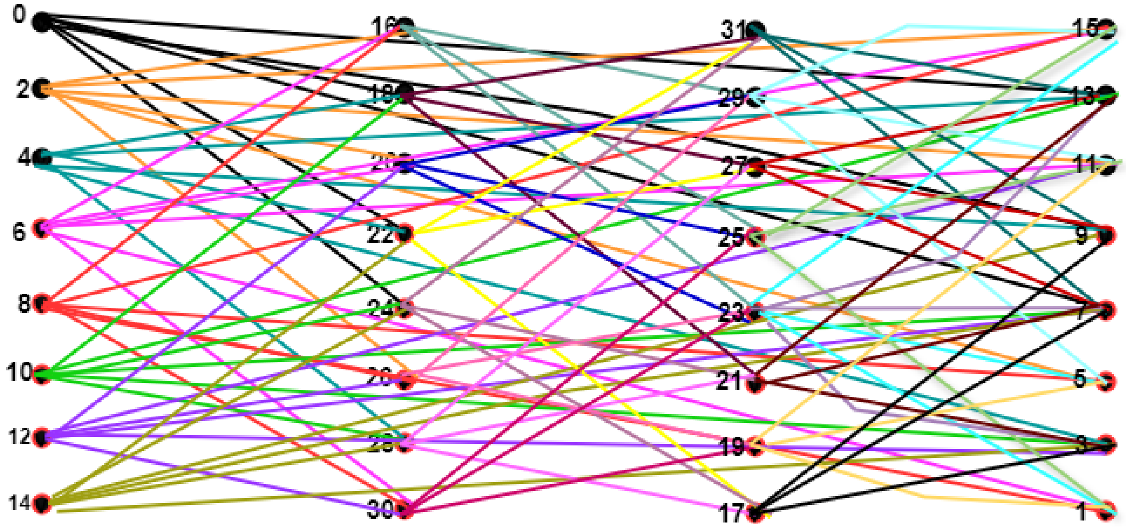

Figure 5.4: Labelling of Direct Product of graph $\mathbb{P}_4 \times C_4$ under $\mathbb{Z}_{32}$

Example.5.0.3. Direct product of graph $\mathbb{P}_5 \times C_4$ under mudulo group $\mathbb{Z}_{40}$ having magic constant 37.

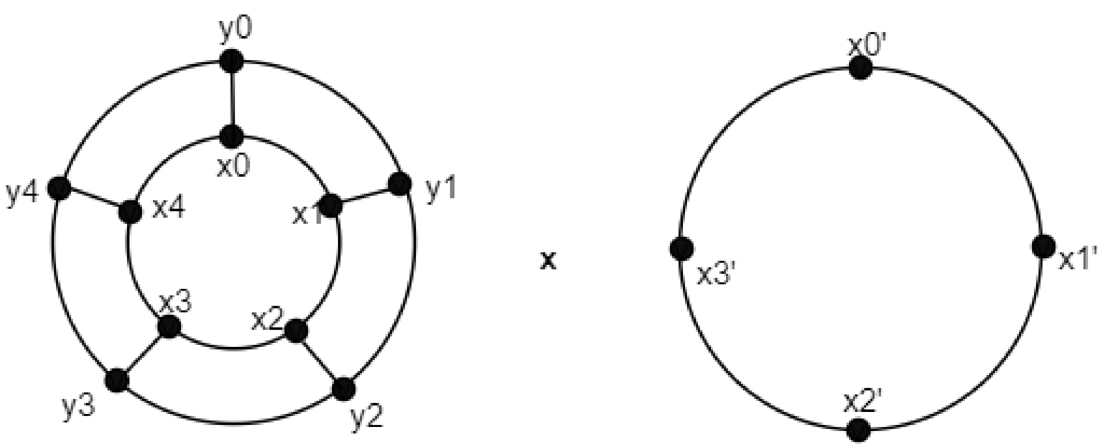

Figure 5.5: $\mathbb{P}_5 \times C_4$

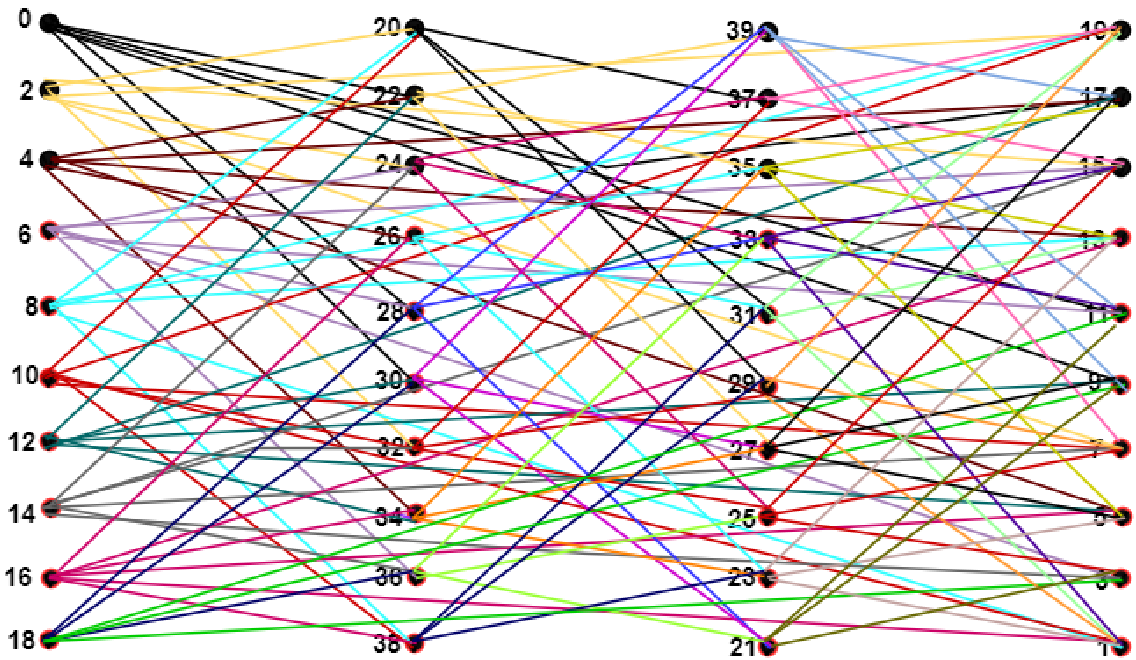

Figure 5.6: Labelling of Direct Product of graph $\mathbb{P}_5 \times C_4$ under $\mathbb{Z}_{40}$

Example.5.0.4. Direct product of graph $\mathbb{P}_8 \times C_4$ under mudulo group $\mathbb{Z}_{64}$ having magic constant 61.

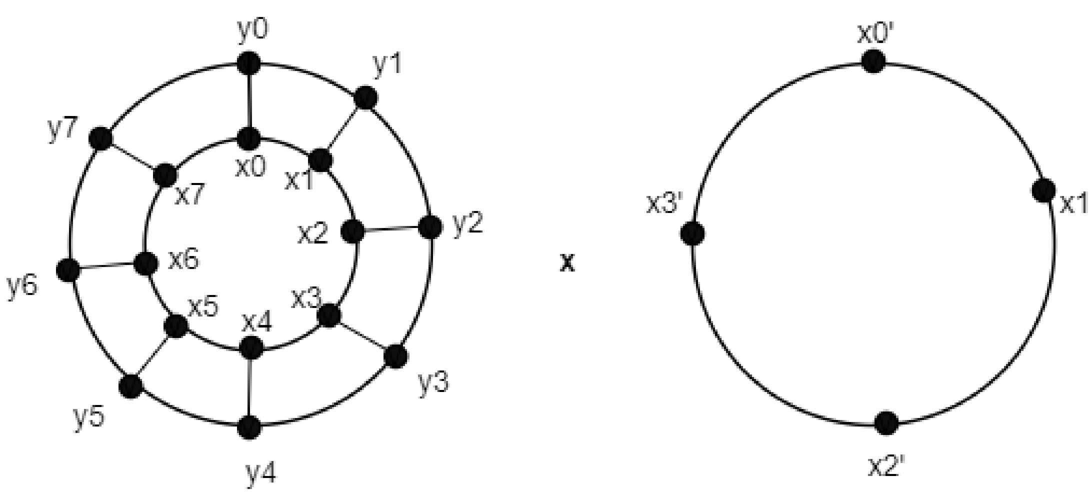

Figure 5.7: $\mathbb{P}_8 \times C_4$

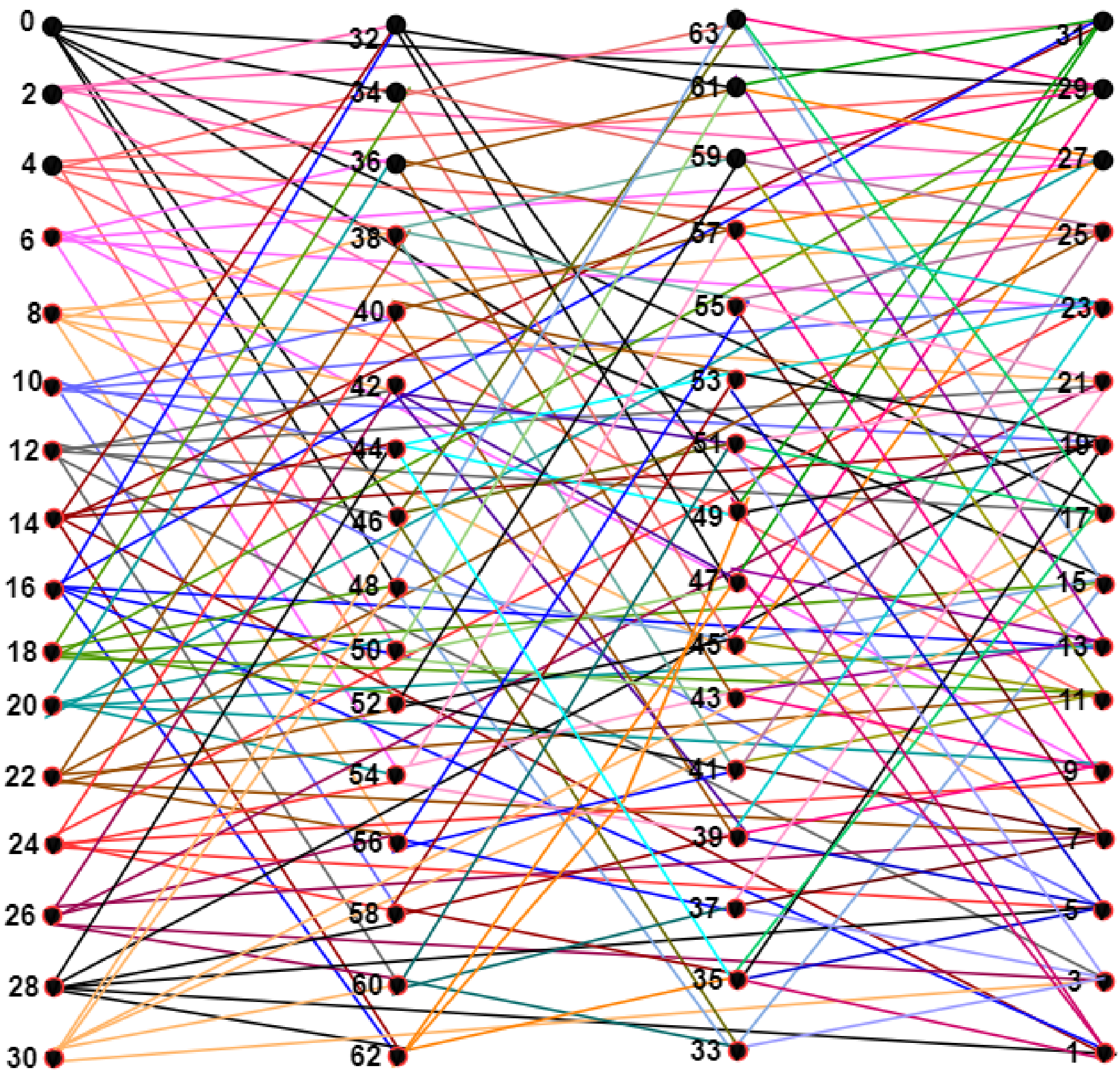

Figure 5.8: Labelling of Direct Product of graph $\mathbb{P}_8 \times C_4$ under $\mathbb{Z}_{64}$

Theorem 5.0.2. *Let $G \cong \mathbb{P}_n$ and $H \cong C_m$, where $\mathbb{P}_n$ is prism graph of order $2n$ and C_m is cycle graph of order m and $\mathbb{Z}_m \times \mathbb{Z}_{2n}$ be the modulo group of order $2nm$ such that $gcd(n,m) \neq 1$.Then the direct product of graph $G \times H$ admits a $\mathbb{Z}_m \times \mathbb{Z}_{2n}$ distance magic labelling $\forall$ n =3 and m = ≥ 4.*

Proof

The vertex and edge set of $\mathbb{P}_n$ and C_m are given below.

V $(\mathbb{P}_n) = \{x_i x_{i+1}, y_i y_{i+1}, x_i y_i, n \leq i \leq m\} \cup \{x_0 x_2, y_0 y_2\}$

E $(C_m) = \{x'_i x'_{i+1}, n \geq i \geq m\} \cup \{x'_0 x'_{i-1}\}$

Given below is the labeling of graph $\mathbb{P}_3 \times C_3$ under $\mathbb{Z}_3 \times \mathbb{Z}_6$ abd $\mathbb{P}_3 \times C_4$ under $\mathbb{Z}_3 \times \mathbb{Z}_8$.

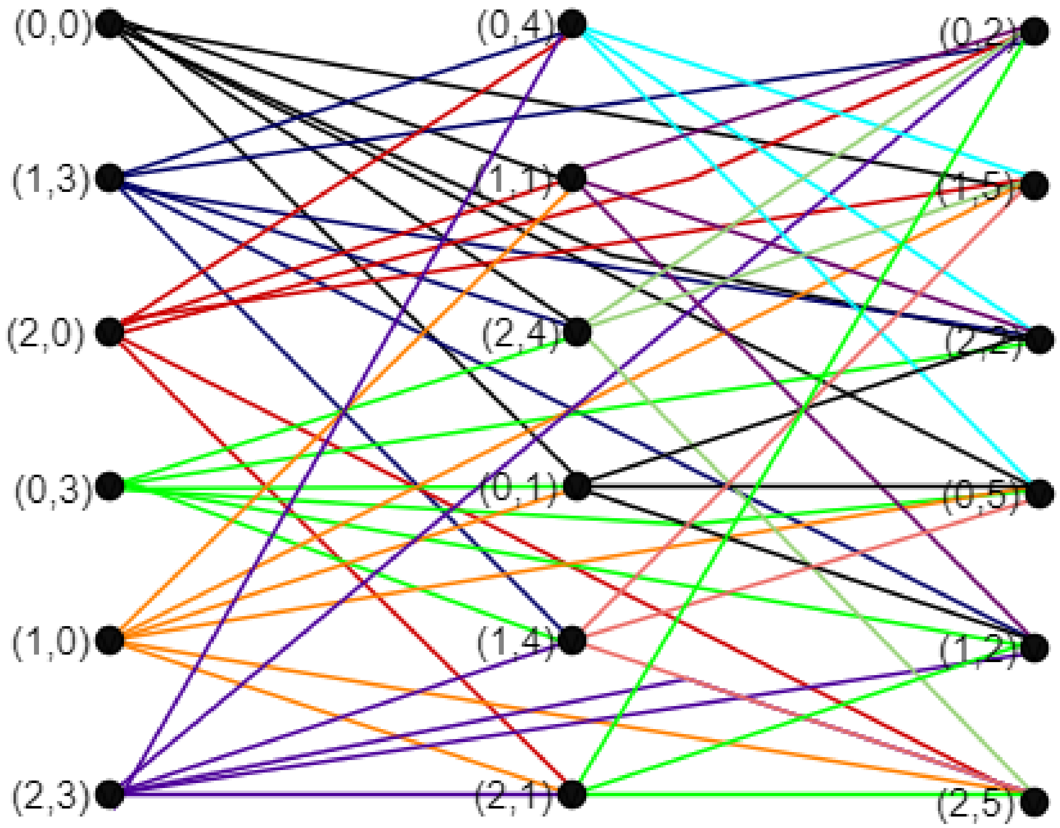

Figure 5.9: Labeling of Direct Product of $\mathbb{P}_3 \times C_3$ under $\mathbb{Z}_3 \times \mathbb{Z}_6$

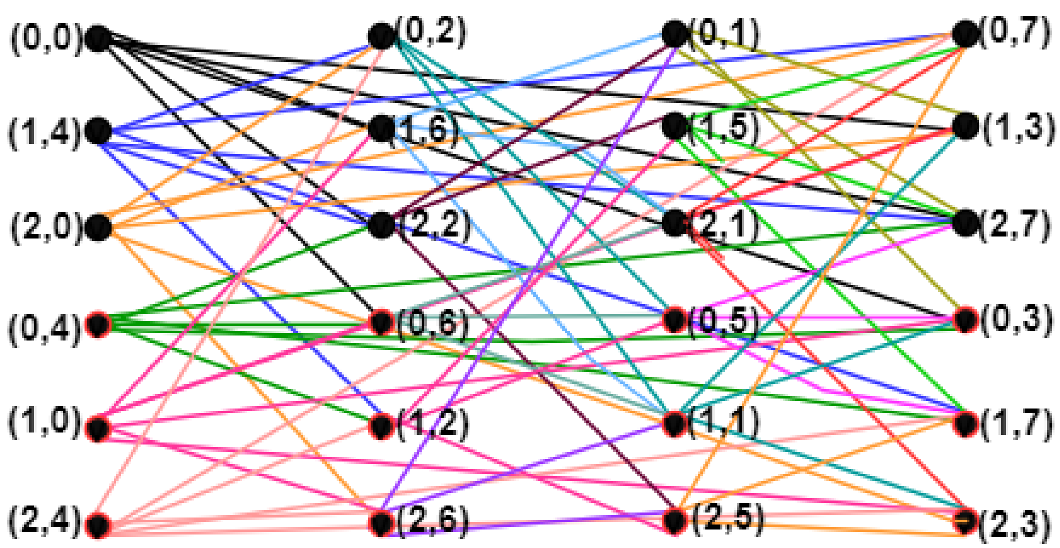

Figure 5.10: Labeling of Direct Product of $\mathbb{P}_3 \times C_4$ under $\mathbb{Z}_3 \times \mathbb{Z}_8$

Chapter 6
Conclusion

In our research we look into labeling techniques of group distance magic labeling.Modulo groups and their products are discussed using GDML. Basically we have tried to eastablish a bridge between graph theory and group theory both.

In the first chapter we have discussed how the subject of graph theory enter in the field of mathematics, what is the motivation behind it. Which type of researches and work have been done so far. What its role in our lives and how it help us. Its different applications in different fields.

In the second chapter we have a brief discussion on what are graphs and different components of graphs. We also studied different types of graphs, different operations and products on them and how they are applicable.

In third chapter we discussed graph labeling in detail.Overview of graph labeling like from where the idea of graph labeling came and how the different researches have been done on it. We also discussed different types of labeling which are graceful labeling,harmonious labeling, magic labeling and anti-magic labelings with the help of different graphs.

In fourth chapter we talked on group based labelings and its types that are group distance magic labeling, group distance anti-magic labeling and orientable group distance magic labeling. We have also discussed some proved results of these labelings and their applications.

In the fifth chapter, which is our main chapter we worked on group distance magic labeling of direct product of prism graphs. We generalized the direct product of prism graph with cycle $\mathbb{P}_n \times C_4$ under modulo group $\mathbb{Z}_{2mn}$. Then we have discussed the examples of it like $\mathbb{P}_3 \times C_4$ under modulo group $\mathbb{Z}_{24}$, $\mathbb{P}_4 \times C_4$ under modulo group $\mathbb{Z}_{32}$ and $\mathbb{P}_5 \times C_4$ under modulo group $\mathbb{Z}_{40}$. We have calculate the magic constant and weight of their vertices.We have also find the direct product of prism graph with cycle in which prism graph is of order 2_n and cycle is of order m under modulo group $\mathbb{Z}_m \times \mathbb{Z}_{2n}$ of order 2_{nm} with $gcd(n,m) \neq 1$ $\forall n = 3$ and $m \geq 4$.

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