Orientable Group Distance Magic Labeling of Regular Graphs and Their Direct Product

MS Thesis

by

Sana Ali

CIIT/FA22-RMT-048/LHR

COMSATS University Islamabad
Pakistan

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Abstract

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Graph labeling provides connectivity operation in networks used in computer networking, chemical structures, circuit design, and database administration.Group Distance Magic Labeling (GDML) combines graph theory with group theory by using Abelian groups. A graph G has a GDML if we use elements of group for the labeling of graph's components in such a way that the weight of each vertex in its neighborhood is contants in that group. The main focus of this work is on digraph orientable group distance magic labeling(OGDML). If an group H exits a digraph G, and if there is a injective map ϕ from G vertex set to the group members, then for every $x \in V$, there exists a set of values such that $\sum_{y \in N^{+}_{G(x)}} \phi^{\rightarrow}(x) - \sum_{y \in N^{-}_{G(x)}} \phi^{\rightarrow}(x)$. We study oriented graphs in this work. In particular, special labeling (OGDML) on directed graphs is the main emphasis of this study on oriented graphs.In this study, we prove that the directed direct product of Prism graphs $\mathbb{P}_n$ and Cycle C_n is OGDML under these non-isomorphic modulo groups $\mathbb{Z}_{2nm}$, $\mathbb{Z}_2 \times \mathbb{Z}_n$, $\mathbb{Z}_n \times \mathbb{Z}_{2m}$, $\mathbb{Z}_2 \times \mathbb{Z}_{n/2} \times \mathbb{Z}_{2m}$, and $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_{n/4} \times \mathbb{Z}_{2m}$.

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Chapter 1
Introduction

A significant area of mathematics that examines the characteristics and uses of graphs structures composed of vertices connected by edges is called graph theory, and it was first introduced by Leonhard Euler in the eighteenth century. By emphasizing structure, flow, and connectedness, this abstract representation aids in the modeling and problem-solving of many systems.Numerous disciplines, including computer science, biology, the social sciences, logistics, and transportation, heavily rely on graph theory. It facilitates the creation of effective network security protocols, search algorithms, and data structures. In addition, it optimizes routes and timetables, simulates intricate relationships in genetic networks and protein structures, and is essential to data mining, machine learning, and artificial intelligence.

The earliest recorded publication on graph theory dates back to 1736 when Leonhard Euler published his research on Königsberg bridges. Euler's work and that of Vandermonde, Cauchy, and L'Huillier established the foundation for topology by investigating the relationships between the faces, vertices, and edges of convex polyhedra [8].Almost two centuries later, Listing and Cayley continued to advance topology. Cayley researched tree graphs, combining chemistry and mathematics principles to develop enumerative graph theory [4], which was improved by Polya in the 1930s. De Bruijn later generalized these discoveries, spanning the fields of chemistry and mathematics.

In 1878, Sylvester created the term "graph" to describe the relationship between algebraic concepts and graph structures. Denes König wrote the first graph theory textbook in 1936, followed by Frank Harary's renowned volume, which expanded the field's appeal to professionals in mathematics, chemistry, and electrical engineering. Effective interaction between engineers and social scientists has been highlighted as critical.The four-color problem, introduced by Francis Guthrie in 1852, is one of the most well-known graph theory issues [12].Is it feasible to color a map with only 4 colors while ensuring that no nearby places have the same hue?. This problem, first mentioned in communication between De Morgan and Hamilton, prompted substantial inquiry and multiple false proofs by mathematicians such as Cayley, Kempe, Heawood, Ramsey, Tait, and Hadwiger [14]. Their research led to the investigation of graph colorings on diverse surfaces, as well as the formulation of factorization problems. The four-color problem, introduced by Francis Guthrie in 1852, is one of the most well-known graph theory issues. Does a map have to be colored in all four hues for any two adjacent areas to have the same color? This problem, first mentioned in communication between De Morgan and Hamilton, prompted substantial inquiry and multiple false proofs by mathematicians such as Tait, Heawood, Cayley, Ramsey, Kempe, and Hadwiger.

Their research led to the investigation of graph colorings on diverse surfaces, as well as the formulation of factorization problems.For over a century, the four-color problem remained unresolved In 1969, Heinrich Heesch suggested using computers to solve it. Kenneth Appel and Wolfgang Haken successfully implemented this strategy in 1976, delivering a computer-assisted proof using Heesch's discharge method, despite early criticism due to its complexity.

Between 1860 and 1930, topology and graph theory developed independently but complemented one another, with key contributions from Jordan, Kuratowski, and Whitney. Topology used new algebraic approaches, as demonstrated by Gustav Kirchhoff's 1845 work on electric circuits, which related voltage and current. Erdős and Rényi's [17] work on probabilistic graph theory led to the development of random graph theory and asymptotic results, significantly expanding the subject.Overall, the history of graph theory is characterized by interdisciplinary collaboration and revolutionary advances that have influenced modern mathematical and scientific research.

1.1 Application of Graph in Diverse Areas

Graphs are an effective tool for expressing relationships and processes in a variety of domains, including physical, biological, social, and informational systems. Graphs, often known as ”networks” when nodes and/or edges include attributes, are especially helpful in computer science for representing communication networks, data structures, computing devices, and computational operations [22]. For example, directed graphs can reflect the structure of a website, with vertices representing web pages and edges representing links between them. Similar approaches are used to address challenges in social networking, travel planning, biology, and computer chip design.

Managing graphs is a major subject in computer science, which has resulted in the creation of graph rewriting systems for defining and visualizing graph changes, as well as graph databases for storing and retrieving durable graph-structured data. Graph-theoretic approaches are also often used in linguistics. The hierarchical structure of syntax and compositional semantics frequently uses tree-based graphs, although directed acyclic graphs are employed in current syntactic theories such as head-driven phrase structure grammar. Furthermore, semantic networks are important in computational linguistics for modeling word meanings within contexts, as demonstrated by initiatives such as WordNet and VerbNet [21]. In chemistry and physics, graph theory is useful for investigating molecular structures and condensed matter. In chemistry, vertices represent atoms and edges indicate bonds inside a molecule, allowing for computer-aided molecular structure processing and database searches [8].

Graphs are used in statistical physics to investigate system components and physical process dynamics, whereas graphs are used in computational neuroscience to explain functional correlations between brain areas, which aids in cognitive function understanding. Similarly, graphs can represent the channels in porous materials, which connect pores via smaller channels.

Graph theory is extremely useful in sociology, particularly in social network analysis for assessing actor status or tracking rumor transmission. Friendship graphs portraying personal contacts, impact graphs expressing individual influence, and cooperation graphs depicting team efforts, such as in film making, are all examples of social network graphs. Graph theory is widely used in biological and environmental conservation, where vertices represent habitats and edges indicate migration patterns, allowing researchers to examine breeding behaviors, disease transmission, and the effects of movement changes on species.

1.2 Graph Valuation

The process of assigning numerical numbers to edges and vertices based on the kind and purpose of the graph is known as graph labeling. Since its introduction in 1960, graph valuation has been the focus of several research. The concept of magic squares, which were initially put out by Sedlacek in 1963, was influenced by this idea—an array of numerical values with equal horizontal, vertical, and diagonal row-line totals. This grid is usually represented by a square matrix. The magic constant or magic sum is the name given to this common sum. A lot of focus was placed on vertex magic total labeling (VMTL), edge magic total labeling (EMTL), harmonic and elegant labeling, as well as other graph labeling approaches like DML, GDML, and OGDML, at the start of the 1990s. Every kind of valuation has distinct features and uses.

The notion of Distance Magic Labeling, or DML, originated in 1994 with Vilfred's sigma labeling, in which the sum of the weights of each vertex is constant. Sugeng et al. later dubbed it DML, referring to magic squares. In the research on distance magic labeling there are some several important findings have been proved. For example,a path graph is DML only of order 1 and 3 .

Similarly, the cycle graph can only be labeled in this manner if it has four vertices. This type of valuation applies to tree graphs only if the tree has a single vertex or a three-vertex path. Also DML is only achievable in wheel graphs with four vertices.

For w is even or uvw is odd, and $u,v,w \geq 1$ then a graph $uK_{v[w]}$ admits DML And for wv is odd, $v \equiv 3(\mod 4)$ and u is even it is not DML. These findings show the precise conditions under which distance magic labeling works across different types of graphs. M. Kashif Shafiq and colleagues extended DML to include unions of distinct graphs, demonstrating its existence under specific conditions [20].

Dalibor Froncek introduced GDML, or Group Distance Magic Labeling, in 2013 [10]. In GDML, labels are assigned from an Abelian group, which includes the magic constant. Under some circumstances, Froncek gave a demonstration of GDML for Cartesian and direct products of cycles. GDML was expanded by Marcin Anholcer et al. [1] to accommodate direct graph products.They showed GDML in $Cn \times Cm$ for $\mathbb{Z}_m$ and $\mathbb{Z}_n$ of orders m and n if $m,n \equiv 0(mod 4)$. Sylwia Cichacz [5] extended GDML to include lexicographic products of regular graphs with cycles and bipartite graphs, defining many requirements for the presence of DML in these graphs.She also develop a formula $\mu = \frac{r(n+1)}{2}$for regular graph G.

Bryan Freyberg et al. proposed OGDML,for directed graphs in 2017. This includes directed cycles, full graphs, regular graphs, and bipartite graphs.

In OGDML, each vertex's weight is computed by subtracting the sum of its outer and inward edges.An directed group DML of directed graph G is described as map $\overrightarrow{\lambda} : V \to (G,+) \forall x \in V$ such that $\sum_{y \in N^+_{G(x)}} \lambda^{\to} l(x) - \sum_{y \in N^-_{G(x)}} \lambda^{\to}(x) = \mu \forall \text{x} \in V.$, where G is the group under $+$. Freyberg's team demonstrated OGDML for Cartesian products of cycles under specified conditions, but left open questions about its application to graphs of certain orders and degrees.

Overall, GDML and OGDML seek to integrate graph theory with group theory by employing group elements to name graph components and compute their weights via group operations. This work aims to further the link between algebra and graph theory by introducing fresh insights and approaches for labeling distinct graph families.

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1.3 Objectives of the Research

Our research primarily focused on the mathematical analysis of Oriented Group Distance Magic Labeling (OGDML), specifically its applications to chemical graphs and network labeling. The main goal was to answer a basic question about graph labeling using modulo groups and their products. Our goal was to create modulo groups to label various graph families using modulo n operations, with a focus on prism graphs. Doing the labeling computation for the oriented direct product of cycles and prism graphs—which involved non-isomorphic modulo groups and their products—was another objective.

Graph labeling with modulo groups has important consequences in coding, networking, design development, and other aspects of computer science. This study uses labeling techniques to connect graphs and modulo groups. The goal of using these labeling approaches is to generate a sequence that correlates graph order with modulo-group characteristics. This study promotes theoretical understanding while simultaneously strengthening practical applications in computing and related fields.

Chapter 2
Basics of Graph Theory

2.1 Fundamental Notations

A graph is an abstract representation of information. It shows the relationship between variables using lines, arrows, or other shapes. In mathematics, a graph is pairwise structure which is consisting of two sets, a set of lines (the edge set) and a collection of points (the vertex set).Furthermore there is a particular relation between vertex set and edge set. Graphs are used to solved hundreds of problems in several fields, such as determine the quickest transportation route between two cities, optimizing traffic flow, to find out any unique location,to assign different channels to a television station, to determine which specific switch is present in an electric circuit, to scheduling tasks and optimizing useful resources in construction projects etc. There are many kinds of graphs, including simple graph multi-graph, null graph, finite graph ,infinite graph ,K-regular graph, irregular graph connected graph, disconnected graph , planner graph ,non-planner graph, tree graph, prism graph, Peterson graph, cycles, banana tree, harray (circulant or gear), wheel graph, fan graph,' snake graph and anti-prism graph, directed and undirected graphs. These graphs are classified by their order, sizes, degrees and characteristics.

2.2 Definition and Terminology

In mathematics, graphs are the combination of points and lines, where those points in a graphical structure are named vertices/points, and the lines between two nodes are known as edges. A graph is a type of data structure made up of an ordered or unordered set of pairs of nodes (edges) and a finite number of nodes (vertices). Networks such as social networks, communication networks, and linked data structures are models that are created using it.

Order of Graph

For most graph-related tasks, including ideas, algorithms, and comparative graph analysis, this locality property is a fundamental feature of graph theory. Counting the vertices in a graph is one technique to estimate its size (very roughly); the graph's order does not account for links!

Definition 2.2.1. The order of a graph, (G), is the total number of nodes in its vertex set; for instance, $|V|$ indicates the order of G and the number of points in the collection V.

Example 2.2.1. *In the graph in the picture below has an order of* 4 *since it has four vertices.*

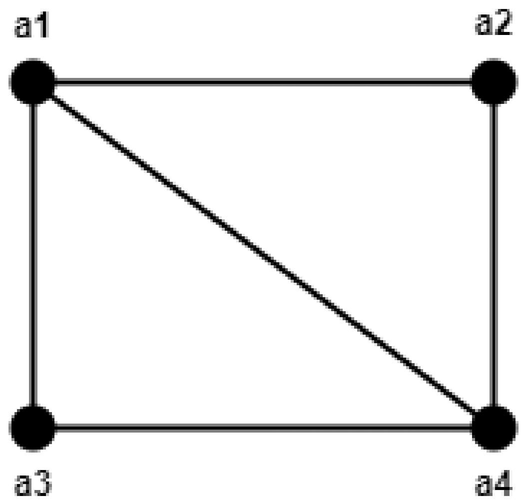

Figure 2.1: Graph

Size of Graph

In graph theory, a graph's order is a basic parameter that is important for many ideas, methods, and analyses connected to graphs. The number of edges in a graph, which indicate the connections or interactions between its vertices, determines the network's size.

Definition 2.2.2. The term *size* of a graph G refers to the count of its edges.To put it mathematically, the cardinality of the set E is $|G|$, or (G), and the size of G is shown as such if $G(V,E)$ is a graph.

Example 2.2.2. *For example, a cycle graph C_n, has size n.In figure 2.1 ,the graph has five edges so the size of graph is* 5 .

Types of Edges

There are different types of edges depending on their characteristics following are defined below

- **Loop:** An edge that joins a vertex to itself and begins and ends at the same vertex is known as a loop in graph theory. Another name for this kind of edge is a loop edge or a self-loop edge. A graph G has to have a loop if its vertex and edge sets are both singleton sets.

  Definition 2.2.3. A Loop is a edge whose initial and end point a is same. Formally, a graph G with a order pair (v,e) ,a loop is a edge $e = v,v$ where both endpoints of the edge is same vertex .

  In the figure 2.2 ,the graph has a edge loop $e8$ which start and end at same vertex $v5$.

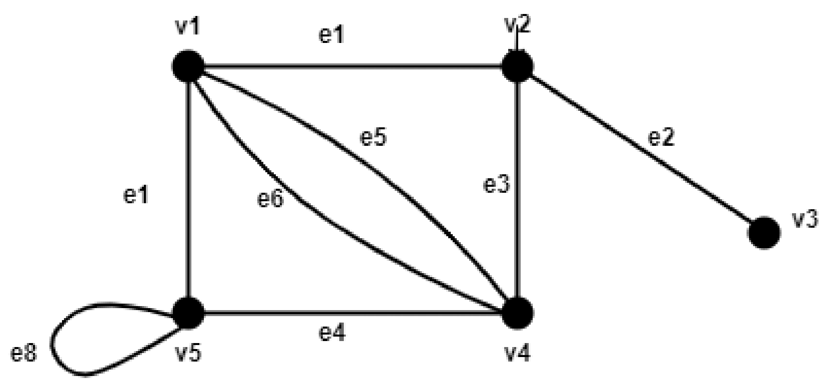

Figure 2.2: Adjacent Edges

- **Adjacent Edges:** If two non-parallel edges share a same vertex, they are considered neighboring.

  Definition 2.2.4. In graph theory, two or more edges in a graph that are adjacent to one another at a common vertex are referred to as neighboring edges.

For adjacent edges in figure 2.2 we see the sets of edges $\{e7, e5, e6, e1\}, \{e3e4, e5, e6\}$ and $\{e1, e2, e3\}$ these are three sets of adjacent edges as they all are sharing some common vertex.

- **Parallel Edges:** When two or more graph edges share ends, they are referred to as parallel edges. In above figure 2.2 , $e5$ and $e6$ are parallel edges as they both have same endpoints $(v1, v4)$.

2.2.1 Types of Vertices :

Here are some important types of vertices defined below

- **Isolated Vertex:** In a graph, an isolated vertex is one that is devoid of any edges. Stated otherwise, it is a vertex with a degree of 0, which indicates that it has no neighbors in the graph. Stated otherwise, an isolated vertex is a vertex in the graph G that is not the last point of any edge. In the below figure ,vertex $v1$ is an isolated vertex as it is not attached to any other points of the graph.

- **Adjacent Vertices:** Two or more edges which are joined by an edge are named as neighbors or adjacent vertices of each other.From figure 2.3 we can see,$v3v5$ are adjacent vertices of each other but $v4, v5$ are not adjacent to each other as their is no edge between them.Also $v2v3, v2v5, v3v4, v3v5, v5v6$ and $v2v6$ are all the adacent vertices from the below graph.

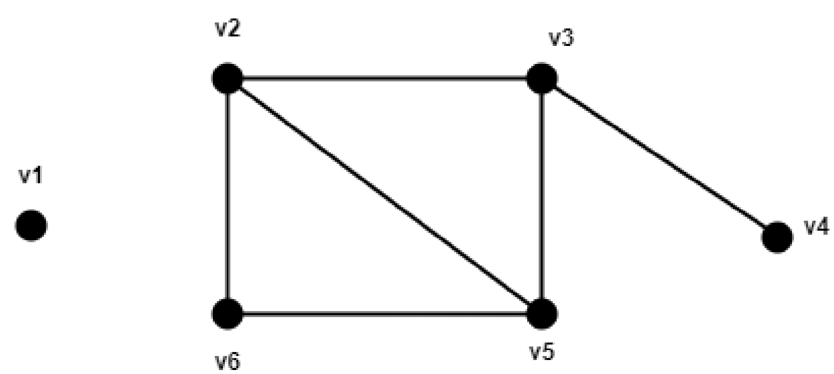

Figure 2.3: Isolated Vertex

- **Pendant Vertex:** A pendant vertex, often referred to as a leaf, is a vertex with a degree of one and just one edge incident to it that is linked to only one other vertex. The vertex $v4$ in above figure 2.3 is a pendant vertex as it is of degree 1.

- **Even Vertex:** A vertex in a graph is named as Even vertex , if it has even degree.For example, in Cycle graph C_n of order n ,each vertex is a even vertex as every vertex has even degree. The vertex v6 in figure 2.3 ,is an even vertex as it has even degree or there are even number of edges attached to it.

- **Odd Vertex:** A vertex in a graph is named as Odd vertex , if it has odd degree.For example,in Prism graph of n order , each vertex is a odd vertex as every vertex has odd degree.i.e.,the vertices $v2, v3, v4$ and $v5$ are odd vertices as all these have odd number of edges attached to them in figure 2.3.

2.2.2 Types of Graphs:

Graphs are categorized in to several kinds depending on their properties:

- **Simple Graph:** An undirected graph with many edges and no loops is called a simple graph.

  Definition 2.2.5. In graph theory, simple graphs are essential because they are undirected, have no loops, and have just one edge.

  In other words simple graphs are those which more than one edges between two vertices or a edge with same end and starting point.In the figure given below **(a)** is simple graph having no loops or multi edges.

- **Multi-graph:** As compared to simple graphs, a multigraph is a kind of graph in graph theory that permits numerous parallel edges connecting vertices. It may have loops and more than one edge joining the same pair of vertices.

  Definition 2.2.6. A graph G is referred to as a multi graph if it has loops or more than one edge connecting two vertices.

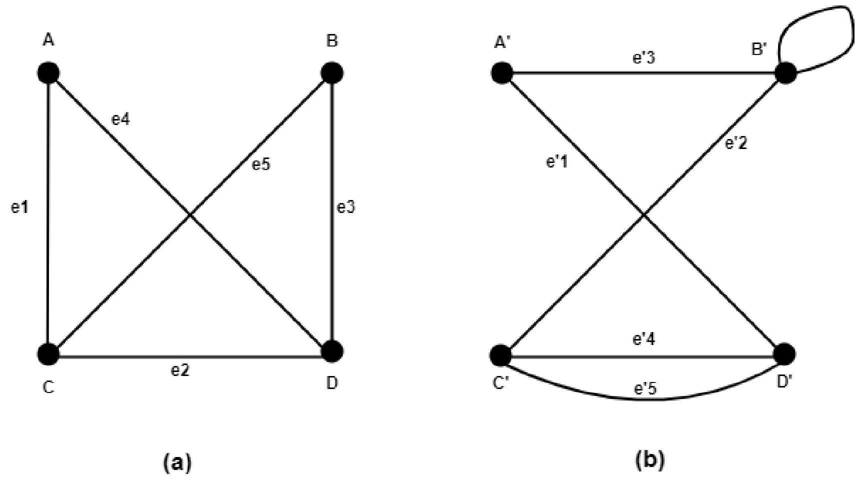

Figure 2.4: Graphs

The part **(b)** of the figure 2.4 is multi graph as it has a loop at vertex **B'** and had multi edges between vertecies **C'** and **D'**.

- **Finite Graphs:** A graph finite if it has the limited number of components. Mathematically ,if the Order $|V(G)| \leq \infty$ and the size $|E(G)| \leq \infty$ of a graph G is restricted and countable then it is named as a Finite graph.

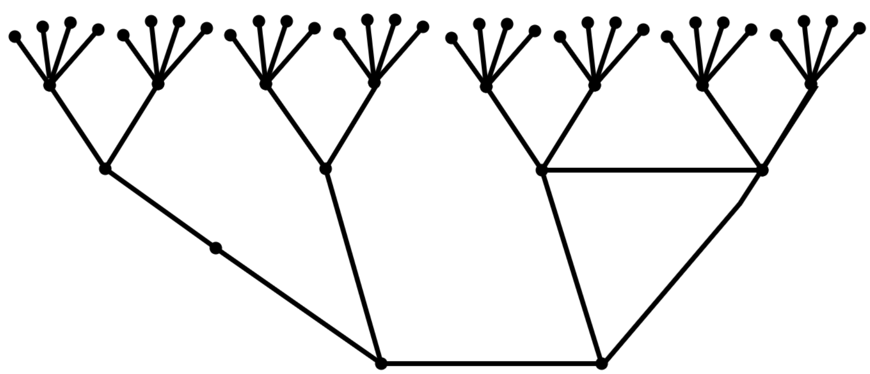

Figure 2.5: Finite Graph

- **Infinte Graphs:** An infinite graph has a never-ending number of components. In terms of mathematics, a graph G is infinite if the cardinalities of the sets that make up graph components are infinite.

- **Complete Graph:** In graph theory, a complete graph is a basic graph in which each pair of different vertices is connected by a single edge.

  A complete graph is a simple graph where every pair of different vertices is connected by a single edge. K_n represents it, where n is the graph's order.The graph shown in Figure 2.6 is complete since every pair of vertices is connected by a distinct edge.

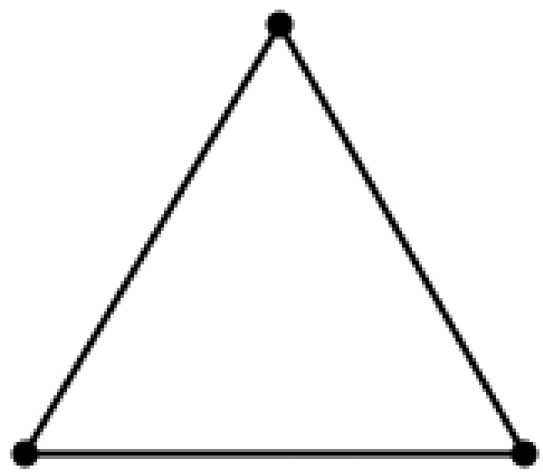

Figure 2.6: Complete Graph (k-Regular Graph)

- **K-Regular graph:** A graph is said to be k-regular if each vertex in it has exactly k edges, or a degree of k; that is, if every vertex is linked to exactly the same number of other vertices.

  Definition 2.2.7. In a graph $G(V,E)$,if k is some positive integer, degree of vertices $d(v) = k$ for every vertex v of graph then G is known as $k-regular$ graph.

  If the degree of each point in a graph G is equal to k, then the graph is considered $k-regular$. For example,complete graph K_4 is 3-regular graph.

- **Weighted Graphs:** Every edge on a weighted graph has a numerical value, or weight. Figure 2.6, which is seen above, is a weighted graph because each edge has a label with a certain weight.From the figure we can calculate the weight of each vertex as the total of all the labels in its neighbourhood as vertex with label 1 has weight $w = 0+2+3 = 5$.

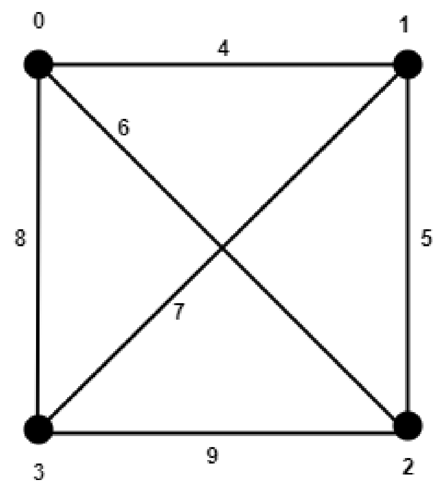

Figure 2.7: Weighted Graph

- **Irregular Graphs:** A graph in which the degrees of every vertex of the graph is different .In the graph given below the degrees of every vertex is not same that's why it is a irregular graph.In the figure 2.7 given below the graph is irregular as the degrees of each vertex is different.

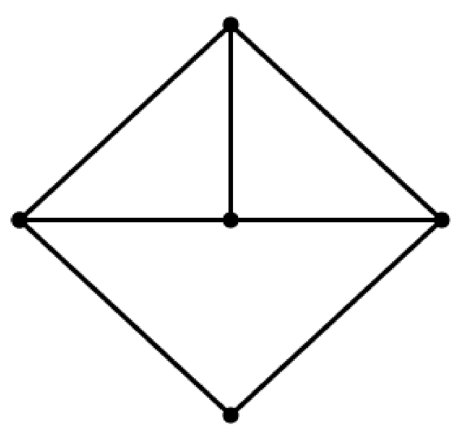

Figure 2.8: Irregular Graph

- **Undirected Graphs:** And graphs without any directions are undirected graphs.In the figure given below the part **(a)** is a directed graph as each edges has a specific orientation which make it a special type of graph, also in part **(b)** of the figure we see that its a simple graph without any direction.

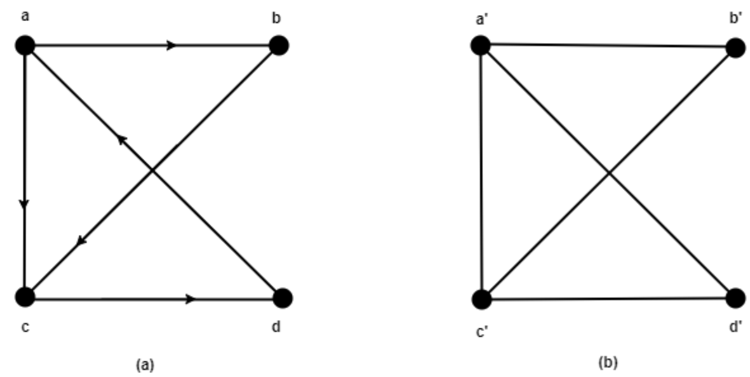

Figure 2.9: Simple Directed/Undirected Graph

- **Directed Graphs:** A graph in which edges have direction associated with it, indicating a one-way relationship between vertices.And graphs without any directions are undirected graphs. An edge that is $\overrightarrow{(}x_0Y_0)$is said to be directed from X_0 to y_0, with x_0 acting as the edge's tail and y_0 acting as its head.The degree of the vertices of is calculated as

$$deg(x) = deg^+(x) + deg^-(x)\forall x \in V(G).$$

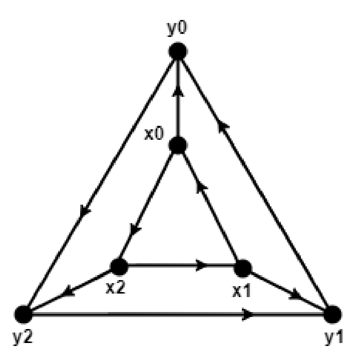

Figure 2.10: Directed Graphs Prism Graph $\mathbb{P}_3$

- **Bipartite Graph:** A graph with vertices that are partitioned into two sets so that each edge joins a point on one set to a point on the other. To put it another way, a bipartite graph $G(U,V,E) = B_{n,m}$ has two vertices, U,V, and E has all of the edges connecting the sets U and V.Furthermore, the vertices of U and V do not have an edge connecting them.

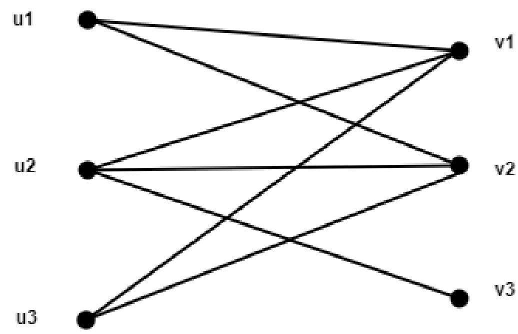

Figure 2.11: Bipartite Graph

- **Complete Bipartite:** The graph in which each vertex from one set is attached with exactly each vertex of second set that type of graph named as **Complete Bipartite**. The figure 2.8 is bipartite graph as its set of vertexes is divided into two parts and

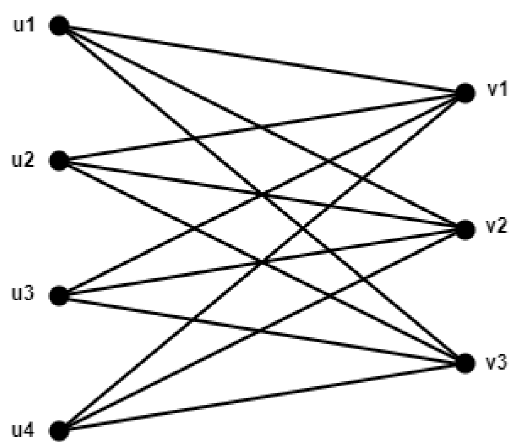

Figure 2.12: Complete Bipartite $K_{4,3}$

both of them are linked with a edge set, also as each vertex from one set is attached with exactly each vertex of other set so it is a complete bipartite graph.

- **Disconnected/Connected Graphs:** If a graph contains two or more disconnected sub-graphs, it is considered disconnected.Furthermore, a graph is said to be connected if a path exists between every pair of vertices.

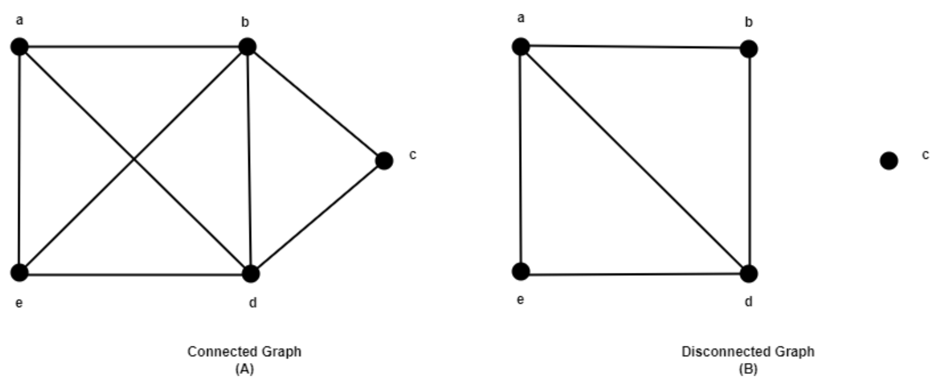

Figure 2.13: Disconnected/Connected Graphs

As a example of Connected and Disconnected graph here is figure given below part (A) is a connected graph as there is complete path which is connecting each vertices in a row. Also part (B) has a disconnection as it is parted into two parts and their is no path that connects each vertex.

- **Planer Graph:** If a graph G can be drawn in a plane without any edges crossing, then it is referred to as a planer graph.The figure 2.10 is K_4 graph which can be embedded into plane, as any graph which can be fits inside of plan is planner.

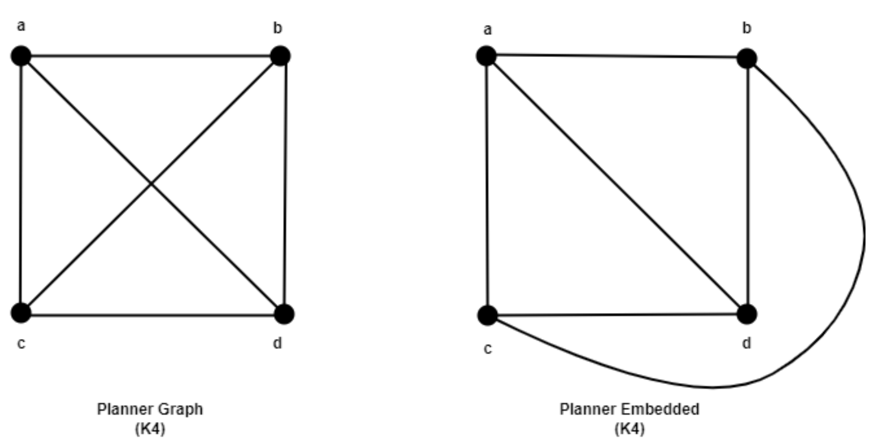

Figure 2.14: Planer Graph

- **Non-Planer:** If a graph G cannot be embedded in a plane, it is referred to as a non-planer graph. For example a complete graph K_5 is non-planer graph.It is impossible to draw on a plane .

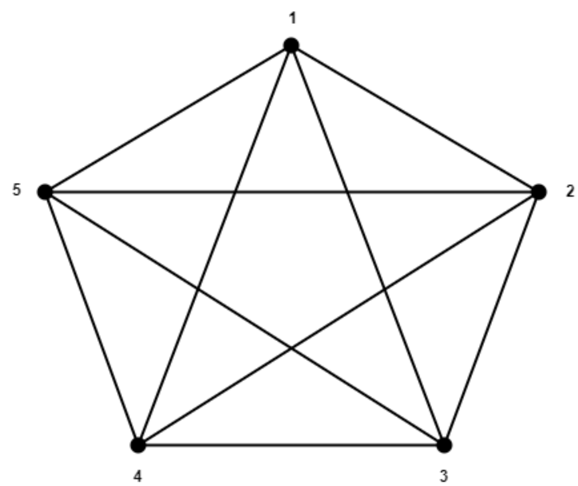

Figure 2.15: Non-Planner

- **Walk :** In a graph, a walk is a series of vertices and edges where the endpoints of each edge are the vertices that come before and after it. A sequence in a graph G such as if $v_0, v_1, v_2, v_3, v_4 \& v_5 \in V(G)$ and $e_0, e_1, e_2, e_3, e_4 \& e_5 \in E(G)$ then it is , $v_0, e_0, v_2, e_2, v_3, e_3, v_1, e_1 \& v_5, e_5 \in G$ is walk in G.In a walk edges and vertices of graph can be repeated.

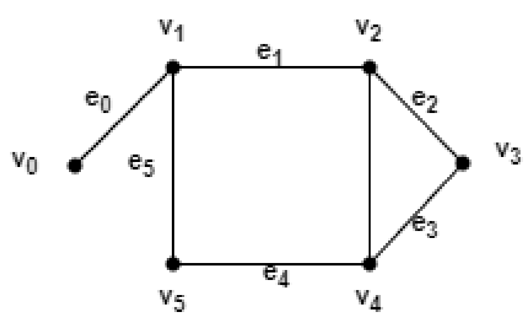

Figure 2.16: Walk/ trail in a Graph

- **Trail In a Graph:** A walk where edges do not repeat but vertices do.For example $v_0 \to e_0 \to v_1 \to e_1 \to v_0$,where $\{v_0, v_1\}$ is set of vertices and $\{e_0, e_1\}$ is a set of edge of graph H

- **Path Graph:** A path graph that forms a linear sequence of vertices connected by edges. More precisely, n vertices in a linear sequence make up a path graph P_n, where $n \geq 2$.

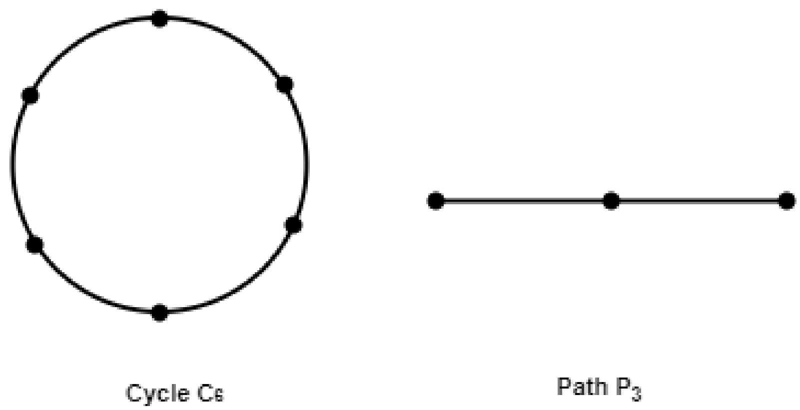

Figure 2.17: Graphs

- **Cycle Graph:** A closed path is named as a cycle. A cycle graph C_n of order n ,consists on n vertices arranged in a cycle,where $n \geq 3$.Every vertex is connected to its two neighboring vertices in the sequence and there are no other edges in the graph.
- **Prism Graph** A Prism graph is a graph formed by the Cartesian product of a path graph P_2 and a cycle graph C_n.

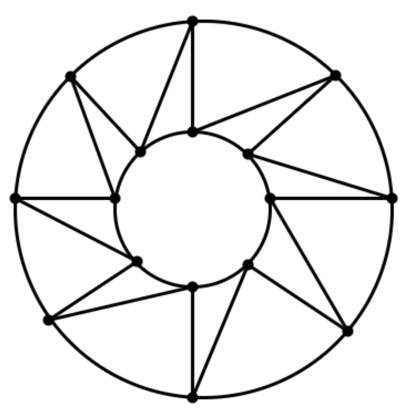

Figure 2.18: Prism Graph

As a example of a prism graph in figure 2.16 the Cartesian product $P_2 \times C_8$ is a prism graph of order 16.

- **Tree Graph:** A simple graph that is both connected and acyclic means that, a connected graph without any cycles is a tree.

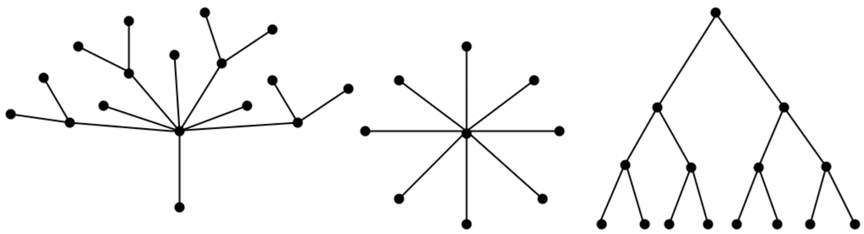

Figure 2.19: Tree Graphs

In figure 2.17 are tree graphs as there is a simple path between any two vertices.

- **Acyclic graph:** ”Acyclic” refers to a graph that is devoid of cycles. There is no closed path in this kind of graph.

- **Eulerian Graph:** A linked graph with an Eulerian circuit—that is, a closed trail that reaches each edge precisely once—is known as an Eulerian graph. Euler’s solution to the Königsberg bridges problem serves as the foundation for this idea. All vertices in a graph must have an even degree in order for it to be considered Eulerian. This type of trail forms a closed loop by traversing the edges and returning to the vertices, as seen in an example of an Eulerian circuit.

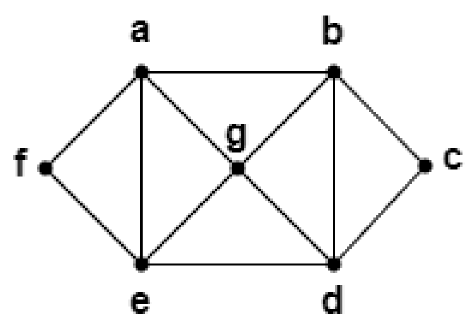

Figure 2.20: Eulerian Graph

- **Semi-Eulerian Graph:** A linked graph with an Eulerian trail—a path that hits each edge exactly once but does not always return to the starting vertex—is known as a semi-Eulerian graph. A semi-Eulerian graph is characterized by having exactly two vertices with odd degrees and all other vertices with even degrees. Practical applications where a non-circular traversal of all edges is necessary are made possible by this kind of graph.

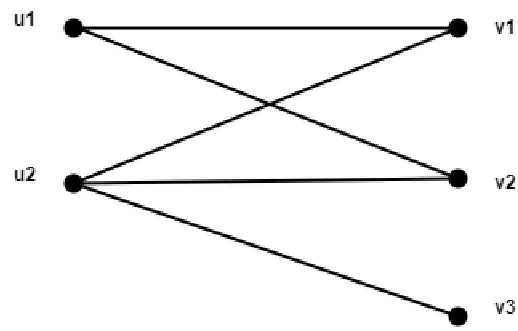

Figure 2.21: Semi-Eulerian Graph

- **Hamiltonian Graph:** The first person to look at the possibility of Hamiltonian cycles in the dodecahedron graph was Sir William Hamilton, even though the Rev. T.P. Kirkman had already addressed a more broader problem. A linked graph is Hamiltonian if every vertex is included in a cycle. The closed cycle abcdegfa in the graph below has edge repetition but no vertex repetition.

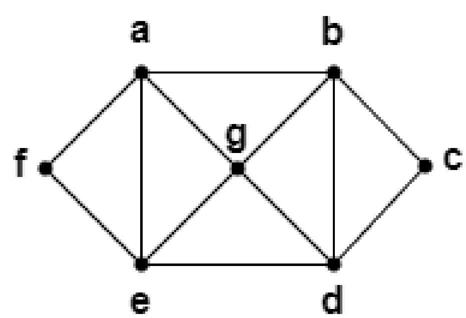

Figure 2.22: Hamiltonian Graph

- **Semi-Hamiltonian Graph** In a graph that is connected but does not have a cycle that passes through each vertex is referred to as semi-Hamiltonian.

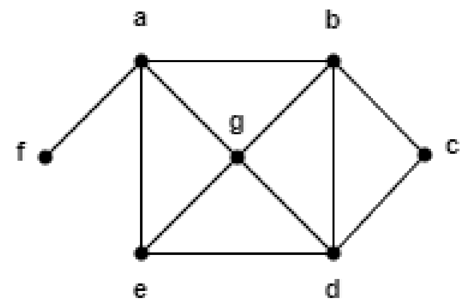

Figure 2.23: Semi-Hamiltonian Graph

Lemma 2.2.1. *The Handshaking Lemma, a fundamental graph theory tool, asserts that the total degree of all vertices in an undirected network is twice the number of edges.*

$$\sum deg(V_i) = 2|E(G)|$$

Lemma 2.2.2. *Handshaking Dilemma states that in an directed graph, the degree of each vertex is equal sum of in-degrees and out-degrees of a vertex.*

$$in \sum deg(V_i) = out \sum deg(V_i)$$

- **Face in a Graph:** A face in a planar graph is an area enclosed by the graph's edges. Faces might be endless or finite (interior).

- **Infinite face:** The unbounded, infinitely long area outside the graph is known as the Infinite Face.

Lemma 2.2.3. *The Handshaking Lemma for planar graphs states that there are twice as many face degrees as edges in every representation of a plane graph.*

- **Euler' Formula:** Let G be a linked planar graph. Let n, m, and f denote the number of vertices, edges, and faces, respectively, in a plane representation of G. $n - m + f = 2$ is the Euler formula. since,

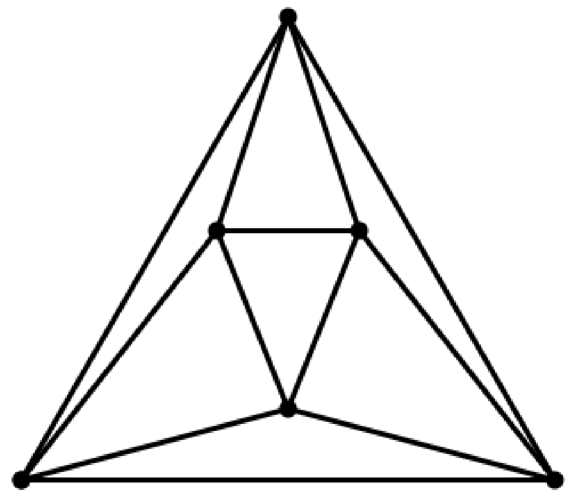

Figure 2.24: Euler' Formula

$$n = 6, m = 12, f = 8$$

$$n - m + f = 2$$

$$6 - 12 + 8 = 2$$

$$2 = 2$$

- **Dual Graph:** An array of graphs where each face of a planar graph has a vertex In the discipline of graph theory mathematics, G is referred to as its dual graph. Every pair of faces in G that are separated by an edge also has an edge in the dual graph; a self-loop is formed when a face occurs on both sides of an edge.

- **Geometrical distance:** The separation of two graph vertices is expressed in terms of edges along a shortest path, or graph geodesic, which connects them. This concept is used in graph theory mathematics. This is also known as the geometrical or shortest-path distance.

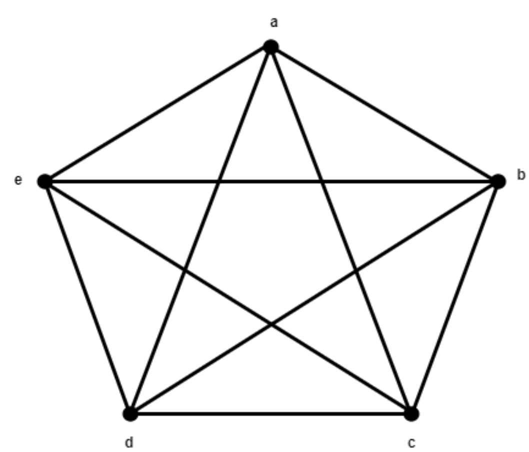

Figure 2.25: Distance b/w two Vertices

2.2.3 **Basic Graph Operations:**

There are some basic operations of graph which are used to produce new graphs.

- **Sum of Graphs:** A graph operation known as the ”sum of two graphs” joins the vertex and edge sets of the two graphs and adds new edges that link each vertex in the graph to each vertex in the second graph.

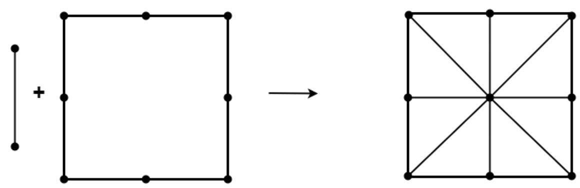

Figure 2.26: Sum Graphs

- **Corona of Graphs:** A graph operation known as the corona of graphs joins two graphs by joining every vertex in one to every vertex in a duplicate of the other. This allows for the construction of intricate, hierarchical graph structures from more basic building blocks.

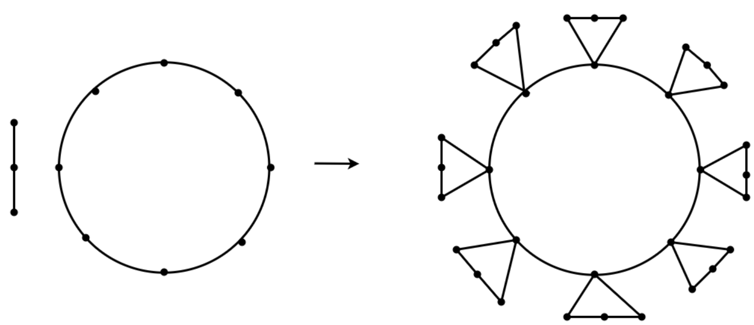

Figure 2.27: Corona of Graphs

- **Union of Graphs:** When two graphs are joined by their vertex and edge sets, an operation known as union is a fundamental one in network construction and theoretical study. Because it shows where the vertex and edge sets of two graphs overlap, it is crucial for expanding smaller networks into bigger ones.

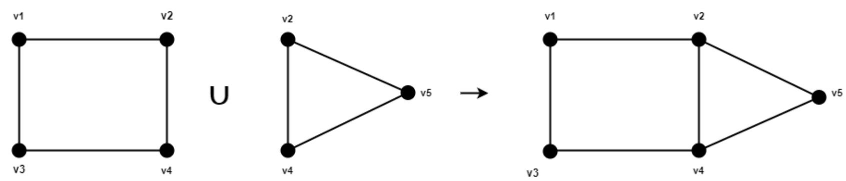

Figure 2.28: Union of Graphs

Simply,union of two graphs is a graph which consists of vertices and edges of both graphs.In the figure 2.26 is the Union of two graphs as it contains the vertices and edges of both and graphs.

- **Intersection of Graphs:** The operation that produces a new graph with shared vertices and edges from both original graphs is known as the intersection of two graphs, and it is represented as $G^1 \cap G^2$. Here's how this is defined:

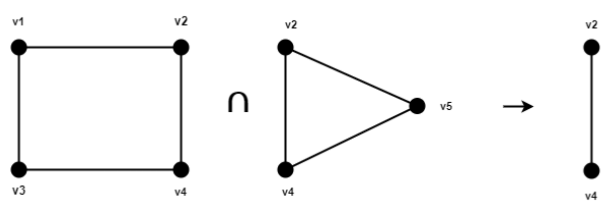

Figure 2.29: Intersection of Graphs

When two graphs intersect, a new graph with shared edges and vertices is produced.

- **Complement:** A graph is said to be its complement if it has the same number of vertices as the original graph but more edges. In the original graph, if two vertices are joined by an edge, they are not connected in the complement. The graph with a set of five vertices is the complement of K_5 in the illustration below.

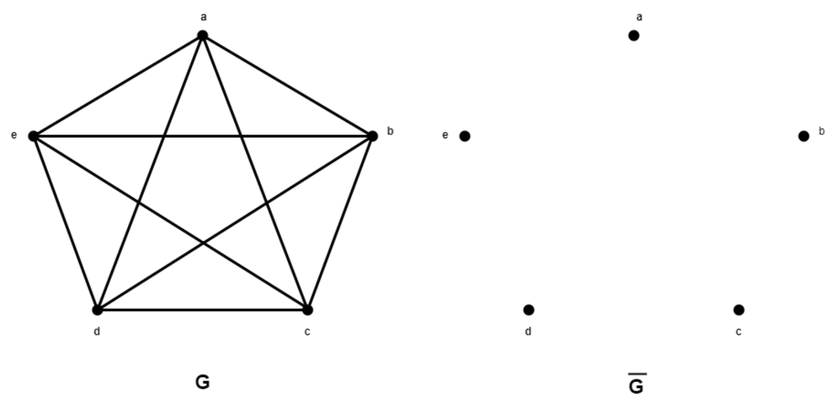

Figure 2.30: Complement of Graph

- **Graph Isomorphism:** According to graph theory, two graphs are isomorphic if their vertex and edge sets correspond exactly, maintaining adjacency, and they have the same structural elements even though they are represented differently.

Definition 2.2.8. A graph A is isomorphic to graph B , if there is a bijective mapping $\lambda : A \to B$ exists so that each a, b belongs to $V(A)$,if ab belongs to $E(A)$ such that $\lambda(a)\lambda(b) \in E(B)$.

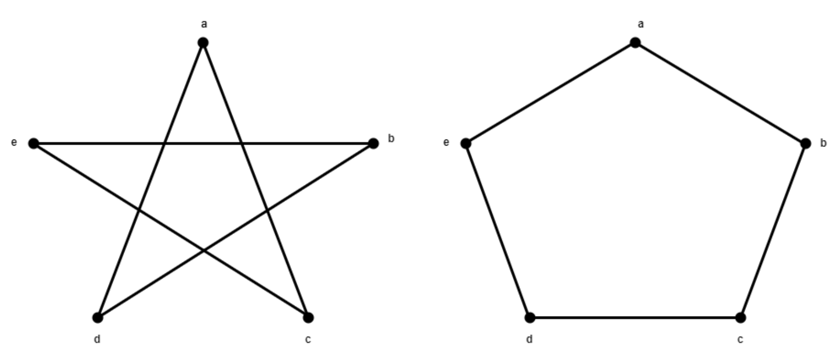

Figure 2.31: Isomorphic Graphs

An isomorphic graph has the same order and size of graphs; additionally, given that specific one-one and onto mapping, an edge connecting any two vertices of H has an edge connecting any pair of vertices in G.

2.2.4 Subgraph:

Given the graph G=(V,E) and a subgraph H = (V(H), E(H)) of G is defined as A graph formed from a subset of the edges and vertices of another graph is called a subgraph. For instance, if G is a graph, then combining its vertices and edges—where the edges join pairs of vertices—forms a subgraph H.

- Every edge in $E(H)$,links two vertices in $V(H)$.

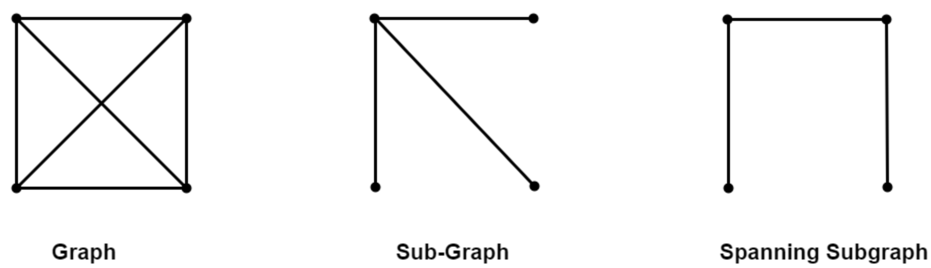

Figure 2.32: Types of Sub-Graphs

There are two main kinds of sub-graphs depending on some properties:

- **Proper subgraph:** An Proper subgraph differs from the main graph. Stated otherwise, it is a subset of nodes and lines that results in a new graph that is smaller than the original.

- **Improper sub-graph :** Considering the graph $G = (V,E)$ is the improper sub-graph of G, where the vertex and edge are respectively set to exactly V and E.Stated otherwise, the improper sub-graph is equal to G because it contains all of the original graph's vertices and edges.

- **Spanning Sub-Graph:** A graph's spanning sub-graph $G = (V,E)$ is a subgraph of G that has all of its vertices but might only contain a portion of its edges.

Here are some useful remarks;

- An edge is called a bridge if, when removed, it makes a disconnected graph within a connected graph.

- It is possible to divide each connected graph into several connected sub-graphs, often known as components.

2.2.5 **Products of Graphs:**

There are different types of products of two graphs:

- **Cartesian product:** The graph product $G\square H$ of two graphs has adjacent vertices g and h if either the second component of g from G is adjacent to the second component of h from H in H, or the second component of G is equal to the second component of H and the first component of G is adjacent to the first component of H in G.

  Definition 2.2.9. The vertices (g,g') and (h,h') are adjacent in Cartesian product graphs G and H if an edge exists between h and h' and $g = g'$ in G, or between h and h' and $g = g'$ in G.

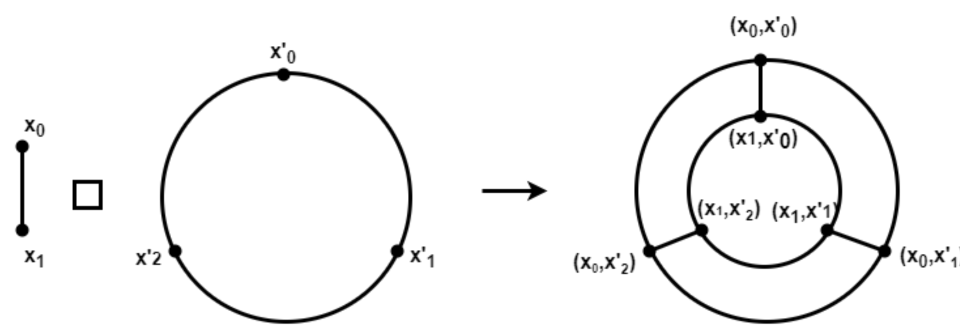

Figure 2.33: Cartesian Product of path and Cycle graphs

When two graphs are Cartesian producted, their edges and vertices are combined in a specific way, resulting in a new graph whose edges connect pairs of vertices based on criteria determined by the edges of the original graphs and whose vertices are pairs of pairs from the original graphs.

- **Direct Product:** The graph with vertex set $V(G) \times V(H)$ is obtained by taking the direct product of two graphs G and H. In this case, the vertices (g,g') and (h,h')

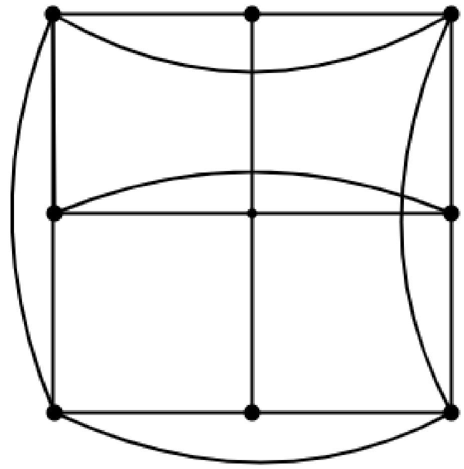

Figure 2.34: Cartesian Product

are adjacent in $G \times H$ if and only if the first component is neighboring. The first component of H in G and the second component of G in H are next to each other.

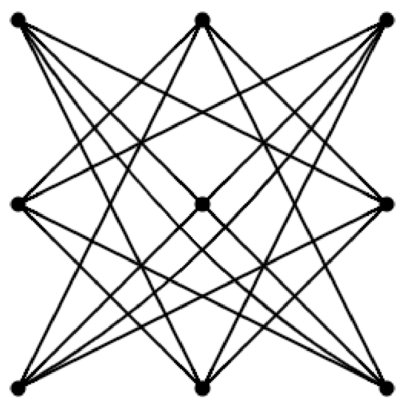

Figure 2.35: Direct Product

As a perfect example, in the figure above, we take the direct product of the third-order cycle with itself using two separate ways. It is essential for a direct product that there must be an edge connecting any two of the vertices for the two graphs that this product involves.

From the above graph is the tensor product of path graph or order 2 and the cycle graph of order 3 such that the newly formed graph fulfill all the conditions of direct product as the edge between vertices (x_0, x'_0) and (x_1, x'_2) is only because the first components $(x_0) and (x_1)$ are adjacent in first graph that is path graph P_2 and the second components $(x'_0) and (x'_2)$ are

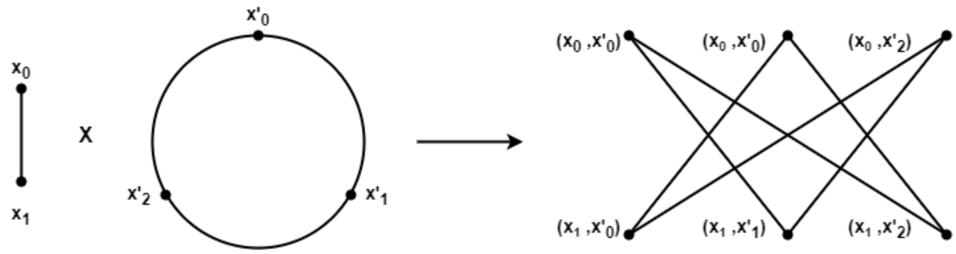

Figure 2.36: Direct Product of Path graph and cycle

adjacent in second graph that is cycle graph C_3.

- **Lexicographic Product:** The lexicographic product of two graphs G and H is defined as the existence of an edge between g and g' in G or $g = g'$ and an edge between h and h' in H.

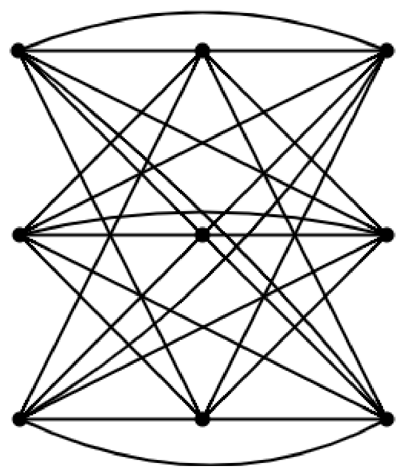

Figure 2.37: Lexicographic Product

In the given figure, we see there is a lexicographic product of the cycle graph of order 3, C_3 with itself. In this type of product, every pair of vertices has an edge between them only if either the first components are same or their is an edge between them in first graph that is G or the second components are same or their is an edge between them in second graph that is H

- **Strong Product:** A graph with a vertex set $V(G) \otimes V(H)$ is called a strong product of two graphs, $G \otimes H$. If there is no edge between two vertices in G and h is next to h' in H, or if there is no edge between two vertices in H and g is next to g' in G, or if g is next to g' in G and h is next to h', then vertices in $G \otimes H$ are considered joining.

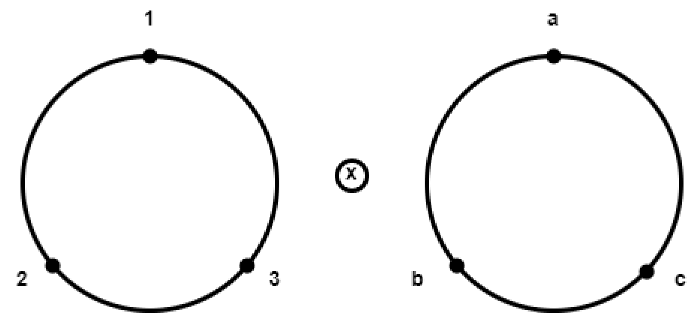

Figure 2.38: Strong product of C_3 and C_3

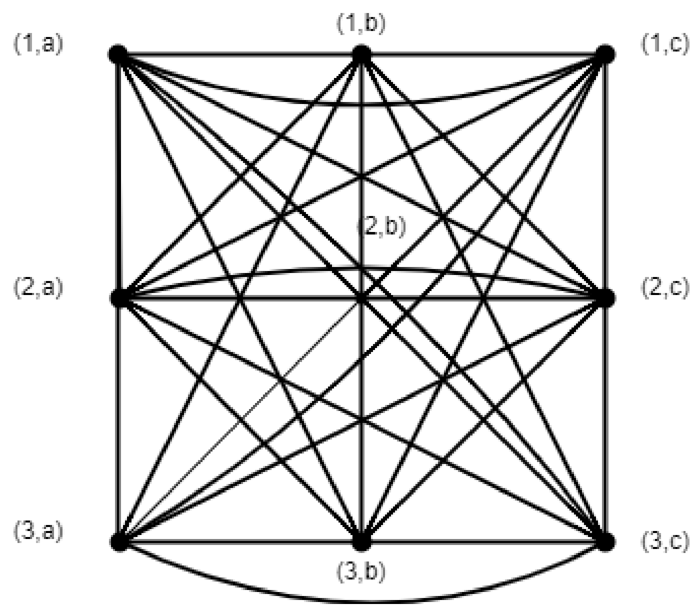

Figure 2.39: Strong Product

As a example of strong product we take product of cycle C_3 of order 3 with itself.In strong product, a pair of vertices $(1,a)$ and $(2,b)$ are right next in $G \otimes H$ if and only if: $1 = 2$ and b is next to a in $H = C_3$, or $a = b$ and 1 is adjacent to 2 in $G = C_3$, or 1 is neighboring to 2 in G and a is adjacent to b in $H = C_3$.

This chapter comprehensively reviews the basics of graph theory, such as definitions, terminology, graph types, representative strategies, and central operations.A strong understanding of these concepts provides a base for further research into advanced graph project areas and their applications in other fields.

Chapter 3
Introduction of Graph Labeling

3.1 Overview of Graph Labeling

Graph valuation is a fundamental part of graph theory that includes a wide range of methods and techniques for providing labels generally numbers to the vertices, edges, or both of a graph.The main goal was often to find a labeling that meets the appropriate requirements of the job at hand.Labeled graphs are useful frameworks for the administration of databases, circuit layout, coding theory, network addressing, and secret sharing methods. The concept of graph labeling was initially introduced in the mid-1960s and most techniques were inspired by Rosa's (1967) β-valuation graphs [18]. The β-valuation of a graph with q edges employs an injective mapping f from vertices to set $\{0,1,,q\}$, resulting in an one-one induced map. For a wide range of applications in coding theory, program design, database administration, communicating entanglement addressing, and private sharing methods, labeled graphs are helpful models.

Definition 3.1.1. Graph valuation is a technique that assigns labels to vertices, edges, or both in a graph using specific rules, with a labeling scheme representing a relation between components and labels.Formally, a labeling scheme of a graph $G(V,E)$ is a relation from graph components to a set of labels under certain conditions.

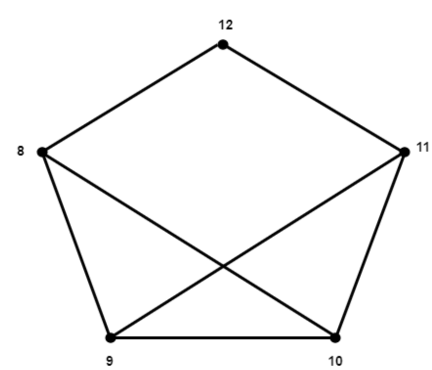

Figure 3.1: Labeled Graph

3.2 Types of Graph Labeling

There are some important types of graph labeling depends on some properties of labels.

3.2.1 Magic Labeling

The method of allocating labels (often numbers) to a graph's edges or vertices so that the labels' sums satisfy certain requirements is known as **magic labeling** in combinatorial mathematics and graph theory. Every vertex in the graph is assigned a label so that the total of the labels on the edges that intersect with each vertex is the same for every vertex.

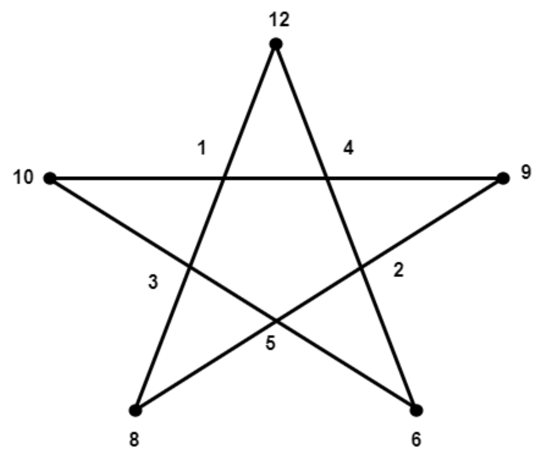

Figure 3.2: Magic Labeled Graph

Edge magic Total Labeling (EMTL):

Magic labeling is a bijection from the union vertex and edge sets to the set of integers $V \cup E$, where $\phi(x) + \phi(y) + \phi(xy)$ is constant for any edges xy. This concept was first introduced by Kotzig and Rosa in 1970 and later rediscovered by Ringel and Llad in 1996. Wallis (2001) named it "edge magic total labeling" (EMTL).

Definition 3.2.1. For any edge $xy \in E(G)$, there exists a positive number k such that

$$\phi(x) + \phi(xy) + \phi(y) = K$$

here,

$$W_\phi(xy) = (x) + \phi(xy) + \phi(y)$$

is termed the weight of edge xy. EMTL is a mapping $\phi : V \cup E \to \{1,2,...,v+e\}$.

We label all the edges and vertices on the graph with a sequence of consecutive integers. The integers from 1 to $n+m$ are the labels used for a graph having m edges and n vertices.Edge-magic labeling occurs if there is a constant k such that, for every edge $(c,d) \in (u,v) \in E$, the sum of the labels of the edge and its two endpoints equals k.

Example 3.2.1. *Let P_n be a prism graph then as map from the union of the components of graph to the set of integers upto n $\mu : V \cup E \to \{1,2,3,,,,,,n\}$ for $n = 11$ we see the weight of each edge is constant $W_\mu(xy)$ for $x = 2$ and $y = 4$ $W_\mu(2,4) = \mu(2) + \mu(2,4) + \mu(4) = 16$ and this weight is constant for all values of x and y.*

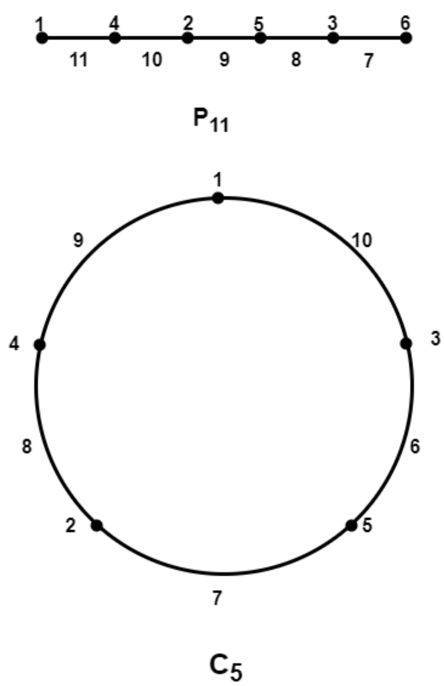

Figure 3.3: Edge Magic Total labeling

Similarly, we have an other example of C_5 the weight of each edge is $K = 14$

Here are some important results on Edge magic total labeling;

Theorem 3.2.1. *Every Path graph of order n $P_n : n \geq 3$ admits EMTL with magic constant* $W = 3n+1$

Theorem 3.2.2. *Every cycle C_n of order n admits EMTL.*

Super Edge Magic Total Labeling (SEMTL):

An EMTL with sequential integer labels for vertices and edges, such as $1,2,3,...,n+m$, is called Super Edge magic total labeling.

Definition 3.2.2. A graph $G(V,E)$ with an EMTL that has the additional property that the vertex labels are from 1 to V.

The vertex labels are arranged in a sequential order, numbered 1 through *n*. Stated otherwise, the vertices bear the labels of the first *n* integers. Super edge-magic total labeling can be more difficult to discover and more limited than edge-magic total labeling. Because of the concept's intriguing characteristics and its uses in fields like network design and combinatorics, it is being researched.

Example 3.2.2. *As an example of SEMTL we have a Cycle graph C_5 having a magic constant $K = 14$ and tree graph with magic constant $K = 15$.*

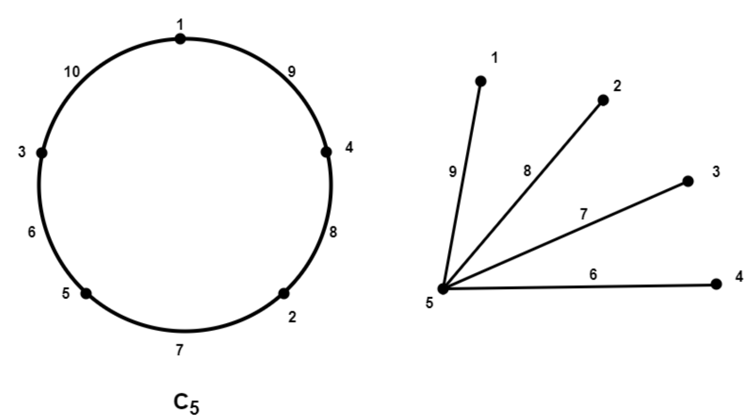

Figure 3.4: SEMTL

Here is an important Observation that; **Every tree admits EMTL / SEMTL**

Duality:

Label the components of the dual graph G^* by the labels found in G. One can get the label of a vertex in G^* (equivalent to a face in G) by taking the label of the edges and vertices that surround that face in G.

Definition 3.2.3. Let ω is a valuation with magic constant K them ω' is a dual labeling which is describe as;

$$\omega'(x_i) = (V+E+1) - \omega(x_i); \forall x \in V$$

$$\omega'(xy) = (V+E+1) - \omega(xy); \forall xy \in E$$

such that magic constant K' then $K' = 3(V+E+1) - K$

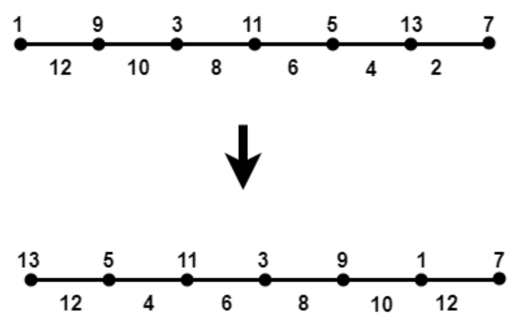

Figure 3.5: Dual Labeling

Dual labeling facilitates a variety of applications in theoretical and applied graph theory by offering an organized method for comprehending and analyzing the link between a graph and its dual.

Vertex magic Labeling:

Vertex-magic total labeling (VMTL) is a concept introduced in the 1980s by R. B. Stanton and C. C. Wallis. It is a one-to-one mapping from vertices and edges onto integers $\{1, 2, 3, ..., v+e\}$ on a graph with v vertices and e edges. The goal is to ensure that the sum of the labels on a vertex and the labels of other vertex is constant. In the 1990s, Lemke and Tarjan began researching exact algorithms for various graph types.

In 1999, MacDougall, Miller, Slamin, and Wallis introduced the concept in a more detailed manner [15]. VMTL has been applied in fields like network architecture, coding theory, and combinatorial designs.

Definition 3.2.4. When $W_\mu(v) = \mu(v) + \sum \mu(vu)$, with k being a constant, and the sum denoting overall vertices adjacent to v, then a VMTL for an injective mapping μ from $V \cup E$ to the set $\{1, 2, ..., |V| + |E|\}$ exists.

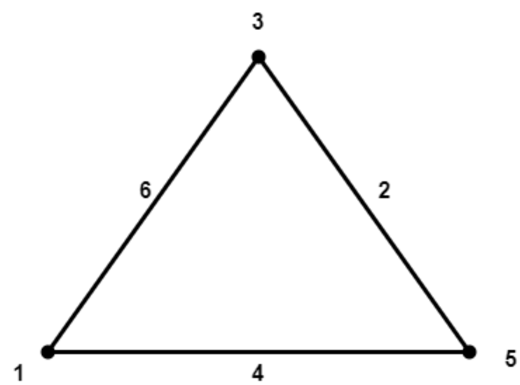

Figure 3.6: Vetex Magic Total Labeling

In a graph, a vertex-magic total labeling gives labels to vertices and edges so that the total number of labels for all vertices is always the same.

Super vertex Magic Total Labeling :

A super vertex-magic total labeling of a graph G satisfies an additional requirement on the vertice labels in addition to the vertex-magic condition. Its a VMTL in which the smallest integers are used to label the vertices

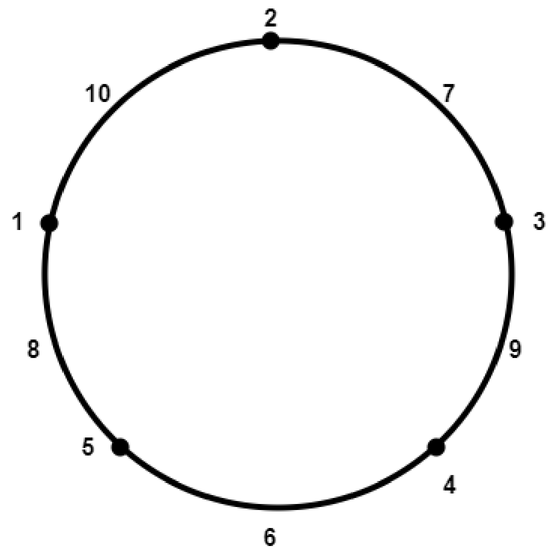

Figure 3.7: Super vertex magic total labeling

Definition 3.2.5. An additional property of a graph $G(V,E)$ with a VMTL is that the vertex labels range from 1 to V.

3.2.2 Antimagic-type labeling

Hatfield and Rigel developed anti-magic graphs in 1990 [13]. A graph with q edges can be labeled with $\{1,2,...,q\}$ without doubles, and the labels of the edges incident to each vertex add up to a distinct value.

Edge Anti-magic Total Labeling (EAMTL):

Edge-antimagic is a concept in graph theory that involves labeling each vertex and edge with a different integer, resulting in a different total of edges' labels and endpoints for each edge. In (a,d)-EAMTL, the minimum edge weight is $a \geq 6$, and the maximum edge weight is $a+(e-1)d \leq 3V+3e-3$.

Definition 3.2.6. Edge Anti-magic Total Labeling is a mapping $\phi : V \cup E \rightarrow \{1,2,..., v+e\}$ to form arithmetic sequence exist for $a>0$ with common difference $d \geq 0$.

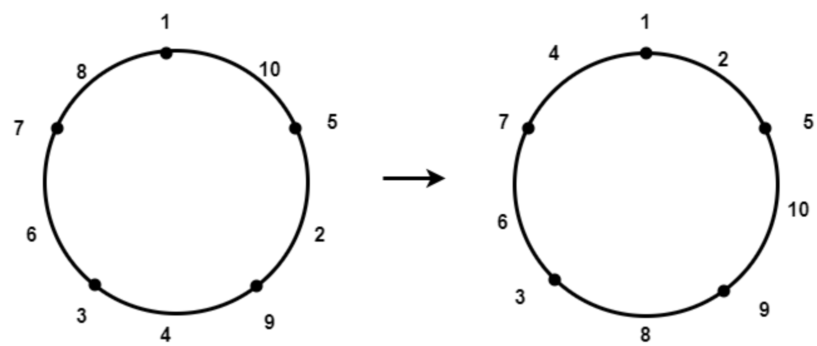

Figure 3.8: EAMTL

Super Edge Anti magic Total Labeling (SEATL):

Distinct vertex and edge label sums are guaranteed by a bijection from a graph's vertices and edges to Super Edge Anti-magic Total Labeling (SEAMTL). The maximum edge weight in (a,d)-SEAMTL is $a+(e-1)d \leq 3V+e+1$ and $d \leq \frac{2V+e-5}{e-1}$. The minimal edge weight in this case is $a = 1+2+(v+1)$.

Definition 3.2.7. Super Edge Anti-magic Total Labeling (SEAMTL) is a bijection between a graph's vertices and edges, ensuring distinct vertex and edge label sums with minimum and maximum edge weights.

Vertex-anti-magic total labeling (VATL)

In graph theory, a labeling system known as Vertex-Antimagic Total Labeling (VATL) labels all of a graph's edges and vertices with integers that meet predetermined criteria. The total label (vertex - label) of all incident edges and vertices in a VATL is unique across all of its vertices. With possible applications in a variety of industries, vertex-antimagic total labeling is a research method that examines the links in graphs between vertices and edges to ensure that each vertex has a unique sum.

A sequence of digits where the first term is a and the common difference A bijection α from a collection of vertices and edges to a set of integers $V+E$ forms d from the weights of its vertices.

Definition 3.2.8. If there exist positive integers a,d and a bijection $f: E \rightarrow \{1,2,...,|E|\}$, then the induced mapping $g_f : V \rightarrow N$ is injective. The greatest edge weight in (a,d)-VAMTL is $a+(v-1)d \leq 3v+e+1$, while the lowest edge weight is $a \geq 1+2+3$.

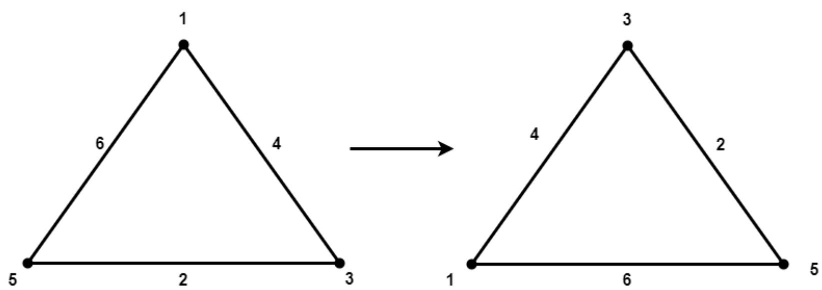

Figure 3.9: Vertex anti-magic total labeling

Super vertex Antimagic Total Labeling (Super-VATL):

An additional set of constraints is added to the standard Vertex-Antimagic Total Labeling (VATL) graph labeling technique by Super Vertex-Antimagic Total Labeling (super-VATL).

Definition 3.2.9. For a 3-regular graph G (a,d)-SVAMTL a is named as minimum vertex weight,

$$a \geq 1+(v+1)+(v+2)+(v+3)$$

$$a \geq 3v+7$$

and maximumm edge weight is

$$a+(v-1)d \leq v+(v+e)+(v+e-1)+(v+e-2)$$

$$d \leq \frac{11V+20}{2(v-1)}$$

3.2.3 **Distance Magic Labeling (DML):**

The process of assigning labels to vertices, edges, or both based on specific criteria involves selecting elements from a set [10]. Dalibor Froncek created the group distance magic labeling (GDML) in 2013[15, 3], drawing inspiration from the concept of distance magic labeling.

Definition 3.2.10. The DML (also called Sigma Labeling) for any graph G of rank n is defined as a bijection $\lambda : V(G) \to \{1,2,3,...,n\}$. Thus, for each $x \in V, w(x) = \sum y \in N_{G(x)} \lambda(y) = k$, where k is the positive integer known as the magic constant and $N_{G(x)}$, the neighborhood of vertex x, is the collection of vertices near x.

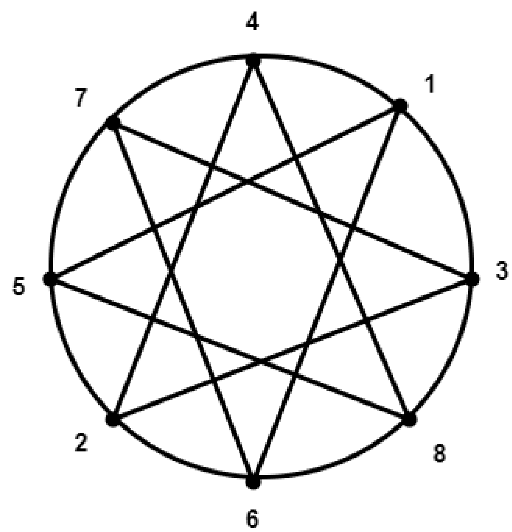

Figure 3.10: Distance magic labeling

Path graphs of order 1&3, cycle graphs of order 4, the existence of a distance magic graph with a magic constant of k for odd integers $k \geq 3$, and a 4-regular DM graph with a magic constant of $2t$ are the main topics of this categorization of graphs on distance magic labeling [11].

3.2.4 Balanced Distance Magic Labeling:

Labels applied to vertices or edges in a graph theory technique known as Balanced Distance Magic Labeling (DML) seek to establish equality or balance in certain qualities throughout the graph. Labels applied to vertices, edges, and their neighbors must be added up. In several sectors, DML is essential for efficient resource allocation, communication, and optimization. A balanced DML graph has balanced labeling and a constant sum k for each vertex in V.

Definition 3.2.11. A DM graph G of order n has balance distance magic labeling if and only if there exists a $v \in N_G(x)$ with $f(v) = |V(G)| + 1 - i$ for every $u \in N_G(x)$ and a bijection $f : V(G) \to \{1,2,...,n\}$ exits.

The direct product of cycle graphs $C_m \times C_n$ has BDM only for $n = 4$ or $m = 4$, according to theorems on balanced DML, but the direct product of regular graphs G and H is balanced distance magic [1].

3.2.5 D-Distance Magic Labeling:

An extension of graph theory known as D-Distance Magic Labeling (D-DML) labels vertices or edges in order to meet certain requirements for the distances between vertices. Its goal is to achieve particular features associated with distances of a given order, represented by D. D-DML is a modified type of Sigma labeling which evaluate the weight of each vertex in open neighborhoods , but in D-DML closed neighborhoods are used to calculate the weights [19].

Definition 3.2.12. The term "closed distance magic" refers to a network $G = (V,E)$ with n vertices and a closed neighborhood of x if, for each $x \in V(G)$, the weight $w(x) = l(x) + y \in N_G(x) l(y) = k'$ is equal to k'. Another name for a $0,1\}$-labeling is a closed distance magic labeling.

Definition 3.2.13. For every vertex x, there exists an exit k such that, in the case where $N_D^G(x) = \{y \in V | d(x,y) \in D\}$, $w'(x) = \sum_{y \in N_D^G(x)} f(y) = k'$. This establishes a magic labeling on the D-distance for the bijection $f : V \to \{1,2,...,n\}$. D-distance magic is the labeling of a graph that admits such a labeling. DML and 1-DML are equivalent.

The basic theorems of D-distance magic labeling state that every connected graph has d-distance magic, no even-order graph can admit both closed-DML and DML, and a closed Dm graph for all $n \geq 4$ does not exist for the cycle C_n [3].

This chapter covers a number of graph labeling techniques, such as distance magic labeling (DML), balance distance magic labeling (BDML), edge magic labeling (EML), super edge magic total labeling (SEMTL), edge anti magic total labeling (EAMTL), vertex magic total labeling (VMTL), and super vertex magic total labeling (SVMTL). In order to calculate graph verity load, it attributes convinced integers to verities, edges, and both. It covers orientable group-based magic labeling and group-based magic labeling in following chapters.

Chapter 4
Group Based Graph Labeling

This introduction examines the fundamental concepts, techniques, and extensive applications of Group-Based Graph Labeling, an intriguing field that blends group algebra and graph theory.Group-based graph labeling is a mathematical technique that combines group theory and graph labeling.It examines the relationships between vertices and edges in a graph and delves into basic group theory ideas such as group homomorphisms, cosets, and subgroups. The idea of group distance magic graphs originated with his concept of distance magic labeling. Froncek made significant advances in GDML applications and graph labeling. Excellent work on GDML of the graph's direct product was done in 2015 by Marcin Anholcer et al. They proved that a r-regular graph G of order n and C_4 are the direct products of GDML. Additionally, they discovered that $G \times H$, in which G is a balanced magic graph and H is a normal graph, is GDML. They did find out, though, that if m,n, then the direct product of C_m and C_n is not GDML.

The study by Cichacz and colleagues demonstrated that the direct product of two regular graphs is $H_1 \times H_2$-DM when they are H_1-DM and H_2-DM.Additionally, they provided a formulation for a regular graph and contributed to the GDML extension for the lexicographic product of regular graphs with cycles. They also proved that C_4, the group Γ of order $4(m+n)$, is a lexicographic product of the whole bipartite graph $K_{m,n}$ with GDML . Dr. John Smith's study on Group Distance Magic Labeling has significantly advanced graph theory. Their article "Group Distance Magic Labeling: A Unified Approach" provides a single process for creating group-based labeling that meets distance magic requirements, demonstrating that H-distance magic labeling is not the Cartesian product of C_m and C_n.

Definition 4.0.1. A graph G has a group distance magic labeling if an Abelian group H of order n exists and there is a one-to-one map $l : V(G) \to H$,such that $\sum_{x \in N(u)} l(x) = \mu$, where $\mu \in H$ is the magic constant.

An Abelian group H of order n and a one-to-one map $l : V(G) \to H$ from the vertex set to group elements are needed for a group distance magic labeling of a graph G. The magic constant in this map is $\mu \in H$.

4.0.1 Group Distance Antimagic labeling:

A notion of Distance anti-magic labeling is a relation where the weights of vertices are not constant, and it maps the set of vertices to the set $\{1, 2, ..., n\}$.By adding anti-magic qualities, GADML is a graph theory idea that expands on the ideas of Distance Magic Labeling (DML).

Definition 4.0.2. Given that A and G have the same cardinality, a A-distance antimagic labeling of G is a relation f from the set of vertices (V(G)) to the group A, where each vertex in G has a pairwise unique weight.

The research shows that for an integer $n \geq 4$, a graph D_n is antimagic at $\mathbb{Z}_{2n}$-distance; for an odd number $n \geq 3$, it is antimagic at C_n-distance.

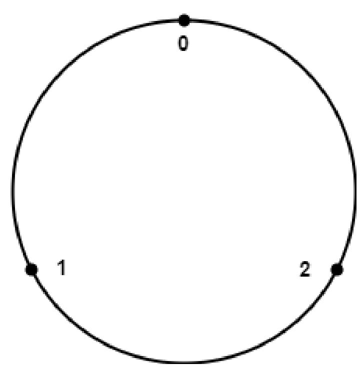

Figure 4.1: C_3 is $\mathbb{Z}_3$-distance antimagic

4.0.2 **Orientable Group Distance Magic labeling:**

In 2017, Brayn Freyberg introduced orientable group distance magic labeling (OGDML), a generalization of group distance magic labeling for oriented graphs. OGDML assigns an orientation to every edge, with unique starting and ending vertex. It has been extensively researched, including examples of OGDML of anti-prism graphs, direct product of anti-prism graphs with cycles, orientable $\mathbb{Z}_n$-distance magic regular graphs, and Cartesian product of two cycles.

Definition 4.0.3. A graph that is directed If there is an Abelian group H and a one-to-one map $l : V \to H$ such that for all $x \in V$,

$$\sum_{y \in N^+_{G(x)}} l^{\to} l(x) - \sum_{y \in N^-_{G(x)}} l^{\to}(x) = \mu$$

. then G has an orientable group distance magic labeling.

Bryan Freyberg et al. proposed Orientable Group Distance Magic Labeling (OGDML), which uses directed graph families like directed cycles, complete graphs, regular graphs, and bipartite graphs. They found that the direct product cycle graph with inself is OGDML for modulo group $\mathbb{Z}_{nm}$, the Cartesian product is orientable $\mathbb{Z}_{mn}$-distance magic labeled, and the complete graph K_n is OGML for the modulo group $\mathbb{Z}_{2n}$ for only odd values of n [5].

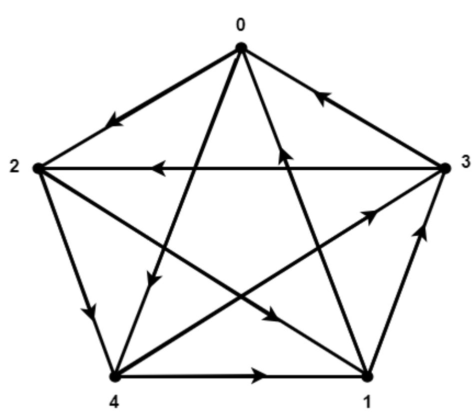

Figure 4.2: K_5 is $\mathbb{Z}_5 - OGDML$

Proposition 4.0.1. *If n is even, a complete bipartite graph $K_{n,n}$ has $\mathbb{Z}_{2n}$-distance magic.*

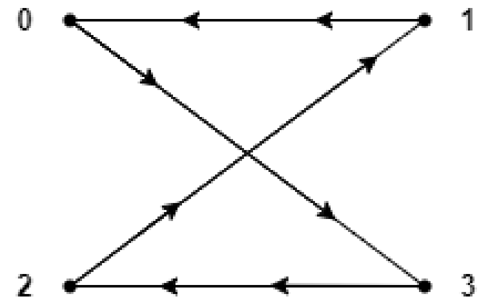

Figure 4.3: $K_{2,2}$ admits orientable $\mathbb{Z}_4$ distance magic

Theorem 4.0.2. *A Prism graph $\mathbb{P}_n$ of order $2n$ is not an OGDML under modulo group $\mathbb{Z}_{2n}$.*

The study reveals that a prism graph is not OGDML due to differing in and out degrees. It also proposes that if G is a $2r$-regular graph of order n, it is orientable $\mathbb{Z}_n$-distance magic. However, using orientable groups for graph valuation changes the properties of a cycle graph. They left open challenges for OGDML, showing that if a graph G of order n accepts GDML, it also accepts OGDML. If the graph has an even degree of order n, it is in OGDML [5].Recently, researchers have focused on OGDML and its potential application to anti-prism graphs and their direct products. Notably, great progress has been made in determining the structural features and labeling limitations of these complex graph structures. One significant outcome is the identification of situations under which anti-prism graphs can be endowed with OGDML, exposing subtle patterns in vertex labeling. These investigations have revealed new insights into the adaptability and usefulness of OGDML in a variety of graph-theoretic contexts, opening the way for future research and applications.

The results of OGDML on anti-prism graphs and their direct product show that a directed anti-prim A_n is orientable A-distance magic. $G \equiv A_m$ and $H \equiv A_n$ are directed antiprism graphs that permit orientable $\mathbb{Z}_{4mn}$-distance magic labeling, and $G \equiv A_n$ admits orientable $\mathbb{Z}_2 \times \mathbb{Z}_n$-distance magic labeling for every even n.The research left open questions about Abelian groups with anti-prism graphs being GDML and OGDML, and the contradiction that a graph of order n is orientable Z_n-distance magic.[2].

Example 4.0.1. *We drew graph directed anti-prism graph A_8 and labeled it under modulo group $\mathbb{Z}_2 \times \mathbb{Z}_8$ with magic constant $(0,3)$ .We take the direction of of edges anticlockwise.*

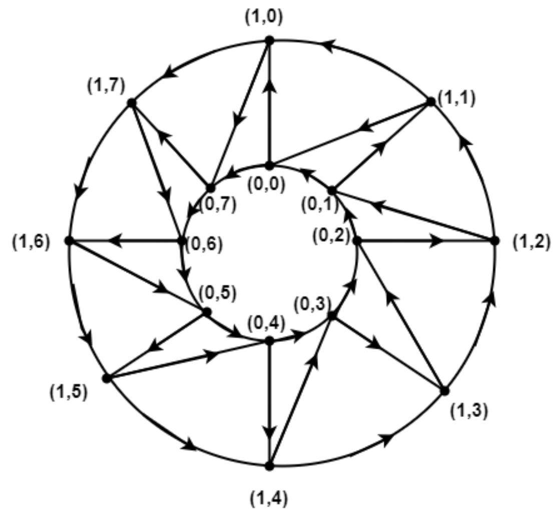

Figure 4.4: Directed Anti-prism graph A_8 is Orientable $\mathbb{Z}_2 \times \mathbb{Z}_8$-DML

In this chapter, we explored the basics of group-based graph valuations and types of group-based labeling and their important results. To sum it up, group-based graph labeling offers a solid foundation for comprehending and working with intricate graph structures using algebraic techniques. Group-based labeling methods have been shown to help resolve complex issues in a variety of disciplines, such as combinatorial optimization, network design, and coding theory.In the next chapter we will discuss some main results and proves of some theorem.

Chapter 5
OGDML of Regular Graphs and Their Direct Product

Orientable Generalized Distance Magic Labeling OGDML is an advanced graph possibility concept involving the naming of labels to vertices in directed graphs under specific algebraical conditions. A vertex's weight in OGDML is determined by subtracting the total of its outgoing and incoming edge labels. OGDML has been proven to exist on certain types of graphs, such as Cartesian product of cycles and complete graphs of order n. However, the study of OGDML is meaningful for applications in entanglement design, coding theory, and cryptography,' offering insights into how graph properties can be controlled by finished labeling. A graph that is orientated When there is an Abelian group, then G is an OGDML H and a one-to-one map from the graph's vertices to the group's elements, where the weight of each vertex remains constant. OGDML is studied for hundreds of families of regular graphs since 2017 [9]. However, not all regulars must be OGDML, as not all regulars must be OGDML. For instance, the Prism graph G of order $2n$ is not an OGDML.

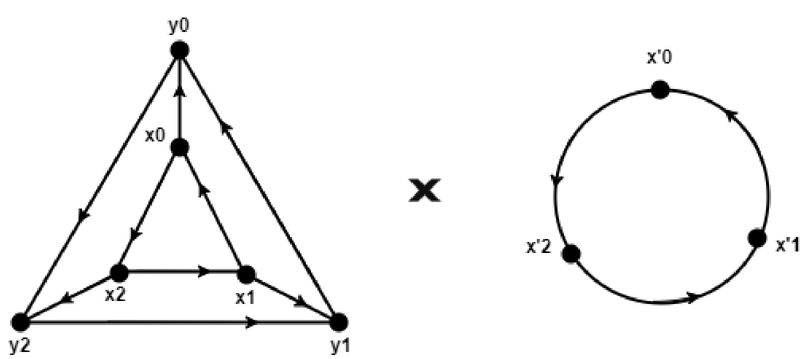

Figure 5.1: Direct Product of $\mathbb{P}_n \times C_n$

In the following theorems, we give the OGDML of the oriented direct product of the prism graph $\mathbb{P}_n$ of order $2n$ and the cycle graph C_n of order n. The vertex and edge representations of the directed prism and cycle graphs' direct product are shown below before we explain our primary findings.

$V\ (\mathbb{P}_n \times C_m) \ = \ \{(\mathrm{x}_i, x'_j), (y_i, x'_j) | 0 \leq i \leq n-1, 0 \leq \mathrm{j} \leq \mathrm{m\text{-}1}\ \}$.

$\mathrm{E}(\mathbb{P}_n \times C_m) \ = \ \{(x_i, x'_j)(y_i, x'_j) | x_i y_j \in V\ (\ \mathbb{P}_n)\ , x'_i, x'_j \in V(C_m), x_i x_j \in E(\mathbb{P}_n),\ x'_i x'_j \in E(C_m) | \ 0 \leq \mathrm{i} \leq n-1,\ 0 \leq j \leq m-1\ \}$,

Theorem 5.0.1. *Let $G \cong \mathbb{P}_n$ and $H \cong C_m$, where $\mathbb{P}_n$ is prism graph of order n and C_m is cycle graph of order m and $\mathbb{Z}_{2nm}$ be the modulo group of order $2nm$.Then, a directed direct product of graph $G \times H$ admits OGML under modulo group $\mathbb{Z}_{2nm}$ $\forall\ n, m \geq 3$.*

Proof. The vertex and edge set of $\mathbb{P}_n$ and C_m are as follows:

$V\ (\mathbb{P}_n)$=$\{\ x_i, y_i | 0 \leq i \leq n-1\ \}$,

$E\ (\mathbb{P}_n)$ = $\{\ x_i x_{i+} 1, y_i y_i + 1, x_i y_i | 0 \leq i \leq n-2\ \} \cup\ \{x_0 x_n - 1, y_0 y_n - 1, x_n - 1 y_{n-1}\ \}$,

$V\ (C_m)$ = $\{\ x'_i, 0 \leq i \leq m-1\ \}$,

$E\ (C_m)$ = $\{\ x_i, x'_i + _1, 0 \leq i \leq m-2\ \} \cup \{x'_0 x'_m - _1\}$,

By using concept of direct product we get the vertex set below representing $\mathbb{P}_n \times C_m$,

$V\ (\mathbb{P}_n \times C_m) \ = \ \{(x'_i x'_j), (y_i, x'_j) | 0 \leq i \leq n-1, 0 \leq j \leq m\text{-}1\ \}$.

Define: $\overrightarrow{l} : V\ (\mathbb{P}_n \times C_m) \rightarrow \mathbb{Z}_{2nm}$ *as follow*

$\overrightarrow{l}\ (x_i, x'_j) \ = \ i + 2nj,\ for\ 0 \leq i \leq n-1\ , 0 \leq j \leq m-1$

$\overrightarrow{l}\ (y_i, x'_j) \ = \ i + n(1+2j), for\ 0 \leq i \leq n-1\ 0 \leq j \leq m-1$

Under $\overrightarrow{l}$, $\mathbb{P}_n \times C_m$ *is Orientable* $\mathbb{Z}_{2nm}$*-distance magic graph and magic constant* μ *is*

$$\sum_{y \in N^+_{G(x)}} \overrightarrow{l(x)} - \sum_{y \in N^-_{G(x)}} \overrightarrow{l(x)} = \mu$$

$$\mu = \begin{cases} (-6n - 2nm) \mod 2mn & \textit{for odd values of n} \\ (-nm - 8n) \mod 2mn & \textit{for even values of n} \end{cases}$$

Example 5.0.1. *We drew graph $P_3 \times C_3$ and labeled it under modulo group $\mathbb{Z}_{18}$ with magic constant* 0.*We take the direction of each copy anticlockwise.*

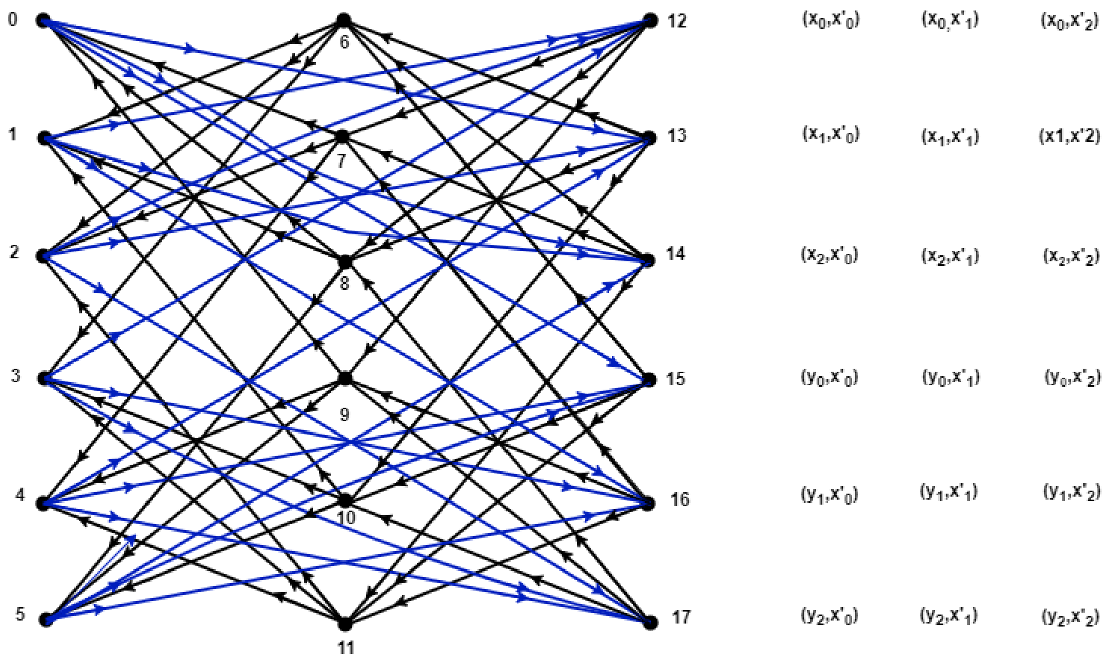

Figure 5.2: Direct Product of prism with Cycle $\mathbb{P}_3 \times C_3$ with Orientable $\mathbb{Z}_{18}$-labeling

Example 5.0.2. *We drew graph $P_4 \times C_4$ and labeled it under modulo group $\mathbb{Z}_{32}$ with magic constant* 0.*We take the direction of each copy anticlockwise.*

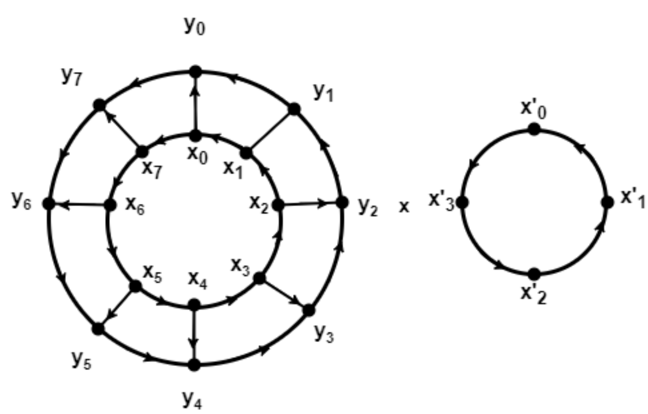

Figure 5.3: Direct Product Of $\mathbb{P}_4 \times C_4$

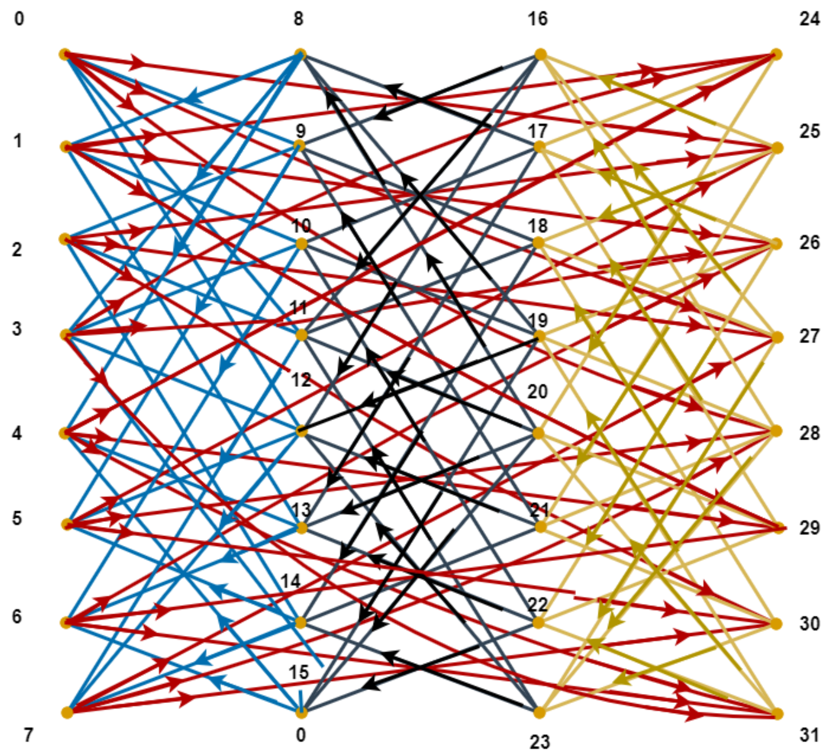

Figure 5.4: Direct Product of prism with Cycle $\mathbb{P}_4 \times C_4$ with Orientable $\mathbb{Z}_{32}$-labeling

Theorem 5.0.2. *Let $G \cong \mathbb{P}_n$ and $H \cong C_m$, where $\mathbb{P}_n$ is prism graph of order $2n$ and C_m is cycle graph of order m and $\mathbb{Z}_m \times \mathbb{Z}_{2n}$ be the modulo group of order $2nm$ such that $gcd(n,m) \neq 1$.Then the direct product of graph $G \times H$ admits a orientable $\mathbb{Z}_m \times \mathbb{Z}_{2n}$ distance magic labeling $\forall\ n,m \geq 3$.*

Proof. *The vertex and edge set of $\mathbb{P}_n$ and C_m are as follows:*

$V\ (\mathbb{P}_n)$=\{ $x_i, y_i | 0 \leq i \leq n-1$ \},

$E\ (\mathbb{P}_n)$ = \{ $x_i x_{i+}1, y_i y_i+1, x_i y_i | 0 \leq i \leq n-2$ \} $\cup$ \{$x_0 x_n-1, y_0 y_n-1, x_n-1 y_n-_1$ \},

$V\ (C_m)$ = \{ $x'_i, 0 \leq i \leq m-1$ \},

$E\ (C_m)$ = \{ $x_i, x'_i+_1, 0 \leq i \leq m-2$ \}$\cup$ \{$x'_0 x'_m-_1$\},

By using concept of direct product we get the vertex set below representing $\mathbb{P}_n \times C_m$,

$V(\mathbb{P}_n \times C_m)\ =\ \{(x_i, x'_j), (y_i, x'_j) | 0 \leq i \leq n-1, 0 \leq j \leq$ *m-1* \}.

Define: $\overrightarrow{l} : V\ (\mathbb{P}_n \times C_m) \rightarrow \mathbb{Z}_m \times \mathbb{Z}_{2n}$ *as follow*

$\overrightarrow{l}(x_i, x'_j) = (j, i),\ for\ 0 \leq i \leq n-1, 0 \leq j \leq m-1$

$\overrightarrow{l}(y_i, x'_j) = (j, n+i),\ for\ 0 \leq i \leq n-1,\ 0 \leq j \leq m-1$

Under $\overrightarrow{l}$, $\mathbb{P}_n \times C_m$ is Orientable $\mathbb{Z}_m \times \mathbb{Z}_{2n}$-distance magic graph and magic constant μ is

$$\sum_{y \in N^+_{G(x)}} \overrightarrow{l(x)} - \sum_{y \in N^-_{G(x)}} \overrightarrow{l(x)} = \mu$$

$$\mu = \begin{cases} (0,4) \mod 2mn & \text{for } m = 4 \\ (0,2) \mod 2mn & \text{for } m = 5 \\ (0, 4n(m-3)) \mod 2mn & \text{for other-values of } n \end{cases}$$

Example 5.0.3. *We drew graph $P_3 \times C_3$ and labeled it under modulo group $\mathbb{Z}_3 \times \mathbb{Z}_6$ with magic constant $(0,0)$. We take the direction of each copy anticlockwise.*

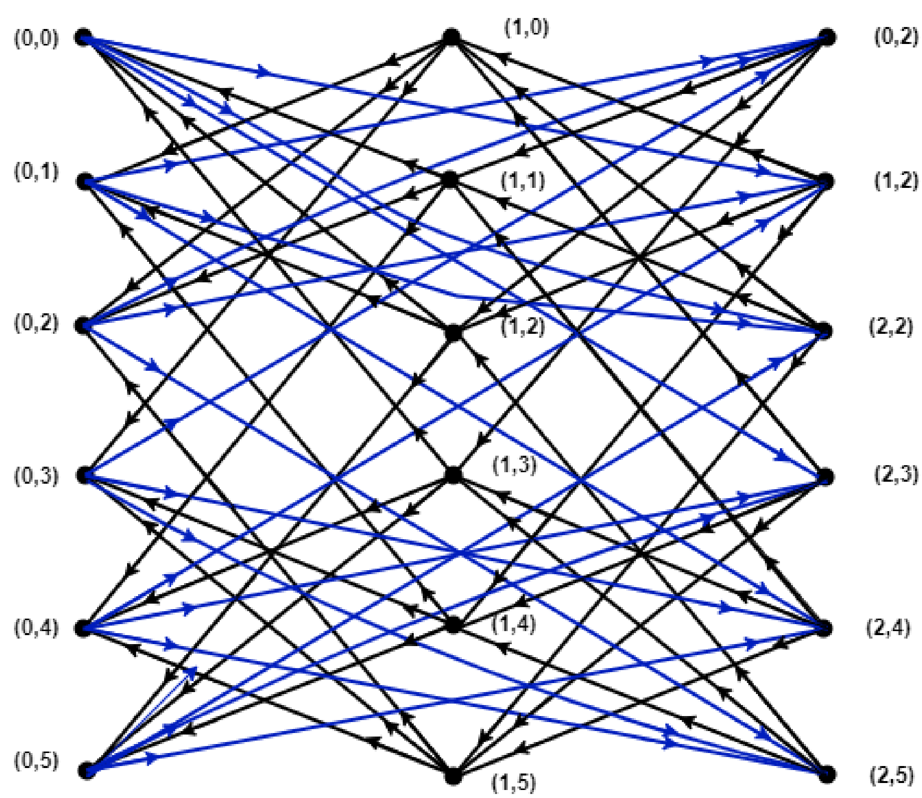

Figure 5.5: Direct Product of prism with Cycle $\mathbb{P}_3 \times C_3$ with Orientable $\mathbb{Z}_3 \times \mathbb{Z}_6$-labeling

Example 5.0.4. *We drew graph $P_4 \times C_3$ and labeled it under modulo group $\mathbb{Z}_3 \times \mathbb{Z}_8$ with magic constant $(0,0,0)$.We take the direction of each copy anticlockwise.*

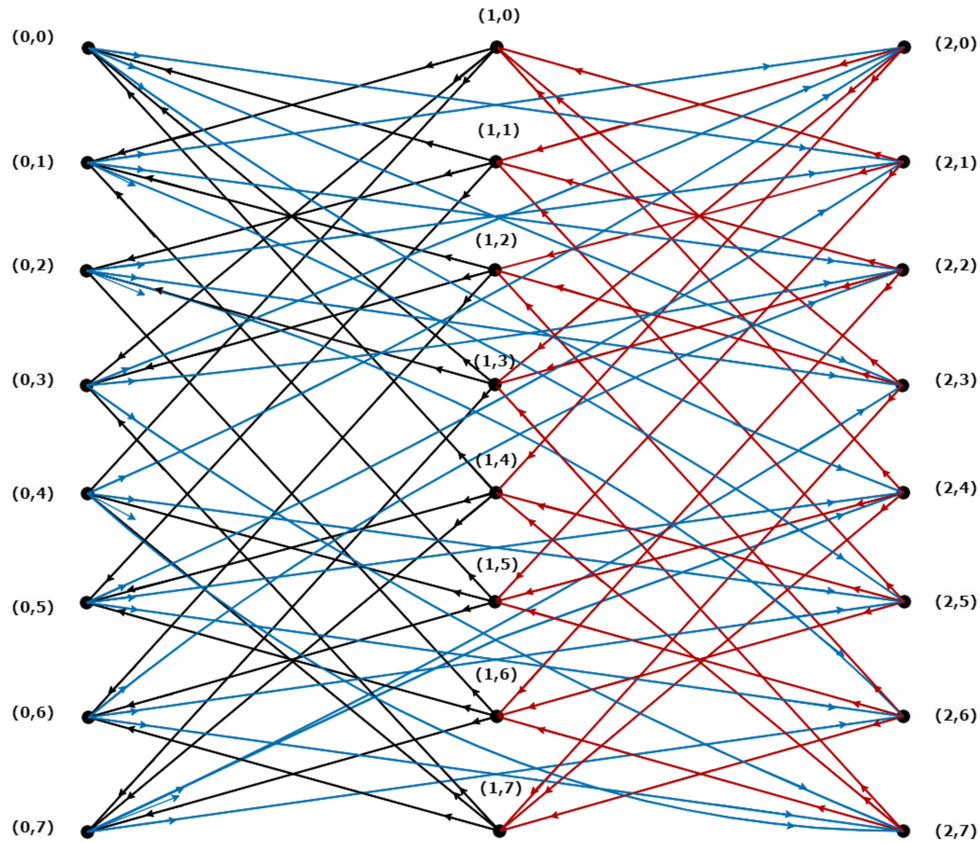

Figure 5.6: Direct Product of prism with Cycle $\mathbb{P}_4 \times C_3$ with Orientable $\mathbb{Z}_3 \times \mathbb{Z}_8$-labeling

Theorem 5.0.3. *Let $G \cong \mathbb{P}_n$ and $H \cong C_m$, where $\mathbb{P}_n$ is prism graph of order $2n$ and C_m is cycle graph of order m and $\mathbb{Z}_2 \times \mathbb{Z}_{n/2} \times \mathbb{Z}_{2m}$ be the modulo group of order $2mn$ such that $gcd(2,n/2,2m) \neq 1$.Then the directed direct product of graph $G \times H$ admits a orientable $\mathbb{Z}_2 \times \mathbb{Z}_{n/2} \times \mathbb{Z}_{2m}$ distance magic labeling $\forall\ n \geq 4, m \geq 3$.*

Proof. The vertex and edge set of $\mathbb{P}_n$ and C_m are as follows:

$V(\mathbb{P}_n)=\{ x_i, y_i | 0 \leq i \leq n-1 \}$,

$E(\mathbb{P}_n) = \{ x_i x_{i+}1, y_i y_i + 1, x_i y_i | 0 \leq i \leq n-2 \} \cup \{x_0 x_n - 1, y_0 y_n - 1, x_n - 1 y_{n-1} \}$,

$V(C_m) = \{ x'_i, 0 \leq i \leq m-1 \}$,

$E(C_m) = \{ x_i, x'_i + _1, 0 \leq i \leq m-2 \} \cup \{x'_0 x'_m - _1\}$,

By using concept of direct product we get the vertex set below representing $\mathbb{P}_n \times C_m$,

$V(\mathbb{P}_n \times C_m) = \{(x'_i x'_j), (y_i, x'_j) | 0 \leq i \leq n-1, 0 \leq j \leq m\text{-}1 \}$.

Define: $\overrightarrow{l} : V(\mathbb{P}_n \times C_m) \to \mathbb{Z}_l \times \mathbb{Z}_m \times \mathbb{Z}_n$ *as follow;*

Case(i) *when n is Even.*

$$\overrightarrow{l}(x_i, x'_j) = \begin{cases} k\,, & \\ j \mod \left(\frac{n}{2}\right), & for\ 0 \leq j \leq n, \\ 2k & for 0 \leq k \leq m-1. \end{cases} \qquad where, k = \begin{cases} 0, & 0 \leq i < \frac{n}{2} \\ 1, & \frac{n}{2} \leq i < n \end{cases}$$

$$\overrightarrow{l}(y_i, x'_j) = \begin{cases} k\,, & \\ j \mod \left(\frac{n}{2}\right), & for\ 0 \leq j \leq n, \\ 2j+1 & for\ 0 \leq j \leq m-1 \end{cases} \qquad where,$$

$$k = \begin{cases} 0, & 0 \leq i < \frac{n}{2} \\ 1, & \frac{n}{2} \leq i < n \end{cases}$$

Case(ii) *When n is odd.* $\overrightarrow{l}\,(x_i,x'_j) = \begin{cases} j \mod \left(\frac{m}{2}\right), & for\ 0 \le j \le m-1, \\ k\,, & \\ i & for 0 \le i \le n-1. \end{cases}$ *where,*

$$k = \begin{cases} 0, & 0 \le i < \frac{m}{2} \\ 1, & \frac{m}{2} \le i \le\ m-1 \end{cases}$$

$\overrightarrow{l}\,(y_i,x'_j) = \begin{cases} j \mod \left(\frac{m}{2}\right), & for\ 0 \le j \le m, \\ k\,, & \\ n+i & for 0 \le i \le n-1. \end{cases}$ *where,* $k = \begin{cases} 0, & 0 \le i < \frac{m}{2} \\ 1, & \frac{m}{2} \le i \le m-1 \end{cases}$

Under $\overrightarrow{l}$, $\mathbb{P}_n \times C_m$ *is Orientable* $\mathbb{Z}_m \times \mathbb{Z}_{2n}$*-distance magic graph and magic constant* μ *is*

$$\sum_{y \in N^+_{G(x)}} \overrightarrow{l(x)} - \sum_{y \in N^-_{G(x)}} \overrightarrow{l(x)} = \mu$$

$$\mu = \begin{cases} (0,0,4n(m-3)) & for\ n = 3 \\ (0,0,2) & for\ n = 5 \\ (0,4n(m-3)+1,0) & for\ odd\ values\ of\ n \end{cases}$$

Example 5.0.5. *We drew graph $P_4 \times C_3$ and labeled it under modulo group $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_6$ with magic constant $(0,0,0)$.We take the direction of each copy anticlockwise.*

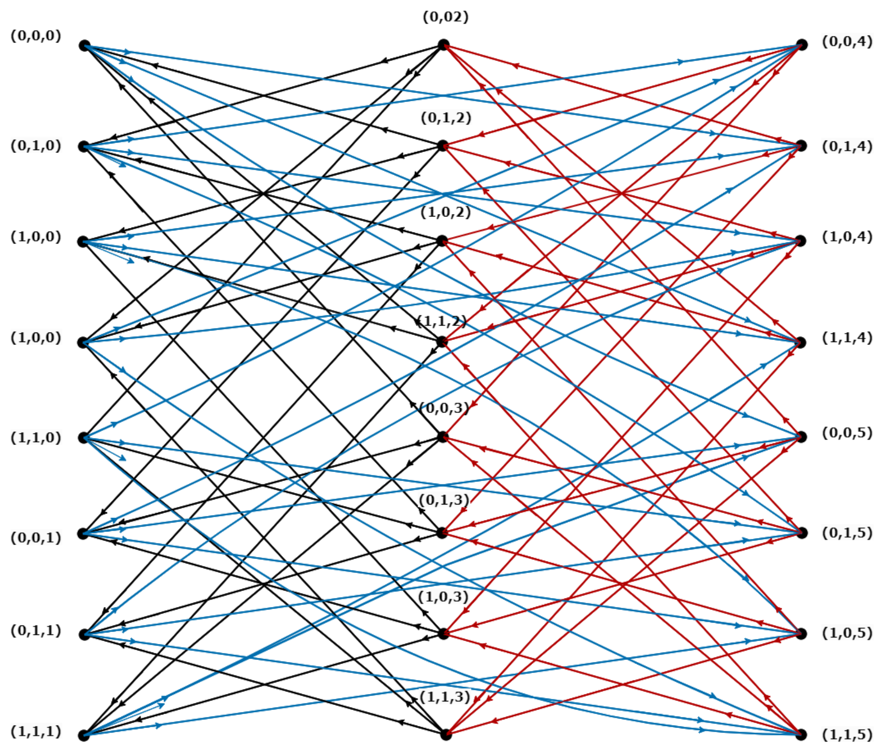

Figure 5.7: Direct Product of prism with Cycle $\mathbb{P}_4 \times C_3$ with Orientable $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_8$-labeling

Theorem 5.0.4. *Let $G \cong \mathbb{P}_n$ and $H \cong C_m$, where $\mathbb{P}_n$ is prism graph of order $2n$ and C_m is cycle graph of order m and $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_{n/2} \times \mathbb{Z}_{2m}$ be the modulo group of order $2mn$ such that $gcd(2,2,n/2,m) \neq 1$.Then the directed direct product of graph $G \times H$ admits a orientable$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_{n/2} \times \mathbb{Z}_m$ distance magic labeling $\forall$ even values of n,m $n \geq 4, m \geq 3$.*

Proof.

The vertex and edge set of $\mathbb{P}_n$ and C_m are as follows:

V ($\mathbb{P}_n$)=$\{ x_i, y_i | 0 \leq i \leq n-1 \}$,

E ($\mathbb{P}_n$) = $\{ x_i x_{i+}1, y_i y_i + 1, x_i y_i | 0 \leq i \leq n-2 \} \cup \{ x_0 x_n - 1, y_0 y_n - 1, x_n - 1 y_n{-}_1 \}$,

V (C_m) = $\{ x'_i, 0 \leq i \leq m-1 \}$,

E (C_m) = $\{ x_i, x'_i{+}_1, 0 \leq i \leq m-2 \} \cup \{ x'_0 x'_m{-}_1 \}$,

By using concept of direct product we get the vertex set below representing $\mathbb{P}_n \times C_m$, *V* ($\mathbb{P}_n \times C_m$) $= \{(x'_i x'_j), (y_i, x'_j) | 0 \leq i \leq n-1, 0 \leq j \leq m\text{-}1\}$.

Define: $\overrightarrow{l} : V(\mathbb{P}_n \times C_m) \rightarrow \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_{n/2} \times \mathbb{Z}_m$ *as follow;*

$\overrightarrow{l}(x_i, x'_j) = (0, k, j \mod 2, i)$

$\overrightarrow{l}(y_i, x'_j) = (1, k, j \mod 2, i)$, *where* $k = \begin{cases} 0, & 0 \leq i < \frac{m}{2} \\ 1, & \frac{m}{2} \leq i \leq m-1 \end{cases}$

Under $\overrightarrow{l}$, $\mathbb{Z}_l \times \mathbb{P}_n \times C_m$ *is Orientable* $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_{n/2} \times \mathbb{Z}_m$*-distance magic graph and magic constant* μ *is*

$$\sum_{y \in N^+_{G(x)}} \overrightarrow{l(x)} - \sum_{y \in N^-_{G(x)}} \overrightarrow{l(x)} = \mu$$

$$\mu = \begin{cases} (0,0,0,0) & \textit{for } n = 3 \\ (0,0,0,2) & \textit{for othervalues of } n \end{cases}$$

Example 5.0.6. *We drew graph $P_4 \times C_4$ and labeled it under modulo group $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_4$ with magic constant $(0,0,0,0)$.We take the direction of each copy anticlockwise.*

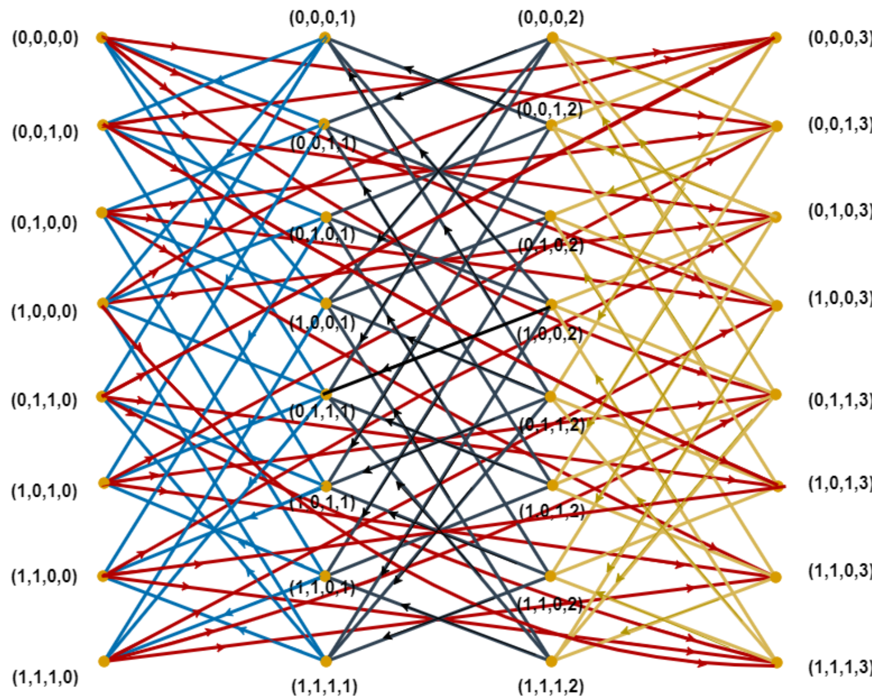

Figure 5.8: Direct Product of Prism with Cycle $\mathbb{P}_4 \times C_4$ with Orientable $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_4$ -distance magic

Theorem 5.0.5. *Let $G \cong \mathbb{P}_n$ and $H \cong C_m$, where $\mathbb{P}_n$ is prism graph of order $2n$ and C_m is cycle graph of order m and $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_{n/4} \times \mathbb{Z}_{2m}$ be the modulo group of order $2mn$ such that $gcd(2,2,n/4,2m) \neq 1$.Then the directed direct product of graph $G \times H$ admits a orientable $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_{n/4} \times \mathbb{Z}_{2m}$ distance magic labeling $\forall$ even values ofn , $n \geq 4$,$m \geq 3$.*

Proof. *The vertex and edge set of $\mathbb{P}_n$ and C_m are as follows:*

V ($\mathbb{P}_n$)={ $x_i, y_i | 0 \leq i \leq n-1$ },

E ($\mathbb{P}_n$) = { $x_i x_{i+}1, y_i y_i + 1, x_i y_i | 0 \leq i \leq n-2$ } $\cup$ {$x_0 x_n - 1, y_0 y_n - 1, x_n - 1 y_{n} {-}_1$ },

V (C_m) = { $x'_i, 0 \leq i \leq m-1$ },

E (C_m) = { $x_i, x'_i{+}_1, 0 \leq i \leq m-2$ }$\cup$ {$x'_0 x'_m {-}_1$},

By using concept of direct product we get the vertex set below representing $\mathbb{P}_n \times C_m$, *V* $(\mathbb{P}_n \times C_m) = \{(x_i, x'_j), (y_i, x'_j) | 0 \leq i \leq n-1, 0 \leq j \leq m\text{-}1\}$.

Define: $\overrightarrow{l} : V\ (\mathbb{P}_n \times C_m) \rightarrow \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_{n/2} \times \mathbb{Z}_m$ *as follow;*

$$\overrightarrow{l}(x_i, x'_j) = ((i \mod 4) \geq 2, k, i \mod 2, 2j)$$

$$\overrightarrow{l}(y_i, x'_j) = ((i \mod 4 \geq 2), k, i \mod (2), 2j+1)$$

$$\textit{where } k = \begin{cases} 0, & 0 \leq i < \frac{n}{2} \\ 1, & \frac{n}{2} \leq i \leq n-1 \end{cases}$$

Under $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_{n/2} \times \mathbb{Z}_m$ *is Orientable*

$$\sum_{y \in N^+_{G(x)}} \overrightarrow{l(x)} - \sum_{y \in N^-_{G(x)}} \overrightarrow{l(x)} = \mu = (0, 4n(m-4)+1, 0, 0)$$

$, \forall m \& n > 3$

Example 5.0.7. *We drew graph $P_8 \times C_5$ and labeled it under modulo group $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_{10}$ with magic constant $(0,0,0,0)$.The direction of each copy anticlockwise .*

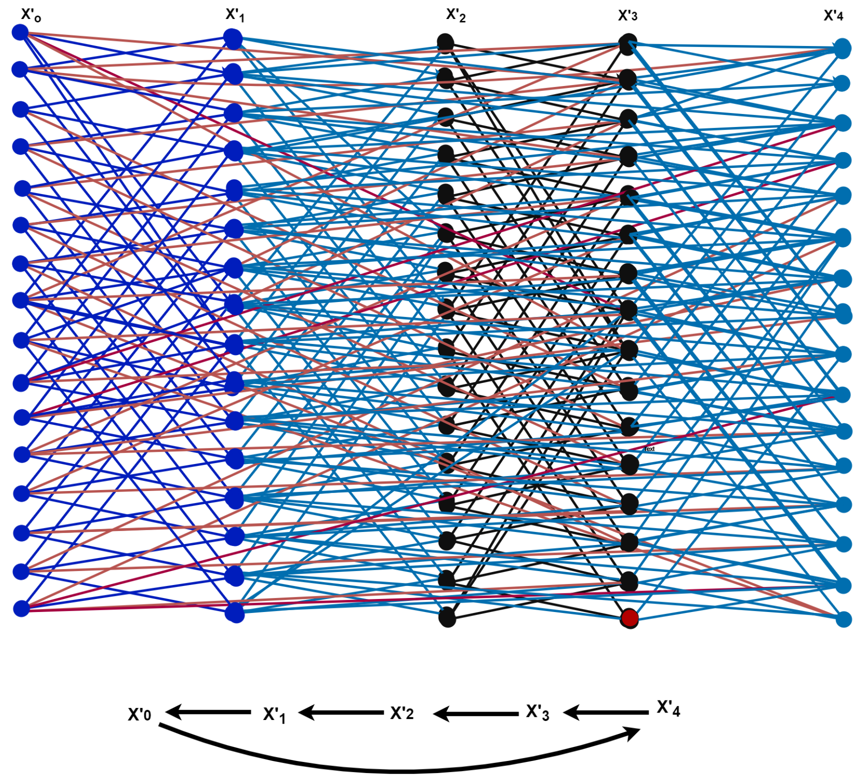

Figure 5.9: Direct product of $\mathbb{P}_8 \times C_5$

Throughout this chapter, we have carried out a comprehensive study on directed families some regular graphs and direct product of prism graphs and cycles for well-defined modulo groups. More precisely we concentrated on the modulo groups that are not isomorphic and used well-known results from group theory to guide our search. We have also investigated direct product of directed direct product of Prism graph $\mathbb{P}_n$ and cycle graph C_n of nth order. We have here used group $\mathbb{Z}_{2nm}, \mathbb{Z}_2 \times \mathbb{Z}_n, \mathbb{Z}_n \times \mathbb{Z}_{2m}, \mathbb{Z}_2 \times \mathbb{Z}_{n/2} \times \mathbb{Z}_{2m}, and \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_{n/4} \times \mathbb{Z}_{2m}$ for the direct product and extended work.

We have used distinct non-isomorphic groups for this work, which guarantees the generalization in a wider aspect. In this study, we also generalized the representation of vertex sets, edge sets, and labeling for the same directed graph in different non-isomorphic groups. Furthermore, we obtained generalized formulae for magic constants of the directed direct product graphs. This formula will be quite useful when you want to find the weight of each vertex in the directed product of graphs. In the case of OGDML, the weight w of any vertex in a directed graph is computed by subtracting the sum of integers at inward rays from outward rays. On the other hand, in GDML each vertex receives weight by considering the sum of all its incident edges. This difference in fundamental nature points to the essential distinction between these two modalities of labeling.

Additionally, we also took into account the orientation of directed graphs and handled both clockwise or anticlockwise but mostly all case studies used anti-clockwise orientation for every directed direct product of the graph. Our results increase a general understanding of directed graph labeling, especially for non isomorphic modulo groups, and will help to develop the research in that direction. The generalized formulae and methodologies presented in this research provide researchers with useful tools to compute the vertex weights and also, further delve into new directions of graph labeling theory.

Chapter 6
Conclusion

We research the labeling methods of Orientable Group Distance Magic Labeling (OGDML) in this work. We aim to find out how these strategies relate to one another to solve an open issue. To reconcile graph theory with group theory, the work contains talks on modulo groups and their products for both labeling. We specifically concentrate on the following conjectures: ”G admits OGDML if it is an even r-regular graph of order n,” and ”GDM graph with n vertices is also an OGDM graph if it is a GDM graph.” These hypotheses remain unsolved issues in the GDML and OGDML mathematical domains. We describe OGDML, which is different from GDML in certain aspects, as the direct product of the cycle graph of the family of order n and the prism graph of order 2n. For OGDML, we offer multiple examples in the form of graph products.

We looked at the family of direct products of cycle graphs and prism graphs for various modulo groups in Chapter 5. We selected those modulo groups that do not have isomorphism. We employ a well-known group theory result for this study. Additionally, we employ graphs that have the form of the direct product of a family of $2n$ order Prism graphs with cycles of order n. For the direct product and extended work, we employ groups $\mathbb{Z}_{2nm}, \mathbb{Z}_2 \times \mathbb{Z}_n, \mathbb{Z}_n \times \mathbb{Z}_{2m}, \mathbb{Z}_2 \times \mathbb{Z}_{\frac{n}{2}} \times \mathbb{Z}_{2m}$, and $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_{\frac{n}{4}} \times \mathbb{Z}_{2m}$-distance magic labeled.In this study, we extended the vertex set, edge set, and labeling representations of the aforementioned graphs for various non-isomorphic groups.

Additionally, we expanded the direct product graph's magic constant formulas. The method used for computation-orientable group labeling differs significantly from the procedure for computing group labeling. Each directed graph vertex's weight in OGDML is equal to the sum of its outward rays minus the sum of its inward rays, whereas, in GDML, each vertex's weight is equal to the sum of all the edges that are present on that specific vertex. This explains why the two labeling phenomena are different. We can take both clockwise and counterclockwise orientation for directed graphs; nevertheless, we took anticlockwise orientation for every directed graph and every directed direct product graph. In this study, we attempted to use labeling to establish a connection between modulo groups and graphs.

6.1 Open Problems

This chapter outlines open issues resulting from the research into Orientable Group Distance Magic Labeling (OGDML).

Table 6.1: Open Problem

Sr.No	**Description**
5.2.1	Comprehensive Analysis of OGDML for the direct product of prism and cycle $\mathbb{P}_n \times C_m$ graphs using $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times Z_{n/8} \times Z_{2m}$.
5.2.2	Using OGDML, calculate the weights of $\mathbb{P}_n \times C_m$ graphs using $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times ...... \times \mathbb{Z}_{n/2!} \times \mathbb{Z}_{2m}$
5.2.3	Examining the triple direct product of graph $\mathbb{P}_m \times C_n$ with P_p in OGDML.
5.2.4	Determine the triple direct product of the graph $P_m \times P_n$ with C_P using OGDML.
5.2.5	OGDML computation of the triple direct product of the graph $P_m \times C_n$ with C_p.
5.2.6	Computation of the triple direct product of the graph $P_m \times C_n$ with C_p using OGDML.
5.2.7	Using GDML, calculate the triple direct product of the graph $P_m \times C_n$ with C_p.

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