On Cubic Pythagorean Fuzzy Topological Spaces and their Properties

By

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CIIT/FA21-RMT-035/LHR

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Declaration

I Farwa Raza, CIIT/FA21-RMT-035/LHR, hereby state that my MS thesis titled "On Cubic Pythagorean Fuzzy Topological Spaces and their Properties" is my own work and has not been submitted previously by me for taking any degree from this University "COMSATS University Islamabad, Lahore Campus" or anywhere else in the country/world. At any time if my statement is found to be incorrect even after my Graduate the university has the right to withdraw my MS degree.

Date:______________ ____________________________

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It is certified that Farwa Raza, CIIT/FA21-RMT-035/LHR has carried out all the work related to this thesis under my supervision at the Department of Mathematics, COMSATS University Islamabad, Lahore Campus and the work fulfills the requirement for award of Ms degree.

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DEDICATION

To My Parents and All Family

ACKNOWLEDGEMENTS

**Praise to be ALLAH, the Cherisher and Lord
of the World, Most gracious and Most Merciful**

First and foremost, I would like to thank ALLAH Almighty (the most beneficent and most merciful) for giving me the strength, knowledge, ability and opportunity to undertake this research study and to preserve and complete it satisfactorily. Without countless blessing of ALLAH Almighty, this achievement would not have been possible. May His peace and blessings be upon His messenger Hazrat Muhammad (PBUH), upon his family, companions and whoever follows him. My insightful gratitude to Hazrat Muhammad (PBUH) Who is forever a track of guidance and knowledge for humanity as a whole. In my journey towards this degree, I have found a teacher, an inspiration, a role model and a pillar of support in my life, my kind.

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ABSTRACT

On Cubic Pythagorean Fuzzy Topological Spaces and their Properties

In the domain of set theory, fuzzy mathematics is an extension of traditional mathematics. From the inspection of literature on fuzzy mathematics, Simple observation reveals that fuzzy set theory hold a broad scope of mathematical contents that were previously known. It has a comprehensive applications, containing automobiles, transport systems, communication, and other activities. The main idea of a FS, provides an appropriate framework for both constructing new region of fuzzy mathematics. The most important region of research is to expand the concepts of PFTSs, which are basically extensions of FTSs and IFTSs. Here we study, to develop the concept of CPFTSs and CPF continuity of a mapping among them. Also, we will discuss some useful features of these concepts. Furthermore, we will design a CPFT on an accessible nonempty reference set using continuity ideas. We will begin by going over some definitions and fundamental concepts, including fuzzy sets, and fuzzy topological spaces. These ideas will be extremely useful to us as we do this research. After that, we'll explore PFTSs. By using this idea, our major main goal to expand and used to distinguish between FTS and IFTS, also aim develop the concept in cubic form and continuity map between them. Furthermore, we will design it on a nonempty reference set using continuity ideas.

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Chapter 1

Introduction

First and foremost, let's delve into the fascinating realm of set theory, where fuzzy mathematics sets itself apart from traditional mathematics. It wasn't long ago when fuzzy mathematics emerged, encompassing various mathematical topics. Its applications span multiple domains, such as automotive engineering and traffic management, where logical circuits control anti-skid handbrakes, communication systems, and other intricate operations. The revolutionary idea of a FS , initially proposed by Zadeh in 1965, gives a solid substructure for developing latst section of fuzzy mathematics and mathematically addressing the well-known fuzzy phenomena that permeate our daily lives.

In this research, we delve into the notion of precision, which surpasses the simplicity of crisp sets that can be either a "yes" or a "no," or represented by "1" or "0." In our physical world, it would be ideal to categorize objects neatly based on clearly defined criteria. However, more often than not, this is a challenging task due to the need for precisely delineated association criteria. Let's consider some examples: when classifying animals, we can quickly identify birds, cats, and deer. However, an uncertain classification arises for entities like bacteria, viruses, starfish, and jellyfish within the animal kingdom. Similarly, numerical values, like the number ten, present uncertainty when considering their "class" within the collection of all real values larger than one. It becomes evident that using expressions like "the class of all real values ¡ 1," "the group of attractive ladies," or "the group of tall gentlemen" fails to conform to the conventional mathematical logic of sets and groups. Nevertheless, it remains true that these loosely termed "classes" play a crucial role in human cognition, particularly in prototype abstraction, effective communication, and pattern recognition.

Since its inception in the mid-1960s, Fuzzy set theory-based research has been rapidly expanding. The concepts and findings associated with fuzzy sets have garnered considerable attention, showcasing their remarkable potential. Furthermore, vigorous research driven by practical applications has yielded exceptional outcomes, further solidifying the significance of fuzzy sets in various fields.

The concepts presented in this essay were foreseen by the American philosopher Black three decades ago. The author put forth a hypothesis with the fundamental premise that fuzzy sets are sets with indistinct boundaries. To maintain brevity, Chang focused primarily on fundamental definitions, theorems, and proofs concerning the meaning of topology on fuzzy sets, also known as topological fuzzy spaces. In their study, Michalek identified and examined an alternative notion of topological fuzzy space that significantly differed from Chang's traditional concept. Lowen contributed thought-provoking principles for constructing topological fuzzy spaces, introducing two new functions and employing fuzzy compression to simplify topological compression. This approach facilitated a more straightforward relationship between topological fuzzy spaces and topological spaces. Cheng-Ming explored product-induced spaces, a distinctive type of fuzzy topological space, demonstrating the topological isomorphism between each topological fuzzy space and a specific topological space. The concept of two points and the construction of a fuzzy point neighbourhood, such as the neighbourhood, were also established as crucial ideas in topological fuzzy sets. Furthermore, the metrization problem in fuzzy on topological fuzzy spaces was investigated, leading to a metrization hypothesis. Zimmermann's book delved into the specifics of FST and its applications. For instance , Papageorgiou proposed several ideas related to fuzzy topological concepts, fuzzy multifunctions, and fuzzy neighborhood points, aiding in understanding typical optimization strategies and fuzzy games.

IFS, pioneered by Atanassov, take fuzzy sets to a new level. They open up Pandora's box of possibilities, IVIFSs, linguistic IVIFS, complex IFS, and complex IVIFS. While IFS and IVIFS follow the rule that the sum of membership and non-membership degrees must be ≤ 1. Atanassov introduced a different breed of IFSs. In this second type, the sum of the squares of membership and non-membership degrees plays by the rule of being ≤ 1. This fresh perspective adds a new layer of versatility to IFSs, allowing us to tackle unreliability and vagueness effectively. The beauty of IFS shines through when applied to real-life scenarios like medical diagnosis, career paths, election processes, and different measures of taking options.

Intuitionistic fuzzy sets, establish by Atanassov, took fuzzy sets to the next level. But you know what? They had their limitations. They couldn't handle situations where both membership grade μ and non-membership grade ν went beyond 1. That's when Pythagorean fuzzy sets came to the rescue! These excellent sets consider $\mu+\nu \leq 1$ and $\mu+\nu \geq 1$. And there's a neat relationship that goes with it: $\mu^2+\nu^2+\pi^2 = 1$. It all started with IFSST, but Pythagorean fuzzy sets took it to a new level. They handle uncertainty like a boss! Garg, being his genius, came up with a scoring function that ranks order in interval-valued Pythagorean fuzzy sets. It's a clever way to determine preference based on similarity to the ideal solution. Just imagine experts' preferences are considered using interval-valued Pythagorean fuzzy decision matrices. Researchers are going gaga over Pythagorean fuzzy sets. They use them left and right in taking right option, aggregation operators, and system of measurement. It's like a treasure trove of possibilities waiting to be explored. So buckle up and get ready to dive into the world of Pythagorean fuzzy sets.

In research, our goal to build upon the existing theories of FSs and IFSs, taking them a step further. The major goal is to expand the Principle of fuzzy cubic sets by considering the degree of rejection and introducing a novel concept called cubic IFSs. These sets incorporate interval-valued and IFSs, bringing afresh aspect to the area. The research delves into the features and characteristics of these cubic intuitionistic fuzzy sets, investigating their behaviour and potential applications.

In decision-making procedures, PFS and IVPFSs hold significant importance. This study introduces a fascinating notion called CPFSs, where the membership degree is expressed as a PFS. In contrast, the non-membership degree is an IVPFS. The research sheds light on their utility and potential advantages by exploring various tasks involving CPF numbers.

Furthermore, this study endeavours to extend the concept of PFTSs, setting them apart from fuzzy and intuitionistic fuzzy topological spaces. The research defines PFTSs based on Chang's theory. It explores the idea of Pythagorean fuzzy continuity, which captures the idea of continuity conecting7 two PFTSs. The investigation also delves into image and pre-image concepts concerning functions and examines their fundamental properties. By establishing rough PFT on a set A (having at least one element), a given function $h : A \to B$ can be transformed into a Pythagorean fuzzy continuous function within a PFTS. These findings carry implications for making decision, Analyse data, and intelligent retrieval, promising more accurate results. Future research can explore the categorical properties of PFTSs, delve deeper into their applications, and explore the realm of PFT Spaces.

CPFTS combine elements from fuzzy set theory, topology, and Pythagorean fuzzy sets.

They gives a substructure for modeling uncertainty and imprecision in topological spaces, allowing for more flexible and nuanced analysis. In a CPFTS, open sets are defined using CPFSs, providing a more nuanced understanding of openness and representing uncertainty in the topological structure. By incorporating degrees of non-membership, membership, and hesitancy, CPFTSs offer a realistic representation of imprecision and uncertainty in the topological analysis. The study of CPFTSs contributes to a deeper understanding of fuzzy topological spaces and provides a significant tool for tackling with complex as well as uncertain data. It enables more robust analysis of topological properties in imprecision, making it useful in various applications i.e designs, directing, and information retrieval, where uncertainty plays a significant role.

The for most intention of the exploration is to explore the realm of CPFTS and investigate the continuity of mappings within these spaces. The study delves into these concepts' fundamental aspects and notable characteristics, drawing upon using PFSs and IVPFSs. In the context of CPFTS, IVPFN number effectively represents the degree of membership. In contrast, the degree of non-membership is expressed using a PFN. The research focuses on defining the notions of continuity and establishing a cubic PFT on a reference set having at least one element.

Chapter 2

An Introduction to fuzzy topological space

2.1 Fuzzy Sets

A FS is a general idea of a classical set that is essential for concernig with ambiguity and unreliability. It is commonly defined as a membership function, whereas the universal set U determine a membership function to the set range between 0 and 1.

Definition 2.1.1. Let $I = [0,1]$ and $X = \phi$, where x stands for any individual member of X. To represent a FS in X, we define $\mu : X \to I$, here $x \to \mu(x)$. In this structure, "$x \in X$ in μ" is represented by $\mu(x)$.

Example 1

Let us use the FS X to show that concept of a real integer close to zero. The following function can be used to model this concept.

$$\mu_X(x) = \frac{1}{1+x^4}$$

$\mu_X(0) = 1,\ \mu_X(1) = 0.5,\ \mu_X(-1) = 0.5,\ \mu_X(2) = 0.05,\ \mu_X(-2) = 0.05$

The given function is symmetric, modeled so that when $x = 0$, it returns the highest membership value, 1. The membership value declines equally on both sides as we travel away from 0 in either direction. The membership values $+x$ and $-x$ are the same. As the number approaches zero, its membership value gradually declines. The membership value becomes 0 if it exceeds a certain threshold. In this set, all additional items that are considerably far from 0 is give a membership value of 0.

2.1.1 Crisp set

It is a set of items that uses binary (yes/no) logic to describe a clearly defined property. Each element in a this set is either a member of the set S, a subset of the more extensive set X or it is not. As a specific instance of fuzzy sets, crisp sets are found on classical set theory and have membership values that can only be 1 (which indicates full membership) or 0 (which indicates no membership). In this sense, crisp sets are a particular kind of fuzzy set that uses bi-valued logic. It denotes

$$\chi_s : X \to \{0,1\}$$

Definition 2.1.2. In crisp set theory, A be the U, a crisp set V makes collection of elements from A where each element either belongs or does not belong to V. It denote a characteristic function $\chi_s : A \to \{0,1\}$ where $\chi_s(a) = 1$ if $a \in V$ and $\chi_v(a) = 0$ if $a \notin V$.

Example

If we have a universal set of integers $A = \{-4,-2,0,2,4\}$, and we define a crisp set V as $V = \{-2,2\}$, the characteristic function would $\chi_v(a) = 1$ for $a = -2$ or $a = 2$, and $\chi_v(a) = 0$ for $a = -4, a = 0$, or $a = 4$.

It's worth noting that crisp sets are different from fuzzy sets, which allow for partial membership or degrees of membership using fuzzy logic.

2.1.2 Classical Set

Definition 2.1.3. A classical set comprises distinct objects that are identified as members or elements of the set. These elements possess unique characteristics and adhere to certain fundamental properties. The set is structured in rather that every member is categorized strictly as either an element or a not an element. For instance, in the case of set A, an element can either belong to A or not belong to A. Partial membership is not a possibility within this framework.

Example

Let's consider the set $\mathbb{N}$ defined as $\{x \in \mathbb{N} \mid x < 10\}$. In this case, the elements of the set are the integers 2, 4, 6, and 8.

$\mathbb{N} = \{2,4,6,8\}$

Now, let's test the membership of some numbers in set A using the notation $\mathbb{N} \in$ or $\mathbb{N} \notin$:

Is 3 an element of set $\mathbb{N}$? $3 \notin \mathbb{N}$ (since 3 is not an even number).

Is 4 an element of set $\mathbb{N}$? $4 \in \mathbb{N}$ (since 4 is an even number).

Is 7 an element of set $\mathbb{N}$? $7 \notin \mathbb{N}$ (since 7 is not an even number).

Is 8 an element of set $\mathbb{N}$? $8 \in \mathbb{N}$ (since 8 is an even number).

In this example, we can see that each number is either a member or a non-member of the set N, with no possibility of partial membership.

2.1.3 Crisp Sets Have a Unique Function

Definition 2.1.4. Consider Y to be U, and $S \subset X$, where S can be $\emptyset$ (also denoted as ϕ) or a non-empty set. The characteristic function for every $y \in Y$ is denoted as 0_S or 1_S and following is the definition:

$$1_S(y) = 0_S(y) = \begin{cases} 0, & \text{if } y \notin S \\ 1, & \text{if } y \in S \end{cases}$$

When $0_S(y) = 1$, it implies that y belongs to set S, and when $1_S(y) = 0$, it implies that x does not belong to set S.

Example

Consider in a universal set $X = \mathbb{Z}$, and the subset $S = \{2,4,6,8,...\}$.
$X = \{\ldots,-4,-3,-2,-1,0,1,2,3,4,\ldots\}$
$S = \{2,4,6,8,\ldots\}$
Now, we can define the characteristic function for S using the notation $1_S(x)$ and $0_S(x)$:
For any integer x:

$$1_s(y) = 1, \text{ if } \; x \in S, \; 0 \; \text{ otherwise}$$

or

$$0_s(y) = 1, \text{ if } \; x \in S, \; 1 \; \text{ otherwise}$$

Let's evaluate the characteristic function for a few integers:
$1_S(3) = 0$ because 3 is not a positive even number.
$1_S(4) = 1$ because 4 is a positive even number.
$0_S(-2) = 0$ because -2 is not a positive even number.
$0_S(10) = 1$ because 10 is a positive even number.
In this example, the characteristic function allows us to determine whether an integer belongs to the subset S (positive even numbers) or not, by assigning a value of 1 or 0 to

corresponding.

2.2 Fuzzy Operations

In FS theory, the set operations are follow:

- complement
- union
- intersection

2.2.1 Fuzzy Union

Definition 2.2.1. let A be a universal set, two FSs X and Y with the union $X \cup Y$, is define as follows:

$$\mu_{X \cup Y} = max\ [\mu_X(a)\ ,\ \mu_Y(a)], \text{for all } a \in A$$

Example

The universal set A, where A represents the set of integers from 1 to 10, then two FSs, X and Y defined on the universal set

$X = \{2,4,6,8,10\}$ denotes the membership degree of every member in A :

$$\mu_X(a) = \begin{cases} 1, & \text{if } a \in \{2,4,6,8,10\} \\ 0, & \text{otherwise} \end{cases}$$

FS "Y" denotes every element in A to the concept of "prime numbers":

$$\mu_Y(a) = \begin{cases} 1, & \text{if } a \text{ is a prime number} \\ 0, & \text{otherwise} \end{cases}$$

Now, let's calculate the fuzzy union of X and Y, written as $X \cup Y$, using the maximum membership principle:

$$\mu_{X \cup Y}(x) = \max[\mu_X(a), \mu_Y(a)],\ \text{for all} a \in A$$

For example, let's evaluate the fuzzy union at $a = 6$:

$$\mu_{X\cup Y}(6) = \max[\mu_X(6), \mu_Y(6)] = \max[1,0] = 1$$

Similarly, we can calculate the fuzzy union for other elements in A. Here are a few more examples:

$$\mu_{X\cup y}(3) = \max[\mu_X(3), \mu_Y(3)] = \max[0,1] = 1$$

$$\mu_{X\cup Y}(7) = \max[\mu_X(7), \mu_Y(7)] = \max[0,1] = 1$$

$$\mu_{X\cup Y}(9) = \max[\mu_X(9), \mu_Y(9)] = \max[0,0] = 0$$

In this example, the fuzzy union combines the membership values of every member in A from the FSs X and Y using the maximum operator. It allows us to obtain a new FS that represent the membership degree to both concepts simultaneously.

2.2.2 Fuzzy Intersection

Definition 2.2.2. The fuzzy intersection of two FSs X and Y on the universal set A, denoted as $X \cap y$, is defined as follows:

$$\mu_{X\cap Y} = min\ [\mu_X(a)\ ,\ \mu_y(a)], \text{for all}\ \ a \in A$$

Example

Consider the universal set A as the set of integers from 1 to 5: $A = 1,2,3,4,5$.

Let's define two FSs, X and Y, on the universal set A, where each FS denotes the membership degree of every member to a specific concept.

Fuzzy set X represent the membership degree of each member in A of the notion of small numbers:

FS “Y” shows member in A to the concept of even numbers:

$$\mu_X(a) = \begin{cases} 1, & \text{if } a \leq 3 \\ 0, & \text{otherwise} \end{cases}$$

FS Y shows the membership degree of every member in A to the idea of even numbers:

$$\mu_Y(a) = \begin{cases} 1, & \text{if } a \text{ is an even number} \\ 0, & \text{otherwise} \end{cases}$$

Now, let’s calculate the fuzzy intersection of X and Y which can be written as $X \cap Y$, using the minimum operator:

$$\mu_{X\cap Y}(x) = \min[\mu_X(a), \mu_Y(a)], \text{ for all} a \in A$$

We can evaluate the fuzzy intersection for every member in A:

$$\mu_{X\cap Y}(1) = \min[\mu_X(1), \mu_Y(1)] = \min[1,0] = 0$$

$$\mu_{X\cap Y}(2) = \min[\mu_X(2), \mu_Y(2)] = \min[1,1] = 1$$

$$\mu_{X\cap Y}(3) = \min[\mu_X(3), \mu_Y(3)] = \min[1,0] = 0$$

$$\mu_{X\cap Y}(4) = \min[\mu_X(4), \mu_Y(4)] = \min[0,1] = 0$$

$$\mu_{X\cap Y}(5) = \min[\mu_X(5), \mu_Y(5)] = \min[0,0] = 0$$

The fuzzy intersection $\mu_{X\cap Y}$ combines the membership values of every member from the FSs X and Y using the minimum operator. It provides a new FS that represents membership degree to both concepts simultaneously. In this example, the fuzzy intersection reflects the minimum membership degree between ”small numbers” and ”even numbers” for each element in the A.

2.2.3 Fuzzy Complement

Definition 2.2.3. If X is a FS defined on Y be a universal set, its complement X^C is another fuzzy set on Y. The complement of FS X is represented by the membership function $\mu_{X^C} :\rightarrow [0,1]$. It can be described as follows:

$$\mu_{X^c} = 1 - \mu_X(y), \ \forall \ y \in Y$$

Example Consider a universal set Y as the set of integers from 1 to 5: $Y = \{2,4,6,8,10\}$. Let X be a FS on Y with the given function $\mu_X(y)$ as below:

$$\mu_X(y) = \begin{cases} 0.2, & \text{if } y = a \\ 0.5, & \text{if } y = b \\ 0.7, & \text{if } y = c \\ 0.9, & \text{if } y = d \\ 0.3, & \text{if } y = e \end{cases}$$

Using the formula for the complement fuzzy set, we can calculate the membership function $\mu_{X^C}(y)$ as follows:

$$\mu_{X^C}(y) = 1 - \mu_X(y)$$

Substituting the membership values of A into the formula, we get:

$$\mu_{X^C}(y) = \begin{cases} 0.8, & \text{if } y = a \\ 0.5, & \text{if } y = b \\ 0.3, & \text{if } y = c \\ 0.1, & \text{if } y = d \\ 0.7, & \text{if } y = e \end{cases}$$

This represents the complement FS X^C for each element y in the universal set Y.

2.3 Topological Space

Topology, a fundamental branch of mathematics, has emerged as a powerful and influential discipline with profound implications in various areas of pure mathematics, including geometry and analysis. Despite the term ”topology” being coined relatively recently in the 1930s, its significance and impact have extended far and wide. “Topology” found from two Greek words, encapsulating, simply define as “the science of place and location.” Modern general topology has become indispensable for undergraduate and graduate mathematics students, serving as a cornerstone of their education. Topology has solidified its position as an entire field within pure mathematics, fostering advancements not only in abstract algebra but also in the realm of fuzzy mathematics. Its rich concepts and principles inspire innovative research and pave the way for discoveries in mathematical theory and applications.

2.3.1 Topology

Definition 2.3.1. Let $X \neq \phi$, and let τ be a class of subsets of X. τ is knows as a topology if it fulfill the given conditions as below:

(i) X and $\phi \in \tau$.

(ii) The finite union of any two sets in τ belongs to τ.

(ii) The arbitrary intersection of A_i and A_j in τ also $A_i \cap A_j \in \tau$.

Where (X, τ) is known as a TS, and class of subsets of $X \in \tau$ are called open sets in X.

Example

Consider subsets of $A = \{\alpha, \beta, \gamma, \sigma, \chi\}$.

$$\tau_a = \{A, \phi, \{\alpha\}, \{\gamma, \sigma\}, \{\alpha, \gamma, \sigma\}, \{\beta, \gamma, \sigma, \chi\}\}$$

$$\tau_b = \{A, \phi, \{\alpha\}, \{\gamma, \sigma\}, \{\alpha, \gamma, \sigma\}, \{\beta, \gamma, \sigma, \}\}$$

$$\tau_c = \{A, \phi, \{\alpha\}, \{\gamma, \sigma\}, \{\beta, \gamma, \sigma\}, \{\beta, \gamma, \sigma, \chi\}\}$$

We observe that τ_a forms a topology on A, while τ_b and τ_c do not. The reason is as follows:

τ_a satisfies the necessary conditions (i) and (ii) to be a topology. Both A and $\emptyset$ are present in τ_a. Additionally, the union of any collection of sets in τ_a is also in τ_a, and the intersection of any finite number of sets in τ_a is in τ_a as well.

On the other hand, τ_b and τ_c fail to satisfy condition (ii) of being a topology. The union $\{\alpha,\gamma,\sigma,\}\cup\{\beta,\gamma,\sigma,\}=\{\alpha,\beta,\gamma,\sigma,\}\notin\tau_b$, and the intersection $\{\alpha,\gamma,\sigma,\}\cap\{\alpha,\beta,\gamma,\sigma,\}=\{\beta,\gamma\}$ of two sets in $\tau_c\notin\tau_c$. Hence, neither τ_b nor τ_c meet the requirement for being a topology on X.

Discrete topology

Definition 2.3.2. On a nonempty set Y, the **discrete topology** is defined as the collection of all subsets of A, as well as ϕ, generating a τ on A. This collection of subsets is known as the power set $P(A)$.

Example

Consider $X=\{p,q\}$. In this case, the discrete topology τ on X is given by $\tau=\{X,\emptyset,\{p\},\{q\}\}$. This means that every subset of X is considered an open set in the discrete topology.

Indiscrete topology

Definition 2.3.3. The indiscrete topology, also known as the trivial topology or the coarsest topology, is describe on any nonempty set X. It consists of only two sets: ϕ and the entire set X itself. This collection of sets generates a τ on X. This collection of sets defines the elements in the set X, known as a topology

Example

Consider $X=\{p,q,r\}$. In this case, the **indiscrete topology** τ on X is given by $\tau=X,\phi$. This means that only the entire set X and ϕ are considered open sets in the indiscrete topology.

2.3.2 Interior, Exterior and Boundary Points

Definition 2.3.4. a point in a set E that is also contained within a larger set B is considered an interior point of B if and only if it is part of both E and B simultaneously. This concept is often represented as x belonging to E and being an interior point of B, denoted as $x \in E \subseteq \text{int}(B)$. Here, $\text{int}(B)$ denotes $\text{int}(B)$ in the context of topology.

Definition 2.3.5. The interior of B^c, write as $\text{ext}(B)$, is known as the exterior of B. It consists of points that are not contained in B.

Definition 2.3.6. The set of points that are neither in the $\text{int}(B)$ nor in the $\text{ext}(B)$ is called the boundary points of B, denoted as $\text{bd}(B)$. It can be expressed as the complement of the union of the interior and the exterior of B

Example

Let $X = \{1,2,3,4\}$ be a TS, and let $B = \{1,2,3\}$ be a set that is a subset of X. Assume that $E = \{1,2\}$ is an open set $\subset B$.
Therefore, $\text{int}(B) = E = \{1,2\}$.
In this example, the $\text{int}(B)$ of B is $\{1,2\}$.
The complement of B is $B^c = \{4\}$, which consists of the point d that is not in B. Thus, the exterior of B is $\text{ext}(B) = \text{int}(B^c) = \text{int}(\{d\}) = \emptyset$, as there are no points in B^c.
In this example, the exterior of B is $\{\ \}$ or $\emptyset$.
Therefore, the boundary of B, express as $\text{bd}(B)$, is the complement of the union of the interior and the exterior:
$\text{bd}(B) = \text{Complement}(\text{int}(B) \cup \text{ext}(B)) = \text{Complement}(\{1,2\} \cup \emptyset)$
$\text{bd}(B) = \text{Complement}\{1,2\} = \{3,4\}$.
In this example, the boundary of B is $\{3,4\}$.

2.3.3 Homomorphism Topological Spaces

Definition 2.3.7. In simpler terms, homeomorphism is a continuous and one-to-one mapping between two topological spaces. A homeomorphism allows us to establish a correspondence between the points of the two spaces i.e their open sets also compare to each other. This means that if spaces X and Y are homomorphism, we can map each point in X to a single point in Y, and the open sets in X will match the open sets in Y according to this mapping.

Example

Let's consider two topological spaces:

$X = \{1,2,3\}$ with the topology $\tau_X = \{\emptyset, \{1\}, \{2\}, \{1,2\}, \{1,2,3\}\}$

$Y = \{l,m,n\}$ with the topology $\tau T_Y = \{\emptyset, l\}, \{m\}, \{l,m\}, \{l,m,n\}\}$

We can define a mapping $g : X \to Y$ as follows:

$$g(1) = l$$

$$g(2) = m$$

$$g(3) = n$$

This mapping is continuous since the g^{-1} of any open set in Y under g is open in X. For example, Consider the open set $\{l,m\}$ in Y. Its pre-image under g is $\{1,2\}$, which is an open set in X.

Furthermore, g is one-to-one and onto, and $g^{-1} : Y \to X$ is also continuous.

Therefore, g is homomorphic between the TS X and Y.

2.4 Fuzzy Topological Spaces

The perspective of FS theory offers us with a broader framework than classical set theory, which generalises numerous topological ideas. FT combines ordered and topological structures.The field of mathematics known as topology was initially pioneered by the esteemed mathematician Ehrenman, who included nearly two dynamic topological properties on a grid that affected everyone. The first person to utilize the FT nomenclature was Chang. Then, other academics continued their research in this area. We found that the characteristic functions in FT substitute the membership functions in the point set topology.

2.4.1 Definitions

Chang began the pioneering effort to construct the fuzzy counterpart of fundamental topology in 1968, from Chang's perspective.

Definition 2.4.1. A FT in X is a family $textbfF = \mu : \mu$ is a FS in X of fuzzy subsets (i.e.$\mathbf{F} \in I^X$) that satisfy the three axioms:

1. $0, 1 \in \mathbf{F}$
2. $\mu_1, \mu_2 \in \mathbf{F}$, then $\mu_1 \wedge \mu_2 \in \mathbf{F}$
3. If $\{\mu_x : x$ blongs to $y\}$subset $\mathbf{F}$, then $\vee \mu_x \in \mathbf{F}$

$\mathbf{F}$ is called a foundation for X, and $(X, \mathbf{F})$ is referred to as a foundation space. The elements of $\mathbf{F}$ are defined as the elements of an $\mathbf{F}$-open foundation set. If the complement of a set σ in $\mathbf{F}$ is $\mathbf{F}$-open, then the set σ is considered closed in the context of the set I^X.

Example

Consider a set $X = \{1, 2, 3\}$ and define a fuzzy topology F on X as follows:

$F = \{\mu_1, \mu_2, \mu_3\}$, where

$$\mu_1 = \{(1, 0.7), (2, 0.1), (3, 0.2)\}$$

$$\mu_2 = \{(1, 0.5), (2, 0.6), (3, 0.8)\}$$

$$\mu_3 = \{(1, 0.2), (2, 0.1), (3, 0.4)\}$$

Here, μ_1, μ_2, μ_3 are FTs on X, and they form the FT F. Each μ_i represents an F-open fuzzy set.

In this example, the FTS is (X, F), in which X is the underlying set and F is the FT defined on X.

Indiscrete Fuzzy topology

Indiscrete fuzzy topology, like general FT, comprises only FSs 0 and 1.

Example

Indiscrete FT $(X, D), D = \{\alpha \in I^X : i$ constant fuzzy set $\}$ i.e $\forall \alpha \in D$, if $\alpha = constant$

So, $\alpha(x) = 0,$ for all x

$$\alpha(x) = 1, \text{for all } x$$

I^X is the FS on X.

Discrete fuzzy topology

The discrete FT is the collection of all possible FT. Discrete FT is the collection of all FSs.

Example

$$X = \{a,b,c\}, p = (0.7, 0.8.0.6),\ q = (0.5, 0.4, 0.3),$$

$$r = (0.2, 0.2, 0.1)\ ,\ D = \{0, a, b, c, 1\}.$$

Now,

$$p \wedge p = (0.7, 0.8.0.6) \wedge (0.5, 0.4, 0.3) = (0.5, 0.4, 0.3) = q \in D$$

$$p \wedge r = (0.7, 0.8.0.6) \wedge (0.2, 0.2, 0.1) = (0.2, 0.2, 0.1) = r \in D$$

$$q \wedge r = (0.5, 0.4, 0.3) \wedge (0.2, 0.2, 0.1) = (0.2, 0.2, 0.1) = r \in D$$

$$q \vee p = (0.5, 0.4, 0.3) \vee (0.5, 0.4, 0.3) = (0.7, 0.8, 0.6) = p \in D$$

$$p \vee r = (0.7, 0.8.0.6) \vee (0.2, 0.2, 0.1) = (0.7, 0.8, 0.6) = p \in D$$

$$p \vee r = (0.5, 0.4, 0.3) \vee (0.2, 0.2, 0.1) = (0.5, 0.4, 0.3) = q \in D$$

also,

$$p \vee q \vee r = (0.7, 0.8, 0.6) \vee (0.5, 0.4, 0.3) \vee (0.2, 0.2, 0.1)$$

$$= (0.7, 0.8, 0.6)$$

$$i \in D$$

It is obvious $0, 1 \in D$.

Therefore, (X, D) is a FTS.

2.4.2 Closure Fuzzy Topological Space

The fuzzy closure of a FS μ in a FTS is represented by $\overline{\mu}$ which is define by the smallest closed FS that encompasses μ.

The fuzzy closure $\overline{\mu}$ is define by:

$$\overline{\mu} = \{x : x \text{ is } F\text{-closed and } \alpha \geq \mu\}$$

It is evident that the fuzzy closure $\overline{\mu}$ is always F-closed. We have explored various properties as following.

Properties of Closure Operator

The closure operation, denoted as $\mu \to \overline{\mu}$, is a mapping from $[0,1]^X$ to $[0,1]^X$. In order for this mapping to qualify as a closure operation, it must satisfy the following four properties for any μ and λ belonging to $[0,1]^X$:

(i) μ is less than or equal to its closure, i.e., $\mu \leq \overline{\mu}$. (ii) The closure of the closure of μ is less than or equal to the closure of μ itself, i.e., $\overline{\overline{\mu}} \leq \overline{\mu}$ (this property is known as idempotence). (iii) The closure of the union of μ and λ is equal to the union of their individual closures, i.e., $\overline{\mu \vee \lambda} = \overline{\mu} \vee \overline{\lambda}$. (iv) The closure of the empty set is equal to the empty set itself, i.e., $\overline{0} = 0$.

Definition 2.4.2. The FT is introduced by the closure operator that is denoted by F_X and is given as below:

F_X is the set of all FSs μ in $[0,1]^X$ such that closure of the complement of $\mu = \mu^c$, i.e., $\overline{1-\mu} = 1-\mu$.

The pair (F_X, X) is referred to as the closure of the FTS generated by the closure operator. In this context, X represents the underlying set and F_X represents the collection of fuzzy sets that satisfy the closure property mentioned above.

Interior of a Fuzzy Topological Space

μ° captures the core or innermost region of the FS μ, encompassing the elements with the highest membership degree. It is the smallest fuzzy set that can be considered open with respect to the fuzzy topology F.

Definition 2.4.3. Consider a FTS (X,F) and $\mu, \alpha \in I^X$ such that $I = [0,1]$ The fuzzy set α is define by an interior fuzzy set of μ iff $\exists$ a FS ρ in the collection F_X s.t $\mu \geq \rho \geq \alpha$. In short ρ lies between μ and α, and it satisfies the condition that it is $\geq \alpha$ and less than or equal to μ.

Example

In the Fuzzy Topological Space (X,F), with X as the underlying set and F as the fuzzy topology, let's consider two fuzzy sets μ and α in $[I^X$, where μ is greater than or equal to α (denoted as $\mu \geq \alpha$).

Suppose we have the following fuzzy sets:

$$\mu = \{(x, 0.8) \mid x \in X\}$$

$$\alpha = \{(x, 0.6) \mid x \in X\}$$

To determine if α is an interior fuzzy set of μ, we need to find a fuzzy set ρ in F_X such that $\mu \geq \rho \geq \alpha$.

Let's consider $\rho = \{(x, 0.7) \mid x \in X\}$. We can observe that $\mu \geq \rho \geq \alpha$, which means that α is indeed an interior FS of μ in the FST (X,F).

Theorem 2.4.1. *Let (X,F) be a FTS,*

1. $0^\circ = 0,\ 1^\circ = 1$
2. $F^\circ \leq F$
3. $F^{\circ\circ} = F^\circ$
4. $(F \wedge G)^\circ = F^\circ \wedge G^\circ$

Theorem 2.4.2. *Suppose f and g be two FSs in FTS. Then $f^\circ \vee g^\circ = (f \vee g)^\circ$*

Proof. f° is open and g° is open.

So $f^\circ = f$, and $g^\circ = g$.

Then, $(f^\circ \vee g^\circ)^\circ = (f \vee g^\circ$

Again $f^\circ \vee g^\circ$ is open, so $(f^\circ \vee g^\circ)^\circ = fa^\circ \vee g^\circ$

Therefore, $f^\circ \vee g^\circ = (f \vee g)^\circ$

Hence, $f^\circ \vee g^\circ = (f \vee g)^\circ$ □

2.4.3 Boundary of Fuzzy Topological Space

Let a FTS (X,F), where X is the set and F is the fuzzy set defined on X.

The fuzzy boundary μ^b of a FS $\mu \in [0,1]^X$ is define by the minimum of all F-closed sets ρ that satisfy $\rho(x) \geq \overline{\mu}(x)$ for all x belong to X, where $\overline{\mu} \wedge (1-\mu) > 0$.

Example

Consider the following FTS: $X = \{a,b,c\}$ (a set with three elements) $F = \{\mu_1, \mu_2, \mu_3\}$, where μ_1, μ_2, μ_3 are membership functions defined on X.

Membership Function Table for F:

X	μ_1	μ_2	μ_3
a	0.8	0.2	0.4
b	0.3	0.6	0.9
c	0.1	0.7	0.5

And find the fuzzy boundary μ^b for a specific $\mu \in [0,1]^X$.

Let's take $\mu = \mu_1$, i.e., $\mu(a) = 0.8$, $\mu(b) = 0.3$, $\mu(c) = 0.1$.

To calculate μ^b, we need to find ρ that satisfy $\rho(x) \geq \overline{\mu}(x)$ for x where $[\overline{\mu} \wedge (1-\mu)](x) > 0$.

Using $\overline{\mu^c}$ defined as $\overline{\mu}(x) = 1-\mu(x)$, we have:

$$\overline{\mu}(a) = 1-\mu(a) = 1-0.8 = 0.2$$

$$\overline{\mu}(b) = 1-\mu(b) = 1-0.3 = 0.7$$

$$\overline{\mu}(c) = 1-\mu(c) = 1-0.1 = 0.9$$

Now, let's calculate $[\overline{\mu} \wedge (1-\mu)](x)$ for each $x \in X$:

$$[\overline{\mu} \wedge (1-\mu)](a) = \min(\overline{\mu}(a), 1-\mu(a)) = \min(0.2, 0.2) = 0.2$$

$$[\overline{\mu} \wedge (1-\mu)](b) = \min(\overline{\mu}(b), 1-\mu(b)) = \min(0.7, 0.7) = 0.7$$

$$[\overline{\mu} \wedge (1-\mu)](c) = \min(\overline{\mu}(c), 1-\mu(c)) = \min(0.9, 0.9) = 0.9$$

Now, we need to find ρ that satisfy $\rho(x) \geq \overline{\mu}(x)$ for x where $[\overline{\mu} \wedge (1-\mu)](x) > 0$.

Let's consider the F-closed sets:

$\rho_1 = \{\mu_1, \mu_2\}$ with

$$\rho_1(a) = \max(\mu_1(a), \mu_2(a)) = \max(0.8, 0.2) = 0.8$$

and

$$\rho_1(b) = \max(\mu_1(b), \mu_2(b)) = \max(0.3, 0.6) = 0.6$$

$\rho_2 = \{\mu_2, \mu_3\}$ with

$$\rho_2(b) = \max(\mu_2(b), \mu_3(b)) = \max(0.6, 0.9) = 0.9$$

and

$$\rho_2(c) = \max(\mu_2(c), \mu_3(c)) = \max(0.7, 0.5) = 0.7$$

$\rho_3 = \{\mu_1, \mu_2, \mu_3\}$ with

$$\rho_3(a) = \mu_1(a) = 0.8, \rho_3(b) = \mu_2(b) = 0.6, \rho_3(c) = \mu_3(c) = 0.5$$

Among these F-closed sets, the minimum is ρ_1 since $\rho_1(a) = 0.8$, $\rho_1(b) = 0.6$, and $\rho_1(c)$ is not defined.

Therefore, $\mu^b = \rho_1 = \{\mu_1, \mu_2\}$.

We can observe that μ^b is F-closed, and $\mu^b \leq \overline{\mu}$.

This example demonstrates the calculation of the fuzzy boundary μ^b for a given fuzzy set μ in a fuzzy topological space.

Fuzzy point

Any FS, $\mu_\beta \in I^X$, fuzzy point is define by;

$$\mu_\beta(x) = \begin{cases} \beta & \text{if } x = x_\circ \\ 0, & \text{otherwise} \end{cases}$$

$x_\circ$ is the hold by of the fuzzy point μ_β.

2.4.4 Neighborhood of Fuzzy Topological Space

Definition 2.4.4. By comparing two fuzzy membership values μ and λ, if μ is greater than or equal to λ, we consider μ to be a fuzzy neighborhood of λ if there exists another fuzzy set γ that lies between μ and λ, meaning that μ is greater than or equal to γ and γ is greater than or equal to λ.

Example

IF (X,F) is a FTS, and $x,y \in F$ so, b is known as fuzzy neighborhood of x iff x less then and equal to b. Consider:

$$A = \{l,m,n\}, F = \{0,1,p,q,r\}, p = \{0.4,0.2,0.1\}, q = \{0.6,0.1,0.2\}, r = \{0.4,0.1,0.0\}$$

$$F_{1p} = (0.4,0.1,0.0) \leq p$$

$$F_{2p} = (0.0,0.2,0.0) \leq p$$

$$F_{3p} = (0.0,0.0,0.1) \leq p$$

where F_{1p}, F_{2p}, F_{3p} and denote fuzzy points.

Here, $F_{1p} \vee F_{2p} \vee F_{3p} = x$, but $F_{1p} \vee F_{2p} \vee F_{3p} \neq q$.

Also since,

$$F_{1p} \leq q$$

$$F_{2p} \leq q$$

$$F_{3p} \leq q$$

So, we conclude that $p \leq q$.

Therefore q is a fuzzy neighborhood of p.

2.4.5 Continuous Function of a Fuzzy Topological Space

Definition 2.4.5. Consider two fuzzy topological spaces, denoted as (A,H_1) and (B,H_2), and let $H : (A,H_1) \to (B,H_2)$ be a function mapping elements from A to B. In this context, we say that H is "continuous at a point" a in A if the inverse image $H^{-1}(\nu)$ of every H_2-open set ν in B is a set that belongs to H_1.

Example

Consider the fuzzy topological space (X, H_1), where X is the set of real numbers and H_1 is a fuzzy topology defined on X. Let's define the fuzzy open sets in H_1 as follows:
Where $x \in X$ and any $\mathbb{R}^+$ i.e ε, the fuzzy open set $U_{x,\varepsilon}$ in H_1 is defined as:
$U_{y,n}$ in H_2 is defined as:

$$U_{y,n}(z) = \begin{cases} 1, & \text{if } |y-x| < \varepsilon \\ 0, & \text{otherwise} \end{cases}$$

Now, consider another fuzzy topological space (Y, H_2), where Y is the set of integers and H_2 is a fuzzy topology defined on Y. Let's define the fuzzy open sets in H_2 as follows:
Where $y \in Y$ and $\mathbb{Z}^+$ i.e n, the fuzzy open set $V_{y,n}$ in H_2 is defined as:

$$V_{y,n}(z) = \begin{cases} 1, & \text{if } |z-y| < n \\ 0, & \text{otherwise} \end{cases}$$

Now, let's define $F : (X, H_1) \to (Y, H_2)$ from X to Y. Suppose we define H as follows:

$$H(x) = \lfloor x \rfloor$$

where $\lfloor x \rfloor$ represents the greatest integer $\leq x$. To show the continuity of H at a point $x \in X$, we need to demonstrate that $H^{-1}(\nu)$ of every fuzzy open subset ν in H_2 is a fuzzy open set in H_1. Let's consider an example to demonstrate the continuity of H at a specific point. Suppose we choose $x = 2.5$. For any fuzzy open set $V_{y,n}$ in F_2, we can calculate the inverse image $H^{-1}(V_{y,n})$:

$$H^{-1}(V_{y,n}) = \{x \in X \mid H(x) \in V_{y,n}\}$$

Let's consider $V_{3,1}$ as a fuzzy open set in H_2. We can calculate the inverse image $H^{-1}(V_{3,1})$ as follows:

$$H^{-1}(V_{3,1}) = \{x \in X \mid H(x) \in V_{3,1}\} = \{x \in X \mid \lfloor x \rfloor \in V_{3,1}\}$$

Since $\lfloor 2.5 \rfloor = 2$, we have:

$$H^{-1}(V_{3,1}) = \{x \in X \mid \lfloor x \rfloor = 2\}$$

This inverse image corresponds to the fuzzy open set $U_{2,1}$ in H_1, which is defined as:

$$U_{2,1}(y) = \begin{cases} 1, & \text{if } |y-2| < 1 \\ 0, & \text{otherwise} \end{cases}$$

Theorem 2.4.3. *Let (X,H_1) and (Y,H_2) be two FTSs. Then a function $H : (X,H_1) \to (Y,H_2)$ is continuous if $H(\overline{\mu}) \leq \overline{H(\mu)}$ for all $\mu \in I^Y$*

2.4.6 Homomorphism of Fuzzy Topological Space

Let a function H. Then (X,H_1) and (Y,H_2) be FTSs.

1. $H : (X,H_1) \to (Y,H_2)$ is called *continuous* if for all $\nu \in H_2$, $H^{-1}(\nu) \in H_1$.

2. $H : (X,H_1) \to (Y,H_2)$ is called *open* if for all $\mu \in H_1$, $H(\mu) \in H_2$.

3. $H : (X,H_1) \to (Y,H_2)$ is called *closed* if for all H_1-closed sets μ, $H(\mu)$ is H_2-closed.

4. $H : (X,H_1) \to (Y,H_2)$ is called *homomorphism* if H is bijective and both H and H^{-1} are continuous.

Example

Let's consider two FTSs: 1. Fuzzy Topological Space (X,H_1):

X	μ_1	μ_2	μ_3
a	1	0	0
b	0	1	0
c	0	0	1

2. Fuzzy Topological Space (Y, H_2):

Y	ν_1	ν_2	ν_3
p	1	0	0
q	0	1	0
r	0	0	1

Now, let's define $H : (X, H_1) \to (Y, H_2)$ as bellow:

$H(a) = p$, $H(b) = q$, $H(c) = r$.

To show that H is a homomorphism, we need to verify bellow points:

1. H is continuous: For all $\nu \in H_2$, $H^{-1}(\nu)$ is in H_1. Let's consider ν_1 in H_2. $H^{-1}(\nu_1)$ should be in H_1. $H^{-1}(\nu_1) = H^{-1}(\{p\}) = \{a\}$ (since $H(a) = p$). The fuzzy set $\{a\}$ with membership function μ_1 is indeed in H_1. Similarly, we can verify that $H^{-1}(\nu_2)$ and $H^{-1}(\nu_3)$ are in H_1. Hence, H is continuous.

2. H is open: For all $\mu \in H_1$, $H(\mu)$ is in H_2. Let's consider μ_1 in H_1. $H(\mu_1)$ should be in H_2. $H(\mu_1) = H(\{a\}) = \{p\}$ (since $H(a) = p$). The fuzzy set $\{p\}$ with membership function ν_1 is indeed in H_2. Similarly, we can verify that $H(\mu_2)$ and $H(\mu_3)$ are in H_2. Hence, H is open.

3. H is closed: For all H_1-closed sets μ, $H(\mu)$ is H_2-closed. Let's consider the H_1-closed set $\mu = \{a, b\}$ (its membership function is $\max(\mu_1, \mu_2)$). $H(\mu) = H(\{a, b\}) = \{p, q\}$ (since $H(a) = p$ and $H(b) = q$). The fuzzy set $\{p, q\}$ with membership function $\max(\nu_1, \nu_2)$ is indeed H_2-closed. Similarly, we can verify that $H(\mu)$ is H_2-closed for other H_1-closed sets. Hence, H is closed.

4. H is a bijection and both H and H^{-1} are continuous: H is a bijection since it is one-to-one and onto. H and H^{-1} are continuous (as shown in conditions (i) and (iii)).

Therefore, $H : (X, H_1) \to (Y, H_2)$ is a homomorphism between the given fuzzy topological spaces.

Dense of fuzzy topological spaces

Consider (X, F) to be a FTS. The following are few definitions for this domain:

Definition 2.4.6. A FS is considered fuzzy dense or everywhere dense iff its membership value μ is equal to 1.

Definition 2.4.7. A FS μ is considered fuzzy-nowhere dense iff its complement's closure, denoted as $\overline{(\mu)^c}$, equals 1. In simpler terms, a fuzzy set is fuzzy-nowhere dense when the closure of its complement is equal to 1.

Example:

If int(μ) of the closure of a FS is an empty set, then we say that μ is nowhere dense in X. In other words, a fuzzy set is classified as nowhere dense in X if there are no non-empty open subsets within the closure of μ.

Definition 2.4.8. If a fuzzy set μ is such that its complement, represented by $1-\mu$, has a closure that includes all possible values (i.e., $\overline{1-\mu} = 1$), then we classify μ as a fuzzy boundary or an F-boundary. In simpler terms, a fuzzy set is considered a fuzzy boundary when its complement, after closure, covers the entire range of possible values.

2.4.7 Base and Subbase of the Fuzzy Topological Spaces

Base

Definition 2.4.9. For B to be a base of F, every member of F can be obtained by taking the maximum value from a collection of elements in P.

Example

let $A = \{p,q\}$ and $l,m,n \in I^X$ where $l = (0.2, 0.5)$, $m = (0.6, 0.4)$, $n = (0.6, 0.5)$ and (A,F) be a FTS, where $F = \{0,1,l,m,n,o\}$ and $X = \{0,1,l,m,o\}$

Then, 0, $l,m,n,1 \in B$.

WE have, $c = (0.6, 0.5) = a \vee b$

Therefore, B forms a base for the fuzzy topology F.

Subbase

Definition 2.4.10. $B \subset F$ is known as subbase of F if, by taking finite intersections of the elements in B, we can obtain a collection of sets that forms a base for F.

Example

Let $A = \{p, q\}$ and $l, m, n, o \in I^X$ where $l = (0.1, 0.5)$, $m = (0.1, 0.4)$, $n = (0.1, 0.4)$ and $o = (0.2, 0.5)$

So, (X, F) be a FTS, where $F = \{0, 1, l, m, n, o\}$. In this space, $B = \{l, m\}$ is a subbase and $P = \{0, 1, l, m, n\}$ is a base of the FT F.

Chapter 3

Pythagorean fuzzy topological spaces

3.1 Intuitionistic Fuzzy Sett

Definition 3.1.1. An IFS P of A is (μ_P, ν_P) of a membership function

$$\mu_P : A \to I$$

and a non-membership function

$$\nu_P : A \to I$$

Where $I \in [o, 1]$, with

$$\mu_P(a) + \nu_P(a) \leq 1 \text{ for any } a \in A$$

For each P in A:

$$\pi_P(a) = 1 - \mu_P(a) - \nu_P(a)$$

The IFS index of a in A refers to the measure of uncertainty associated with the membership status of a in the set A. It is written as $\pi_P(a)$ and represents the degree of non-determinacy or lack of certainty regarding whether a belongs to A or not. Thus:

$$\mu_P(a) + \nu_P(a) + \pi_P(a) = 1$$

Example

Consider a set $P = \{v, w, x, y, z\}$. Let's define an intuitionistic fuzzy subset P on A as (μ_P, ν_P), where:

$$\mu_P(v) = 0.6 \ , \ \mu_P(w) = 0.4 \ , \ \mu_P(x) = 0.1 \ , \ \mu_P(y) = 0.9 \ , \ \mu_P(z) = 0.4$$

$$\nu_P(v) = 0.3 \ , \ \nu_P(w) = 0.4 \ , \ \nu_P(x) = 0.3 \ , \ \nu_P(y) = 0.1 \ , \ \nu_P(z) = 0.6$$

To fulfil the criteria for an intuitionistic fuzzy subset,

$$\mu_P(a) + \nu_P(a) \leq 1$$

holds for each element $a \in A$. Let's verify this for $p = v$:

$$\mu_A(a) + \nu_A(a) = 0.6 + 0.3 = 0.9$$

Let's verify this for $p = w$:

$$\mu_A(a) + \nu_A(a) = 0.4 + 0.4 = 0.9$$

Similarly, you can verify the condition for $(p = x)$, $(p = y)$ and, $(p = z)$ to ensure that $\mu_P(a) + \nu_P(a) \leq 1$ holds for all elements in A.

3.2 A Pythagorean Fuzzy Set

Definition 3.2.1. A PFS P in a set A, where P is not empty, can be represented as a pair (μ_P, ν_P), where μ_P denotes the membership function associated with P.

$$\mu_P \ : \ A \ \rightarrow \ I$$

and a non-membership function

$$\nu_P \ : \ A \ \rightarrow \ I$$

Where $I = [0, 1]$;

$$\mu_P^2(a) \ + \ \mu_P^2(a) \ = \ r_P^2(a)$$

for any $x \in X$ where $r_A : X \rightarrow I$ is a function which is known as the strength of commitment at point x.

Supposing

$$(\mu_P(a))^2 + (\nu_P(a))^2 \leq 1,$$

there is a degree of indeterminacy of $a \in A$ to P defined by

$$\pi_P(a) = \sqrt{1 - [(\mu_P(a))^2 + (\nu_P(a))^2]}$$

and

$$\pi_P(a) \in I$$

In what follows,

$$(\mu_P(a))^2 + (\nu_P(a))^2 + (\pi_P(a))^2 = 1$$

Otherwise, $\pi_P(a) = 0$ whenever $(\mu_P(a))^2 + (\nu_P(a))^2 = 1$.

Example

Consider a non-empty set $P = \{p, q, r, s\}$. Let's define a Pythagorean fuzzy subset P on a as (μ_P, ν_P), where:

$$\mu_P(p) = 0.8\ ,\ \mu_P(q) = 0.5\ ,\ \mu_P(r) = 0.4\ ,\ \mu_P(s) = 0.2$$

$$\nu_P(p) = 0.2\ ,\ \nu_P(q) = 0.2\ ,\ \nu_P(r) = 0.3\ ,\ \nu_P(s) = 0.5$$

To satisfy the definition, we need to check that

$$\mu_P^2(a) + \nu_P^2(a) = r_P^2(a)$$

holds for each element a in A.

Let's verify this for $p = p$:

$$\mu_P^2(p) + \nu_P^2(p) = 0.8^2 + 0.2^2 = 0.64 + 0.04 = 0.68$$

Now, let's calculate $r_P^2(p)$:

$$(r_A^2(p) = 0.68)$$

now verify this for $p = q$:

$$\mu_P^2(q) + \nu_P^2(q) = 0.5^2 + 0.2^2 = 0.25 + 0.04 = 0.29$$

Now, let's calculate $r_P^2(q)$:

$$(r_A^2(q) = 0.29)$$

We can perform the same verification for $p = r$ and $p = s$ to ensure that the Pythagorean fuzzy subset property holds for all elements in X.

3.2.1 Difference between Intuitionistic Fuzzy Sets and Pythagorean Fuzzy sets

The difference between PFSs and IFSs is given as following.

intuitionistic fuzzy sets	Pythagorean fuzzy sets
$\mu+\nu \leq 1$	$\mu+\nu \leq 1$ or $\mu+\nu \geq 1$
$0 \leq \mu+\nu \leq 1$	$0 \leq \mu^2+\nu^2 \leq 1$
$\pi = 1-(\mu+\nu)$	$\pi = \sqrt{1-(\mu^2+\nu^2)}$
$\mu+\nu+\pi = 1$	$\mu^2+\nu^2+\pi^2 = 1$

Table 3.1: Difference between PFSs and IFSs

Mendel extensively researched a type of non-standard fuzzy set known as the IVFS. In contrast, Yager initiate the concept of PFS in 2014, that is belongs to a novel set of non-standard fuzzy subsets with numerous practical applications in fields like natural and social sciences.

3.3 An Interval-Valued Fuzzy Set

Definition 3.3.1. Taking A a fixed set. Then an IVPFS on A is represented as $\tilde{P}$ which is defined as:

$$\tilde{P} \; = \; \{\langle a, \; \mu_{\tilde{P}}(a), \; \nu_{\tilde{P}}(a)\rangle \mid a \; \in \; A\}$$

where,

$$\mu_{\tilde{P}}(a) \; = \; [\mu^L_{\tilde{P}}(a), \; \mu^U_{\tilde{P}}(a)] \; \subset \; I$$

and

$$\nu_{\tilde{P}}(a) \; = \; [\nu^L_{\tilde{P}}(a), \; \nu^U_{\tilde{P}}(a)] \; \subset \; I,$$

treated as intervals, with $\mu^L_{\tilde{P}}(a) = inf\,\mu_{\tilde{P}}(a)$ and $\mu^U_{\tilde{P}}(a) = sup \; \mu_{\tilde{P}}(a)$ likewise $\nu^L_{\tilde{P}}(a) = inf\,\nu_{\tilde{P}}(a)$ and $\nu^U_{\tilde{P}}(a) = sup \; \nu_{\tilde{P}}(a)$ for all $a \; \in \; A$.

Example

For example, let's assume the following intervals for $\mu_{\tilde{P}}(a)$ and $\nu_{\tilde{P}}(a)$:

$$\mu_{\tilde{P}}(l) = [0.4, 0.6], \quad \mu_{\tilde{P}}(m) = [0.1, 0.6], \quad \mu_{\tilde{P}}(n) = [0.2, 0.4]$$

and

$$\nu_{\tilde{P}}(l) = [0.4, 0.5], \quad \nu_{\tilde{P}}(m) = [0.2, 0.6], \quad \nu_{\tilde{P}}(n) = [0.1, 0.5]$$

$$\tilde{P} \ = \ \{\langle l,\ [0.4, 0.6], [0.4, 0.5]\rangle\ ,\ \langle m,\ [0.1, 0.6], [0.2, 0.6]\rangle\ ,\ \langle n,\ [0.2, 0.4], [0.1, 0.5]\rangle\}$$

In this example, we have an IVPFS $\tilde{P}$ defined on A with specific membership and non-membership intervals for every member a in A.

3.4 Set operations over Pythagorean fuzzy subsets

Now we evaluate the operations performed on Pythagorean fuzzy subsets.

Definition 3.4.1. Suppose we have a Pythagorean fuzzy subset P of a set A, represented as $P = (\mu_P, \nu_P)$. In this case P^c is defined as

$$P^c := (\nu_P, \mu_P)$$

Example

Let $P \ = \ \{(0.2, 0.5), (0.1, 0.7), (0.4, 0.8)\}$ be a Pythagorean fuzzy subset of a set A then,

$$P^c \ = \ \{(0.5, 0.2), (0.7, 0.1), (0.8, 0.4)\}$$

Definition 3.4.2. Consider two Pythagorean fuzzy subsets of a set A, denoted as $P = (\mu_P, \nu_P)$ and $Q = (\mu_Q, \nu_Q)$. The intersection of P and Q, denoted as $P \cap Q$, can be defined as follows.

$$P \cap Q := (min\{\mu_P, \mu_Q\}, max\{\nu_P, \nu_Q\})$$

Example

Suppose two Pythagorean fuzzy subsets of a set A, denoted as P and Q.

Let $P \ = \ \{(0.2, 0.5), (0.1, 0.7), (0.4, 0.8)\}$ and $Q \ = \ \{(0.4, 0.5), (0.9, 0.5), (0.8, 0.3)\}$.

Then,

$$P \cap Q = \{(0.2, 0.5), (0.1, 0.7), (0.4, 0.8)\}$$

Definition 3.4.3. Consider two Pythagorean fuzzy subsets of a set A, denoted as $P = (\mu_P, \nu_P)$ and $Q = (\mu_Q, \nu_Q)$. The union of P and Q, denoted as $P \cup Q$, can be defined as follows.

$$P \cup Q := (max\{\mu_P, \mu_Q\}, min\{\nu_P, \nu_Q\})$$

Example

Suppose two Pythagorean fuzzy subsets of a set A, denoted as P and Q.
Let $P = \{(0.2, 0.5), (0.1, 0.7), (0.4, 0.8)\}$ and $Q = \{(0.4, 0.5), (0.9, 0.5), (0.8, 0.3)\}$.
Then,

$$P \cup Q = \{(0.4, 0.5), (0.9, 0.5), (0.8, 0.3)\}$$

Definition 3.4.4. Let $P = (\mu_P, \nu_P)$ and $Q = (\mu_Q, \nu_Q)$ be two Pythagorean fuzzy subsets of a set A then $P \subset Q$ or $P \supset Q$ if $\mu_P \leq \mu_Q$ and $\nu_P \geq \nu_Q$.

Example

Suppose two Pythagorean fuzzy subsets of a set A, denoted as P and Q.
Let $P = \{(0.2, 0.5), (0.1, 0.7), (0.4, 0.8)\}$ and $Q = \{(0.4, 0.5), (0.9, 0.5), (0.8, 0.3)\}$
We say P is a subset of Q because $\mu_P \leq \mu_Q$ and $\nu_P \geq \nu_Q$.

3.5 Pythagorean Fuzzy Topological Space

Definition 3.5.1. Let $X \neq \phi$ be a set and let τ be a family of Pythagorean fuzzy subsets of X. If

(1) $1_X,\ 0_X \in \tau$

(2) for any $P_1,\ P_2 \in \tau$, we have $P_1 \cap P_2 \in \tau$

(3) for any $\{A_i\}_{i \in I} \subset \tau$, we have $\bigcup_{i \in I} A_i \in \tau$ so, τ is called a PFT on X.

In the context of PFTS, defined by PFTS as a pair (X, τ), where X is a set and τ is a collection of open Pythagorean fuzzy subsets. These open Pythagorean fuzzy subsets represent the elements that define the topology of the space. Closed Pythagorean fuzzy subset is defined by complement of an open Pythagorean fuzzy subset.

Similar to traditional or FTS, we have two special cases in PFTS. The first one is the indiscreet PFTS, denoted by $\{1_X, 0_X\}$, where every subset of the set X is considered open. The second one is the discrete PFTS, where the topology τ contains all possible Pythagorean fuzzy subsets, making each subset open.

Furthermore, when comparing two Pythagorean fuzzy topologies, τ_1 is said to be coarser than τ_2 if τ_1 is a subset of τ_2. In other words, if every open Pythagorean fuzzy subset in τ_1 is also an open Pythagorean fuzzy subset in τ_2, then τ_1 is considered coarser than τ_2.

Example

Let a set X with two elements i.e $X = \{1,2\}$. We have a family of Pythagorean fuzzy subsets denoted by τ, which consists of the following elements, the crisp subsets 1_X and 0_X, as well as the Pythagorean fuzzy subsets A_1, A_2, A_3, A_4, and A_5.

$$\begin{array}{llll}
\mu_1(1) = 0.5 & \nu_1(1) = 0.7 & \mu_2(1) = 0.6 & \nu_2(1) = 0.5 \\
\mu_1(2) = 0.2 & \nu_1(2) = 0.4 & \mu_2(2) = 0.3 & \nu_2(2) = 0.9 \\
\mu_3(1) = 0.4 & \nu_3(1) = 0.8 & \mu_4(1) = 0.6 & \nu_4(1) = 0.5 \\
\mu_3(2) = 0.1 & \nu_3(2) = 0.9 & \mu_4(2) = 0.3 & \nu_4(2) = 0.4 \\
\mu_5(1) = 0.5 & \nu_5(1) = 0.7 & \mu_5(2) = 0.2 & \nu_5(2) = 0.8
\end{array}$$

In the given example, the functions μ_i and ν_i represent the membership and non-membership functions, accordingly, associated with each A_i where i ranges from 1 to 5. It can be observed that (X, τ) forms a PFTS.

In classical topology, the concept of a neighborhood plays a crucial role, as it is used to define or characterize various concepts like continuity, closure, and convergence. However, Chang introduced a different approach by defining the neighborhood of a fuzzy open subset instead of the neighborhood of a point.

3.6 Neighborhood of Fuzzy Set

Definition 3.6.1. In a PFTS, consider two Pythagorean fuzzy subsets V and N, we say that N is a neighborhood of V if $\exists$ an open Pythagorean fuzzy subset H s.t $V \subset H$, and $H \subset N$ which can be denoted by:

$$V \subset H \subset N$$

This shows that H consists of A and H exists between V and N.

Example $X = \{1,2\}$ and $Y = \{0_X, 1_X, V, N\}$ where V, in a PFTS N be a Pythagorean fuzzy subsets.

$$V = \{(0.2, 0.5), (0.1, 0.7)\}$$

$$N = \{(0.4, 0.5), (0.9, 0.5)\}$$

To show that N is a neighborhood of V as define above, we have find a Pythagorean fuzzy subset H s.t $V \subset H$ and $H \subset N$
let

$$H = \{(0.2, 0.4), (0.9, 0.5)\}$$

A set N is considered a neighborhood of A if there exists H s.t V is a subset of H, and H is a subset of N.

Proposition 1

In PFTS, a Pythagorean fuzzy subset V is open. if, for every possible subset within V, there exists a neighborhood around that subset that is entirely contained within V.

Example

Consider the set $X = \{1,2,3\}$ and the PFTS $Y = \{0_X, 1_X, A, U\}$, where

$$A = \{(0.2, 0.6), (0.4, 0.3), (0.7, 0.5)\}$$

and

$$U = \{(0.1, 0.7), (0.6, 0.4), (0.8, 0.6)\}$$

To show that A is open in V, we need to show that for each point $x \in A$, there exists a Pythagorean fuzzy subset V such that $x \in V$ and $V \subset A$.
Let's consider each point in A: For $x = 1$,

$$V_1 = \{(0.2, 0.6), (0.4, 0.3)\} \text{ We see that } V_1 \subset A$$

For $x = 2$,

$$V_2 = \{(0.4, 0.3), (0.7, 0.5)\} \text{ We see that } V_2 \subset A$$

For $x = 3$,

$$V_3 = \{(0.2, 0.6), (0.7, 0.5)\} \text{ We see that } V_3 \subset A$$

Therefore, for each point x in A, we have found a Pythagorean fuzzy subset V that contains x and is a subset of A. This indicates that in the provided PFTS A is an open Pythagorean fuzzy subset.

3.7 Continuity of Pythagorean fuzzy set

Definition

Let us define the membership and non-membership functions for the image of set A under the function g, denoted as $g[A]$, when A and B are non-empty sets and $g : A \to B$ is a function. Suppose X and Y are Pythagorean fuzzy subsets of A and B respectively. The definitions of the membership and non-membership functions are as follows: For any $b \in B$,

$$\mu_{g[X]}(b) = \begin{cases} \underset{z \in g^{-1}(b)}{sup} \ \mu_X(z), & \text{if } g^{-1}(b) \neq \phi \\ 0, & \text{otherwise} \end{cases}$$

$$\nu_{g[X]}(b) = \begin{cases} \underset{z \in g^{-1}(b)}{inf} \ \nu_X(z), & \text{if } g^{-1}(b) \neq \phi \\ 0, & \text{otherwise} \end{cases}$$

Similarly, we can define the membership and non-membership functions for the pre-image of Y under g, denoted as $g^{-1}[Y]$: For any $a \in A$,

$$\mu_{g^{-1}[Y]}(a) = \mu_Y(g(a))$$

$$\nu_{g^{-1}[Y]}(a) = \nu_Y(g(a))$$

Note that $g[X]$ and $g^{-1}[Y]$ are Pythagorean fuzzy subsets. This can be observed from the properties of the membership and non-membership functions. Specifically, when μ_X and ν_Y are non-negative functions, we have:

$$\begin{aligned}
\mu^2_{g[X]}(b) + \nu^2_{g[X]}(b) &= \left(\sup_{z \in g^{-1}(b)} \mu_X(z)\right)^2 + \left(\inf_{z \in g^{-1}(b)} \nu_X(z)\right)^2 \\
&= \sup_{z \in g^{-1}(b)} \mu^2_X(z) + \inf_{z \in g^{-1}(b)} \nu^2_X(z) \\
&= \sup_{z \in g^{-1}(b)} (1 - \nu^2_X(z)) + \inf_{z \in g^{-1}(b)} \nu^2_X(z)
\end{aligned}$$

This holds whenever $g^{-1}(b) \neq phi$. Additionally, if $g^{-1}(b) = \emptyset$, we get:

$$\mu^2_{g[X]}(b) + \nu^2_{g[X]}(b) = 1$$

Example

Let we have two sets $A = \{a, b, c\}$ and $B = \{1, 2, 3\}$, and defined by a function $g : A \to B$ as $g(a) = a$. Also, let X be a Pythagorean fuzzy subset of A defined by

$$\mu_X(a) = 0.7,\ \mu_X(b) = 0.5,\ \text{and } \mu_X(c) = 0.4$$

and let Y be a Pythagorean fuzzy subset of B defined by

$$\mu_Y(1) = 0.9,\ \mu_Y(2) = 0.7 \text{ and } \mu_Y(3) = 0.5$$

.

We can calculate for the functions of $g[X]$ as follows:

$$\mu_{g[X]}(1) = \sup_{z\in g^{-1}(1)} \mu_X(z) = \mu_X(a) = 0.7,$$

$$\mu_{g[X]}(2) = \sup_{z\in g^{-1}(2)} \mu_X(z) = \mu_X(b) = 0.5,$$

$$\mu_{g[X]}(3) = \sup_{z\in g^{-1}(3)} \mu_X(z) = \mu_X(c) = 0.4,$$

and

$$\nu_{g[X]}(1) = \inf_{z\in g^{-1}(1)} \nu_X(z) = \nu_X(1) = 0,$$

$$\nu_{g[X]}(2) = \inf_{z\in g^{-1}(2)} \nu_X(z) = \nu_X(2) = 0,$$

$$\nu_{g[X]}(3) = \inf_{z\in g^{-1}(3)} \nu_X(z) = \nu_X(3) = 0.$$

Similarly, we can calculate for the functions of $g^{-1}[B]$ as:

$$\mu_{g^{-1}[Y]}(1) = \mu_Y(g(a)) = \mu_Y(1) = 0.9,$$

$$\mu_{g^{-1}[Y]}(2) = \mu_Y(g(b)) = \mu_Y(2) = 0.7,$$

$$\mu_{g^{-1}[Y]}(3) = \mu_Y(g(c)) = \mu_Y(3) = 0.5,$$

and

$$\nu_{g^{-1}[B]}(a) = \nu_Y(g(a)) = \nu_Y(1) = 0,$$

$$\nu_{g^{-1}[B]}(b) = \nu_Y(g(b)) = \nu_Y(2) = 0,$$

$$\nu_{g^{-1}[B]}(c) = \nu_Y(g(c)) = \nu_Y(3) = 0.$$

Therefore, the Pythagorean fuzzy subset $g[X]$ can be define as follows:

$$g[Y] = \{(4,0.7),(5,0.5),(6,0.4)\}.$$

Similarly, the Pythagorean fuzzy subset $g^{-1}[Y]$ can be represented as:

$$g^{-1}[Y] = \{(1,0.9),(2,0.7),(3,0.5)\}.$$

Proposition 2

Suppose we have two sets, A and B, where both sets are not empty. let us consider a function h that maps elements from A to corresponding elements in B.

1. $h^{-1}[Y^c] = h^{-1}[Y]^c$.
2. $h[X]^c \subset h[X^c]$.
3. if $Y_1 \subset Y_2$ then $h^{-1}[Y_1] \subset h^{-1}[Y_2]$ where Y_1 and Y_2 are Pythagorean fuzzy subsets of B.
4. if $X_1 \subset X_2$, then $h[X_1] \subset h[X_2]$ where X_1 and X_2 are Pythagorean fuzzy subsets of A.
5. $h[h^{-1}[Y]] \subset Y$.
6. $X \subset h^{-1}[h[X]]$.

Example

Let $A = \{l,m,n\}$ and $Y = \{p,q,r\}$, and let $g : A \to B$ be define by bellow:

$$g(l) = p\ ,\ h(m) = q\ ,\ h(n) = r$$

We will demonstrate each of the propositions using examples with Pythagorean fuzzy subsets X and Y of A and B, respectively.
Let X be a Pythagorean fuzzy subset of A defined as:

$$\mu_X, \nu_X(l) = (0.2,0.5)\ ,\ \mu_X, \nu_X(m) = (0.4,0.7)\ ,\ \mu_X, \nu_X(n) = (0.6,0.8)$$

Let B be a Pythagorean fuzzy subset of Y defined as:

$$\mu_Y, \nu_Y(p) = (0.3,0.6)\ ,\ \mu_Y, \nu_Y(q) = (0.1,0.4)\ ,\ \mu_Y, \nu_Y(r) = (0.5,0.7)$$

Now, we can demonstrate each of the propositions using these examples:

(i) $g^{-1}[Y^c] = g^{-1}[Y]^c$:

$Y^c = \{\emptyset\}$.

$g^{-1}[Y^c] = g^{-1}[\{\emptyset\}] = \{\emptyset\}$.

$g^{-1}[Y] = g^{-1}[\{p,q,r\}] = \{l,m,n\}$.

Therefore, $g^{-1}[Y^c] = \{\emptyset\} = \{l,m,m\}^c = g^{-1}[Y]^c$.

(ii) $g[X]^c \subset g[X^c]$:

$X^c = \{\emptyset\}$.

$g[X^c] = g[\{\emptyset\}] = \{\emptyset\}$.

$g[X] = g[\{l,m,n\}] = \{p,q,r\}$.

Therefore, $g[X]^c = \{\emptyset\} \subset \{\emptyset\} = g[X^c]$.

(iii) if $Y_1 \subset Y_2$ then $g^{-1}[Y_1] \subset g^{-1}[Y_2]$:

Let $Y_1 = \{q\}$ and $Y_q = \{p,q,r\}$.

Since $Y_1 \subset Y_2$, the proposition holds.

$g^{-1}[Y_1] = g^{-1}[\{q\}] = \{m\}$.

$g^{-1}[Y_2] = g^{-1}[\{p,q,r\}] = \{l,m,n\}$.

Therefore, $h^{-1}[Y_1] = \{m\} \subset \{l,m,n\} = g^{-1}[Y_2]$.

(iv) if $X_1 \subset X_2$, then $g[X_1] \subset g[X_2]$:

Let $X_1 = \{l,m\}$ and $X_2 = \{l,m,n\}$.

Since $X_1 \subset X_2$, the proposition holds.

$g[X_1] = g[\{l,m\}] = \{p,q\}$.

$g[X_2] = g[\{l,m,n\}] = \{p,q,r\}$.

Therefore, $g[X_1] = \{p,q\} \subset \{p,q,r\} = g[X_2]$

(v) $g[g^{-1}[Y]] \subset Y$:

Let $Y = \{p,r\}$.

$g^{-1}[Y] = g^{-1}[\{q,r\}] = \{m,n\}$.

$g[g^{-1}[Y]] = g[\{m,n\}] = \{p,q\}$.

Therefore, $g[g^{-1}[Y]] = \{q,r\} \subset \{q,r\} = Y$.

(vi) $X \subset g^{-1}[g[X]]$: Let $X = \{l\}$.
$g[X] = g[\{l\}] = \{q\}$.
$g^{-1}[g[X]] = g^{-1}[\{q\}] = \{l\}$.
Therefore, $X = \{p\} \subset \{p\} = g^{-1}[g[X]]$.

Definition 3.7.1. Consider two PFTSs denoted as (X,τ_1) and (Y,τ_2). Let define a function $h: X \to Y$. We say that h is Pythagorean fuzzy cont. if, for any A of X and any neighborhood U containing the corresponding elements of A in the image of h, $\exists$ a neighborhood N of A s.t the corresponding elements of U in the image of h are entirely contained within U.

Example

Let $A = \{a,b,c\}$ and $B = \{1,2,3\}$ be PFTSs with the respective fuzzy topologies τ_1 and τ_2. Consider the function $h: A \to B$ defined as follows: $h(a) = 1$, $h(b) = 2$, $h(c) = 3$.

Let X be a Pythagorean fuzzy subset of A defined as:

$$\mu_X(a) = (0.1, 0.4)\ ,\ \mu_X(b) = (0.3, 0.6)\ ,\ \mu_X(c) = (0.5, 0.7)$$

Now, Consider a neighborhood U of $h[X]$ in B, say $U = \{1,2\}$.
We have to find a neighborhood N of X in A such that $h[U] \subset U$.
Since $h[X] = \{1,2\}$, we can choose N to be a Pythagorean fuzzy subset of A such that

$$\mu_N(a) = (0.2, 0.5)\ ,\ \mu_N(2) = (0.3, 0.6)\ ,\ \mu_N(c) = (0.2, 0.3)$$

Now we compute $h[U]$:

$$h[U] = h[\{1,2\}] = \{h(1), h(2)\} = \{1,2\}$$

Since $h[U] = \{1,2\}$ is a subset of U, we have shown that for the given Pythagorean fuzzy subset X and neighborhood U of $h[X]$, there exists a neighborhood N of X such that $h[U]$ is a subset of U.

Therefore, the function h is Pythagorean fuzzy continuous.

Chapter 4

Cubic Pythagorean Fuzzy Topological Space

4.1 Cubic Pythagorean Fuzzy Set

We want to introduce the idea of CPFS, which stands for Cubic Pythagorean Fuzzy Set, an extension of both Cubic and intuitionistic fuzzy sets initially proposed by Jun.

Definition 4.1.1. Consider a nonempty set X. A CPFS in Y is a structured representation denoted by:

$$C_Q = \{(y,\ \tilde{Q}(y),\ Q(y)) \mid y \in Y\}$$

where $\tilde{Q}$ is an IVPFS in Y, and Q is a PFS in Y.

Example

For Y={1,2,3,4,5,6,7,8}, $C_Q = ((y,\ \tilde{Q}(y),\ Q(y))$ is a CPFS.

Y	$\tilde{Q} = \langle \mu_{\tilde{Q}},\ \nu_{\tilde{Q}} \rangle$	$Q = \langle \mu_Q,\ \nu_Q \rangle$
1	$\langle [0.4, 0.5]\,[0.7, 0.8] \rangle$	(0.6,0.3)
2	$\langle [0.5, 0.6]\,[0.6, 0.7] \rangle$	(0.7,0.4)
3	$\langle [0.2, 0.9]\,[0.3, 0.4] \rangle$	(0.8,0.5)
4	$\langle [0.7, 0.8]\,[0.4, 0.5] \rangle$	(0.8,0.3)
5	$\langle [0.3, 0.4]\,[0.6, 0.7] \rangle$	(0.5,0.2)
6	$\langle [0.4, 0.5]\,[0.5, 0.6] \rangle$	(0.6,0.3)
7	$\langle [0.1, 0.8]\,[0.4, 0.6] \rangle$	(0.7,0.4)
8	$\langle [0.6, 0.7]\,[0.3, 0.4] \rangle$	(0.8,0.3)

4.2 Set Operation on Cubic Pythagorean Fuzzy Subset

We are examining the set operations performed on Pythagorean fuzzy subsets in greater detail.

Definition 4.2.1. Consider a CPF subset $C_1 = (\mu_{C_1}, \nu_{C_1})$ of a set X. Its complement is $C_1^c := (\nu_{C_1}, \mu_{C_1})$.

Example

Let $C_1 = \{\{1, \langle [0.3, 0.4], [0.5, 0.6] \rangle (0.8, 0.4)\}, \{2, \langle [0.3, 0.7], [0.4, 0.6] \rangle (0.2, 0.7)\}\}$ be the

CPF subset of a set X,

$$C_1^c = \{\{1,\langle[0.5,0.6],[0.3,0.4]\rangle(0.4,0.8)\},\{2,\langle[0.4,0.6],[0.3,0.7]\rangle(0.7,0.2)\}\}$$

Definition 4.2.2. Let $C_1 = (\mu_{C_1}, \nu_{C_1})$ and $C_2 = (\mu_{C_2}, \nu_{C_2})$ be subsets of two CPF subsets of the set X. The intersection of C_1 and C_2 is defined by the membership function and non-membership function as follows:

$$C_1 \cap C_2 := \begin{pmatrix} \langle[min\{\mu^L_{C_1},\mu^L_{C_2}\},\ min\{\nu^L_{C_1},\nu^L_{C_2}\}],\ [min\{\mu^U_{C_1},\mu^U_{C_2}\},\ min\{\nu^U_{C_1},\nu^U_{C_2}\}]\rangle; \\ (min\{\mu_{C_1},\mu_{C_2}\}, max\{\nu_{C_1},\nu_{C_2}\}) \end{pmatrix}$$

Example

Let $C_1 = \{\{1,\langle[0.3,0.4],[0.5,0.6]\rangle(0.8,0.4)\},\{2,\langle[0.3,0.7],[0.4,0.6]\rangle(0.2,0.7)\}\}$ and $C_2 = \{\{1,\langle[0.4,0.7],[0.4,0.6]\rangle(0.2,0.7)\},\{2,\langle[0.5,0.9],[0.2,0.3]\rangle(0.7,0.3)\}\}$ be two CPF subsets of a set X. Then,

$$C_1 \cap C_2 = \{\{1,\langle[0.3,0.4],[0.7,0.6]\rangle(0.2,0.7)\},\{2,\langle[0.3,0.2],[0.9,0.6]\rangle(0.2,0.7)\}\}$$

Definition 4.2.3. Let $C_1 = (\mu_{C_1}, \nu_{C_1})$ and $C_2 = (\mu_{C_2}, \nu_{C_2})$ be subsets of two CPF subsets of the set X. The union of C_1 and C_2 is defined by the membership function and non-membership function as follows:

$$C_1 \cup C_2 := \begin{pmatrix} \langle[max\{\mu^L_{C_1},\mu^L_{C_2}\},\ max\{\nu^L_{C_1},\nu^L_{C_2}\}],\ [max\{\mu^U_{C_1},\mu^U_{C_2}\},\ max\{\nu^U_{C_1},\nu^U_{C_2}\}]\rangle; \\ (max\{\mu_{C_1},\mu_{C_2}\}, min\{\nu_{C_1},\nu_{C_2}\}) \end{pmatrix}$$

Example

Let $C_1 = \{\{1,\langle[0.3,0.4],[0.5,0.6]\rangle(0.8,0.4)\},\{2,\langle[0.3,0.7],[0.4,0.6]\rangle(0.2,0.7)\}\}$ and $C_2 = \{\{1,\langle[0.4,0.7],[0.4,0.6]\rangle(0.2,0.7)\},\{2,\langle[0.5,0.9],[0.2,0.3]\rangle(0.7,0.3)\}\}$ be two CPF subsets of a set X. Then,

$$C_1 \cup C_2 = \{\{1,\langle[0.4,0.5],[0.4,0.5]\rangle(0.8,0.4)\},\{2,\langle[0.5,0.4],[0.7,0.3]\rangle(0.7,0.6)\}\}$$

Definition 4.2.4. Let $C_1 = (\mu_{C_1}, \nu_{C_1})$ and $C_2 = (\mu_{C_2}, \nu_{C_2})$ be two CPF subsets of a set X defined C_1 is a subset of C_2 / C_2 contains C_1 which can be written as $C_1 \subset C_2$ / $C_2 \supset C_1$ if $\mu_{C_1} \leq \mu_{C_2}$ and $\nu_{C_1} \geq \nu_{C_2}$

Example

Let $C_1 = \{\{1, \langle[0.3,0.4],[0.5,0.6]\rangle(0.8,0.4)\}, \{2, \langle[0.3,0.7],[0.4,0.6]\rangle(0.2,0.7)\}\}$ and
$C_2 = \{\{1, \langle[0.4,0.7],[0.4,0.6]\rangle(0.2,0.7)\}, \{2, \langle[0.5,0.9],[0.2,0.3]\rangle(0.7,0.3)\}\}$
In this example we see that $C_1 \subset C_2$ or $C_2 \supset C_1$ as define in the definition.

4.3 Cubic Pythagorean Fuzzy topological space

Definition 4.3.1. Let $X \neq \phi$ be a set and let τ be a family of CPF subsets of X. If

1- 1 $1_X,\ 0_X \in \tau$,

2- for any $C_i,\ C_j \in \tau$, we have $C_i \cap C_j \in \tau$,

3- for any $\{C_i\}_{i \in I} \subset \tau$, we have $\bigcup\limits_{i \in I} C_i \in \tau$ then τ is known as a PFT on X.

(X, τ) is known as PFTS.

Example

Let $X = \{1,2,3\}$

$\tau = \{1_X, 0_X, C_1, C_2, C_3, C_4, C_5, C_6\}$

$C_1(1) = \{1, \langle[0.3,0.4],[0.5,0.6]\rangle, (0.8,0.4)\}$ $C_1(2) = \{2, \langle[0.3,0.7],[0.4,0.6]\rangle, (0.2,0.7)\}$

$C_1(3) = \{3, \langle[0.2,0.5],[0.4,0.7]\rangle, (0.3,0.8)\}$ $C_2(1) = \{1, \langle[0.4,0.7],[0.4,0.6]\rangle, (0.2,0.7)\}$

$C_2(2) = \{2, \langle[0.5,0.9],[0.2,0.3]\rangle, (0.7,0.3)\}$ $C_2(3) = \{3, \langle[0.5,0.6],[0.3,0.6]\rangle, (0.4,0.2)\}$

$C_3(1) = \{1, \langle[0.4,0.5],[0.4,0.5]\rangle, (0.8,0.4)\}$ $C_3(2) = \{2, \langle[0.3,0.4],[0.2,0.3]\rangle, (0.7,0.6)\}$

$C_3(3) = \{3, \langle[0.3,0.4],[0.5,0.6]\rangle, (0.4,0.2)\}$ $C_4(1) = \{1, \langle[0.3,0.4],[0.5,0.6]\rangle, (0.2,0.7)\}$

$C_4(2) = \{2, \langle[0.2,0.2],[0.3,0.6]\rangle, (0.2,0.7)\}$ $C_4(3) = \{3, \langle[0.2,0.3],[0.6,0.7]\rangle, (0.3,0.8)\}$

$C_5(1) = \{1, \langle[0.3,0.4],[0.3,0.4]\rangle, (0.7,0.3)\}$ $C_5(2) = \{2, \langle[0.2,0.3],[0.1,0.2]\rangle, (0.6,0.5)\}$

$C_5(3) = \{3, \langle[0.2,0.3],[0.6,0.8]\rangle, (0.3,0.1)\}$ $C_6(1) = \{1, \langle[0.2,0.5],[0.6,0.7]\rangle, (0.1,0.6)\}$

$C_6(2) = \{2, \langle[0.4,0.8],[0.3,0.6]\rangle, (0.4,0.2)\}$ $C_4(3) = \{3, \langle[0.1,0.3],[0.5,0.7]\rangle, (0.3,0.8)\}$

The $\tilde{P}_i$ and P_i are the corresponding membership and non-membership of the function C_i for each $i = 1,2,3,4$ respectively observe (X,τ) is a CPFTS.

4.4 Neighborhood of Cubic Pythagorean fuzzy set

In classical topology, the concept of a neighborhood plays a significant role as it is utilized to define or characterize various concepts like continuity, closure, and convergence. However, Chang introduced a different perspective by defining a neighborhood of a fuzzy open subset instead of a neighborhood of a point.

Definition 4.4.1. In a CPFTS, consider two Cubic Pythagorean fuzzy subsets V and N, we say that N is a neighborhood of V if $\exists$ an open Cubic Pythagorean fuzzy subset H s.t $V \subset H$, and $H \subset N$ which can be denoted by:

$$V \subset H \subset N$$

This shows that U consists of A and E exists between A and U.

Example

Let $X = \{1,2\}$ and $Y = \{0_X, 1_X, A, U\}$, where A and U are CPF subsets in a CPFTS.

$$A = \{\{1, \langle[0.3,0.4],[0.5,0.6]\rangle,(0.8,0.4)\},\{2,\langle[0.3,0.7],[0.4,0.6]\rangle,(0.2,0.7)\}\}$$

$$U = \{\{1, \langle[0.4,0.7],[0.4,0.6]\rangle,(0.2,0.7)\},\{2,\langle[0.5,0.9],[0.2,0.3]\rangle,(0.7,0.3)\}\}$$

To show that U is a neighborhood of A according to the definition, we need to find an open CPF subset H such that $A \subset H$ and $H \subset U$.

Let's take

$$H = \{\{1, \langle[0.3,0.5],[0.5,0.7]\rangle,(0.7,0.5)\},\{2,\langle[0.3,0.8],[0.4,0.6]\rangle,(0.2,0.7)\}\}$$

. We can see that every element in A is also present in H, so $A \subset H$. Additionally, every element in H is also present in U, so $H \subset U$.

Therefore, we have $A \subset H \subset U$, satisfying the condition stated in the definition. Thus, U is a neighborhood of A.

Example

$X = \{1,2\}$ and $Y = \{0_X, 1_X, A, U\}$ where A and U are CPF subsets in a CPFTS.

$A = \{\{1, \langle[0.3,0.4],[0.5,0.6]\rangle,(0.8,0.4)\},\{2,\langle[0.3,0.7],[0.4,0.6]\rangle,(0.2,0.7)\}\}$

$U = \{\{1, \langle[0.4,0.7],[0.4,0.6]\rangle,(0.2,0.7)\},\{2,\langle[0.5,0.9],[0.2,0.3]\rangle,(0.7,0.3)\}\}$

$H = \{\{1, \langle[0.2,0.4],[0.5,0.6]\rangle,(0.9,0.7)\},\{2,\langle[0.9,0.8],[0.5,0.9]\rangle,(0.9,0.2)\}\}$ (a CPF subset)

In this case, H is given as

$$H = \{\{1, \langle[0.2,0.4],[0.5,0.6]\rangle,(0.9,0.7)\},\{2,\langle[0.9,0.8],[0.5,0.9]\rangle,(0.9,0.2)\}\}$$

We can see that A is a subset of H since every element in A is also present in H. Therefore, $A \subset H$ holds.

However, H is not a subset of U because there are elements in H that are not present in U. For example, $2, \langle[0.9,0.8],[0.5,0.9]\rangle,(0.9,0.2)$ is in H but not in U. Hence, $H \not\subset U$.

This example does not satisfy the condition stated in the definition.

Proposition 1

In CPFTS, a Cubic Pythagorean fuzzy subset V is open. if, for every possible subset within V, there exists a neighborhood around that subset that is entirely contained within V.

Example

Consider an example that satisfies the condition for a CPF subset to be open in a CPFTS.

Let $X = \{1,2\}$ be the underlying set and $Y = \{0_X, 1_X, C, N\}$ be CPFTS of X.

Consider the following CPF subsets:

$$C = \{\{1, \langle[0.3,0.4],[0.5,0.6]\rangle,(0.8,0.4)\},\{2,\langle[0.3,0.7],[0.4,0.6]\rangle,(0.2,0.7)\}\}$$

$$N = \{\{1, \langle[0.4,0.7],[0.4,0.6]\rangle,(0.2,0.7)\},\{2,\langle[0.5,0.9],[0.2,0.3]\rangle,(0.7,0.3)\}\}$$

To show that C is an open CPF subset, we need to demonstrate that it contains a neighborhood of each of its subsets.

Let's consider the subset $\{1, \langle[0.3,0.4],[0.5,0.6]\rangle,(0.8,0.4)\}$ of C.

To find a neighborhood of this subset, we can choose

$$H = \{\{1, \langle[0.25, 0.5], [0.4, 0.6]\rangle, (0.7, 0.4)\}, \{2, \langle[0.3, 0.7], [0.4, 0.6]\rangle, (0.2, 0.7)\}\}$$

We can see that every element in $\{1, \langle[0.3, 0.4], [0.5, 0.6]\rangle, (0.8, 0.4)\}$ is also present in H
So $\{1, \langle[0.3, 0.4], [0.5, 0.6]\rangle, (0.8, 0.4)\} \subset H$.
Similarly, we can choose $H' = \{\{1, \langle[0.4, 0.7], [0.4, 0.6]\rangle, (0.2, 0.7)\}, \{2, \langle[0.3, 0.7], [0.4, 0.6]\rangle, (0.2, 0.7)\}\}$ as a neighborhood of $\{2, \langle[0.3, 0.7], [0.4, 0.6]\rangle, (0.2, 0.7)\}$. Thus, we have shown that C contains a neighborhood of each of its subsets. Therefore, C is an open CPF subset in this example.

Definition

Consider $X \neq \phi$ and $Y \neq \phi$, and a function $g : A \to B$. Suppose C_1 and C_2 are CPF subsets of A and B, respectively. We want to determine the membership and non-membership functions of the image of C_1 under the mapping g, denoted as $g[C_1]$. The membership function $\mu_{g[C_1]}(b)$ of $g[C_1]$ for each $b \in B$ is defined as follows:

$$\mu_{g[C_1]}(b) = \begin{cases} \sup_{z \in g^{-1}(b)} \mu_{C_1}(z), & \text{if } g^{-1}(b) \neq \phi, \\ 0, & \text{otherwise.} \end{cases}$$

Similarly, the non-membership function $\nu_{g[C_1]}(b)$ of $g[C_1]$ for each $b \in B$ is defined as follows:

$$\nu_{g[C_1]}(b) = \begin{cases} \inf_{z \in g^{-1}(b)} \nu_{C_1}(z), & \text{if } g^{-1}(b) \neq \phi, \\ 0, & \text{otherwise.} \end{cases}$$

$g^{-1}[C_2]$ is defined as follows:

Membership function:

$$\mu_{g^{-1}[C_2]}(a) = \mu_{C_2}(g(a))$$

Non-membership function:

$$\nu_{g^{-1}[C_2]}(a) = \nu_{C_2}(g(a))$$

It is important to note that $g[C_1]$ and $g^{-1}[C_2]$ are Pythagorean fuzzy subsets. We can observe the following:

$$\begin{aligned}
\mu^2_{g[C_1]}(b) + \nu^2_{g[C_1]}(b) &= \left(\sup_{z \in g^{-1}(b)} \mu_{C_1}(z)\right)^2 + \left(\inf_{z \in g^{-1}(b)} \nu_{C_1}(z)\right)^2 \\
&= \sup_{z \in g^{-1}(b)} \mu^2_{C_1}(z) + \inf_{z \in g^{-1}(b)} \nu^2_{C_1}(z) \\
&= \sup_{z \in g^{-1}(b)} (1 - \nu^2_{C_1}(z)) + \inf_{z \in g^{-1}(b)} \nu^2_{C_1}(z)
\end{aligned}$$

This holds whenever $g^{-1}(b) \neq \phi$. Similarly, if $g^{-1}(b) = \phi$, Since:

$$\mu^2_{g[C_1]}(b) + \nu^2_{g[C_1]}(b) = 1$$

The proof for $g^{-1}[C_2]$ follows a similar argument.

Example

Let the sets $X = \{1,2,3\}$ and $Y = \{4,5,6\}$, and define the function $g : X \to Y$ as $g(x) = x$. Also, let A be a CPF subset of X defined by

let C_1 be a CPF subset of X defined by

$$C_1(1) = \{1, \langle [0.2, 0.4], [0.3, 0.5] \rangle, (0.4, 0.6)\},$$

$$C_1(2) = \{2, \langle [0.4, 0.6], [0.5, 0.7] \rangle, (0.7, 0.8)\},$$

$$C_1(3) = \{3, \langle [0.6, 0.8], [0.7, 0.9] \rangle, (0.9, 0.1)\}.$$

let C_2 be a CPF subset of Y defined by

$$C_2(a) = \{a, \langle [0.3, 0.5], [0.4, 0.6] \rangle, (0.5, 0.7)\},$$

$$C_2(b) = \{b, \langle [0.1, 0.3], [0.2, 0.4] \rangle, (0.6, 0.5)\},$$

$$C_2(c) = \{c, \langle [0.5, 0.7], [0.6, 0.8] \rangle, (0.6, 0.9)\})$$

We can calculate for $g[C_1]$ as bellow:

$$\mu_{g[C_1]}(4) = \sup_{z\in g^{-1}(4)} \mu_{C_1}(z) = \mu_{C_1}(1) = (0.2, 0.4, 0.4),$$
$$\mu_{g[C_1]}(5) = \sup_{z\in g^{-1}(5)} \mu_{C_1}(z) = \mu_{C_1}(2) = (0.4, 0.6, 0.7),$$
$$\mu_{g[C_1]}(6) = \sup_{z\in g^{-1}(6)} \mu_{C_1}(z) = \mu_{C_1}(3) = (0.7, 0.9, 0.1),$$

and

$$\nu_{g[C_1]}(4) = \inf_{z\in g^{-1}(4)} \nu_{C_1}(z) = \nu_{C_1}(1) = (0.3, 0.5, 0.6),$$
$$\nu_{g[C_1]}(5) = \inf_{z\in g^{-1}(5)} \nu_{C_1}(z) = \nu_{C_1}(2) = (0.5, 0.7, 0.8),$$
$$\nu_{g[C_1]}(6) = \inf_{z\in g^{-1}(6)} \nu_{C_1}(z) = \nu_{C_1}(3) = (0.6, 0.8, 0.9).$$

Similarly, $g^{-1}[C_2]$ define as:

$$\mu_{g^{-1}[C_2]}(1) = \mu_{C_2}(g(1)) = \mu_{C_2}(4) = (0.3, 0.5, 0.5),$$
$$\mu_{g^{-1}[C_2]}(2) = \mu_{C_2}(g(2)) = \mu_{C_2}(5) = (0.1, 0.3, 0.6),$$
$$\mu_{g^{-1}[C_2]}(3) = \mu_{C_2}(g(3)) = \mu_{C_2}(6) = (0.5, 0.7, 0.6),$$

and

$$\nu_{g^{-1}[C_2]}(1) = \nu_{C_2}(g(1)) = \nu_{C_2}(4) = (0.4, 0.6, 0.7),$$
$$\nu_{g^{-1}[C_2]}(2) = \nu_{C_2}(g(2)) = \nu_{C_2}(5) = (0.2, 0.4, 0.5),$$
$$\nu_{g^{-1}[C_2]}(3) = \nu_{C_2}(g(3)) = \nu_{C_2}(6) = (0.6, 08, 0.9).$$

Therefore, the CPF subset $g[C_1]$ can be represented as follows:

$$g[C_1] = \{(4, (0.2, 0.4, 0.4)), (5, (0.4, 0.6, 0.7)), (6, (0.7, 0.9, 0.1))\}.$$

Similarly, the CPF subset $g^{-1}[C_2]$ can be represented as:

$$g^{-1}[C_2] = \{(1,(0.3,0.5,0.5)),(2,(0.1,0.3,0.6)),(3,(0.5,0.7,0.6))\}.$$

4.5 Properties of Continuity

The proposition below presents several characteristic of the continuity:

Proposition 2

Consider $A \neq \phi$ and $B \neq \phi$ and a function $g : A \to B$, then

(i) For any CPF subset C_2 of B, the pre-image of its complement, represented as $g^{-1}[C_2^c]$, is equal to the complement of the pre-image of C_2, i.e., $g^{-1}[C_2^c] = g^{-1}[C_2]^c$.

(ii) The complement of the image of a CPF subset C_1 in A, denoted as $g[C_1]^c$, is a subset of the image of the complement of C_1, i.e., $g[C_1]^c \subseteq g[C_1^c]$.

(iii) if $C_{2_A} \subset C_{2_B}$ then $g^{-1}[C_{1_A}] \subset g^{-1}[C_{2_B}]$.

(iv) if $C_{1_A} \subset C_{2_B}$, then $g[C_{1_A}] \subset g[C_{2_B}]$ where C_{1_A} and C_{2_B} are CPF subsets of A.

(v) $g[g^{-1}[C_2]] \subset C_2$

(vi) For any CPF subset C_1 of A, C_1 is a subset of the preimage of its image, i.e., $C_1 \subset g^{-1}[g[C_1]]$.

Proof

(i) By considering any element a belonging to set A and any compound Pythagorean fuzzy subset C_2 of set B, we can deduce the following based on the definition of complement that

$$\begin{aligned}
\mu_{g^{-1}[C_2^c]}(a) &= \mu_{C_2^c}(g(a)) \\
&= \nu_{C_2}(g(a)) \\
&= \mu_{g^{-1}[C_2]}(a) \\
&= \mu_{g^{-1}[C_2]^c}
\end{aligned}$$

Since, $\nu_{g-1[C_2^c]}(a) = \nu_{g^{-1}[C_2]^c}(a)$. So, $g^{-1}[C_2^c] = g^{-1}[C_2]^c$.

(ii) b belong to B s.t $g(b) = \phi$ and CPF subset C_1 of A, written as;

$$\begin{aligned}
r^2_{g[C_1]}(b) &= \mu^2_{g[C_1]}(b) + \nu^2_{g[C_1]}(b) \\
&= \sup_{z\in g^{-1}(b)} \mu^2_{C_1}(z) + \inf_{z\in g^{-1}(b)} \nu^2_{C_1}(z) \\
&= \sup_{z\in g^{-1}(b)} (r^2_{C_1}(z) - \nu^2_{C_1}(z)) + \inf_{z\in g^{-1}(b)} \nu^2_{C_1}(z) \\
&\leq \sup_{z\in g^{-1}(b)} r^2_{C_1}(z) - \inf_{z\in g^{-1}(b)} \nu^2_{C_1}(z) + \inf_{z\in g^{-1}(b)} \nu^2_{C_1}(z) \\
&= \sup_{z\in g^{-1}(b)} r^2_{C_1}(z)
\end{aligned}$$

Since;

$$\begin{aligned}
\mu_{g[C_1{}^c}(b) &= \sup_{z\in g^{-1}(b)} \mu_{C_1c}(z) \\
&= \sup_{z\in g^{-1}(b)} \nu_{C_1}(z) \\
&= \sup_{z\in g^{-1}(b)} \sqrt{r^2_{C_1}(z) + \mu^2_{C_1}(z)} \\
&\geq \sqrt{\sup_{z\in g^{-1}(b)} r^2_{C_1}(z) - \sup_{z\in g^{-1}(b)} \mu^2_{C_1}(z)} \\
&\geq \sqrt{r^2_{g[C_1]}(b) - \mu^2_{g[C_1]}(b)} \\
&= \nu_{g[C_1]}(b)
\end{aligned}$$

$$= \mu_{g[C_1]^c}(b)$$

The proof is straightforward for every element b in set B where $g(b) = \phi$.

Therefore, $\nu_{g[C_1^c]}(b) \leq \nu_{g[C_1]^c}(b)$ applying the same concept. Since, we get $g[C_1]^c \subset g[C_1^c]$.

(iii) Consider $C_{2_A} \subset C_{2_B} \forall a \in A$;

$$\begin{aligned}\mu_{g^{-1}[C_{2_A}]}(a) &= \mu_{C_{2_A}}(g(a)) \\ &\leq \mu_{C_{2_B}}(g(a)) \\ &= \mu_{g^{-1}[C_{2_B}]}(a)\end{aligned}$$

Similarly, $\mu_{g^{-1}[C_{2_A}]} \leq \mu_{g^{-1}[C_{2_B}]}$. Likewise, demonstrating or proving that is not a challenging task. $\nu_{g^{-1}[C_{2_A}]} \geq \nu_{g^{-1}[C_{2_B}]}$.

(iv) Consider $C_{1_A} \subset C_{1_B}$ and $b \in B$. If $g(b) = \phi$, then the proof is straightforward for every element . Suppose $g(b) = \phi$. Since;

$$\begin{aligned}\mu_{g[C_{1_A}]}(b) &= \sup_{z \in g^{-1}(b)} \mu_{C_{1_A}}(z) \\ &\leq \sup_{z \in g^{-1}(b)} \\ &\mu_{C_{1_B}}(z) \\ &= \mu_{g[C_{1_B}]}(b)\end{aligned}$$

So, $\mu_{g[C_{1_A}]} \leq \mu_{g[C_{1_B}}$.

Since $\nu_{g[C_{1_A}]} \geq \nu_{g[C_{1_B}]}$.

(v) $b \in B$ s.t $g(b) \neq \phi$, written as;

$$\begin{aligned}\mu_{g[g^{-1}[C_2]]}(b) &= \sup_{z \in g^{-1}(b)} \mu_{g^{-1}[C_2]}(z) \\ &= \sup_{z \in g^{-1}(y)} \mu_{C_2}(g(z)) \\ &= \mu_{C_2}(b)\end{aligned}$$

If $g(b) = \phi$, then we get, $\mu_{g[g^{-1}[C_2]]}(b) = 0 \leq \mu_{C_2}(b)$. Similarly, we have $\nu_{g[g^{-1}[C_2]]}(b) = 0 \geq \nu_{C_2}(b)$.

(vi) For any $a \in A$, we have

$$\mu_{g[g^{-1}[C_1]]}(a) = \mu_{g[C_1]}(g(a))$$
$$= \sup_{z \in g^{-1}(b)} \mu_{C_1}(z)$$
$$\geq \mu_{C_1}(a).$$

Therefore, we get $\nu_{g[g^{-1}[C_1]} \leq \nu_{C_1}$.

Example

Let $X = \{1,2,3\}$ and $Y = \{a,b,c\}$, and let $f : X \to Y$ be defined as below:

$$f(1) = a, \quad f(2) = b, \quad f(3) = c$$

Now, let's define the CPF subsets A and B of X and Y.

For C_1, the membership values of each element are defined as intervals in the cubic domain. Let's define:

$$C_1(1) = \{1, \langle [0.2,0.4],[0.3,0.5] \rangle, (0.4,0.6)\},$$

$$C_1(2) = \{2, \langle [0.4,0.6],[0.5,0.7] \rangle, (0.6,0.8)\},$$

$$C_1(3) = \{3, \langle [0.6,0.8],[0.7,0.9] \rangle, (0.8,0.1)\}.$$

For C_2, the membership values of each element are also defined as intervals in the cubic domain. Let's define:

$$C_2(a) = \{a, \langle [0.3,0.5],[0.4,0.6] \rangle, (0.5,0.7)\},$$

$$C_2(b) = \{b, \langle [0.1,0.3],[0.2,0.4] \rangle, (0.3,0.5)\},$$

$$C_2(c) = \{c, \langle [0.5,0.7],[0.6,0.8] \rangle, (0.7,0.9)\})$$

Now, we can demonstrate each of the propositions using these CPF subsets:

(i) $f[C_2]^c \subset f[C_2^c]$:

$C_2^c = \emptyset$.

$f[C_2^c] = f[\emptyset] = \phi$.

$f[C_2] = f[\{1,2,3\}] = \{a,b,c\}$.

Therefore, $f[C_2]^c = \phi \subset \phi = f[C_2^c]$.

(ii) $f[C_1]^c \subset f[C_1^c]$:

$C_1^c = \{\}$ (empty set).

$f[C_1^c] = f[\phi] = \phi$.

$f[C_1] = f[\{1,2,3\}] = \{a,b,c\}$.

Therefore, $f[C_1]^c = \phi \subset \phi = f[C_1^c]$.

(iii) if $C_{2_A} \subset C_{2_B}$ then $f^{-1}C_{2_A}] \subset f^{-1}[C_{2_B}]$:

Let $C_{2_A} = \{b\}$ and $C_{2_B} = \{a,b,c\}$.

Since $C_{2_A} \subset C_{2_B}$, the proposition holds.

$f^{-1}[C_{2_A}] = f^{-1}[\{b\}] = \{2\}$.

$f^{-1}[C_{2_B}] = f^{-1}[\{a,b,c\}] = \{1,2,3\}$.

Therefore, $f^{-1}[C_{2_A}] = \{2\} \subset \{1,2,3\} = f^{-1}[C_{2_B}]$.

(iv) if $C_{1_A} \subset C_{1_B}$, then $f[C_{1_A}] \subset f[A_2]$:

Let $C_{1_A} = \{1,2\}$ and $C_{1_B} = \{1,2,3\}$.

Since $C_{1_A} \subset C_{1_B}$, the proposition holds.

$f[C_{1_A}] = f[\{1,2\}] = \{a,b\}$.

$f[C_{1_B}] = f[\{1,2,3\}] = \{a,b,c\}$.

Therefore, $f[C_{1_A}] = \{a,b\} \subset \{a,b,c\} = f[C_{1_B}]$.

(v) $f[f^{-1}[C_2]] \subset C_2$:

Let $C_2 = \{b,c\}$.

$f^{-1}[C_2] = f^{-1}[\{b,c\}] = \{2,3\}$.

$f[f^{-1}[C_2]] = f[\{2,3\}] = \{b,c\}$.

Therefore, $f[f^{-1}[C_2]] = \{b,c\} \subset \{b,c\} = C_2$.

(vi) $C_1 \subset f^{-1}[f[C_1]]$:

Let $C_1 = \{1\}$.

$f[C_1] = f[\{1\}] = \{a\}$.

$f^{-1}[f[C_1]] = f^{-1}[\{a\}] = \{1\}$.

Therefore, $C_1 = \{1\} \subset \{1\} = f^{-1}[f[C_1]]$.

4.6 Cubic Pythagorean fuzzy continuity

Definition 4.6.1. Consider $g : A \to B$ be a function between two CPFTSs (A, τ_1) and (B, τ_2). We say that g is CPF cont. if, for any CPF subset X of A and any neighborhood U of $g[C_1]$, $\exists$ a neighborhood N of C_1 s.t $g[N] \subset U$. This property ensures that a preimage of a CPF subset under g remains within a neighborhood of the original subset.

Theorem 4.6.1. *Consider two PFTSs, denoted as (A, τ_1) and (B, τ_2), and let $g : A \to B$ be a mapping. The following statements are equivalent:*

(1) The Mapping g is the CPF cont.

(2) For any CPF subset C_1 of A and any neighborhood U of $g[C_1]$, $\exists$ a neighborhood N of C_1 s.t, for any CPF subset C_2 contained in N, the image $g[C_2]$ is entirely contained within U.

(3) For any CPF subset C_1 of A and any neighborhood U of $g[C_1]$, $\exists$ a neighborhood N of C_1 s.t all elements in N are mapped by g into U.

(4) For any CPF subset C_1 of A and any neighborhood U of $g[C_1]$, the preimage $g^{-1}[U]$ forms a neighborhood of C_1.

Proof. (1) $\Rightarrow$ (2):

Consider (A, τ_1) and (B, τ_2) be two CPFTSs, and $g : A \to B$ be a CPF cont. function. For any CPF subset C_1 of P and any neighborhood U of $g[C_1]$, $\exists$ a neighborhood N of C_1

s.t $g[N]$ ia a subset of V. Now, consider any CPF subset C_2 contained in N. Since $C_2 \subset N$. So, $g[C_2] \subset g[N] \subset U$.

(2) $\Rightarrow$ (3):

Assuming the implication (2) holds true. Let C_1 be a CPF subset of A and U be the neighborhood of $g[C_1]$. According to (2), $\exists$ a neighborhood N of C_1 s.t for any CPF subset C_2 contained in N, the image $g[C_2]$ is entirely contained within U. This implies that $\forall x \in N$, $g(x) \in V$. Hence N is a neighborhood of C_1, we can conclude that all elements in N are mapped by g into U.

(3) $\Rightarrow$ (4):

Consider (A, τ_1) and (B, τ_2) as two CPFTSs, and let $g : A \to B$ be a function satisfying condition (3). Take a CPF subset C_1 of A and a neighborhood U of $g[C_1]$. According to (3), $\exists$ a neighborhood N of C_1 s.t all elements in N are mapped by g into U. Since N is a neighborhood of C_1, we have an open CPF subset H of A s.t $C_1 \subset H \subset U$. Furthermore, since all elements in H are mapped by g into U, we have $C_1 \subset H \subset g^{-1}[U]$, indicating that $g^{-1}[U]$ is a neighborhood of C_1.

(4) $\Rightarrow$ (1):

Assuming (4) holds, let (A, τ_1) and (B, τ_2) be two CPFTSs, and $g : A \to B$ be a function. Suppose C_1 is a CPF subset of A and U is a neighborhood of $g[C_1]$. We find $g^{-1}[U]$ is a neighborhood of C_1. Since, $\exists$ an open CPF subset H of A s.t $C_1 \subset H \subset g^{-1}[U]$. This $\implies$ $g[H] \subset g[g^{-1}[U]] \subset V$. Moreover, hence H is open, it is also a neighborhood of C_1. Hence, we can concluded that g is CPF cont. $\square$

Theorem 4.6.2. *Consider two CPFTSs, (A, τ_1) and (B, τ_2), and let $g : A \to B$ be a function. We can state that g is CPF cont. iff for every open CPF subset C_2 of B, the pre-image $g^{-1}[C_2]$ is an open CPF subset of A.*

Proof. Let C_2 be an open Cubic Pythagorean fuzzy subset of B, and suppose C_1 is a subset of $g^{-1}[C_2]$. Since g is cont. Now, $g[C_1] \subset C_2$. By Proposition 1, $\exists$ a neighborhood U of $g[C_1]$ s.t $U \subset C_2$. From the CPF cont. of g and (iv) of Thm 1, concluded that $g^{-1}[U]$ is a neighborhood of C_1. Moreover, from (iii) of Proposition 2, conclude $g^{-1}[U] \subset g^{-1}[C_2]$. $g^{-1}[C_2]$ is also a neighborhood of C_1. Since C_1 is an arbitrary subset of $g^{-1}[C_2]$, we can deduce from Proposition 1 that $g^{-1}[C_2]$ is an open CPF subset.

Conversely, suppose C_1 is a CPF subset of A, and U be a neighborhood of $g[C_1]$. $\exists$ an open CPF subset Q of B s.t $g[C_1]$ is a subset of Q is a subset of U. By the assumption that $g^{-1}[Q]$ is open, we have C_1 is a subset of $g^{-1}[g[C_1]]$ is a subset of $g^{-1}[Q]$ is a subset of $g^{-1}[U]$. So, $g^{-1}[U]$ is a neighborhood of C_1, that demonstrates the CPF cont. of g.

Therefore, conclude that $g : A \to B$ is CPF cont iff, for every open CPF subset C_2 of B, the preimage $g^{-1}[C_2]$ is an open CPF subset of A. $\square$

We can construct a PFT on a set X by using a specific PFTS B and $g : X \to B$. Validity of this claim can be established through the below thm:

Theorem 4.6.3. *let a set $X \neq \phi$, a CPFTS (B, τ), and $g : X \to Y$. We want to show that $\exists$ a coarsest CPFT τ^* on X s.t g is CPF continuous.*

Proof. We define the class of CPF subsets τ^* of X as follows:

$$\tau^* := \{g^{-1}[U] : U \in \tau\}$$

To verify the properties of a topology for τ^*:

(T1) We observe that $g^{-1}[0_Y] = 0_X$ and $g^{-1}[1_Y] = 1_X$, satisfying the conditions of being a CPF subset. Therefore, $0_X, 1_X \in \tau^*$.

(T2) Suppose $U_1, U_2 \in \tau^*$, $\exists$ $C_{2_1}, C_{2_2} \in \tau$ s.t $g^{-1}[C_{2_1}] = U_1$ and $g^{-1}[C_{2_2}] = U_2$. We can show that $U_1 \cap U_2 = g^{-1}[C_{2_1} \cap C_{2_2}]$, satisfying the minimum operation in the definition of a CPF subset. Therefore, $U_1 \cap U_2 \in \tau^*$.

(T3) Consider U_i $i \in I$ of τ^*. For each $i \in [0,1]$, $\exists C_{2_i} \in \tau$ s.t $g^{-1}[C_{2_i}] = U_i$. We can show that $\bigcup U_i = g^{-1}\left[\bigcup C_{2_i}\right]$, satisfying the supremum operation in the definition of a CPF subset. Thus, $\bigcup U_i \in \tau^*$.

Furthermore, the continuity of g with respect to τ^* follows trivially from the definition.

To complete the proof, we need to show that τ^* is the rough CPFT over X in which g is CPF cont.

Assume that $(\tau^*)^* \subset \tau^*$ is a CPFT on X in which g is CPF cont. If $C_2 \in \tau^*$, then $\exists\ U \in \tau$ s.t $g^{-1}[U] = C_2$. Since g is CPF cont. w.r.t $(\tau^*)^*$, then $B = g^{-1}[U] \in \tau^{**}$. So, concluded that $(\tau^*)^* = \tau^*$.

Therefore, we have proven that τ^* is the rough CPFT over X s.t g is CPF cont. $\square$

Chapter 5

Conclusion

The aim of this thesis is to assist the reader in understanding the connection between the FS and FTSs. Yager developed the notion of PFS as a generalisation of IFS. Peng and Yang pioneered the notion of IVPFS. First, we present the concept of PFTS, which expands the concepts of FTS and IFTS. In this study, we developed the notion of CPFS, in which the membership degree is an IVPFS and the non-membership degree is a PFS. We described several fundamental operations and explored certain features of the suggested operation. Subsequently, we establish the definitions of continuity of Pythagorean fuzzy subsets. We also acquire features of these ideas. We present and characterise CPF continuity of functions. Further show that utilising the idea of CPF continuity, one may get a CPFTS on a non-empty set.

Chapter 6

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