# An Introduction to Projective Geometry and the Klein Quadric

*By*

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CIIT/FA21-RMT-101/LHR

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In partial fulfillment

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MS Mathematics

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A Post Graduate Thesis submitted to the name of Department of Mathematics as partial fulfilment of the requirement for the award of the degree of MS Mathematics


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# Declaration

I **Ammara Rashid, CIIT/FA21-RMT-101/LHR**, hereby state that my MS thesis titled “An Introduction to Projective Geometry and the Klein Quadric” is my own work and has not been submitted previously by me for taking any degree from this University ”COMSATS University Islamabad, Lahore Campus” or anywhere else in the country/world. At any time if my statement is found to be incorrect even after my Graduate the university has the right to withdraw my MS degree.

Date:______________ ____________________________

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# Certificate

It is certified that Ammara Rashid, CIIT/FA21-RMT-101/LHR has carried out all the work related to this thesis under my supervision at the Department of Mathematics, COMSATS University Islamabad, Lahore Campus and the work fulfils the requirement for the award of MS degree.

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# DEDICATION

To

My beloved family and esteemed mentors

# ACKNOWLEDGEMENTS

With profound gratitude, I begin by acknowledging the infinite wisdom and benevolence of **Allah Almighty**, the Creator of all universes, for granting me strength, guidance, and wisdom throughout this journey and expressing my deepest reverence to His **Prophet Muhammad (PBUH)**, whose teachings and exemplary life continue to inspire me. Their divine guidance has illuminated my path and bestowed me with the strength and perseverance to embark on this scholarly endeavour.
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At this point, how can I forget my family, who has been my pillar of strength, I am eternally grateful to my **parents** and siblings, whose unconditional love, support and unwavering belief in my abilities have been a constant source of motivation and have been the driving force behind my accomplishments. Lastly, I extend my heartfelt thanks to my friends, whose unwavering support, encouragement, and camaraderie have made this journey a memorable one. Their presence has brought joy, laughter, and shared experiences that have kept me motivated and focused throughout this endeavour.
Finally, I am once again grateful to my Lord who has given me consciousness and without whose help the completion of this journey would not have been possible.

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# ABSTRACT

## An Introduction to Projective Geometry and the Klein Quadric

In this thesis, the main topic is the Klein quadric and its properties. In order to study this important geometric object, we will need to discuss the basics of projective geometry. We will see how the Klein quadric is defined by a quadratic equation in a projective space which has dimension 5. The main focus of the thesis will be to show that the Klein quadric is an important example of parameter space and in particular, it parameterizes all the lines in the projective space of dimension 3.

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# Chapter 1

# Introduction

Algebraic geometry is a branch of mathematics that deals with the study of geometric objects defined by polynomial equations, known as algebraic varieties. In this thesis, our focus lies specifically on the projective setting, where we investigate solutions of homogeneous polynomial equations in projective spaces, known as projective varieties We have an interplay between algebra and geometry, that is, using the algebra of polynomial equations, we can understand the geometric properties of algebraic varieties. The classification problem is a fundamental task in many areas of mathematics, and in algebraic geometry, it revolves around the classification of algebraic varieties. The desire to classify these varieties arises from the need to gain a deeper understanding of their structures and properties. However, the classification problem is often complex and challenging.

To approach the classification problem, one effective strategy is to construct a parameter space that parametrizes a specific class of algebraic varieties. This parameter space itself becomes an algebraic variety, whose points are in one-to-one correspondence with the varieties that we try to classify, providing a convenient framework to study and analyze the desired class of algebraic varieties. The geometric properties of the parameter space offer valuable insights into the properties of the algebraic varieties within the class.

It is essential that the parameter space be an algebraic variety and not merely a set. The algebraic variety structure allows us to leverage its geometric properties to infer properties about the class of algebraic varieties we aim to study. This algebraic structure grants us a better understanding and control over the behaviour of families of algebraic varieties.

In algebraic geometry, we often require more than simple one-to-one correspondence between points in a parameter space and algebraic varieties in a collection. To gain better control over families of algebraic varieties, mathematicians have refined the notion of parameter space and arrived at the construction of moduli spaces. Moduli spaces are fundamental objects in various mathematical disciplines, from hyperbolic and birational geometry to mathematical physics and string theory.[10][Lectures on birational geometry by C.Birkar]

This thesis focuses on laying the first basic mathematical foundations of moduli spaces by studying the first examples of parameter spaces in detail. One such example is the Klein quadric, which parametrizes lines in the three-dimensional projective space. By delving into the Klein quadric and its properties, we aim to provide a comprehensive understanding of parameter spaces and their role in algebraic geometry.

By exploring the concepts outlined above, this thesis seeks to contribute to the field of projective geometry and the study of algebraic varieties as parameter spaces. The subsequent chapters delve into specific aspects and propertiies of the Klein quadric and its implications for understanding families of algebraic varieties.

Keeping in view this general structure of the thesis, a brief description of each chapter is following.

In Chapter 2, titled “Preliminary Results,” we provide a concise introduction to the fundamental concepts of projective geometry. This chapter serves as the foundation for the rest of the thesis. We cover topics such as projective spaces, projective transformations, projective subspaces, incidence and collinearity, projective duality, and homogeneous coordinates. By establishing these core ideas, we prepare the reader for the subsequent exploration of the Klein quadric and its connection to projective geometry.

In Chapter 3, titled “Quadrices and Conics,” we explore the geometric representation of symmetric bilinear forms in projective geometry. This chapter provides a brief overview of quadrics and conics, which are fundamental geometric objects in projective geometry. By understanding these concepts, we establish a connection between linear algebra and projective geometry, laying the foundation for the subsequent discussions on the Klein quadric.

Chapter 4 focuses on two main topics: exterior algebra and the Klein quadric. In this chapter, we explore exterior algebra and its application in studying geometric properties. It introduces decomposable vectors and presents the main theorem related to them. The chapter then focuses on the Klein quadric, defining it as a parameter space for lines in three-dimensional projective space. The significance of the Klein quadric in the study of algebraic varieties is discussed.

# Chapter 2

# Preliminary Results

In this chapter, we introduce some essential notations and definitions in projective geometry that will serve as a foundation for all the material that will be discussed in this thesis.

## 2.1 Preliminaries

To be able to define the main objects of study, we shall briefly introduce some basic definitions.

**Definition 2.1.1.** Let $\mathbb{K}$ be a field and $V$ be a $(n+1)$-dimensional $\mathbb{K}$-vector space. The **Projective space** $\mathbb{P}^n_{\mathbb{K}}$ or the **Projectivization of** $V$ $\mathbb{P}(V)$ is the set of 1-dimensional vector subspace of $V$, that is, all the lines in $V$ passing through the origin.

$$\mathbb{P}^n_{\mathbb{K}} = \mathbb{P}(V) = \{\text{Vector subspaces of V having dimension 1}\}$$

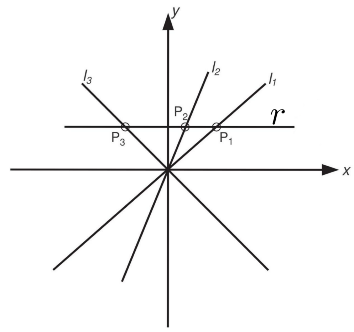
Or equivalently, projectivization of $V$ can be defined as $V\backslash\{0\}$ the quotient by an equivalence relation.

$$\mathbb{P}(V) = V\backslash\{0\}/\sim_\lambda \quad \text{(w.r.t. } \sim_\lambda)$$

$$v, w \in V \quad \text{and } v \sim_\lambda w \Longleftrightarrow \exists \quad \lambda \subset K^* \text{such that } v = \lambda w$$

**Definition 2.1.2.** 1- dimensional projective space is called **a projective line**.
We explain in detail the construction of the real projective line i.e $\mathbb{P}^1_{\mathbb{R}}$
Starting from the definition:

$$\mathbb{P}^1_{\mathbb{R}} = \{\text{lines through the origin in } \mathbb{R}^2\}$$




We can first start by picturing all the lines passing through the origin in $\mathbb{R}^2$. Then if we consider an affine line *r* that is parallel to the x-axis, we see that every line that is different from the x-axis would intersect exactly at one point on the affine line *r*. in this way, we can build a one-to-one correspondence between points in the line *r* and the lines through the origin that are different from the x-axis. So, we are missing only the x-axis, we need to add this extra point, this is the point we called the point at infinity. Conseqiuently, the real projective line is the union of the affine line and the point at infinity.

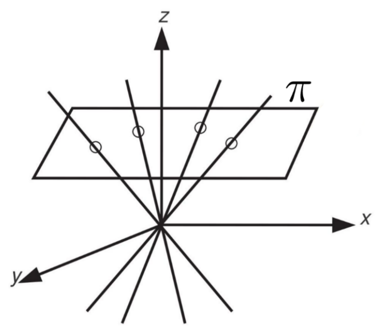
$$\mathbb{P}^1_{\mathbb{R}} = \mathbb{A}^1_{\mathbb{R}} \cup \{\infty\}$$

Here $\{\infty\}$ means point at infinity, where the x-axis and the affine line *r* meet.
Topologically, the circle $S1$ and the real projective line are homomorphic.

**Definition 2.1.3.** A projective space of dimension 2 is called **projective plane**.

We explain in detail the construction of the real projective plane i.e $\mathbb{P}^2_{\mathbb{R}}$
From the definition:

$$\mathbb{P}^2_{\mathbb{R}} = \{\text{lines through the origin in } \mathbb{R}^3\}$$




We can first start by picturing all the lines passing through the origin in $\mathbb{R}^3$. Then, if we consider an affine plane $\pi$ that is parallel to the xy-plane, we see that all the lines not contained in the plane $z = 0$ would intersect exactly at one point in the affine plane $\pi$. In this way, we can build a one-to-one correspondence between points in the plane $\pi$ and the lines through the origin that are not contained in the plane $z = 0$. So, we are missing only the lines through the origin in the xy-plane, we need to add these lines and these lines are actually $\mathbb{P}^1_{\mathbb{R}}$. The lines not contained in $\mathbb{R}^3$ are parameterized by $\pi$ and consequently, $\mathbb{P}^2_{\mathbb{R}}$ is the combination of affine plane and the real projective line.

$$\mathbb{P}^2_{\mathbb{R}} = \mathbb{A}^2_{\mathbb{R}} \cup \mathbb{P}^1_{\mathbb{R}}$$

Here, $\mathbb{A}^2_{\mathbb{R}}$ is an affine plane of all the lines which are not in the plane $z = 0$ and $\mathbb{P}^1_{\mathbb{R}}$ is the real projective line having all the lines through the origin in the xy-plane $z = 0$

Now, move towards another important definition of homogenous coordinate. And before moving on to this definition, we define another important terminology in $\mathbb{P}(V)$

**Definition 2.1.4.** The idea of a representative vector for a point in the Projective space is important for our purposes. For any non-zero vector $v$, the 1-dimensional vector subspace generated by $V$ is a set of all non-zero multiples of the vector $v \in V$. The point $[v] \in \mathbb{P}(V)$ is therefore said to have $v$ as a **representative vector**. It is obvious that if $\lambda \neq 0$ then $\lambda v$ is a another represents vector and therefore c

**Definition 2.1.5.** Consider we pick a basis $\{u_0, ..., u_n\}$ for $V$. we can write the vector $u$ as

$$\sum_{i=1}^{n} x_i u_i$$

and the coordinates of $u \in V$ are provided by the $n+1$ tuple $\{x_0, x_1, \ldots, x_n\}$. If $u \neq 0$, we can see the respective point $[u] \in \mathbb{P}(V)$ as $[u] = [x_0, x_1, \ldots, x_n]$ and we called these as **homogeneous coordinates** of a given point in the projective space.

The following two properties characterize the homogeneous coordinates:
$[u] = [x_0, x_1, \ldots, x_n]$ are homogeneous coordinates of a given point $[u] \in \mathbb{P}(V)$ if

1. $\exists x_i$ for $0 \leq i \leq n$ such that $x_i \neq 0$

2. The coordinates are defined up to rescaling.
   i.e for $\lambda \neq 0$

$$[\lambda x_0, \lambda x_1, \ldots, \lambda x_n] = [x_0, x_1, \ldots, x_n]$$

**Definition 2.1.6.** we can define **affine charts** as;

$$\begin{aligned} \bigsqcup\nolimits_0 &:= \{[x_0 : x_1 : \cdots : x_n] \in \mathbb{P}^n | x_0 \neq 0\} \\ &\iff \left(\frac{x_1}{x_0}, \ldots, \frac{x_n}{x_0}\right) \in \mathbb{A}^n_{\mathbb{K}} \end{aligned}$$

Similarly, we can define $\bigsqcup_i$ for every $i = 1, 2 \ldots, n$

$$\begin{aligned} \mathbb{P}^n_{\mathbb{K}} \backslash \bigsqcup\nolimits_0 &:= \{[0 : x_1 : \cdots : x_n] \in \mathbb{P}^n\} = \mathbb{P}(\{x_0 = 0\}) \\ &= \mathbb{P}^{n-1}_{\mathbb{K}} \end{aligned}$$

Thus for every $n$ in $\mathbb{P}^n_{\mathbb{K}}$, we can define $n+1$ affine charts. We obtain that:

$$\mathbb{P}^n_{\mathbb{K}} = \mathbb{A}^n_{\mathbb{K}} \cup \mathbb{P}^{n-1}_{\mathbb{K}}$$

where $\mathbb{A}^n_{\mathbb{K}} \cong \bigsqcup_0$ and $\mathbb{P}^{n-1}_{\mathbb{K}} \cong \mathbb{P}^n \backslash \bigsqcup_0$

Thus, the projective space is covered by the affine charts.

## 2.2 Linear Subspaces

**Definition 2.2.1.** The collection of vector subspaces of dimension 1 in a vector or linear subspace $W \subset V$ constitutes a linear or vector subspace of the projective space $\mathbb{P}(V)$.
i.e $\mathbb{P}(W)$ is linear subspace of the $\mathbb{P}(V)$.

**Example 2.1.** Let $V$ be $n+1$ dimensional vector space and $H = \{x_0 = 0\}$ is hyperplane of $V$. Then, $\mathbb{P}(H)$ of dimension $n-1$ is the linear subspace of $\mathbb{P}(V)$ of dimension $n$.

**Proposition 2.1.** *Let $V$ be a $(n+1)$ dimensional vector space and $\mathbb{P}^n_{\mathbb{K}}$ be the projective space associated to $V$. Given two distinct points $p$ and $q$ in $\mathbb{P}^n_{\mathbb{K}}$, there is a unique line passing through them.*

*Proof.* Let $\mathbb{P}^n_{\mathbb{K}}$ be the projective space and let $p = [v]$ and $q = [w]$ be two points in projective space. Here, $v$ and $w$ are representative vectors in $V$. Since $p$ and $q$ are distinct points, $v$ and $w$ are linearly independent. Take, $U$, the plane in $V$ spanned by $v$ and $w$ and so $\mathbb{P}(U)$ is the line joining $p$ and $q$. Suppose that $\mathbb{P}(U')$ is another such line,

$$\Rightarrow v, w \in U'$$

Since $U$ is spanned by $v$ and $w$, any other vector subspace that contains $v$ and $w$, it also contains $U$, because the span is minimal. So,

$$\Rightarrow U \subset U'$$

and since dimension $U$ is equal to dimension $U'$ which is equal to 2, we can conclude that $U = U'$ □

**Proposition 2.2.** *At the point at which two distinct lines intersect in a projective plane, this must be unique.*

*Proof.* Let $V$ be a vector space of dimension 3 and $\mathbb{P}(V)$ be the associated projective plane. Let $l = \mathbb{P}(R)$ and $r = \mathbb{P}(R')$ be the two lines in $\mathbb{P}^2$, where $R$ and $R'$ are 2-dimensional vector subspaces of $V$.

Since, $l$ and $r$ are distinct lines $\Rightarrow R$ and $R'$ are distinct planes.

Two distinct planes through the origin in a 3-dimensional vector space intersect in a line.

Now from the Grassmann formula of linear algebra[11],

$$dimV \geq dim(R+R') = dimR + dimR' - dim(R \cap R')$$

i.e.

$$3 \geq 2+2-dim(R \cap R')$$

$$dim(R \cap R') \geq 1$$

But as we have 2-dimensional vector subspaces $R$ and $R'$, so

$$dim(R \cap R') \leq 2$$

And since $R$ and $R'$ are distinct so equality does not occur, therefore

$$\Rightarrow dim(R \cap R') = 1$$

$$R \cap R' \subset V$$

and $R \cap R'$ is the point of intersection.

Then, we can conclude that $l$ and $r$ meet in the point $\mathbb{P}(R \cap R')$ $\square$

**Example 2.2.** Let $p = [1:2:7]$ and $q = [2:-1:0]$ be the two points in $\mathbb{P}^2$. We want to find the equation of the line passing through $p$ and $q$.

Consider a point $[x_0 : x_1 : x_2]$ that belongs to the line passing through $p$ and $q$. Take

$$A = \begin{bmatrix} 1 & 2 & 7 \\ 2 & -1 & 0 \\ x_0 & x_1 & x_2 \end{bmatrix}$$

We want to impose that $rank(A) = 2$, because this would mean that the point $[x_0 : x_1 : x_2]$ belongs to the span of $p$ and $q$. Hence, the rank must be 2.

This is equivalent to imposing that,

$$detA = 0$$

$$7x_0 + 14x_1 - 3x_2 = 0$$

Thus, $\{7x_0 + 14x_1 - 3x_2 = 0\}$ is the required equation of line passing through $p$ and $q$.

**Example 2.3.** Let $p = [1 : -1 : 2 : 0]$ and $q = [2 : -1 : 0 : 3]$ be the two points in $\mathbb{P}^3$. We want to find the equation of the line passing through $p$ and $q$.
Consider a point $[x_0 : x_1 : x_2 : x_3]$ that belongs to the line passing through $p$ and $q$. Take

$$B = \begin{bmatrix} 1 & -1 & 2 & 0 \\ 2 & -1 & 0 & 3 \\ x_0 & x_1 & x_2 & x_3 \end{bmatrix}$$

Like before, imposing the condition that $rank(B) = 2$

$$\iff detB = 0$$

That is,

$$\det \begin{bmatrix} 1 & -1 & 2 \\ 2 & -1 & 0 \\ x_0 & x_1 & x_2 \end{bmatrix} = 0 \Rightarrow 2x_0 + 4x_1 + x_2 = 0$$

and

$$\det \begin{bmatrix} 1 & -1 & 0 \\ 2 & -1 & 3 \\ x_0 & x_1 & x_3 \end{bmatrix} = 0 \Rightarrow -3x_0 - 3x_1 + x_3 = 0$$

Thus,

$$\begin{cases} 2x_0 + 4x_1 + x_2 = 0 \\ -3x_0 - 3x_1 + x_3 = 0 \end{cases}$$

is the required equation of line passing through $p$ and $q$ in $\mathbb{P}(V)^3$.

Now we can generalize it to the case of finding the equation of a line passing through two points $p$ and $q$ in $\mathbb{P}^n$. As in $\mathbb{P}^2$, the equation of the line passing through two points is given by 1 linear equation and in $\mathbb{P}^3$ it is given by 2 linear equations, so in general for $\mathbb{P}^n$, equation of the line passing through two points is given by $n-1$ equations.

**Example 2.4.** Let $\mathbb{P}(V)$ be an $n+1$ dimensional vector space and let $w_1$ and $w_2$ be distinct linear subspaces of $V$. Let $L_i = \mathbb{P}(w_i)$ be the projectivization of $w_i$. We want to show that the following formula holds:

$$dim(L_1 + L_2) = dimL_1 + dimL_2 - dim(L_1 \cap L_2)$$

Let $w_1$ and $w_2$ be distinct linear subspaces of $V$ and consider $dim(w_1) = r+1$, $dim(w_2) = m+1$ that is $dim(L_1) = r$, $dim(L_2) = m$ and $dim(L_1 \cap L_2) = p$.
Notice that

$$p \leq min(r,m)$$

We need to show that

$$dim(L_1 + L_2) = r + m - p$$

Fix the basis $B = \{v_1, \dots, v_p\}$ for $L_1 \cap L_2$. By the theorem of completion of basis in linear algebra, we can find $(r-p)$ vectors $w_1, \dots, w_{r-p}$ such that

$$B_1 = \{v_1, \dots, v_p, w_1, \dots, w_{r-p}\}$$

is a basis for $L_1$. Similarly, we can find $(m-p)$ vectors $z_1, \dots, z_{m-p}$ such that

$$B_2 = \{v_1, \dots, v_p, z_1, \dots, z_{m-p}\}$$

is a basis for $L_2$. Then, notice that

$$
\begin{aligned}
L_1 + L_2 &= Span(v_1, \ldots, v_p, w_1, \ldots, w_{r-p}, v_1, \ldots, v_p, z_1, \ldots, z_{m-p}) \\
&= Span(v_1, \ldots, v_p, w_1, \ldots, w_{r-p}, 0, \ldots 0, z_1, \ldots, z_{m-p}) \\
&= Span(v_1, \ldots, v_p, w_1, \ldots, w_{r-p}, z_1, \ldots, z_{m-p})
\end{aligned}
$$

Therefore, it remains shown that

$$v_1, \ldots, v_p, w_1, \ldots, w_{r-p}, z_1, \ldots, z_{m-p}$$

are linearly independent. Consider then the general equation,

$$\sum_{i=1}^{p} \lambda_i v_i + \sum_{j=1}^{r-p} \mu_j w_j + \sum_{k=1}^{m-p} \eta_k z_k = 0 \tag{2.2.1}$$

and notice that this implies

$$\sum_{i=1}^{p} \lambda_i v_i + \sum_{j=1}^{r-p} \mu_j w_j = -\sum_{k=1}^{m-p} \eta_k z_k$$

$$\implies -\sum_{k=1}^{m-p} \eta_k z_k \in L_1 \cap L_2$$

Therefore, $\exists \alpha_1, \ldots, \alpha_p \in \mathbb{K}$ such that

$$-\sum_{k=1}^{m-p} \eta_k z_k = \sum_{q=1}^{p} \alpha_q v_q$$

$$\implies \sum_{q=1}^{p} \alpha_q v_q + \sum_{k=1}^{m-p} \eta_k z_k = 0$$

$$\implies \alpha_1 = \cdots = \alpha_p = v_1 = \cdots = v_{m-p} = 0$$

Going back to (2.2.1), we obtain

$$\sum_{i=1}^{p} \lambda_i v_i + \sum_{j=1}^{r-p} \mu_j w_j + 0 = 0$$

$$\Longrightarrow \lambda_1 = \cdots = \lambda_p = \mu_1 = \cdots = \mu_{r-p} = 0$$

Thus;

$dim(L_1 + L_2) = r + m - p$ So, it proved that

$$dim(L_1 + L_2) = dimL_1 + dimL_2 - dim(L_1 \cap L_2).$$

Now, we will move towards another important concept of projective geometry which is parameter space. But before defining parameter space, we have to define the algebraic variety.

**Definition 2.2.2.** A geometrical object specified by a set of polynomial equations is an **algebraic variety** over a field $\mathbb{K}$. A set of points in some affine or projective space that satisfy a number of polynomial equations are what it is more officially known as.

An affine algebraic variety is a set of points in the affine space that satisfy a set of polynomial equations. These equations are functions on the affine space, so the variety consists of all points in the affine space where these functions are zero.

Similarly, a projective algebraic variety is a set of points in the projective space that satisfy a set of homogeneous polynomial equations. These equations are polynomials where all the monomials have the same degree. Therefore, the projective variety consists of all points in the projective space where these polynomials are zero.

**Example 2.5.** Lines and conics in $\mathbb{P}^2$ are examples of a projective variety. A linear homogeneous equation of the type $ax_0 + bx_1 + cx_2 = 0$, where $a$, $b$, and $c$ are coefficients, can be used to describe a line in mathbbP2. Any point that fits this equation, $[x_0 : x_1 : x_2]$, is on the line.

Contrarily, conics are curves in the $MathbbP2$ space that are defined by homogeneous quadratic equations. Circles, ellipses, parabolas, and hyperbolas are examples of conics. A conic equation has the generic form $ax_02 + bx_12 + cx_22 + dx_0x_1 + ex_0x_2 + fx_1x_2 = 0$, where $a$, $b$, $c$, $d$, $e$, and $f$ are coefficients.

**Definition 2.2.3.** Given a collection $\mathscr{C}$ of algebraic varieties, a **parameter space** for all objects in $\mathscr{C}$ is an algebraic variety whose points are in 1-1 correspondence with elements of $\mathscr{C}$.

**Remark 1.** *Projective space is itself an example of parameter space. In projective space, each point represents a line passing through the origin. The coordinates of a point in projective space can be seen as parameters that define the direction of the corresponding line. Thus, projective space serves as a parameter space for lines passing through the origin in the underlying vector space.*

## 2.3 Projective Transformations

Now, we shall define another important concept of projective geometry i.e. projective transformations.

**Definition 2.3.1.** The term **Projective transformation** from projective space $\mathbb{P}(V)$ to projective space $\mathbb{P}(W)$ where $V$ and $W$ are the vector space, is the map $f$, defined by a linear transformation whose inverse exists, $\psi$ from $V$ to $W$ that is $\Psi : V \to W$ such that

$$f([v]) = [\psi(v)] \quad \forall \quad v \in V/\{0\}$$

If $\Psi$ is a linear transformation inducing $f$, the set of linear transformations from $V$ to $W$ inducing $f$ coincides with family of set $\{c\Psi \mid c \in \mathbb{K}^*\}$ That is,

The linear transformation which induces projective transformation is determined only upto non-zero scaler multiplication

$$[(cf)(v)] = [c(f(v))] = [f(v)]$$

**Example 2.6.** Let us investigate the possibility of a projective transformation.
$w : \mathbb{P}^3 \to \mathbb{P}^3$ such that

$$w(L_1) = L_2 \text{ and } w(M_1) = M_2$$

in the following 2 cases:

**Case: 1**

$$L_1 = \{x_0 = x_1 = 0\}, L_2 = \{x_0 = x_2 = 0\} \text{ and}$$

$$M_1 = \{x_2 = x_3 = 0\}, M_2 = \{x_0 + x_1 = 5x_2 + x_1 - x_0 = 0\}$$

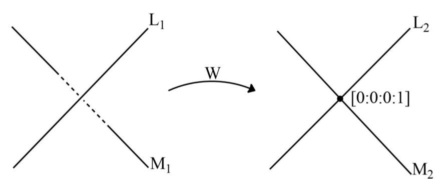
Since

$$L_1 \cap M_1 = \phi$$

and

$$L_2 \cap M_2 = \{[0:0:0:1]\}$$

Suppose on the contrary that there exists such projective transformation $w$. Since $w$ is




invertible, So

$$L_1 \cap M_1 = \phi$$

Since the map is injective, so it implies that;

$$w(L_1) \cap w(M_1) = \phi$$

But

$$w(L_1) = L_2 \text{ and } w(M_1) = M_2$$

and we have seen that

$$L_2 \cap M_2 \neq \phi$$

This is a contradiction, So, such $w$ cannot exist.

**Case 2:**

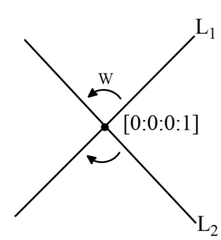
$$L_1 = \{x_0 = x_1 = 0\}; \quad L_2 = \{x_0 = x_2 = 0\} \text{ and}$$
$$M_1 = \{x_2 = x_3 = 0\} \text{ and} \quad M_2 = \{x_0 + x_1 = 2x_2 + x_1 - x_3 = 0\}$$

Here, we noticed that

$$L_1 \cap M_1 = \phi$$
$$L_2 \cap M_2 = \phi$$

So, there are chances that projective transformation may exist. To find that, consider.

$$L_1 = \left\{[0:0:s:t] \in \mathbb{P}^3 \mid s,t \in \mathbb{R}/\{0\}, s = 0 \text{ or } t = 0\right\}$$
$$L_2 = \left\{[0:s:0:t] \in \mathbb{P}^3\right\}$$
$$M_1 = \left\{[s:t:0:0] \in \mathbb{P}^3\right\}$$
$$M_2 = \left\{[s:-s:t:2t-s] \in \mathbb{P}^3\right\}$$




Since

$$L_1 \cap L_2 = \{[0;0;0:1]\}$$

and

$$w(L_1) = L_2$$

take as simpler case

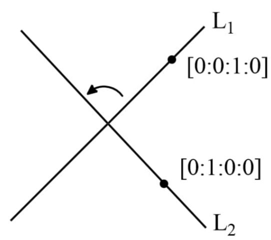
$$w([0:0:0:1]) = [0:0:0:1]$$

By its matrix of transformation, we will get,

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & 0 \\ a_{21} & a_{22} & a_{23} & 0 \\ a_{31} & a_{32} & a_{32} & 0 \\ a_{41} & a_{42} & a_{43} & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

and let take another simpler point

$[0:0:1:0]$ maps on $[0:1:0:0]$ That is



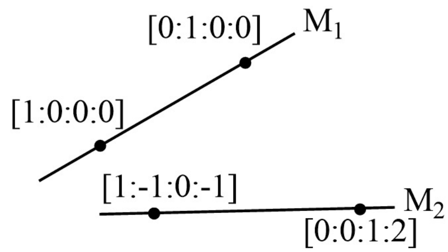

$$\begin{bmatrix} a_{11} & a_{12} & 0 & 0 \\ a_{21} & a_{22} & 1 & 0 \\ a_{31} & a_{32} & 0 & 0 \\ a_{41} & a_{42} & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

Now, to find remaining the first two columns of the matrix of transformation, we will use $w(M_1) = M_2$ Since,

$$M_1 \cap M_2 = \phi$$

take a simpler point on $M_1$, that is

$[0:1:0.0]$, that maps on $[0:0:1:2]$ and




$$\begin{bmatrix} a_{11} & 0 & 0 & 0 \\ a_{21} & 0 & 1 & 0 \\ a_{31} & 1 & 0 & 0 \\ a_{41} & 2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 2 \end{bmatrix}$$

and similarly for the remaining column.

Thus; for any general point $[x_0 : x_1 : x_2 : x_3]$ from $L_1$ or $M_1$, we have

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 2 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_0 \\ -x_0 + x_2 \\ x_1 \\ -x_0 + 2x_1 + x_3 \end{bmatrix}$$

i.e

$$w([x_0 : x_1 : x_2 : x_3]) = [x_0 : -x_0 + x_2 : x_1 : -x_0 + 2x_1 + x_3]$$

is the required projective transformation.

**Definition 2.3.2.** If $n+1$ points in an $n$-dimensional projective space $\mathbb{P}(V)$ have representative vectors in $V$ that are linearly independent, then it is possible to add a new point to the space such that the resulting set of $n+2$ points are said to be in **general position**.

**Example 2.7.** Consider the points in $\mathbb{P}^3_{\mathbb{R}}$

$$p_1 = [1:0:1:2]$$
$$p_2 = [0:1:1:1]$$
$$p_3 = [2:1:2:2]$$
$$p_4 = [1:1:2:3]$$

We will examine whether the points are in general position or not.

Since

$$\det \begin{bmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & 1 & 1 \\ 2 & 1 & 2 & 2 \\ 1 & 1 & 2 & 3 \end{bmatrix} = 0$$

Since the determinant of the matrix is zero, so by the knowledge of basic linear algebra, means that the vectors are linearly dependent. So, these points are not in general position in $\mathbb{P}^3_{\mathbb{R}}$.

**Theorem 2.3.1.** *Let R and S are the vector spaces and consider $p_1, \ldots, p_{n+2}$ and $q_1, \ldots, q_{n+2}$ be two $(n+2)$- tuple of points which are in general position in $\mathbb{P}(R)$ and $\mathbb{P}(S)$ respectively, then there is exist a projective transformation $f : \mathbb{P}(R) \to \mathbb{P}(S)$ that is unique, such that*

$$f(p_j) = q_j \quad , \quad 1 \leqslant j \leqslant n+2$$

*[1][Theorem 3 from notes of 3rd chapter of Nigel Hitchin].*

## 2.4 Dual Projective Space and Duality

**Definition 2.4.1.** Let $V$ be a vector space and $V^*$ be its dual space.Then **dual projective space** refers to the projective space $\mathbb{P}(V^*)$, also denoted by $\mathbb{P}(V)^*$.

Since $dimV = dimV^*$, So, if $dimV = n+1$ then $\mathbb{P}(V^*)$ has dimension $n$. From linear algebra, as we know that $V$ and $V^*$ are linearly isomorphic to each other, therefore, $\mathbb{P}(V^*)$

is projectively isomorphic to $\mathbb{P}(V)$.

Here being isomorphic means the map between them is linear and bijective.

**Proposition 2.3.** *There is a natural one-to-one correspondence between the points of the dual projective space $\mathbb{P}(V^*)$ and the hyperplanes in the projectivization of vector space $Vi.e.$ $\mathbb{P}(V)$. [1]*

i.e naturally, a point $[f]$ in $\mathbb{P}(V^*)$ defines a linear vector subspace $\mathbb{P}(R)$ of $\mathbb{P}(V)$ and a vector space $R$ inside $V$ with dimension one less than that of $V$.

The general idea of the proof of this proposition (2.3) is that; to establish the one-to-one correspondence between the points of the dual projective space $\mathbb{P}(V^*)$ and the hyperplanes in the projectivization of vector space $Vi.e.$ $\mathbb{P}(V)$, The idea of annihilators can be used. Given a hyperplane $H$ in $\mathbb{P}(V)$, we can associate it with the annihilator of $H$ in $V^*$, denoted as $H^0$. The annihilator consists of all linear functionals in $V^*$ that vanish on every point of $H$. By definition, $H^0$ is a subspace of $V^*$.

Conversely, for any subspace $W^*$ in $V^*$, we can associate it with the hyperplane $W$ in $\mathbb{P}(V)$, where $W$ is the set of all points in $V$ that are annihilated by every linear functional in $W^*$. Again, by definition, $W$ is a hyperplane in $\mathbb{P}(V)$.

**Remark 2.** *We can deduce from the aforementioned statement that the dual projective space functions as a parameter space for hyperplanes. We defined a hyperplane in $\mathbb{P}(V)$ as the collection of solutions to homogeneous linear equations. A point corresponds to a hyperplane in the dual projective space $\mathbb{P}(V^*)$, on the other hand. We can connect each hyperplane in $\mathbb{P}(V)$ with a point in $\mathbb{P}(V^*)$ using this correspondence, and vice versa.*
*As a result, it is possible to think of the dual projective space $\mathbb{P}(V^*)$ as a parameter space for the hyperplanes in the projective space $\mathbb{P}(V)$. By changing the point in $\mathbb{P}(V^*)$, we can explore several hyperplanes in $\mathbb{P}(V)$. Each point in $\mathbb{P}(V)$ represents a particular hyperplane. This connection illustrates the dual projective space's function as a natural parameter.*

**Proposition 2.4.** *A linear subspace $\mathbb{P}(R)$ of dimension $p$ in the dual projective space $\mathbb{P}(V^*)$ of dimension $n$ consists of the hyperplanes in $\mathbb{P}(V)$ that intersect with a fixed linear sub-*

*space $\mathbb{P}(S)$ of dimension $n-p-1$ in $\mathbb{P}(V)$.*

*For the proof, see [1][Proposition 8 from notes of 1st chapter of Nigel Hitchin]*

Now, we will discuss some examples related to duality. But before this, we will define the “duality correspondence.”

### 2.4.1 Duality Correspondence

Consider a projectivization of $V$ $\mathbb{P}(V)$ and its $m$ dimensional subspace $S=\mathbb{P}(U)$. The annihilator $Ann(U)$ consists of all linear functionals $f$ in the dual space $V^*$ such that $f$ applied to every element of $U$ is equal to zero. $Ann(U)$ is a linear subspace of $V^*$ and has a dimension of $n-m$, where $n$ is the dimension of $V$.

Now, we will define the **duality correspondence map $\phi$**. This map takes subspaces of dimension $m$ in $\mathbb{P}(V)$ as input and maps them to subspaces of dimension $n-m-1$ in $\mathbb{P}(V^*)$ as output. Specifically, it associates the subspace $S=\mathbb{P}(U)$ with the subspace $\mathbb{P}(Ann(U))$. The duality correspondence map $\phi$ is a bijection, meaning it establishes a one-to-one correspondence between the subspaces in the domain and target.

Moreover, the duality correspondence map has some key properties. First, it reverses inclusions, which means if one subspace is contained within another in $\mathbb{P}(V)$, their corresponding subspaces in $\mathbb{P}(V^*)$ have their inclusion order reversed. Second, the map $\phi$ preserves the intersection of subspaces. That is, the intersection of two subspaces $S_1$ and $S_2$ in $\mathbb{P}(V)$ corresponds to the direct sum of their corresponding subspaces $\phi(S_1)$ and $\phi(S_2)$ in $\mathbb{P}(V^*)$.

**Example 2.8.** Let us find the dual of linear space

$$L=\{x_0-x_1=0, x_2-2x_4=0\}\in\mathbb{P}^4$$

then we want to show that for any $p\in\mathbb{P}^4\backslash L$, $\exists$ a unique hyperplane of $\mathbb{P}^4$ that one contains $L$ and passes through $p$. In the end, we will find the equation for one passing through $p=[1:2:1:3:0]$.

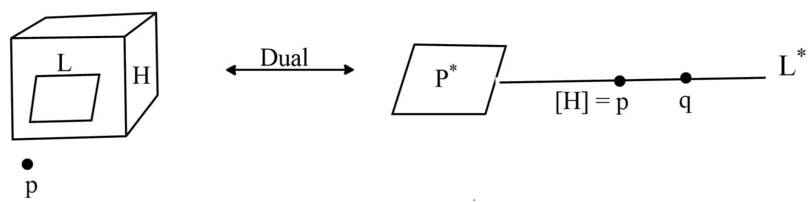
To find the dual of the linear space in $\mathbb{P}^4$

$$L = \begin{cases} x_0 - x_1 = 0 \\ x_2 - 2x_4 = 0 \end{cases}$$

$L$ is a plane in $\mathbb{P}^4$. The dimension of the dual of $L$ i.e. $L^*$ is

$$dim L^* = 4 - 1 - 2 = 1$$

To find this $L^*$, see that




Thus, dual of $L$ i.e is the point corresponding to hyperplane that contains $L$ (since the duality correspondence is inclusion reversing) i.e.

$$L^* = \{[H] = p \in (\mathbb{P}^4)^* | \, H \text{ hyperplane in } \mathbb{P}^4 \text{ such that } L \subseteq H\}$$

such that we have to find $p$ and $q$

$$L^* = < p, q >$$

where $p = [1 : -1 : 0 : 0 : 0]$ and $q = [0 : 0 : 1 : 0 : -2]$
To find this line $L^*$, consider a point $y = [y_0 : y_1 : y_2 : y_3 : y_4]$ belong to line passing through $p$ and $q$.
Consider

$$A = \begin{bmatrix} 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & -2 \\ y_0 & y_1 & y_2 & y_3 & y_4 \end{bmatrix}$$

We want to impose that $rank(A) = 2$

$$\iff detA = 0$$

That is,

$$\det \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \\ y_0 & y_1 & y_2 \end{bmatrix} = 0 \Rightarrow y_0 + y_1 = 0$$

$$\det \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ y_1 & y_2 & y_3 \end{bmatrix} = 0 \Rightarrow y_3 = 0$$

and

$$\det \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & -2 \\ y_1 & y_2 & y_4 \end{bmatrix} = 0 \Rightarrow 2y_2 + y_4 = 0$$

Thus,

$$L^* = \begin{cases} y_0 + y_1 = 0 \\ y_3 = 0 \\ 2y_2 + y_4 = 0 \end{cases}$$

is the required equation of line $L^*$. Now, for the second part of the example, for our convenience firstly we will translate the statement into dual i.e."$\forall p \in \mathbb{P}^4 \backslash L$ where $p = [1 : 2 : 1 : 3 : 0]$ $\exists$ a unique hyperplane of $\mathbb{P}^4$ that contains $L$ and passes through $p$"
The dual statement of the above statement is:

"For any hyperplane $P^* = \{y_0 + 2y_1 + y_2 + 3y_3\}$ such that $L^* \not\subset P^*$, show that $\exists$ unique point $q'$ such that $q' \in L^*$ and $q' \in P^* \leftrightarrow L^* \cap P^* = \{q'\}$"
i.e. we want to show that $L^* \cap P^* \neq \phi$ and $dim(L^* \cap P^*) = 0$. To show this, using the Grassmanian formula:

$$1 \geq dim(L^* \cap P^*) \geq dimP^* + dimL^* - dim(\mathbb{P}^4)$$

$$= 3 + 1 - 4 = 0$$

i.e

$$1 \geq dim(L^* \cap P^*) \geq 0$$

It means that for sure, $L^* \cap P^* \neq \phi$ and if $dim(L^* \cap P^*) = 1$

$\Rightarrow L^* \subseteq P^*$ which contradicts the assumption that $L^* \not\subset P^*$. Thus,

$$dim(L^* \cap P^*) = 0$$

i.e $L^*$ and $P^*$ intersect in a unique point $q'$.

To find this $q$, we will solve;

$$L^* \cap P^* = \begin{cases} y_0 + y_1 = 0 \\ y_3 = 0 \\ 2y_2 + y_4 = 0 \\ y_0 + 2y_1 + y_2 + 3y_3 = 0 \end{cases}$$

Solving this we see that

$$y_3 = 0$$

$$y_0 = -y_1$$

$$y_1 = -y_2 \quad \text{and} \quad y_4 = 2y_1$$

$$\text{i.e if } y_0 = 1 \Rightarrow y_1 = -1 \text{ and } y_4 = -2$$

i.e

$$q' = [1 : -1 : 1 : 0 : -2]$$

Thus, the required equation of hyperplane corresponding to this $q'$ is

$$\{x_0 - x_1 + x_2 - 2x_4 = 0\}$$

For the third part of the exercise, i.e to find the equation for one passing through

$$P = [1,2,1,3,0]$$

i.e It is simply;

$$P^* = \{y_0 + 2y_1 + y_2 + 3y_3 = 0\}$$

**Example 2.9.** we want to find the dual statement to
"Given any two points in $\mathbb{P}^2$, $\exists$ a unique line passing through them"
So,
Its dual statement is
"Given two points on a line in $\mathbb{P}^2$, we obtain by duality; two concurrent lines"

that is:

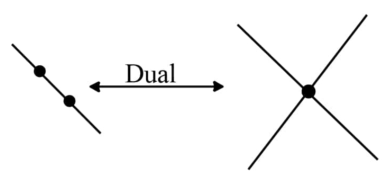



Figure 2.1: Two Points in $\mathbb{P}^2$ Two Concurrent Lines

Indeed, take into account the two points $A$ and $B$ in $\mathbb{P}^2$. Each point in the dual projective space, indicated by the symbols $A^*$ and $B^*$, corresponds to a line. The lines $A^*$ and $B^*$ in the dual projective space that corresponds to the separate points $A$ and $B$ are also distinct. Let's now think about the point where $A^*$ and $B^*$ connect. A common point must be found where the separate lines $A^*$ and $B^*$ intersect because they are not parallel. A point in the projective space $\mathbb{P}^2$ corresponds to this common point of intersection. This point will be designated as $C$. As a result, we have demonstrated that the lines that correspond to $A$ and $B$ in the dual projective space become concurrent at point $C$ in $\mathbb{P}^2$.

# Chapter 3

# Quadrics and Conics

The projective geometry of quadrics is the geometric representation of linear algebra that deals with symmetric bilinear forms. We recall:

## 3.1 Quadratic Forms

**Definition 3.1.1.** Let V be an $(n+1)$-dimensional vector space over a field $\mathbb{K}$. A **Symmetric bilinear form** on $V$ is a map $B : V \times V \to \mathbb{K}$ such that:

- $B(v,w) = B(w,v)$ i.e $B$ is symmetric.
- $B(\lambda_1 v_1 + \lambda_2 v_2, w) = \lambda_1 B(v_1,w) + \lambda_2 B(v_2,w)$ i.e $B$ is linear.

This form is **non-degenerate** if for all $v_2 \neq 0$ the form $B(v_1,v_2) = 0$ gives that $v_1 = 0$ for all $v_1, v_2 \in V$.

If we pick a basis for $V$ i.e $B = \{v_0, v_q, \ldots, v_n\}$, then $v = \sum_{i=0}^{n} x_i v_i$ and $w = \sum_{j=0}^{n} y_j v_j$, we have that

$$B = (v,w) = v = \sum_{i,j=0}^{n} B(v_i,v_j) x_i y_j$$

that is $B$ is uniquely determined by the symmetric matrix

$$mat_B B := [B_{ij}] = [B(v_i,v_j)]$$

**Remark 3.** *The symmetric bilinear form can be added and multiplied by a scaler, that is;*

$$(B_1 + B_2)(v,w) = B_1(v,w) + B_2(v,w)$$

$$(\mu B)(v,w) = \mu B(v,w)$$

*Thus, it creates a vector space which is equivalent to the space of $n \times n$ symmetric matrices.*

**Definition 3.1.2.** Let $g =< .,. >$ be a scaler product on $V$. By **Quadratic form** determined by $g$ mean a map $Q : V \to \mathbb{K}$ such that :

$$Q(v) = g(v,v) =< v,v >$$

The quadratic form associated with the symmetric bilinear form determines it completely i.e

$$Q(v) = B(v,v)$$

with specifying the condition that $char\mathbb{K} \neq 2$.
Indeed, as

$$\begin{aligned} Q(u+v) &= B(u+v,u+v) \\ &= 2B(u,v) + B(u,u) + B(v,v) \\ &= 2B(u,v) + Q(u) + Q(v) \\ \implies B(u,v) &= \frac{1}{2}(Q(u+v) - Q(u) - Q(v)) \end{aligned}$$

Therefore, the quadratic form associated with the symmetric bilinear form can be used to uniquely identify it, where $char\mathbb{K} \neq 2$.

**Theorem 3.1.1.** ***(Sylvester's Theorem)*** *Let $\mathbb{F}$ be a field and $V$ be a vector space of dimension $m$ over the field $\mathbb{F}$. Consider that $B$ be a quadratic form on $V$, then*

- *If $\mathbb{F} = \mathbb{C}$, there exist basis such that if $v = \sum_i z_i v_i$*

$$B(v,v) = \sum_{i=1}^{n} {z_i}^2$$

*where $n$ is the rank of $B$.(that diagonal matrix associated to quadratic form)*

- *If $\mathbb{F} = \mathbb{R}$, there exist basis such that*

$$B(v,v) = \sum_{i=1}^{g} {z_i}^2 - \sum_{i=j}^{h} {z_j}^2$$

*If B is non-degenrate then $m = n = g + h = rankB$. [1][Theorem 10 from notes of 1st chapter of Nigel Hitchin]*

Here, $(g,h)$, which is the number of positive term and negative term sequences, is called the signature.

**Example 3.1.** Consider the quadratic form in $\mathbb{R}^3$ having basis $\{v_1.v_2,v_3\}$ such that

$$v = \sum_{i=1}^{3} x_i v_i$$

$$Q(v) = x_1x_2 + x_2x_3 + x_3x_1$$

Thanks to a change of coordinates, we put $y_1 = (x_1+x_2)/2$, $y_2 = (x_1-x_2)/2$ to get

$$Q(v) = y_1^2 - y_2^2 + x_3(2y_1)$$

Completing the square, we get

$$Q(v) = (y_1+x_3)^2 - y_2^2 - x_3^2$$

so that $z_1 = y_1 + x_3$, $z_2 = y_2$ and $z_3 = x_3$

$$\implies Q(v) = z_1^2 - z_2^2 - z_3^2$$

here, signature is $g = 1$, $h = 2$ and rank is $g + h = m = 3$.

## 3.2 The Conics and Quadrics

**Definition 3.2.1.** In a projective space, a **quadric** connected to $\mathbb{B}$, for a quadratic form $Q$, i.e.

$$Q_B = \{[u] \in \mathbb{P}(U) | B(u,u) = Q(u) = 0\}$$

is the set of the points $p = [u]$ in $\mathbb{P}(U)$ that fulfil the equation $Q(u) = 0$.

When B is not degenerate, the above-defined quadric is not singular. The quadric has the dimension dim$\mathbb{P}(V)$-1.

The quadric is non singular if $B$ is non degenrate. The dimension of the quadric is dim$\mathbb{P}(V)$-1.

**Definition 3.2.2.** A quadric in $\mathbb{P}^2$ (quadric of dimension 1) is called a **conic**.

A quadric in $\mathbb{P}^2$ with dimension 1 is known as a ”conic” quadric because of its relationship to a cone in three-dimensional space.

When a cone is thought of in three dimensions, the intersection of the cone and a plane can result in a variety of shapes known as conic sections. They consist of well-known curves like parabolas, circles, ellipses, and hyperbolas. These conic sections in three-dimensional space are extended to the projective plane $\mathbb{P}^2$ to form the concept of a conic in projective geometry. The locus of points that satisfy a specific quadratic equation in homogeneous coordinates is represented by a conic in $\mathbb{P}^2$.
Different conics are all equivalent in the projective space.[1][Example at page 26 of Hitchin notes chapter 3]

**Remark 2**

- $Q_B$ is well defined, since

$$B(\lambda v, \lambda v) = \lambda^2 B(v, v)$$

- If we pick a basis for $V$ i.e. $\{w_0, w_1, \ldots, w_n\}$ and fixing a system of homogeneous coordinate $[x_0, x_1, \ldots, x_n] \in \mathbb{P}^4$, we have

$$v = \sum_{i=0}^{n} x_i w_i$$

  then, $Q(v) = B(v, v) = v = \sum_{i,j} B(w_i, w_j) x_i x_j$
  since $A = [B(w_i, w_j)]$, so,

$$Q(v) = x^t A x$$

  Consequently, the quadric associated with $B$ will become

$$Q_B = \{[x_b mats : x_n] = x \in \mathbb{P}^4 | x^t A x = 0\}$$

**Example 3.2.** Let $V$ be a 3-dimensional vector space over $\mathbb{R}$ having basis

$$B = \left\{ e_0 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, e_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, e_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

and

$$A = [B(v_i, v_j)] = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Then the quadric associated to $B$ will be:

$$\begin{aligned} Q_B &= \left\{ [x_0 : x_1 : x_2] \in \mathbb{P}^2 \mid x^t A x = 0 \right\} \\ &= \left\{ \begin{bmatrix} x_0 & x_1 & x_2 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_0 \\ x_1 \\ x_2 \end{bmatrix} \right\} \\ &= \left\{ x_0^2 + 2x_0x_1 + 2x_1^2 = 0 \right\} \end{aligned}$$

## 3.3 Quadrics in $\mathbb{P}^1_{\mathbb{C}}$

Consider the complex projective line i.e. $\mathbb{P}^1_{\mathbb{C}}$ with the homogeneous coordinates $[x_0 : x_1]$ and quadratic form is given by

$$Q(x_0, x_1) = a{x_0}^2 + bx_0x_1 + c{x_1}^2$$

The quadric in $\mathbb{P}^1_{\mathbb{C}}$ is given by the geometric locus of $\{Q(x_0, x_1) = 0\}$.

To find this geometric locus, we have the following cases:

**Case 1:**

If $b=c=0$ i.e.

$$A=\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}$$

then

$$\{Q(x_0,x_1)=0\}=\{x_0^2=0\}=\{[0:1]\}$$

that is, one point with multiplicity 2.

**Case 2:**

If $b=0$ and $c\neq 0$ i.e.

$$A=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$$

then

$$\{Q(x_0,x_1)=0\}=\{x_0^2+x_1^2=0\}=\{[0:1],[1:0]\}$$

that is two distinct points.

So, to summarize, the quadrics in $\mathbb{P}^1_{\mathbb{C}}$ are either:

- One point with multiplicity 2 i.e. $\{x_0^2=0\}$ (rank 1)
- Two distinct points i.e. $\{x_0^2+x_1^2=0\}$ (rank 2, maximal rank)

### 3.3.1 Quadrics in $\mathbb{P}^1_{\mathbb{R}}$

Similarly, in case of quadrics of $\mathbb{P}^1_{\mathbb{R}}$, we have one more case i.e.

- $\{x_0^2=0\}$
- $\{x_0^2-x_1^2=0\}$
- $\{x_0^2+x_1^2=0\}$

## 3.4 Quadrics in $\mathbb{P}^2$: Conics

A conic in $\mathbb{P}^2$ with homogeneous coordinates $[x_0 : x_1 : x_2]$ has generic equation:

$$\{a_0x_0^2 + a_1x_1^2 + a_2x_2^2 + a_3x_0x_1 + a_4x_0x_2 + a_5x_1x_2 = 0\}$$

homogeneous polynomial of degree 2 in the variables $x_0, x_1, x_2$.
We can write it as:

$$\{Conics\} \longleftrightarrow \{[a_0 : a_1 : \cdots : a_5] \in \mathbb{P}^5\}$$

That is: "$\mathbb{P}^5$ is a parameter space for $\mathscr{C}$=all the conics in $\mathbb{P}^2$"

### 3.4.1 Projective Classification of Conics of $\mathbb{P}^2_{\mathbb{K}}$

1. Projectively, each conic in $\mathbb{P}^2_{\mathbb{C}}$ corresponds to one conic from the following list:

   - $\{x_0^2 + x_1^2 + x_2^2 = 0\}$ (rank 3, maximal rank)
   - $\{x_0^2 + x_1^2 = 0\}$ (rank 2)
   - $\{x_0^2 = 0\}$ (rank 1)

   This descends from the Sylvester theorem (3.1.1).
   Indeed, due to the fact that it divides conics into groups according to their rank, which is the dimension of the vector space spanned by the fundamental constituents of the quadratic form.

   The equation $x_0^2 + x_1^2 + x_2^2 = 0$ denotes a conic of rank 3 in this instance. This indicates that the corresponding quadratic form has a rank of 3 and is non-degenerate.

   The corresponding quadratic form is degenerate and has a rank of 2 according to the equation $x_0^2 + x_1^2 = 0$, which corresponds to a conic of rank 2.

   Last but not least, the equation $x_0^2 = 0$ denotes a conic of rank 1, indicating that the corresponding quadratic form is extremely degenerate with a rank of 1.

   Similarly,

2. Projectively, each conic in $\mathbb{P}^2_{\mathbb{R}}$ corresponds to one conic from the following list:

   - $\{x_0^2 + x_1^2 + x_2^2 = 0\}$
   - $\{x_0^2 + x_1^2 - x_2^2 = 0\}$
   - $\{x_0^2 + x_1^2 = 0\}$
   - $\{x_0^2 - x_1^2 = 0\}$
   - $\{x_0^2 = 0\}$

[2][Theorem 1.8.2 form the book of Projective geometry by Fortuna, E., Frigerio, R., & Pardini, R.]

At the end of this chapter, we can discuss an example to understand how the parameter space can help infer the properties of varieties.

**Example 3.3.** consider the example of finding a unique conic passing through five points in a general position. In projective space, the condition of a conic passing through a point can be reperesented as a hyperplane in $\mathbb{P}^5$. Each point corresponds to a hyperplane, and when five points are in general position, their corresponding hyperplanes intersect at a unique point.

Imagine that we are trying to locate a conic that passes through all five of our projective space points. Each point has a corresponding hyperplane, which denotes the state of the conic flowing over it. These five hyperplanes now come together at a single place when we intersect them. The conic that crosses through all five points is symbolized by this special point.

This example shows how we can deduce attributes of variety by using the parameter space, in this case, the hyperplanes. We may ascertain the existence and uniqueness of the required conic by looking at the intersection of these hyperplanes. We can examine and analyze the conic's attributes as it passes through the specified places using the hyperplanes as parameters.

# Chapter 4

# Exterior Algebra and the Klein Quadric

In order to handle the space of all the lines in a projective space $\mathbb{P}(U)$, new notions must now be introduced. Lines found in the spaces which have high dimensions behave differently from those in the projective plane, as we have seen duality is capable of handling. We study the linear algebraic properties of the two-dimensional vector subspaces $W \subset U$.
To guide our work, think about how we define a subspace of $\mathbb{R}^3$ which has dimension 2, in Euclidean geometry. The vector cross product of two vectors in the space, $u$ and $w$, which do not depend on each other, that are linearly independent, could be described by utilising its standard normal, $n$, having magnitude 1 and in parallel with $u \times w$. Following are the properties of vector product:

- $v \times w = -w \times v$
- $(\mu_1 v_1 + \mu_2 v_2) \times w = \mu_1 v_1 \times w + \mu_2 v_2 \times w$

We will generalise these features to vectors in any vector space $V$; nevertheless, the product will not be a vector in $V$; it will be a vector in another vector space.

**Definition 4.0.1.** A map $B : V \times V \to \mathbb{F}$ is an **alternating bilinear form** on a vector space $V$ such that it satisfy the following:

- $B(u,v) = -B(v,u)$
- $B(\mu_1 u_1 + \mu_2 u_2, v) = \mu_1 B(u_1,v) + \mu_2 B(u_2,v)$

This represents the skew-symmetric version of a symmetric bilinear form that we used to define the quadrics. The skew symmetric matrix $B(v_i,v_j)$ determines $B$ uniquely, given a basis $\{v_1,\dots,v_n\}$.To create a vector space that is isomorphic to the space of skew-symmetric $n \times n$ matrices, the addition of alternating forms and the multiplication of scalars are possible. This vector space is spanned by the basis elements $E^{pq}$ for $p < q$ and has dimension $n(n-1)/2$.

## 4.1 Second Exterior Power

**Definition 4.1.1.** On a vector space $U$ which has a finite dimension, the **second external power**, represented by the symbol $\Lambda^2 U$, is the dual space of the above-defined alternating bilinear form's vector space on $U$.

The elements of second exterior power are called **2-vectors** or **bivectors**. Using this space as our starting point, now, we can generalize the usual cross-product, sometimes referred to as the external product or also known as the wedge product of given two vectors.

**Definition 4.1.2.** Let $v$ and $w$ be elements of vector space $V$. The **exterior product** $v \wedge w \in \Lambda^2 V$ is the linear map that assigns a value to $F$ for an alternating bilinear form $B$, as follows:

$$(v \wedge w)(B) = B(v,w)$$

### 4.1.1 Fundamental Properties of Exterior Product

- $(v \wedge w)(B) = B(v,w) = -B(w,v) = -(w \wedge v)(B)$ so that

$$w \wedge v = -v \wedge w$$

- Specifically, $v \wedge v = 0$ as by using above first property;
  $(v \wedge v)(B) = B(v,v) = -B(v,v) = 0$

- $((\mu_1 v_1 + \mu_2 v_2) \wedge w)(B) = B(\mu_1 v_1 + \mu_2 v_2, w) = \mu_1 B(v_1,w) + \mu_2 B(v_2,w)$ which implies

$$(\mu_1 v_1 + \mu_2 v_2) \wedge w = \mu_1 v_1 \wedge w + \mu_2 v_2 \wedge w.$$

- If $\{u_1,...,u_n\}$ is a basis set for vector space $V$ then the set of all possible products of two elements from the set $\{u_1,...,u_n\}$, i.e. $u_m \wedge u_n$ for $m < n$, forms a basis set for the vector space of all 2-linear forms on V i.e. for $\Lambda^2 V$.

**Proposition 4.1.** *For any non-zero vector $v \in V$, the exterior product of $v$ and $w$ is zero if and only if $w$ can be expressed as the scalar multiplication of $v$ by some scalar $\mu$ i.e $w = \mu v$.*

*Proof.* Consider the case if $w = \mu v$ is given, then

$$v \wedge w = v \wedge (\mu v) = \mu(v \wedge v) = \mu(0) = 0$$

For the converse,
we will prove this part using a contrapositive statement. i.e if $w \neq \mu v$ which means that $w$ and $v$ are linearly independent and so extendable to the basis of $V$ so then $v \wedge w$ is a basis vector and therefore non zero. □

We will focus on the elements of $\Lambda^2 V$ expressed as $v \wedge w$. This is of particular interest when considering a 2-dimensional vector subspace $W \subset V$, where $\{v, w\}$ forms a basis for $W$. Any other basis can be represented as $\{pv + qw, rv + sw\}$. We can derive the following, by utilizing the properties $v \wedge w = v \wedge w = 0$.

$$(pv + qw) \wedge (rv + sw) = (ps - qr) v \wedge w$$

The matrix

$$\begin{bmatrix} p & q \\ r & s \end{bmatrix}$$

must be invertible i.e. $ps - qr \neq 0$ because if it is not invertible i.e. $ps - qr = o$ then the above wedge product is zero and by proposition 4.1 the vectors are linearly dependent but they cannot be, because they are basis elements.
As a consequence of the matrix being invertible, the 1-dimensional subspace of $\Lambda^2 V$, which is spanned by $v \wedge w$ and serves as a basis for $W$, is uniquely determined by $W$ itself. This determination remains consistent regardless of the chosen basis. Consequently, we can associate a point in $\mathbb{P}(\Lambda^2 V)$ with each line in $\mathbb{P}(V)$.

Now, we will generalize the concept of second exterior power and exterior product to higher exterior powers and $q-$th exterior product respectively. Before generalizing this concept, we will define alternating multilinear form as following:

**Definition 4.1.3.** A map $M : V \times \cdots \times V \to \mathbb{F}$ is an **alternating multilinear form** which has degree $q$ on a given vector space $V$ such that it satisfy the following:

- $M(v_1, \ldots, v_i, \ldots, v_j, \ldots, v_q) = -M(v_1, \ldots, v_j, \ldots, v_i, \ldots, v_q)$
- $M(\mu_1 v_1 + \mu_2 v_2, w_2, \ldots, w_q) = \mu_1 M(v_1, w_2, \ldots, w_q) + \mu_2 B(v_2, w_2, \ldots, w_q)$

## 4.2 Higher Exterior Powers

**Definition 4.2.1.** On a vector space $V$ which has finite dimension, the $q$**-th exterior power**, represented by $\Lambda^q V$, is the dual space of the above-defined alternating multilinear form's vector space on the $V$.

The elements of second exterior power are called $q$**-vectors**.

**Definition 4.2.2.** Let $v_1, v_2, \ldots, v_q \in V$. The **exterior product** $v_1 \wedge v_2 \wedge \cdots \wedge v_q \in \Lambda^q V$ is the linear map that assigns a value to $F$ for an alternating multilinear form $M$, as follows:

$$(v_1 \wedge v_2 \wedge \cdots \wedge v_q)(M) = M(v_1, v_2, \ldots, v_q)$$

As we have stated the fundamental properties of exterior products, so also $q$-th exterior power has:

- It demonstrates linearity with respect to each variable $v_i$ individually.
- Swapping two variables results in a change of sign for the exterior product.
- When two variables are the same, the above-defined exterior product equals zero.

Proposition 4.1 can be generalized in the following useful manner:

**Proposition 4.2.** *The $q$ vectors $v_i \in V$ form an exterior product $v_1 \wedge v_2 \wedge \cdots \wedge v_q$ that equals zero iff one vector can be written as a linear combination of the other vectors i.e. independent vectors.*

*Proof.* Firstly, we will suppose that the vectors are linearly dependent and will show that their exterior product vanishes. So, if the vectors are linearly dependent then for some scaler $\mu_i \neq 0$ we can write them as

$$\mu_1 v_1 + \cdots + \mu_q v_q = 0$$

Then, by the property of linear dependence, $v_i$ can be written as

$$v_i = \sum_{i \neq j} \eta_j v_j$$

and so the exterior product will become

$$v_1 \wedge v_2 \wedge \cdots \wedge v_q = v_1 \wedge v_2 \wedge (\sum_{i \neq j} \eta_j v_j) \wedge v_{i+1} \cdots \wedge v_q$$

Expanding this expression using linearity, every term that contains a repeated variable $v_j$ will vanish.

For the converse,

we will prove this part using a contrapositive statement. i.e. if $v_1, \ldots, v_q$ are linearly independent, they can be augmented to form a basis. Consequently, $v_1 \wedge v_2 \wedge \ldots \wedge v_q$ becomes a basis vector for $\Lambda^q V$ and therefore remains non-zero. □

**Basic characteristics of general exterior product**

Let $V$ be the vector space, $v_1, v_2, v_3 \in V$ and $\Lambda^p V$ and $\Lambda^q V$ be the $p$-th and $q$-th exterior powers respectively. We have the following main properties:

- $v_1 \wedge (v_2 + v_3) = v_1 \wedge v_2 + v_1 \wedge v_3$
- $(v_1 \wedge v_2) \wedge v_3 = v_1 \wedge (v_2 \wedge v_3)$
- If $v_1 \in \Lambda^p V$ and $v_2 \in \Lambda^q V$ then $v_1 \wedge v_2 = (-1)^{pq} v_2 \wedge v_1$

**Example 4.1.** Given the following $a, b$ and $v_1, v_2, v_3, v_4 \in V$ are linearly independent, we will calculate the exterior product of $a$ and $b$ that is $a \wedge b$:

- $a = v_1 \wedge v_2 + v_3 \wedge v_1$ ; $\quad b = v_2 \wedge v_3 \wedge v_4$

$$\begin{aligned} a \wedge b &= (v_1 \wedge v_2 + v_3 \wedge v_1) \wedge (v_2 \wedge v_3 \wedge v_4) \\ &= v_1 \wedge v_2 \wedge v_2 \wedge v_3 \wedge v_4 + v_3 \wedge v_1 \wedge v_2 \wedge v_3 \wedge v_4 \\ &= 0 \qquad \text{(because of repeated factor } v_i\text{)} \end{aligned}$$

- $a = v_1 + v_2 + v_3$ ; $\quad b = v_1 \wedge v_2 + v_2 \wedge v_3 + v_3 \wedge v_1$

$$\begin{aligned} a \wedge b &= (v_1 + v_2 + v_3) \wedge (v_1 \wedge v_2 + v_2 \wedge v_3 + v_3 \wedge v_1) \\ &= v_1 \wedge v_2 \wedge v_3 + v_2 \wedge v_3 \wedge v_1 + v_3 \wedge v_1 \wedge v_2 \quad \text{(rest of terms became zero again by repeated } v_i\text{)} \\ &= v_1 \wedge v_2 \wedge v_3 + v_1 \wedge v_2 \wedge v_3 + v_1 \wedge v_2 \wedge v_3 \qquad \text{(using properties of exterior product)} \\ &= 3v_1 \wedge v_2 \wedge v_3 \end{aligned}$$

## 4.3 Decomposable 2-Vectors

**Definition 4.3.1.** An element in $\Lambda^2 V$ is **decomposable** if it can be written as $v \wedge w$ for $v, w \in V$.

A point in the projective space $\mathbb{P}(\Lambda^2 V)$ is determined by a decomposable 2-vector $p = v \wedge w$, where the line in $\mathbb{P}(V)$ corresponds to that point.
To algebraically describe the decomposability, we can utilize the following theorem, which precisely accomplishes this task.

**Theorem 4.1.** *If we consider a non-zero element $p \in \Lambda^2 V$, it can be decomposed if and only if $p \wedge p = 0 \in \Lambda^4 V$.*

*Proof.* Firstly, we will assume that $p$ is decomposable and will prove that $p \wedge p = 0$.
Since $p$ is decomposable so $p = x \wedge y$, where $x$ and $y$ are two vectors, the expression

$$p \wedge p = x \wedge y \wedge x \wedge y$$

equals zero due to the presence of the repeated factor x (or y). That is

$\Longrightarrow p \wedge p = 0.$

To establish the converse, we utilize an inductive proof based on the dimension of $V$. We begin by considering the cases where $dimV = 0$ or 1, which results in $\Lambda^2 V = 0$ and $p \in \Lambda^2 V$ but the $p$ we are considering should be the non-zero element of $p \in \Lambda^2 V$. Thus, we focus on the case where $dimV = 2$.

When $dimV = 2$, we have $dim\Lambda^2 V = 1$. If $v_1$ and $v_2$ form a basis for $V$, then $v_1 \wedge v_2$ represents a non-zero element. Consequently, any $p = v_1 \wedge v_2$ in this scenario is decomposable.

Now let's focus specifically on the case where the dimension of $V$ is 3. In this case, given a non-zero element "p" belonging to $\Lambda^2 V$, we can define a mapping $P : V \to \Lambda^3 V$ as follows:

$$P(u) = p \wedge u$$

for $u \in V$ and $p \in \Lambda^2 V$. Since the dimension of $\Lambda^3 V$ is 1, so by rank-nullity theorem of linear algebra, $dim(KerP) \geq 2$, therefore, we can select two linearly independent vectors $v_1$ and $v_2$ from the kernel and expand them to form a basis $v_1$, $v_2$, $v_3$ for $V$. With this basis, we can express the situation as follows:

$$p = \mu_1 v_2 \wedge v_3 + \mu_2 v_3 \wedge v_1 + \mu_3 v_1 \wedge v_2. \tag{4.3.1}$$

By definition, now $p \wedge p = 0$, so we can take

$$0 = p \wedge v_1 = (\mu_1 v_2 \wedge v_3 + \mu_2 v_3 \wedge v_1 + \mu_3 v_1 \wedge v_2) \wedge v_1$$

$$0 = \mu_1 v_2 \wedge v_3 \wedge v_1$$

$$0 = \mu_1$$

Similarly,

$$0 = p \wedge v_2 = (\mu_1 v_2 \wedge v_3 + \mu_2 v_3 \wedge v_1 + \mu_3 v_1 \wedge v_2) \wedge v_2$$
$$0 = \mu_2 v_3 \wedge v_1 \wedge v_2$$
$$0 = \mu_2$$

Consequently, it can be deduced that

$$p = \mu_3 v_1 \wedge v_2$$

which can be decomposed.

Assume, based on the principle of induction, that the above-stated theorem is valid for vector spaces having dimensions less than equal to $n-1$ dimensions and suppose that the scenario where the dimension of $dimV = n$. By utilizing the basis $u_1, \ldots, u_n$ of $V$, we can express $p$ in the following manner:

$$\begin{aligned} p &= \sum_{1\leq i\leq j}^{n} p_{ij} u_i \wedge u_j \\ &= (\sum_{i=1}^{n-1} p_{in} u_i) \wedge u_n + \sum_{1\leq i\leq j}^{n-1} p_{ij} u_i \wedge u_j \\ &= v \wedge u_n + p' \end{aligned} \tag{4.3.2}$$

where $v \in U$, $p' \in \Lambda^2 U$ for $U$ be $n-1$ dimensional space which is spanned by $u_1, \ldots, u_{n-1}$. Now, by using the given condition;

$$0 = p \wedge p = (v \wedge u_n + p') \wedge (v \wedge u_n + p') = 2v \wedge p' \wedge u_n + p' \wedge p'$$

However, $u_n$ does not appear in the expansion of $v \wedge p'$ or $p' \wedge p'$. As a result, we obtain the following separately:

$$v \wedge p' = 0 \tag{4.3.3}$$

and

$$p' \wedge p' = 0$$

By the inductive step, $p' \wedge p' = 0$ implies that $p'$ is decomposable that is $p' = v_1 \wedge v_2$. Put this value of $p'$ in equation (4.3.3),

$$v \wedge v_1 \wedge v_2 = 0$$

By proposition 4.2. above equation implies that

$$\mu v + \lambda_1 v_1 + \lambda_2 v_2 = 0 \tag{4.3.4}$$

Now we have two cases

Case-1 (if $\mu = 0$):

Then equation (4.3.4) implies that $v_1$ and $v_2$ are linearly dependent so by proposition 4.1. $p' = v_1 \wedge v_2 = 0$. Putting back this value of $p'$ in (4.3.2), we get

$$p = v \wedge u_n$$

So, $p$ is decomposable in this case.

Case-2 (if $\mu \neq 0$):

Then equation (4.3.4) implies that $v = \mu_1 v_1 + \mu_2 v_2$. So, by (4.3.2);

$$\begin{aligned} p &= (\mu_1 v_1 + \mu_2 v_2) \wedge u_n + P' \\ &= (\mu_1 v_1 + \mu_2 v_2) \wedge u_n + v_1 \wedge v_2 \\ &= \mu_1 v_1 \wedge u_n + \mu_2 v_2 \wedge u_n + v_1 \wedge v_2 \end{aligned}$$

and as demonstrated earlier, this represents the three-dimensional scenario, which is always decomposable.

Therefore, we can deduce that $p$ is decomposable in every case.

□

**Example 4.2.** Given the following 2-vectors, with the help of the above theorem, we will check which of them is decomposable:

- $a = v_1 \wedge v_2 + v_2 \wedge v_3 + v_3 \wedge v_4$

$$\begin{aligned} a \wedge a &= (v_1 \wedge v_2 + v_2 \wedge v_3 + v_3 \wedge v_4) \wedge (v_1 \wedge v_2 + v_2 \wedge v_3 + v_3 \wedge v_4) \\ &= v_1 \wedge v_2 \wedge v_3 \wedge v_4 + v_1 \wedge v_2 \wedge v_3 \wedge v_4 \\ &= 2v_1 \wedge v_2 \wedge v_3 \wedge v_4 \\ &\neq 0 \end{aligned}$$

Since, $a \wedge a \neq 0$, so by theorem (4.1), $a$ is not decomposable.

- $b = v_1 \wedge v_2 + v_2 \wedge v_3 + v_3 \wedge v_4 + v_4 \wedge v_1$

$$\begin{aligned} b \wedge b &= (v_1 \wedge v_2 + v_2 \wedge v_3 + v_3 \wedge v_4 + v_4 \wedge v_1) \wedge (v_1 \wedge v_2 + v_2 \wedge v_3 + v_3 \wedge v_4 + v_4 \wedge v_1) \\ &= v_1 \wedge v_2 \wedge v_3 \wedge v_4 - v_1 \wedge v_2 \wedge v_3 \wedge v_4 + v_1 \wedge v_2 \wedge v_3 \wedge v_4 - v_1 \wedge v_2 \wedge v_3 \wedge v_4 \\ &= 0 \end{aligned}$$

Since, $b \wedge b = 0$, so by theorem (4.1), $b$ is decomposable.

## 4.4 Motivation: the Importance of the Klein Quadric

The dual projective plane, which acts as a parameter space for lines in $\mathbb{P}^2$ (two-dimensional projective space), is where the concept of parametrizing lines in projective space first emerges. In other words, if we consider a line in $\mathbb{P}^2$, it is parameterized by a point in the dual projective plane $(\mathbb{P}^2)^*$. However, the dual projective plane is no longer sufficient to parametrize lines when we move to a higher-dimensional space, notably $\mathbb{P}^3$ (three-dimensional projective space). We need to propose the Klein quadric as an alternative. We can easily define and parameterize lines in $mathbbP3$ using the Klein quadric.

This urge to comprehend and categorize geometric objects, particularly lines, in three-dimensional projective space is the driving force behind the Klein quadric. These lines are represented geometrically by the Klein quadric, which enables a better comprehension of their characteristics.

The motivation is related to the decomposable 2-vectors in the preceding theorem (4.1), which provides context. An element of the exterior algebra $\Lambda^2 V$, where $V$ is a vector space, is a decomposable 2-vector. The wedge product of two vectors $v \wedge w$, where $v$ and $w$ are components of $V$, can be used to express it. The quadric equation $a \wedge a = 0$ holds true for any decomposable 2-vector, according to the theorem.

This theorem explains how quadric equations and 2-vectors are related. One way to think of the quadratic form $awedgea = 0$ is as a geometric requirement that defines a quadric. It depicts a group of projective space points that are solutions to this quadratic equation.

One particular quadric that results from this link is the Klein quadric. The Klein quadric is actually a quadric in the projectivization of $\Lambda^2 V$, or $\mathbb{P}(\Lambda^2 V)$, where $\Lambda^2 V$ is the same as the exterior algebra of the provided vector space $V$ as previously stated. The locus of points in $\mathbb{P}(\Lambda^2 V)$ that fulfil the quadratic equation $a \wedge a = 0$, where $a$ denotes a decomposable 2-vector, is known as the Klein quadric.

## 4.5 The Klein Quadric

The Klein quadric arises as a natural construction when studying lines in three-dimensional projective space, $\mathbb{P}^3(V)$, where $V$ is a four-dimensional vector space. By considering the exterior algebra $\Lambda^2 V$, which has dimension one in this case, we can represent projective lines as decomposable 2-vectors, such as $a = x \wedge y$, where $x$ and $y$ are vectors in $V$.

For any generic element $a \in \Lambda^2 V$, we write

$$a = \mu_1 v_0 \wedge v_1 + \mu_2 v_0 \wedge v_2 + \mu_3 v_0 \wedge v_3 + \eta_1 v_2 \wedge v_3 + \eta_2 v_1 \wedge v_3 + \eta_3 v_1 \wedge v_2$$

and then we apply the theorem if we want to characterise the decomposable one;

$$\begin{aligned}
a\wedge a &= (\mu_1 v_0\wedge v_1+\mu_2 v_0\wedge v_2+\mu_3 v_0\wedge v_3+\eta_1 v_2\wedge v_3+\eta_2 v_1\wedge v_3+\eta_3 v_1\wedge v_2)\wedge \\
&\quad (\mu_1 v_0\wedge v_1+\mu_2 v_0\wedge v_2+\mu_3 v_0\wedge v_3+\eta_1 v_2\wedge v_3+\eta_2 v_1\wedge v_3+\eta_3 v_1\wedge v_2) \\
&= \mu_1\eta_1(v_0\wedge v_1\wedge v_2\wedge v_3)+\mu_2\eta_2(v_0\wedge v_2\wedge v_1\wedge v_3)+\mu_3\eta_3(v_0\wedge v_3\wedge v_1\wedge v_2)+ \\
&\quad \mu_1\eta_1(v_2\wedge v_3\wedge v_0\wedge v_1)+\mu_2\eta_2(v_1\wedge v_3\wedge v_0\wedge v_2)+\mu_3\eta_3(v_1\wedge v_2\wedge v_0\wedge v_3) \\
&= 2(\mu_1\eta_1-\mu_2\eta_2+\mu_3\eta_3)v_0\wedge v_1\wedge v_2\wedge v_3 \\
&= B(a,a)v_0\wedge v_1\wedge v_2\wedge v_3 \qquad (4.5.1)
\end{aligned}$$

where

$$B(a,a)=2(\mu_1\eta_1-\mu_2\eta_2+\mu_3\eta_3) \qquad (4.5.2)$$

Since $a$ is decomposable, so $B(a,a)$ is a quadratic form which is non-degenerate and therefore, by the above theorem, a quadric which is non-singular is defined by $B(a,a)=0$, denoted as $Q$, in the projective space $\mathbb{P}(\Lambda^2 V)$ that is $Q\subset\mathbb{P}(\Lambda^2 V)$. The quadric $Q$ is well-defined, independent of the choice of basis, and serves as a geometric object parametrizing the projective lines in $\mathbb{P}(V)$.

Felix Klein, motivated by his supervisor's work named Julius Plücker, provided a comprehensive description of this concept. As a result, the quadric $\mathbf{Q}$ is commonly referred to as the notion of our main topic, the **Klein quadric**. The (4.5.2) equation, of the described quadric, illustrates that it contains linear subspaces of maximum dimension 2, regardless of the field considered.
In short, since $Q\subset\mathbb{P}(\Lambda^2 V)$ which is isomorphic to $\mathbb{P}^5$, so $Q$, the Klein quadric is actually the solution of quadratic equation in a 5-dimensional projective space $\mathbb{P}^5$.

Now, we will move toward one of the main properties of the Klein quadric that is;

**Proposition 4.3.** *There exists a one-to-one correspondence between lines L in a 3-dimensional* *projective space $\mathbb{P}(V)$ and points P in the 4-dimensional quadric $Q \subset \mathbb{P}(\Lambda^2 V)$.*

*Proof.* To prove the correspondence between points inside the quadric $Q$ subset of $\mathbb{P}(\Lambda^2 V)$ of dimension 4 and lines inside in a projective space of vector space $V$ i.e. $\mathbb{P}(V)$ of dimension three, we can use the concept of decomposable 2-vectors.

Let's start with a line $L$ in $\mathbb{P}(V)$. By Theorem (4.1), we know that every line can be represented by a decomposable 2-vector. So, we can write the line $L$ as $L = x \wedge y$, where $x$ and $y$ are vectors in $V$. Now, consider the point $P = (x \wedge y) \wedge (x \wedge y)$ in $Q$. This point $P$ lies on the quadric $Q$ because the wedge product of a decomposable 2-vector with itself is always zero. In other words, $(x \wedge y) \wedge (x \wedge y) = 0$. So, the line $L$ in $\mathbb{P}(V)$ corresponds a point $P$ in $Q \subset \mathbb{P}(\Lambda^2 V)$.

For the converse part, it is proven straightforward by the statement of theorem (4.1), which says that there is a decomposable 2-vector that is used to represent each point on the Klein quadric $Q$.

So, it is proved the correspondence between points inside the quadric $Q$ subset of $\mathbb{P}(\Lambda^2 V)$ of dimension four and lines inside in a projective space of vector space $V$ of dimension three. □

Now, we have a few more properties of the Klein quadric in the form of the following propositions:

**Proposition 4.4.** *Two lines $L_1$ and $L_2$ intersect in the projective space of vector space V iff the line that connects their respective points $P_1$ and $P_2$ in the Klein quadric Q lies entirely within Q.*

*Proof.* Let $W_1$ and $W_2$ be the two-dimensional subspaces of $V$ defined by the lines $L_1$ and $L_2$, respectively. Suppose the lines intersect at a point $K$, represented by the vector $v \in V$. We can extend $v$ to bases $\{v, v_1\}$ for $W_1$ and $\{v, v_2\}$ for $W_2$. The line in $\mathbb{P}(\Lambda^2 V)$ joining $P_1$ and $P_2$ corresponds to the subspace spanned by $v \wedge v_1$ and $v \wedge v_2$.

Any 2-vector in this subspace can be written as

$$\mu_1 v \wedge v_1 + \mu_2 v \wedge v_2 = v \wedge (\mu_1 v_1 + \mu_2 v_2)$$

which is a decomposable 2-vector and symbolizes a point in $Q$.

On the other hand, if the taken lines $L_1$ and $L_2$ do not meet, then their intersection $W_1 \cap W_2 = 0$ that is the zero subspace, indicating that $V = W_1 \oplus W_2$. In this case, we can choose bases $\{v_1, u_1\}$ for $W_1$ and $\{v_2, u_2\}$ for $W_2$. Then $\{v_1, u_1, v_2, u_2\}$ is a basis of $V$. The 4-vector $v_1 \wedge u_1 \wedge v_2 \wedge u_2$ is non-zero since it corresponds to a nonzero element in the exterior algebra $\Lambda^4 V$, where $V$ is 4-dimensional.

A point on the line joining $P_1$ and $P_2$ is represented by a 2-vector

$$a = \mu_1 v_1 \wedge u_1 + \mu_2 v_2 \wedge u_2$$

Computing $a \wedge a$,

$$a \wedge a = 2\mu_1 \mu_2 v_1 \wedge u_1 \wedge v_2 \wedge u_2$$

we find that it vanishes only if $\mu_1$ or $\mu_2$ is zero, which cannot be. Thus, the line only intersects the Klein quadric Q at the points $P_1$ and $P_2$.

Therefore, we conclude that the two lines $L_1$ and $L_2$ intersect in the projective space $\mathbb{P}(V)$ iff the line that connects their respective points $P_1$ and $P_2$ in the Klein quadric $Q$ lies entirely within $Q$. □

**Proposition 4.5.** *Given a fixed point X in projective space, the collection of lines L passing through X corresponds to the set of points P in the Klein quadric that lies on a fixed plane contained within the quadric*
*[1][Proposition 20 from 3rd chapter of Nigel Hitchin]*

**Remark 4.** *Within the domain of algebraic geometry, there exists another natural example of a parameter space known as* ***"Grassmannians."*** *A Grassmannian denoted as* $\mathbb{G}(k,n)$*, is defined as the collection of all k-dimensional vector subspaces within an n-dimensional vector space. One can use the idea of Plücker embedding to offer a concrete representation of the Grassmannian as an algebraic variety. Each point in the Grassmannian is mapped to a point in projective space by this embedding. The Klein quadric, precisely* $\mathbb{G}(1,3)$*, is a prime example of a Grassmannian. It captures the parameterization of the projective lines in a projective space of dimension 3. The geometry of lines can be understood by relating* *each line to a point on the Klein quadric and by using Grassmannian's algebraic structure. The relationship between Grassmannians and the Klein quadric demonstrates how closely geometry and algebraic varieties are related. It emphasizes how research into parameter spaces and their embeddings can shed light on projective spaces' geometric features and structures.*

[4][Lecture 6 of Joe Harris book of algebraic geometry]

# Chapter 5

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